Ind-Sheaves

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Introduction

If $\mathcal{C}$ is an abelian category, the category $\mathrm{I}(\mathcal{C})$ of ind-objects of $\mathcal{C}$ has many remarkable properties: it contains $\mathcal{C}$ and admits small inductive limits, it is abelian and the natural functor $\mathcal{C} \longrightarrow \mathrm{I}(\mathcal{C})$ is exact and fully faithful. Moreover, $\mathrm{I}(\mathcal{C})$ is, in a certain sense, "dual" to $\mathcal{C}$.

We introduce here the category of ind-sheaves on a locally compact space as the category of ind-objects of the category of sheaves with compact supports. This construction has some analogy with that of distributions: the space of distributions is bigger than that of functions, and is dual to the space of functions with compact supports. This last condition implies the local nature of distributions, and similarly, we shall prove that ind-sheaves form a stack (a sheaf of categories).

As we shall see, ind-sheaves allows us to treat in the formalism of sheaves (the "six operations" of Grothendieck) functions with growth conditions. For example, on a complex manifold X, we can define the ind-sheaf of "tempered holomorphic functions" $\mathcal{O}_X^t$, or the ind-sheaf of "Whitney holomorphic functions" $\mathcal{O}_X^{\mathrm{w}}$, and obtain for example the sheaves of distributions (or of C^∞-functions) using Sato's construction of hyperfunctions, simply replacing $\mathcal{O}_X$ with $\mathcal{O}_X^t$ (or $\mathcal{O}_X^{\mathrm{w}}$).

The contents of these Notes is as follows.

Chapter I is a short review, without proofs, of the theory of ind-objects with some applications to derived categories. Of course, both theories, ind-objects and derived categories (both invented by Grothendieck), are now classical. However, we shall also recall some technical statements extracted from [10] which are new.

Chapter II is devoted to stacks on a locally compact space X. We introduce the notion of a proper stack, show that this notion is stronger than the usual one of a stack, although its axioms are quite easy to check, and prove that the indization of a proper stack is a proper stack. There are new functors: ι which sends stacks to ind-stacks, and its left inverse, α_X. Under reasonable conditions which will be satisfied by sheaves, α_X also admits a left adjoint β_X.

Aside from the third chapter, we restrict our study to the stack of ind-sheaves over a field k, a situation in which the functor β_X is exact. We introduce first the internal operations: tensor product denoted by $\otimes$, and internal hom denoted by

$\mathcal{I}hom$. Then we define the external operations, inverse image f^{-1}, direct image f_*, proper direct image $f_{!!}$, and study the various relations among all these functors. The proper direct image functor is different from the other functors. The proper direct image of a sheaf is not the same in general whether we calculate it in sheaf theory or ind-sheaf theory.

Chapter IV is the derived version of the previous one. We show how to derive all the functors we have constructed, and the relations among the derived functors. Moreover, similarly to the classical case, the functor $Rf_{!!}$ admits a right adjoint $f^!$, and we study its main properties. One of the difficulties in the study is that we do not know if there are enough injective ind-sheaves. In this chapter, we also introduce the notions of ind-sheaves of rings and modules. This will be necessary for applications. For example, the ind-sheaf $\mathcal{O}_X^t$ of "tempered holomorphic functions" cannot be defined in the derived category $\mathrm{I}(\mathcal{D}_X)$ of ind-sheaves of $\mathcal{D}_X$-modules, and one has to replace $\mathcal{D}_X$ with the ind-sheaf of rings $\beta_X(\mathcal{D}_X)$. As we shall see, this does not cause much trouble.

In Chapter V, we give tools to construct ind-sheaves. Consider a family $\mathcal{T}$ of open subsets of X, with some conditions (for example, the family of relatively compact subanalytic open subsets of a real analytic manifold), and a presheaf F of k-vector spaces on $\mathcal{T}$. We prove that F extends to an ind-sheaf on X as soon as it is left exact, (that is, for any $U, V \in \mathcal{T}$, the sequence $0 \longrightarrow F(U \cup V) \longrightarrow F(U) \oplus F(V) \longrightarrow F(U \cap V)$ is exact). As an application, we construct various ind-sheaves, first on a real analytic manifold, then in the complex case. We mainly concentrate ourselves on the ind-sheaves $\mathcal{O}_X^t$ and $\mathcal{O}_X^{\mathrm{w}}$ of "tempered holomorphic functions" and "Whitney holomorphic functions", respectively. The study of their functoriality (inverse and direct images, product) in the $\mathcal{D}_X$-module framework is postponed to another paper, in preparation.

Chapter 1

Indization of categories: a review

In this chapter, we recall some particular results on indization and localization of categories that we shall need in the sequel. References are made to [6] [2] for the theory of ind-objects, to [8] or [3] for an exposition of homological algebra and derived categories and to [10] for an exposition of the theory of ind-objects and localization as well as for some complementary results.

1.1 Ind-objects

Let $\mathcal{U}$ be a universe (see [11]). A set is called $\mathcal{U}$-small if it is isomorphic to a set belonging to $\mathcal{U}$. Recall that a $\mathcal{U}$-category $\mathcal{C}$ is a category, such that for any $X, Y \in \mathcal{C}$, the set $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ is $\mathcal{U}$-small. If moreover the family of objects (a set in a bigger universe) of $\mathcal{C}$ is $\mathcal{U}$-small, then one says that the category is $\mathcal{U}$-small.

In these notes, we fix a universe $\mathcal{U}$ and we shall not refer to $\mathcal{U}$. We shall often abusively refer to a $\mathcal{U}$-category as a category. A category without the $\mathcal{U}$-small condition is called a big category. We shall often simply denote by **Set** the category of $\mathcal{U}$-small sets.

Definition 1.1.1. Let $\mathcal{C}$ be a category. One sets

$$\mathcal{C}^\vee \quad : \quad \text{the big category of functors from } \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}\,.$$

One shall beware that $\mathcal{C}^\vee$ is not a $\mathcal{U}$-category in general (unless $\mathcal{C}$ is small).

One defines the functor

$$h^\vee \quad : \mathcal{C} \longrightarrow \mathcal{C}^\vee : X \mapsto \mathrm{Hom}_{\mathcal{C}}(\,\cdot\,, X)\,.$$

Then for $G \in \mathcal{C}^\vee$ and $X \in \mathcal{C}$,

$$\mathrm{Hom}_{\mathcal{C}^\vee}(h^\vee(X), G) \simeq G(X). \tag{1.1.1}$$

In particular,
$$\mathrm{Hom}_{\mathcal{C}^\vee}(h^\vee(X), h^\vee(Y)) \simeq \mathrm{Hom}_{\mathcal{C}}(X,Y),$$
and $h^\vee$ is fully faithful. We shall identify $\mathcal{C}$ with a full subcategory of $\mathcal{C}^\vee$ by $h^\vee$. We denote by $\mathcal{C}^\wedge$ the big category $((\mathcal{C}^{\mathrm{op}})^\vee)^{\mathrm{op}}$. Then $\mathcal{C}$ is also embedded into $\mathcal{C}^\wedge$.

The big category $\mathcal{C}^\vee$ admits small inductive limits, but in general, even if $\mathcal{C}$ also admits small inductive limits, the functor $h^\vee$ does not commute with $\varinjlim$. In order to avoid confusion, we denote by "$\varinjlim$" the inductive limit in $\mathcal{C}^\vee$ and by $\varinjlim$ the inductive limit in $\mathcal{C}$. If I is small and $\alpha : I \longrightarrow \mathcal{C}$ is a functor, we set "$\varinjlim$" $\alpha = \varinjlim(h^\vee \circ \alpha)$. In other words, "$\varinjlim$" α is the object of $\mathcal{C}^\vee$ defined by:
$$\text{“}\varinjlim\text{”}\,\alpha \,:\, \mathcal{C} \ni X \mapsto \varinjlim_i \mathrm{Hom}_{\mathcal{C}}(X, \alpha(i))\,.$$
With this convention
$$\varinjlim_i \mathrm{Hom}_{\mathcal{C}}(X, \alpha(i)) \;=\; \mathrm{Hom}_{\mathcal{C}^\vee}(h^\vee(X), \text{“}\varinjlim\text{”}\,\alpha).$$
Similarly we denote by " $\oplus$ " the direct sum in $\mathcal{C}^\vee$.

Definition 1.1.2. Let $\mathcal{C}$ be a category. An ind-object in $\mathcal{C}$ is an object $A \in \mathcal{C}^\vee$ which is isomorphic to "$\varinjlim$" α for some functor $\alpha : I \longrightarrow \mathcal{C}$ with I filtrant and small. One denotes by $\mathrm{Ind}(\mathcal{C})$ the full subcategory of $\mathcal{C}^\vee$ consisting of ind-objects, and calls it the indization of $\mathcal{C}$.

Note that $\mathrm{Ind}(\mathcal{C})$ is a $\mathcal{U}$-category.
For $A \in \mathcal{C}^\vee$, we define the category $\mathcal{C}_A$ and the functor $\alpha_A : \mathcal{C}_A \longrightarrow \mathcal{C}$ by:
$$\begin{aligned}
\mathrm{Ob}(\mathcal{C}_A) &= \{(X,a); X \in \mathcal{C}, a \in A(X)\},\\
\mathrm{Hom}_{\mathcal{C}_A}((X,a),(Y,b)) &= \{f : X \longrightarrow Y; a = b \circ f\},\\
\alpha_A &: (X,a) \mapsto X.
\end{aligned}$$
One proves easily that $A \in \mathrm{Ind}(\mathcal{C})$ if and only if $\mathcal{C}_A$ is filtrant and cofinally small, and $A \simeq$ "$\varinjlim$" α_A in this case.

One extends a functor $F : \mathcal{C} \longrightarrow \mathcal{C}'$ to a functor $IF : \mathrm{Ind}(\mathcal{C}) \longrightarrow \mathrm{Ind}(\mathcal{C}')$ as follows. For $A \in \mathrm{Ind}(\mathcal{C})$ one defines $IF(A) \in \mathrm{Ind}(\mathcal{C}')$ by
$$IF(A) = \text{“}\varinjlim_{X \in \mathcal{C}_A}\text{”}\, F(X).$$

For $B \in \mathrm{Ind}(\mathcal{C})$ and a morphism $f : A \longrightarrow B$ in $\mathrm{Ind}(\mathcal{C})$, f defines a functor $\mathcal{C}_A \longrightarrow \mathcal{C}_B$ $(A(X) \ni a \mapsto f \circ a \in B(X))$. Hence we get a morphism

$$IF(f) : \text{“}\varinjlim_{X \in \mathcal{C}_A}\text{”}\, F(X) \longrightarrow \text{“}\varinjlim_{Y \in \mathcal{C}_B}\text{”}\, F(Y),$$

and one checks that IF is a functor.

When $A \simeq \text{“}\varinjlim\limits_{i}\text{”}\, \alpha(i)$, $B \simeq \text{“}\varinjlim\limits_{j}\text{”}\, \beta(j)$, one has

$$\mathrm{Hom}\,(A, B) \simeq \varprojlim_i \varinjlim_j \mathrm{Hom}\,(\alpha(i), \beta(j))\,,$$

and the map $IF : \mathrm{Hom}\,(A, B) \longrightarrow \mathrm{Hom}\,(IF(A), IF(B))$ is defined by

$$\varprojlim_i \varinjlim_j \mathrm{Hom}_{\mathcal{C}}(\alpha(i), \beta(j)) \longrightarrow \varprojlim_i \varinjlim_j \mathrm{Hom}_{\mathcal{C}'}(F(\alpha(i)), F(\beta(j))).$$

Proposition 1.1.3. *Let $F : \mathcal{C} \longrightarrow \mathcal{C}'$. Then*

(i) *the diagram below commutes*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d] & \mathcal{C}' \arrow[d] \\ \mathrm{Ind}(\mathcal{C}) \arrow[r, "IF"] & \mathrm{Ind}(\mathcal{C}'), \end{tikzcd}$$

(ii) *the functor $IF : \mathrm{Ind}(\mathcal{C}) \longrightarrow \mathrm{Ind}(\mathcal{C}')$ commutes with “$\varinjlim$”,*

(iii) *if F is faithful (resp. fully faithful), so is IF.*

1.2 Indization and localization

Let $\mathcal{C}$ be a category and let $\mathcal{S}$ be a family of morphisms in $\mathcal{C}$.

Definition 1.2.1. A localization of $\mathcal{C}$ by $\mathcal{S}$ is the data of a big category $\mathcal{C}_{\mathcal{S}}$ and a functor $Q : \mathcal{C} \longrightarrow \mathcal{C}_{\mathcal{S}}$ satisfying:

(i) for every $s \in \mathcal{S}$, $Q(s)$ is an isomorphism,

(ii) for any functor $F : \mathcal{C} \longrightarrow \mathcal{C}'$ such that $F(s)$ is an isomorphism for all $s \in \mathcal{S}$, there exist a functor $F_{\mathcal{S}} : \mathcal{C}_{\mathcal{S}} \longrightarrow \mathcal{C}'$ and an isomorphism $F \simeq F_{\mathcal{S}} \circ Q$,

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "Q"'] & \mathcal{C}' \\ \mathcal{C}_{\mathcal{S}} \arrow[ur, dotted, "F_{\mathcal{S}}"'] & \end{tikzcd}$$

(iii) if G_1 and G_2 are two functors from $\mathcal{C}_\mathcal{S}$ to a big category $\mathcal{C}'$, then the natural map $\mathrm{Hom}\,(G_1, G_2) \longrightarrow \mathrm{Hom}\,(G_1 \circ Q, G_2 \circ Q)$ is bijective.

Note that (iii) implies that $\mathcal{C}_\mathcal{S}$ is unique up to equivalence and $F_\mathcal{S}$ in (ii) is unique up to unique isomorphism. It is well-known that if $\mathcal{S}$ is "a multiplicative system", then the localization exists. Using $\mathrm{Ind}(\mathcal{C})$, the localization is constructed as follows.

For any object $X \in \mathcal{C}$ let us define the categories $\mathcal{S}^l_X, \mathcal{S}^r_X$ and the functors α_X, β_X as follows.

$$
\begin{aligned}
\mathrm{Ob}(\mathcal{S}^r_X) &= \{s : X \to X'; s \in \mathcal{S}\} \\
\mathrm{Hom}_{\mathcal{S}^r_X}((s : X \to X'), (s' : X \to X'')) &= \{h : X' \to X''; h \circ s = s'\} \\
\alpha_X : \mathcal{S}^r_X \to \mathcal{C} \quad &\text{is} \quad \alpha_X(X \underset{s}{\to} X') = X', \\
\mathrm{Ob}(\mathcal{S}^l_X) &= \{s : X' \to X; s \in \mathcal{S}\} \\
\mathrm{Hom}_{\mathcal{S}^l_X}((s : X' \to X), (s' : X'' \to X)) &= \{h : X'' \to X'; s \circ h = s'\} \\
\beta_X : (\mathcal{S}^l_X)^{\mathrm{op}} \to \mathcal{C} \quad &\text{is} \quad \beta_X(X' \underset{s}{\to} X) = X'.
\end{aligned}
$$

By the definition, $\mathcal{S}$ is a multiplicative system if and only if the categories $\mathcal{S}^{l,\mathrm{op}}_X (:= (\mathcal{S}^l_X)^{\mathrm{op}})$ and $\mathcal{S}^r_X$ are filtrant and contains $X \xrightarrow{\mathrm{id}_X} X$.

In the sequel, we shall have to consider the inductive limit $\varinjlim \mathrm{Hom}\,(Y, \alpha_X)$ with $X, Y \in \mathcal{C}$. We shall often denote it by

$$\varinjlim_{X \underset{s}{\to} X',\, s \in \mathcal{S}} \mathrm{Hom}\,(Y, X'),$$

and similarly for β_X. If there is no ambiguity, we shall not write "$s \in \mathcal{S}$". One has:

$$
\begin{aligned}
\mathrm{Hom}_{\mathcal{C}_\mathcal{S}}(X, Y) &\simeq \varinjlim \mathrm{Hom}_\mathcal{C}(X, \alpha_Y) = \varinjlim_{Y \underset{t}{\to} Y',\, t \in \mathcal{S}} \mathrm{Hom}_\mathcal{C}(X, Y') \\
&\simeq \varinjlim \mathrm{Hom}_\mathcal{C}(\beta_X, \alpha_Y) = \varinjlim_{X' \underset{s}{\to} X,\, Y \underset{t}{\to} Y',\, s,t \in \mathcal{S}} \mathrm{Hom}_\mathcal{C}(X', Y') \\
&\simeq \varinjlim \mathrm{Hom}_\mathcal{C}(\beta_X, Y) = \varinjlim_{X' \underset{s}{\to} X,\, s \in \mathcal{S}} \mathrm{Hom}_\mathcal{C}(X', Y).
\end{aligned}
$$

One defines the functor

$$\alpha : \mathcal{C} \longrightarrow \mathrm{Ind}(\mathcal{C}) \subset \mathcal{C}^\vee$$

by setting

$$\alpha(X) = \text{“}\varinjlim\text{”}\, \alpha_X.$$

If $f : X \longrightarrow Y$ is a morphism in $\mathcal{C}$, one constructs $\alpha(f) : \alpha(X) \longrightarrow \alpha(Y)$ using the axioms of multiplicative systems, and one obtains a functor $\alpha : \mathcal{C} \longrightarrow \mathrm{Ind}(\mathcal{C})$.

Proposition 1.2.2. (i) *The functor α factorizes through $\mathcal{C}_{\mathcal{S}}$, hence defines a functor $\alpha_{\mathcal{S}} : \mathcal{C}_{\mathcal{S}} \longrightarrow \mathrm{Ind}(\mathcal{C})$.*

(ii) *The functor $\alpha_{\mathcal{S}}$ is fully faithful.*

One shall beware that the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "Q"] \arrow[dr, "h^{\vee}"'] & \mathcal{C}_{\mathcal{S}} \arrow[d, "\alpha_{\mathcal{S}}"] \\ & \mathrm{Ind}(\mathcal{C}) \end{tikzcd}$$

is *not* commutative in general. However, there is a natural morphism:

$$h^{\vee} \longrightarrow \alpha = \alpha_{\mathcal{S}} \circ Q,\ X \longrightarrow \text{“}\varinjlim\text{”}\, \alpha_X. \tag{1.2.1}$$

Localization of functors

Let $\mathcal{C}$ be a category, $\mathcal{S}$ a multiplicative system and $F : \mathcal{C} \longrightarrow \mathcal{C}'$ a functor. In general, F does not send morphisms in $\mathcal{S}$ to isomorphisms in $\mathcal{C}'$, hence, does not factorizes through $\mathcal{C}_{\mathcal{S}}$. It is however possible in some cases to define a localization of F as follows.

Definition 1.2.3. (i) A right localization of F (if it exists) is a functor $F_{\mathcal{S}} : \mathcal{C}_{\mathcal{S}} \longrightarrow \mathcal{C}'$ and a morphism of functors $\tau : F \longrightarrow F_{\mathcal{S}} \circ Q$ such that, for any morphism $G : \mathcal{C}_{\mathcal{S}} \longrightarrow \mathcal{C}'$, the map $\mathrm{Hom}\,(F_{\mathcal{S}}, G) \longrightarrow \mathrm{Hom}\,(F, G \circ Q)$ is bijective. We say that F is right localizable if it admits a right localization.

(ii) We say that F is universally right localizable, if for any functor $K : \mathcal{C}' \longrightarrow \mathcal{C}''$, the functor $K \circ F$ is right localizable and moreover $(K \circ F)_{\mathcal{S}} \xrightarrow{\sim} K \circ F_{\mathcal{S}}$.

Proposition 1.2.4. *Let $\mathcal{C}$ be a category, $\mathcal{I}$ a full subcategory, $\mathcal{S}$ a multiplicative system in $\mathcal{C}$, and $\mathcal{T}$ the family of morphisms in $\mathcal{I}$ which belong to $\mathcal{S}$. Let $F : \mathcal{C} \longrightarrow \mathcal{C}'$ be a functor. Assume that*

(i) *$\mathcal{T}$ is a multiplicative system in $\mathcal{I}$,*

(ii) *for any $X \in \mathcal{C}$ there exists $s : X \longrightarrow W$ with $W \in \mathcal{I}$ and $s \in \mathcal{S}$,*

(iii) *for any $t \in \mathcal{T}$, $F(t)$ is an isomorphism.*

Then F is universally right localizable.

Indeed, the restriction of F to $\mathcal{I}$ is localizable, and the natural functor $\mathcal{I}_{\mathcal{T}} \longrightarrow \mathcal{C}_{\mathcal{S}}$ is an equivalence. This is visualized by the diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "Q"] & & \mathcal{C}_{\mathcal{S}} \arrow[dd, dotted, "F_{\mathcal{S}}"] \\
\mathcal{I} \arrow[u, "\iota"] \arrow[r, "Q"] \arrow[drr, "F\circ\iota"'] & \mathcal{I}_{\mathcal{T}} \arrow[ur, "\sim", "\iota_Q"'] \arrow[dr, "F_{\mathcal{T}}"] & \\
& & \mathcal{C}'
\end{tikzcd}$$

Definition 1.2.5. Let $X \in \mathcal{C}$. One says that F is right localizable at X if "$\varinjlim$"$(F \circ \alpha_X)$ is representable in $\mathcal{C}'$.

Recall that

$$\text{“}\varinjlim\text{”}(F \circ \alpha_X) = \underset{X \to X'}{\text{“}\varinjlim\text{”}} F(X'),\ (X \to X' \in \mathcal{S}).$$

Proposition 1.2.6. *Let $F : \mathcal{C} \longrightarrow \mathcal{C}'$ be a functor and $\mathcal{S}$ a multiplicative system in $\mathcal{C}$. The two conditions below are equivalent:*

(i) *F is right localizable at each $X \in \mathcal{C}$,*

(ii) *F is universally right localizable.*

1.3 Indization of abelian categories

From now on, $\mathcal{C}$ is an abelian category. One denotes by $\mathcal{C}^{\vee,add}$ the big category of additive functors from $\mathcal{C}^{\mathrm{op}}$ to $\mathrm{Mod}(\mathbb{Z})$. This big category is clearly abelian. One denotes by $\mathcal{C}^{\vee,add,l}$ the full big subcategory consisting of left exact functors. The functor $h^{\vee} : \mathcal{C} \longrightarrow \mathcal{C}^{\vee}$ makes $\mathcal{C}$ a full abelian subcategory of $\mathcal{C}^{\vee,add}$ and this functor is left exact, but not exact.

As in §1.1, an ind-object in $\mathcal{C}$ is an object $A \in \mathcal{C}^{\vee}$ which is isomorphic to "$\varinjlim$" α for some functor $\alpha : I \longrightarrow \mathcal{C}$ with I filtrant and small. Hence, $\mathrm{Ind}(\mathcal{C})$ is a full additive subcategory of $\mathcal{C}^{\vee,add,l}$. Recall that it is a $\mathcal{U}$-category. If $\mathcal{C}$ is small, then $\mathrm{Ind}(\mathcal{C}) \simeq \mathcal{C}^{\vee,add,l}$.

The category $\mathrm{Ind}(\mathcal{C})$ admits kernels and cokernels. Indeed, if $f : A \longrightarrow B$ is a morphism in $\mathrm{Ind}(\mathcal{C})$, one may construct a small filtrant category I, $\alpha, \beta : I \longrightarrow \mathcal{C}$ two functors, and $\varphi : \alpha \longrightarrow \beta$ a morphism of functors such that $A \simeq$ "$\varinjlim$" α, $B \simeq$ "$\varinjlim$" β and $f =$ "$\varinjlim$" φ. Then "$\varinjlim$" $\ker\varphi$ and "$\varinjlim$" $\operatorname{coker}\varphi$ will be a kernel and a cokernel of f, respectively.

Theorem 1.3.1 ([10, Proposition 3.3.4]). (i) *The category* $\mathrm{Ind}(\mathcal{C})$ *is abelian.*

(ii) *The natural functor* $\mathcal{C} \longrightarrow \mathrm{Ind}(\mathcal{C})$ *is fully faithful and exact, and the natural functor* $\mathrm{Ind}(\mathcal{C}) \longrightarrow \mathcal{C}^{\vee,add}$ *is fully faithful and left exact.*

(iii) *The category* $\mathrm{Ind}(\mathcal{C})$ *admits exact small filtrant inductive limits.*

(iv) *Assume that for any family* $\{X_i\}_{i\in I}$ *of objects of* $\mathcal{C}$ *indexed by a small set* I*, the product* $\prod_i X_i$ *(which is well-defined in* $\mathcal{C}^\vee$*) belongs to* $\mathrm{Ind}(\mathcal{C})$*. Then the category* $\mathrm{Ind}(\mathcal{C})$ *admits small projective limits, and the functor* $\varprojlim$ *is left exact.*

As a consequence of the preceding results, one gets that if $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ is an exact sequence in $\mathrm{Ind}(\mathcal{C})$, then one may construct a filtrant and small category I and an exact sequence of functors from I to $\mathcal{C}$, $0 \longrightarrow \alpha' \longrightarrow \alpha \longrightarrow \alpha'' \longrightarrow 0$ such that $0 \longrightarrow$ "$\varinjlim$" $\alpha' \longrightarrow$ "$\varinjlim$" $\alpha \longrightarrow$ "$\varinjlim$" $\alpha'' \longrightarrow 0$ is isomorphic to $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$.

This immediately implies that if $F : \mathcal{C} \longrightarrow \mathcal{C}'$ is an additive functor of abelian categories and $IF : \mathrm{Ind}(\mathcal{C}) \longrightarrow \mathrm{Ind}(\mathcal{C}')$ is the associated functor, then IF is left (resp. right) exact as soon as F is left (resp. right) exact.

Proposition 1.3.2. *Let* $f : A \longrightarrow B$ *be a morphism in* $\mathrm{Ind}(\mathcal{C})$*. The two conditions below are equivalent.*

(i) f *is an epimorphism,*

(ii) *for any diagram in* $\mathrm{Ind}(\mathcal{C})$ *with* $Y \in \mathcal{C}$

$$\begin{tikzcd} X \arrow[r, dotted, "g"] \arrow[d, dotted] & Y \arrow[d] \\ A \arrow[r, "f"] & B, \end{tikzcd}$$

the dotted arrows may be completed to a commutative diagram with $X \in \mathcal{C}$ *and* g *an epimorphism.*

Construction of ind-objects

Let $\mathcal{C}$ be an abelian category and $\mathcal{J}$ a full additive subcategory.

Definition 1.3.3. We say that $\mathcal{J}$ is generating (resp. cogenerating) in $\mathcal{C}$ if for any $X \in \mathcal{C}$ there exists an epimorphism $Y \twoheadrightarrow X$ (resp. a monomorphism $X \rightarrowtail Y$) with $Y \in \mathcal{J}$.

We denote by $\mathcal{J}^{\vee,add,l}$ the big category of additive functors $F : \mathcal{J}^{\mathrm{op}} \longrightarrow \mathrm{Mod}(\mathbb{Z})$ with the property that for any exact sequence $Y' \longrightarrow Y \longrightarrow Y'' \longrightarrow 0$ in $\mathcal{C}$ with Y', Y, Y'' in $\mathcal{J}$, the sequence $0 \longrightarrow F(Y'') \longrightarrow F(Y) \longrightarrow F(Y')$ is exact. There is a natural forgetful functor

$$for : \mathcal{C}^{\vee,add,l} \longrightarrow \mathcal{J}^{\vee,add,l}. \tag{1.3.1}$$

Theorem 1.3.4. *Let $\mathcal{C}$ be an abelian category and $\mathcal{J}$ a full additive subcategory. Assume that $\mathcal{J}$ is generating in $\mathcal{C}$. Then the forgetful functor for in* (1.3.1) *is an equivalence of categories. In particular, if $A : \mathcal{J} \longrightarrow \mathrm{Mod}(\mathbb{Z})$ is a left exact functor, there exists a unique* (*up to isomorphism*) *left exact functor $A^+ : \mathcal{C} \longrightarrow \mathrm{Mod}(\mathbb{Z})$ such that $A^+(Y) \simeq A(Y)$ for all $Y \in \mathcal{J}$.*

1.4 Derived categories

In this subsection, $\mathcal{C}$, $\mathcal{C}'$, etc. are abelian categories.

One denote by $C(\mathcal{C})$ the abelian category of complexes in $\mathcal{C}$. By regarding morphisms in $C(\mathcal{C})$ which are homotopic to 0 as the zero morphism, one obtains the triangulated category $K(\mathcal{C})$, whose distinguished triangles are those isomorphic to $X \underset{f}{\longrightarrow} Y \longrightarrow M(f) \underset{+1}{\longrightarrow}$, where $M(f)$ denotes the mapping cone of the morphism f in $C(\mathcal{C})$.

A morphism $f : X \longrightarrow Y$ in $K(\mathcal{C})$ (or in $C(\mathcal{C})$) is called a quasi-isomorphism (a qis for short) if it induces an isomorphism $H^j(f) : H^j(X) \longrightarrow H^j(Y)$ for all $j \in \mathbb{Z}$.

The derived category $D(\mathcal{C})$ is the localization of $K(\mathcal{C})$ by the multiplicative system of quasi-isomorphisms. It is naturally a triangulated category. It is a big category in general.

One denotes as usual by $D^*(\mathcal{C})$ ($* = +, -$ or b) the full triangulated category of $D(\mathcal{C})$ consisting of complexes bounded from below, from above, or bounded.

Now consider a left exact functor $F : \mathcal{C} \longrightarrow \mathcal{C}'$. It defines a functor $K(F) : K(\mathcal{C}) \longrightarrow K(\mathcal{C}')$. We shall often write F instead of $K(F)$ for short.

Definition 1.4.1. Let $\mathcal{J}$ be a full additive subcategory of $\mathcal{C}$. One says that $\mathcal{J}$ is F-injective if:

(i) the category $\mathcal{J}$ is cogenerating in $\mathcal{C}$,

(ii) for any $X \in K^+(\mathcal{J})$ such that X is qis to 0, $F(X)$ is qis to 0.

By considering $\mathcal{C}^{\mathrm{op}}$, one obtains the notion of an F-projective subcategory, when F is right exact.

Proposition 1.4.2. *Let $\mathcal{J}$ be an F-injective subcategory of $\mathcal{C}$. Then F is right derivable, i.e. $F : K^+(\mathcal{C}) \longrightarrow D^+(\mathcal{C}')$ is universally right localizable with respect to the multiplicative system of quasi-isomorphisms. In particular F admits a right derived functor $RF : D^+(\mathcal{C}) \longrightarrow D^+(\mathcal{C}')$. Moreover we have $R^kF(X) = 0$ for any $X \in \mathcal{J}$ and $k \neq 0$.*

Recall that if $X \in K^+(\mathcal{C})$, then

$$RF(X) = \text{“}\varinjlim_{X \xrightarrow[qis]{} X'}\text{”} \; F(X')$$

in $\mathrm{Ind}(D^+(\mathcal{C}'))$.

There is a useful tool to check that a subcategory is F-injective.

Theorem 1.4.3 ([10]). *Let $\mathcal{J}$ be a full additive subcategory of $\mathcal{C}$. Assume:*

(i) *the category $\mathcal{J}$ is cogenerating in $\mathcal{C}$,*

(ii) *for any monomorphism $Y' \rightarrowtail X$ with $Y' \in \mathcal{J}$ there exists an exact sequence $0 \longrightarrow Y' \longrightarrow Y \longrightarrow Y'' \longrightarrow 0$ with Y, Y'' in $\mathcal{J}$ such that $Y' \longrightarrow Y$ factorizes through $Y' \longrightarrow X$ and such that the sequence $0 \longrightarrow F(Y') \longrightarrow F(Y) \longrightarrow F(Y'') \longrightarrow 0$ is exact.*

Then the category $\mathcal{J}$ is F-injective.

As a corollary, we recover a classical result:

Corollary 1.4.4. *Let $\mathcal{J}$ be a full additive subcategory of $\mathcal{C}$. Assume:*

(i) *the category $\mathcal{J}$ is cogenerating in $\mathcal{C}$,*

(ii) *for any exact sequence $0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$ in $\mathcal{C}$, if $X', X \in \mathcal{J}$, then $X'' \in \mathcal{J}$,*

(iii) *for any exact sequence $0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$ in $\mathcal{C}$ with $X', X \in \mathcal{J}$, the sequence $0 \longrightarrow F(X') \longrightarrow F(X) \longrightarrow F(X'') \longrightarrow 0$ is exact.*

Then the category $\mathcal{J}$ is F-injective.

We shall also have to derive bifunctors. Consider three abelian categories $\mathcal{C}$, $\mathcal{C}'$, $\mathcal{C}''$ and an additive bifunctor $F : \mathcal{C} \times \mathcal{C}' \longrightarrow \mathcal{C}''$. We shall assume that F is left exact with respect to each of its arguments.

Definition 1.4.5. Let $\mathcal{J}$ and $\mathcal{J}'$ be full additive subcategories of $\mathcal{C}$ and $\mathcal{C}'$ respectively. One says that $\mathcal{J} \times \mathcal{J}'$ is F-injective if

(i) $\mathcal{J}$ and $\mathcal{J}'$ are cogenerating,

(ii) for any $Y \in \mathcal{J}$, $\mathcal{J}'$ is $F(Y, \cdot\,)$-injective,

(iii) for any $Y' \in \mathcal{J}'$, $\mathcal{J}$ is $F(\,\cdot\,, Y')$-injective.

Proposition 1.4.6. *Assume that there exists $\mathcal{J}$ and $\mathcal{J}'$ such that $\mathcal{J} \times \mathcal{J}'$ is F-injective. Then F is right derivable and for $(X, X') \in K^+(\mathcal{C}) \times K^+(\mathcal{C}')$ one has:*

$$RF(X, X') \simeq \underset{X \xrightarrow[qis]{} Y, X' \xrightarrow[qis]{} Y'}{\text{“}\varinjlim\text{”}} F(Y, Y').$$

1.5 Indization and derivation

We shall study the derived category $D(\mathrm{Ind}(\mathcal{C}))$ of the category $\mathrm{Ind}(\mathcal{C})$ associated with an abelian category $\mathcal{C}$. In such a study, the universe $\mathcal{U}$ plays an important role. Even if $\mathcal{C}$ has enough injectives, we do not know whether $\mathrm{Ind}(\mathcal{C})$ has enough injectives. Instead, we shall use the following notion of quasi-injectives. Recall that unless otherwise specified, when we consider a category $\mathcal{C}$, it is a $\mathcal{U}$-category. We shall say that a category is essentially small if it is equivalent to a small category.

Definition 1.5.1. Let $A \in \mathrm{Ind}(\mathcal{C})$. We say that A is quasi-injective if the functor

$$\begin{array}{ccl} \mathcal{C} & \to & \mathrm{Mod}(\mathbb{Z}), \\ X & \mapsto & A(X) \ (= \mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(X, A)) \end{array}$$

is exact.

Assuming that $\mathcal{C}$ has enough injectives, one proves easily that A is quasi-injective if and only if there exist a small filtrant category I and $\alpha : I \to \mathcal{C}$ such that $A \simeq \text{“}\varinjlim\text{”}\, \alpha$ and $\alpha(i)$ is injective in $\mathcal{C}$ for all $i \in I$.

Definition 1.5.2. Let $\mathcal{C}$ be an abelian category. A system of strict $\mathcal{U}$-generators in $\mathcal{C}$ is a family $\{G_a; a \in A\}$ of objects of $\mathcal{C}$ such that A is $\mathcal{U}$-small and:

(i) for all $X \in \mathcal{C}$ and all $a \in A$, the object $G_a^{\oplus \mathrm{Hom}\,(G_a, X)}$ exists,

(ii) for all $X \in \mathcal{C}$, there exists $a \in A$, such that the morphism $G_a^{\oplus \mathrm{Hom}\,(G_a, X)} \to X$ is an epimorphism.

A system of strict $\mathcal{U}$-generators is a system of $\mathcal{U}$-generators in the sense of Grothendieck ([4]).

Theorem 1.5.3 ([10, Theorem 6.5.8]). *Let $\mathcal{C}$ be an abelian category with a system of strict generators and let $S \subset \mathrm{Ob}(\mathrm{Ind}(\mathcal{C}))$ be a small subset.*

(a) *There exists an essentially small full abelian subcategory $\mathcal{C}_0$ of $\mathcal{C}$ such that $S \subset \mathrm{Ob}(\mathrm{Ind}(\mathcal{C}_0))$ with the properties:*

  (i) *$\mathcal{C}_0$ is stable under sub-object, quotient and extension in $\mathcal{C}$,*

  (ii) *for any epimorphism $X \twoheadrightarrow Y''$ with $Y'' \in \mathcal{C}_0$, there exists a morphism $Y' \longrightarrow X$ with $Y' \in \mathcal{C}_0$ such that the composition $Y' \longrightarrow Y''$ is an epimorphism,*

  (iii) *$\mathrm{Ind}(\mathcal{C}_0)$ is stable by sub-object, quotient and extension in $\mathrm{Ind}(\mathcal{C})$,*

  (iv) *for any epimorphism $X \twoheadrightarrow Y''$ with $Y'' \in \mathrm{Ind}(\mathcal{C}_0)$, there exists a morphism $Y' \longrightarrow X$ with $Y' \in \mathrm{Ind}(\mathcal{C}_0)$ such that the composition $Y' \longrightarrow Y''$ is an epimorphism,*

  (v) *$\mathrm{Ind}(\mathcal{C}_0)$ has enough injectives.*

(b) *Assume moreover that $\mathcal{C}$ has enough injectives. Then we may choose $\mathcal{C}_0$ having the above properties and such that the injective objects of $\mathrm{Ind}(\mathcal{C}_0)$ are quasi-injective in $\mathrm{Ind}(\mathcal{C})$.*

Corollary 1.5.4. *Assume that $\mathcal{C}$ has enough injectives and a system of strict generators. Then $\mathrm{Ind}(\mathcal{C})$ admits enough quasi-injectives.*

We denote by $\mathcal{I}_q$ the category of quasi-injective objects in $\mathrm{Ind}(\mathcal{C})$.

As above, denote by $F : \mathcal{C} \longrightarrow \mathcal{C}'$ a left exact functor, and by $IF : \mathrm{Ind}(\mathcal{C}) \longrightarrow \mathrm{Ind}(\mathcal{C}')$ the associated left exact functor.

Theorem 1.5.5. *Assume that $\mathcal{C}$ has enough injectives and a system of strict generators.*

(i) *The category $\mathcal{I}_q$ is IF-injective.*

(ii) *The diagram below commutes :*

$$\begin{tikzcd} D^+(\mathcal{C}) \arrow[r, "RF"] \arrow[d] & D^+(\mathcal{C}') \arrow[d] \\ D^+(\mathrm{Ind}(\mathcal{C})) \arrow[r, "RIF"] & D^+(\mathrm{Ind}(\mathcal{C}')). \end{tikzcd}$$

(iii) *The functor $R^kIF : \mathrm{Ind}(\mathcal{C}) \longrightarrow \mathrm{Ind}(\mathcal{C}')$ commutes with "$\varinjlim$". In other words, if I is small and filtrant and $\alpha : I \longrightarrow \mathrm{Ind}(\mathcal{C})$, then*

$$R^kIF(\text{“}\varinjlim\text{”}\,\alpha) \simeq \text{“}\varinjlim\text{”}(R^kIF \circ \alpha).$$

Chapter 2

Stacks

In this whole chapter, we work in a fixed universe $\mathcal{U}$. Hence, otherwise stated, a category means a $\mathcal{U}$-category and all sets are $\mathcal{U}$-small. In this chapter, X is a topological space and k is a commutative ring.

In all these notes, a ring means an associative unitary ring, and the action of a ring on a module is unitary. For a commutative ring k, a k-ring is a ring A endowed with a morphism $k \longrightarrow A$ of rings. A k-ring A is called a k-algebra if the image of k is contained in the center of A.

2.1 Definition of stacks

Definition 2.1.1. A prestack $\mathcal{C}$ on a topological space X is the data of:

(i) a category $\mathcal{C}(U)$ for each open subset U of X,

(ii) for each open inclusion $V \subset U$, a functor (sometimes called the restriction functor) $\rho_{VU} : \mathcal{C}(U) \longrightarrow \mathcal{C}(V)$,

(iii) for each open inclusions $W \subset V \subset U$, an isomorphism of functors $\lambda_{WVU} : \rho_{WV} \circ \rho_{VU} \xrightarrow{\sim} \rho_{WU}$,

these data satisfying:

(i) $\rho_{UU} = \mathrm{id}_{\mathcal{C}(U)}$,

(ii) for each open inclusions $U_1 \subset U_2 \subset U_3 \subset U_4$, the diagram below is commutative (in this diagram, we shall write ρ_{ij} instead of $\rho_{U_iU_j}$ and λ_{ijk} instead of $\lambda_{U_iU_jU_k}$):

$$\begin{tikzcd}
\rho_{12} \circ \rho_{23} \circ \rho_{34} \arrow[r, "\lambda_{234}"] \arrow[d, "\lambda_{123}"] & \rho_{12} \circ \rho_{24} \arrow[d, "\lambda_{124}"] \\
\rho_{13} \circ \rho_{34} \arrow[r, "\lambda_{134}"] & \rho_{14}.
\end{tikzcd}$$

It follows from the axioms that $\lambda_{VVU} = \mathrm{id}$ and $\lambda_{VUU} = \mathrm{id}$ for $V \subset U$.

A prestack of k-additive (resp. k-abelian) categories is a prestack such that for each U, $\mathcal{C}(U)$ is k-additive (resp. k-abelian) and the functors ρ_{VU} are k-linear.

Notation 2.1.2. (i) For $F, G \in \mathcal{C}(X)$ one denotes by $\mathcal{H}om_{\mathcal{C}}(F, G)$ the presheaf of sets on X: $U \mapsto \mathrm{Hom}_{\mathcal{C}(U)}(\rho_{UX} F, \rho_{UX} G)$ (U open). One sets $\mathcal{E}nd_{\mathcal{C}}(F) = \mathcal{H}om_{\mathcal{C}}(F, F)$.

(ii) For $F \in \mathcal{C}(U)$ one often writes $F|_V$ instead of $\rho_{VU}(F)$ for short, and one calls it the restriction of F to V. Hence, for $W \subset V \subset U$, λ_{WVU} defines an isomorphism $\lambda_{WVU} : (F|_V)|_W \xrightarrow{\sim} F|_W$.

(iii) If $\mathcal{C}$ is a prestack on X, one defines in an obvious way the prestack $\mathcal{C}|_U$ on U, its restriction to U.

(iv) If $\{U_i; i \in I\}$ is a family of open subsets, one writes $U_{ij} = U_i \cap U_j, U_{ijk} = U_i \cap U_j \cap U_k$, etc.

Definition 2.1.3. We say that a prestack $\mathcal{C}$ satisfies the axiom ST1, if for any open subset U of X and any $F, G \in \mathcal{C}(U)$, the presheaf $\mathcal{H}om_{\mathcal{C}|_U}(F, G)$ is a sheaf on U.

If $\mathcal{C}$ is a prestack of additive categories satisfying ST1 and $F \in \mathcal{C}(U)$, one defines the support of F as the complementary set in U of the union of all open subsets $V \subset U$ such that $F|_V \simeq 0$. This coincides with the support of $\mathrm{id}_F \in \Gamma(X; \mathcal{E}nd_{\mathcal{C}}(F))$. Note that if $V = \bigcup_{i \in I} V_i$ and $F|_{V_i} \simeq 0$ for all i, then $F|_V \simeq 0$.

Definition 2.1.4. We say that a prestack $\mathcal{C}$ satisfies the axiom ST2, if for any open subset $U \subset X$, any open covering $U = \bigcup_{i \in I} U_i$, any family $F_i \in \mathcal{C}(U_i)$, any family of isomorphisms $\theta_{ji} : F_i|_{U_{ji}} \xrightarrow{\sim} F_j|_{U_{ji}}$ such that:

$$\theta_{ij}|_{U_{ijk}} \circ \theta_{jk}|_{U_{ijk}} = \theta_{ik}|_{U_{ijk}}, \tag{2.1.1}$$

there exist $F \in \mathcal{C}(U)$ and isomorphisms $\theta_i : F|_{U_i} \xrightarrow{\sim} F_i$ such that

$$\theta_{ij} \circ (\theta_j|_{U_{ij}}) = \theta_i|_{U_{ij}}. \tag{2.1.2}$$

More precisely, (2.1.1) means that the diagram below (in which we do not write explicitly the morphisms λ_{ijk}) commutes:

$$\begin{tikzcd}
F_k|_{U_{jk}}|_{U_{ijk}} \arrow[r, "\theta_{jk}"] \arrow[d, "\sim"] & F_j|_{U_{jk}}|_{U_{ijk}} \arrow[r, "\sim"] & F_j|_{U_{ijk}} \\
F_k|_{U_{ijk}} & & F_j|_{U_{ij}}|_{U_{ijk}} \arrow[u, "\sim"'] \arrow[d, "\theta_{ij}"] \\
F_k|_{U_{ik}}|_{U_{ijk}} \arrow[u, "\sim"'] \arrow[d, "\theta_{ik}"] & & F_i|_{U_{ij}}|_{U_{ijk}} \arrow[d, "\sim"'] \\
F_i|_{U_{ik}}|_{U_{ijk}} \arrow[rr, "\sim"] & & F_i|_{U_{ijk}}
\end{tikzcd}$$

and the equation (2.1.2) means that the diagram below commutes

$$\begin{tikzcd}
F|_{U_j}|_{U_{ij}} \arrow[r, "\sim"] \arrow[d, "\theta_j"] & F|_{U_{ij}} & F|_{U_i}|_{U_{ij}} \arrow[l, "\sim"'] \arrow[d, "\theta_i"] \\
F_j|_{U_{ij}} \arrow[rr, "\theta_{ij}"] & & F_i|_{U_{ij}}.
\end{tikzcd}$$

Definition 2.1.5. A stack is a prestack which satisfies the axioms ST1 and ST2.

Note that if $\mathcal{C}$ is a stack and if F is defined as in ST2, then F is unique up to isomorphism. Indeed, if (F', θ'_i) is another candidate, the isomorphisms $\alpha_i : \theta_i'^{-1} \circ \theta_i : F|_{U_i} \xrightarrow{\sim} F'|_{U_i}$ will glue as an isomorphism $\alpha : F \xrightarrow{\sim} F'$ by ST1.

Note also that for a stack of additive categories, $\mathcal{C}(\varnothing)$ is equivalent to 0, the category consisting of the zero object.

Definition 2.1.6. Let $\mathcal{C}$ and $\mathcal{C}'$ be two prestacks on X. We shall denote by ρ_{VU}, λ_{WVU} (resp. ρ'_{VU}, λ'_{WVU}) the associated functors and morphisms of functors on $\mathcal{C}$ and $\mathcal{C}'$, respectively. A functor of prestacks $\varphi : \mathcal{C} \longrightarrow \mathcal{C}'$ is the data of:

(i) for each open subset U, a functor $\varphi_U : \mathcal{C}(U) \longrightarrow \mathcal{C}'(U)$,

(ii) for each open inclusion $V \subset U$, an isomorphism of functors $\theta_{VU} : \varphi_V \circ \rho_{VU} \xrightarrow{\sim} \rho'_{VU} \circ \varphi_U$,

such that for each open inclusions $W \subset V \subset U$, the diagram below commutes:

$$\begin{tikzcd}
\varphi_W \circ \rho_{WV} \circ \rho_{VU} \arrow[r, "\lambda_{WVU}"] \arrow[d, "\theta_{WV}"'] & \varphi_W \circ \rho_{WU} \arrow[dd, "\theta_{WU}"'] \\
\rho'_{WV} \circ \varphi_V \circ \rho_{VU} \arrow[d, "\theta_{VU}"'] & \\
\rho'_{WV} \circ \rho'_{VU} \circ \varphi_U \arrow[r, "\lambda'_{WVU}"] & \rho'_{WU} \circ \varphi_U .
\end{tikzcd}$$

A functor of stacks is a functor of the underlying prestacks. One defines naturally the notion of a functor of k-additive or k-abelian stacks.

Definition 2.1.7. Let $\mathcal{C}$ and $\mathcal{C}'$ be two prestacks on X. We shall use the same notations ρ_{VU} and ρ'_{VU} as in the preceding Definition 2.1.6. Let $\varphi : \mathcal{C} \longrightarrow \mathcal{C}'$ and $\varphi' : \mathcal{C} \longrightarrow \mathcal{C}'$ be two functors of prestacks. We denote by θ_{VU} (resp. θ'_{VU}) the associated morphism to φ (resp. φ'). A morphism $f : \varphi \longrightarrow \varphi'$ of functors of prestacks is the data which associate, to each open subset U, a morphism $f_U : \varphi_U \longrightarrow \varphi'_U$ of functors from $\mathcal{C}(U)$ to $\mathcal{C}'(U)$, such that for each open inclusion $V \subset U$ and $X \in \mathcal{C}(U)$, the following diagram commutes:

$$\begin{tikzcd}
\varphi_V(\rho_{VU}X) \arrow[r, "f_V(\rho_{VU}X)"] \arrow[d, "\theta_{VU}(X)"'] & \varphi'_V(\rho_{VU}X) \arrow[d, "\theta'_{VU}(X)"] \\
\rho'_{VU}(\varphi_U X) \arrow[r, "\rho'_{VU}(f_U X)"] & \rho'_{VU}(\varphi'_U X) .
\end{tikzcd}$$

Hence, one has the notion of equivalence of stacks.

Examples 2.1.8. (i) If $\mathcal{C}$ is a stack on X, then $\mathcal{C}|_U$ is a stack on U.

(ii) Let $\mathcal{A}$ be a sheaf of unitary k-rings on X. Then $U \mapsto \mathrm{Mod}(\mathcal{A}|_U)$ is a stack of k-abelian categories on X.

2.2 Proper stacks

From now on, we assume that the topological space X is locally compact (in particular X is Hausdorff) and countable at infinity. In this section, $\mathcal{C}$ is a prestack of abelian categories on X.

Notation 2.2.1. For an open subset U of X, we denote by i_U the open embedding $U \hookrightarrow X$. We often write i_U^{-1} instead of ρ_{UX} to denote the restriction functor $\mathcal{C}(X) \longrightarrow \mathcal{C}(U)$. Hence, for $F \in \mathcal{C}(X)$, we have three notations:

$$\rho_{UX}F = F|_U = i_U^{-1}F.$$

Definition 2.2.2. A proper stack $\mathcal{C}$ on X is a prestack $\mathcal{C}$ of abelian categories satisfying the following axioms:

(i) $\mathcal{C}$ satisfies ST1,

(ii) for all open subsets $V \subset U \subset X$, the functor ρ_{VU} is exact,

(iii) for all open subset $U \subset X$, $\mathcal{C}(U)$ admits small filtrant inductive limits, and the functor $\varinjlim$ is exact over such limits and commutes with ρ_{VU},

(iv) for all open subset $U \subset X$, $\mathcal{C}(U)$ admits small filtrant projective limits, and the functor $\varprojlim$ commutes with ρ_{VU},

(v) for all open subset $U \subset X$, the functor i_U^{-1} admits a left adjoint, and denoting this functor by $i_{U!}$, it satisfies:

(a) $\mathrm{id}_{\mathcal{C}(U)} \xrightarrow{\sim} i_U^{-1} \circ i_{U!}$,

(b) for all $F \in \mathcal{C}(U)$, $\operatorname{supp} i_{U!}F \subset \overline{U}$.

We shall prove later that a proper stack is actually a stack.

Lemma 2.2.3. *Consider a stack $\mathcal{C}$ which satisfies all hypotheses of* Definition 2.2.2, *except* (v)(b). *Then $\mathcal{C}$ is a proper stack (i.e. the hypothesis* (v)(b) *is a consequence of the other axioms).*

Proof. Let $W = X \setminus \overline{U}$ and $V = U \cup W$. Define $G \in \mathcal{C}(V)$ by $G|_U = F$, $G|_W = 0$. The object G is well-defined in view of ST2 since $U \cap W = \varnothing$. For any $K \in \mathcal{C}(X)$ one has

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}(X)}(i_{V!}G, K) &\simeq \mathrm{Hom}_{\mathcal{C}(V)}(G, i_V^{-1}K) \simeq \Gamma(V\,;\mathcal{H}om_{\mathcal{C}|_V}(G, i_V^{-1}K)) \\ &\simeq \Gamma(U\,;\mathcal{H}om_{\mathcal{C}|_V}(G, i_V^{-1}K)) \simeq \mathrm{Hom}_{\mathcal{C}(U)}(G|_U, i_U^{-1}K) \\ &\simeq \mathrm{Hom}_{\mathcal{C}(U)}(F, i_U^{-1}K) \simeq \mathrm{Hom}_{\mathcal{C}(X)}(i_{U!}F, K). \end{aligned}$$

Hence, $i_{U!}F \simeq i_{V!}G$, and we get $G \xrightarrow{\sim} i_V^{-1}i_{V!}G \simeq i_V^{-1}i_{U!}F$. Since $G|_W = 0$, we obtain $i_W^{-1}i_{U!}F = 0$. q.e.d.

In the sequel let $\mathcal{C}$ be a proper stack.

Lemma 2.2.4. *For any small filtrant inductive system $\{F_i\}_i$ in $\mathcal{C}(X)$ and for any small filtrant projective system $\{G_j\}_j$ in $\mathcal{C}(X)$, we have*

$$\mathcal{H}om_{\mathcal{C}}(\varinjlim_i F_i, \varprojlim_j G_j) \simeq \varprojlim_{i,j} \mathcal{H}om_{\mathcal{C}}(F_i, G_j).$$

Proof. This immediately follows from the fact that inductive and projective limits commute with the restriction functors. q.e.d.

Lemma 2.2.5. *For any $F \in \mathcal{C}(U)$, one has* $\operatorname{supp}(i_{U!}F) \subset \overline{\operatorname{supp} F}$.

Proof. It is enough to show

(2.2.1) $\quad \operatorname{supp}(i_{U!}F) \subset \overline{V} \quad$ for any open subset V such that $\operatorname{supp} F \subset V \subset U$.

For any $G \in \mathcal{C}(X)$ we have $\operatorname{supp} \mathcal{H}om_{\mathcal{C}|_U}(F, G|_U) \subset \operatorname{supp} F \subset V$, and hence

$$\begin{aligned} \operatorname{Hom}_{\mathcal{C}(X)}(i_{U!}F, G) &\simeq \operatorname{Hom}_{\mathcal{C}(U)}(F, G|_U) \\ &\simeq \Gamma(U; \mathcal{H}om_{\mathcal{C}|_U}(F, G|_U)) \simeq \Gamma(V; \mathcal{H}om_{\mathcal{C}|_U}(F, G|_U)) \\ &\simeq \operatorname{Hom}_{\mathcal{C}(V)}(F|_V, G|_V) \simeq \operatorname{Hom}_{\mathcal{C}(X)}(i_{V!}(F|_V), G). \end{aligned}$$

Hence we have $i_{U!}F \simeq i_{V!}(F|_V)$ and (2.2.1) follows from (v)(b). q.e.d.

Lemma 2.2.6. *For $F \in \mathcal{C}(U)$, we have $i_{U!}F \simeq \varinjlim_{V \subset\subset U} i_{V!}(F|_V)$.*

Proof. Let $G \in \mathcal{C}(X)$. Then

$$\begin{aligned} &\operatorname{Hom}_{\mathcal{C}(X)}(\varinjlim_{V \subset\subset U} i_{V!}(F|_V), G) \simeq \varprojlim_{V \subset\subset U} \operatorname{Hom}_{\mathcal{C}(X)}(i_{V!}(F|_V), G) \\ &\quad \simeq \varprojlim_{V \subset\subset U} \operatorname{Hom}_{\mathcal{C}(V)}(F|_V, i_V^{-1}G) \simeq \varprojlim_{V \subset\subset U} \Gamma(V\,; \mathcal{H}om_{\mathcal{C}(X)|_U}(F, i_U^{-1}G)) \\ &\quad \simeq \Gamma(U\,; \mathcal{H}om_{\mathcal{C}(X)|_U}(F, i_U^{-1}G)) \simeq \operatorname{Hom}_{\mathcal{C}(U)}(F, i_U^{-1}G) \simeq \operatorname{Hom}_{\mathcal{C}(X)}(i_{U!}F, G). \end{aligned}$$

q.e.d.

Lemma 2.2.7. *Let $F \in \mathcal{C}(X)$ with* $\operatorname{supp} F \subset U$. *Then $i_{U!}i_U^{-1}F \xrightarrow{\sim} F$.*

Proof. Let $W = X \setminus \operatorname{supp} F$. Since $\operatorname{supp} i_U^{-1} i_{U!}F \subset \operatorname{supp} F$, we get $(i_{U!}i_U^{-1}F)|_W = 0$. On the other hand $(i_{U!}i_U^{-1}F)|_U \simeq F|_U$. By ST1, it is an isomorphism on $X = U \cup W$. q.e.d.

Lemma 2.2.8. *For $F \in \mathcal{C}(U)$ and a monomorphism $G \rightarrowtail i_{U!}F$, we have $i_{U!}i_U^{-1}G \xrightarrow{\sim} G$.*

Proof. Recall the isomorphism $i_{U!}F \simeq \varinjlim_{V \subset\subset U} i_{V!}(F|_V)$. Define $G_V \in \mathcal{C}(X)$ by the Cartesian square

$$\begin{tikzcd} G_V \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & G \arrow[d] \\ i_{V!}(F|_V) \arrow[r] & i_{U!}F. \end{tikzcd}$$

Since the functor $\varinjlim_{V \subset\subset U}$ is exact, we get the isomorphism $\varinjlim_{V \subset\subset U} G_V \xrightarrow{\sim} G$. Hence, we get the isomorphisms

$$i_{U!}i_U^{-1}G \xrightarrow{\sim} i_{U!}i_U^{-1} \varinjlim_{V \subset\subset U} G_V \xrightarrow{\sim} \varinjlim_{V \subset\subset U} i_{U!}i_U^{-1}G_V.$$

Indeed, i_U^{-1} commutes with $\varinjlim$ by the axioms, and $i_{U!}$ commutes with $\varinjlim$ since it has a right adjoint. On the other-hand, the monomorphism $G_V \rightarrowtail i_{V!}(F|_V)$ implies that $\operatorname{supp} G_V$ is contained in $\overline{V} \subset U$. Applying Lemma 2.2.7, we get the isomorphism $i_{U!}i_U^{-1}G_V \xrightarrow{\sim} G_V$, and then $i_{U!}i_U^{-1}G \xrightarrow{\sim} \varinjlim_{V \subset\subset U} G_V \xrightarrow{\sim} G$. q.e.d.

Lemma 2.2.9. *For $F \in \mathcal{C}(X)$, the natural morphism $i_{U!}i_U^{-1}F \longrightarrow F$ is a monomorphism.*

Proof. Define N by the exact sequence $0 \longrightarrow N \longrightarrow i_{U!}i_U^{-1}F \longrightarrow F$. Applying the exact functor i_U^{-1} and using the isomorphism $i_U^{-1}i_{U!}i_U^{-1} \simeq i_U^{-1}$, we get $i_U^{-1}N = 0$. Since N is a subobject of $i_{U!}i_U^{-1}F$, we find $N = 0$ by Lemma 2.2.8. q.e.d.

Proposition 2.2.10. *The functor $i_{U!}$ is exact.*

Proof. This functor is right exact since it admits a right adjoint. Consider a monomorphism $G \rightarrowtail F$ in $\mathcal{C}(U)$, and define N by the exact sequence $0 \longrightarrow N \longrightarrow i_{U!}G \longrightarrow i_{U!}F$. Applying the exact functor i_U^{-1}, we find $i_U^{-1}N = 0$. Since $N \rightarrowtail i_{U!}G$, we have $N \simeq i_{U!}i_U^{-1}N$ by Lemma 2.2.8. Hence $N = 0$. q.e.d.

Proposition 2.2.11. *Let $V \subset U \subset X$ be open inclusions. Then $i_U^{-1}i_{V!}$ is a left adjoint to ρ_{VU}:*

$$\operatorname{Hom}_{\mathcal{C}(U)}(i_U^{-1}i_{V!}G, F) \simeq \operatorname{Hom}_{\mathcal{C}(V)}(G, \rho_{VU}F) \quad \text{for } G \in \mathcal{C}(V) \text{ and } F \in \mathcal{C}(U).$$

Proof. Recall the isomorphism $i_{V!}G \simeq \varinjlim_{W \subset\subset V} i_{W!}(G|_W)$, which implies $i_U^{-1}i_{V!}G \simeq \varinjlim_{W \subset\subset V} i_U^{-1}i_{W!}(G|_W)$. Using the fact that for $W \subset\subset V$, $\operatorname{supp} i_{W!}(G|_W)$ is contained in $\overline{W} \subset V$, we have

$$\Gamma(U; \mathcal{H}om_{\mathcal{C}|_U}(i_U^{-1}i_{V!}G, F)) \simeq \Gamma(V; \mathcal{H}om_{\mathcal{C}|_U}(i_U^{-1}i_{V!}G, F)).$$

Hence we get the chain of isomorphisms

$$\begin{aligned}
\operatorname{Hom}_{\mathcal{C}(U)}(i_U^{-1}i_{V!}G, F) &\simeq \varprojlim_{W \subset\subset V} \operatorname{Hom}_{\mathcal{C}(U)}(i_U^{-1}i_{W!}(G|_W), F) \\
&\simeq \varprojlim_{W \subset\subset V} \operatorname{Hom}_{\mathcal{C}(V)}(i_V^{-1}i_{W!}(G|_W), \rho_{VU}F) \\
&\simeq \operatorname{Hom}_{\mathcal{C}(V)}(i_V^{-1} \varinjlim_{W \subset\subset V} i_{W!}(G|_W), \rho_{VU}F) \\
&\simeq \operatorname{Hom}_{\mathcal{C}(V)}(G, \rho_{VU}F)\,.
\end{aligned}$$

q.e.d.

Proposition 2.2.12. *Let $U \subset X$. Then $\mathcal{C}|_U$ is a proper stack.*

Proof. By the preceding proposition, ρ_{VU} admits a left adjoint $i_U^{-1} i_{V!}$. The axioms are now easily checked. q.e.d.

Proposition 2.2.13. *Let U and V be open subsets, and let $F \in \mathcal{C}(U)$. Then*

$$i_{V!} i_V^{-1} i_{U!} F \simeq i_{U\cap V!}(F|_{U\cap V}).$$

Proof. For $U' \subset\subset U$ and $V' \subset\subset V$, set

$$\begin{aligned} &G_{U'V'} = i_{V'!} i_{V'}^{-1} i_{U'!}(F|_{U'}), \\ &G = \varinjlim_{U',V'} G_{U'V'}. \end{aligned}$$

Then we have

$$G \simeq i_{V!} i_V^{-1} i_{U!} F. \tag{2.2.2}$$

Since $\operatorname{supp} G_{U'V'} \subset \overline{U'} \cap \overline{V'} \subset U \cap V$,

$$i_{U\cap V!} i_{U\cap V}^{-1} G_{U'V'} \xrightarrow{\sim} G_{U'V'}.$$

Taking the inductive limit with respect to (U', V'), we obtain

$$i_{U\cap V!} i_{U\cap V}^{-1} G \xrightarrow{\sim} G. \tag{2.2.3}$$

On the other hand, (2.2.2) implies $i_{U\cap V}^{-1} G \simeq F|_{U\cap V}$, and we get

$$i_{U\cap V!} i_{U\cap V}^{-1} G \simeq i_{U\cap V!}(F|_{U\cap V}). \tag{2.2.4}$$

Then the assertion follows from (2.2.2), (2.2.3) and (2.2.4). q.e.d.

Proposition 2.2.14. *Let $F \in \mathcal{C}(U), G \in \mathcal{C}(X)$. Then*

$$\mathcal{H}om_{\mathcal{C}}(i_{U!}F, G) \simeq i_{U*} \mathcal{H}om_{\mathcal{C}|_U}(F, i_U^{-1} G).$$

Proof. Let $V \subset X$. Then

$$\begin{aligned} \Gamma(V; \mathcal{H}om_{\mathcal{C}}(i_{U!}F, G)) &\simeq \operatorname{Hom}_{\mathcal{C}(V)}(i_V^{-1} i_{U!} F, i_V^{-1} G) \simeq \operatorname{Hom}_{\mathcal{C}(X)}(i_{V!} i_V^{-1} i_{U!} F, G) \\ &\simeq \operatorname{Hom}_{\mathcal{C}(X)}(i_{U\cap V!}(F|_{U\cap V}), G) \simeq \operatorname{Hom}_{\mathcal{C}(U\cap V)}(F|_{U\cap V}, i_{U\cap V}^{-1} G) \\ &\simeq \Gamma(U\cap V; \mathcal{H}om_{\mathcal{C}|_U}(F, i_U^{-1} G)) \simeq \Gamma(V; i_{U*}\mathcal{H}om_{\mathcal{C}|_U}(F, i_U^{-1} G)). \end{aligned}$$

q.e.d.

Definition 2.2.15. For $F \in \mathcal{C}(X)$, we set $F_U := i_{U!} i_U^{-1} F$.

The functor $\mathcal{C}(X) \ni G \mapsto \Gamma(U; \mathcal{H}om_{\mathcal{C}}(F, G))$ is representable by F_U.

Let $V \subset U$. The morphism $\Gamma(U; \mathcal{H}om_{\mathcal{C}}(F, G)) \longrightarrow \Gamma(V; \mathcal{H}om_{\mathcal{C}}(F, G))$ defines a morphism

$$F_V \longrightarrow F_U.$$

Now, consider an open covering $U = \bigcup_{i \in I} U_i$. The families of morphisms $F_{U_{ij}} \longrightarrow F_{U_i}$ and $F_{U_{ij}} \longrightarrow F_{U_j}$ define the two morphisms α and β:

$$\bigoplus_{i,j \in I} F_{U_{ij}} \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \bigoplus_{k \in I} F_{U_k}. \tag{2.2.5}$$

Here we have used the convention in Notation 2.1.2 (iv).

Lemma 2.2.16. *Let $F \in \mathcal{C}(X)$. There is a natural isomorphism* $\operatorname{coker}(\alpha - \beta) \xrightarrow{\sim} F_U$.

Proof. Set for short $F_0 := \bigoplus_{k \in I} F_{U_k}$ and $F_1 := \bigoplus_{ij} F_{U_{ij}}$. Let $G \in \mathcal{C}(X)$ and set $\mathcal{H} := \mathcal{H}om_{\mathcal{C}}(F, G)$. The two sequences

$$\begin{gathered} 0 \longrightarrow \Gamma(U; \mathcal{H}) \longrightarrow \prod_k \Gamma(U_k; \mathcal{H}) \longrightarrow \prod_{i,j \in I} \Gamma(U_{ij}; \mathcal{H}) \quad \text{and} \\ 0 \longrightarrow \operatorname{Hom}_{\mathcal{C}(X)}(F_U, G) \longrightarrow \operatorname{Hom}_{\mathcal{C}(X)}(F_0, G) \longrightarrow \operatorname{Hom}_{\mathcal{C}(X)}(F_1, G) \end{gathered}$$

are isomorphic. Since the first one is exact, the result follows. q.e.d.

Theorem 2.2.17. *Let $\mathcal{C}$ be a proper stack. Then $\mathcal{C}$ is a stack.*

Proof. Let $U = \bigcup_{i \in I} U_i$ be an open covering of $U \subset X$. Let $F_i \in \mathcal{C}(U_i)$ and let $\theta_{ji} : F_i|_{U_{ij}} \xrightarrow{\sim} F_j|_{U_{ij}}$, and assume that these isomorphisms satisfy the conditions of ST2.

Let us introduce the following notations:

$$\begin{array}{lll} f_i : U_i \hookrightarrow X, & f_{i!} = i_{U_i!}, & f_i^{-1} = i_{U_i}^{-1}, \\ f_{ij} : U_{ij} \hookrightarrow X, & f_{ij!} = i_{U_{ij}!}, & f_{ij}^{-1} = i_{U_{ij}}^{-1}, \\ g_{ij} : U_{ij} \hookrightarrow U_i, & g_{ij!} = i_{U_i}^{-1} \circ i_{U_{ij}!}, & g_{ij}^{-1} = \rho_{U_{ij} U_i}. \end{array}$$

and define

$$F_{ij} := g_{ij}^{-1} F_i = F_i|_{U_{ij}}.$$

The morphism $g_{ij!} g_{ij}^{-1} \longrightarrow \mathrm{id}$ defines the morphism

$$\alpha_{ij} : f_{ij!} F_{ij} \longrightarrow f_{i!} F_i. \tag{2.2.6}$$

Denote by β_{ij} the composition of the morphisms

$$f_{ij\,!}F_{ij} \xrightarrow[\theta_{ij}]{} f_{ij\,!}F_{ji} \xrightarrow[\alpha_{ji}]{} f_{j\,!}F_j.$$

Then:

$$\beta_{ij} : f_{ij\,!}F_{ij} \to f_{j\,!}F_j. \tag{2.2.7}$$

We thus get two morphisms in $\mathcal{C}(U)$:

$$\oplus_{i,j\in I} f_{ij\,!}F_{ij} \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \oplus_k f_{k!}F_k.$$

Set $F := \mathrm{coker}(\alpha - \beta)$ and define $\theta_i : F_i \to f_i^{-1}F$ by the natural morphism $f_{i!}F_i \to \oplus_k f_{k!}F_k \to F$. It remains to show that θ_{i_0} is an isomorphism for any $i_0 \in I$.

We may assume from the beginning that $U = U_{i_0}$. Set $F^0 = F_{i_0}$, $F_i^0 = F^0|_{U_i}$, $F_{ij}^0 = F^0|_{U_{ij}}$. The isomorphisms $\theta_{i_0 i} : F_i \xrightarrow{\sim} F_i^0$ define isomorphisms

$$\begin{aligned} \Theta^0 = \oplus_k \theta_{i_0 k} \quad &: \quad \oplus_k f_{k!}F_k \xrightarrow{\sim} \oplus_k f_{k!}F_k^0 \\ \Theta^1 = \oplus_{ij} \theta_{i_0 i} \quad &: \quad \oplus_{ij} f_{ij\,!}F_{ij} \xrightarrow{\sim} \oplus_{ij} f_{ij\,!}F_{ij}^0. \end{aligned}$$

Consider the diagram below where α^0 and β^0 are defined as in Lemma 2.2.16 for F_0, so that $F^0 \simeq \mathrm{coker}(\alpha^0 - \beta^0)$:

$$\begin{tikzcd} \oplus_{ij} f_{ij\,!}F_{ij} \arrow[r, "\alpha-\beta"] \arrow[d, "\Theta^1", "\sim"'] & \oplus_k f_{k!}F_k \arrow[d, "\Theta^0", "\sim"'] \\ \oplus_{ij} f_{ij\,!}F_{ij}^0 \arrow[r, "\alpha^0-\beta^0"] & \oplus_k f_{k!}F_k^0. \end{tikzcd}$$

This diagram commutes, thanks to the commutativity of the diagrams

$$\begin{tikzcd} f_{ij\,!}F_{ij} \arrow[r] \arrow[d, "\theta_{i_0 i}"] & f_{i!}F_i \arrow[d, "\theta_{i_0 i}"] & f_{ij\,!}F_{ij} \arrow[r, "\theta_{ji}"] \arrow[d, "\theta_{i_0 i}"] & f_{j\,!}F_j \arrow[d, "\theta_{i_0 j}"] \\ f_{ij\,!}F_{ij}^0 \arrow[r] & f_{i!}F_i^0 & f_{ij\,!}F_{ij}^0 \arrow[r] & f_{j\,!}F_j^0. \end{tikzcd}$$

Hence, F is isomorphic to F^0. q.e.d.

We shall extend classical construction in sheaf theory to proper stacks.

Let $F \in \mathcal{C}(X)$. For an open subset U, we have already defined the object $F_U \in \mathcal{C}(X)$. For a closed subset S, we define the object F_S by the exact sequence

$$0 \to F_{X\setminus S} \to F \to F_S \to 0. \tag{2.2.8}$$

For any locally closed subset Z, one can find an open subset U and a closed subset S such that $Z = U \cap S$. We shall see below that $(F_U)_S$ depends only on Z, and we shall denote this object by F_Z.

Proposition 2.2.18. *Let $F \in \mathcal{C}(X)$, and let Z be a locally closed subset of X. Then*

(i) *the functor $\mathcal{C}(X) \ni G \mapsto \Gamma_Z(X; \mathcal{H}om_{\mathcal{C}}(F, G))$ is representable by F_Z,*

(ii) *one has $\mathcal{H}om_{\mathcal{C}}(F_Z, G) \simeq \Gamma_Z \mathcal{H}om_{\mathcal{C}}(F, G)$,*

(iii) *the functor $F \mapsto F_Z$ is exact and commutes with inductive limits,*

(iv) *if Z_1 and Z_2 are two locally closed subsets, then $(F_{Z_1})_{Z_2} \simeq F_{Z_1 \cap Z_2}$,*

(v) *if $Z' \subset Z$ is closed in Z, the sequence $0 \longrightarrow F_{Z \setminus Z'} \longrightarrow F_Z \longrightarrow F_{Z'} \longrightarrow 0$ is exact.*

(vi) *if U is open, then we have $i_U^{-1}(F_Z) \simeq (i_U^{-1}F)_{Z \cap U}$ i.e. $\mathcal{C}(U) \ni F \mapsto F_{Z \cap U} \in \mathcal{C}(U)$ is a functor of stacks.*

(vii) $\operatorname{supp}(F_Z) \subset \overline{Z} \cap \operatorname{supp} F$.

Proof. (i) The formula $\Gamma_Z(X; \mathcal{H}om_{\mathcal{C}}(F, G)) \simeq \operatorname{Hom}_{\mathcal{C}(X)}(F_Z, G)$ is true when Z is open. Applying the left exact functor $\operatorname{Hom}_{\mathcal{C}(X)}(\,\cdot\,, G)$ to the exact sequence (2.2.8), we find that this formula remains true when Z is closed. Now let $Z = U \cap S$. Applying Proposition 2.2.14, we get

$$\begin{aligned}\Gamma_Z(X; \mathcal{H}om_{\mathcal{C}}(F, G)) &\simeq \Gamma_S(X; i_{U*} i_U^{-1} \mathcal{H}om_{\mathcal{C}}(F, G)) \\ &\simeq \Gamma_S(X; \mathcal{H}om_{\mathcal{C}}(F_U, G)) \simeq \Gamma(X; \mathcal{H}om_{\mathcal{C}}((F_U)_S, G)).\end{aligned}$$

(ii) The formula is true when Z is open. Since $\mathcal{H}om_{\mathcal{C}}$ is left exact, this formula remains true when Z is closed. Assume $Z = U \cap S$. Then the result follows from $\Gamma_Z \simeq \Gamma_S \circ \Gamma_U$.

(iii) Let us first show that $F \mapsto F_Z$ is exact. The functor $F \mapsto F_U \simeq i_{U!} i_U^{-1} F$ is exact. Hence, we may assume that $Z = S$ is closed. By the definition of F_S in (2.2.8), this functor is exact.

The functor $F \mapsto \Gamma_Z(X; \mathcal{H}om_{\mathcal{C}}(F, G))$ sends inductive limits to projective limits. Therefore, the functor $F \mapsto F_Z$ commutes with inductive limits.

(iv) For any $G \in \mathcal{C}(X)$, one has

$$\begin{aligned}\operatorname{Hom}_{\mathcal{C}(X)}(F_{Z_1 \cap Z_2}, G) &\simeq \Gamma_{Z_1 \cap Z_2}(X; \mathcal{H}om_{\mathcal{C}}(F, G)) \simeq \Gamma_{Z_1}(X; \Gamma_{Z_2} \mathcal{H}om_{\mathcal{C}}(F, G)) \\ &\simeq \Gamma_{Z_1}(X; \mathcal{H}om_{\mathcal{C}}(F_{Z_2}, G)) \simeq \operatorname{Hom}_{\mathcal{C}(X)}((F_{Z_2}))_{Z_1}, G)).\end{aligned}$$

(v) We may assume $Z' = S \cap Z$, with S closed in X. Then the result follows from the exact sequence (2.2.8) applied to F_Z and (iv).

(vi) For any $G \in \mathcal{C}(U)$, one has

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}(U)}(i_U^{-1}(F_Z), G) &\simeq \Gamma(U; \mathcal{H}om_{\mathcal{C}}(F_Z, i_{U!}G)) \simeq \Gamma_{Z\cap U}(U; \mathcal{H}om_{\mathcal{C}}(F, i_{U!}G)) \\ &\simeq \Gamma_{Z\cap U}(U; \mathcal{H}om_{\mathcal{C}|_U}(i_U^{-1}F, G)) \simeq \mathrm{Hom}_{\mathcal{C}(U)}((i_U^{-1}F)_{Z\cap U}, G).\end{aligned}$$

Hence we have $i_U^{-1}(F_Z) \simeq (i_U^{-1}F)_{Z\cap U}$.

(vii) For any $G \in \mathcal{C}(X)$, we have $\mathrm{supp}\,\mathcal{H}om_{\mathcal{C}}(F, G) \subset \mathrm{supp}\,F$, and hence

$$\mathrm{supp}\,\mathcal{H}om_{\mathcal{C}}(F_Z, G) = \mathrm{supp}\,\Gamma_Z\mathcal{H}om_{\mathcal{C}}(F, G) \subset \overline{Z} \cap \mathrm{supp}\,F.$$

Hence setting $G = F_Z$, one has $\mathrm{supp}\,F = \mathrm{supp}(\mathcal{E}nd_{\mathcal{C}}(F)) \subset \overline{Z} \cap \mathrm{supp}\,F$. q.e.d.

Proposition 2.2.19. *Let $F \in \mathcal{C}(X)$. Then*

$$\varinjlim_{U\subset\subset X} F_U \xrightarrow{\sim} F \xrightarrow{\sim} \varprojlim_K F_K.$$

As usual U is open, and K ranges through the family of compact subsets of X.

Proof. Since $\mathcal{C}$ is a stack it is enough to show that they are isomorphisms on any relatively compact open subset of X. This is obvious because inductive limits and projective limits commute with the restriction functors. q.e.d.

Definition 2.2.20. Let U be an open subset of X and let $G \in \mathcal{C}(U)$. One sets

$$i_{U*}G = \varprojlim_{K\subset U} i_{U!}(G_K), \ K \text{ compact}.$$

Proposition 2.2.21. *The functor i_{U*} is a right adjoint to the functor i_U^{-1}.*

Proof. For $F \in \mathcal{C}(X)$ and $G \in \mathcal{C}(U)$, we have

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}(X)}(F, \varprojlim_{K\subset U} i_{U!}G_K) &\simeq \varprojlim_{K\subset U} \mathrm{Hom}_{\mathcal{C}(X)}(F, i_{U!}G_K) \\ &\simeq \varprojlim_{K\subset U} \Gamma(X; \mathcal{H}om_{\mathcal{C}}(F, i_{U!}G_K)).\end{aligned}$$

Since $\mathrm{supp}(\mathcal{H}om_{\mathcal{C}}(F, i_{U!}G_K)) \subset K \subset U$ we have

$$\begin{aligned}\Gamma(X; \mathcal{H}om_{\mathcal{C}}(F, i_{U!}G_K)) &\simeq \Gamma(U; \mathcal{H}om_{\mathcal{C}}(F, i_{U!}G_K)) \\ &\simeq \mathrm{Hom}_{\mathcal{C}(U)}(i_U^{-1}F, G_K).\end{aligned}$$

Hence we have

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}(X)}(F, \varprojlim_{K\subset U} i_{U!}G_K) &\simeq \varprojlim_{K\subset U} \mathrm{Hom}_{\mathcal{C}(U)}(i_U^{-1}F, G_K) \\ &\simeq \mathrm{Hom}_{\mathcal{C}(U)}(i_U^{-1}F, \varprojlim_{K\subset U} G_K) \simeq \mathrm{Hom}_{\mathcal{C}(U)}(i_U^{-1}F, G).\end{aligned}$$

q.e.d.

Note that the morphisms $G \longrightarrow G_K$ define the morphism

$$i_{U!}G \longrightarrow i_{U*}G.$$

If G has compact support in U, this morphism is an isomorphism.

Proposition 2.2.22. *Let Z be a locally closed subset of X. The functor $F \mapsto F_Z$ admits a right adjoint.*

We shall denote this adjoint functor by $\Gamma_Z(\,\cdot\,)$. Hence, for F and G in $\mathcal{C}(X)$, one has:

$$\mathrm{Hom}_{\mathcal{C}(X)}(G_Z, F) \simeq \mathrm{Hom}_{\mathcal{C}(X)}(G, \Gamma_Z(F)).$$

Proof. (i) If Z is open, $i_{U*}i_U^{-1}$ is a right adjoint to $i_{U!}i_U^{-1}$.

(ii) If $Z = S$ is closed, set $U = X \setminus S$ and define $\Gamma_S(F)$ by the exact sequence $0 \longrightarrow \Gamma_S(F) \longrightarrow F \longrightarrow \Gamma_U(F)$. Since $\mathrm{Hom}_{\mathcal{C}(X)}(G,\,\cdot\,)$ is left exact, the result follows in this case.

(iii) Now assume that $Z = S \cap U$ with S closed and U open. Set $\Gamma_Z = \Gamma_U \circ \Gamma_S$. q.e.d.

As an immediate consequence of the properties of the functor $F \mapsto F_Z$, one gets:

Corollary 2.2.23. (i) *The functor Γ_Z is left exact and commutes with projective limits,*

(ii) *one has $\Gamma_{Z_1} \circ \Gamma_{Z_2} \simeq \Gamma_{Z_1 \cap Z_2}$,*

(iii) *if Z' is closed in Z, there is an exact sequence of functors $0 \longrightarrow \Gamma_{Z'} \longrightarrow \Gamma_Z \longrightarrow \Gamma_{Z \setminus Z'}$.*

(iv) *$\Gamma_Z(F)$ represents the contravariant functor $G \longrightarrow \Gamma_Z(X; \mathcal{H}om_{\mathcal{C}}(G, F))$.*

2.3 Indization of proper stacks

As in section 2.2, we assume that X is a locally compact space countable at infinity. Let $\mathcal{C}$ be a proper stack of abelian categories on X. We define the full abelian subcategory $\mathcal{C}_c(X)$ of $\mathcal{C}(X)$ by

$$\mathrm{Ob}(\mathcal{C}_c(X)) = \{F \in \mathcal{C}(X)\,;\, \mathrm{supp}\, F \text{ is compact}\}.$$

We denote for short

$$\mathrm{I}\mathcal{C}(X) = \mathrm{Ind}(\mathcal{C}_c(X)).$$

The reason why we consider the ind-objects of $\mathcal{C}_c(X)$ is that the correspondence $U \mapsto \mathrm{I}\mathcal{C}(U)$ is a stack as we shall see later, contrarily to the correspondence $U \mapsto \mathrm{Ind}(\mathcal{C}(U))$ as we shall see now.

Example 2.3.1. Let $X = \mathbb{R}$, and consider the stack $\mathrm{Mod}(k_X)$ of sheaves of k-modules on X. Let $F = k_X$, $G_n = k_{[n,+\infty[}$, $G = \text{“}\varinjlim_n\text{”}\, G_n$. Then $G|_U = 0$ for any relatively compact open subset U of X. On the other hand, $\mathrm{Hom}_{\mathrm{Ind}(\mathbf{Mod}(k_X))}(k_X, G) \simeq \varinjlim_n \mathrm{Hom}_{k_X}(k_X, G_n) \simeq k$.

Lemma 2.3.2. $\mathrm{I}\mathcal{C}(X)$ *admits small projective limits.*

Proof. By the general result in Theorem 1.3.1, it is enough to show that for a small family $\{F_i \in \mathcal{C}_c(X)\}$, its product (which is well-defined in the category $\mathcal{C}_c(X)^\vee$ of contravariant functors on $\mathcal{C}_c(X)$) belongs to $\mathrm{I}\mathcal{C}(X)$. For $G \in \mathcal{C}_c(X)$ we have

$$\begin{aligned}\prod_i \mathrm{Hom}_{\mathcal{C}(X)}(G, F_i) &\xleftarrow{\sim} \varinjlim_{U \subset\subset X} \mathrm{Hom}_{\mathcal{C}(X)}(G, \prod_i (F_{iU}))\\ &\simeq \mathrm{Hom}_{\mathrm{I}\mathcal{C}(X)}(G, \text{“}\varinjlim_U\text{”} \prod_i (F_{iU})).\end{aligned}$$

Here the first arrow is an isomorphism because $\mathrm{Hom}_{\mathcal{C}(X)}(G, F_{iU}) \to \mathrm{Hom}_{\mathcal{C}(X)}(G, F_i)$ is an isomorphism whenever $\mathrm{supp}(G) \subset U$. Thus the product of the F_i's is represented by $\text{“}\varinjlim_U\text{”} \prod_i (F_{iU})$. q.e.d.

We introduce the functor:

$$\begin{aligned}\iota_X : \mathcal{C}(X) &\to \mathrm{I}\mathcal{C}(X)\\ F &\mapsto (\mathcal{C}_c(X) \ni G \mapsto \mathrm{Hom}_{\mathcal{C}(X)}(G, F)).\end{aligned} \tag{2.3.1}$$

This functor is well-defined by the lemma below.

Lemma 2.3.3. *For* $F \in \mathcal{C}(X)$ *one has the isomorphisms:*

$$\iota_X F \simeq \text{“}\varinjlim_{U \subset\subset X}\text{”} F_U \simeq \varprojlim_{K \subset X} F_K\,,$$

where U *ranges through the family of relatively compact open subsets of* X *and* K *through that of compact subsets.*

Proof. Let $G \in \mathcal{C}_c(X)$. Then:

$$\mathrm{Hom}_{\mathcal{C}(X)}(G, F) \simeq \varinjlim_U \mathrm{Hom}_{\mathcal{C}(X)}(G, F_U) \simeq \varprojlim_K \mathrm{Hom}_{\mathcal{C}(X)}(G, F_K).$$

q.e.d.

Proposition 2.3.4. *The functor* ι_X *of* (2.3.1) *is fully faithful, exact, and commutes with* $\varprojlim$.

Proof. Let F, G belong to $\mathcal{C}(X)$. One has the chain of isomorphisms:

$$\begin{aligned}\mathrm{Hom}_{\mathrm{I}\mathcal{C}(X)}(\iota_X F, \iota_X G) &\simeq \varprojlim_{U\subset\subset X} \mathrm{Hom}_{\mathrm{I}\mathcal{C}(X)}(F_U, \iota_X G)\\ &\simeq \varprojlim_{U\subset\subset X} \mathrm{Hom}_{\mathcal{C}(X)}(F_U, G) \simeq \mathrm{Hom}_{\mathcal{C}(X)}(F, G).\end{aligned}$$

Hence, ι_X is fully faithful. It is exact since the functor $F \mapsto F_U$ is exact as well as "$\varinjlim_U$". Finally let us prove that ι_X commutes with projective limits. Let $\{F_i\}_i$ be a small projective system in $\mathcal{C}(X)$. Then we have for $G \in \mathcal{C}_c(X)$

$$\begin{aligned}\mathrm{Hom}_{\mathrm{I}\mathcal{C}(X)}(G, \iota_X(\varprojlim F_i)) &\simeq \mathrm{Hom}_{\mathcal{C}(X)}(G, \varprojlim F_i) \simeq \varprojlim \mathrm{Hom}_{\mathcal{C}(X)}(G, F_i)\\ &\simeq \varprojlim \mathrm{Hom}_{\mathrm{I}\mathcal{C}(X)}(G, \iota_X(F_i)) \simeq \mathrm{Hom}_{\mathrm{I}\mathcal{C}(X)}(G, \varprojlim(\iota_X F_i)).\end{aligned}$$

q.e.d.

Notation 2.3.5. In the sequel, we shall identify $\mathcal{C}(X)$ with a full subcategory of $\mathrm{I}\mathcal{C}(X)$ by ι_X and often write F instead of $\iota_X F$.

Definition 2.3.6. Let U be an open subset of X. We introduce the functor ρ_{UX} : $\mathrm{I}\mathcal{C}(X) \longrightarrow \mathrm{I}\mathcal{C}(U)$, also denoted by $F \mapsto F|_U$, by the formula:

$$\mathrm{Hom}_{\mathrm{I}\mathcal{C}(U)}(G, F|_U) = \mathrm{Hom}_{\mathrm{I}\mathcal{C}(X)}(i_{U!}G, F) \quad \text{for } G \in \mathcal{C}_c(U).$$

This functor is well-defined by the following lemma.

Lemma 2.3.7. *Let $F =$ "$\varinjlim_i$" F_i, $F_i \in \mathcal{C}_c(X)$. Then:*

$$F|_U \simeq \text{“}\varinjlim_{i, V\subset\subset U}\text{”}\, (F_{iV}|_U) \simeq \varprojlim_{K\subset U} \text{“}\varinjlim_i\text{”}\, F_{iK}|_U \quad V \textit{ open, } K \textit{ compact.}$$

Proof. Let $G \in \mathcal{C}_c(U)$. Then:

$$\begin{aligned}\mathrm{Hom}_{\mathrm{I}\mathcal{C}(U)}(G, F|_U) &= \mathrm{Hom}_{\mathrm{I}\mathcal{C}(X)}(i_{U!}G, F)\\ &\simeq \varinjlim_i \mathrm{Hom}_{\mathcal{C}(X)}(i_{U!}G, F_i) \simeq \varinjlim_i \mathrm{Hom}_{\mathcal{C}(U)}(G, F_i|_U).\end{aligned}$$

Furthermore we have

$$\varinjlim_i \mathrm{Hom}_{\mathcal{C}(U)}(G, F_i|_U) \simeq \varinjlim_i \varinjlim_{V\subset\subset U} \mathrm{Hom}_{\mathcal{C}(U)}(G, F_{iV}|_U)$$

and

$$\begin{aligned}\varinjlim_i \mathrm{Hom}_{\mathcal{C}(U)}(G, F_i|_U) &\simeq \varprojlim_{K\subset U} \varinjlim_i \mathrm{Hom}_{\mathcal{C}(U)}(G, F_{iK}|_U)\\ &\simeq \mathrm{Hom}_{\mathrm{I}\mathcal{C}(U)}(G, \varprojlim_{K\subset U} \text{“}\varinjlim_i\text{”}\, F_{iK}|_U).\end{aligned}$$

q.e.d.

Lemma 2.3.8. *For $F \in \mathcal{C}(X)$, we have $\iota_U(F|_U) \simeq (\iota_X F)|_U$.*

Proof. For any $G \in \mathcal{C}_c(U)$ we have

$$\begin{aligned}\mathrm{Hom}_{\mathrm{I}\mathcal{C}(U)}(G, \iota_U(F|_U)) &\simeq \mathrm{Hom}_{\mathcal{C}(U)}(G, F|_U) \simeq \mathrm{Hom}_{\mathcal{C}(X)}(i_{U!}G, F)\\ &\simeq \mathrm{Hom}_{\mathrm{I}\mathcal{C}(X)}(i_{U!}G, \iota_X F) \simeq \mathrm{Hom}_{\mathrm{I}\mathcal{C}(U)}(G, (\iota_X F)|_U).\end{aligned}$$

q.e.d.

Lemma 2.3.9. *The correspondence $U \mapsto \mathrm{I}\mathcal{C}(U)$ is a prestack of abelian categories.*

The proof is obvious.

Proposition 2.3.10. *The functor $\rho_{UX} : \mathrm{I}\mathcal{C}(X) \to \mathrm{I}\mathcal{C}(U)$ is exact and commutes with "$\varinjlim$" and $\varprojlim$.*

Proof. The functor ρ_{UX} commutes with $\varprojlim$ since the functor $\mathrm{Hom}_{\mathrm{I}\mathcal{C}(X)}(i_{U!}G, \,\cdot\,)$ does. The exactness is obvious, and the commutativity with "$\varinjlim$" follows from Lemma 2.3.7. q.e.d.

Recall that if F and G belong to $\mathrm{I}\mathcal{C}(X)$, the presheaf $U \mapsto \mathrm{Hom}_{\mathrm{I}\mathcal{C}(U)}(F|_U, G|_U)$ is denoted by $\mathcal{H}om_{\mathrm{I}\mathcal{C}}(F, G)$.

Proposition 2.3.11. *Let F, G belong to $\mathrm{I}\mathcal{C}(X)$.*

(i) *The presheaf $\mathcal{H}om_{\mathrm{I}\mathcal{C}}(F, G)$ is a sheaf.*

(ii) *For $F \simeq$ "$\varinjlim\limits_i$" F_i and $G \simeq$ "$\varinjlim\limits_j$" G_j with F_i, $G_j \in \mathcal{C}_c(X)$, one has*

$$\mathcal{H}om_{\mathrm{I}\mathcal{C}}(F, G) \simeq \varprojlim_i \varinjlim_j \mathcal{H}om_{\mathcal{C}}(F_i, G_j).$$

Proof. One has the chain of isomorphisms:

$$\begin{aligned}\mathrm{Hom}_{\mathrm{I}\mathcal{C}(U)}(F|_U, G|_U) &\simeq \mathrm{Hom}_{\mathrm{I}\mathcal{C}(U)}(\text{“}\varinjlim_{i, V\subset\subset U}\text{”}\ F_{iV}|_U,\ \text{“}\varinjlim_{j, W\subset\subset U}\text{”}\ G_{jW}|_U)\\ &\simeq \varprojlim_{i, V\subset\subset U}\ \varinjlim_{j, W\subset\subset U}\ \mathrm{Hom}_{\mathcal{C}(U)}(F_{iV}|_U, G_{jW}|_U)\\ &\simeq \varprojlim_{i, V\subset\subset U}\ \varinjlim_{j} \mathrm{Hom}_{\mathcal{C}(U)}(F_{iV}|_U, G_j|_U)\\ &\simeq \varprojlim_{i, V\subset\subset U}\ \varinjlim_{j} \Gamma(V; \mathcal{H}om_{\mathcal{C}}(F_i, G_j)).\end{aligned}$$

On the other hand we have

$$\begin{aligned}\varprojlim_{V\subset\subset U}\varinjlim_{j}\Gamma(V;\mathcal{H}om_{\mathcal{C}}(F_i,G_j)) &\simeq \varprojlim_{V,V'\subset\subset U}\varinjlim_{j}\Gamma(V;\mathcal{H}om_{\mathcal{C}}(F_i,G_j)_{\overline{V'}})\\ &\simeq \varprojlim_{V'\subset\subset U}\varinjlim_{j}\Gamma(U;\mathcal{H}om_{\mathcal{C}}(F_i,G_j)_{\overline{V'}})\end{aligned}$$

Finally we obtain

$$\begin{aligned}\mathrm{Hom}_{\mathrm{I}\mathcal{C}(U)}(F|_U,G|_U) &\simeq \varprojlim_{i,\,V\subset\subset U}\varinjlim_{j}\Gamma(\overline{V};\mathcal{H}om_{\mathcal{C}}(F_i,G_j))\\ &\simeq \varprojlim_{i,\,V\subset\subset U}\Gamma(\overline{V};\varinjlim_{j}\mathcal{H}om_{\mathcal{C}}(F_i,G_j))\\ &\simeq \varprojlim_{i}\Gamma(U;\varinjlim_{j}\mathcal{H}om_{\mathcal{C}}(F_i,G_j))\\ &\simeq \Gamma(U;\varprojlim_{i}\varinjlim_{j}\mathcal{H}om_{\mathcal{C}}(F_i,G_j)).\end{aligned}$$

q.e.d.

Corollary 2.3.12. *Let $\{F_i\}$ be a small filtrant inductive system in $\mathrm{I}\mathcal{C}(X)$.*

(i) *For any $G \in \mathrm{I}\mathcal{C}(X)$, we have*

$$\mathcal{H}om_{\mathrm{I}\mathcal{C}}(\text{“}\varinjlim_{i}\text{”}\, F_i, G) \simeq \varprojlim_{i}\mathcal{H}om_{\mathrm{I}\mathcal{C}}(F_i,G).$$

(ii) *For any $G \in \mathcal{C}(X)$, we have*

$$\mathcal{H}om_{\mathrm{I}\mathcal{C}}(G, \text{“}\varinjlim_{i}\text{”}\, F_i) \simeq \varinjlim_{i}\mathcal{H}om_{\mathrm{I}\mathcal{C}}(G,F_i).$$

Note that (ii) does not hold in general for $G \in \mathrm{I}\mathcal{C}(X)$ (see Exercise 2.5).

Lemma 2.3.13. *The functor i_U^{-1} admits a left adjoint. In other words, denoting this adjoint by $i_{U!!}$, we have for $F \in \mathrm{I}\mathcal{C}(U)$ and $G \in \mathrm{I}\mathcal{C}(X)$:*

$$\mathrm{Hom}_{\mathrm{I}\mathcal{C}(X)}(i_{U!!}F,G) \simeq \mathrm{Hom}_{\mathrm{I}\mathcal{C}(U)}(F,i_U^{-1}G).$$

Moreover, if $F = \text{“}\varinjlim_{i}\text{”}\, F_i$ with $F_i \in \mathcal{C}_c(U)$, then

$$i_{U!!}\,\text{“}\varinjlim_{i}\text{”}\, F_i \simeq \text{“}\varinjlim_{i}\text{”}\, i_{U!}F_i.$$

Proof. We may assume that $F = \text{“}\varinjlim_i\text{”}\, F_i$ with $F_i \in \mathcal{C}_c(U)$ and $G = \text{“}\varinjlim_j\text{”}\, G_j$ with $G_j \in \mathcal{C}_c(X)$. One defines $i_{U!!}F \in \mathrm{I}\mathcal{C}(X)$ as in the statement. Then

$$\begin{aligned}\mathrm{Hom}_{\mathrm{I}\mathcal{C}(X)}(i_{U!!}F, G) &\simeq \varprojlim_i \varinjlim_j \mathrm{Hom}_{\mathcal{C}(X)}(i_{U!}F_i, G_j) \simeq \varprojlim_i \varinjlim_j \mathrm{Hom}_{\mathcal{C}(U)}(F_i, i_U^{-1}G_j) \\ &\simeq \varprojlim_i \varinjlim_{j, V\subset\subset U} \mathrm{Hom}_{\mathcal{C}(U)}(F_i, (i_U^{-1}G_j)_V) \simeq \mathrm{Hom}_{\mathrm{I}\mathcal{C}(U)}(F, i_U^{-1}G).\end{aligned}$$

q.e.d.

One shall beware that the natural morphism $i_{U!!}\iota_U F \longrightarrow \iota_X i_{U!}F$ is not an isomorphism in general. Here this morphism is defined by the morphisms

$$i_{U!!}F \simeq \text{“}\varinjlim_{V\subset\subset U}\text{”}\, i_{U!}(F_V) \longrightarrow \text{“}\varinjlim_{W\subset\subset X}\text{”}\, (i_{U!}F)_W$$

(we have not written the functors ι_U, ι_X for short). This is the reason why we employ the different notation $i_{U!!}$. Of course, if $\operatorname{supp} F$ is compact in U, then $i_{U!!}F \xrightarrow{\sim} i_{U!}F$.

Theorem 2.3.14. *The prestack* $\mathrm{I}\mathcal{C}$ *is a proper stack.*

Proof. By the preceding results, it remains to check properties (v)(a) and (v)(b) of Definition 2.2.2 for the functor $i_{U!!}$.
(v)(a) Let $F = \text{“}\varinjlim_i\text{”}\, F_i \in \mathrm{I}\mathcal{C}(U)$ with $F_i \in \mathcal{C}_c(U)$. Since i_U^{-1} commutes with inductive limits, one has

$$i_U^{-1} i_{U!!}F \simeq i_U^{-1}\, \text{“}\varinjlim_i\text{”}\, i_{U!}F_i \simeq \text{“}\varinjlim_i\text{”}\, i_U^{-1} i_{U!}F_i \simeq \text{“}\varinjlim_i\text{”}\, F_i.$$

(v)(b) Let $F \in \mathrm{I}\mathcal{C}(U)$. We may assume $F = \text{“}\varinjlim_i\text{”}\, F_i$ with $F_i \in \mathcal{C}_c(U)$. Set $V = X \setminus \overline{U}$. Then we have $i_V^{-1} i_{U!!}F \simeq \text{“}\varinjlim_{i, V'\subset\subset V}\text{”}\, (i_{V'!} i_{V'}^{-1} i_{U!}F_i)|_V$, and $i_{V'}^{-1} i_{U!}F_i = 0$. q.e.d.

Let Z be a locally closed subset of X and let $F \in \mathrm{I}\mathcal{C}(X)$. Since $\mathrm{I}\mathcal{C}$ is a proper stack, the objects F_Z and $\Gamma_Z(F)$ are well-defined in $\mathrm{I}\mathcal{C}(X)$. For $F \in \mathcal{C}(X)$, we have

(2.3.2) $$\iota_X(F_Z) \not\simeq (\iota_X(F))_Z \text{ in general, and } \iota_X(\Gamma_Z(F)) \simeq \Gamma_Z(\iota_X F),$$

as we shall see later. Therefore we shall use another notation ${}_Z F$ instead of F_Z.

Definition 2.3.15. For a locally closed subset Z and $F \in \mathrm{I}\mathcal{C}(X)$, let ${}_Z F$ be an object of $\mathrm{I}\mathcal{C}(X)$ that represents the functor $G \mapsto \Gamma_Z(X; \mathcal{H}om_{\mathrm{I}\mathcal{C}}(F, G))$.

This functor is representable by Proposition 2.2.18, and the functor $F \longrightarrow {}_Z F$ shares the properties in the same proposition. In particular we have

$$\begin{aligned} \mathcal{H}om_{\mathrm{I}\mathcal{C}}({}_Z F, G) &\simeq \Gamma_Z \mathcal{H}om_{\mathrm{I}\mathcal{C}}(F, G)), \\ \mathrm{Hom}_{\mathrm{I}\mathcal{C}(X)}({}_Z F, G) &\simeq \Gamma_Z(X; \mathcal{H}om_{\mathrm{I}\mathcal{C}}(F, G)). \end{aligned}$$

Proposition 2.3.16. *Let Z be a locally closed subset and $F \in \mathrm{I}\mathcal{C}(X)$.*

(i) *If $F = \text{“}\varinjlim\limits_i\text{”}\, F_i$ with $F_i \in \mathcal{C}(X)$ and if $Z = U \cap S$ for a closed subset S and an open subset U, then we have*

$$_Z F \simeq \text{“}\varinjlim_{V \subset\subset U, W \supset S}\text{”}\, (F_i)_{V \cap \overline{W}}.$$

(ii) *For $G \in \mathcal{C}(X)$ and $F \in \mathrm{I}\mathcal{C}(X)$, we have the isomorphism*

$$\mathcal{H}om_{\mathrm{I}\mathcal{C}}(G, F)_Z \simeq \mathcal{H}om_{\mathrm{I}\mathcal{C}}(G, {}_Z F).$$

(iii) *The functor $\mathcal{C}_c(X) \ni G \mapsto \Gamma(X; \mathcal{H}om_{\mathrm{I}\mathcal{C}}(G, F)_Z)$ is representable by ${}_Z F$.*

Proof. (i) For any $G = \text{“}\varinjlim\text{”}\, G_j \in \mathrm{I}\mathcal{C}(X)$ we have

$$\begin{aligned} \mathrm{Hom}_{\mathrm{I}\mathcal{C}(X)}(\text{“}\varinjlim_{i, V \subset\subset U}\text{”}\, F_{iV}, G) &\simeq \varprojlim_{i, V \subset\subset U} \varinjlim_j \mathrm{Hom}_{\mathcal{C}(X)}(F_{iV}, G_j) \\ &\simeq \varprojlim_{i, V \subset\subset U} \varinjlim_j \Gamma(V; \mathcal{H}om_{\mathcal{C}}(F_i, G_j)) \simeq \varprojlim_{V \subset\subset U} \varinjlim_j \Gamma(\overline{V}; \mathcal{H}om_{\mathcal{C}}(F_i, G_j)) \\ &\simeq \varprojlim_{i, V \subset\subset U} \Gamma(\overline{V}; \varinjlim_j \mathcal{H}om_{\mathcal{C}}(F_i, G_j)) \simeq \varprojlim_{i, V \subset\subset U} \Gamma(\overline{V}; \mathcal{H}om_{\mathrm{I}\mathcal{C}}(F_i, G)) \\ &\simeq \varprojlim_i \Gamma(U; \mathcal{H}om_{\mathrm{I}\mathcal{C}}(F_i, G)) \simeq \Gamma(U; \mathcal{H}om_{\mathrm{I}\mathcal{C}}(F, G)). \end{aligned}$$

Hence we have the desired result. Similarly if $Z = S$ is closed, then we have

$$\begin{aligned} \mathrm{Hom}_{\mathrm{I}\mathcal{C}(X)}(\text{“}\varinjlim_{i, W \supset S}\text{”}\, F_{i\overline{W}}, G) &\simeq \varprojlim_{i, W \supset S} \varinjlim_j \mathrm{Hom}_{\mathcal{C}(X)}(F_{i\overline{W}}, G_j) \\ &\simeq \varprojlim_{i, W \supset S} \varinjlim_j \Gamma_{\overline{W}}(X; \mathcal{H}om_{\mathcal{C}}(F_i, G_j)) \simeq \varprojlim_{i, W \supset S} \varinjlim_j \Gamma(X; (\mathcal{H}om_{\mathcal{C}}(F_i, G_j))_W) \\ &\simeq \varprojlim_{i, W \supset S} \Gamma(X; \varinjlim_j (\mathcal{H}om_{\mathcal{C}}(F_i, G_j))_W) \simeq \varprojlim_{i, W \supset S} \Gamma(X; (\mathcal{H}om_{\mathrm{I}\mathcal{C}}(F_i, G))_W) \\ &\simeq \varprojlim_i \Gamma_S(X; \mathcal{H}om_{\mathrm{I}\mathcal{C}}(F_i, G)) \simeq \Gamma_S(X; \mathcal{H}om_{\mathrm{I}\mathcal{C}}(F, G)). \end{aligned}$$

The general case follows from ${}_U({}_S F) \simeq {}_{S \cap U} F$.

(ii) First assume that $Z = U$ is open. Then the formula follows from the isomorphisms

$$\begin{aligned}\mathcal{H}om_{\mathrm{I}\mathcal{C}}(G,F)_U &\simeq \varinjlim_{V'\subset\subset U,i} \mathcal{H}om_{\mathrm{I}\mathcal{C}}(G,F_i)_{V'} \simeq \varinjlim_{V,V'\subset\subset U,i} \mathcal{H}om_{\mathrm{I}\mathcal{C}}(G,F_{iV})_{V'}\\ &\simeq \varinjlim_{V\subset\subset U,i} \mathcal{H}om_{\mathrm{I}\mathcal{C}}(G,F_{iV}) \simeq \mathcal{H}om_{\mathrm{I}\mathcal{C}}(G,\text{“}\varinjlim_{V\subset\subset U,i}\text{”}\, F_{iV}).\end{aligned}$$

Here the last isomorphism follows from Corollary 2.3.12 (ii).

Next, assume that $Z = S$ is closed. In this case, the formula follows from the isomorphisms

$$\begin{aligned}\mathcal{H}om_{\mathrm{I}\mathcal{C}}(G,{}_SF) &\simeq \varinjlim_{W\supset S,i} \mathcal{H}om_{\mathrm{I}\mathcal{C}}(G,F_{i\overline{W}}) \simeq \varinjlim_{W,W'\supset S,i} \mathcal{H}om_{\mathrm{I}\mathcal{C}}(G,F_{i\overline{W}})_{\overline{W'}}\\ &\simeq \varinjlim_{W'\supset S,i} \mathcal{H}om_{\mathrm{I}\mathcal{C}}(G,F_i)_{\overline{W'}} \simeq \mathcal{H}om_{\mathrm{I}\mathcal{C}}(G,F)_S.\end{aligned}$$

The general case follows, since ${}_{U\cap S}F \simeq {}_S({}_UF)$.

(iii) follows from (ii) by applying the functor $\Gamma(X;\,\cdot\,)$. q.e.d.

Lemma 2.3.17. *Let Z be a locally closed subset and $F \in \mathrm{I}\mathcal{C}(X)$. Then we have $\iota_X(\Gamma_Z(F)) \simeq \Gamma_Z(\iota_X F)$.*

Proof. For any $G \in \mathcal{C}_c(X)$, we have

$$\begin{aligned}\mathrm{Hom}_{\mathrm{I}\mathcal{C}(X)}(G,\iota_X(\Gamma_Z F)) &\simeq \mathrm{Hom}_{\mathcal{C}(X)}(G,\Gamma_Z F) \simeq \Gamma_Z(X;\mathcal{H}om_{\mathcal{C}}(G,F))\\ &\simeq \Gamma_Z(X;\mathcal{H}om_{\mathrm{I}\mathcal{C}}(G,\iota_X F)) \simeq \mathrm{Hom}_{\mathrm{I}\mathcal{C}(X)}(G,\Gamma_Z(\iota_X F)).\end{aligned}$$

q.e.d.

We shall introduce new functors between the categories $\mathcal{C}(X)$ and $\mathrm{I}\mathcal{C}(X)$.

Definition 2.3.18. One defines the functor $\alpha_X : \mathrm{I}\mathcal{C}(X) \to \mathcal{C}(X)$ by

$$\alpha_X(\text{“}\varinjlim_i\text{”}\, F_i) = \varinjlim_i F_i.$$

Hence, we have the chain of functors

$$\mathcal{C}(X) \xrightarrow[\iota_X]{} \mathrm{I}\mathcal{C}(X) \xrightarrow[\alpha_X]{} \mathcal{C}(X)$$

and

$$\alpha_X \circ \iota_X \simeq \mathrm{id}_{\mathcal{C}(X)},$$

since for $F \in \mathcal{C}(X)$, one has $F \simeq \varinjlim_{U\subset\subset X} F_U$.

Proposition 2.3.19. *The functor ι_X is a right adjoint to α_X, that is, for $F \in \mathcal{C}(X)$ and $G \in \mathrm{I}\mathcal{C}(X)$ one has the isomorphism:*

$$\mathrm{Hom}_{\mathcal{C}(X)}(\alpha_X G, F) \simeq \mathrm{Hom}_{\mathrm{I}\mathcal{C}(X)}(G, \iota_X F).$$

Proof. Let $G \simeq$ "$\varinjlim\limits_j$" G_j. then

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}(X)}(\alpha_X G, F) &\simeq \mathrm{Hom}_{\mathcal{C}(X)}(\varinjlim_j G_j, F) \simeq \varprojlim_j \mathrm{Hom}_{\mathcal{C}(X)}(G_j, F) \\ &\simeq \varprojlim_j \mathrm{Hom}_{\mathrm{I}\mathcal{C}(X)}(G_j, \iota_X F) \simeq \mathrm{Hom}_{\mathrm{I}\mathcal{C}(X)}(\text{“}\varinjlim_j\text{”}\, G_j, \iota_X F).\end{aligned}$$

q.e.d.

Corollary 2.3.20. *For any open subset U, we have*

$$\alpha_U i_U^{-1} \simeq i_U^{-1}\alpha_X,$$

or equivalently, $\alpha : \mathrm{I}\mathcal{C} \longrightarrow \mathcal{C}$ is a functor of stacks.

Proof. If $F =$ "$\varinjlim\limits_i$" F_i with $F_i \in \mathcal{C}_c(X)$ then

$$\begin{aligned}\alpha_U i_U^{-1} F &\simeq \alpha_U\Big(\text{“}\varinjlim_{i, V\subset\subset U}\text{”}\, (F_{iV})|_U\Big) \simeq \varinjlim_{i, V\subset\subset U} (F_{iV})|_U \\ &\simeq \varinjlim_{V\subset\subset U} ((\alpha_X F)_V)|_U \simeq i_U^{-1}(\alpha_X F).\end{aligned}$$

q.e.d.

Corollary 2.3.21. *For $F \in \mathcal{C}(X)$ and $G \in \mathrm{I}\mathcal{C}(X)$, one has the isomorphism $\mathcal{H}om_{\mathcal{C}}(\alpha_X G, F) \simeq \mathcal{H}om_{\mathrm{I}\mathcal{C}}(G, \iota_X F)$.*

It follows from Proposition 2.3.19 that there is a natural morphism of functors $\mathrm{id} \longrightarrow \iota_X \circ \alpha_X$. Recall that the natural morphism $\alpha_X \circ \iota_X \longrightarrow \mathrm{id}$ is an isomorphism.

Proposition 2.3.22. *The functor α_X is exact and commutes with small $\varinjlim$ and small $\varprojlim$.*

Proof. (i) Let $0 \longrightarrow F' \longrightarrow F \longrightarrow F'' \longrightarrow 0$ be an exact sequence in $\mathrm{I}\mathcal{C}(X)$. There exists an inductive system of exact sequences $0 \longrightarrow F_i' \longrightarrow F_i \longrightarrow F_i'' \longrightarrow 0$, whose "$\varinjlim$" is the above exact sequence. Then the sequence $0 \longrightarrow \varinjlim\limits_i F_i' \longrightarrow \varinjlim\limits_i F_i \longrightarrow \varinjlim\limits_i F_i'' \longrightarrow 0$ is exact.

(ii) Since α_X admits a right adjoint, it commutes with $\varinjlim$.

(iii) Let us show that α_X commutes with $\varprojlim$. Since this functor is exact, it it is enough to show that it commutes with products. Let A be a small set and let $F_a \in \mathrm{I}\mathcal{C}(X), a \in A$. For each $a \in A$, there exists a small filtrant category I_a such that $F_a \simeq \text{“}\varinjlim\limits_{i \in I_a}\text{”}\, F_{a,i}$ with $F_{a,i} \in \mathcal{C}(X)$. Define the small set

$$B := \{\varphi : A \longrightarrow \bigsqcup I_a; \varphi(a) \in I_a\}.$$

We have then the chain of isomorphisms

$$\begin{aligned}
\alpha_X(\prod_{a\in A} F_a) &\simeq \alpha_X(\prod_{a\in A} \text{“}\varinjlim_{i\in I_a}\text{”}\, F_{a,i}) \simeq \alpha_X(\text{“}\varinjlim_{\varphi\in B}\text{”} \prod_{a\in A} F_{a,\varphi(a)}) \\
&\simeq \varinjlim_{\varphi\in B} \prod_{a\in A} F_{a,\varphi(a)} \simeq \prod_{a\in A} \varinjlim_{i\in I_a} F_{a,i} \\
&\simeq \prod_{a\in A} \alpha_X(\text{“}\varinjlim_{i\in I_a}\text{”}\, F_{a,i}) \simeq \prod_{a\in A} \alpha_X(F_a).
\end{aligned}$$

q.e.d.

Corollary 2.3.23. *Assume that $F \in \mathcal{C}(X)$ is injective. Then $\iota_X F \in \mathrm{I}\mathcal{C}(X)$ is injective.*

Proof. By Proposition 2.3.22, the functor

$$G \mapsto \mathrm{Hom}_{\mathrm{I}\mathcal{C}(X)}(G, \iota_X F) = \mathrm{Hom}_{\mathcal{C}(X)}(\alpha_X(G), F)$$

is exact. q.e.d.

In order to construct a left adjoint to the functor α_X, we need a hypothesis.

Definition 2.3.24. (i) Let $F \in \mathcal{C}(X)$. We say that F is light on $U \subset X$ if, for any small filtrant inductive system $i \mapsto G_i$ in $\mathcal{C}(X)$, the natural morphism $\varinjlim\limits_i \mathcal{H}om_{\mathcal{C}}(F, G_i)|_U \longrightarrow \mathcal{H}om_{\mathcal{C}}(F, \varinjlim\limits_i G_i)|_U$ is an isomorphism. This condition is equivalent to saying that $\mathcal{H}om_{\mathrm{I}\mathcal{C}}(F, G)|_U \longrightarrow \mathcal{H}om_{\mathcal{C}}(F, \alpha_X G)|_U$ is an isomorphism for any $G \in \mathrm{I}\mathcal{C}(X)$.

(ii) We denote by $\mathcal{L}(X)$ the full additive subcategory of $\mathcal{C}(X)$ whose objects are the direct sums $\bigoplus_{i\in I}(F_i)_{U_i}$ with I small and F_i light on U_i, U_i open in X.

(iii) If $\mathcal{L}(X)$ is generating in $\mathcal{C}(X)$ (i.e. for any $F \in \mathcal{C}(X)$, there exists an epimorphism $G \twoheadrightarrow F$ with $G \in \mathcal{L}(X)$), we say that $\mathcal{C}(X)$ has enough light objects.

Example 2.3.25. If $\mathcal{A}$ is a sheaf of ring on X and $\mathcal{C}(X) = \mathrm{Mod}(\mathcal{A})$, the sheaf $\mathcal{A}_U$ is light on U, and the category $\mathcal{C}(X)$ has enough light objects.

Theorem 2.3.26. *Assume that $\mathcal{C}(X)$ has enough light objects. Then the functor α_X admits a left adjoint β_X:*

$$\mathrm{Hom}_{\mathrm{I}\mathcal{C}(X)}(\beta_X F, G) \simeq \mathrm{Hom}_{\mathcal{C}(X)}(F, \alpha_X G).$$

Moreover, if F is light on U, then $\beta_X(F_U) \simeq {}_U F \simeq \text{“}\varinjlim_{V \subset\subset U}\text{”}\, F_V$.

Proof. It is equivalent to saying that the object $\beta_X F \in \mathrm{I}\mathcal{C}(X)^\wedge$ defined by

$$(\beta_X F)(G) = \mathrm{Hom}_{\mathcal{C}(X)}(F, \alpha_X G),\ G \in \mathrm{I}\mathcal{C}(X)$$

is representable.

Now remark that if $F_1 \longrightarrow F_0 \longrightarrow F \longrightarrow 0$ is an exact sequence and $\beta_X F_1$, $\beta_X F_0$ are representable, then $\beta_X F$ is representable by $\mathrm{coker}(\beta_X F_1 \longrightarrow \beta_X F_0)$. Similarly, if $\{F_i\}$ is a small family of objects of $\mathcal{C}(X)$ such that each $\beta_X F_i$ is representable, then $\beta_X(\oplus_i F_i)$ is representable by “$\bigoplus\limits_i$” $\beta_X F_i$. Hence, it is enough to prove the result for F_U when F is light on U. For any $G \in \mathrm{I}\mathcal{C}(X)$, one has

$$\begin{aligned} \mathrm{Hom}_{\mathrm{I}\mathcal{C}(X)}({}_U F, G) &\simeq \Gamma(U; \mathcal{H}om_{\mathrm{I}\mathcal{C}}(F, G)) \\ &\xrightarrow{\sim} \Gamma(U; \mathcal{H}om_{\mathcal{C}}(F, \alpha_X G)) \simeq \mathrm{Hom}_{\mathrm{I}\mathcal{C}(X)}(F_U, \alpha_X G)\,. \end{aligned}$$

Here the first isomorphism follows from by Definition 2.3.15. and the second isomorphism from the fact that F is light on U. q.e.d.

Until the end of this section, we assume that $\mathcal{C}(X)$ has enough light objects. The functor β_X has the following properties.

Proposition 2.3.27. (i) *β_X commutes with $\varinjlim$ (i.e. $\beta_X \circ \varinjlim \simeq$ “$\varinjlim$” $\circ \beta_X$),*

(ii) *β_X is right exact,*

(iii) *$\alpha_X \circ \beta_X \simeq \mathrm{id}$,*

(iv) *β_X is fully faithful.*

Proof. (i) and (ii) follows from the fact that β_X has a right adjoint.

(iii) For F and G in $\mathcal{C}(X)$, we have

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}(X)}(\alpha_X \beta_X F, G) &\simeq \mathrm{Hom}_{\mathrm{I}\mathcal{C}(X)}(\beta_X F, \iota_X G) \\ &\simeq \mathrm{Hom}_{\mathcal{C}(X)}(F, \alpha_X \iota_X G) \simeq \mathrm{Hom}_{\mathcal{C}(X)}(F, G). \end{aligned}$$

(iv) For F and G in $\mathcal{C}(X)$, we have

$$\mathrm{Hom}_{\mathcal{C}(X)}(F,G) \simeq \mathrm{Hom}_{\mathcal{C}(X)}(F,\alpha_X \circ \beta_X G) \simeq \mathrm{Hom}_{\mathrm{I}\mathcal{C}(X)}(\beta_X F, \beta_X G).$$

q.e.d.

Lemma 2.3.28. *For any open subset U, one has $i_U^{-1}\beta_X \simeq \beta_U i_U^{-1}$, that is, $\beta : \mathcal{C} \to \mathrm{I}\mathcal{C}$ is a functor of stacks.*

Proof. By Corollary 2.3.20 we have $i_U^{-1} \simeq i_U^{-1}\alpha_X\beta_X \simeq \alpha_U i_U^{-1}\beta_X$. By adunction we obtain $\beta_U i_U^{-1} \to i_U^{-1}\beta_X$.

Let us show that $\beta_U i_U^{-1}F \to i_U^{-1}\beta_X F$ is an isomorphism for any $F \in \mathcal{C}(X)$. Since both sides are right exact and commute with inductive limits, it is enough to prove when $F = L_V$ for L light on V. In this case the assertion follows from Proposition 2.2.18 (vi). q.e.d.

This lemma along with Corollary 2.3.20 implies the following result.

Lemma 2.3.29. *For $F \in \mathcal{C}(X)$ and $G \in \mathrm{I}\mathcal{C}(X)$, we have*

$$\mathcal{H}om_{\mathrm{I}\mathcal{C}}(\beta_X F, G) \simeq \mathcal{H}om_{\mathcal{C}}(F, \alpha_X G).$$

Remark 2.3.30. The morphism of functors $\beta_X \circ \alpha_X \to \mathrm{id}$ defines

$$\beta_X \simeq \beta_X \circ \alpha_X \circ \iota_X \to \iota_X.$$

This morphism is not an isomorphism in general, even when $X = \{pt\}$ and $\mathcal{C} = \mathrm{Mod}(k_X)$. Indeed, if F is a k-vector space, $\beta_X F = \text{“}\varinjlim_i\text{”}\, F_i$, where F_i ranges through the family of finite-dimensional subspaces of F.

Exercises to Chapter 2

Exercise 2.1. Let $X = \bigcup_{i\in I} U_i$ be an open covering and let $\mathcal{C}_i$ be a stack on U_i. Assume to be given equivalences of stacks $\varphi_{ij} : \mathcal{C}_i|_{U_{ij}} \to \mathcal{C}_j|_{U_{ij}}$ and isomorphisms of functors $\lambda_{ijk} : \varphi_{ij} \circ \varphi_{ik} \xrightarrow{\sim} \varphi_{ik}$ from $\mathcal{C}_i|_{U_{ijk}} \to \mathcal{C}_k|_{U_{ijk}}$. Assume that for any i,j,k,l, the diagram below commutes:

$$\begin{tikzcd}
\varphi_{ij} \circ \varphi_{jk} \circ \varphi_{kl} \arrow[r, "\lambda_{jkl}"] \arrow[d, "\lambda_{ijk}"] & \varphi_{ij} \circ \varphi_{jl} \arrow[d, "\lambda_{ijl}"] \\
\varphi_{ik} \circ \varphi_{kl} \arrow[r, "\lambda_{ikl}"] & \varphi_{il}.
\end{tikzcd}$$

Prove that there exists a stack $\mathcal{C}$ on X and equivalence of stacks $\varphi_i : \mathcal{C}|_{U_i} \to \mathcal{C}_i$ satisfying the natural conditions (i.e. if φ_i^{-1} is a quasi-inverse to φ_i, then $\varphi_{ij} \simeq \varphi_j \circ \varphi_i^{-1}$ etc.).

Prove that $\mathcal{C}$ is unique up to equivalence.

Exercise 2.2. Give an example of F such that $\beta_X F \longrightarrow F$ is not a monomorphism (resp. epimorphism).

Exercise 2.3. Let $\mathcal{C}$ be a proper stack on X, and let $K' \underset{\varphi}{\longrightarrow} K \underset{\psi}{\longrightarrow} K''$ be a complex in $\mathrm{I}\mathcal{C}(X)$. Prove that this complex is exact when the sequence of sheaves

$$\mathcal{H}om_{\mathrm{I}\mathcal{C}}(F, K') \to \mathcal{H}om_{\mathrm{I}\mathcal{C}}(F, K) \to \mathcal{H}om_{\mathrm{I}\mathcal{C}}(F, K'')$$

is exact for any $F \in \mathcal{C}_c(X)$.

Exercise 2.4. Let $\mathcal{C}$ denote a proper stack of abelian categories. By definition,

$$\begin{aligned}\mathrm{End}\,(\mathrm{id}_{\mathcal{C}}) \quad = \quad & \{\{\xi_U\}_U;\ U \text{ ranging over the family of open sets,} \\ & \xi_U \in \mathrm{End}\,(\mathrm{id}_{\mathcal{C}(U)}),\ \text{ with } \rho_{VU} \circ \xi_U = \xi_V \circ \rho_{VU}\}.\end{aligned}$$

Clearly, the presheaf $U \mapsto \mathrm{End}\,(\mathrm{id}_{\mathcal{C}|_U})$ is a sheaf of commutative rings. One denotes this sheaf by $\mathcal{E}nd(\mathrm{id}_{\mathcal{C}})$.

Let $\mathcal{B}$ be a sheaf of commutative rings on X. Let us say that $\mathcal{C}$ is a $\mathcal{B}$-stack if one is given a morphism of rings $\mathcal{B} \longrightarrow \mathcal{E}nd(\mathrm{id}_{\mathcal{C}})$.

Let $\mathcal{A}$ be a sheaf of (not necessarily commutative) rings. One defines the stack $\mathrm{Mod}(\mathcal{A}, \mathcal{C})$ by

$$\begin{aligned}&\mathrm{Ob}(\mathrm{Mod}(\mathcal{A}, \mathcal{C})(U)) = \{(F, \xi_F)\,;\ F \in \mathcal{C}(U),\ \xi_F \in \mathrm{Hom}_{k_U}(\mathcal{A}|_U, \mathcal{E}nd_{\mathcal{C}|_U}(F))\}, \\ &\mathrm{Hom}_{\mathrm{Mod}(\mathcal{A},\mathcal{C})(U)}((F, \xi_F), (G, \xi_G)) = \{f \in \mathrm{Hom}_{\mathcal{C}(U)}(F, G)\,; \xi_G(a) \circ f = f \circ \xi_F(a) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for all } V \subset U, a \in \mathcal{A}(V)\}.\end{aligned}$$

(i) Prove that $\mathrm{Mod}(\mathcal{A}, \mathcal{C})$ is a proper stack of abelian categories, and if $\mathcal{A}$ is commutative, it is an $\mathcal{A}$-stack.

(ii) Prove that the natural functor of abelian categories $\mathrm{Mod}(\mathcal{A}, \mathcal{C})(X) \longrightarrow \mathcal{C}(X)$ is exact and faithful.

(iii) Let $\mathcal{A}$ be a sheaf of rings on X. Show that the stack $\mathrm{Mod}(\mathcal{A}, \mathrm{Mod}(\mathbb{Z}_X))$ is equivalent to $\mathrm{Mod}(\mathcal{A})$.

Exercise 2.5. Let $F \in \mathrm{I}\mathcal{C}(X)$.

(i) Prove that the functor $\mathrm{I}\mathcal{C}(X) \ni G \mapsto \mathcal{H}om_{\mathrm{I}\mathcal{C}}(F, G) \in \mathrm{Mod}(\mathbb{Z}_X)$ commutes with filtrant inductive limits if and only if $F \in \mathcal{C}(X)$.

(ii) Prove that the functor $\mathrm{I}\mathcal{C}(X) \ni G \mapsto \mathrm{Hom}_{\mathrm{I}\mathcal{C}}(F, G) \in \mathrm{Mod}(\mathbb{Z})$ commutes with filtrant inductive limits if and only if $F \in \mathcal{C}_c(X)$.

Chapter 3

Ind-sheaves

As in the preceding chapter, we work in a given universe $\mathcal{U}$. Hence, otherwise specified, a category means a $\mathcal{U}$-category and all sets are $\mathcal{U}$-small. Moreover, all topological spaces are assumed to be locally compact (in particular, Hausdorff) and countable at infinity. We denote by k a commutative ring.

3.1 The stack of ind-sheaves

In this section, $\mathcal{A}$ will be a sheaf of k-algebras on a topological space X (with the image of k_X contained in the center of $\mathcal{A}$).

We denote by $\mathrm{Mod}(\mathcal{A})$ the abelian category of $\mathcal{A}$-modules, and by $\mathrm{Mod}^c(\mathcal{A})$ its full abelian subcategory consisting of sheaves with compact support. Recall that $\mathrm{Mod}(\mathcal{A})$ and $\mathrm{Mod}^c(\mathcal{A})$ have enough injectives and also have systems of strict generators. Moreover, the injective objects of $\mathrm{Mod}^c(\mathcal{A})$ are injective in $\mathrm{Mod}(\mathcal{A})$. Also recall that $U \mapsto \mathrm{Mod}(\mathcal{A}|_U)$ is a proper stack.

Notation 3.1.1. We shall use the functors $\mathcal{H}om_{\mathcal{A}}$ and $\otimes_{\mathcal{A}}$ on $\mathrm{Mod}(\mathcal{A})$. When $\mathcal{A} = k_X$ we shall simply denote these functors by $\mathcal{H}om$ and $\otimes$. Moreover, we write as usual $\mathrm{Hom}_{\mathcal{A}}$ instead of $\mathrm{Hom}_{\mathrm{Mod}(\mathcal{A})}$.

Definition 3.1.2. We call an object of $\mathrm{Ind}(\mathrm{Mod}^c(\mathcal{A}))$ an ind-sheaf (of $\mathcal{A}$-modules) on X.

We set for short:

$$\mathrm{I}(\mathcal{A}) := \mathrm{Ind}(\mathrm{Mod}^c(\mathcal{A})).$$

Hence, if $\mathcal{C}$ is the proper stack $U \mapsto \mathrm{Mod}(\mathcal{A}(U))$, $\mathrm{I}(\mathcal{A})$ is the category $\mathrm{I}\mathcal{C}(X)$, and we may apply the preceding results. In particular,

$$U \mapsto \mathrm{I}(\mathcal{A}|_U) \text{ is a proper stack.}$$

Let us denote by $\mathcal{F}(\mathcal{A})$ the full additive subcategory of $\mathrm{Mod}(\mathcal{A})$ consisting of objects isomorphic to $\bigoplus_{i\in I}\mathcal{A}_{U_i}$ with I small and U_i open in X. Then $\mathcal{F}(\mathcal{A})$ is generating. Since $\mathcal{A}$ is light on X, $\mathcal{F}(\mathcal{A})$ is contained in $\mathcal{L}(X)$. Applying the constructions of section 2.3, we get the functors

$$\begin{aligned}\iota_X &: \mathrm{Mod}(\mathcal{A}) \longrightarrow \mathrm{I}(\mathcal{A}),\\ \alpha_X &: \mathrm{I}(\mathcal{A}) \longrightarrow \mathrm{Mod}(\mathcal{A}),\\ \beta_X &: \mathrm{Mod}(\mathcal{A}) \longrightarrow \mathrm{I}(\mathcal{A}).\end{aligned}$$

These functors satisfy the following properties:

(i) ι_X is exact, fully faithful, and commutes with $\varprojlim$,

(ii) α_X is exact and commutes with $\varprojlim$ and $\varinjlim$,

(iii) β_X is right exact, fully faithful and commutes with $\varinjlim$,

(iv) α_X is left adjoint to ι_X,

(v) α_X is right adjoint to β_X,

(vi) $\alpha_X \circ \iota_X \simeq \mathrm{id}_{\mathrm{Mod}(\mathcal{A})}$ and $\alpha_X \circ \beta_X \simeq \mathrm{id}_{\mathrm{Mod}(\mathcal{A})}$.

We have also defined the functor $F \mapsto F|_U$ from $\mathrm{I}(\mathcal{A})$ to $\mathrm{I}(\mathcal{A}|_U)$, as well as the functor $(F,G) \mapsto \mathcal{H}om_{\mathrm{I}(\mathcal{A})}(F,G)$ from $\mathrm{I}(\mathcal{A})^{\mathrm{op}} \times \mathrm{I}(\mathcal{A})$ to $\mathrm{Mod}(k_X)$. Recall that if $i \mapsto F_i$ is a small filtrant inductive system with $F_i \in \mathrm{Mod}^c(\mathcal{A})$ and $F \simeq \text{“}\varinjlim\limits_i\text{”}\, F_i$, then

$$F|_U \simeq \text{“}\varinjlim_{i, V\subset\subset U}\text{”}\,(F_{iV}|_U),$$

and if $j \mapsto G_j$ is a small filtrant inductive system with $G_j \in \mathrm{Mod}^c(\mathcal{A})$, then

$$\mathcal{H}om_{\mathrm{I}(\mathcal{A})}(\text{“}\varinjlim_i\text{”}\, F_i, \text{“}\varinjlim_j\text{”}\, G_j) \simeq \varprojlim_i \varinjlim_j \mathcal{H}om_{\mathcal{A}}(F_i, G_j).$$

Note that we have

$$\beta_X(\mathcal{A}_U) \simeq {}_U\mathcal{A} \simeq \text{“}\varinjlim_{V\subset\subset U}\text{”}\, \mathcal{A}_V\,.$$

Notation 3.1.3. If $\mathcal{A} = k_X$, we shall simply denote the functor $\mathcal{H}om_{k_X}$ by $\mathcal{H}om$.

Remark 3.1.4. Denote for a while by for one of the natural functors $\mathrm{Mod}(\mathcal{A}) \longrightarrow \mathrm{Mod}(k_X)$ or $\mathrm{I}(\mathcal{A}) \longrightarrow \mathrm{I}(k_X)$. Then, clearly, α_X and ι_X commute with for. One shall beware that β_X *does not* commute with for.

The functor β_X

In Chapter 2, we have introduced the functor β_X. This functor has no classical counterpart in sheaf theory, and we shall study here some of its properties.

Notation 3.1.5. (i) In view of Remark 3.1.4, we shall use the notation $\beta_{\mathcal{A}}$ when working in $\mathrm{Mod}(\mathcal{A})$. Hence: $\beta_{\mathcal{A}} : \mathrm{Mod}(\mathcal{A}) \longrightarrow \mathrm{I}(\mathcal{A})$.

(ii) We shall often identify a sheaf $F \in \mathrm{Mod}(\mathcal{A})$ and its image $\iota_X F \in \mathrm{I}(\mathcal{A})$.

Proposition 3.1.6. *Let F and G be in* $\mathrm{Mod}(\mathcal{A})$.

(i) *There is a canonical morphism*

$$\mathcal{H}om_{\mathcal{A}}(G, \mathcal{A}) \otimes_{\mathcal{A}} F \longrightarrow \mathcal{H}om_{\mathrm{I}(\mathcal{A})}(G, \beta_{\mathcal{A}} F). \tag{3.1.1}$$

(ii) *The canonical morphism $\mathcal{H}om_{\mathcal{A}}(G, \mathcal{A}) \otimes_{\mathcal{A}} F \longrightarrow \mathcal{H}om_{\mathcal{A}}(G, F)$ factors as*

$$\mathcal{H}om_{\mathcal{A}}(G, \mathcal{A}) \otimes_{\mathcal{A}} F \longrightarrow \mathcal{H}om_{\mathrm{I}(\mathcal{A})}(G, \beta_{\mathcal{A}} F) \longrightarrow \mathcal{H}om_{\mathcal{A}}(G, F), \tag{3.1.2}$$

where the first arrow is given by (3.1.1) *and the second arrow is induced by $\beta_{\mathcal{A}} F \longrightarrow F$.*

(iii) *For $F \in \mathcal{F}(\mathcal{A})$, the morphism* (3.1.1) *is an isomorphism.*

Proof. (i) (ii) Let U be an open subset of X. For $\varphi \in \mathrm{Hom}_{\mathcal{A}|_U}(G|_U, \mathcal{A}|_U)$ and $s \in F(U) \simeq \mathrm{Hom}_{\mathcal{A}|_U}(\mathcal{A}|_U, F|_U)$, we define $\psi \in \mathrm{Hom}_{\mathrm{I}(\mathcal{A}|_U)}(G|_U, \beta_{\mathcal{A}}(F)|_U)$ by

$$G|_U \xrightarrow{\varphi} \mathcal{A}|_U \xleftarrow{\sim} \beta_{\mathcal{A}} \mathcal{A}|_U \xrightarrow{s} \beta_{\mathcal{A}}(F)|_U.$$

Then $\varphi \otimes s \mapsto \psi$ defines $\mathrm{Hom}_{\mathcal{A}|_U}(G|_U, \mathcal{A}|_U) \otimes_{\mathcal{A}(U)} F(U) \longrightarrow \mathrm{Hom}_{\mathrm{I}(\mathcal{A}|_U)}(G|_U, \beta_{\mathcal{A}} F|_U)$.

(iii) Since both sides of (3.1.1) commute with direct sums with respect to F, we may assume that $F = \mathcal{A}_U$ for some open subset U. In this case the assertion follows from Proposition 2.3.16. q.e.d.

Lemma 3.1.7. *Let $F \in \mathcal{F}(\mathcal{A})$, $G \in \mathrm{Mod}(\mathcal{A})$.*

(i) *Let $\psi \in \mathrm{Hom}_{\mathrm{I}(\mathcal{A})}(G, \beta_{\mathcal{A}} F)$. For each $x \in X$, there exist an open neighborhood U of x, $g : G|_U \longrightarrow \mathcal{A}^n|_U$ and $f : \mathcal{A}^n|_U \longrightarrow F|_U$ such that $\psi|_U$ factorizes as*

$$G|_U \underset{g}{\longrightarrow} \mathcal{A}^n|_U \xleftarrow{\sim} \beta_{\mathcal{A}} \mathcal{A}^n|_U \underset{\beta_{\mathcal{A}}(f)}{\longrightarrow} \beta_{\mathcal{A}}(F|_U).$$

(*Here, we write $\beta_{\mathcal{A}}$ instead of $\beta_{\mathcal{A}|_U}$ for short in view of* Lemma 2.3.28.)

(ii) *Assume that* $\beta_{\mathcal{A}}(f)\circ g = 0$ *in* $\mathrm{I}(\mathcal{A}|_U)$, *where* f *and* g *are as above. Then for any* $x \in U$, *there exist an open neighborhood* V *of* x *with* $V \subset U$ *and a commutative diagram*

$$\begin{tikzcd} & \mathcal{A}_V^k \arrow[r, "b"] \arrow[d, "a"] & \mathcal{A}_V^m \arrow[d, "h"] \\ G_V \arrow[ru, "\varphi"] \arrow[r, "g"'] & \mathcal{A}_V^n \arrow[r, "f"'] & F_V \end{tikzcd}$$

such that $b \circ \varphi = 0$.

Proof. (i) follows from Proposition 3.1.6 (iii).

(ii) Let $\sum_i g_i \otimes f_i \in \mathcal{H}om_{\mathcal{A}}(G,\mathcal{A})_x \otimes_{\mathcal{A}_x} F_x$, and assume this section vanishes. There exists $\varphi_j \in \mathcal{H}om_{\mathcal{A}}(G,\mathcal{A})_x, h_k \in F_x$ and $a_{ij}, b_{jk} \in \mathcal{A}_x$ with

$$\begin{aligned} g_i &= \sum_j \varphi_j a_{ji} \text{ for all } i, \\ \sum_k b_{jk}h_k &= \sum_i a_{ji} f_i \text{ for all } j, \\ \sum_j \varphi_j b_{jk} &= 0 \text{ for all } k. \end{aligned}$$

q.e.d.

Proposition 3.1.8. *Let* F *be an* $\mathcal{A}$*-module.*

(i) $\beta_{\mathcal{A}} F \longrightarrow F$ *is an epimorphism if and only if* F *is locally of finite type.*

(ii) $\beta_{\mathcal{A}} F \longrightarrow F$ *is an isomorphism if and only if* F *is locally of finite presentation.*

Proof. (i) (a) Assume that $\beta_{\mathcal{A}} F \longrightarrow F$ is an epimorphism. Let us choose an epimorphism $L = \oplus_{i\in I}\mathcal{A}_{U_i} \twoheadrightarrow F$. Let $s_i \in F(U_i)$ be the image of $1_{U_i} \in \Gamma(U_i;\mathcal{A}_{U_i})$. Since $\beta_X L \longrightarrow \beta_X F$ and $\beta_X F \longrightarrow F$ are epimorphisms, $\beta_X L \simeq$ "$\underset{i\in I}{\oplus}$" ${}_{U_i}\mathcal{A} \longrightarrow F$ is an epimorphism. Hence there exist a finite subset J of I and $V_i \subset\subset U_i$ such that $\oplus_{i\in J}\mathcal{A}_{V_i} \longrightarrow F$ is an epimorphism. Hence $F = \sum_{i\in J}\mathcal{A}_{V_i}s_i$. For any $x \in X$, set $J(x) = \{i \in J\,;\, x \in U_i\}$. Then $W = \bigcap_{i\in J(x)} U_i \setminus (\bigcup_{i\in J\setminus J(x)} \overline{V_i})$ is an open neighborhood of x. Since $V_i \cap W = \varnothing$ for any $i \in J \setminus J(x)$, we have $F|_W = \sum_{i\in J(x)}(\mathcal{A}|_W)s_i$.

(i) (b) Conversely, assume that F is locally of finite type. There exists locally an epimorphism $g : \mathcal{A}^n \twoheadrightarrow F$. This morphism factors through $\mathcal{A}^n \xleftarrow{\sim} \beta_{\mathcal{A}}(\mathcal{A}^n) \xrightarrow[\beta_{\mathcal{A}}(g)]{} \beta_{\mathcal{A}} F \longrightarrow F$. Hence $\beta_{\mathcal{A}} F \longrightarrow F$ is an epimorphism.

(ii) (a) Assume that $\beta_{\mathcal{A}} F \longrightarrow F$ is an isomorphism. By (i), F is locally of finite type. Hence there exists locally an exact sequence $0 \longrightarrow N \longrightarrow \mathcal{A}^n \longrightarrow F \longrightarrow 0$.

Consider the commutative exact diagram

$$\begin{tikzcd} & \beta_{\mathcal{A}}N \arrow[r] \arrow[d] & \beta_{\mathcal{A}}\mathcal{A}^n \arrow[r] \arrow[d, "\sim"] & \beta_{\mathcal{A}}F \arrow[r] \arrow[d, "\sim"] & 0 \\ 0 \arrow[r] & N \arrow[r] & \mathcal{A}^n \arrow[r] & F \arrow[r] & 0. \end{tikzcd}$$

It shows that $\beta_{\mathcal{A}}N \longrightarrow N$ is an epimorphism, and hence N is locally of finite type. This implies that F is locally of finite presentation.

(ii) (b) Conversely, assume that F is locally of finite presentation and let us show that $\beta_{\mathcal{A}}F \longrightarrow F$ is an isomorphism. There exists locally an exact sequence $\mathcal{A}^m \longrightarrow \mathcal{A}^n \longrightarrow F \longrightarrow 0$. Then the assertion is obvious by the following diagram with exact rows

$$\begin{tikzcd} \beta_{\mathcal{A}}\mathcal{A}^m \arrow[r] \arrow[d, "\sim"] & \beta_{\mathcal{A}}\mathcal{A}^n \arrow[r] \arrow[d, "\sim"] & \beta_{\mathcal{A}}F \arrow[r] \arrow[d] & 0 \\ \mathcal{A}^m \arrow[r] & \mathcal{A}^n \arrow[r] & F \arrow[r] & 0. \end{tikzcd}$$

q.e.d.

Corollary 3.1.9. *$F \in \mathrm{Mod}(\mathcal{A})$ is light if and only if F is locally of finite presentation.*

Proposition 3.1.10. *The following two conditions are equivalent:*

(i) *the functor $\beta_{\mathcal{A}} : \mathrm{Mod}(\mathcal{A}) \longrightarrow \mathrm{I}(\mathcal{A})$ is exact,*

(ii) *the sheaf of rings $\mathcal{A}$ is (left) coherent,*

Proof. (i) $\Rightarrow$ (ii). Consider an exact sequence $0 \longrightarrow F \longrightarrow \mathcal{A}^m \longrightarrow \mathcal{A}$. Applying the exact functor $\beta_{\mathcal{A}}$, we find the isomorphism $\beta_{\mathcal{A}}F \xrightarrow{\sim} F$. Then Proposition 3.1.8 implies that F is locally of finite type.

In order to prove (ii) $\Rightarrow$ (i), we need a lemma.

Lemma 3.1.11. *Assume that $\mathcal{A}$ is coherent and consider an exact sequence $0 \longrightarrow N \longrightarrow L_1 \longrightarrow L_0$ in $\mathrm{Mod}(\mathcal{A})$ with L_1 and L_0 in $\mathcal{F}(\mathcal{A})$. Then the sequence $\beta_{\mathcal{A}}N \longrightarrow \beta_{\mathcal{A}}L_1 \longrightarrow \beta_{\mathcal{A}}L_0$ is exact.*

Proof. By the result of Exercise 2.3, it is enough to show that for any $F \in \mathrm{Mod}(\mathcal{A})$, the sequence below is exact:

$$\mathcal{H}om_{\mathrm{I}(\mathcal{A})}(F, \beta_{\mathcal{A}}N) \longrightarrow \mathcal{H}om_{\mathrm{I}(\mathcal{A})}(F, \beta_{\mathcal{A}}L_1) \longrightarrow \mathcal{H}om_{\mathrm{I}(\mathcal{A})}(F, \beta_{\mathcal{A}}L_0). \tag{3.1.3}$$

Let $x \in X$. In a neighborhood of x, take $u : F \longrightarrow \beta_{\mathcal{A}} L_1$ such that the composition $F \longrightarrow \beta_{\mathcal{A}} L_1 \longrightarrow \beta_{\mathcal{A}} L_0$ vanishes. Applying Lemma 3.1.7, this morphism $F \longrightarrow \beta_{\mathcal{A}} L_1$ factorizes through $\mathcal{A}^n$, and we get a commutative diagram

$$\begin{tikzcd}
 & \mathcal{A}^k \arrow[rr, "b"] \arrow[d] & & \mathcal{A}^m \arrow[d] \\
F \arrow[ur, "\varphi"] \arrow[r] & \mathcal{A}^n \arrow[r] & \beta_{\mathcal{A}} L_1 \arrow[r] & \beta_{\mathcal{A}} L_0
\end{tikzcd}$$

with $b \circ \varphi = 0$.

Set $G = \ker(b)$ and consider the diagram:

$$\begin{tikzcd}
 & F \arrow[r] \arrow[dr, "u"] & G \arrow[d, "\psi"] \arrow[dr, "0"] & \\
 & \beta_{\mathcal{A}} N \arrow[r, "\beta_{\mathcal{A}} v"] \arrow[d] & \beta_{\mathcal{A}} L_1 \arrow[r] \arrow[d] & \beta_{\mathcal{A}} L_0 \arrow[d] \\
0 \arrow[r] & N \arrow[r, "v"] & L_1 \arrow[r] & L_0 .
\end{tikzcd}$$

In order to prove that the morphism $u : F \longrightarrow \beta_{\mathcal{A}} L_1$ factorizes through $\beta_{\mathcal{A}} N$, it is enough to prove this result for the morphism $\psi : G \longrightarrow \beta_{\mathcal{A}} L_1$. Since the composition $G \longrightarrow L_1 \longrightarrow L_0$ is 0, we get a morphism $\xi : G \longrightarrow N$. Since G is coherent, we obtain a morphism $h : G \longrightarrow \beta_{\mathcal{A}} N$ as the composition $G \xleftarrow{\sim} \beta_{\mathcal{A}} G \xrightarrow[\beta_{\mathcal{A}} \xi]{} \beta_{\mathcal{A}} N$.

It remains to show that the composition

$$G \xrightarrow[h]{} \beta_{\mathcal{A}} N \xrightarrow[\beta_{\mathcal{A}} v]{} \beta_{\mathcal{A}} L_1$$

coincides with ψ. This follows from the fact that $\beta_{\mathcal{A}} L_1 \longrightarrow L_1$ is a monomorphism, and the commutative diagram below:

$$\begin{tikzcd}
\beta_{\mathcal{A}} G \arrow[r, "\beta_{\mathcal{A}} \xi"] \arrow[d, "\sim"] & \beta_{\mathcal{A}} N \arrow[r, "\beta_{\mathcal{A}} v"] \arrow[d] & \beta_{\mathcal{A}} L_1 \arrow[d] \\
G \arrow[r, "\xi"'] \arrow[ur, "h"] \arrow[urr, "\psi"] & N \arrow[r, "v"] & L_1 .
\end{tikzcd}$$

q.e.d.

We can now complete the proof of Proposition 3.1.10.

Consider an exact sequence $0 \longrightarrow F' \longrightarrow F \longrightarrow F'' \longrightarrow 0$ in $\mathrm{Mod}(\mathcal{A})$. We can find an exact commutative diagram as below such that the sheaves L'_j, L_j, L''_j $(j = 0, 1)$

are in $\mathcal{F}(\mathcal{A})$, and moreover such that the second and third rows split:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & N' \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & N'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L'_1 \arrow[r] \arrow[d] & L_1 \arrow[r] \arrow[d] & L''_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L'_0 \arrow[r] \arrow[d] & L_0 \arrow[r] \arrow[d] & L''_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F' \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & F'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Let us apply the right exact functor $\beta_{\mathcal{A}}$ to the diagram. The second and third rows being split exact, they will remain exact. Then the result follows from Lemma 3.1.11. q.e.d.

3.2 Internal operations

Definition 3.2.1. We define the functors internal tensor product, denoted by $\otimes_{\mathcal{A}}$, and internal hom, denoted by $\mathcal{I}hom_{\mathcal{A}}$

$$\begin{aligned} \otimes_{\mathcal{A}} &: \mathrm{I}(\mathcal{A}^{\mathrm{op}}) \times \mathrm{I}(\mathcal{A}) \longrightarrow \mathrm{I}(k_X) \\ \mathcal{I}hom_{\mathcal{A}} &: \mathrm{I}(\mathcal{A})^{\mathrm{op}} \times \mathrm{I}(\mathcal{A}) \longrightarrow \mathrm{I}(k_X) \end{aligned}$$

by the formulas:

$$\begin{aligned} (\text{“}\varinjlim_i\text{”}\, F_i) \otimes_{\mathcal{A}} (\text{“}\varinjlim_j\text{”}\, G_j) &= \text{“}\varinjlim_{i,j}\text{”}(F_i \otimes_{\mathcal{A}} G_j), \\ \mathcal{I}hom_{\mathcal{A}}(\text{“}\varinjlim_i\text{”}\, F_i, \text{“}\varinjlim_j\text{”}\, G_j) &= \varprojlim_i \text{“}\varinjlim_j\text{”}\, \mathcal{H}om_{\mathcal{A}}(F_i, G_j). \end{aligned}$$

Similarly we define $\otimes$ and $\mathcal{I}hom$ using $\otimes_{k_X}$ and $\mathcal{H}om_{k_X}$. Then those functors $\otimes$ and $\mathcal{I}hom$ induce

$$\begin{aligned} \otimes &: \mathrm{I}(k_X) \times \mathrm{I}(\mathcal{A}) \longrightarrow \mathrm{I}(\mathcal{A}), \\ \otimes &: \mathrm{I}(\mathcal{A}) \times \mathrm{I}(k_X) \longrightarrow \mathrm{I}(\mathcal{A}), \\ \mathcal{I}hom &: \mathrm{I}(k_X)^{\mathrm{op}} \times \mathrm{I}(\mathcal{A}) \longrightarrow \mathrm{I}(\mathcal{A}), \\ \mathcal{I}hom &: \mathrm{I}(\mathcal{A}^{\mathrm{op}})^{\mathrm{op}} \times \mathrm{I}(k_X) \longrightarrow \mathrm{I}(\mathcal{A}). \end{aligned}$$

In the course of the proofs below, we shall sometimes write $\otimes$, $\mathcal{H}om$ and $\mathcal{I}hom$ instead of $\otimes_{\mathcal{A}}$, $\mathcal{H}om_{\mathcal{A}}$ and $\mathcal{I}hom_{\mathcal{A}}$, for short.

Proposition 3.2.2. (i) *The functor $\otimes_{\mathcal{A}}$ is right exact and commutes with "$\varinjlim$". Moreover, $\mathcal{A} \otimes_{\mathcal{A}} F \simeq F$.*

(ii) *The functor $\mathcal{I}hom_{\mathcal{A}}$ is left exact. Moreover, $\mathcal{I}hom_{\mathcal{A}}(\mathcal{A}, F) \simeq F$.*

The proof is obvious.

Proposition 3.2.3. *The diagram below commutes:*

$$\begin{tikzcd}
\mathrm{Mod}(\mathcal{A}^{\mathrm{op}}) \times \mathrm{Mod}(\mathcal{A}) \arrow[r, "\otimes_{\mathcal{A}}"] \arrow[d, "\iota_X \times \iota_X"] & \mathrm{Mod}(k_X) \arrow[d, "\iota_X"] \\
\mathrm{I}(\mathcal{A}^{\mathrm{op}}) \times \mathrm{I}(\mathcal{A}) \arrow[r, "\otimes_{\mathcal{A}}"] \arrow[d, "\alpha_X \times \alpha_X"] & \mathrm{I}(k_X) \arrow[d, "\alpha_X"] \\
\mathrm{Mod}(\mathcal{A}^{\mathrm{op}}) \times \mathrm{Mod}(\mathcal{A}) \arrow[r, "\otimes_{\mathcal{A}}"] & \mathrm{Mod}(k_X).
\end{tikzcd}$$

Proof. Let F and G in $\mathrm{Mod}(k_X)$. Then:

$$\begin{aligned}
\iota_X(F \otimes G) &= \text{“}\varinjlim_U\text{”}(F \otimes G)_U \simeq \text{“}\varinjlim_U\text{”}(F_U \otimes G_U) \\
&\simeq \text{“}\varinjlim_{U,U'}\text{”}(F_U \otimes G_{U'}) \simeq (\text{“}\varinjlim_U\text{”}\, F_U) \otimes (\text{“}\varinjlim_{U'}\text{”}\, G_{U'})\,.
\end{aligned}$$

Let $F = \text{“}\varinjlim_i\text{”}\, F_i$ and $G = \text{“}\varinjlim_j\text{”}\, G_j$. Then

$$\begin{aligned}
\alpha_X((\text{“}\varinjlim_i\text{”}\, F_i) \otimes (\text{“}\varinjlim_j\text{”}\, G_j)) &\simeq \alpha_X(\text{“}\varinjlim_{i,j}\text{”}(F_i \otimes G_j)) \\
&\simeq \varinjlim_{i,j}(F_i \otimes G_j) \simeq (\varinjlim_i F_i) \otimes (\varinjlim_j G_j)\,.
\end{aligned}$$

q.e.d.

Proposition 3.2.4. *The diagram below commutes:*

$$\begin{tikzcd}
\mathrm{Mod}(\mathcal{A})^{\mathrm{op}} \times \mathrm{Mod}(\mathcal{A}) \arrow[r, "\mathcal{H}om_{\mathcal{A}}"'] \arrow[d, "\iota_X \times \iota_X"] & \mathrm{Mod}(k_X) \arrow[d, "\iota_X"] \\
\mathrm{I}(\mathcal{A})^{\mathrm{op}} \times \mathrm{I}(\mathcal{A}) \arrow[r, "\mathcal{I}hom_{\mathcal{A}}"'] \arrow[dr, "\mathcal{H}om_{I(\mathcal{A})}"'] & \mathrm{I}(k_X) \arrow[d, "\alpha_X"] \\
& \mathrm{Mod}(k_X).
\end{tikzcd}$$

Proof. (i) Let $F, G \in \mathrm{Mod}(k_X)$. Then

$$\begin{aligned}\mathcal{I}hom\,(\iota_X F, \iota_X G) &\simeq \varprojlim_U \text{“}\varinjlim_V\text{”}\, \mathcal{H}om\,(F_U, G_V) \\ &\simeq \varprojlim_U \mathcal{H}om\,(F_U, G) \simeq \iota_X(\mathcal{H}om\,(F, G)).\end{aligned}$$

(ii) Let $F = \text{“}\varinjlim_i\text{”}\, F_i, G = \text{“}\varinjlim_j\text{”}\, G_j$. Since α_X commutes with $\varprojlim$, we get the chain of isomorphisms

$$\begin{aligned}\alpha_X(\mathcal{I}hom\,(F, G)) &\simeq \alpha_X(\varprojlim_i \text{“}\varinjlim_j\text{”}\, \mathcal{H}om\,(F_i, G_j)) \\ &\simeq \varprojlim_i \varinjlim_j \mathcal{H}om\,(F_i, G_j) \simeq \mathcal{H}om\,(F, G).\end{aligned}$$

q.e.d.

Proposition 3.2.5. *Let $K \in \mathrm{I}(k_X)$ and $F,\ G \in \mathrm{I}(\mathcal{A})$. Then:*

$$\begin{aligned}\mathrm{Hom}_{\mathrm{I}(\mathcal{A})}(K \otimes F, G) &\simeq \mathrm{Hom}_{\mathrm{I}(k_X)}(K, \mathcal{I}hom_{\mathcal{A}}(F, G)) \\ &\simeq \mathrm{Hom}_{\mathrm{I}(\mathcal{A})}(F, \mathcal{I}hom\,(K, G)).\end{aligned}$$

In particular, the functors $K \otimes \cdot$ is a left adjoint of $\mathcal{I}hom\,(K, \cdot)$.

Proof. One has the chain of isomorphisms:

$$\begin{aligned}&\mathrm{Hom}_{\mathrm{I}(\mathcal{A})}(\text{“}\varinjlim_k\text{”}\, K_k \otimes \text{“}\varinjlim_i\text{”}\, F_i\,,\ \text{“}\varinjlim_j\text{”}\, G_j) \\ &\simeq \mathrm{Hom}_{\mathrm{I}(\mathcal{A})}(\text{“}\varinjlim_{i,k}\text{”}(K_k \otimes F_i),\ \text{“}\varinjlim_j\text{”}\, G_j) \\ &\simeq \varprojlim_{i,k} \varinjlim_j \mathrm{Hom}_{\mathcal{A}}(K_k \otimes F_i, G_j) \\ &\simeq \varprojlim_{i,k} \varinjlim_j \mathrm{Hom}_{k_X}(K_k, \mathcal{H}om_{\mathcal{A}}(F_i, G_j)) \\ &\simeq \varprojlim_i \mathrm{Hom}_{\mathrm{I}(k_X)}(\text{“}\varinjlim_k\text{”}\, K_k,\ \text{“}\varinjlim_j\text{”}\, \mathcal{H}om_{\mathcal{A}}(F_i, G_j)) \\ &\simeq \mathrm{Hom}_{\mathrm{I}(k_X)}(\text{“}\varinjlim_k\text{”}\, K_k, \varprojlim_i \text{“}\varinjlim_k\text{”}\, \mathcal{H}om_{\mathcal{A}}(F_i, G_j)) \\ &\simeq \mathrm{Hom}_{\mathrm{I}(k_X)}\Big(\text{“}\varinjlim_k\text{”}\, K_k, \mathcal{I}hom_{\mathcal{A}}(\text{“}\varinjlim_i\text{”}\, F_i,\ \text{“}\varinjlim_j\text{”}\, G_j)\Big).\end{aligned}$$

The second isomorphism is proved similarly. q.e.d.

Corollary 3.2.6. *For $F, G \in \mathrm{I}(\mathcal{A})$ there is a canonical morphism*

$$F \otimes \mathcal{I}hom_{\mathcal{A}}(F, G) \to G.$$

Corollary 3.2.7. *For F, $G \in \mathrm{I}(\mathcal{A})$ and $K \in \mathrm{I}(k_X)$, there is a canonical morphism*

$$\mathcal{I}hom_{\mathcal{A}}(F,G) \otimes K \to \mathcal{I}hom_{\mathcal{A}}(F, G \otimes K).$$

Proof. By the preceding corollary there is a morphism $F \otimes \mathcal{I}hom_{\mathcal{A}}(F,G) \otimes K \to G \otimes K$, and we obtain the desired morphism by adjunction (Proposition 3.2.5). q.e.d.

Corollary 3.2.8. *Let $K \in \mathrm{I}(\mathcal{A})$ and let $i \mapsto F_i \in \mathrm{I}(\mathcal{A})$ be a small filtrant inductive system of ind-sheaves, and $j \mapsto G_j \in \mathrm{I}(\mathcal{A})$ a small filtrant projective system of ind-sheaves. Then, we have the isomorphisms*

(i) $\mathcal{I}hom_{\mathcal{A}}(K, \varprojlim_j G_j) \simeq \varprojlim_j \mathcal{I}hom_{\mathcal{A}}(K, G_j)$,

(ii) $\mathcal{I}hom_{\mathcal{A}}(\text{“}\varinjlim_i\text{”}\, F_i, K) \simeq \varprojlim_i \mathcal{I}hom_{\mathcal{A}}(F_i, K)$.

(iii) *If $K \in \mathrm{Mod}(\mathcal{A})$, then one has* $\mathcal{I}hom_{\mathcal{A}}(K, \text{“}\varinjlim_i\text{”}\, F_i) \simeq \text{“}\varinjlim_i\text{”}\, \mathcal{I}hom_{\mathcal{A}}(K, F_i)$,

Proof. (i) Let $S \in \mathrm{Mod}^c(k_X)$. Then

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{I}(k_X)}(S, \mathcal{I}hom_{\mathcal{A}}(K, \varprojlim_j G_j)) &\simeq \mathrm{Hom}_{\mathrm{I}(\mathcal{A})}(S \otimes K, \varprojlim_j G_j) \\
&\simeq \varprojlim_j \mathrm{Hom}_{\mathrm{I}(\mathcal{A})}(S \otimes K, G_j) \\
&\simeq \varprojlim_j \mathrm{Hom}_{\mathrm{I}(k_X)}(S, \mathcal{I}hom_{\mathcal{A}}(K, G_j)) \\
&\simeq \mathrm{Hom}_{\mathrm{I}(k_X)}(S, \varprojlim_j \mathcal{I}hom_{\mathcal{A}}(K, G_j)).
\end{aligned}$$

(ii) Let $S \in \mathrm{Mod}^c(k_X)$. Then

$$\begin{aligned}
&\mathrm{Hom}_{\mathrm{I}(k_X)}(S, \mathcal{I}hom_{\mathcal{A}}(\text{“}\varinjlim_i\text{”}\, F_i, K)) \simeq \mathrm{Hom}_{\mathrm{I}(\mathcal{A})}(S \otimes \text{“}\varinjlim_i\text{”}\, F_i, K) \\
&\quad \simeq \mathrm{Hom}_{\mathrm{I}(\mathcal{A})}(\text{“}\varinjlim_i\text{”}(S \otimes F_i), K) \simeq \varprojlim_i \mathrm{Hom}_{\mathrm{I}(\mathcal{A})}(S \otimes F_i, K) \\
&\quad \simeq \varprojlim_i \mathrm{Hom}_{\mathrm{I}(k_X)}(S, \mathcal{I}hom_{\mathcal{A}}(F_i, K)) \simeq \mathrm{Hom}_{\mathrm{I}(k_X)}(S, \varprojlim_i \mathcal{I}hom_{\mathcal{A}}(F_i, K)).
\end{aligned}$$

(iii) Let $S \in \mathrm{Mod}^c(k_X)$. Then

$$\begin{aligned}
&\mathrm{Hom}_{\mathrm{I}(k_X)}(S, \mathcal{I}hom_{\mathcal{A}}(K, \text{“}\varinjlim_i\text{”}\, F_i)) \simeq \mathrm{Hom}_{\mathrm{I}(\mathcal{A})}(S \otimes K, \text{“}\varinjlim_i\text{”}\, F_i) \\
&\quad \simeq \varinjlim_i \mathrm{Hom}_{\mathrm{I}(\mathcal{A})}(S \otimes K, F_i) \simeq \varinjlim_i \mathrm{Hom}_{\mathrm{I}(k_X)}(S, \mathcal{I}hom_{\mathcal{A}}(K, F_i)) \\
&\quad \simeq \mathrm{Hom}_{\mathrm{I}(k_X)}(S, \text{“}\varinjlim_i\text{”}\, \mathcal{I}hom_{\mathcal{A}}(K, F_i)).
\end{aligned}$$

q.e.d.

Corollary 3.2.9. *Let $K \in \mathrm{I}(k_X)$ and $F,\ G \in \mathrm{I}(\mathcal{A})$. Then:*

$$\begin{aligned} \mathcal{I}hom_{\mathcal{A}}(K \otimes F, G) &\simeq \mathcal{I}hom\,(K, \mathcal{I}hom_{\mathcal{A}}(F,G)) \\ &\simeq \mathcal{I}hom_{\mathcal{A}}(F, \mathcal{I}hom\,(K,G)). \end{aligned}$$

Proof. For $S \in \mathrm{Mod}^c(k_X)$ we have

$$\begin{aligned} \mathrm{Hom}_{\mathrm{I}(k_X)}(S, \mathcal{I}hom_{\mathcal{A}}(K \otimes F, G)) &\simeq \mathrm{Hom}_{\mathrm{I}(\mathcal{A})}(S \otimes K \otimes F, G) \\ &\simeq \mathrm{Hom}_{\mathrm{I}(k_X)}(S \otimes K, \mathcal{I}hom_{\mathcal{A}}(F,G)) \\ &\simeq \mathrm{Hom}_{\mathrm{I}(k_X)}(S, \mathcal{I}hom\,(K, \mathcal{I}hom_{\mathcal{A}}(F,G)))\,. \end{aligned}$$

This shows the first isomorphism. The proof of the second isomorphism is similar. q.e.d.

We shall now study the relations among the functors $\otimes$, $\mathcal{I}hom$ and β_X. Here β_X is the functor from $\mathrm{Mod}(k_X) \longrightarrow \mathrm{I}(k_X)$.

Proposition 3.2.10. *Let $K \in \mathrm{Mod}(k_X)$ and $F,\ G \in \mathrm{I}(\mathcal{A})$. Then:*

$$\begin{aligned} \mathrm{Hom}_{k_X}(K, \mathcal{H}om_{\mathcal{A}}(F,G)) &\simeq \mathrm{Hom}_{\mathrm{I}(\mathcal{A})}(\beta_X K \otimes F, G), \\ \mathcal{H}om\,(K, \mathcal{H}om_{\mathcal{A}}(F,G)) &\simeq \mathcal{H}om_{\mathcal{A}}(\beta_X K \otimes F, G). \end{aligned}$$

Proof. Consider the chain of isomorphisms:

$$\begin{aligned} \mathrm{Hom}_{k_X}(K, \mathcal{H}om_{\mathcal{A}}(F,G)) &\simeq \mathrm{Hom}_{\mathrm{I}(k_X)}(K, \alpha_X(\mathcal{I}hom_{\mathcal{A}}(F,G))) \\ &\simeq \mathrm{Hom}_{\mathrm{I}(k_X)}(\beta_X K, \mathcal{I}hom_{\mathcal{A}}(F,G)) \\ &\simeq \mathrm{Hom}_{\mathrm{I}(k_X)}(\beta_X K \otimes F, G). \end{aligned}$$

The second formula follows. q.e.d.

Proposition 3.2.11. *Let F and G be in $\mathrm{Mod}(k_X)$. Then:*

$$\beta_X F \otimes \beta_X G \simeq \beta_X(F \otimes G).$$

Proof. Let $K \in \mathrm{I}(k_X)$. Consider the chain of isomorphisms:

$$\begin{aligned} &\mathrm{Hom}_{\mathrm{I}(k_X)}(\beta_X F \otimes \beta_X G, K) \simeq \mathrm{Hom}_{\mathrm{I}(k_X)}(\beta_X F, \mathcal{I}hom\,(\beta_X G, K)) \\ &\quad \simeq \mathrm{Hom}_{k_X}(F, \mathcal{H}om\,(\beta_X G, K)) \simeq \mathrm{Hom}_{k_X}(F, \mathcal{H}om\,(G, \alpha_X K)) \\ &\quad \simeq \mathrm{Hom}_{k_X}(F \otimes G, \alpha_X K) \simeq \mathrm{Hom}_{\mathrm{I}(k_X)}(\beta_X(F \otimes G), K). \end{aligned}$$

Here the third isomorphism follows from Lemma 2.3.29. q.e.d.

Remark 3.2.12. For $F \in \mathrm{I}(\mathcal{A})$, $\mathcal{H}om_{\mathcal{A}}(F,\,\cdot\,)$ and $\mathcal{I}hom_{\mathcal{A}}(F,\,\cdot\,)$ commute with inductive limits if and only if $F \in \mathrm{Mod}(\mathcal{A})$. $\mathrm{Hom}_{\mathrm{I}(\mathcal{A})}(F,\,\cdot\,)$ commutes with inductive limits if and only if $F \in \mathrm{Mod}^c(\mathcal{A})$. See Exercise 2.5.

Sheaves associated with locally closed subsets

In this subsection, k is a commutative ring and the base ring is k_X. Recall that if Z is a locally closed subset and $F \in \mathrm{I}(k_X)$, we have constructed the ind-sheaf ${}_ZF$ in Definition 2.3.15, by setting, for $Z = U \cap S$, U open, S closed:

$$ {}_ZF = \underset{V \subset\subset U, W \supset S}{\text{“}\varinjlim\text{”}} F_{V \cap \overline{W}}. $$

When there is no risk of confusion, we often write k_Z and ${}_Zk$ instead of $(k_X)_Z$ and ${}_Z(k_X)$, respectively.

If $Z = \{x\}$ with $x \in X$, we shall write ${}_xk$ and k_x instead of ${}_{\{x\}}k$ and $k_{\{x\}}$, respectively.

Proposition 3.2.13. (i) *One has the isomorphisms*

$$ {}_Zk \simeq \beta_X k_Z, \ \ {}_ZF \simeq F \otimes {}_Zk. $$

(ii) *Let U be open and let $F, G \in \mathrm{I}(k_X)$. Then*

$$ \mathrm{Hom}_{\mathrm{I}(k_U)}(F|_U, G|_U) \simeq \mathrm{Hom}_{\mathrm{I}(k_X)}(F \otimes {}_Uk, G). $$

(iii) *Let Z_1 and Z_2 be two locally closed subsets. Then*

$$ {}_{Z_1 \cap Z_2}k \simeq {}_{Z_1}k \otimes {}_{Z_2}k. $$

(iv) *Assume that Z' is closed in Z. Then there is an exact sequence*

$$ 0 \longrightarrow {}_{Z \setminus Z'}k \longrightarrow {}_Zk \longrightarrow {}_{Z'}k \longrightarrow 0. $$

(v) *Let Z be a locally closed subset. Then for $G \in \mathrm{Mod}(k_X)$ and $F \in \mathrm{I}(k_X)$, one has*

$$ \mathcal{H}om\,(G, F \otimes {}_Zk) \simeq \mathcal{H}om\,(G, F) \otimes k_Z. $$

Proof.
(ii) By Proposition 3.2.10, one has $\mathrm{Hom}_{\mathrm{I}(k_X)}(F \otimes {}_Uk, G) \simeq \mathrm{Hom}_{k_X}(k_U, \mathcal{H}om\,(F, G))$.
(v) is a particular case of Proposition 2.3.16.
The other results are obvious. q.e.d.

Corollary 3.2.14. (i) *If U is open, the morphism ${}_Uk \longrightarrow k_U$ is a monomorphism.*

(ii) *If S is closed, the morphism ${}_Sk \longrightarrow k_S$ is an epimorphism.*

Proposition 3.2.15. *For any locally closed subset Z and $F \in \mathrm{I}(k_X)$, one has $\Gamma_Z F \simeq \mathcal{I}hom\,({}_Z k, F)$.*

Proof. For any $G \in \mathrm{I}(k_X)$, one has

$$\begin{aligned}\mathrm{Hom}_{\mathrm{I}(k_X)}(G, \Gamma_Z F) &\simeq \Gamma_Z(X\,;\mathcal{H}om\,(G,F)) \simeq \mathrm{Hom}_{\mathrm{I}(k_X)}(G_Z, F)\\ &\simeq \mathrm{Hom}_{\mathrm{I}(k_X)}({}_Z k \otimes G, F) \simeq \mathrm{Hom}_{\mathrm{I}(k_X)}(G, \mathcal{I}hom\,({}_Z k, F)).\end{aligned}$$

q.e.d.

Example 3.2.16. Let X be a real manifold of dimension $n \geq 1$ and let $x \in X$. Define $N_x \in \mathrm{I}(k_X)$ by the exact sequence

$$0 \longrightarrow N_x \longrightarrow {}_x k \longrightarrow k_x \longrightarrow 0. \tag{3.2.1}$$

Since $\mathrm{Hom}\,(k_x, {}_x k) \simeq \varinjlim \mathrm{Hom}\,(k_x, k_{\overline{V}})$, where V ranges through a neighborhood system of x, we find $\mathrm{Hom}\,(k_x, {}_x k) = 0$, hence $N_x \neq 0$. On the other hand,

$$\mathrm{Hom}\,(k_U, N_x) = 0$$

for any open neighborhood U of x.

Quasi-injective ind-sheaves

In this subsection, the base ring is k_X for a commutative ring k.

Recall that an object $K \in \mathrm{I}(k_X)$ is called a quasi-injective ind-sheaf if the functor $G \mapsto \mathrm{Hom}_{\mathrm{I}(k_X)}(G, K)$ is exact on $\mathrm{Mod}^c(k_X)$, or equivalently, if $K \simeq \text{“}\varinjlim_i\text{”}\, K_i$ with $K_i \in \mathrm{Mod}^c(k_X)$ and K_i injective in $\mathrm{Mod}^c(k_X)$.

Proposition 3.2.17. *Let $K \in \mathrm{I}(k_X)$ and consider the following properties:*

(i) *K is quasi-injective,*

(ii) *$K \simeq \text{“}\varinjlim_i\text{”}\, K_i$ with $K_i \in \mathrm{Mod}(k_X)$ and K_i injective,*

(iii) *the functor $\mathrm{Hom}_{\mathrm{I}(k_X)}(\,\cdot\,, K)$ is exact on $\mathrm{Mod}(k_X)$,*

(iv) *the functor $\mathcal{H}om\,(\,\cdot\,, K)$ is exact on $\mathrm{Mod}(k_X)$, and for any $F \in \mathrm{Mod}(k_X)$, the sheaf $\mathcal{H}om\,(F, K)$ is soft,*

(v) *the functor $\mathcal{I}hom\,(\,\cdot\,, K)$ is exact on $\mathrm{Mod}(k_X)$.*

Then (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) $\Rightarrow$ (v).

Proof. (i) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (i). One has the isomorphism $K \simeq \text{“}\varinjlim_{Z,i}\text{”}\, \Gamma_Z(K_i)$, where Z ranges over the family of compact subsets of X. Since $\Gamma_Z(K_i)$ is injective in $\mathrm{Mod}^c(k_X)$, the result follows.

(ii) $\Rightarrow$ (v) follows from the isomorphism $\mathcal{I}hom\,(F, K) \simeq \text{“}\varinjlim_{i}\text{”}\, \mathcal{H}om\,(F, K_i)$, and the fact that $\mathcal{H}om\,(\,\cdot\,, K_i)$ is exact on $\mathrm{Mod}(k_X)$.

(ii)+(v) $\Rightarrow$ (iv). By applying the exact functor α_X to $\mathcal{I}hom\,(\,\cdot\,, K)$ we get that $\mathcal{H}om_{\mathrm{I}(k_X)}(\,\cdot\,, K)$ is exact. Then $\mathcal{H}om_{k_X}(F, K_i)$ is flabby, and $\mathcal{H}om_{\mathrm{I}(k_X)}(F, K) \simeq \varinjlim_{i} \mathcal{H}om\,(F, K_i)$ is soft.

(iv) $\Rightarrow$ (iii) follows from $\mathrm{Hom}_{\mathrm{I}(k_X)}(\,\cdot\,, K) = \Gamma(X; \mathcal{H}om\,(\,\cdot\,, K))$.

(iii) $\Rightarrow$ (i) is obvious. q.e.d.

Corollary 3.2.18. *Let $0 \to K' \to K \to K'' \to 0$ be an exact sequence in $\mathrm{I}(k_X)$, and assume that K' is quasi-injective. Then for any $F \in \mathrm{Mod}(k_X)$ the following sequences are exact:*

$$\begin{array}{ccccccccc}
0 & \to & \mathcal{I}hom\,(F, K') & \to & \mathcal{I}hom\,(F, K) & \to & \mathcal{I}hom\,(F, K'') & \to & 0, \\
0 & \to & \mathcal{H}om\,(F, K') & \to & \mathcal{H}om\,(F, K) & \to & \mathcal{H}om\,(F, K'') & \to & 0, \\
0 & \to & \mathrm{Hom}_{\mathrm{I}(k_X)}(F, K') & \to & \mathrm{Hom}_{\mathrm{I}(k_X)}(F, K) & \to & \mathrm{Hom}_{\mathrm{I}(k_X)}(F, K'') & \to & 0\,.
\end{array}$$

Proof. There exists a filtrant inductive system of exact sequences $0 \to K'_i \to K_i \to K''_i \to 0$ in $\mathrm{Mod}(k_X)$ such that every K'_i is injective and its inductive limit gives $0 \to K' \to K \to K'' \to 0$. Then $0 \to \mathcal{H}om\,(F, K'_i) \to \mathcal{H}om\,(F, K_i) \to \mathcal{H}om\,(F, K''_i) \to 0$ is exact. Hence we obtain the exactness of the first and second sequences. Since $\mathcal{H}om\,(F, K')$ is soft by the preceding proposition, the last sequence is exact by applying $\Gamma(X;\,\cdot\,)$ to the second. q.e.d.

Proposition 3.2.19. *Assume that k is a field. Let $K \in \mathrm{I}(k_X)$ be quasi-injective. Let $F \in \mathrm{Mod}(k_X)$. Then $\beta_X F \otimes K$ is quasi-injective.*

Proof. (i) Let us first prove the result when $F = k_Z$ for a locally close set Z. By the result of Proposition 3.2.13 (v), we have for any $G \in \mathrm{Mod}^c(k_X)$:

$$\mathrm{Hom}_{\mathrm{I}(k_X)}(G, \beta_X k_Z \otimes K) \;\simeq\; \Gamma(X; k_Z \otimes \mathcal{H}om\,(G, K))\,,$$

and this functor is exact with respect to G.

(ii) Let $F \in \mathrm{Mod}(k_X)$. There exists an epimorphism $\bigoplus_{i\in I} k_{U_i} \twoheadrightarrow F$. For a finite subset J of I, denote by F_J the image of $\bigoplus_{i\in J} k_{U_i}$. Then $\beta_X F \simeq \text{“}\varinjlim_{J}\text{”}(\beta_X F_J)$, and it is enough to prove the result with F replaced by F_J. If $|J| = 1$, then F is isomorphic to k_Z for some locally closed subset Z, and the result follows by (i). Then the proof goes by induction on the cardinal of J. Indeed, if one has an exact sequence $0 \longrightarrow F' \longrightarrow F \longrightarrow F'' \longrightarrow 0$ then $0 \longrightarrow \beta_X F' \otimes K \longrightarrow \beta_X F \otimes K \longrightarrow \beta_X F'' \otimes K \longrightarrow 0$ is exact, and if $\beta_X F' \otimes K$ and $\beta_X F'' \otimes K$ are quasi-injective then $\beta_X F \otimes K$ will be quasi-injective. q.e.d.

Proposition 3.2.20. *Assume that k is a field. Let F, $G \in \mathrm{Mod}(k_X)$ and let $K \in \mathrm{I}(k_X)$. Then there are natural isomorphisms*

$$\begin{aligned} \mathcal{I}hom\,(G,K) \otimes \beta_X F &\xrightarrow{\sim} \mathcal{I}hom\,(G, K \otimes \beta_X F)\,, \\ \mathcal{H}om\,(G,K) \otimes F &\xrightarrow{\sim} \mathcal{H}om\,(G, K \otimes \beta_X F)\,, \\ \Gamma(X; F \otimes \mathcal{H}om\,(G, k_X)) &\xrightarrow{\sim} \mathrm{Hom}_{\mathrm{I}(k_X)}(G, \beta_X F)\,. \end{aligned}$$

Note that the third formula gives an alternative definition of the ind-object $\beta_X F$ when $\mathcal{A} = k_X$.

Proof. The first morphism is obtained by Corollary 3.2.7, the second by applying α to the first, and the last by applying $\Gamma(X;\,\cdot\,)$ to the second.

First let us prove that the second morphism is an isomorphism. Remark that both sides are left exact with respect to K. Hence we may assume that K is quasi-injective. Since the functor $\mathcal{H}om\,(G,\,\cdot\,)$ sends exact sequence of quasi-injective objects to exact sequences, we get by Proposition 3.2.19 that both sides are exact with respect to F. Since both sides commute with $\varinjlim$ with respect to F, we may assume $F = k_U$ for an open subset U. Then the second formula follows from Proposition 3.2.13.

The third formula follows from the second one by applying the functor $\Gamma(X;\,\cdot\,)$ with $K = k_X$.

Finally let us prove the first formula. For any $S \in \mathrm{Mod}^c(k_X)$, one has

$$\begin{aligned} \mathrm{Hom}_{\mathrm{I}(k_X)}(S, \mathcal{I}hom\,(G,K) \otimes \beta_X F) &\simeq \Gamma\Big(X\,;\,\mathcal{H}om\,\big(S, \mathcal{I}hom\,(G,K) \otimes \beta_X F\big)\Big) \\ &\simeq \Gamma\Big(X\,;\,\mathcal{H}om\,\big(S, \mathcal{I}hom\,(G,K)\big) \otimes F\Big) \\ &\simeq \Gamma\Big(X\,;\,\mathcal{H}om\,(S \otimes G, K) \otimes F\Big) \\ &\simeq \Gamma\Big(X\,;\,\mathcal{H}om\,(S \otimes G, K \otimes \beta_X F)\Big) \\ &\simeq \Gamma\Big(X\,;\,\mathcal{H}om\,\big(S, \mathcal{I}hom\,(G, K \otimes \beta_X F)\big)\Big) \\ &\simeq \mathrm{Hom}_{\mathrm{I}(k_X)}(S, \mathcal{I}hom\,(G, K \otimes \beta_X F))\,. \end{aligned}$$

q.e.d.

3.3 External operations

Recall that all spaces are Hausdorff, locally compact and countable at infinity. Let $f : X \longrightarrow Y$ be a continuous map.

From now on, for the sake of simplicity, *we assume that k is a field,* and we consider only the base rings k_X, k_Y, etc. However, we could have worked in a more general setting, considering a sheaf of rings $\mathcal{B}$ on Y and the sheaf $\mathcal{A} := f^{-1}\mathcal{B}$ on X.

Inverse image

Let $G \in \mathrm{I}(k_Y)$. We define $f^{-1}G \in \mathrm{I}(k_X)$ as follows. If $G = \text{“}\varinjlim_i\text{”}\, G_i$ for $G_i \in \mathrm{Mod}^c(k_Y)$, we set:

$$f^{-1}G = \text{“}\varinjlim_{i, U\subset\subset X}\text{”}\,(f^{-1}G_i)_U.$$

Note that one also has $f^{-1}G = \text{“}\varinjlim_i\text{”}\, f^{-1}G_i$, where $f^{-1}G_i \in \mathrm{Mod}(k_X)$.

Note that if $j : U \hookrightarrow X$ is an open inclusion, and $F \in \mathrm{I}(k_X)$, then $j^{-1}F \simeq F|_U$, by Lemma 2.3.7.

Proposition 3.3.1. *The diagram below commutes:*

$$\begin{tikzcd}
\mathrm{Mod}(k_Y) \arrow[r, "f^{-1}"] \arrow[d, "\iota_Y"'] & \mathrm{Mod}(k_X) \arrow[d, "\iota_X"'] \\
\mathrm{I}(k_Y) \arrow[r, "f^{-1}"] \arrow[d, "\alpha_Y"'] & \mathrm{I}(k_X) \arrow[d, "\alpha_X"'] \\
\mathrm{Mod}(k_Y) \arrow[r, "f^{-1}"] & \mathrm{Mod}(k_X).
\end{tikzcd}$$

Proof. (i) Let $G \in \mathrm{Mod}(k_Y)$. Then $\iota_Y(G) = \text{“}\varinjlim_{V\subset\subset Y}\text{”}\, G_V$ and

$$\begin{aligned}
f^{-1}\iota_Y(G) &\simeq \text{“}\varinjlim_{V\subset\subset Y, U\subset\subset X}\text{”}\,(f^{-1}G_V)_U \\
&\simeq \text{“}\varinjlim_{V\subset\subset Y, U\subset\subset X}\text{”}\,(f^{-1}G)_{f^{-1}(V)\cap U} \simeq \text{“}\varinjlim_{U\subset\subset X}\text{”}(f^{-1}G)_U.
\end{aligned}$$

(ii) Let $G = \text{“}\varinjlim_i\text{”}\, G_i \in \mathrm{I}(k_Y)$. Then $\alpha_Y(G) = \varinjlim_i G_i$ and

$$f^{-1}\alpha_Y(G) \simeq f^{-1}\varinjlim_i G_i \simeq \varinjlim_i f^{-1}G_i \simeq \alpha_X(\text{“}\varinjlim_i\text{”}\, f^{-1}G_i)\,.$$

q.e.d.

Proposition 3.3.2. *The functor $f^{-1}\colon \mathrm{I}(k_Y) \longrightarrow \mathrm{I}(k_X)$ is exact and commutes with "$\varinjlim$" and $\otimes$.*

Proof. Recall that $f^{-1}G = \text{“}\varinjlim_{i,\,U\subset\subset X}\text{”}\,(f^{-1}G_i)_U$, and the functors "$\varinjlim$", f^{-1} and $(\,\cdot\,)_U$ are exact and commute with "$\varinjlim$" and $\otimes$. q.e.d.

Proposition 3.3.3. *For $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow Z$, one has $(g\circ f)^{-1} \simeq f^{-1}\circ g^{-1}$.*

The proof is straightforward.

We shall prove later (Corollary 3.3.7) that the functors f^{-1} and β commute.

Direct image

Let $F = \text{“}\varinjlim_i\text{”}\, F_i \in \mathrm{I}(k_X)$ for $F_i \in \mathrm{Mod}^c(k_X)$. One defines $f_*F \in \mathrm{I}(k_Y)$ by the formula:

$$f_*(\text{“}\varinjlim_i\text{”}\, F_i) = \varprojlim_K \text{“}\varinjlim_i\text{”}\, f_*(F_{iK}) \simeq \varprojlim_{U\subset\subset X} \text{“}\varinjlim_i\text{”}\, f_*(\Gamma_U(F_i)).$$

Here, K (resp. U) ranges through the family of compact (resp. relatively compact open) subsets of X. The isomorphism above is described by the morphisms

$$\begin{aligned}\varprojlim_K \text{“}\varinjlim_i\text{”}\, f_*(F_{iK}) \quad &\xrightarrow{\sim}\quad \varprojlim_{K,U} \text{“}\varinjlim_i\text{”}\, f_*(\Gamma_U F_i)_K \\ &\xleftarrow{\sim}\quad \varprojlim_U \text{“}\varinjlim_i\text{”}\, f_*(\Gamma_U F_i).\end{aligned}$$

Proposition 3.3.4. *The two functors f^{-1} and f_* are adjoint. More precisely, let $F \in \mathrm{I}(k_X)$ and let $G \in \mathrm{I}(k_Y)$. Then*

$$\mathrm{Hom}_{\mathrm{I}(k_X)}(f^{-1}G, F) \simeq \mathrm{Hom}_{\mathrm{I}(k_Y)}(G, f_*F).$$

Proof. Let $F = \text{“}\varinjlim_i\text{”}\, F_i$ and $G = \text{“}\varinjlim_j\text{”}\, G_j$. One has the chain of isomorphisms

$$\begin{aligned}\mathrm{Hom}_{\mathrm{I}(k_X)}(f^{-1}G, F) \;&\simeq\; \mathrm{Hom}_{\mathrm{I}(k_X)}\big(\text{“}\varinjlim_{j,\,U\subset\subset X}\text{”}\,(f^{-1}G_j)_U, \text{“}\varinjlim_i\text{”}\, F_i\big) \\ &\simeq\; \varprojlim_{j,U\subset\subset X}\ \varinjlim_i \mathrm{Hom}_{\mathrm{Mod}(k_X)}((f^{-1}G_j)_U, F_i) \\ &\simeq\; \varprojlim_{j,U\subset\subset X}\ \varinjlim_i \mathrm{Hom}_{\mathrm{Mod}(k_X)}(f^{-1}G_j, \Gamma_U F_i) \\ &\simeq\; \varprojlim_{j,U\subset\subset X}\ \varinjlim_i \mathrm{Hom}_{\mathrm{Mod}(k_X)}(G_j, f_*\Gamma_U F_i) \\ &\simeq\; \mathrm{Hom}_{\mathrm{I}(k_X)}(G, \varprojlim_{U\subset\subset X} \text{“}\varinjlim_i\text{”}\, f_*\Gamma_U F_i).\end{aligned}$$

q.e.d.

Corollary 3.3.5. *The functor* $f_* : \mathrm{I}(k_X) \to \mathrm{I}(k_Y)$ *is left exact and commutes with* $\varprojlim$.

Proposition 3.3.6. *The diagram below commutes:*

$$\begin{tikzcd}
\mathrm{Mod}(k_X) \arrow[r, "f_*"] \arrow[d, "\iota_X"'] & \mathrm{Mod}(k_Y) \arrow[d, "\iota_Y"] \\
\mathrm{I}(k_X) \arrow[r, "f_*"] \arrow[d, "\alpha_X"'] & \mathrm{I}(k_Y) \arrow[d, "\alpha_Y"] \\
\mathrm{Mod}(k_X) \arrow[r, "f_*"] & \mathrm{Mod}(k_Y).
\end{tikzcd}$$

Proof. (i) $f_*\iota_X \simeq \iota_Y f_*$ follows from $\alpha_X f^{-1} \simeq f^{-1}\alpha_Y$ and the fact that f_* and ι are right adjoint of f^{-1} and α.

(ii) Let $F \simeq$ "$\varinjlim\limits_i$" F_i with $F_i \in \mathrm{Mod}^c(k_X)$. Then

$$\begin{aligned}
f_*(\alpha_X F) &\simeq \varprojlim_K f_*((\alpha_X F)_K) \simeq \varprojlim_K f_*(\varinjlim_i F_{iK}) \\
&\simeq \varprojlim_K f_!(\varinjlim_i F_{iK}) \simeq \varprojlim_K \varinjlim_i f_!(F_{iK}) \\
&\simeq \varprojlim_K \alpha_Y(\text{“}\varinjlim_i\text{”}\, f_! F_{iK}) \simeq \alpha_Y(\varprojlim_K(\text{“}\varinjlim_i\text{”}\, f_* F_{iK})) \simeq \alpha_Y(f_*F).
\end{aligned}$$

q.e.d.

Corollary 3.3.7. *The diagram below commutes:*

$$\begin{tikzcd}
\mathrm{Mod}(k_Y) \arrow[r, "f^{-1}"] \arrow[d, "\beta_Y"'] & \mathrm{Mod}(k_X) \arrow[d, "\beta_X"] \\
\mathrm{I}(k_Y) \arrow[r, "f^{-1}"] & \mathrm{I}(k_X).
\end{tikzcd}$$

Proof. By adjunction, using Proposition 3.3.6. q.e.d.

Proposition 3.3.8. *Let* $F \in \mathrm{I}(k_X)$ *and* $G \in \mathrm{I}(k_Y)$. *Then*

$$\begin{aligned}
\mathcal{I}hom\,(G, f_*F) &\simeq f_*\mathcal{I}hom\,(f^{-1}G, F), \\
\mathcal{H}om_{\mathrm{I}(k_Y)}(G, f_*F) &\simeq f_*\mathcal{H}om_{\mathrm{I}(k_X)}(f^{-1}G, F).
\end{aligned}$$

Proof. The second formula follows from the first one by applying the functor α_Y. To prove the first formula, consider $K \in \mathrm{I}(k_Y)$. Then

$$\begin{aligned}
&\mathrm{Hom}_{\mathrm{I}(k_Y)}(K, \mathcal{I}hom\,(G, f_*F)) \simeq \mathrm{Hom}_{\mathrm{I}(k_Y)}(K \otimes G, f_*F) \\
&\quad \simeq \mathrm{Hom}_{\mathrm{I}(k_X)}(f^{-1}(K \otimes G), F) \simeq \mathrm{Hom}_{\mathrm{I}(k_X)}(f^{-1}K \otimes f^{-1}G, F) \\
&\quad \simeq \mathrm{Hom}_{\mathrm{I}(k_X)}(f^{-1}K, \mathcal{I}hom\,(f^{-1}G, F)) \simeq \mathrm{Hom}_{\mathrm{I}(k_Y)}(K, f_*\mathcal{I}hom\,(f^{-1}G, F)).
\end{aligned}$$

q.e.d.

Proposition 3.3.9. *For $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$, one has $(g \circ f)_* \simeq g_* \circ f_*$.*

This follows from Proposition 3.3.3 and adjunction.

Note that f_* does not commute with β in general.

Proper direct image

Let $F = \text{“}\varinjlim_i\text{”}\, F_i \in \mathrm{I}(k_X)$ for $F_i \in \mathrm{Mod}^c(k_X)$. One defines $f_{!!}F \in \mathrm{I}(k_Y)$ by the formula

$$f_{!!}\,\text{“}\varinjlim_i\text{”}\, F_i = \text{“}\varinjlim_i\text{”}\, f_! F_i.$$

If $i_U : U \hookrightarrow X$ is an open injection, $i_{U!!}$ coincides with the previous construction of section 2.3.

Note that the natural morphism $f_{!!}\iota_X F \longrightarrow \iota_Y f_! F$ is not an isomorphism for $F \in \mathrm{Mod}(k_X)$, in general. Here, this morphism is described by the morphisms (we do not write the functors ι_X, ι_Y for short)

$$f_{!!}F \simeq \text{“}\varinjlim_{U \subset\subset X}\text{”}\, f_!(F_U) \longrightarrow \text{“}\varinjlim_{V \subset\subset Y}\text{”}(f_! F)_V.$$

To avoid the confusion, we use the different notation $_{!!}$.

Lemma 3.3.10. *If the support of $F \in \mathrm{Mod}(k_X)$ is proper over Y, then $f_{!!}\iota_X F \xrightarrow{\sim} \iota_Y f_! F$.*

Proof. We have

$$\begin{aligned} f_{!!}\iota_X F &\simeq \text{“}\varinjlim_{U \subset\subset X}\text{”}\, f_!(F_U) \simeq \text{“}\varinjlim_{U \subset\subset X,\, V \subset\subset Y}\text{”}\, (f_!(F_U))_V \\ &\simeq \text{“}\varinjlim_{V \subset\subset Y}\text{”}(f_! F)_V \simeq \iota_Y f_! F. \end{aligned}$$

q.e.d.

Proposition 3.3.11. *The functor $f_{!!}$ is left exact and commutes with “$\varinjlim$”.*

The proof is evident.

Proposition 3.3.12. *The diagram below commutes:*

$$\begin{tikzcd} \mathrm{I}(k_X) \arrow[r, "f_{!!}"] \arrow[d, "\alpha_X"'] & \mathrm{I}(k_Y) \arrow[d, "\alpha_Y"'] \\ \mathrm{Mod}(k_X) \arrow[r, "f_!"] & \mathrm{Mod}(k_Y). \end{tikzcd}$$

Proof. One has the chain of isomorphisms

$$\begin{aligned}\alpha_Y f_{!!}(\text{“}\varinjlim_i\text{”}\, F_i) &\simeq \alpha_Y(\text{“}\varinjlim_i\text{”}\, f_! F_i) \simeq \varinjlim_i f_! F_i\\ &\simeq f_!(\varinjlim_i F_i) \simeq f_!\alpha_X(\text{“}\varinjlim_i\text{”}\, F_i).\end{aligned}$$

q.e.d.

Theorem 3.3.13. *For $G \in \mathrm{I}(k_Y)$ and let $F \in \mathrm{I}(k_X)$, one has*

$$G \otimes f_{!!}F \simeq f_{!!}(f^{-1}G \otimes F).$$

Proof. Let $G = \text{“}\varinjlim_j\text{”}\, G_j$ for $G_j \in \mathrm{Mod}^c(k_Y)$ and $F = \text{“}\varinjlim_i\text{”}\, F_i$ for $F_i \in \mathrm{Mod}^c(k_X)$. Then we have

$$f^{-1}G \otimes F \simeq \text{“}\varinjlim_{i,j}\text{”}(f^{-1}G_j) \otimes F_i\,,$$

Since the F_i's have compact support, $(f^{-1}G_j) \otimes F_i \in \mathrm{Mod}^c(k_X)$, and

$$\begin{aligned}f_{!!}(f^{-1}G \otimes F) &\simeq \text{“}\varinjlim_{i,j}\text{”}\, f_!(f^{-1}G_j \otimes F_i) \simeq \text{“}\varinjlim_{i,j}\text{”}\, G_j \otimes f_! F_i\\ &\simeq (\text{“}\varinjlim_j\text{”}\, G_j) \otimes (\text{“}\varinjlim_i\text{”}\, f_! F_i) \simeq G \otimes f_{!!}F.\end{aligned}$$

q.e.d.

Corollary 3.3.14. *Let $G \in \mathrm{I}(k_Y)$ and let $F \in \mathrm{I}(k_X)$.*

(i) *There are natural morphisms*

$$\begin{aligned}f_{!!}\mathcal{I}hom\,(f^{-1}G, F) &\to \mathcal{I}hom\,(G, f_{!!}F)\,,\\ f_!\mathcal{H}om_{\mathrm{I}(k_X)}(f^{-1}G, F) &\to \mathcal{H}om_{\mathrm{I}(k_Y)}(G, f_{!!}F)\,.\end{aligned}$$

(ii) *If $G \in \mathrm{Mod}(k_Y)$, these morphisms are isomorphisms.*

Proof. Since the assertions for the second morphism follows from the first by applying α, we shall prove the assertion for the first morphism.

(i) Applying Corollary 3.2.6, we have a natural morphism

$$f^{-1}G \otimes \mathcal{I}hom\,(f^{-1}G, F) \to F\,.$$

Hence, by Theorem 3.3.13, we get the morphism

$$G \otimes f_{!!}\mathcal{I}hom\,(f^{-1}G, F) \simeq f_{!!}(f^{-1}G \otimes \mathcal{I}hom\,(f^{-1}G, F)) \to f_{!!}F.$$

Applying Proposition 3.2.5, we get the desired morphism.

(ii) Let $F \simeq \text{“}\varinjlim_i\text{”}\, F_i$ with the F_i's in $\mathrm{Mod}^c(k_X)$. One has the chain of isomorphisms

$$\begin{aligned}
\mathcal{I}hom\,(G, f_{!!}F) &\simeq \mathcal{I}hom\,(G, f_{!!}\,\text{“}\varinjlim_i\text{”}\, F_i) \\
&\simeq \mathcal{I}hom\,(G, \text{“}\varinjlim_i\text{”}\, f_!F_i) \simeq \text{“}\varinjlim_i\text{”}\, \mathcal{H}om\,(G, f_*F_i) \\
&\simeq \text{“}\varinjlim_i\text{”}\, f_*\mathcal{H}om\,(f^{-1}G, F_i) \simeq \text{“}\varinjlim_i\text{”}\, f_!\mathcal{H}om\,(f^{-1}G, F_i) \\
&\simeq f_{!!}\,\text{“}\varinjlim_i\text{”}\, \mathcal{H}om\,(f^{-1}G, F_i) \simeq f_{!!}\mathcal{I}hom\,(f^{-1}G, F).
\end{aligned}$$

Note that we have used the fact that $\mathcal{I}hom\,(K,\,\cdot\,)$ commutes with “$\varinjlim$” when K is a sheaf, and this does not hold when K is an ind-sheaf. q.e.d.

Consider now a Cartesian square of topological spaces

$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\square"] & Y' \arrow[d, "g"] \\
X \arrow[r, "f"] & Y.
\end{tikzcd}$$

Theorem 3.3.15. *There is a natural isomorphism*

$$f'_{!!}g'^{-1} \simeq g^{-1}f_{!!}.$$

Proof. Let $F = \text{“}\varinjlim_i\text{”}\, F_i$ with $F_i \in \mathrm{Mod}^c(k_X)$. Since $\operatorname{supp} F_i$ is compact, the map $f' : \operatorname{supp}(g'^{-1}F_i) \longrightarrow Y'$ is proper. Hence $f'_{!!}g'^{-1}F_i \simeq f'_!g'^{-1}F_i$, and we get

$$\begin{aligned}
f'_{!!}g'^{-1}F &\simeq f'_{!!}(\text{“}\varinjlim_i\text{”}\, g'^{-1}F_i) \simeq \text{“}\varinjlim_i\text{”}\, f'_!g'^{-1}F_i \\
&\simeq \text{“}\varinjlim_i\text{”}\, g^{-1}f_!F_i \simeq g^{-1}\,\text{“}\varinjlim_i\text{”}\, f_!F_i \simeq g^{-1}f_{!!}F.
\end{aligned}$$

q.e.d.

Corollary 3.3.16. *Let V be an open subset of Y. Set $U = f^{-1}(V)$, $f_V := f|_V : U \longrightarrow V$, and recall that i_V (resp. i_U) denotes the open inclusion $V \hookrightarrow Y$, (resp. $U \hookrightarrow X$). Let $F \in \mathrm{I}(k_X)$. Then*

$$i_V^{-1}f_*F \simeq f_{V*}i_U^{-1}F.$$

Proof. Let $G \in \mathrm{Mod}^c(k_V)$. Then

$$\begin{aligned}\mathrm{Hom}_{\mathrm{I}(k_V)}(G, i_V^{-1} f_* F) &\simeq \mathrm{Hom}_{\mathrm{I}(k_Y)}(i_{V!!}G, f_*F) \simeq \mathrm{Hom}_{\mathrm{I}(k_X)}(f^{-1}i_{V!!}G, F)\\ &\simeq \mathrm{Hom}_{\mathrm{I}(k_X)}(i_{U!!}f_V^{-1}G, F) \simeq \mathrm{Hom}_{\mathrm{I}(k_Y|_V)}(G, f_{V*}i_U^{-1}F).\end{aligned}$$

q.e.d.

In general $f_{!!}$ does not commute with β. However they commute when f is an open embedding as seen in the following.

Proposition 3.3.17. *The diagram below commutes:*

$$\begin{tikzcd} \mathrm{Mod}(k_U) \arrow[r, "i_{U!}"] \arrow[d, "\beta_U"'] & \mathrm{Mod}(k_X) \arrow[d, "\beta_X"'] \\ \mathrm{I}(k_U) \arrow[r, "i_{U!!}"] & \mathrm{I}(k_X). \end{tikzcd}$$

Proof. The functor $i_{U!!} \circ \beta_U$ is left adjoint to $\alpha_U \circ i_U^{-1} \simeq i_U^{-1} \circ \alpha_X$ which is right adjoint to $\beta_X \circ i_{U!}$. q.e.d.

Ind-stalk

Let $x \in X$, and denote by j_x the embedding $\{x\} \hookrightarrow X$. For $F \in \mathrm{I}(k_X)$, it is natural to define its stalk at x by setting $F_x = j_x^{-1}F$. However, there exist non-zero ind-sheaves on X whose stalk at every $x \in X$ vanishes.

Example 3.3.18. We keep the same notation as in Example 3.2.16, that is, we define the ind-sheaf N_x by the exact sequence $0 \to N_x \to {}_xk \to k_x \to 0$. Since j_x^{-1} commutes with inductive limits, we get that $j_y^{-1}N_x = 0$ for all $y \in X$.

Hence, we define the ind-stalk of F at x as ${}_xF$. This is an ind-sheaf whose support is contained in $\{x\}$. By Proposition 2.3.16, for $F \in \mathrm{Mod}(k_X)$ and $K \in \mathrm{I}(k_X)$ one has the isomorphism:

$$\mathcal{H}om_{\mathrm{I}(k_X)}(F, K)_x \simeq \mathrm{Hom}_{\mathrm{I}(k_X)}(F, {}_xK).$$

The functor which associates to F the family of its ind-stalks is faithful:

Proposition 3.3.19. *Let $f : F \to G$ be a morphism in $\mathrm{I}(k_X)$, and assume that for each $x \in X$ the induced morphism ${}_xf : {}_xF \to {}_xG$ is the zero morphism. Then f is the zero morphism.*

Proof. One has the chain of equivalences

$f = 0 \Leftrightarrow$ for all $K \in \mathrm{Mod}^c(k_X)$, the induced morphism $\mathrm{Hom}_{\mathrm{I}(k_X)}(K, F) \to \mathrm{Hom}_{\mathrm{I}(k_X)}(K, G)$ is zero $\Leftrightarrow$ for all $x \in X$ and all K, $\mathcal{H}om(K, F)_x \to \mathcal{H}om(K, G)_x$ is zero $\Leftrightarrow$ for all $x \in X$ and all K, $\mathrm{Hom}_{\mathrm{I}(k_X)}(K, {}_xF) \to \mathrm{Hom}_{\mathrm{I}(k_X)}(K, {}_xG)$ is zero $\Leftrightarrow$ for all $x \in X$ ${}_xF \to {}_xG$ is zero. q.e.d.

Exercises to Chapter 3

Exercise 3.1. Let $Y = \mathbb{R}, X = \{0\}$ and denote by $f : X \longrightarrow Y$ the embedding. Let $G_n = k_{]\frac{-1}{n},\frac{1}{n}[} \in \mathrm{Mod}(k_Y)$. Prove that $f^{-1}(\prod_{n\in\mathbb{N}} G_n) \simeq k_X^{\oplus\mathbb{N}}$ and $\prod_{n\in\mathbb{N}} f^{-1}(G_n) \simeq k_X^{\mathbb{N}}$. Hence, f^{-1} does not commute with $\varprojlim$.

Exercise 3.2. Let $f : X \longrightarrow Y$ be as in Exercise 3.1. By choosing $F = k_X \in \mathrm{Mod}(k_X)$, prove that both f_* and $f_{!!}$ do not commute with β.

Exercise 3.3. Let $X =]-1,1[$, $Y = \mathbb{R}$ and $f : X \hookrightarrow Y$ the embedding. Let $F_n = k_{]-1+\frac{1}{n},1-\frac{1}{n}[} \in \mathrm{Mod}(k_X)$. Show that "$\varinjlim_n$" $F_n \simeq k_X$, "$\varinjlim_n$" $f_*F_n \simeq {}_X(k_Y)$ and $f_*($"$\varinjlim_n$" $F_n) \simeq (k_Y)_X$, Hence, f_* does not commute with $\varinjlim$.

Exercise 3.4. Let $f : X \longrightarrow Y$ be as in Exercise 3.3. Prove that $f_{!!}\iota_X k_X \simeq {}_X(k_Y)$ and $f_! k_X \simeq (k_Y)_X$. Hence, $f_{!!}\iota_X \not\simeq \iota_Y f_!$ in general.

Exercise 3.5. Let $X = \{\mathrm{pt}\}$, $F_n = k$. Prove that $\beta_X(\prod_{n\in\mathbb{Z}} F_n)$ is not isomorphic to $\prod_{n\in\mathbb{Z}} \beta_X(F_n)$. Hence, β_X does not commute with $\varprojlim$.

Exercise 3.6. Let $X = \mathbb{R}$, $Y = \{0\}$, and denote by $f : X \longrightarrow Y$ the projection. Let $F_n = k_{\{n\}} \in \mathrm{Mod}(k_X)$. Prove that $\prod_{n\in\mathbb{N}} F_n \simeq \bigoplus_{n\in\mathbb{N}} F_n \simeq$ "$\bigoplus_{n\in\mathbb{N}}$" F_n. Deduce that $f_{!!}(\prod_{n\in\mathbb{N}} F_n) \simeq k^{\text{“}\oplus\text{”}\,\mathbb{N}}$ and $\prod_{n\in\mathbb{N}} f_{!!}F_n \simeq k^{\mathbb{N}}$. Hence, $f_{!!}$ does not commute with $\varprojlim$.

Exercise 3.7. Assume that k is a field. Let $K \in \mathrm{Mod}(k_X)$ be a soft sheaf. Prove that $F \otimes K$ is soft for any sheaf $F \in \mathrm{Mod}(k_X)$.
(Hint: adapt the proof of Proposition 3.2.19.)

Chapter 4

Derived categories of ind-sheaves

In this chapter, as in the preceding one, we work in a given universe $\mathcal{U}$. All topological spaces (denoted by X, Y etc.) are assumed to be locally compact and countable at infinity, and k is a field.

Notation 4.0.20. If $\varphi : \mathcal{C} \longrightarrow \mathcal{C}'$ is a left exact functor of abelian categories, we shall usually still denote by φ instead of $K^+(\varphi)$ the associated functor from $K^+(\mathcal{C})$ to $K^+(\mathcal{C}')$, and similarly with bifunctors. For example, we still denote by $\mathcal{I}hom$ the functor $K^-(\mathrm{I}(k_X))^{\mathrm{op}} \times K^+(\mathrm{I}(k_X)) \longrightarrow K^+(\mathrm{I}(k_X))$ associated with $\mathcal{I}hom$.

4.1 Internal operations

Recall that the categories $\mathrm{Mod}(k_X)$ and $\mathrm{Mod}^c(k_X)$ have enough injectives and systems of strict generators.

As usual, we denote by $D(k_X)$ the derived category of the category $\mathrm{Mod}(k_X)$ of sheaves of k-vector spaces on X. We denote by $D(\mathrm{I}(k_X))$ the derived category of the category $\mathrm{I}(k_X)$ of ind-objects of $\mathrm{Mod}^c(k_X)$.

Proposition 4.1.1. *The natural functor $D^+(k_X) \longrightarrow D^+(\mathrm{I}(k_X))$ is fully faithful. In particular, $D^+(k_X)$ is the full triangulated subcategory of $D^+(\mathrm{I}(k_X))$ consisting of objects F such that $H^j(F) \in \mathrm{Mod}(k_X)$ for all j.*

Proof. Let $F, G \in K^+(\mathrm{Mod}(k_X))$. We have

$$\begin{aligned}
\mathrm{Hom}_{D^+(\mathrm{I}(k_X))}(\iota_X G, \iota_X F) &\simeq \varinjlim_{G' \xrightarrow[qis]{} G} \mathrm{Hom}_{K^+(\mathrm{I}(k_X))}(G', \iota_X F)\,, \qquad G' \in K^+(\mathrm{I}(k_X)) \\
&\simeq \varinjlim_{G' \xrightarrow[qis]{} G} \mathrm{Hom}_{K^+(\mathrm{Mod}(k_X))}(\alpha_X G', F)
\end{aligned}$$

The last term is isomorphic to $\varinjlim_{G' \xrightarrow[qis]{} G} \mathrm{Hom}_{K^+(\mathrm{Mod}(k_X))}(G', F)$, where $G' \longrightarrow G$ ranges over the family of quasi-isomorphisms in $K^+(\mathrm{Mod}(k_X))$. This is isomorphic to $\mathrm{Hom}_{D^+(\mathrm{Mod}(k_X))}(G, F)$.

Let us prove that if $F \in D^+(\mathrm{I}(k_X))$ satisfies $H^j(F) \in \mathrm{Mod}(k_X)$ for any j then $F \in D^+(k_X)$. There is a natural morphism $F \longrightarrow \iota_X \alpha_X F$, and it is an isomorphism because it induces isomorphisms in cohomologies. q.e.d.

We introduce the categories

$$\begin{aligned} \mathcal{P}(k_X) &:= \{\text{“}\bigoplus_j\text{”}\, G_j\,;\, G_j \in \mathrm{Mod}(k_X)\}, \\ \mathcal{I}_q(k_X) &:= \{F \in \mathrm{I}(k_X)\,;\, F \text{ is quasi-injective}\}. \end{aligned}$$

If there is no risk of confusion, we shall write $\mathcal{P}$ and $\mathcal{I}_q$ instead of $\mathcal{P}(k_X)$ and $\mathcal{I}_q(k_X)$, respectively. Note that $\mathcal{P}$ is generating and $\mathcal{I}_q$ is cogenerating.

Lemma 4.1.2. (i) *The category $\mathcal{P}$ is stable by $\otimes$ and the category $\mathcal{I}_q$ is stable by $\prod$,*

(ii) *if $G \in \mathcal{P}$ and $F \in \mathcal{I}_q$, then $\mathcal{I}hom\,(G, F) \in \mathcal{I}_q$,*

(iii) *if $G \in \mathcal{P}$ and $F \in \mathcal{I}_q$, then $\mathcal{H}om\,(G, F)$ is soft.*

Proof. (i) is obvious.

(ii) Since $\mathcal{I}hom\,(\,\cdot\,, F)$ sends direct sums to direct products, we may assume by (i) that $G \in \mathrm{Mod}(k_X)$. In this case, $\mathrm{Hom}_{\mathrm{I}(k_X)}(\,\cdot\,, \mathcal{I}hom\,(G, F)) \simeq \mathrm{Hom}_{\mathrm{I}(k_X)}(\,\cdot\, \otimes G, F)$ is an exact functor on $\mathrm{Mod}^c(k_X)$.

(iii) follows from (ii) and Proposition 3.2.17. Note that

$$\mathcal{H}om\,(G, F) \simeq \mathcal{H}om\,(k_X, \mathcal{I}hom\,(F, G)).$$

q.e.d.

Theorem 4.1.3. (a) *The category $\mathcal{P}^{\mathrm{op}} \times \mathcal{I}_q$ is injective with respect to the functors $\mathrm{Hom}_{\mathrm{I}(k_X)}$, $\mathcal{I}hom$, $\mathcal{H}om$.*

(b) *The functors below are well-defined:*

$$\begin{aligned} \mathrm{RHom}_{\mathrm{I}(k_X)} &: D^-(\mathrm{I}(k_X))^{\mathrm{op}} \times D^+(\mathrm{I}(k_X)) \to D^+(\mathrm{Mod}(k)) \\ R\mathcal{I}hom &: D^-(\mathrm{I}(k_X))^{\mathrm{op}} \times D^+(\mathrm{I}(k_X)) \to D^+(\mathrm{I}(k_X)) \\ R\mathcal{H}om &: D^-(\mathrm{I}(k_X))^{\mathrm{op}} \times D^+(\mathrm{I}(k_X)) \to D^+(k_X). \end{aligned}$$

Proof. (b) follows from (a) by Proposition 1.4.6.

Set

$$\mathcal{P}_c := \{\text{“}\bigoplus_j\text{”}\, G_j\,;\, G_j \in \mathrm{Mod}^c(k_X)\}\,. \tag{4.1.1}$$

This is obviously generating in $\mathrm{I}(k_X)$.
(a1) We shall show first that $\mathcal{P}_c^{\mathrm{op}} \times \mathcal{I}_q$ is injective with respect to $\mathrm{Hom}_{\,\mathrm{I}(k_X)}$.

(i) Let $G = \text{“}\bigoplus_j\text{”}\, G_j \in \mathcal{P}$. In order to see that $\mathcal{I}_q$ is injective with respect to the functor $\mathrm{Hom}_{\,\mathrm{I}(k_X)}(G,\,\cdot\,)$, we shall apply Corollary 1.4.4. Consider an exact sequence $0 \longrightarrow F' \underset{f}{\longrightarrow} F \underset{g}{\longrightarrow} F'' \longrightarrow 0$ in $\mathrm{I}(k_X)$ with F', F, F'' in $\mathcal{I}_q$. Let us prove that the functor $\mathrm{Hom}_{\,\mathrm{I}(k_X)}(G,\,\cdot\,)$ applied to this exact sequence gives an exact sequence. We have

$$\mathrm{Hom}_{\,\mathrm{I}(k_X)}(G, F) \;\simeq\; \prod_j \mathrm{Hom}_{\,\mathrm{I}(k_X)}(G_j, F) \tag{4.1.2}$$

and similarly with F replaced by F' or F''. Since the functor $\prod_j \mathrm{Hom}\,(G_j,\,\cdot\,)$ is exact on quasi-injective sheaves by Corollary 3.2.18, the result follows.

(ii) Let $F \in \mathcal{I}_q$. In order to see that $\mathcal{P}_c$ is injective with respect to the functor $\mathrm{Hom}_{\,\mathrm{I}(k_X)}(\,\cdot\,, F)$, we shall apply Theorem 1.4.3. Consider an epimorphism $H \twoheadrightarrow G''$ in $\mathrm{I}(k_X)$ with $G'' = \text{“}\bigoplus_j\text{”}\, G''_j \in \mathcal{P}_c$. By Proposition 1.3.2, for each j, there exist $G_j \in \mathrm{Mod}^c(k_X)$ and an epimorphism $G_j \twoheadrightarrow G''_j$ such that the composition $G_j \longrightarrow G''_j \longrightarrow G''$ factors through H. Let $G'_j = \ker(G_j \longrightarrow G''_j)$. We get an exact sequence

$$0 \longrightarrow \text{“}\bigoplus_j\text{”}\, G'_j \longrightarrow \text{“}\bigoplus_j\text{”}\, G_j \longrightarrow \text{“}\bigoplus_j\text{”}\, G''_j \longrightarrow 0 \tag{4.1.3}$$

such that the morphism $\text{“}\bigoplus_j\text{”}\, G_j \longrightarrow G''$ factors through H. It remains to show that (4.1.3) will remain exact after applying the functor $\mathrm{Hom}_{\,\mathrm{I}(k_X)}(\,\cdot\,, F)$. This is clear by (4.1.2).

(a2) Next let us show that $\mathcal{P}^{\mathrm{op}} \times \mathcal{I}_q$ is injective with respect to the functor $\mathcal{I}hom$. For $G \in \mathcal{P}$, let us prove that $\mathcal{I}hom\,(G, I^\bullet)$ is an exact sequence if $I^\bullet$ is an exact sequence in $\mathcal{I}_q$ bounded from below. It is enough to prove that for any $S \in \mathrm{Mod}^c(k_X)$, $\mathrm{Hom}_{\,\mathrm{I}(k_X)}(S, \mathcal{I}hom\,(G, I^\bullet))$ is exact. This follows from (a1) and the formula

$$\mathrm{Hom}_{\,\mathrm{I}(k_X)}(S, \mathcal{I}hom\,(G, I^\bullet)) \simeq \mathrm{Hom}_{\,\mathrm{I}(k_X)}(S \otimes G, I^\bullet)$$

and $S \otimes G \in \mathcal{P}_c$. The exactness in G is similarly proved.

(a3) The case of the functor $\mathcal{H}om$ follows from (a2) by applying the exact functor α_X.

(a4) Finally let us prove that $\mathcal{P}^{\mathrm{op}} \times \mathcal{I}_q$ is injective with respect to $\mathrm{Hom}_{\mathrm{I}(k_X)}$. Note that $\mathcal{H}om(F, G)$ is soft for $G \in \mathcal{P}$ and $F \in \mathcal{I}_q$ by Lemma 4.1.2 (iii). Hence the assertion follows from (a3) and $\mathrm{Hom}_{\mathrm{I}(k_X)}(G, F) \simeq \Gamma(X\,; \mathcal{H}om(G, F))$. q.e.d.

Proposition 4.1.4. *Let $G, K \in D^-(\mathrm{I}(k_X))$ and let $F \in D^+(\mathrm{I}(k_X))$. Then*

(i) $\alpha_X R\mathcal{I}hom(G, F) \simeq R\mathcal{H}om(G, F)$,

(ii) $R\mathcal{I}hom(G \otimes K, F) \simeq R\mathcal{I}hom(G, R\mathcal{I}hom(K, F))$.

Proof. (i) follows from the isomorphism $\alpha_X \circ \mathcal{I}hom \simeq \mathcal{H}om$ and the exactness of α_X.

(ii) We may assume $G, K \in K^-(\mathcal{P})$ and $F \in K^+(\mathcal{I}_q)$. Since $G \otimes K \in K^-(\mathcal{P})$, and $\mathcal{I}hom(K, F) \in K^+(\mathcal{I}_q)$, we get the isomorphisms

$$\begin{aligned} R\mathcal{I}hom(G \otimes K, F) &\simeq \mathcal{I}hom(G \otimes K, F) \\ &\simeq \mathcal{I}hom(G, \mathcal{I}hom(K, F)) \simeq R\mathcal{I}hom(G, R\mathcal{I}hom(K, F)). \end{aligned}$$

q.e.d.

Proposition 4.1.5. *Let $G \in D^-(\mathrm{I}(k_X))$ and $F \in D^+(\mathrm{I}(k_X))$. Then*

$$\mathrm{RHom}_{\mathrm{I}(k_X)}(G, F) \simeq \mathrm{R}\Gamma(X; R\mathcal{H}om(G, F)).$$

Proof. We may assume that $G \in K^-(\mathcal{P})$ and $F \in K^+(\mathcal{I}_q)$. Then

$$\begin{aligned} \mathrm{RHom}_{\mathrm{I}(k_X)}(G, F) &\simeq \mathrm{Hom}_{\mathrm{I}(k_X)}(G, F) \\ &\simeq \Gamma(X; \mathcal{H}om(G, F)) \simeq \mathrm{R}\Gamma(X; R\mathcal{H}om(G, F)). \end{aligned}$$

Note that the last isomorphism follows since $\mathcal{H}om(G, F)$ is soft. q.e.d.

One difficulty of the theory of ind-sheaves is that one does not know whether the category $\mathrm{I}(k_X)$ has enough injectives. This difficulty is partly overcome by the use of Theorem 1.5.3.

Theorem 4.1.6. *Let S be a small set contained in $\mathrm{Ob}(\mathrm{I}(k_X))$. Then there exists a small full abelian subcategory $\mathcal{C}_0$ of $\mathrm{Mod}^c(k_X)$ with the properties below.*

(i) $S \subset \mathrm{Ind}(\mathcal{C}_0)$.

(ii) *$\mathcal{C}_0$ satisfies the properties in (a) of* Theorem 1.5.3. *In particular, $\mathcal{C}_0$ as well as $\mathrm{Ind}(\mathcal{C}_0)$ are stable by sub-objects and quotients,*

(iii) $\mathrm{Ind}(\mathcal{C}_0)$ *has enough injectives and such objects are quasi-injective in* $\mathrm{I}(k_X)$.

(iv) $\mathcal{C}_0$ *is stable by* $\otimes$, $\mathcal{H}om$. *Moreover,* $k_U \in \mathcal{C}_0$ *for all open* $U \subset\subset X$.

(v) *If* $F \in S$, *then* $\mathcal{I}hom\,(F, \cdot\,)$ *sends* $\mathrm{Ind}(\mathcal{C}_0)$ *to* $\mathrm{Ind}(\mathcal{C}_0)$.

(vi) *Let* $K \in \mathrm{Ind}(\mathcal{C}_0)$ *and assume* K *is injective in this category.*

(a) *If* $F \in S$, *then* $\mathcal{I}hom\,(F, K)$ *is quasi-injective in* $\mathrm{I}(k_X)$.

(b) *If* $F \in S$, *then* $H^k R\mathcal{I}hom\,(F, K) = 0$ *for* $k \neq 0$.

(c) *If* $F \in \mathrm{Ind}(\mathcal{C}_0)$, *then* $\mathcal{H}om\,(F, K)$ *is injective in* $\mathrm{Mod}(k_X)$.

(d) *If* $F \in \mathrm{Ind}(\mathcal{C}_0)$, *then* $H^k R\mathcal{H}om\,(F, K) = 0$ *and* $H^k \mathrm{RHom}_{\mathrm{I}(k_X)}(F, K) = 0$ *for* $k \neq 0$.

Proof. (i)–(iv) There exists a small set B contained in $\mathrm{Ob}(\mathrm{Mod}^c(k_X))$ such that S consists of F_s's and $F_s \simeq \text{“}\varinjlim_{i \in I_s}\text{”}\, F_{s,i}$ with I_s small and filtrant and $F_{s,i} \in B$. Set $I = \bigsqcup_s I_s$. We may assume from the beginning that S contains the sheaves k_U, for all open subsets U of X.

With slight modifications of the proof of Theorem 1.5.3 (see [10]), we may construct a small full subcategory $\mathcal{C}_0$ satisfying conditions (a) and (b) of this theorem and such that $\mathcal{C}_0$ contains B and is stable by $\otimes$, $\mathcal{H}om$. Moreover, we may assume that $\mathcal{C}_0$ is stable by products indexed by I of sheaves in $\mathcal{C}_0$ with supports contained in a fixed compact subset. Then (i)–(iv) follow from this statement.

(v) Let $F = F_s \simeq \text{“}\varinjlim_{i \in I_s}\text{”}\, F_i$ and $G \simeq \text{“}\varinjlim_{j \in J}\text{”}\, G_j$ with $G_j, F_i \in \mathcal{C}_0$. Then $\mathcal{I}hom\,(F, G)$ is a sub-object of $\prod_{i \in I_s} \text{“}\varinjlim_{j \in J}\text{”}\, \mathcal{H}om\,(F_i, G_j)$. One has

$$\prod_{i \in I_s} \text{“}\varinjlim_{j \in J}\text{”}\, \mathcal{H}om\,(F_i, G_j) \simeq \text{“}\varinjlim_{\varphi, U}\text{”} \big(\prod_{i \in I_s} \mathcal{H}om\,(F_i, G_{\varphi(i)})_U\big),$$

where φ ranges over the set of maps $J \longrightarrow I_s$ and U over the sets of open $U \subset\subset X$. Then (v) follows since $\mathcal{C}_0$ is stable by $\mathcal{H}om$ and by product indexed by I_s.

(vi) (a) Let $G \in \mathrm{Ind}(\mathcal{C}_0)$. Then

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Ind}(\mathcal{C}_0)}(G, \mathcal{I}hom\,(F, K)) &\simeq \mathrm{Hom}_{\mathrm{I}(k_X)}(G, \mathcal{I}hom\,(F, K)) \\ &\simeq \mathrm{Hom}_{\mathrm{I}(k_X)}(G \otimes F, K) \simeq \mathrm{Hom}_{\mathrm{Ind}(\mathcal{C}_0)}(G \otimes F, K). \end{aligned}$$

and this functor is exact with respect to G. Therefore, $\mathcal{I}hom\,(F, K)$ is injective in $\mathrm{Ind}(\mathcal{C}_0)$, hence quasi-injective in $\mathrm{I}(k_X)$.

(vi) (b) Let $F_\bullet \longrightarrow F$ be a resolution of F with the components of $F_\bullet$ in $\mathcal{P}$. We may assume from the beginning that the F_j's belong to S. We may also assume

that S is stable by kernels and cokernels.
Since K is quasi-injective, $R\mathcal{I}hom(F,K)$ is represented by $\mathcal{I}hom(F_\bullet,K)$. Hence, it is enough to prove that this complex is qis to $\mathcal{I}hom(F,K)$. By standard arguments, it is enough to show that $\mathcal{I}hom(\,\cdot\,,K)$ is exact on S. This follows from the fact that $\mathrm{Hom}_{\mathrm{Ind}(\mathcal{C}_0)}(G,\mathcal{I}hom(H,K))\simeq\mathrm{Hom}_{\mathrm{Ind}(\mathcal{C}_0)}(G\otimes H,K)$ is exact with respect to $H\in S$ for any $G\in\mathcal{C}_0$.

(vi) (c) It is enough to prove that $\mathcal{H}om(F,K)$ is flabby. Let U be an open subset of X. The monomorphism $F\otimes{}_Uk\rightarrowtail F$ gives rise to the epimorphism

$$\begin{aligned}\Gamma(X;\mathcal{H}om(F,K))&\simeq\mathrm{Hom}_{\mathrm{Ind}(\mathcal{C}_0)}(F,K)\\&\twoheadrightarrow\mathrm{Hom}_{\mathrm{Ind}(\mathcal{C}_0)}(F\otimes{}_Uk,K)\simeq\Gamma(U;\mathcal{H}om(F,K)).\end{aligned}$$

(vi) (d) Consider a resolution $F_\bullet\longrightarrow F$ as in (vi) (b). Then $R\mathcal{H}om(F,K)$ is represented by the complex $\mathcal{H}om(F_\bullet,K)$. Hence, it is enough to show that $\mathcal{H}om(\,\cdot\,,K)$ is exact on $\mathrm{Ind}(\mathcal{C}_0)$. This follows from the formula

$$\Gamma(U;\mathcal{H}om(F,K))\simeq\mathrm{Hom}_{\mathrm{Ind}(\mathcal{C}_0)}({}_Uk\otimes F,K).\tag{4.1.4}$$

The case of $\mathrm{RHom}_{\mathrm{I}(k_X)}(F,K)$ is similar. q.e.d.

Corollary 4.1.7. *For any small subset S of $\mathrm{Ob}(\mathrm{I}(k_X))$, the subcategory $\mathcal{J}(S)$ of $\mathrm{I}(k_X)$ consisting of objects F with the following properties* (i)–(iii) *is cogenerating in $\mathrm{I}(k_X)$.*

(i) *For any $G\in S$, $\mathcal{I}hom(G,F)$ is in $\mathcal{I}_q(k_X)$.*
(ii) *$H^kR\mathcal{I}hom(G,F)=0$, $H^kR\mathcal{H}om(G,F)=0$ and $H^kR\mathrm{Hom}_{\mathrm{I}(k_X)}(G,F)=0$ for any $G\in S$ and any $k\neq 0$.*
(iii) *$\mathcal{H}om(G,F)$ is an injective sheaf for any $G\in S$.*

Corollary 4.1.8. *Let $F\in D^+(\mathrm{I}(k_X))$ and $G\in D^-(\mathrm{I}(k_X))$. Then*

(i) *$R\mathcal{I}hom(G,F)\simeq\text{“}\varinjlim_{F\xrightarrow[qis]{}F'}\text{”}\,\mathcal{I}hom(G,F')$ in $\mathrm{Ind}(D^+(\mathrm{I}(k_X)))$,*
(ii) *$R\mathcal{H}om(G,F)\simeq\text{“}\varinjlim_{F\xrightarrow[qis]{}F'}\text{”}\,\mathcal{H}om(G,F')$ in $\mathrm{Ind}(D^+(k_X))$,*
(iii) *$R\mathrm{Hom}_{\mathrm{I}(k_X)}(G,F)\simeq\text{“}\varinjlim_{F\xrightarrow[qis]{}F'}\text{”}\,\mathrm{Hom}_{\mathrm{I}(k_X)}(G,F')$ in $\mathrm{Ind}(D^+(\mathrm{Mod}(k)))$.*

Proof. (i) Choose a qis $G' \longrightarrow G$ with $G' \in K^-(\mathcal{P})$ and a qis $F \longrightarrow F'$ with $F' \in K^+(\mathcal{I}_q)$. Then, by Theorem 4.1.3, $R\mathcal{I}hom\,(G,F) \simeq \mathcal{I}hom\,(G',F')$. Hence it is enough to prove that, given a qis $G' \longrightarrow G$ as above, there exists a qis $F \longrightarrow F'$ with $F' \in K^+(\mathcal{I}_q)$ such that the morphism $\mathcal{I}hom\,(G,F') \longrightarrow \mathcal{I}hom\,(G',F')$ is a qis. Let G'' denote the mapping cone of $G' \longrightarrow G$. We choose a small set S in $\mathrm{Ob}(\mathrm{I}(k_X))$ such that S contains all objects of the complexes F, G'' and is stable by kernels and cokernels. Let us apply Corollary 4.1.7 and use the notation $\mathcal{J}(S)$ there. There exists a qis $F \longrightarrow F'$ with $F' \in K^+(\mathcal{J}(S))$. By Corollary 4.1.7 (ii), $\mathcal{I}hom\,(G'',F')$ is qis to 0.

(ii) (iii) The proof is similar. q.e.d.

This corollary means that, for a fixed $G \in K^-(\mathrm{I}(k_X))$, the functor $\mathcal{I}hom\,(G,\,\cdot\,)$ is right derivable, and its right derived functor coincides with the right derived functor $R\mathcal{I}hom\,(\,\cdot\,,\,\cdot\,)$ of the bifunctor $\mathcal{I}hom\,(\,\cdot\,,\,\cdot\,)$.

Proposition 4.1.9. *Let $K \in D^-(k_X)$, $K' \in D^+(k_X)$, $G \in D^-(\mathrm{I}(k_X))$ and $F \in D^+(\mathrm{I}(k_X))$. Then*

(i) $R\mathcal{H}om\,(\beta_X K, F) \simeq R\mathcal{H}om\,(K, \alpha_X F)$,

(ii) $R\mathcal{H}om\,(\beta_X K \otimes G, F) \simeq R\mathcal{H}om\,(K, R\mathcal{H}om\,(G,F))$,

(iii) $\mathrm{RHom}_{D^+(k_X)}(K, R\mathcal{H}om_{\mathrm{I}(k_X)}(G,F)) \simeq \mathrm{RHom}_{D^+(\mathrm{I}(k_X))}(\beta_X K \otimes G, F)$,

(iv) $R\mathcal{I}hom\,(K, F \otimes \beta_X K') \simeq R\mathcal{I}hom\,(K,F) \otimes \beta_X K'$,

(v) $R\mathcal{H}om\,(K, F \otimes \beta_X K') \simeq R\mathcal{H}om\,(K,F) \otimes K'$.

Proof. (i) By Corollary 4.1.7, we can find some quasi-injective $F' \in K^+(\mathrm{I}(k_X))$ and some quasi-isomorphism $F \longrightarrow F'$ such that $R\mathcal{H}om\,(\beta_X K, F)$ is represented by $\mathcal{H}om\,(\beta_X K, F') \simeq \mathcal{H}om\,(K, \alpha_X F')$. Moreover we may assume that $\alpha_X(F') \simeq \mathcal{H}om\,(k_X, F')$ is injective. Hence $\mathcal{H}om\,(K, \alpha_X F')$ represents $R\mathcal{H}om\,(K, \alpha_X F)$.

(ii) By (i) we have

$$\begin{aligned} R\mathcal{H}om\,(\beta_X K \otimes G, F) &\simeq R\mathcal{H}om\,(\beta_X K, R\mathcal{I}hom\,(G,F)) \\ &\simeq R\mathcal{H}om\,(K, \alpha_X R\mathcal{I}hom\,(G,F)) \\ &\simeq R\mathcal{H}om\,(K, R\mathcal{H}om\,(G,F)). \end{aligned}$$

(iii) follows from (ii) by applying $\mathrm{R}\Gamma(X;\,\cdot\,)$.

(iv) We may assume that $K \in \mathcal{P}$ and $F \in \mathcal{I}_q$. Then the result follows from Proposition 3.2.20 since $F \otimes \beta_X K \in \mathcal{I}_q$ by Proposition 3.2.19.

(v) follows from (iv) by applying the functor α_X. q.e.d.

Proposition 4.1.10. *Let $G \in D^-(k_X)$. Let $\{F_i\}_i$ be a small filtrant inductive system in $\mathrm{I}(k_X)$. Then we have*

$$\begin{aligned} \text{“}\varinjlim_i\text{”}\, H^k R\mathcal{I}hom\,(G, F_i) &\xrightarrow{\sim} H^k R\mathcal{I}hom\,(G, \text{“}\varinjlim_i\text{”}\, F_i)\,, \\ \varinjlim_i H^k R\mathcal{H}om\,(G, F_i) &\xrightarrow{\sim} H^k R\mathcal{H}om\,(G, \text{“}\varinjlim_i\text{”}\, F_i)\,. \end{aligned}$$

Proof. We may assume that $G \in \mathrm{Mod}(k_X)$. Then the assertion follows from Theorem 1.5.5 and Corollary 4.1.8. q.e.d.

4.2 External operations

Let $f : X \longrightarrow Y$ be a continuous map. Recall that X and Y are assumed to be locally compact and countable at infinity.

Direct images

Proposition 4.2.1. (i) *The category $\mathcal{I}_q(k_X)$ is f_*-injective.*

(ii) *If $F \in \mathcal{I}_q(k_X)$, then $f_*F \in \mathcal{I}_q(k_Y)$.*

Proof. (i) Consider an exact sequence $0 \longrightarrow F' \longrightarrow F \longrightarrow F'' \longrightarrow 0$ in $\mathrm{I}(k_X)$, with F' quasi-injective. In order to prove that the sequence obtained by applying the functor f_* remains exact, it is enough to prove that for $H \in \mathrm{Mod}^c(k_Y)$, the sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathrm{I}(k_Y)}(H, f_*F') \longrightarrow \mathrm{Hom}_{\mathrm{I}(k_Y)}(H, f_*F) \longrightarrow \mathrm{Hom}_{\mathrm{I}(k_Y)}(H, f_*F'') \longrightarrow 0$$

is exact. This follows immediately from the adjunction formula of Proposition 3.3.4 and Corollary 3.2.18.

(ii) The same argument proves the exactness of the functor $\mathrm{Hom}_{\mathrm{I}(k_Y)}(\,\cdot\,, f_*F)$ on the category $\mathrm{Mod}^c(k_Y)$ by Proposition 3.2.17. q.e.d.

Corollary 4.2.2. (i) *The derived functor $Rf_* : D^+(\mathrm{I}(k_X)) \longrightarrow D^+(\mathrm{I}(k_Y))$ is well-defined.*

(ii) *If $g : Y \longrightarrow Z$ is another continuous map, then $R(g \circ f)_* \simeq Rg_* \circ Rf_*$.*

Proposition 4.2.3. *For $G \in D^-(\mathrm{I}(k_Y))$ and $F \in D^+(\mathrm{I}(k_X))$ and one has the isomorphisms*

$$\begin{aligned} \mathrm{RHom}_{\mathrm{I}(k_Y)}(G, Rf_*F) &\simeq \mathrm{RHom}_{\mathrm{I}(k_X)}(f^{-1}G, F), \\ R\mathcal{H}om\,(G, Rf_*F) &\simeq Rf_*R\mathcal{H}om\,(f^{-1}G, F), \\ R\mathcal{I}hom\,(G, Rf_*F) &\simeq Rf_*R\mathcal{I}hom\,(f^{-1}G, F). \end{aligned}$$

Proof. We may assume that $G \in K^-(\mathcal{P}(k_Y))$. We can take a quasi-isomorphism $F \to F'$ with a quasi-injective F'. Then applying Theorem 4.1.3, we obtain the result from the corresponding result in the non derived case. q.e.d.

Theorem 4.2.4. *The functors Rf_* and f^{-1} are adjoint. More precisely, for $F \in D^+(\mathrm{I}(k_X))$ and $G \in D^+(\mathrm{I}(k_Y))$ one has the isomorphism*

$$\mathrm{Hom}_{D^+(\mathrm{I}(k_Y))}(G, Rf_*F) \simeq \mathrm{Hom}_{D^+(\mathrm{I}(k_X))}(f^{-1}G, F).$$

Proof. We have the chain of isomorphisms

$$\begin{aligned}
\mathrm{Hom}_{D^+(\mathrm{I}(k_Y))}(G, Rf_*F) &\simeq \varinjlim_{F \underset{qis}{\longrightarrow} F',\, G' \underset{qis}{\longrightarrow} G} \mathrm{Hom}_{K^+(\mathrm{I}(k_Y))}(G', f_*F') \\
&\simeq \varinjlim_{F \underset{qis}{\longrightarrow} F',\, G' \underset{qis}{\longrightarrow} G} \mathrm{Hom}_{K^+(\mathrm{I}(k_X))}(f^{-1}G', F') \\
&\simeq \mathrm{Hom}_{D^+(\mathrm{I}(k_X))}(f^{-1}G, F).
\end{aligned}$$

q.e.d.

Proposition 4.2.5. *There are natural isomorphisms and morphisms*

(i) $\iota_Y Rf_* \xrightarrow{\sim} Rf_*\iota_X$,

(ii) $\alpha_Y Rf_* \xrightarrow{\sim} Rf_*\alpha_X$,

(iii) $\beta_Y Rf_* \to Rf_*\beta_X$.

Proof. (i) To prove that for $F \in D^+(k_X)$, $\iota_Y Rf_*F \simeq Rf_*\iota_X F$, we may assume that $F \in K^+(\mathrm{Mod}(k_X))$ and that F is injective. Then $\iota_X F \in K^+(\mathcal{I}_q(k_X))$, and the result follows.

(ii) To prove that for $F \in D^+(\mathrm{I}(k_X))$, $\alpha_Y Rf_*F \simeq Rf_*\alpha_X F$, we may assume that $F \in K^+(\mathcal{I}_q(k_X))$. Then $\alpha_X F$ is soft, hence f_*-acyclic and the result follows.

(iii) By Theorem 4.2.4, we get a morphism $f^{-1}Rf_* \to \mathrm{id}$. This defines $f^{-1}\beta_Y Rf_* \simeq \beta_X f^{-1}Rf_* \to \beta_X$, and the result follows by adjunction (i.e. again by Theorem 4.2.4). q.e.d.

Proper direct images

Proposition 4.2.6. (i) *The category $\mathcal{I}_q(k_X)$ is injective with respect to $f_{!!}$, and the functor $f_{!!}$ admits a right derived functor $Rf_{!!} : D^+(\mathrm{I}(k_X)) \to D^+(\mathrm{I}(k_Y))$. Moreover, for each k, $R^kf_{!!}$ commutes with "$\varinjlim$".*

(ii) *If* $K \in \mathrm{Mod}(k_X)$ *is soft, then* $R^k f_{!!}(F \otimes K) = 0$ *for* $k \neq 0$ *and* $F \in \mathrm{I}(k_X)$.

(iii) *If* $F \in D^+(k_X)$ *and* f *is proper on* $\mathrm{supp}\, F$, *then* $Rf_{!!}F \simeq Rf_!F$.

(iv) *The functor* $f_{!!}$ *sends* $\mathcal{I}_q(k_X)$ *to* $\mathcal{I}_q(k_Y)$. *In particular, if* $g : Y \longrightarrow Z$ *is another continuous map, then* $R(g \circ f)_{!!} \simeq Rg_{!!} \circ Rf_{!!}$.

(v) *One has the isomorphism* $\alpha_Y \circ Rf_{!!} \xrightarrow{\sim} Rf_! \circ \alpha_X$.

(vi) *One has a natural morphism* $\beta_Y Rf_! \longrightarrow Rf_{!!}\beta_X$.

Proof. (i) The functor $f_{!!} : \mathrm{I}(k_X) \longrightarrow \mathrm{I}(k_Y)$ is of the type $If_!$, where one denotes again by $f_! : \mathrm{Mod}^c(k_X) \longrightarrow \mathrm{Mod}^c(k_Y)$ the restriction of the usual functor $f_!$. Hence we can apply Theorem 1.5.5.

(ii) Let $F = \text{“}\varinjlim_i\text{”}\, F_i \in \mathrm{I}(k_X)$ with $F_i \in \mathrm{Mod}^c(k_X)$. One has

$$R^k f_{!!}(F \otimes K) \simeq \text{“}\varinjlim_i\text{”}\, R^k f_!(F_i \otimes K) = 0 \quad \text{for } k \neq 0.$$

because $F_i \otimes K$ is soft by Exercise 3.7.

(iii) We may assume that F is injective (keeping the hypothesis that f is proper on $\mathrm{supp}\, F$). Then the result follows from $f_{!!}F \simeq f_!F$.

(iv) Let $F = \text{“}\varinjlim_i\text{”}\, F_i$ with the F_i's injective with compact support. Then $f_{!!}F \simeq \text{“}\varinjlim_i\text{”}\, f_!F_i$, and the sheaves f_*F_i are injective.

(v) If $F \in \mathcal{I}_q(k_X)$, then $\alpha_X F$ is soft, hence $f_!$-acyclic.

(vi) The isomorphism $\mathrm{id} \xrightarrow{\sim} \alpha_X\beta_X$ defines $Rf_! \longrightarrow Rf_!\alpha_X\beta_X \simeq \alpha_Y Rf_{!!}\beta_X$, and the result follows by adjunction (i.e. by Theorem 2.3.26). q.e.d.

Theorem 4.2.7. *Let* $F \in D^+(\mathrm{I}(k_X))$ *and* $G \in D^+(\mathrm{I}(k_Y))$. *Then*

$$G \otimes Rf_{!!}F \simeq Rf_{!!}(f^{-1}G \otimes F).$$

Proof. We may assume that F is quasi-injective. In this case, the left hand side is isomorphic to $G \otimes f_{!!}F$. By Proposition 3.3.13, $G \otimes f_{!!}F \simeq f_{!!}(f^{-1}G \otimes F)$. Hence $f_{!!}(f^{-1}G \otimes F)$ represents $Rf_{!!}(f^{-1}G \otimes F)$ by Proposition 4.2.6 (ii). q.e.d.

The next result will be a useful tool.

Lemma 4.2.8. (i) *For* $G \in D^-(\mathrm{I}(k_Y))$ *and* $F \in D^+(\mathrm{I}(k_X))$, *there are natural morphisms*

$$\begin{aligned} Rf_{!!}R\mathcal{I}hom\,(f^{-1}G, F) &\longrightarrow R\mathcal{I}hom\,(G, Rf_{!!}F)\,, \\ Rf_!R\mathcal{H}om\,(f^{-1}G, F) &\longrightarrow R\mathcal{H}om\,(G, Rf_{!!}F)\,. \end{aligned}$$

(ii) *If $G \in D^-(k_Y)$, these morphisms are isomorphisms.*

Proof. The second morphism being obtained from the first by applying α, we shall prove the assertions for the first morphism. We have a chain of morphisms:

$$G \otimes Rf_{!!}R\mathcal{I}hom\,(f^{-1}G, F) \simeq Rf_{!!}(f^{-1}G \otimes R\mathcal{I}hom\,(f^{-1}G, F)) \longrightarrow Rf_{!!}F.$$

Then by the adjunction, we obtain the first morphism.

In order to see (ii), we may assume $G \in \mathrm{Mod}(k_Y)$ and $F \in \mathcal{I}_q(k_X)$. In this case, the morphisms above reduce to $f_{!!}\mathcal{I}hom\,(f^{-1}G, F) \longrightarrow \mathcal{I}hom\,(G, f_{!!}F)$. This morphism is an isomorphism by Corollary 3.3.14. q.e.d.

Consider now a Cartesian square of topological spaces

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\square"] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y. \end{tikzcd}$$

Theorem 4.2.9. *There is a natural isomorphism of functors*

$$Rf'_{!!}g'^{-1} \simeq g^{-1}Rf_{!!}\,.$$

Proof. By Proposition 3.3.15, there is an isomorphism $f'_{!!}g'^{-1} \simeq g^{-1}f_{!!}$. The right hand side admits a right derived functor, and $R(g^{-1}f_{!!}) \simeq g^{-1}Rf_{!!}$. Hence we get the morphism of functors $g^{-1}Rf_{!!} \longrightarrow Rf'_{!!}g'^{-1}$. Hence, it is enough to prove that for each k and each $F \in \mathrm{I}(k_X)$, it induces an isomorphism $g^{-1}R^kf_{!!}F \longrightarrow R^kf'_{!!}g'^{-1}F$. Since both sides commute with "$\varinjlim$", we may assume that $F \in \mathrm{Mod}^c(k_X)$. Then the result follows from the corresponding one for sheaves, in view of Proposition 4.2.6 (iii). q.e.d.

4.3 Duality

Consider as in the preceding section a continuous map $f : X \longrightarrow Y$. We shall extend the classical Poincaré-Verdier duality to ind-sheaves and construct a right adjoint functor to $f_{!!}$. We refer to ([8] Chapter III) for the classical case.

From now on, we shall make the following hypothesis:

(4.3.1) the functor $f_!$ has finite cohomological dimension.

This means that there exists an integer d such that for all $F \in \mathrm{Mod}(k_X)$, $R^jf_!F = 0$ if $j > d$. The smallest d with this property is called the cohomological dimension of $f_!$.

Lemma 4.3.1. *The two functors $f_{!!}$ and $f_!$ have the same cohomological dimension.*

Proof. Denote by d the cohomological dimension of $f_!$. Let $F \simeq \text{“}\varinjlim_i\text{”}\, F_i$, with $F_i \in \mathrm{Mod}^c(k_X)$. Then $R^k f_{!!}F \simeq \text{“}\varinjlim_i\text{”}\, R^k f_! F_i = 0$ for $j > d$. Hence, the cohomological dimension of $f_{!!}$ is less than or equal to that of $f_!$.

The other estimate follows from $R^k f_! F \simeq \alpha_Y R^k f_{!!} F$ for $F \in \mathrm{Mod}(k_X)$. q.e.d.

Let K be a soft sheaf on X and let $G \in \mathrm{Mod}(k_Y)$. One introduces the presheaf on X:

$$f_K^! G : U \mapsto \mathrm{Hom}_{k_Y}(f_! K_U, G)\,.$$

Proposition 4.3.2. (i) *For $F \in \mathrm{Mod}(k_X)$, $F \otimes K$ is soft. In particular, the functor $F \mapsto f_!(F \otimes K_U)$ is exact,*

(ii) *if G is injective, the presheaf $f_K^! G$ is an injective sheaf,*

(iii) *if G is injective and $F \in \mathrm{Mod}^c(k_X)$, then*

$$\mathrm{Hom}_{k_X}(F, f_K^! G) \simeq \mathrm{Hom}_{k_Y}(f_!(K \otimes F), G).$$

We refer to loc. cit. for a proof.

We shall extend the functor $f_K^!$ to ind-sheaves. Let $G \in \mathrm{I}(k_Y)$. We define $f_K^! G \in \mathrm{Mod}^c(k_X)^{\vee,add}$ by the formula:

$$f_K^! G(F) = \mathrm{Hom}_{\mathrm{I}(k_Y)}(f_!(K \otimes F), G) \quad \text{for } F \in \mathrm{Mod}^c(k_X).$$

Lemma 4.3.3. (i) *$f_K^! G$ belongs to $\mathrm{I}(k_X)$.*

(ii) *The functor $f_K^! : \mathrm{I}(k_Y) \longrightarrow \mathrm{I}(k_X)$ commutes with filtrant inductive limits and with projective limits (in particular, it is left exact).*

(iii) *If G is quasi-injective, then $f_K^! G$ is quasi-injective.*

(iv) *The category $\mathcal{I}_q(k_Y)$ is $f_K^!$-injective.*

(v) *For $F \in \mathrm{I}(k_X)$ and $G \in \mathrm{I}(k_Y)$, one has*

$$\mathrm{Hom}_{\mathrm{I}(k_X)}(F, f_K^! G) \simeq \mathrm{Hom}_{\mathrm{I}(k_Y)}(f_{!!}(K \otimes F), G).$$

Proof. (i) Clearly, the functor $f_K^! : \mathrm{I}(k_Y) \longrightarrow \mathrm{Mod}^c(k_X)^{\vee,add}$ commutes with filtrant inductive limits and with projective limits. Hence, to prove (i), we may assume that G is an injective sheaf. Then the result follows from Proposition 4.3.2.

(ii) is obvious.

(iii) follows from the fact that the functor $F \mapsto f_!(K \otimes F)$ is exact.

(iv) Consider an exact sequence $0 \longrightarrow G' \longrightarrow G \longrightarrow G'' \longrightarrow 0$ of quasi-injective sheaves, and let us apply the functor $f^!_K$ to this sequence. To check that the sequence we obtain is exact, it is enough to prove that it is exact after applying the functor $\mathrm{Hom}_{\mathrm{I}(k_X)}(S, \cdot)$, with $S \in \mathrm{Mod}^c(k_X)$. This is clear by the definition of $f^!_K$.

(v) Let $F \simeq$ "$\varinjlim_i$" F_i with $F_i \in \mathrm{Mod}^c(k_X)$. One has the chain of isomorphisms

$$\begin{aligned}\mathrm{Hom}_{\mathrm{I}(k_X)}(F, f^!_K G) &\simeq \varprojlim_i \mathrm{Hom}_{\mathrm{I}(k_X)}(F_i, f^!_K G) \simeq \varprojlim_i \mathrm{Hom}_{\mathrm{I}(k_Y)}(f_!(K \otimes F_i), G) \\ &\simeq \mathrm{Hom}_{\mathrm{I}(k_Y)}(\text{“}\varinjlim_i\text{”}\, f_!(K \otimes F_i), G) \simeq \mathrm{Hom}_{\mathrm{I}(k_Y)}(f_{!!}(K \otimes F), G).\end{aligned}$$

q.e.d.

Theorem 4.3.4. *The functor $Rf_{!!} : D^+(\mathrm{I}(k_X)) \longrightarrow D^+(\mathrm{I}(k_Y))$ admits a right adjoint, denoted by $f^!$, that is:*

$$\mathrm{Hom}_{D^+(\mathrm{I}(k_Y))}(Rf_{!!}F, G) \simeq \mathrm{Hom}_{D^+(\mathrm{I}(k_X))}(F, f^!G).$$

Moreover, $\iota_X f^! \simeq f^! \iota_Y$. (In other words, the restriction of this new functors to the derived category of sheaves coincides with the classical functor $f^!$.)

Proof. There exists a complex of soft sheaves $K \in K^b(\mathrm{Mod}(k_X))$ quasi-isomorphic to the sheaf k_X (see (loc. cit., Lemma 3.1.1). For $F \in K^+(\mathrm{I}(k_X))$ and $G \in K^+(\mathrm{I}(k_Y))$, one has by Lemma 4.3.3 the isomorphism

$$\mathrm{Hom}_{K^+(\mathrm{I}(k_X))}(F, f^!_K G) \simeq \mathrm{Hom}_{K^+(\mathrm{I}(k_Y))}(f_{!!}(F \otimes K), G).$$

Let $G \in K^+(\mathrm{I}(k_Y))$ and assume that all components of the complex G are quasi-injective. Let us prove that $f^!_K G$ represents $f^!G$. We have the chain of isomorphisms

$$\begin{aligned}\mathrm{Hom}_{D^+(\mathrm{I}(k_Y))}(Rf_{!!}F, G) &\simeq \varinjlim_{F' \xrightarrow[qis]{} F,\, G \xrightarrow[qis]{} G'} \mathrm{Hom}_{K^+(\mathrm{I}(k_Y))}(f_{!!}(F' \otimes K), G') \\ &\quad (G' \text{ quasi-injective}) \\ &\simeq \varinjlim_{F' \xrightarrow[qis]{} F,\, G \xrightarrow[qis]{} G'} \mathrm{Hom}_{K^+(\mathrm{I}(k_X))}(F', f^!_K G') \\ &\simeq \mathrm{Hom}_{D^+(\mathrm{I}(k_X))}(F, f^!_K G).\end{aligned}$$

Here, we have used the fact that $f_{!!}(F' \otimes K)$ represents $Rf_{!!}(F \otimes K)$ and the fact that $f^!_K G'$ is qis to $f^!_K G$.

By its construction the new functor $f^!$ coincides with the classical one when restricted to the derived category of sheaves. q.e.d.

Corollary 4.3.5. *Let $F \in D^b(\mathrm{I}(k_X))$ and $G \in D^+(\mathrm{I}(k_Y))$. Then*

(i) $R\mathcal{I}hom\,(Rf_{!!}F, G) \simeq Rf_* R\mathcal{I}hom\,(F, f^! G)$,

(ii) $R\mathcal{H}om\,(Rf_{!!}F, G) \simeq Rf_* R\mathcal{H}om\,(F, f^! G)$.

Proof. Let $S \in D^+(\mathrm{I}(k_Y))$. One has the chain of isomorphisms

$$\begin{aligned}
\mathrm{Hom}_{D^+(\mathrm{I}(k_Y))}(S, R\mathcal{I}hom\,(Rf_{!!}F, G)) &\simeq \mathrm{Hom}_{D^+(\mathrm{I}(k_Y))}(S \otimes Rf_{!!}F, G)\\
&\simeq \mathrm{Hom}_{D^+(\mathrm{I}(k_Y))}(Rf_{!!}(f^{-1}S \otimes F), G)\\
&\simeq \mathrm{Hom}_{D^+(\mathrm{I}(k_X))}(f^{-1}S \otimes F, f^! G)\\
&\simeq \mathrm{Hom}_{D^+(\mathrm{I}(k_X))}(f^{-1}S, R\mathcal{I}hom\,(F, f^! G))\\
&\simeq \mathrm{Hom}_{D^+(\mathrm{I}(k_Y))}(S, Rf_* R\mathcal{I}hom\,(F, f^! G)).
\end{aligned}$$

The second formula follows by applying α_X. q.e.d.

Proposition 4.3.6. (i) *Assume that $f : X \longrightarrow Y$ is an open embedding. Then $f^! \simeq f^{-1}$.*

(ii) *Assume that $f : X \longrightarrow Y$ is a closed embedding. Then*

$$f^! \simeq f^{-1} R\mathcal{I}hom\,((k_Y)_X,\ \cdot\ )\,.$$

Proof. (i) In this case, $f_{!!}$ is exact and f^{-1} is a right adjoint by Lemma 2.3.13.

(ii) Note that $f_{!!}$ is also exact in this case. For $F = \text{“}\varinjlim_i\text{”}\, F_i \in \mathrm{I}(k_X)$ with $F_i \in \mathrm{Mod}^c(k_X)$ and $G = \text{“}\varinjlim_j\text{”}\, G_j$ with $G_j \in \mathrm{Mod}^c(k_Y)$, we have the chain of isomorphisms:

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{I}(k_Y)}(f_{!!}F, G) &\simeq \varprojlim_i \varinjlim_j \mathrm{Hom}_{k_Y}(f_! F_i, G_j)\\
&\simeq \varprojlim_i \varinjlim_j \mathrm{Hom}_{k_X}(F_i, f^{-1}\mathcal{H}om\,((k_Y)_X, G_j))\\
&\simeq \mathrm{Hom}_{\mathrm{I}(k_X)}(F, f^{-1}\mathcal{I}hom\,((k_Y)_X, G)).
\end{aligned}$$

In particular if G is quasi-injective, then $f^{-1}\mathcal{I}hom\,((k_Y)_X, G)$ is quasi-injective. Hence for $F \in D^+(\mathrm{I}(k_X))$ and $G \in D^+(\mathrm{I}(k_Y))$ we have the chain of isomorphisms:

$$\begin{aligned}
\mathrm{Hom}_{D^+(\mathrm{I}(k_X))}(F, f^! G) &\simeq \mathrm{Hom}_{D^+(\mathrm{I}(k_Y))}(f_{!!}F, G)\\
&\simeq \varinjlim_{F' \xrightarrow[qis]{} F,\, G \xrightarrow[qis]{} G'} \mathrm{Hom}_{K^+(\mathrm{I}(k_Y))}(f_{!!}F', G')\\
&\simeq \varinjlim_{F' \xrightarrow[qis]{} F,\, G \xrightarrow[qis]{} G'} \mathrm{Hom}_{K^+(\mathrm{I}(k_X))}(F', f^{-1}\mathcal{I}hom\,((k_Y)_X, G'))\\
&\simeq \mathrm{Hom}_{D^+(\mathrm{I}(k_X))}(F, f^{-1}R\mathcal{I}hom\,((k_Y)_X, G)).
\end{aligned}$$

Here we have used the fact that $f_{!!}F' \longrightarrow f_{!!}F$ is a quasi-isomorphism. This completes the proof. q.e.d.

Proposition 4.3.7. *Let $\{G_i\}_i$ be a small filtrant inductive system in $\mathrm{I}(k_Y)$. Then for $k \in \mathbb{Z}$ one has*

$$H^k(f^!(\text{“}\varinjlim_i\text{”}\, G_i)) \simeq \text{“}\varinjlim_i\text{”}\, H^k(f^!(G_i)).$$

Proof. Since each G_i is a small inductive limit of sheaves, we may assume from the beginning that all G_i's are sheaves. Denote by $G_i^\bullet$ the canonical injective resolution of G_i. Then $\{G_i^\bullet\}_i$ is an inductive system of complexes of injective sheaves. Denote by K a bounded soft resolution of the constant sheaf k_X, as in the proof of Theorem 4.3.4. Then

$$\begin{aligned} H^k(f^!(\text{“}\varinjlim_i\text{”}\, G_i)) &\simeq H^k(f^!(\text{“}\varinjlim_i\text{”}\, G_i^\bullet)) \simeq H^k(f^!_K(\text{“}\varinjlim_i\text{”}\, G_i^\bullet)) \\ &\simeq \text{“}\varinjlim_i\text{”}\, H^k(f^!_K(G_i^\bullet)) \simeq \text{“}\varinjlim_i\text{”}\, H^k(f^!(G_i)). \end{aligned}$$

Here we have used the fact that if G is quasi-injective then $f^!_K(G)$ represents $f^!(G)$, and $f^!_K$ commutes with “$\varinjlim$”. q.e.d.

Proposition 4.3.8. *Let $K \in D^-(\mathrm{I}(k_Y))$ and $G \in D^+(\mathrm{I}(k_Y))$. Then*

$$R\mathcal{I}hom\,(f^{-1}K, f^!G) \xrightarrow{\sim} f^!R\mathcal{I}hom\,(K, G).$$

Proof. For any $F \in D^+(\mathrm{I}(k_X))$, one has the chain of isomorphisms

$$\begin{aligned} &\mathrm{Hom}_{D^+(\mathrm{I}(k_X))}(F, f^!R\mathcal{I}hom\,(K,G)) \simeq \mathrm{Hom}_{D^+(\mathrm{I}(k_Y))}(Rf_{!!}F, R\mathcal{I}hom\,(K,G)) \\ &\quad\simeq \mathrm{Hom}_{D^+(\mathrm{I}(k_Y))}(K \otimes Rf_{!!}F, G) \simeq \mathrm{Hom}_{D^+(\mathrm{I}(k_Y))}(Rf_{!!}(f^{-1}K \otimes F), G) \\ &\quad\simeq \mathrm{Hom}_{D^+(\mathrm{I}(k_X))}(f^{-1}K \otimes F, f^!G) \simeq \mathrm{Hom}_{D^+(\mathrm{I}(k_X))}(F, R\mathcal{I}hom\,(f^{-1}K, f^!G)). \end{aligned}$$

q.e.d.

Proposition 4.3.9. *There are natural morphisms and isomorphisms*

(i) $\alpha_X f^! \longrightarrow f^!\alpha_Y$,

(ii) $f^{-1}\beta_Y(\,\cdot\,) \otimes f^!k_Y \xrightarrow{\sim} f^!\beta_Y$.

Proof. (i) By Theorem 4.3.4, there is a natural morphism $Rf_{!!}f^! \longrightarrow \mathrm{id}$. This defines $Rf_!\alpha_X f^! \simeq \alpha_Y Rf_{!!}f^! \longrightarrow \alpha_Y$, and the result follows by adjunction.

(ii) Let $G \in D^+(k_Y)$. Using the morphism $Rf_{!!}f^!k_Y \longrightarrow k_Y$ we have the chain of morphisms

$$\begin{aligned} Rf_{!!}(f^{-1}\beta_Y G \otimes f^! k_Y) &\simeq \beta_Y G \otimes \mathbb{R}f_{!!}f^!k_Y \\ &\longrightarrow \beta_Y G \otimes k_Y. \end{aligned}$$

We get the morphism $f^{-1}\beta_Y G \otimes f^! k_Y \longrightarrow f^! \beta_Y G$ by adjunction. To prove it is an isomorphism, we take $S \in D^b(\mathrm{Mod}^c(k_X))$, and check the chain of isomorphisms

$$\begin{aligned} \mathrm{Hom}_{\mathrm{I}(k_X)}(S, f^{-1}\beta_Y G \otimes f^! k_Y) &\simeq H^0\mathrm{R}\Gamma\Big(X; R\mathcal{H}om\,(S, \beta_X f^{-1}G \otimes f^! k_Y)\Big) \\ &\simeq H^0\mathrm{R}\Gamma\Big(X; R\mathcal{H}om\,(S, f^! k_Y) \otimes f^{-1}G\Big) \\ &\simeq H^0\mathrm{R}\Gamma\Big(Y; Rf_!\big(R\mathcal{H}om\,(S, f^! k_Y) \otimes f^{-1}G\big)\Big) \\ &\simeq H^0\mathrm{R}\Gamma\Big(Y; Rf_! R\mathcal{H}om\,(S, f^! k_Y) \otimes G\Big) \\ &\simeq H^0\mathrm{R}\Gamma\Big(Y; Rf_* R\mathcal{H}om\,(S, f^! k_Y) \otimes G\Big) \\ &\simeq H^0\mathrm{R}\Gamma\Big(Y; R\mathcal{H}om\,(Rf_{!!}S, k_Y) \otimes G\Big) \\ &\simeq H^0\mathrm{R}\Gamma\Big(Y; R\mathcal{H}om\,(Rf_{!!}S, \beta_Y G)\Big) \\ &\simeq \mathrm{Hom}_{\mathrm{I}(k_Y)}(Rf_{!!}S, \beta_Y G) \simeq \mathrm{Hom}_{\mathrm{I}(k_X)}(S, f^! \beta_Y G). \end{aligned}$$

q.e.d.

Consider a Cartesian square of topological spaces

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\Box"] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y. \end{tikzcd}$$

Theorem 4.3.10. *There is a natural isomorphism of functors*

$$Rf'_* g'^! \xrightarrow{\sim} g^! Rf_*.$$

Proof. It is enough to prove the isomorphism

$$\mathrm{RHom}_{\mathrm{I}(k_{Y'})}(K, Rf'_* g'^! F) \simeq \mathrm{RHom}_{\mathrm{I}(k_{Y'})}(K, g^! Rf_* F)$$

for $K \in D^+(\mathrm{I}(k_{Y'}))$ and $F \in D^+(\mathrm{I}(k_X))$. By adjunction, this follows from the isomorphism $Rg'_{!!}f'^{-1}K \simeq f^{-1}Rg_{!!}K$, which is given by Theorem 4.2.9. q.e.d.

Theorem 4.3.11. *There is a natural isomorphism of functors*

$$Rf'_{!!}g'^{!} \xrightarrow{\sim} g^{!}Rf_{!!}.$$

Note that this isomorphism has no counterpart in sheaf theory.

Proof. The morphism is constructed by the chain of morphisms $Rg_{!!}Rf'_{!!}g'^{!} \simeq Rf_{!!}Rg'_{!!}g'^{!} \longrightarrow Rf_{!!}$ and by adjunction. Hence, it is enough to prove the isomorphism

$$\mathrm{RHom}_{\mathrm{I}(k_{Y'})}(K, Rf'_{!!}g'^{!}F) \xrightarrow{\sim} \mathrm{RHom}_{\mathrm{I}(k_{Y'})}(K, g^{!}Rf_{!!}F)$$

for $K \in \mathrm{Mod}^c(k_{Y'})$ and $F \in D^+(\mathrm{I}(k_X))$. Since K has compact support, one has the isomorphisms

$$\begin{aligned}
\mathrm{RHom}_{\mathrm{I}(k_{Y'})}(K, Rf'_{!!}g'^{!}F) &\simeq \mathrm{R}\Gamma(Y'; R\mathcal{H}om\,(K, Rf'_{!!}g'^{!}F)) \\
&\simeq \mathrm{R}\Gamma(Y; Rg_{!}R\mathcal{H}om\,(K, Rf'_{!!}g'^{!}F)).
\end{aligned}$$

Now, using Lemma 4.2.8 and the fact that g' is proper on $\mathrm{supp}(f'^{-1}K)$, we find the chain of isomorphisms

$$\begin{aligned}
Rg_{!}R\mathcal{H}om\,(K, Rf'_{!!}g'^{!}F) &\simeq Rg_{!}Rf'_{!}R\mathcal{H}om\,(f'^{-1}K, g'^{!}F) \\
&\simeq Rf_{!}Rg'_{!}R\mathcal{H}om\,(f'^{-1}K, g'^{!}F) \\
&\simeq Rf_{!}Rg'_{*}R\mathcal{H}om\,(f'^{-1}K, g'^{!}F) \\
&\simeq Rf_{!}R\mathcal{H}om\,(Rg'_{!!}f'^{-1}K, F) \\
&\simeq Rf_{!}R\mathcal{H}om\,(f^{-1}Rg_{!!}K, F) \\
&\simeq R\mathcal{H}om\,(Rg_{!!}K, Rf_{!!}F) \\
&\simeq Rg_{*}R\mathcal{H}om\,(K, g^{!}Rf_{!!}F).
\end{aligned}$$

Summarizing, we get

$$\begin{aligned}
\mathrm{RHom}_{\mathrm{I}(k_{Y'})}(K, Rf'_{!!}g'^{!}F) &\simeq \mathrm{R}\Gamma(Y; Rg_{!}R\mathcal{H}om\,(K, Rf'_{!!}g'^{!}F)) \\
&\simeq \mathrm{R}\Gamma(Y; Rg_{*}R\mathcal{H}om\,(K, g^{!}Rf_{!!}F)) \\
&\simeq \mathrm{R}\Gamma(Y'; R\mathcal{H}om\,(K, g^{!}Rf_{!!}F)) \\
&\simeq \mathrm{RHom}_{\mathrm{I}(k_{Y'})}(K, g^{!}Rf_{!!}F).
\end{aligned}$$

q.e.d.

We may summarize the commutativity of the various functors we have introduced in the table below. Here, "$\circ$" means that the functors commute, and "$\times$" they do not. Examples showing that the functors do not commute are given in Exercises 3.2, 3.3, 3.4, 3.5, 3.6, 4.1, 4.2.

	ι	α	β	$\varinjlim$	$\varprojlim$
f^{-1}	$\circ$	$\circ$	$\circ$	$\circ$	$\times$
f_*	$\circ$	$\circ$	$\times$	$\times$	$\circ$
$f_{!!}$	$\times$	$\circ$	$\times$	$\circ$	$\circ$
$f^!$	$\circ$	$\times$	$\times$	$\circ$	
$\varinjlim$	$\times$	$\circ$	$\circ$		
$\varprojlim$	$\circ$	$\circ$	$\times$		

4.4 Ring action

For the reader's convenience, we define the standard notions of a ring or a module in the case of the category $\mathrm{I}(k_X)$, although such constructions make sense and are classical in the more general framework of tensor categories. Recall that k is assumed to be a field.

Definition 4.4.1. (i) A ring in $\mathrm{I}(k_X)$ (or "an ind-k_X-algebra", or simply "an ind-ring") is the data of an object $\mathcal{A} \in \mathrm{I}(k_X)$ and morphisms $\mu_{\mathcal{A}} : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ and $\varepsilon_{\mathcal{A}} : k_X \to \mathcal{A}$, such that the diagrams below are commutative.

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\sim"] \arrow[rrr, bend right, "\mathrm{id}_{\mathcal{A}}"'] & k_X \otimes \mathcal{A} \arrow[r, "\varepsilon_{\mathcal{A}}\otimes\mathcal{A}"] & \mathcal{A}\otimes\mathcal{A} \arrow[r, "\mu_{\mathcal{A}}"] & \mathcal{A}
\end{tikzcd}
\quad
\begin{tikzcd}
\mathcal{A} \arrow[r, "\sim"] \arrow[rrr, bend right, "\mathrm{id}_{\mathcal{A}}"'] & \mathcal{A} \otimes k_X \arrow[r, "\mathcal{A}\otimes\varepsilon_{\mathcal{A}}"] & \mathcal{A}\otimes\mathcal{A} \arrow[r, "\mu_{\mathcal{A}}"] & \mathcal{A}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A} \arrow[r, "\mu_{\mathcal{A}}\otimes\mathcal{A}"] \arrow[d, "\mathcal{A}\otimes\mu_{\mathcal{A}}"'] & \mathcal{A}\otimes\mathcal{A} \arrow[d, "\mu_{\mathcal{A}}"] \\
\mathcal{A}\otimes\mathcal{A} \arrow[r, "\mu_{\mathcal{A}}"] & \mathcal{A}\,.
\end{tikzcd}$$

(ii) A left $\mathcal{A}$-module (or simply, an $\mathcal{A}$-module or "an ind-module") is the data of an object $M \in \mathrm{I}(k_X)$ and a morphism $\mu_M : A \otimes M \to M$ such that the diagrams below are commutative.

$$\begin{tikzcd}
\mathcal{A}\otimes\mathcal{A}\otimes M \arrow[r, "\mu_{\mathcal{A}}\otimes M"] \arrow[d, "\mathcal{A}\otimes\mu_M"'] & \mathcal{A}\otimes M \arrow[d, "\mu_M"] \\
\mathcal{A}\otimes M \arrow[r, "\mu_M"] & M
\end{tikzcd}
\quad
\begin{tikzcd}
M \arrow[r, "\sim"] \arrow[rrr, bend right, "\mathrm{id}_{\mathcal{A}}"'] & k_X \otimes M \arrow[r, "\varepsilon_{\mathcal{A}}\otimes M"] & \mathcal{A}\otimes M \arrow[r, "\mu_M"] & M
\end{tikzcd}$$

Notation 4.4.2.

(i) One denotes by $\nu_M : M \longrightarrow \mathcal{I}hom\,(\mathcal{A}, M)$ the morphism deduced from μ_M by the isomorphism

$$\mathrm{Hom}_{\mathrm{I}(k_X)}(\mathcal{A} \otimes M, M) \simeq \mathrm{Hom}_{\mathrm{I}(k_X)}(M, \mathcal{I}hom\,(\mathcal{A}, M)). \tag{4.4.1}$$

(ii) One denotes by $e_M : M \longrightarrow \mathcal{A}\otimes M$ the composition $M \xrightarrow{\sim} k_X \otimes M \xrightarrow[\varepsilon_{\mathcal{A}}\otimes M]{} \mathcal{A}\otimes M$.

(iii) One denotes by $e_M^* : \mathcal{I}hom\,(\mathcal{A}, M) \longrightarrow M$ the composition

$$\mathcal{I}hom\,(\mathcal{A}, M) \xrightarrow[\mathcal{I}hom\,(\varepsilon_{\mathcal{A}}, M)]{} \mathcal{I}hom\,(k_X, M) \simeq M\,.$$

Remark 4.4.3. In the classical case of a module M over a ring A, the analogous of the morphisms μ_M, ν_M, e_M and e_M^* are the morphisms $a\otimes m \mapsto am$, $m \mapsto (a \mapsto am)$, $m \mapsto 1 \otimes m$ and $\varphi \mapsto \varphi(1)$, respectively.

Definition 4.4.4. A morphism of $\mathcal{A}$-modules from M to N is a morphism $u \in \mathrm{Hom}\,(M, N)$ such that the diagram below commutes

$$\begin{tikzcd}
\mathcal{A} \otimes M \arrow[r, "\mathcal{A}\otimes u"] \arrow[d, "\mu_M"] & \mathcal{A} \otimes N \arrow[d, "\mu_N"] \\
M \arrow[r, "u"] & N.
\end{tikzcd}$$

If u is such a morphism, we shall say that "u is $\mathcal{A}$-linear".

By considering the $\mathcal{A}$-modules and the morphisms of $\mathcal{A}$-modules, one gets a category, that we denote by $\mathrm{I}(\mathcal{A})$.

Lemma 4.4.5. (i) *The correspondence $U \mapsto \mathrm{I}(\mathcal{A}|_U)$ is a proper stack of k-abelian categories.*

(ii) *The natural functor $\mathrm{I}(\mathcal{A}) \longrightarrow \mathrm{I}(k_X)$ is exact and faithful.*

The proof is left as an exercise.
As usual, one denotes by

$$\mathcal{H}om_{\mathrm{I}(\mathcal{A})} : \mathrm{I}(\mathcal{A})^{\mathrm{op}} \times \mathrm{I}(\mathcal{A}) \longrightarrow \mathrm{Mod}(k_X)$$
$$\mathrm{Hom}_{\mathrm{I}(\mathcal{A})} : \mathrm{I}(\mathcal{A})^{\mathrm{op}} \times \mathrm{I}(\mathcal{A}) \longrightarrow \mathrm{Mod}(k)$$

the natural functors. We sometimes write $\mathcal{H}om_{\mathcal{A}}$ instead of $\mathcal{H}om_{\mathrm{I}(\mathcal{A})}$.

Definition 4.4.6. (a) One denotes by $\mathcal{A}^{\mathrm{op}}$ the object $\mathcal{A}$ endowed with the morphisms $\varepsilon_{\mathcal{A}^{\mathrm{op}}} := \varepsilon_{\mathcal{A}}$ and $\mu_{\mathcal{A}^{\mathrm{op}}} := \mu_{\mathcal{A}} \circ v$, where $v : \mathcal{A} \otimes \mathcal{A} \longrightarrow \mathcal{A} \otimes \mathcal{A}$ is the morphism corresponding to $a \otimes b \mapsto b \otimes a$.

(b) An $\mathcal{A}^{\mathrm{op}}$-module is called a right $\mathcal{A}$-module.

Note that $\mathcal{A}$ is both a left and right $\mathcal{A}$-module.

Example 4.4.7. (i) If $\mathcal{A}$ is a sheaf of k_X-algebras, then $\mathrm{I}(\mathcal{A}) \simeq \mathrm{Ind}(\mathrm{Mod}^{\mathrm{c}}(\mathcal{A}))$ (see Exercise 4.3).

(ii) If $\mathcal{A}$ is a sheaf of k_X-algebras, then $\beta_X\mathcal{A}$ is a ring in $\mathrm{I}(k_X)$, and if M is a sheaf of $\mathcal{A}$-modules, then $\beta_X M$ is a $\beta_X\mathcal{A}$-module. This follows immediately from the fact that β_X commutes with $\otimes$. Note that, with the notations of Exercise 2.4, one has the equivalence $\mathrm{Mod}(\mathcal{A}, \mathrm{I}(k_X)) \simeq \mathrm{I}(\beta_X\mathcal{A})$.

Consider the two sequences of morphisms

$$\mathcal{A}\otimes\mathcal{A}\otimes M \underset{d}{\to} \mathcal{A}\otimes M \underset{\mu_M}{\longrightarrow} M \longrightarrow 0, \quad \text{where } d = \mu_{\mathcal{A}}\otimes M - \mathcal{A}\otimes\mu_M, \tag{4.4.2}$$

$$0 \longrightarrow M \underset{\nu_M}{\longrightarrow} \mathcal{I}hom\,(\mathcal{A}, M) \underset{d}{\to} \mathcal{I}hom\,(\mathcal{A}\otimes\mathcal{A}, M), \quad \text{where } d = \mathcal{I}hom\,(\mu_{\mathcal{A}}, M) - \mathcal{I}hom\,(\mathcal{A}, \nu_M). \tag{4.4.3}$$

Clearly, $\mu_M \circ d = 0$ in (4.4.2), and $d \circ \nu_M = 0$ in (4.4.3).

Lemma 4.4.8. *The two complexes* (4.4.2) *and* (4.4.3) *are exact.*

Proof. (i) The exactness of the complex (4.4.2) follows from the existence of e_M. In fact,

$$\mu_M \circ e_M = \mathrm{id}_M,\ d \circ e_{\mathcal{A}\otimes M} + e_M \circ \mu_M = \mathrm{id}_{\mathcal{A}\otimes M}\,.$$

(Here, we regard $\mathcal{A}\otimes M$ as an $\mathcal{A}$-module via the $\mathcal{A}$-module structure of $\mathcal{A}$.)

(ii) Similarly,

$$e^*_M \circ \nu_M = \mathrm{id}_M,\ e^*_{\mathcal{A}\otimes M} \circ d + \nu_M \circ e^*_M = \mathrm{id}_{\mathcal{I}hom\,(\mathcal{A},M)}\,.$$

q.e.d.

Definition 4.4.9. (i) One defines the bifunctor

$$\begin{aligned} \cdot \otimes_{\mathcal{A}} \cdot \quad &: \quad \mathrm{I}(\mathcal{A}^{\mathrm{op}}) \times \mathrm{I}(\mathcal{A}) \longrightarrow \mathrm{I}(k_X) \\ M \otimes_{\mathcal{A}} N \quad &:= \quad \mathrm{coker}(M\otimes\mathcal{A}\otimes N \underset{d}{\to} M\otimes N) \\ \text{where} \quad & \quad\ \ d = \mu_M\otimes N - M\otimes\mu_N. \end{aligned}$$

(ii) One defines the bifunctor

$$\begin{aligned} \mathcal{I}hom_{\mathcal{A}}(\,\cdot\,,\,\cdot\,) \quad &: \quad \mathrm{I}(\mathcal{A})^{\mathrm{op}} \times \mathrm{I}(\mathcal{A}) \longrightarrow \mathrm{I}(k_X) \\ \mathcal{I}hom_{\mathcal{A}}(M,N) \quad &:= \quad \ker(\mathcal{I}hom\,(M,N) \underset{d}{\to} \mathcal{I}hom\,(\mathcal{A}\otimes M, N)) \\ \text{where} \quad & \quad\ \ d = \mathcal{I}hom\,(\mu_M, N) - \mathcal{I}hom\,(M, \nu_N). \end{aligned}$$

Note that the functor $\otimes_{\mathcal{A}}$ is right exact, the functor $\mathcal{I}hom_{\mathcal{A}}$ is left exact, and by Lemma 4.4.8, one has the isomorphisms

$$\begin{aligned} \mathcal{A}\otimes_{\mathcal{A}} M &\simeq M, \\ \mathcal{I}hom_{\mathcal{A}}(\mathcal{A}, M) &\simeq M. \end{aligned}$$

Proposition 4.4.10. *One has the isomorphism*

$$\alpha_X \mathcal{I}hom_{\mathcal{A}}(M,N) \simeq \mathcal{H}om_{\mathrm{I}(\mathcal{A})}(M,N).$$

Proof. The left hand side is isomorphic to $\ker(\mathcal{H}om(M,N) \underset{d}{\longrightarrow} \mathcal{H}om(\mathcal{A}\otimes M,N))$ where $d = \mathcal{H}om(\mu_M,N) - \mathcal{H}om(M,\nu_N)$. Let $\lambda : \mathcal{H}om_{\mathcal{A}}(M,N) \longrightarrow \mathcal{H}om(M,N)$. It is enough to check that λ induces an isomorphism $\Gamma(U;\mathcal{H}om_{\mathrm{I}(\mathcal{A})}(M,N)) \longrightarrow \ker(d|_U)$ on each open $U \subset X$. We may assume $U = X$. In this case, one checks the isomorphism

$$\mathrm{Hom}_{\mathrm{I}(\mathcal{A})}(M,N) \simeq \ker(\mathrm{Hom}_{\mathrm{I}(k_X)}(M,N) \underset{d}{\longrightarrow} \mathrm{Hom}_{\mathrm{I}(k_X)}(\mathcal{A}\otimes M,N)).$$

Indeed, d is the morphism $u \mapsto (\mu_M \circ u - u \circ \nu_N)$, and u is $\mathcal{A}$-linear if and only if $u \circ \nu_N = \mu_N \circ \mathrm{id}_{\mathcal{A}} \otimes u$. This is better visualized by the diagram

$$\begin{tikzcd}
 & & \mathrm{Hom}_{\mathrm{I}(k_X)}(\mathcal{A}\otimes M,N) \arrow[dd, dash, "\sim"] \\
\mathrm{Hom}_{\mathrm{I}(\mathcal{A})}(M,N) \arrow[r] & \mathrm{Hom}_{\mathrm{I}(k_X)}(M,N) \arrow[ur] \arrow[dr] & \\
 & & \mathrm{Hom}_{\mathrm{I}(k_X)}(M,\mathcal{I}hom(\mathcal{A},N)).
\end{tikzcd}$$

q.e.d.

Sometimes, one has to consider various rings in $\mathrm{I}(k_X)$.

Proposition 4.4.11. *Let $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ be three rings in $\mathrm{I}(k_X)$.*

(i) *The functor $\otimes_{\mathcal{A}_2}$ induces a functor*

$$\mathrm{I}(\mathcal{A}_1\otimes\mathcal{A}_2^{\mathrm{op}}) \times \mathrm{I}(\mathcal{A}_2\otimes\mathcal{A}_3^{\mathrm{op}}) \longrightarrow \mathrm{I}(\mathcal{A}_1\otimes\mathcal{A}_3^{\mathrm{op}}).$$

(ii) *The functor $\mathcal{I}hom_{\mathcal{A}_1}$ induces a functor*

$$\mathrm{I}(\mathcal{A}_1\otimes\mathcal{A}_2)^{\mathrm{op}} \times \mathrm{I}(\mathcal{A}_1\otimes\mathcal{A}_3) \longrightarrow \mathrm{I}(\mathcal{A}_2^{\mathrm{op}}\otimes\mathcal{A}_3).$$

The proof is straightforward.

Proposition 4.4.12. *Let $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4$ be four rings in* $\mathrm{I}(k_X)$. *There is a natural isomorphism of $\mathcal{A}_4^{\mathrm{op}} \otimes A_3$-modules*

$$\mathcal{I}hom_{\mathcal{A}_1}({}_1L_4, \mathcal{I}hom_{\mathcal{A}_2}({}_2M_1, {}_2N_3)) \simeq \mathcal{I}hom_{\mathcal{A}_2}({}_2M_1 \otimes_{\mathcal{A}_1} {}_1L_4, {}_2N_3)$$

where ${}_iF_j$ means that F is an $\mathcal{A}_i \otimes A_j^{\mathrm{op}}$-module.

The proof is straightforward.

We shall now construct the derived functors of $\otimes_{\mathcal{A}}$ and $\mathcal{I}hom_{\mathcal{A}}$.

Definition 4.4.13. (i) $\mathcal{P}_c(\mathcal{A})$ is the full subcategory of $\mathrm{I}(\mathcal{A})$ consisting of the objects isomorphic to $\mathcal{A} \otimes P$ for some $P \in \mathcal{P}_c(k_X) = \{\text{“}\bigoplus_j\text{”}\, G_j\,;\, G_j \in \mathrm{Mod}^c(k_X)\}$.

(ii) $\mathcal{I}_q(\mathcal{A})$ is the full subcategory of $\mathrm{I}(\mathcal{A})$ consisting of the objects which are quasi-injective in $\mathrm{I}(k_X)$.

Lemma 4.4.14. (i) *The category $\mathcal{P}_c(\mathcal{A})$ is generating in* $\mathrm{I}(\mathcal{A})$.

(ii) *The category $\mathcal{I}_q(\mathcal{A})$ is cogenerating in* $\mathrm{I}(\mathcal{A})$.

Proof. (i) Let $M \in \mathrm{I}(\mathcal{A})$. There exists an epimorphism $P \twoheadrightarrow M$ in $\mathrm{I}(k_X)$, with $P \in \mathcal{P}_c(k_X)$. Then $\mathcal{A} \otimes P \to \mathcal{A} \otimes M$ is an $\mathcal{A}$-linear epimorphism, as well as the composition $\mathcal{A} \otimes P \to A \otimes M \xrightarrow[\mu_M]{} M$.

(ii) Let us introduce the additive category $\mathcal{I}_q^{\mathcal{A}}$ of $\mathrm{I}(k_X)$ consisting of objects F such that both F and $\mathcal{I}hom(\mathcal{A}, F)$ belong to $\mathcal{I}_q(k_X)$. It follows from Corollary 4.1.7 that $\mathcal{I}_q^{\mathcal{A}}$ is cogenerating in $\mathrm{I}(k_X)$. Let $M \in \mathrm{I}(\mathcal{A})$. There exists a monomorphism $M \rightarrowtail I$ in $\mathrm{I}(k_X)$, with $I \in \mathcal{I}_q^{\mathcal{A}}(k_X)$. Then the composition $M \to \mathcal{I}hom(\mathcal{A}, M) \to \mathcal{I}hom(\mathcal{A}, I)$ is an $\mathcal{A}$-linear monomorphism . q.e.d.

Lemma 4.4.15. *For an $\mathcal{A}$-linear epimorphism $M \twoheadrightarrow \mathcal{A} \otimes F''$ with $F'' \in \mathrm{Mod}^c(k_X)$, there exist an exact sequence $0 \to F' \to F \to F'' \to 0$ in $\mathrm{Mod}^c(k_X)$ and a commutative exact diagram in* $\mathrm{I}(\mathcal{A})$*:*

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{A} \otimes F' \arrow[r] & \mathcal{A} \otimes F \arrow[r] \arrow[d] & \mathcal{A} \otimes F'' \arrow[r] \arrow[d, equal] & 0 \\
 & & M \arrow[r] & \mathcal{A} \otimes F'' \arrow[r] & 0.
\end{tikzcd}$$

Proof. The morphism $\varepsilon_{\mathcal{A}}$ defines the morphism $F'' \to \mathcal{A} \otimes F''$. Applying Proposition 1.3.2, we find an epimorphism $F \twoheadrightarrow F''$ in $\mathrm{Mod}^c(k_X)$ such that the composition $F \to F'' \to \mathcal{A} \otimes F''$ factors through M. Setting $F' = \ker(F \to F'')$, and tensoring the exact sequence $0 \to F' \to F \to F'' \to 0$ by $\mathcal{A}$, we get the result. q.e.d.

Theorem 4.4.16. (a) *The category* $\mathcal{P}_c(\mathcal{A}_1 \otimes \mathcal{A}_2^{\mathrm{op}}) \times \mathcal{P}_c(\mathcal{A}_2 \otimes \mathcal{A}_3^{\mathrm{op}})$ *is projective with respect to the functor* $\otimes_{\mathcal{A}_2}$.

(b) *The functor below is well-defined:*

$$\overset{L}{\underset{\mathcal{A}_2}{\otimes}} : D^-(\mathrm{I}(\mathcal{A}_1 \otimes \mathcal{A}_2^{\mathrm{op}})) \times D^-(\mathrm{I}(\mathcal{A}_2 \otimes \mathcal{A}_3^{\mathrm{op}})) \longrightarrow D^-(\mathrm{I}(\mathcal{A}_1 \otimes \mathcal{A}_3^{\mathrm{op}})).$$

Proof. We shall only give the proof in case $\mathcal{A}_1 = \mathcal{A}_3 = k_X$, and we set $\mathcal{A}_2 := \mathcal{A}$. The general case is only notationally more involved.

(a) Let $G = F \otimes \mathcal{A} \in \mathcal{P}_c(\mathcal{A}^{\mathrm{op}})$ with $F \in \mathcal{P}_c(k_X)$, and let us prove that the category $\mathcal{P}_c(\mathcal{A})$ is projective with respect to the functor $G \otimes_{\mathcal{A}} \cdot\,$. This follows from $(G \otimes \mathcal{A}) \otimes_{\mathcal{A}} L \simeq G \otimes L$ and Corollary 1.4.4.

(b) follows from (a). q.e.d.

Theorem 4.4.17. (a) *The category* $\mathcal{P}_c(\mathcal{A}_1 \otimes \mathcal{A}_2)^{\mathrm{op}} \times \mathcal{I}_q(\mathcal{A}_1 \otimes \mathcal{A}_3)$ *is injective with respect to the functors* $\mathrm{Hom}_{\mathrm{I}(\mathcal{A}_1)}$, $\mathcal{I}hom_{\mathcal{A}_1}$, $\mathcal{H}om_{\mathrm{I}(\mathcal{A}_1)}$.

(b) *The functors below are well-defined:*

$$\begin{aligned}
\mathrm{RHom}_{\mathrm{I}(\mathcal{A}_1)} &: D^-(\mathrm{I}(\mathcal{A}_1 \otimes \mathcal{A}_2))^{\mathrm{op}} \times D^+(\mathrm{I}(\mathcal{A}_1 \otimes \mathcal{A}_3)) \longrightarrow D^+(\mathrm{Mod}(k))\,,\\
R\mathcal{I}hom_{\mathcal{A}_1} &: D^-(\mathrm{I}(\mathcal{A}_1 \otimes \mathcal{A}_2))^{\mathrm{op}} \times D^+(\mathrm{I}(\mathcal{A}_1 \otimes \mathcal{A}_3)) \longrightarrow D^+(\mathrm{I}(\mathcal{A}_2^{\mathrm{op}} \otimes \mathcal{A}_3))\,,\\
R\mathcal{H}om_{\mathrm{I}(\mathcal{A}_1)} &: D^-(\mathrm{I}(\mathcal{A}_1 \otimes \mathcal{A}_2))^{\mathrm{op}} \times D^+(\mathrm{I}(\mathcal{A}_1 \otimes \mathcal{A}_3))\\
&\qquad\qquad\longrightarrow D^+(\mathrm{Mod}(\alpha_X(\mathcal{A}_2^{\mathrm{op}} \otimes \mathcal{A}_3))).
\end{aligned}$$

Proof. We shall only give the proof in case $\mathcal{A}_1 = \mathcal{A}_3 = k_X$, and we set $\mathcal{A}_2 := \mathcal{A}$. The general case is only notationally more involved.

We shall only treat (a) for the functor $\mathcal{I}hom_{\mathcal{A}}$, the other cases being similar. Moreover, (b) follows from (a).

(i) Let $F \in \mathcal{I}_q(\mathcal{A})$, and let us show that $\mathcal{P}_c(\mathcal{A})$ is projective with respect to the functor $\mathcal{I}hom_{\mathcal{A}}(\,\cdot\,, F)$. We shall apply Theorem 1.4.3. Let $M \twoheadrightarrow \mathcal{A} \otimes P''$ with $P'' = \text{“}\bigoplus_j\text{”}\, P''_j$ and $P''_j \in \mathrm{Mod}^c(k_X)$. By Lemma 4.4.15 there exists an exact sequence $0 \longrightarrow P' \longrightarrow P \longrightarrow P'' \longrightarrow 0$ in $\mathcal{P}_c(k_X)$ such that $\mathcal{A} \otimes P \longrightarrow \mathcal{A} \otimes P''$ factors through $M \longrightarrow \mathcal{A} \otimes P''$. Consider the exact sequence $0 \longrightarrow \mathcal{A} \otimes P' \longrightarrow \mathcal{A} \otimes P \longrightarrow \mathcal{A} \otimes P'' \longrightarrow 0$. It is enough to show that the functor $\mathcal{I}hom_{\mathcal{A}}(\,\cdot\,, F)$ applied to this exact sequence will give an exact sequence. This follows from the isomorphism $\mathcal{I}hom_{\mathcal{A}}(\mathcal{A} \otimes H, F) \simeq \mathcal{I}hom\,(H, F)$ and the fact that $F \in \mathcal{I}_q(k_X)$.

(ii) Let $P \in \mathcal{P}_c(k_X)$. Let us prove that the category $\mathcal{I}_q(\mathcal{A})$ is injective with respect to the functor $\mathcal{I}hom_{\mathcal{A}}(\mathcal{A} \otimes P, \,\cdot\,)$. This category is cogenerating by Lemma 4.4.14. Recall that $\mathcal{I}hom_{\mathcal{A}}(\mathcal{A} \otimes P, \,\cdot\,) \simeq \mathcal{I}hom\,(P, \,\cdot\,)$. Let $0 \longrightarrow F' \longrightarrow F \longrightarrow F'' \longrightarrow 0$ be an exact sequence in $\mathcal{I}_q(\mathcal{A})$. Then, this sequence is exact in $\mathcal{I}_q(k_X)$ and the sequence obtained when applying $\mathcal{I}hom\,(P, \,\cdot\,)$ will remain exact by Theorem 4.1.3. Hence, we may apply Corollary 1.4.4. q.e.d.

For a small subset S of $\mathrm{I}(k_X)$, let us denote by $\mathcal{I}_q(\mathcal{A};S)$ the set of $M \in \mathrm{I}(\mathcal{A})$ such that $M \in \mathcal{I}_q(k_X)$ and $\mathcal{I}hom\,(F,M) \in \mathcal{I}_q(k_X)$ for any $F \in S$. Then by Corollary 4.1.7 and by a similar argument to the proof in Lemma 4.4.14, $\mathcal{I}_q(\mathcal{A};S)$ is cogenerating in $\mathrm{I}(\mathcal{A})$. Hence $\mathcal{P}_c(\mathcal{A})^{\mathrm{op}} \times \mathcal{I}_q(\mathcal{A};S)$ is injective with respect to the functors $\mathrm{Hom}_{\mathrm{I}(\mathcal{A}_1)}$, $\mathcal{I}hom_{\mathcal{A}_1}$, $\mathcal{H}om_{\mathrm{I}(\mathcal{A}_1)}$. Moreover for three ind-rings $\mathcal{A}_1$, A_2, $\mathcal{A}_3$, and $M \in \mathcal{P}_c(\mathcal{A}_1 \otimes \mathcal{A}_2^{\mathrm{op}})$ and $N \in \mathcal{I}_q(\mathcal{A}_1 \otimes \mathcal{A}_3^{\mathrm{op}};\{\mathcal{A}_2\})$ $\mathcal{I}hom_{\mathcal{A}_1}(M,N)$ belongs to $\mathcal{I}_q(\mathcal{A}_2 \otimes \mathcal{A}_3^{\mathrm{op}})$.

Indeed , we have

$$\mathcal{I}hom_{\mathcal{A}_1}(\mathcal{A}_1 \otimes \mathcal{A}_2 \otimes P, N) \simeq \mathcal{I}hom\,(\mathcal{A}_2 \otimes P, N) \simeq \mathcal{I}hom\,(P, \mathcal{I}hom\,(\mathcal{A}_2, N)).$$

Theorem 4.4.18. *Let $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4$ be four rings in $\mathrm{I}(k_X)$. Then for ${}_1L_4 \in D^-(\mathrm{I}(\mathcal{A}_1 \otimes \mathcal{A}_4^{\mathrm{op}}))$, ${}_2M_1 \in D^-(\mathrm{I}(\mathcal{A}_2 \otimes \mathcal{A}_1^{\mathrm{op}}))$ and ${}_2N_3 \in D^+(\mathrm{I}(\mathcal{A}_2 \otimes \mathcal{A}_3^{\mathrm{op}}))$, there is a natural isomorphism in $D^+(\mathrm{I}(\mathcal{A}_4^{\mathrm{op}} \otimes A_3))$*

$$R\mathcal{I}hom_{\mathcal{A}_1}({}_1L_4, R\mathcal{I}hom_{\mathcal{A}_2}({}_2M_1, {}_2N_3)) \simeq R\mathcal{I}hom_{\mathcal{A}_2}({}_2M_1 \overset{L}{\underset{\mathcal{A}_1}{\otimes}} {}_1L_4, {}_2N_3).$$

Proof. We may choose ${}_1L_4 \in K^-(\mathcal{P}_c(\mathcal{A}_1 \otimes \mathcal{A}_4^{\mathrm{op}}))$, ${}_2M_1 \in K^-(\mathcal{P}_c(\mathcal{A}_2 \otimes \mathcal{A}_1^{\mathrm{op}}))$, and ${}_2N_3 \in K^+(\mathcal{I}_q(\mathcal{A}_2 \otimes \mathcal{A}_3^{\mathrm{op}};\{\mathcal{A}_1\}))$. Then we have ${}_2M_1 \otimes_{\mathcal{A}_1} {}_1L_4 \in K^-(\mathcal{P}_c(\mathcal{A}_2 \otimes \mathcal{A}_4^{\mathrm{op}}))$ and $\mathcal{I}hom_{\mathcal{A}_2}({}_2M_1, {}_2N_3) \in K^+(\mathcal{I}_q(\mathcal{A}_1 \otimes \mathcal{A}_3^{\mathrm{op}}))$. Then the result follows from Theorems 4.4.17 and 4.4.16. q.e.d.

Exercises to Chapter 4

Exercise 4.1. Let $Y = \mathbb{R}$, $X = \{0\}$ and denote by $f : X \longrightarrow Y$ the embedding. Let $G = {}_{\{0\}}(k_Y) \in \mathrm{I}(k_Y)$. Prove that $f^!G \simeq k_X[-1]$ and $f^!\alpha_Y G \simeq k_X$. Hence, $f^!$ does not commute with α.

Exercise 4.2. Let $f : X \longrightarrow Y$ be as in Exercise 4.1 and let $G = k_{\{0\}} \in \mathrm{Mod}(k_Y)$. Prove that $f^!\beta_Y G \simeq k[-1]$ and $\beta_X f^! G \simeq k$. Hence, $f^!$ does not commute with β.

Exercise 4.3. Let $\mathcal{A}$ denote a sheaf of k_X-algebras. We also regard $\mathcal{A}$ as a ring in $\mathrm{I}(k_X)$, and we denote by $\varphi : \mathrm{Ind}(\mathrm{Mod}^c(\mathcal{A})) \longrightarrow \mathrm{I}(\mathcal{A})$ the canonical functor.
(i) Construct the functor $\tau : \mathrm{I}(k_X) \longrightarrow \mathrm{Ind}(\mathrm{Mod}^c(\mathcal{A}))$ such that $\varphi \circ \tau(F) \simeq \mathcal{A} \otimes F$ for $F \in \mathrm{I}(k_X)$.
(ii) For $M \in \mathrm{I}(\mathcal{A})$, construct a morphism $\tilde{d}(M) : \tau(\mathcal{A} \otimes M) \longrightarrow \tau(M)$ in $\mathrm{Ind}(\mathrm{Mod}^c(\mathcal{A}))$ such that $\varphi(\tilde{d}(M))$ coincides with d in (4.4.2).
(iii) Define $\psi : \mathrm{I}(\mathcal{A}) \longrightarrow \mathrm{Ind}(\mathrm{Mod}^c(\mathcal{A}))$ as $\psi = \mathrm{coker}\,\tilde{d}$. Prove that the two functors φ and ψ are inverse to each other.

Chapter 5

Construction of ind-sheaves

5.1 Construction I

As in the preceding chapter, X is a locally compact topological space countable at infinity, and k is a field. Recall that if Z is locally closed in X we often write k_Z instead of $(k_X)_Z$, for short.

Let OP(X) denote the family of all open subsets of X. We keep the notation OP(X) to denote the category with the open sets as objects and with the inclusions of open sets as morphisms. Similarly, if $\mathcal{T}$ is a family of open subsets of X, we still denote by $\mathcal{T}$ the full subcategory of $\mathrm{OP}(X)$ whose objects are the elements of $\mathcal{T}$.

We introduce the category $\mathcal{K}(\mathcal{T})$ by setting

$$\begin{aligned} \mathrm{Ob}(\mathcal{K}(\mathcal{T})) &= \{(I, \{U_i\}_{i\in I}); I \text{ finite }, U_i \in \mathcal{T}\}. \\ \mathrm{Hom}_{\mathcal{K}(\mathcal{T})}((I, \{U_i\}_i), (J, \{V_j\}_j)) &= \{(a_{ji})_{j\in J, i\in I}; a_{ji} \in k, a_{ji} \neq 0 \Rightarrow \varnothing \neq U_i \subset V_j\}. \end{aligned}$$

The composition is defined as follows. Let $\varphi = (a_{ji})_{j\in J,i\in I} : (I, \{U_i\}_i) \longrightarrow (J, \{V_j\}_j)$ and $\psi = (b_{kj})_{k\in K,j\in J} : (J, \{V_j\}_j) \longrightarrow (K, \{W_k\}_k)$. Then $\psi \circ \varphi = (c_{ki})_{k\in K,i\in I}$, with

$$c_{ki} = \sum_j b_{kj} a_{ji}.$$

The functor $\mathcal{K}(\mathcal{T}) \longrightarrow \mathrm{Mod}(k_X)$ which associates the sheaf $\bigoplus_i k_{U_i}$ to $(I, \{U_i\}_i)$ is faithful and we shall identify $\mathcal{K}(\mathcal{T})$ with a subcategory of $\mathrm{Mod}(k_X)$. Note that this functor is not fully faithful in general since the open sets which belong to $\mathcal{T}$ are not necessarily connected.

Let us denote by $\mathrm{PSH}(k_{\mathcal{T}})$ the abelian category $(\mathrm{Mod}(k))^{\mathcal{T}^{\mathrm{op}}}$ of contravariant functors from $\mathcal{T}$ to $\mathrm{Mod}(k)$, and by

$$for : \mathrm{Mod}(k_X) \longrightarrow \mathrm{PSH}(k_{\mathcal{T}})$$

the forgetful functor. We call an object of $\mathrm{PSH}(k_{\mathcal{T}})$ a presheaf on $\mathcal{T}$.

First, remark that if F and G belong to $\mathrm{PSH}(k_{\mathcal{T}})$, then one has an exact sequence:

$$0 \longrightarrow \mathrm{Hom}_{\mathrm{PSH}(k_{\mathcal{T}})}(F,G) \longrightarrow \prod_{W} \mathrm{Hom}_{k}(F(W),G(W)) \xrightarrow[a-b]{} \prod_{U\subset V} \mathrm{Hom}_{k}(F(V),G(U)), \tag{5.1.1}$$

where U,V,W belong to $\mathcal{T}$, a is associated to the morphism $F(V) \longrightarrow F(U)$ and b is associated to $G(V) \longrightarrow G(U)$.

One extends the functor for to the category $\mathrm{I}(k_X)$ by defining:

$$\begin{aligned} for &: \mathrm{I}(k_X) \longrightarrow \mathrm{PSH}(k_{\mathcal{T}}), \\ for(G)(U) &= \mathrm{Hom}_{\mathrm{I}(k_X)}(k_U, G). \end{aligned} \tag{5.1.2}$$

Proposition 5.1.1. *The functor "for" in* (5.1.2) *admits a left adjoint, denoted by $F \mapsto F^+$. In other words, if $F \in \mathrm{PSH}(k_{\mathcal{T}})$ and $G \in \mathrm{I}(k_X)$, one has*

$$\mathrm{Hom}_{\mathrm{I}(k_X)}(F^+,G) \simeq \mathrm{Hom}_{\mathrm{PSH}(k_{\mathcal{T}})}(F, for(G)).$$

Proof. In the proof, for $E \in \mathrm{Mod}(k)$, we shall write $\beta_X(E)$ instead of $\beta_X(E_X)$, where as usual E_X denotes the constant sheaf on X with stalk E.

Let $G \in \mathrm{I}(k_X)$ and consider the two morphisms

$$\prod_W \mathrm{Hom}_k(F(W), \mathrm{Hom}_{\mathrm{I}(k_X)}(k_W,G)) \underset{b}{\overset{a}{\rightrightarrows}} \prod_{U\subset V} \mathrm{Hom}_k(F(V), \mathrm{Hom}_{\mathrm{I}(k_X)}(k_U,G)).$$

where U,V,W range through the family of objects of $\mathcal{T}$, a is associated to $F(V) \longrightarrow F(U)$ and b is associated to $k_U \longrightarrow k_V$. By (5.1.1), we find

$$\mathrm{Hom}_{\mathrm{PSH}(k_{\mathcal{T}})}(F, for(G)) \simeq \ker(a-b).$$

On the other hand, one has the isomorphisms

$$\begin{aligned} \mathrm{Hom}_k(F(V), \mathrm{Hom}_{\mathrm{I}(k_X)}(k_U,G)) &\simeq \mathrm{Hom}_{k_X}((F(V))_X, \mathcal{H}om_{\mathrm{I}(k_X)}(k_U,G)) \\ &\simeq \mathrm{Hom}_{\mathrm{I}(k_X)}(k_U \otimes \beta_X(F(V)), G). \end{aligned} \tag{5.1.3}$$

Now consider the two morphisms, denoted by a' and b':

$$\text{“}\bigoplus_{U\subset V}\text{”}(k_U \otimes \beta_X(F(V))) \underset{b'}{\overset{a'}{\rightrightarrows}} \text{“}\bigoplus_{W}\text{”}(k_W \otimes \beta_X(F(W)))$$

where U,V,W range through the family of objects of $\mathcal{T}$, a' is associated to $k_U \otimes \beta_X(F(V)) \longrightarrow k_U \otimes \beta_X(F(U))$ and b' is associated to $k_U \otimes \beta_X(F(V)) \longrightarrow k_V \otimes \beta_X(F(V))$. We define

$$F^+ = \mathrm{coker}(a'-b').$$

By (5.1.3),

$$\ker(a-b) \simeq \operatorname{Hom}_{\mathrm{I}(k_X)}(F^+, G).$$

Hence, the adjunction formula follows. q.e.d.

Now let U be an open subset of X. We denote by $\mathcal{T}|U$ the family of open sets $V \in \mathcal{T}$ with $V \subset U$. We keep the same notations for and $^+$ to denote the adjoint functors of Proposition 5.1.1 when X is replaced with U. Then a natural question is whether $^+$ commutes with the restriction to U. We need hypotheses on $\mathcal{T}$. Consider the conditions:

$$(5.1.4)\quad \begin{cases} \text{(i)} & \text{if } U \text{ and } V \text{ belong to } \mathcal{T}, \text{ then } U \cap V \text{ belongs to } \mathcal{T}, \\ \text{(ii)} & \text{for any } x \in X,\ \{U \in \mathcal{T}; x \in U\} \text{ is a neighborhood system of } x \\ & \text{(i.e. } \mathcal{T} \text{ is a basis of the topology).} \end{cases}$$

Proposition 5.1.2. *Assume* (5.1.4) (i). *Then for $U \in \mathcal{T}$ and $F \in \mathrm{PSH}(k_{\mathcal{T}})$, we have*

$$(F^+)|_U \simeq (F|_{\mathcal{T}|U})^+.$$

Proof. Denote by $j : U \hookrightarrow X$ the open embedding and let $K \in \mathrm{I}(k_U)$. First, let us prove that the natural morphism

$$(5.1.5)\qquad \operatorname{Hom}_{\mathrm{PSH}(k_{\mathcal{T}})}(F, for(j_*K)) \longrightarrow \operatorname{Hom}_{\mathrm{PSH}(k_{\mathcal{T}|U})}(F|_U, for(K))$$

is an isomorphism by constructing an inverse. Let $\varphi \in \operatorname{Hom}_{\mathrm{PSH}(k_{\mathcal{T}|U})}(F|_U, for(K))$ and let $V \in \mathcal{T}$. We define the morphism $F(V) \longrightarrow \operatorname{Hom}_{\mathrm{I}(k_X)}(k_V, j_*K)$ by the sequence of morphisms

$$F(V) \longrightarrow F(U \cap V) \xrightarrow[\varphi(U\cap V)]{} \operatorname{Hom}_{\mathrm{I}(k_U)}(k_{U\cap V}, K) \simeq \operatorname{Hom}_{\mathrm{I}(k_X)}(k_V, j_*K).$$

Hence, applying Proposition 5.1.1, we get the chain of isomorphisms

$$\begin{aligned} \operatorname{Hom}_{\mathrm{I}(k_U)}((F^+)|_U, K) &\simeq \operatorname{Hom}_{\mathrm{I}(k_X)}(F^+, j_*K) \simeq \operatorname{Hom}_{\mathrm{PSH}(k_{\mathcal{T}})}(F, for(j_*K)) \\ &\simeq \operatorname{Hom}_{\mathrm{PSH}(k_{\mathcal{T}|U})}(F|_U, for(K)) \simeq \operatorname{Hom}_{\mathrm{I}(k_U)}((F|_U)^+, K). \end{aligned}$$

q.e.d.

Now consider a sheaf $\mathcal{A}$ of unitary k-algebras on X. Let $\widetilde{\mathcal{A}}$ be the presheaf on $\mathcal{T}$ that associates $\Gamma(\overline{U}; \mathcal{A})$ to $U \in \mathcal{T}$. Let F be a presheaf on $\mathcal{T}$ and assume that for each $U \in \mathcal{T}$, $F(U)$ is a $\Gamma(\overline{U}; \mathcal{A})$-module and the restriction morphisms are $\mathcal{A}$-linear, that is, for $V \subset U$, $U, V \in \mathcal{T}$, the diagram below is commutative:

$$\begin{tikzcd} \Gamma(\overline{U}; \mathcal{A}) \otimes F(U) \arrow[r] \arrow[d] & F(U) \arrow[d] \\ \Gamma(\overline{V}; \mathcal{A}) \otimes F(V) \arrow[r] & F(V). \end{tikzcd}$$

In such a situation, we shall say that F is a presheaf of $\widetilde{\mathcal{A}}$-modules on $\mathcal{T}$.

Corollary 5.1.3. *Assume* (5.1.4). *Let $\mathcal{A}$ be a sheaf of k-algebras on X, and let F be a presheaf of $\widetilde{\mathcal{A}}$-modules on $\mathcal{T}$. Then $F^+ \in \mathrm{Mod}(\mathcal{A}, \mathrm{I}(k_X)) \simeq \mathrm{I}(\beta_X \mathcal{A})$.*

Proof. Let $U \in \mathcal{T}$ and let $a \in \Gamma(\overline{U}; \mathcal{A})$. Then a defines a morphism $F|_U \longrightarrow F|_U$, hence a morphism $(F^+)|_U \simeq (F|_U)^+ \longrightarrow (F|_U)^+ \simeq (F^+)|_U$. Therefore, we have a morphism of presheaves on $\mathcal{T}$, $\widetilde{\mathcal{A}} \longrightarrow \mathcal{E}nd(F^+)$. Since $\mathcal{E}nd(F)$ is a sheaf and $\mathcal{T}$ is a basis of the topology on X, this defines the morphism of sheaves $\mathcal{A} \longrightarrow \mathcal{E}nd(F^+)$. q.e.d.

5.2 $\mathcal{T}$-coherent sheaves

We keep the notations of section 5.1

First, recall that if $\mathcal{C}$ is an abelian category, $\mathcal{J}$ an additive subcategory, and $F \in \mathcal{C}$, one says that:

(i) F is $\mathcal{J}$-finite if there exists an epimorphism $G \twoheadrightarrow F$ with $G \in \mathcal{J}$,

(ii) F is $\mathcal{J}$-pseudo-coherent if if for any morphism $G \underset{\varphi}{\longrightarrow} F$ with $G \in \mathcal{J}$, $\ker \varphi$ is $\mathcal{J}$-finite,

(iii) F is $\mathcal{J}$-coherent if F is both $\mathcal{J}$-finite and $\mathcal{J}$-pseudo-coherent.

Note that (ii) is equivalent to the same condition with "$G \in \mathcal{J}$" replaced by "G is $\mathcal{J}$-finite".

One denotes by $\mathrm{coh}(\mathcal{J})$ the full subcategory of $\mathcal{C}$ consisting of $\mathcal{J}$-coherent objects.

Then, one easily proves that the category $\mathrm{coh}(\mathcal{J})$ is additive and stable by kernels (see [10]).

We shall apply these construction with $\mathcal{C} = \mathrm{Mod}(k_X)$ and $\mathcal{J} = \mathcal{K}(\mathcal{T})$. We shall say that a sheaf F is $\mathcal{T}$-finite (resp. pseudo-coherent, resp. coherent) instead of $\mathcal{K}(\mathcal{T})$-finite (resp. pseudo-coherent, resp. coherent).

One denotes by $\mathrm{coh}(\mathcal{T})$ the full subcategory of $\mathrm{Mod}(k_X)$ consisting of $\mathcal{T}$-coherent objects.

We consider the following hypotheses on $\mathcal{T}$:

(5.2.1)
$$\begin{cases} \text{(i)} & \varnothing \in \mathcal{T}, \\ \text{(ii)} & U \text{ and } V \text{ belong to } \mathcal{T} \text{ if and only if } U \cap V \text{ and } U \cup V \text{ belong to } \mathcal{T}, \\ \text{(iii)} & \text{for } U, V \in \mathcal{T}, U \setminus V \text{ has finitely many connected components}, \\ \text{(iv)} & U \in \mathcal{T} \text{ implies that } U \text{ is relatively compact.} \end{cases}$$

Note that if $\mathcal{T}$ satisfies (i)–(iii), then each $U \in \mathcal{T}$ has finitely many connected components, each of which belongs to $\mathcal{T}$.

Theorem 5.2.1. *Assume that $\mathcal{T}$ satisfies* (i)–(iii) *in* (5.2.1). *Then the subcategory* $\mathrm{coh}(\mathcal{T})$ *is stable by kernels, cokernels and finite direct sums. Moreover, it contains* $\mathcal{K}(\mathcal{T})$.

We know that $\mathrm{coh}(\mathcal{T})$ is stable by kernels and finite direct sums. The proof that it is stable by cokernels is given in Lemmas 5.2.2–5.2.7 below. In all these lemmas, we assume that hypotheses (i)–(iii) are satisfied.

Lemma 5.2.2. *Let $0 \longrightarrow F' \longrightarrow F \longrightarrow F''$ be an exact sequence and assume that F' and F'' are $\mathcal{T}$-pseudo-coherent. Then F is $\mathcal{T}$-pseudo-coherent.*

Proof. We shall show that for any morphism $\varphi : G \longrightarrow F$ with a $\mathcal{T}$-finite G, $\ker(\varphi)$ is $\mathcal{T}$-finite. It follows easily from $\ker(\varphi) = \ker(\ker(G \longrightarrow F'') \longrightarrow F')$. q.e.d.

Notation 5.2.3. Let Z and Z' be a pair of locally closed subsets such that $Z \cap Z'$ is closed in Z and open in Z'. The morphism $k_Z \longrightarrow k_{Z'}$ defined as the composition $k_Z \longrightarrow k_{Z\cap Z'} \longrightarrow k_{Z'}$ will be denoted by $1_{Z\longrightarrow Z'}$. Note that $1_{Z'\longrightarrow Z''} \circ 1_{Z\longrightarrow Z'} = 1_{Z\longrightarrow Z''}$ does not hold in general.

Lemma 5.2.4. *Consider a morphism $\varphi : k_U \longrightarrow k_{X\setminus V}$ with $U, V \in \mathcal{T}$. Then there exists a finite open covering $U = \bigcup_i U_i$ with $U_i \in \mathcal{T}$ and a commutative diagram*

$$\begin{tikzcd} \bigoplus_i k_{U_i} \arrow[d] \arrow[dr, "\psi"] & \\ k_U \arrow[r, "\varphi"] & k_{X\setminus V} \end{tikzcd}$$

where the vertical arrow is given by $1_{U_i\longrightarrow U}$ and ψ is given by $a_i 1_{U_i\longrightarrow X\setminus V}$ for some $a_i \in k$.

Proof. Let $U \setminus V = \bigsqcup_{i\in I} Z_i$ be the decomposition into connected components. Then I is finite, and the Z_i's are connected and closed in U. Set $U_i = (U \cap V) \cup Z_i = U \setminus (\cap_{j\neq i} Z_j)$. Hence U_i is open, $U_i \cup \bigcup_{j\neq i} U_j = U$ and $U_i \cap (\bigcup_{j\neq i} U_j) = V$. Therefore $U_i \in \mathcal{T}$ for all i. Consider the composition $k_{U_i} \longrightarrow k_U \longrightarrow k_{X\setminus V}$. Since $U_i \setminus V = Z_i$, it factors through k_{Z_i}. Since Z_i is connected, it is given by $a_i 1_{U_i\longrightarrow X\setminus V}$ for some $a_i \in k$. q.e.d.

Lemma 5.2.5. *Let F be a subsheaf of k_U with $U \in \mathcal{T}$, and assume that F is $\mathcal{T}$-finite. Then there exists $V \in \mathcal{T}$, $V \subset U$, such that $F \simeq k_V$.*

Proof. By the hypothesis, there exists an epimorphism $\bigoplus_i k_{U_i} \twoheadrightarrow F$ with $U_i \in \mathcal{T}$ and we may assume each U_i connected. The composition $k_{U_i} \longrightarrow F \longrightarrow k_U$ is given by $a_i 1_{U_i \longrightarrow U}$ with $a_i \in k$. Let $V = \bigcup_{i, a_i \neq 0} U_i$. Then V belongs to $\mathcal{T}$ and $F \simeq k_V$. q.e.d.

Lemma 5.2.6. *Let V and W belong to $\mathcal{T}$. Then $k_{V \setminus W}$ is $\mathcal{T}$-coherent.*

Proof. Clearly $k_{V \setminus W}$ is $\mathcal{T}$-finite. Let $S \in \mathcal{K}(\mathcal{T})$ and $\varphi : S \longrightarrow k_{V \setminus W}$. Let us show that $\ker \varphi$ is $\mathcal{T}$-finite. By Lemma 5.2.4, there exist $S' = \bigoplus_{i \in I} k_{U_i}$ with a finite index set I and $U_i \in \mathcal{T}$, and an epimorphism $f : S' \twoheadrightarrow S$ such that $\psi = \varphi \circ f$ is given $a_i 1_{U_i \longrightarrow V \setminus W}$ for $a_i \in k$. As there is an epimorphism $\ker \psi \longrightarrow \ker \varphi$, in order to see that $\ker \varphi$ is $\mathcal{T}$-finite, it is enough to show that $\ker \psi$ is $\mathcal{T}$-finite. Set $J_0 = \{i \in I\,;\, a_i = 0\}$ and $J_1 = J \setminus J_0$. We define the morphism

$$g : \bigoplus_{i \in J_0} k_{U_i} \oplus \bigoplus_{i \in J_1} k_{U_i \cap W} \oplus \bigoplus_{i \neq j} k_{U_i \cap U_j} \longrightarrow S'$$

as follows: $g|_{k_{U_i}}$ is the natural morphism $k_{U_i} \longrightarrow S'$ for $i \in J_0$, $g|_{k_{U_i \cap W}}$ is given by $1_{U_i \cap W \longrightarrow U_i}$ for $i \in J_1$, and $g|_{k_{U_i \cap U_j}}$ is the composition of $k_{U_i \cap U_j} \longrightarrow k_{U_i} \oplus k_{U_j}$ given by $(a_j 1_{U_i \cap U_j \longrightarrow U_i}, -a_i 1_{U_i \cap U_j \longrightarrow U_j})$ and $k_{U_i} \oplus k_{U_j} \longrightarrow S'$.

Clearly the image of g coincides with $\ker \psi$. Therefore, $\ker \psi$ is $\mathcal{T}$-finite. q.e.d.

Lemma 5.2.7. *Let F be the quotient of a $\mathcal{T}$-coherent sheaf by a $\mathcal{T}$-coherent subsheaf. Then F is $\mathcal{T}$-coherent.*

Proof. It is enough to show that F is $\mathcal{T}$-pseudo-coherent. Let us consider an exact sequence $0 \longrightarrow N \longrightarrow L \longrightarrow F \longrightarrow 0$ with $\mathcal{T}$-coherent N and L. There exists an epimorphism $L_0 = \oplus_{i=1}^n k_{U_i} \longrightarrow L$ with $U_i \in \mathcal{T}$. Since the kernel N_0 of $L_0 \longrightarrow F$ is the image of $\ker(N \oplus L_0 \longrightarrow L)$, N_0 is $\mathcal{T}$-finite and hence $\mathcal{T}$-coherent. Hence we may assume $L = \oplus_{i=1}^n k_{U_i}$ with $U_i \in \mathcal{T}$ from the beginning. We shall argue by induction on n.

Case $n = 1$. N is of the form $k_{U_1 \setminus V}$ for some $V \in \mathcal{T}$ by Lemma 5.2.5 and hence F is $\mathcal{T}$-coherent by Lemma 5.2.6.

Case $n > 1$. Set $L_1 = k_{U_1}$ and $L_2 = \oplus_{i=2}^n k_{U_i}$ respectively. Let F_1 be the image of $L_1 \longrightarrow F$. Then we have a commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G_1 & \longrightarrow & G & \longrightarrow & G_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L_1 & \longrightarrow & L & \longrightarrow & L_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_1 & \longrightarrow & F & \longrightarrow & F_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Since G_1 is the kernel of $L_1 \oplus G \longrightarrow L$, G_1 is $\mathcal{T}$-coherent. The sheaf G_2 is $\mathcal{T}$-finite because it is a quotient of G, and it is $\mathcal{T}$-pseudo-coherent because it is a submodule of L_2. Hence G_2 is also $\mathcal{T}$-coherent. Therefore F_1 and F_2 are $\mathcal{T}$-pseudo-coherent by the induction hypothesis, and Lemma 5.2.2 implies that F is $\mathcal{T}$-pseudo-coherent. q.e.d.

This completes the proof of Theorem 5.2.1.

5.3 Construction II

We keep the notations of §5.1

We set for short:

$$\mathrm{I}(\mathrm{coh}(\mathcal{T})) := \mathrm{Ind}(\mathrm{coh}(\mathcal{T})).$$

Theorem 5.3.1. *Let F be a presheaf on $\mathcal{T}$. Assume:*

(i) *hypotheses* (i)–(iii) *in* (5.2.1) *are satisfied,*

(ii) $F(\varnothing) = 0$,

(iii) *for any U and V in $\mathcal{T}$, the sequence $0 \longrightarrow F(U \cup V) \longrightarrow F(U) \oplus F(V) \longrightarrow F(U \cap V)$ is exact.*

Then there exists a unique $F_{\mathcal{T}}^{+} \in \mathrm{I}(coh(\mathcal{T}))$ such that

$$\mathrm{Hom}_{\mathrm{I}(coh(\mathcal{T}))}(k_U, F_{\mathcal{T}}^{+}) \simeq F(U) \quad \textit{functorially in } U \in \mathcal{T}. \tag{5.3.1}$$

Proof. Let us introduce

$$\mathcal{T}_0 = \{U \in \mathcal{T}; U \text{ is connected}\}. \tag{5.3.2}$$

Since the natural functor $\mathcal{K}(\mathcal{T}_0) \longrightarrow \mathrm{Mod}(k_X)$ is fully faithful, we shall identify $\mathcal{K}(\mathcal{T}_0)$ with a full subcategory of $\mathrm{Mod}(k_X)$.

Note that the functor $\mathcal{K}(\mathcal{T}) \longrightarrow \mathcal{K}(\mathcal{T}_0)$ is faithful, and $\mathrm{coh}(\mathcal{T}) = \mathrm{coh}(\mathcal{T}_0)$.

Let $F' \longrightarrow F \longrightarrow F''$ be a complex in $\mathcal{K}(\mathcal{T}_0)$. We shall say that this complex is an exact sequence if it is an exact sequence in $\mathrm{Mod}(k_X)$.

Let $\mathcal{C}$ be an abelian category and let $G : \mathcal{K}(\mathcal{T}_0) \longrightarrow \mathcal{C}$ be an additive contravariant functor. We shall say that G is left exact if for any exact sequence $F' \longrightarrow F \longrightarrow F'' \longrightarrow 0$ with F', F, F'' in $\mathcal{K}(\mathcal{T}_0)$, the sequence $0 \longrightarrow G(F'') \longrightarrow G(F) \longrightarrow G(F')$ is exact in $\mathcal{C}$.

Let F be a presheaf on $\mathcal{T}$ such that $F(\varnothing) = 0$. We define the functor $F_{\mathcal{T}_0} : (\mathcal{K}(\mathcal{T}_0))^{\mathrm{op}} \longrightarrow \mathrm{Mod}(k)$ by setting

$$F_{\mathcal{T}_0}(\oplus_i k_{U_i}) = \oplus_i F(U_i).$$

When there is no risk of confusion, we write F instead of $F_{\mathcal{T}_0}$.

Denote by $\mathrm{coh}(\mathcal{T})^{\vee,add,l}$ the category of additive left exact contravariant functors on $\mathrm{coh}(\mathcal{T})$ and by $\mathcal{K}(\mathcal{T}_0)^{\vee,add,l}$ the category of additive contravariant functors on $\mathcal{K}(\mathcal{T}_0)$ which are left exact with respect to exact sequences in $\mathrm{Mod}(k_X)$. By Theorem 1.3.4, the forgetful functor $coh(\mathcal{T})^{\vee,add,l} \longrightarrow \mathcal{K}(\mathcal{T}_0)^{\vee,add,l}$ admits a quasi-inverse. Moreover, since $\mathcal{K}(\mathcal{T}_0)$ is $\mathcal{U}$-small, we have the equivalence $\mathrm{coh}(\mathcal{T})^{\vee,add,l} \simeq \mathrm{I}(\mathrm{coh}(\mathcal{T}))$.

Hence, we get the equivalences of categories

$$\mathrm{I}(\mathrm{coh}(\mathcal{T})) \xrightarrow{\sim} \mathrm{coh}(\mathcal{T})^{\vee,add,l} \xrightarrow{\sim} \mathcal{K}(\mathcal{T}_0)^{\vee,add,l}. \tag{5.3.3}$$

Hence, in order to prove Theorem 5.3.1, it is enough to check that the functor $F_{\mathcal{T}_0}$ is left exact. This will be done in Lemmas 5.3.2–5.3.6 below. q.e.d.

Lemma 5.3.2. *Let $\{U_j; 1 \leq j \leq n\}$ be a finite family of open sets in $\mathcal{T}$. Then the sequence below is exact:*

$$0 \to F(\bigcup_{1\leq k\leq n} U_k) \to \bigoplus_{1\leq k\leq n} F(U_k) \to \bigoplus_{1\leq i<j\leq n} F(U_{ij}).$$

Proof. For $n \leq 1$ this is trivial, and for $n = 2$ this is the hypothesis. For $n > 2$ assume the result is proved for $j \leq n-1$. Setting $U' = \cup_{1\leq k<n} U_k$, the commutative diagram below is exact by the induction hypothesis.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & F(U' \cup U_n) \arrow[r] \arrow[d] & F(U') \oplus F(U_n) \arrow[r] \arrow[d] & F(U' \cap U_n) \arrow[d] \\
0 \arrow[r] & \oplus_{i<n} F(U_i \cup U_n) \arrow[r] \arrow[d] & \oplus_{i<n}(F(U_i) \oplus F(U_n)) \arrow[r] \arrow[d] & \oplus_{i<n} F(U_{in}) \arrow[d] \\
0 \arrow[r] & \oplus_{i<j<n} F(U_{ij} \cup U_n) \arrow[r] & \oplus_{i<j<n}(F(U_{ij}) \oplus F(U_n)) \arrow[r] & \oplus_{i<j<n} F(U_{ijn})
\end{tikzcd}$$

Here, the second column is the direct sum of the two exact sequences $0 \longrightarrow F(U') \longrightarrow \oplus_{i<n} F(U_i) \longrightarrow \oplus_{i<j<n} F(U_{ij})$ and $0 \longrightarrow F(U_n) \longrightarrow \oplus_{i<n} F(U_n) \longrightarrow \oplus_{i<j<n} F(U_n)$. The assertion easily follows from this. q.e.d.

Lemma 5.3.3. *Consider an epimorphism $\varphi : S \twoheadrightarrow k_U$ with $S \in \mathcal{K}(\mathcal{T}_0)$ and $U \in \mathcal{T}_0$. Then the sequence $0 \longrightarrow F(U) \longrightarrow F(S)$ is exact.*

Proof. We may assume $S = \bigoplus_{i\in I} k_{U_i}$ with the U_i's in $\mathcal{T}_0$. Let $J = \{j \in I; \varphi|_{U_j} : k_{U_j} \longrightarrow k_U \text{ is non zero}\}$, and let $S' = \bigoplus_{j\in J} k_{U_j}$. We may assume that $\varphi|_{k_{U_j}}$ is given by $1_{U_j \longrightarrow U}$ for $j \in J$. Since $U = \bigcup_{j\in J} U_j$, the preceding lemma implies that $F(k_U) \longrightarrow F(S')$ is a monomorphism. q.e.d.

Lemma 5.3.4. *Consider an epimorphism $\varphi : S_1 \twoheadrightarrow S_0$ with S_0, $S_1 \in \mathcal{K}(\mathcal{T}_0)$. Then the sequence $0 \longrightarrow F(S_0) \longrightarrow F(S_1)$ is exact.*

Proof. Let $S_0 = \oplus_{i\in I} k_{U_i}$. For each i choose a morphism $K_i \twoheadrightarrow S_1 \times_{S_0} k_{U_i}$ with $K_i \in \mathcal{K}(\mathcal{T}_0)$ such that the composition $K_i \longrightarrow S_1 \times_{S_0} k_{U_i} \longrightarrow k_{U_i}$ is an epimorphism. Then the result follows from the diagram

$$\begin{tikzcd} F(S_0) \arrow[r] \arrow[d, "\sim"'] & F(S_1) \arrow[d] \\ \oplus_i F(k_{U_i}) \arrow[r, tail] & \oplus_i F(K_i)\,. \end{tikzcd}$$

q.e.d.

Lemma 5.3.5. *Consider an exact sequence $S_1 \longrightarrow S_0 \underset{\varphi}{\longrightarrow} k_U \longrightarrow 0$ with $S_0, S_1 \in \mathcal{K}(\mathcal{T}_0)$ and $U \in \mathcal{T}_0$. Then the sequence $0 \longrightarrow F(U) \longrightarrow F(S_0) \longrightarrow F(S_1)$ is exact.*

Proof. Let $S_0 = \oplus_{i\in I} k_{U_i}$ with the U_i's in $\mathcal{T}_0$.

(a) First, we assume that the morphisms $\varphi|_{k_{U_i}}$ are non-zero. Hence, $U_i \subset U$ and $U = \cup_i U_i$. We may assume that these morphisms are the morphisms $1_{U_i \longrightarrow U}$. Set $S_1' = \bigoplus_{i,j\in I} k_{U_i \cap U_j}$. Then $S_1' \times_{S_0} S_1 \longrightarrow S_1'$ is an epimorphism. Choose $G \in \mathcal{K}(\mathcal{T}_0)$ together with epimorphisms $G \twoheadrightarrow S_1' \times_{S_0} S_1$. Consider the exact commutative diagram

$$\begin{tikzcd} & S_1' \arrow[r] & S_0 \arrow[r] \arrow[dd, equal] & k_U \arrow[r] \arrow[dd, equal] & 0 \\ G \arrow[ur] \arrow[dr] & & & & \\ & S_1 \arrow[r] & S_0 \arrow[r] & k_U \arrow[r] & 0. \end{tikzcd}$$

Applying the functor F we find the diagram:

$$\begin{tikzcd} 0 \arrow[r] & F(k_U) \arrow[r] \arrow[dd, equal] & F(S_0) \arrow[r] \arrow[dd, equal] & F(S_1') \arrow[dr, tail] & \\ & & & & F(G) \\ 0 \arrow[r] & F(k_U) \arrow[r] & F(S_0) \arrow[r] & F(S_1) \arrow[ur] & \end{tikzcd}$$

By Lemma 5.3.2, the row in the top is exact. By Lemma 5.3.4, $F(S_1') \to F(G)$ is a monomorphism. Hence, the row in the bottom is exact.

(b) We treat the general case. Let $I^{(2)} = \{i \in I; k_{U_i} \to k_U \text{ is zero}\}$, $I^{(1)} = I \setminus I^{(2)}$. Define $S_0^a = \bigoplus_{i\in I^{(a)}} k_{U_i}$, $(a = 1, 2)$ and define S_1^a as $S^1 \times_{S_0} S_0^a$. Then S_1^a is $\mathcal{T}_0$-coherent and we have exact sequences

$$\begin{array}{ccccccc} S_1^1 & \to & S_0^1 & \to & k_U & \to & 0 \\ S_1^2 & \to & S_0^2 & \to & 0. & & \end{array}$$

Choose epimorphisms $S_2^a \to S_1^a$ with $S_2^a \in \mathcal{K}(\mathcal{T}_0)$. We get the exact sequences

$$\begin{array}{ccccccc} S_2^1 & \to & S_0^1 & \to & k_U & \to & 0 \\ S_2^2 & \to & S_0^2 & \to & 0. & & \end{array}$$

Applying F to these sequences, we get the exact sequences

$$\begin{aligned} 0 \to F(k_U) \to F(S_0^1) \to F(S_2^1) \\ 0 \to F(S_0^2) \to F(S_2^2). \end{aligned}$$

Therefore, the commutative diagram below is exact, and the right vertical arrow is a monomorphism:

$$\begin{tikzcd} 0 \arrow[r] & F(k_U) \arrow[r] \arrow[d, equal] & F(S_0^1) \oplus F(S_0^2) \arrow[r] \arrow[d, equal] & F(S_2^1) \oplus F(S_2^2) \\ 0 \arrow[r] & F(k_U) \arrow[r] & F(S_0) \arrow[r] & F(S_1). \arrow[u] \end{tikzcd}$$

q.e.d.

Lemma 5.3.6. *The contravariant functor $F : \mathcal{K}(\mathcal{T}_0) \to \mathrm{Mod}(k)$ is left exact. In other words, if a sequence $S' \to S \to S'' \to 0$ in $\mathcal{K}(\mathcal{T})$ is exact, then the sequence $0 \to F(S'') \to F(S) \to F(S')$ is exact.*

Proof. We write $S'' = \oplus_{i=1}^n k_{U_i}$ with $U_i \in \mathcal{T}$. If $n = 1$, this is Lemma 5.3.5. We argue by induction on n and assume the result is proved for $n - 1$. Set $S_1'' = k_{U_1}$ and $S_2'' = \oplus_{i=2}^n k_{U_i}$, and consider the exact commutative diagram:

$$\begin{tikzcd} S_1' \arrow[r, dotted] \arrow[d, dotted] & S_1 \arrow[r, dotted] \arrow[d, dotted] & S_1'' \arrow[r] \arrow[d] & 0 \\ S' \arrow[r] & S \arrow[r] \arrow[d] & S'' \arrow[r] \arrow[d] & 0 \\ & S_2'' \arrow[r, equal] \arrow[d] & S_2'' \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & \end{tikzcd}$$

In order to complete the dotted arrows and to obtain a commutative diagram in $\mathcal{K}(\mathcal{T}_0)$, one first chooses an epimorphism $S_1 \twoheadrightarrow S \times_{S''} S''_1$ with $S_1 \in \mathcal{K}(\mathcal{T}_0)$, then one chooses an epimorphism $S'_1 \twoheadrightarrow S' \times_S \ker(S_1 \longrightarrow S''_1)$ with $S'_1 \in \mathcal{K}(\mathcal{T}_0)$. Applying F to the exact commutative diagram above, we find the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & F(S''_2) \arrow[r, equal] \arrow[d] & F(S''_2) \arrow[d] & \\
0 \arrow[r] & F(S'') \arrow[r] \arrow[d] & F(S) \arrow[r] \arrow[d] & F(S') \arrow[d] \\
0 \arrow[r] & F(S''_1) \arrow[r] \arrow[d] & F(S_1) \arrow[r] & F(S'_1) \\
 & 0 & &
\end{tikzcd}$$

Since the left and middle columns as well as the bottom row are exact, the second row is exact. q.e.d.

5.4 Construction III

The ind-sheaf F^+ associated with F in Proposition 5.1.1 is difficult to calculate in general. We shall give conditions in order that its restriction to $\mathcal{T}$ is F.

From now on, we assume that the open sets in $\mathcal{T}$ are relatively compact. We thus have a fully faithful exact functor $\mathrm{coh}(\mathcal{T}) \longrightarrow \mathrm{Mod}^c(k_X)$ which induces a fully faithful exact functor

$$\iota : \mathrm{I}(\mathrm{coh}(\mathcal{T})) \longrightarrow \mathrm{I}(k_X).$$

Let us denote by $for_{\mathcal{T}} : \mathcal{K}(\mathcal{T})^{\vee,add,l} \longrightarrow \mathrm{PSH}(k_{\mathcal{T}})$ the forgetful functor defined by

$$for_{\mathcal{T}}(F)(U) = \mathrm{Hom}_{\mathrm{I}(\mathrm{coh}(\mathcal{T}))}(k_U, F).$$

We thus have $for_{\mathcal{T}} \simeq for \circ \iota$, that is, the diagram below commutes:

$$\begin{tikzcd}
\mathrm{I}(coh(\mathcal{T})) \arrow[r, "\iota"] \arrow[d, "for_{\mathcal{T}}"'] & \mathrm{I}(k_X) \arrow[dl, "for"] \\
\mathrm{PSH}(k_{\mathcal{T}}) &
\end{tikzcd}$$

Now, recall the functor $^+ : \mathrm{PSH}(k_{\mathcal{T}}) \longrightarrow \mathrm{I}(k_X)$ constructed in Proposition 5.1.1.

Theorem 5.4.1. *Let F be a presheaf on $\mathcal{T}$. Assume:*

(i) *hypotheses* (5.2.1) *are all satisfied,*

(ii) $F(\varnothing) = 0$,

(iii) *for any U and V in $\mathcal{T}$, the sequence $0 \to F(U \cup V) \to F(U) \oplus F(V) \to F(U \cap V)$ is exact.*

Then the ind-sheaf F^+ constructed in Proposition 5.1.1 *coincides with $\iota(F^+_{\mathcal{T}})$, where $F^+_{\mathcal{T}}$ is constructed in* Theorem 5.3.1. *In particular we have*

$$\operatorname{Hom}_{\mathrm{I}(k_X)}(k_U, F^+) \simeq F(U) \quad \textit{for any } U \in \mathcal{T}. \tag{5.4.1}$$

Proof. By the definition of the functor $^+$, the assertion means that for any $G \in \mathrm{I}(k_X)$ one has the isomorphism

$$\operatorname{Hom}_{\mathrm{PSH}(k_{\mathcal{T}})}(for_{\mathcal{T}}(F^+_{\mathcal{T}}), for(G)) \simeq \operatorname{Hom}_{\mathrm{I}(k_X)}(\iota(F^+_{\mathcal{T}}), G).$$

This follows from the lemma below, by setting $H = F^+_{\mathcal{T}}$. q.e.d.

Lemma 5.4.2. *For $H \in \mathrm{I}(\mathrm{coh}(\mathcal{T}))$ and $G \in \mathrm{I}(k_X)$ one has the isomorphism*

$$\operatorname{Hom}_{\mathrm{PSH}(k_{\mathcal{T}})}(for_{\mathcal{T}}(H), for(G)) \simeq \operatorname{Hom}_{\mathrm{I}(k_X)}(\iota(H), G).$$

Proof. Recall that $\mathcal{T}_0 = \{U \in \mathcal{T}; U \text{ connected}\}$. By the same letter for_0, we denote the forgetful functors $\mathrm{I}(k_X) \to \mathcal{K}(\mathcal{T}_0)^{\vee,add,l}$ and also $\mathrm{I}(\mathrm{coh}(\mathcal{T})) \xrightarrow{\sim} \mathcal{K}(\mathcal{T}_0)^{\vee,add,l}$. Then we have

$$\operatorname{Hom}_{\mathrm{PSH}(k_{\mathcal{T}})}(for_{\mathcal{T}}(H), for(G)) \simeq \operatorname{Hom}_{\mathcal{K}(\mathcal{T}_0)^{\vee,add,l}}(for_0(H), for_0(G)) .$$

Now we shall show the isomorphism

$$\operatorname{Hom}_{\mathrm{I}(k_X)}(\iota(H), G) \xrightarrow{\sim} \operatorname{Hom}_{\mathrm{PSH}(k_{\mathcal{T}_0})}(for_0(H), for_0(G)) . \tag{5.4.2}$$

Since both sides send inductive limits with respect to H to projective limits, we may assume that $H \in \mathrm{coh}(\mathcal{T})$. Since both sides are left exact in H, we may assume $H = k_U$ with $U \in \mathcal{T}_0$. Since $\mathcal{K}(\mathcal{T}_0)^{\vee,add,l}$ is a full subcategory of $(\mathcal{K}(\mathcal{T}_0))^\vee$, (1.1.1) implies

$$\operatorname{Hom}_{\mathcal{K}(\mathcal{T}_0)^{\vee,add,l}}(for_0(H), for_0(G)) \simeq for_0(G)(k_U) \simeq \operatorname{Hom}_{\mathrm{I}(k_X)}(k_U, G).$$

Hence we obtain (5.4.2). q.e.d.

Proposition 5.4.3. *In the situation of* Theorem 5.4.1, *assume moreover that for any U, V in $\mathcal{T}$ with $U \subset V$, the sequence $F(V) \to F(U) \to 0$ is exact. Then $F^+_{\mathcal{T}}$ is quasi-injective in* $\mathrm{I}(\mathrm{coh}(\mathcal{T}))$.

Proof. If $G \in \operatorname{coh}(\mathcal{T})$, we shall write $F_{\mathcal{T}}^{+}(G)$ instead of $\operatorname{Hom}_{\mathrm{I}(\operatorname{coh}(\mathcal{T}))}(G, F_{\mathcal{T}}^{+})$ for short.

Let $G' \rightarrowtail G$ be a monomorphism in $\operatorname{coh}(\mathcal{T})$ and let us prove that $F_{\mathcal{T}}^{+}(G) \to F_{\mathcal{T}}^{+}(G')$ is an epimorphism. There exists an epimorphism $\bigoplus_{i=1}^{n} k_{U_i} \twoheadrightarrow G$, with $U_i \in \mathcal{T}$. Define $G'_j := \operatorname{im}\Big(G' \oplus (\bigoplus_{i=1}^{j} k_{U_i}) \to G\Big)$. It is enough to prove that the monomorphisms $G'_j \rightarrowtail G'_{j+1}$ give rise to the epimorphisms $F_{\mathcal{T}}^{+}(G'_{j+1}) \to F_{\mathcal{T}}^{+}(G'_j)$. Hence, we may assume from the beginning that there exists $U \in \mathcal{T}$ and a morphism $\varphi : k_U \to G$ such that $G' \oplus k_U \to G$ is an epimorphism. Consider the commutative exact diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & G' \arrow[r, equal] \arrow[d] & G' \arrow[d] & \\
0 \arrow[r] & K \arrow[r] \arrow[d, equal] & G' \oplus k_U \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K \arrow[r] & k_U \arrow[r] \arrow[d] & G'' \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Then G'' is $\mathcal{T}$-coherent, as well as K. Since K is a sub-object of k_U, it is isomorphic to k_V for some $V \in \mathcal{T}$. Applying the left exact functor $\operatorname{Hom}_{\mathrm{I}(\operatorname{coh}(\mathcal{T}))}(\,\cdot\,, F_{\mathcal{T}}^{+})$ to this diagram, the middle column will remain split exact, as well as the rows, in view of the hypothesis. Hence, the whole diagram will remain exact. q.e.d.

5.5 Ind-sheaves on real manifolds

We shall apply the preceding constructions and define various ind-sheaves associated with the spaces of distributions, or C^∞-functions. This is a reformulation in the language of ind-sheaves, of previous results of [7] and [9] that we shall briefly recall.

In this section, the base field k is $\mathbb{C}$ Denote by X a real analytic manifold. We do not recall here the notions of subanalytic subsets of X and $\mathbb{R}$-constructible sheaves on X, and refer to [8] for an exposition.

As usual, we denote by $\mathcal{C}_X^\infty$ (resp. $\mathcal{C}_X^\omega$) the sheaf of complex functions of class $\mathcal{C}^\infty$ (resp. real analytic), by $\mathcal{D}b_X$ (resp. $\mathcal{B}_X$) the sheaf of Schwartz's distributions (resp. Sato's hyperfunctions), and by $\mathcal{D}_X$ the sheaf of analytic finite order differential operators.

We denote by $\mathbb{R}\text{-}\mathrm{C}(\mathbb{C}_X)$ the abelian category of $\mathbb{R}$-constructible sheaves of $\mathbb{C}$ vector spaces on X, and by $\mathbb{R}\text{-}\mathrm{C}^c(\mathbb{C}_X)$ the full abelian subcategory of sheaves with compact supports. If one denotes by $\mathcal{T}$ the family of open subanalytic relatively compacts subsets of X, then hypotheses (5.2.1) are satisfied, and the category $coh(\mathcal{T})$ coincides with $\mathbb{R}\text{-C}^c(\mathbb{C}_X)$.

An important property of subanalytic subsets is given by the lemma below.

Lemma 5.5.1. *Let U and V be two open subanalytic subsets of X, and K a compact subset. Then there exist a positive integer N and $C > 0$ such that*

$$\operatorname{dist}\big(x, K \setminus (U \cup V)\big)^N \leq C\big(\operatorname{dist}(x, K \setminus U) + \operatorname{dist}(x, K \setminus V)\big) \quad \textit{for any } x \in K.$$

Here, $\operatorname{dist}(x, K \setminus U)$ is the distance from x to $K \setminus U$.

Let U be an open subset of X. One sets $\mathcal{C}^\infty(U) = \Gamma(U; \mathcal{C}^\infty_X)$.

Definition 5.5.2. Let $f \in \mathcal{C}^\infty(U)$. One says that f has *polynomial growth* at $p \in X$ if it satisfies the following condition. For a local coordinate system $(x_1, \ldots, x_n)$ around p, there exist a sufficiently small compact neighborhood K of p and a positive integer N such that

$$\sup_{x \in K \cap U} \big(\operatorname{dist}(x, K \setminus U)\big)^N |f(x)| < \infty \,. \tag{5.5.1}$$

It is obvious that f has polynomial growth at any point of U. We say that f is *tempered* at p if all its derivatives have polynomial growth at p. We say that f is tempered on an open set Ω if it is tempered at any point of Ω.

For U open subanalytic in X, denote by $\mathcal{C}^{\infty,t}_X(U)$ the subspace of $\mathcal{C}^\infty(U)$ consisting of tempered functions. Denote by $\mathcal{D}b^t_X(U)$ the space of tempered distributions on U. This space is defined by the exact sequence

$$0 \longrightarrow \Gamma_{X \setminus U}(X; \mathcal{D}b_X) \longrightarrow \Gamma(X; \mathcal{D}b_X) \longrightarrow \mathcal{D}b^t_X(U) \longrightarrow 0.$$

For S closed subanalytic in X, denote by $\mathcal{I}^\infty_{X,S}$ the subsheaf of ideals of $\mathcal{C}^\infty_X$ consisting of functions which vanish up to infinite order on S,

In [7], [9], one introduces the sheaves:

$$\begin{aligned}
T\mathcal{H}om\,(\mathbb{C}_U, \mathcal{C}^\infty) &:= V \mapsto \mathcal{C}^{\infty,t}_V(U \cap V), \\
T\mathcal{H}om\,(\mathbb{C}_U, \mathcal{D}b_X) &:= V \mapsto \mathcal{D}b^t_V(U \cap V), \\
\mathbb{C}_U \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_X &:= V \mapsto \Gamma(V; \mathcal{I}^\infty_{V, V \setminus U}).
\end{aligned}$$

As a consequence of the theorems of Lojaciewicz (see also Malgrange [12]) one gets the following

Lemma 5.5.3. *Let U and V be two open subanalytic subsets of X. Set for short $W(U) = \Gamma(X; \mathbb{C}_U \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_X)$. Then the sequences below are exact:*

$$0 \to \mathcal{C}^{\infty,t}(U \cup V) \to \mathcal{C}^{\infty,t}(U) \oplus \mathcal{C}^{\infty,t}(V) \to \mathcal{C}^{\infty,t}(U \cap V),$$
$$0 \to \mathcal{D}b^t_X(U \cup V) \to \mathcal{D}b^t_X(U) \oplus \mathcal{D}b^t_X(V) \to \mathcal{D}b^t_X(U \cap V) \to 0,$$
$$0 \to W(U \cap V) \to W(U) \oplus W(V) \to W(U \cup V) \to 0.$$

Consider the functors

$$\begin{array}{rcl} \mathcal{C}^{\infty,t}_X(\,\cdot\,) : \mathcal{T}^{\mathrm{op}} & \to & \mathrm{Mod}(\mathbb{C}) \\ U & \mapsto & \mathcal{C}^{\infty,t}(U) \\ \mathcal{D}b^t_X(\,\cdot\,) : \mathcal{T}^{\mathrm{op}} & \to & \mathrm{Mod}(\mathbb{C}) \\ U & \mapsto & \mathcal{D}b^t_X(U). \end{array}$$

Applying Theorem 5.4.1, these functors extend as left exact functors on the category $\mathbb{R}\text{-}\mathrm{C}^c(\mathbb{C}_X)^{\mathrm{op}}$. Moreover, the functor $\mathcal{D}b^t_X(\,\cdot\,)$ is exact on this category. Set

$$\mathrm{I}(\mathbb{R}\text{-}\mathrm{C}^c(\mathbb{C}_X)) = \mathrm{Ind}(\mathbb{R}\text{-}\mathrm{C}^c(\mathbb{C}_X)).$$

Then $\mathcal{C}^{\infty,t}_X, \mathcal{D}b^t_X \in \mathrm{I}(\mathbb{R}\text{-}\mathrm{C}^c(\mathbb{C}_X)) \subset \mathrm{I}(\mathbb{C}_X)$.

With slight modifications in Theorem 5.4.1, one also gets that the functor

$$\begin{array}{rcl} \cdot \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_X : \mathcal{T} & \to & \mathrm{Mod}(\mathcal{D}_X) \\ U & \mapsto & \mathbb{C}_U \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_X \end{array}$$

extends as an exact functor on the category $\mathbb{R}\text{-}\mathrm{C}^c(\mathbb{C}_X)$. We may thus define the left exact functor:

$$\begin{array}{rcl} \mathcal{C}^{\infty,\mathrm{w}}_X(\,\cdot\,) : \mathbb{R}\text{-}\mathrm{C}^c(\mathbb{C}_X)^{\mathrm{op}} & \to & \mathrm{Mod}(\mathbb{C}) \\ \mathcal{C}^{\infty,\mathrm{w}}_X(F) & = & \Gamma(X; H^0(D'F) \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_X), \end{array}$$

where we set

$$D'F = R\mathcal{H}om\,(F, \mathbb{C}_X),$$

and hence $H^0(D'F) = \mathcal{H}om\,(F, \mathbb{C}_X)$. Therefore, $\mathcal{C}^{\infty,\mathrm{w}}_X$ is an ind-sheaf.

Recall that $\mathrm{Mod}(\mathcal{D}_X, \mathrm{I}(\mathbb{C}_X)) \simeq \mathrm{I}(\beta_X(\mathcal{D}_X))$. In the sequel, we shall write $\beta\mathcal{D}_X$ instead of $\beta_X(\mathcal{D}_X)$ for short.

By Proposition 5.1.1, we get the ind-sheaves:

$$\mathcal{C}^{\infty,t}_X, \mathcal{D}b^t_X, \mathcal{C}^{\infty,\mathrm{w}}_X \in \mathrm{I}(\beta\mathcal{D}_X).$$

These ind-sheaves belong to $\mathrm{I}(\mathbb{R}\text{-}\mathrm{C}^{c}(\mathbb{C}_X))$. They satisfy for $F \in \mathbb{R}\text{-}\mathrm{C}^{c}(\mathbb{C}_X)$

$$\begin{aligned}\mathrm{Hom}_{\mathrm{I}(\beta\mathcal{D}_X)}(F, \mathcal{C}_X^{\infty,t}) &\simeq \Gamma(X; T\mathcal{H}om(F, \mathcal{C}_X^{\infty}))\\ \mathrm{Hom}_{\mathrm{I}(\beta\mathcal{D}_X)}(F, \mathcal{D}b_X^{t}) &\simeq \Gamma(X; T\mathcal{H}om(F, \mathcal{D}b_X))\\ \mathrm{Hom}_{\mathrm{I}(\beta\mathcal{D}_X)}(F, \mathcal{C}_X^{\infty,\mathrm{w}}) &\simeq \Gamma(X; H^0(D'F) \overset{\mathrm{w}}{\otimes} \mathcal{C}_X^{\infty}).\end{aligned}$$

Replacing the "Whitney tensor product" $\cdot \overset{\mathrm{w}}{\otimes} \mathcal{C}_X^{\infty}$ by the classical one, we get the left exact functor (defined on the whole category $\mathrm{Mod}^c(\mathbb{C}_X)$):

$$\mathcal{C}_X^{\infty,\omega}(F) = \Gamma(X; H^0(D'F) \otimes \mathcal{C}_X^{\infty}).$$

Hence we get the ind-sheaf, $\mathcal{C}_X^{\infty,\omega}$ which is nothing but the ind-sheaf $\beta_X(\mathcal{C}_X^{\infty})$ and

$$\mathcal{C}_X^{\infty,\omega} \in \mathrm{I}(\beta\mathcal{D}_X).$$

Lemma 5.5.4. *In* (i) *and* (ii) *below, we assume* $F \in D^b(\mathbb{R}\text{-}\mathrm{C}(\mathbb{C}_X))$. *Then*

(i) $R\mathcal{H}om(F, \mathcal{D}b_X^t) \simeq T\mathcal{H}om(F, \mathcal{D}b_X)$,

(ii) $R\mathcal{H}om(F, \mathcal{C}_X^{\infty,\mathrm{w}}) \simeq (D'F) \overset{\mathrm{w}}{\otimes} \mathcal{C}_X^{\infty}$,

(iii) $R\mathcal{H}om(F, \mathcal{C}_X^{\infty,\omega}) \simeq (D'F) \otimes \mathcal{C}_X^{\infty}$.

Proof. (i) follows from the fact that $T\mathcal{H}om(\,\cdot\,, \mathcal{D}b_X)$ is exact on the category $\mathbb{R}\text{-}\mathrm{C}^{c}(\mathbb{C}_X)$.

(ii) Using the triangulation theorem for subanalytic sets, we may represent F by a bounded complex of sheaves, each one being a finite direct sum of sheaves of type $\mathbb{C}_U$, with $U \in \mathcal{T}$ and such that $D'\mathbb{C}_U \simeq \mathbb{C}_{\overline{U}}$ (see [7]).

(iii) follows from Proposition 4.1.9 (v). q.e.d.

5.6 Ind-sheaves associated to holomorphic functions

Let X be a complex manifold with structure sheaf $\mathcal{O}_X$. Denote by $\overline{X}$ the complex conjugate manifold and by $X^{\mathbb{R}}$ the underlying real analytic manifold, identified with the diagonal of $X \times \overline{X}$.

Let $F \in D^b(\mathbb{R}\text{-}\mathrm{C}(\mathbb{C}_X))$. Recall that in [7] and [9], one introduces the objects:

$$\begin{aligned}T\mathcal{H}om(F, \mathcal{O}_X) &= R\mathcal{H}om_{\mathcal{D}_{\overline{X}}}(\mathcal{O}_{\overline{X}}, T\mathcal{H}om(F, \mathcal{D}b_X))\\ F \overset{\mathrm{w}}{\otimes} \mathcal{O}_X &= R\mathcal{H}om_{\mathcal{D}_{\overline{X}}}(\mathcal{O}_{\overline{X}}, F \overset{\mathrm{w}}{\otimes} \mathcal{C}_X^{\infty}).\end{aligned}$$

Moreover, one proves the canonical isomorphism

$$R\mathcal{H}om_{\mathcal{D}_{\overline{X}}}(\mathcal{O}_{\overline{X}}, T\mathcal{H}om(F, \mathcal{C}^\infty_X)) \xrightarrow{\sim} T\mathcal{H}om(F, \mathcal{O}_X). \tag{5.6.1}$$

For $\lambda = t, \mathrm{w}, \omega$, one defines the objects $\mathcal{O}^\lambda_X \in D^b(\mathrm{I}(\beta\mathcal{D}_X))$ by the formulas:

$$\begin{aligned}
\mathcal{O}^t_X &:= R\mathcal{H}om_{\beta\mathcal{D}_{\overline{X}}}(\beta\mathcal{O}_{\overline{X}}, \mathcal{D}b^t_{X^\mathbb{R}}) \\
&\simeq R\mathcal{H}om_{\beta\mathcal{D}_{\overline{X}}}(\beta\mathcal{O}_{\overline{X}}, \mathcal{C}^{\infty,t}_X), \\
\mathcal{O}^\mathrm{w}_X &:= R\mathcal{H}om_{\beta\mathcal{D}_{\overline{X}}}(\beta\mathcal{O}_{\overline{X}}, \mathcal{C}^{\infty,\mathrm{w}}_X), \\
\mathcal{O}^\omega_X &:= R\mathcal{H}om_{\beta\mathcal{D}_{\overline{X}}}(\beta\mathcal{O}_{\overline{X}}, \mathcal{C}^{\infty,\omega}_X).
\end{aligned}$$

Of course, $\mathcal{O}^\omega_X \simeq \beta_X\mathcal{O}_X$. Moreover the first isomorphism follows from (5.6.1).

Proposition 5.6.1. *Let $F \in D^b(\mathbb{R}\text{-}\mathrm{C}(\mathbb{C}_X))$. Then*

$$\begin{aligned}
R\mathcal{H}om(F, \mathcal{O}^t_X) &\simeq T\mathcal{H}om(F, \mathcal{O}_X), \\
R\mathcal{H}om(F, \mathcal{O}^\mathrm{w}_X) &\simeq (D'F) \overset{\mathrm{w}}{\otimes} \mathcal{O}_X, \\
R\mathcal{H}om(F, \mathcal{O}^\omega_X) &\simeq (D'F) \otimes \mathcal{O}_X.
\end{aligned}$$

Proof. This follows immediately from Lemma 5.5.4. q.e.d.

Notation 5.6.2. Let $\mathcal{L}$ be a locally finite free $\mathcal{O}_X$-module, and let $\lambda = t, \mathrm{w}, \omega$. We set

$$\mathcal{L}^\lambda = (\beta_X\mathcal{L}) \underset{\beta_X\mathcal{O}_X}{\otimes} \mathcal{O}^\lambda_X.$$

As an application, let us give in the framework of ind-sheaves the adjunction formula for integral transforms of [9]. (For the case of sheaves and $\mathcal{D}$-modules, without growth conditions, refer to [1].)

We shall follow the notations of [9] with an exception. If $f : X \longrightarrow Y$ is a morphism of complex manifolds, we denote by Df^{-1} and $Df_!$ the inverse and proper direct images in the sense of $\mathcal{D}$-modules.

We denote by $D^b_{q-good}(\mathcal{D}_X)$ the full triangulated subcategory of $D^b(\mathcal{D}_X)$ consisting of objects with quasi-good cohomologies and by $D^b_{r-h}(\mathcal{D}_X)$ the full triangulated subcategory of $D^b(\mathcal{D}_X)$ consisting of objects with regular holonomic cohomologies.

Consider three complex manifolds X, S, Y of dimension d_X, d_S, d_Y, respectively, and a correspondence:

$$\begin{tikzcd} & S \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd}$$

Let $\mathcal{M} \in D^b_{q-good}(\mathcal{D}_X)$ and $\mathcal{L} \in D^b_{r-h}(\mathcal{D}_S)$. We set $L = R\mathcal{H}om_{\mathcal{D}_S}(\mathcal{L}, \mathcal{O}_S)$. Of course, L has constructible cohomologies.

We make the hypothesis:

(5.6.2) $$f^{-1}\operatorname{supp}(\mathcal{M}) \cap \operatorname{supp}(\mathcal{L}) \text{ is proper over } Y.$$

Let $G \in D^b(\mathrm{I}(C_Y))$. We define

$$\begin{aligned} \mathcal{M} \circ \mathcal{L} &= Dg_!(Df^{-1}\mathcal{M} \overset{L}{\underset{\mathcal{O}_S}{\otimes}} \mathcal{L}), \\ L \circ G &= Rf_{!!}(L \otimes g^{-1}G). \end{aligned}$$

Theorem 5.6.3. *One has the isomorphism*

$$\mathrm{RHom}_{\mathrm{I}(\mathbb{C}_X)}(L \circ G, \Omega_X^t[d_X] \overset{L}{\underset{\beta_X \mathcal{D}_X}{\otimes}} \beta_X(\mathcal{M}))) \simeq \mathrm{RHom}_{\mathrm{I}(\mathbb{C}_Y)}(G, \Omega_Y^t[d_S] \overset{L}{\underset{\beta_Y \mathcal{D}_Y}{\otimes}} \beta_Y(\mathcal{M} \circ \mathcal{L})).$$

The proof is based on the results of [9] and will not be given here.

Bibliography

[1] A. D'Agnolo and P. Schapira, *Leray's quantization of projective duality,* Duke Math. Journ. 84, 453-496 (1996)

[2] P. Deligne, *Cohomologie à supports propres,* in S-G-A 4,

[3] S.I. Gelfand and Yu.I. Manin, *Methods of homological algebra,* Springer (1996)

[4] A. Grothendieck, *Sur quelques points d'algèbre homologique* Tohoku Math. Journ. 119-183 (1957)

[5] A. Grothendieck, *Elements de géométrie algébrique III,* Publ. IHES 11 (1961), 17 (1963)

[6] A. Grothendieck and J-L. Verdier, *Préfaisceaux* in S-G-A 4

[7] M. Kashiwara, *The Riemann-Hilbert problem for holonomic systems,* Publ. RIMS, Kyoto Univ. 20, 319-365 (1984)

[8] M. Kashiwara and P. Schapira, *Sheaves on manifolds,* Grundlehren der Math. Wiss. 292, Springer (1990)

[9] M. Kashiwara and P. Schapira, *Moderate and formal cohomology associated with constructible sheaves,* Mémoires Soc. Math. France 64 (1996)

[10] M. Kashiwara and P. Schapira, *An introduction to ind-objects and derived categories,* to appear (2000)

[11] S. MacLane, *Categories for the working mathematician,* Graduate Texts in Math. 5, Springer (1971)

[12] B. Malgrange, *Ideals of differentiable functions,* Tata Institute, Oxford Univ. Press (1967)

[13] S-G-A 4, Sém. Géom. Algébrique (1963-64) by M. Artin, A. Grothendieck and J-L. Verdier,*Théorie des topos et cohomologie étale des schémas,* Lecture Notes in Math. 269, 270, 305 (1972/73)

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