BLOCH'S CONJECTURE, DELIGNE COHOMOLOGY AND HIGHER CHOW GROUPS

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Let X be a smooth proper complex algebraic variety such that $p_g(X) \neq 0$, where $p_g(X) = \dim \Gamma(X, \Omega_X^2)$. D. Mumford [8] showed that the kernel of the Albanese map

$$\mathrm{CH}_0(X)^0 \to \mathrm{Alb}(X)$$

is 'huge', where $\mathrm{CH}_0(X)^0$ is the subgroup of the Chow group of X consisting of zero cycles of degree zero, and $\mathrm{Alb}(X)$ is the Albanese variety. Then S. Bloch [2] obtained conversely the following:

0.1. Conjecture. The Albanese map is injective if $p_g(X) = 0$.

This conjecture was proved in [4] if X is not of general type, but the general case still remains open. It was show in [11] that (0.1) is closely related to the following:

0.2. Conjecture. The cycle map $\mathrm{CH}^2(U,1)_{\mathbb{Q}} \to H^3_{\mathcal{D}}(U, \mathbb{Q}(2))$ is surjective for any open subvarieties U of X if $p_g(X) = 0$.

Here $\mathrm{CH}^2(U,1)_{\mathbb{Q}}$ is Bloch's higher Chow group [3] tensored with $\mathbb{Q}$, and $H^3_{\mathcal{D}}(U, \mathbb{Q}(2))$ is Deligne cohomology. The cycle map is constructed in [12]. This conjecture is further related to

0.3. Conjecture. $\mathrm{ind}\lim_U H^3_{\mathcal{D}}(U, \mathbb{Q}(2)) = 0$ if $p_g(X) = 0$.

Here U runs over nonempty open subvarieties of X. In this paper we prove

0.4. Theorem. *Conjectures (0.1), (0.2) and (0.3) are all equivalent.*

Originally, this was considered in order to prove Bloch's conjecture (0.1) after [11] and [12] were written. It is expected that (0.4) would clarify the Hodge theoretic aspect of Bloch's conjecture. Using the theory of mixed Hodge Modules, the equivalence of (0.2) and (0.3) is reduced to

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0.5. Proposition. *For a complex variety Y of pure dimension m, the cycle map induces an isomorphism*

$$cl : \mathrm{CH}^1(Y,1) \to \mathrm{Ext}^{1-2m}(\mathbb{Q}_Y^H, \mathbb{D}_Y^H(1-m)).$$

See (1.1.3) for $\mathbb{Q}_Z^H, \mathbb{D}_Z^H$, and (2.5) for the proof. (A sketch of the proof can be found in [11].) To show the equivalence of (0.1) and (0.3), we calculate $H^3_{\mathcal{D}}(U, \mathbb{Q}(2))$ using the weight spectral sequence [6]. See (3.1).

In Sect. 1, we review some elementary facts from the theories of mixed Hodge Modules and Deligne cohomology which are needed in this paper. In Sect. 2, we recall the definition of Bloch's higher Chow groups and the cycle map, and prove (0.5) together with the equivalence of (0.2) and (0.3). In Sect. 3, we calculate the mixed Hodge structure of a smooth open surface, and prove the equivalence of (0.1) and (0.3).

In this paper a variety means a separated scheme of finite type over $\mathbb{C}$.

1. Mixed Hodge Module and Deligne cohomology

1.1. *Mixed Hodge Modules* (see [10]). For a variety X we denote by $\mathrm{MHM}(X)$ the abelian category of mixed $\mathbb{Q}$-Hodge Modules on X, and $D^b\mathrm{MHM}(X)$ its derived category consisting of bounded complexes of mixed $\mathbb{Q}$-Hodge Modules. There is a natural functor $\mathrm{rat} : D^b\mathrm{MHM}(X) \to D^b_c(X,\mathbb{Q})$ assigning the underlying $\mathbb{Q}$-complexes where $D^b_c(X,\mathbb{Q})$ denotes the full subcategory of $D^b_c(X^{\mathrm{an}},\mathbb{Q})$ consisting of $\mathbb{Q}$-complexes whose cohomology sheaves are algebraically constructible. We denote by $H^i : D^b\mathrm{MHM}(X) \to \mathrm{MHM}(X)$ the usual cohomology functor.

For morphisms f of algebraic varieties we have canonically defined functors $f_*, f_!, f^*, f^!$ between the derived categories of mixed $\mathbb{Q}$-Hodge Modules. They are compatible with the corresponding functors of $\mathbb{Q}$-complexes via the functor rat. For a closed embedding $i : X \to Y$, the direct image i_* will be omitted sometimes in order to simplify the notation, because

$$i_* : D^b\mathrm{MHM}(X) \to D^b\mathrm{MHM}(Y) \tag{1.1.1}$$

is an equivalence of categories.

If $X = \mathrm{Spec}\,\mathbb{C}$ we have naturally an equivalence of categories

$$\mathrm{MHM}(\mathrm{Spec}\,\mathbb{C}) = \mathrm{MHS}. \tag{1.1.2}$$

Here the right-hand side is the category of graded-polarizable mixed $\mathbb{Q}$-Hodge structures [6]. So $\mathrm{MHM}(\mathrm{Spec}\,\mathbb{C})$ will be identified with MHS.

We denote by $\mathbb{Q}(j)$ the mixed Hodge structure of type $(-j,-j)$ whose underlying $\mathbb{Q}$-vector space is $(2\pi i)^j\mathbb{Q} \subset \mathbb{C}$. See [6]. For a variety X with structure morphism $a_X : X \to \mathrm{Spec}\,\mathbb{C}$, we define

$$\mathbb{Q}_X^H(j) = a_X^*\mathbb{Q}(j), \quad \mathbb{D}_X^H(j) = a_X^!\mathbb{Q}(j), \tag{1.1.3}$$

so that $\mathbb{D}_X^H(j)$ is the dual of $\mathbb{Q}_X^H(-j)$. We will write $\mathbb{Q}_X^H$ for $\mathbb{Q}_X^H(0)$, and similarly for $\mathbb{D}_X^H$. If X is smooth of pure dimension n, we have a canonical isomorphism

$$\mathbb{D}_X^H = \mathbb{Q}_X^H(n)[2n]. \tag{1.1.4}$$

1.2. *Deligne cohomology* (see [7], [1], etc.) Let X be a smooth variety, and $\overline{X}$ a smooth compactification of X such that $D := \overline{X}\backslash X$ is a divisor with normal crossings. Let $j : X \to \overline{X}$ denote the inclusion morphism. Then $\mathbb{Q}$-Deligne cohomology $H^i_{\mathcal{D}}(X, \mathbb{Q}(k))$ is defined to be the i-th cohomology group of

$$C^{\bullet}_{\overline{X}\langle D\rangle}(k) := C(\mathbf{R}j_*\mathbb{Q}_X \oplus \sigma_{\geq k}\Omega^{\bullet}_{\overline{X}}(\log D) \to \mathbf{R}j_*\Omega^{\bullet}_X)[-1]. \tag{1.2.1}$$

We have a short exact sequence

$$\begin{aligned} 0 \to H^{i-1}(X,\mathbb{C})/(H^{i-1}(X,\mathbb{Q}) + F^kH^{i-1}(X,\mathbb{C})) \\ \to H^i_{\mathcal{D}}(X,\mathbb{Q}(k)) \to H^i(X,\mathbb{Q}) \cap F^kH^i(X,\mathbb{C}) \to 0 \end{aligned} \tag{1.2.2}$$

using the distinguished triangle

$$\to C^{\bullet}_{\overline{X}\langle D\rangle}(j) \to \mathbf{R}j_*\mathbb{Q}_X \to C(\sigma_{\geq k}\Omega^{\bullet}_{\overline{X}}(\log D) \to \mathbf{R}j_*\Omega^{\bullet}_X) \to \tag{1.2.3}$$

together with Hodge theory [6].

If X satisfies the condition

$$H^{i-1}(X,\mathbb{Q}) \text{ and } H^i(X,\mathbb{Q}) \text{ have weights } \leq 2k, \tag{1.2.4}$$

then (1.2.2) becomes

$$0 \to \mathrm{Ext}^1(\mathbb{Q}, H^{i-1}(X,\mathbb{Q})(k)) \to H^i_{\mathcal{D}}(X,\mathbb{Q}(k)) \to \mathrm{Hom}(\mathbb{Q}, H^i(X,\mathbb{Q})(k)) \to 0 \tag{1.2.5}$$

by [5], where Ext^1 and Hom are taken in the abelian category of mixed $\mathbb{Q}$-Hodge structures (not necessarily graded-polarizable). These can be taken in MHS, the category of graded-polarizable mixed $\mathbb{Q}$-Hodge structures, if X satisfies further the condition

$$\mathrm{Gr}^W_{2k}H^{i-1}(X,\mathbb{Q}) \text{ is isomorphic to a direct sum of } \mathbb{Q}(-k). \tag{1.2.6}$$

So we have a canonical morphism

$$H^i_{\mathcal{D}}(X,\mathbb{Q}(k)) = \mathrm{Ext}^i(\mathbb{Q}, (a_X)_*\mathbb{Q}_X^H(k)), \tag{1.2.7}$$

if (1.2.4) and (1.2.6) are satisfied. See [1].

We define in general

$$\begin{aligned} H^i_{\mathcal{D}}(X,\mathbb{Q}(k))' &= \mathrm{Ext}^i(\mathbb{Q}, (a_X)_*\mathbb{Q}_X^H(k)), \\ H_i^{\mathcal{D}}(X,\mathbb{Q}(k))' &= \mathrm{Ext}^{-i}(\mathbb{Q}, (a_X)_*\mathbb{D}_X^H(-k)). \end{aligned} \tag{1.2.8}$$

If X is smooth of pure dimension n, then (1.1.4) implies

$$H_i^{\mathcal{D}}(X,\mathbb{Q}(k))' = H^{2n-i}_{\mathcal{D}}(X,\mathbb{Q}(n-k)). \tag{1.2.9}$$

1.3. Proposition. *Let X be a reduced variety of pure dimension n, and X_i be the irreducible components of X. Let $\mathrm{Rat}(X)^* = \prod \mathrm{Rat}(X_i)^*$ with $\mathrm{Rat}(X_i)$ the rational function field of X_i. Then we have a canonical isomorphism*

$$H^{\mathcal{D}}_{2n-1}(X, \mathbb{Q}(n-1)) = \{g \in \mathrm{Rat}(X)^* : \mathrm{div}\, g = 0\}, \tag{1.3.1}$$

where $\mathrm{div}\, g = \sum \mathrm{div}\, g_i$ if $g = (g_i)$ with $g_i \in \mathrm{Rat}(X_i)^*$.

Proof. If X is smooth, the left-hand side of (1.3.1) is $\mathrm{Ext}^1(\mathbb{Q}^H_X, \mathbb{Q}^H_X(1))$, and the assertion is related with the theory of 1-motives [6], and is more or less well-known. Indeed, if X is a point, the assertion is verified by calculation the period of the mixed Hodge structure on $H^1(\mathbb{A}^1\backslash\{0\}, \{1\} \cup \{x\})$ for $x \in \mathbb{C}\backslash\{0,1\}$, i.e., by using the integral of dt/t on the relative cycle connecting $\{1\}$ and $\{x\}$, where t is the coordinate of $\mathbb{A}^1$.

In the general case, a sketch of proof is given in [11, 3.12]. Let $Z = \mathrm{Sing}\, X$, and $U = X\backslash Z$ with the inclusion morphisms $i : Z \to X, j : U \to X$. Then we have an exact sequence

$$0 \to \mathrm{Ext}^{1-2n}(\mathbb{Q}^H_X, \mathbb{D}^H_X(1-n)) \to \mathrm{Ext}^1(\mathbb{Q}^H_U, \mathbb{Q}^H_U(1)) \xrightarrow{\partial} \mathrm{Ext}^{2-2n}(\mathbb{Q}^H_Z, \mathbb{D}^H_Z(1-n)) \tag{1.3.2}$$

induced by the distinguished triangle

$$\to i_*\mathbb{D}^H_Z \to \mathbb{D}^H_X \to j_*\mathbb{Q}^H_U \to$$

together with the adjoint relations

$$\mathrm{Ext}^1(\mathbb{Q}^H_X, j_*\mathbb{Q}^H_U(1)) = \mathrm{Ext}^1(\mathbb{Q}^H_U, \mathbb{Q}^H_U(1)), \text{etc}\ .$$

Note that the first morphism of (1.3.2) is injective, because

$$\mathrm{Ext}^{1-2n}(\mathbb{Q}^H_Z, \mathbb{D}^H_Z(1-n)) = 0$$

by $H^k\mathbb{Q}^H_Z = H^{-k}\mathbb{D}^H_Z = 0$ for $k > \dim Z$.

Let Z_i be the irreducible components of Z with dimension $n-1$. Then we have a canonical isomorphism

$$\mathrm{Ext}^{2-2n}(\mathbb{Q}^H_Z, \mathbb{D}^H_Z(1-n)) = \oplus_i \mathrm{Ext}^{2-2n}(\mathbb{Q}^H_{Z_i}, \mathbb{D}^H_{Z_i}(1-n)) = \oplus_i \mathbb{Q}.$$

See [10]. We have to show that the morphism ∂ in (1.3.2) is naturally identified with

$$\mathrm{div} : \Gamma(U, \mathcal{O}^*_U) \to \oplus_i \mathbb{Q} \tag{1.3.3}$$

(possibly up to a sign depending only on n) using that isomorphism. Here it is not necessary to assume that $Z = \mathrm{Sing}\, X$, and we may assume that Z is a divisor of X containing $\mathrm{Sing}\, X$. Then, using the normalization of X together with a smooth variety transversal to Z, the assertion is reduced to the case X is a smooth curve and $n = 1$.

Since ∂ is additive and the assertion is etale-local and is clear in the case $g \in \Gamma(X, \mathcal{O}_X^*)$, it is further reduced to the case $X = \mathbb{A}^1, Z = \{0\}$, and g is the identity on $\mathbb{A}^1$. In this case we can identify ∂ with the canonical morphism

$$\mathrm{Ext}^1(\mathbb{Q}_U^H, \mathbb{Q}_U^H(1)) \to \mathrm{Hom}(\mathbb{Q}, H^1(U, \mathbb{Q})(1)),$$

using $H^1(U, \mathbb{Q}(1)) = H^2_{\{0\}}(X, \mathbb{Q}(1))$. Let M be the extension of $\mathbb{Q}_U^H$ by $\mathbb{Q}_U^H(1)$ corresponding to g. Then the monodromy of the underlying local system of M is $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. So the assertion is clear.

2. Higher Chow groups and cycle maps

2.1. *Higher Chow groups* ([3]). Let $\Delta^n = \mathrm{Spec}(\mathbb{C}[t_0, \dots, t_n]/(\sum t_i - 1))$. For a subset I of $\{0, \dots, n\}$, let $\Delta^n_I = \{t_i = 0\,(i \in I)\} \subset \Delta^n$. It is naturally isomorphic to Δ^m with $m = n - |I|$ (fixing the order of the coordinates), and is called a face of Δ^n. For $0 \le i \le n$, we have inclusions

$$\iota_i : \Delta^{n-1} \to \Delta^n,$$

such that its image is $\Delta^n_{\{i\}}$.

Let X be an equidimensional variety. Then $X \times \Delta^n_I$ is also called a face of $X \times \Delta^n$. Following Bloch, we define $z^p(X, n)$ to be the free abelian group with generators the irreducible closed subvarieties of $X \times \Delta^n$ of codimension p, intersecting all the faces of $X \times \Delta^n$ properly. We have face maps

$$\partial_i : z^p(X, n) \to z^p(X, n-1),$$

induced by ι_i. Let $\partial = \sum (-1)^i \partial_i$. Then $\partial^2 = 0$, and $\mathrm{CH}^p(X, n)$ is defined to be $\mathrm{Ker}\,\partial/\mathrm{Im}\,\partial$ which is a subquotient of $z^p(X, n)$. By [3] it is isomorphic to

$$\frac{\bigcap_{0 \le i \le n} \mathrm{Ker}(\partial_i : z^p(X, n) \to z^p(X, n-1))}{\partial_{n+1}(\bigcap_{0 \le i \le n} \mathrm{Ker}(\partial_i : z^p(X, n+1) \to z^p(X, n)))} \tag{2.1.1}$$

2.2. *Functoriality.* Let $f : X \to Y$ be a proper morphism of varieties, and put $r = \dim X - \dim Y$. Then we have the pushforward functor

$$f_* : \mathrm{CH}^p(X, n) \to \mathrm{CH}^{p-r}(Y, n).$$

In fact, for a face map $\iota : \Delta^m \to \Delta^n$, Bloch showed the commutative diagram

$$\begin{array}{ccc} z^p(X, n) & \xrightarrow{\iota^*} & z^p(X, m) \\ {\scriptstyle f_*}\downarrow & & {\scriptstyle f_*}\downarrow \\ z^{p-r}(X, n) & \xrightarrow{\iota^*} & z^{p-r}(X, m) \end{array}$$

As for the pull-back, we have $f^*:\mathrm{CH}^p(Y,n)\to\mathrm{CH}^p(X,n)$ if f is flat or X,Y are quasi-projective and Y is smooth. See loc. cit. In the latter case, we have a quasi-isomorphic subcomplex $z^p_f(Y,n)$ of $z^p(Y,n)$ on which the pull-back f^* is naturally defined.

2.3. *Cycle map.* Let X be an equidimensional variety. By [12] we have a cycle map

$$cl:\mathrm{CH}^p(X,n)\to H^{\mathcal{D}}_{2d+n}(X,\mathbb{Q}(d)),\tag{2.3.1}$$

where $d=\dim X-p$, and the target becomes $H^{2p-n}_{\mathcal{D}}(X,\mathbb{Q}(p))$ by if X is smooth.

Indeed, let $S^{n-1}=\bigcup\Delta^n_{\{i\}}\subset\Delta^n, U=\Delta^n\backslash S^{n-1}$ with the inclusion morphisms $i:S^{n-1}\to\Delta^n, j:U\to\Delta^n$. Then

$$(a_{\Delta^n})_*j_!\mathbb{Q}^H_U=\mathbb{Q}^H_{pt}[-n],\tag{2.3.2}$$

where $a_{\Delta^n}:\Delta^n\to pt:=\mathrm{Spec}\,\mathbb{C}$ is the structure morphism. Let $\zeta=\sum_k n_k[Z_k]\in\bigcap_{0\le i\le n}\mathrm{Ker}\,\partial_i\subset z^p(X,n)$ (see (2.1)), where Z_k are irreducible closed subvarieties of $X\times\Delta^n$. Let $d'=\dim Z_k=d+n$. Put $Z=\bigcup_k Z_k$. Then the coefficients n_k of Z_k induces a morphism

$$\mathbb{Q}^H_Z\to\oplus_k\mathrm{IC}_{Z_k}\mathbb{Q}^H[-d']\to\mathbb{D}^H_Z(-d-n)[-2d']\to\mathbb{D}^H_{X\times\Delta^n}(-d')[-2d'],\tag{2.3.3}$$

where $\mathrm{IC}_{Z_k}\mathbb{Q}^H$ denotes the mixed Hodge Module whose underlying perverse sheaf is the intersection complex $\mathrm{IC}_{Z_k}\mathbb{Q}$. Let $\pi:X\times\Delta^n\to X$ be the first projection, and let j denote also $\mathrm{id}\times j:X\times U\to X\times\Delta^n$ (and the same for i). Then

$$\pi_*j_!\mathbb{D}^H_{X\times U}=\mathbb{D}^H_X(n)[n]$$

by (1.1.4) and (2.3.2). So it is enough to show that (2.3.3) is uniquely lifted to

$$\mathbb{Q}^H_Z\to j_!\mathbb{D}^H_{X\times U}(-d')[-2d'],$$

i.e., the composition of (2.3.3) with

$$\mathbb{D}^H_{X\times\Delta^n}(-d')[-2d']\to i_*i^*\mathbb{D}^H_{X\times\Delta^n}(-d')[-2d']$$

is zero and $\mathrm{Hom}(\mathbb{Q}^H_Z,i_*i^*\mathbb{D}^H_{X\times\Delta^n}(-d')[-2d'-1])=0$. But they follow from the condition on proper intersection together with $\zeta\in\bigcap_{0\le i\le n}\mathrm{Ker}\,\partial_i$. For the well-definedness of the cycle map, it is enough to show its invariance under a deformation of cycle parametrized by $\mathbb{P}^1$. See [12] for detail.

Remark. If X is smooth proper and $n=0$, then the cycle map (2.3.1) induces Griffiths' Abel-Jacobi map (tensored with $\mathbb{Q}$)

$$\mathrm{CH}^p(X)^0\to\mathrm{Ext}^1(\mathbb{Q},H^{2p-1}(X,\mathbb{Q})(p)).\tag{2.3.4}$$

See [10, (4.5.20)]. Here Ext^1 is taken in MHS, and the target of (2.3.4) is isomorphic to Griffiths' intermediate Jacobian tensored with $\mathbb{Q}$ by [5]. If $p = \dim X$, (2.3.4) is the Albanese map (tensored with $\mathbb{Q}$).

2.4. *Compatibility.* The cycle map (2.3.1) is compatible with f_* for a proper morphism f and also with f^* for a morphism $f : X \to Y$ such that X, Y are quasi-projective and Y is smooth. Indeed, this is reduced to the case of the usual Chow groups by (2.2) and the construction of the cycle map (2.3), and follows from LNM 1479, pp. 196–205.

2.5. *Proof of* (0.5). By (1.3), the cycle map becomes

$$cl : \mathrm{CH}^1(Y,1) \to \{g \in \mathrm{Rat}(Y)^* : \mathrm{div}\, g = 0\}. \tag{2.5.1}$$

Then we have a right inverse

$$\iota : \{g \in \mathrm{Rat}(Y)^* : \mathrm{div}\, g = 0\} \to \mathrm{CH}^1(Y,1), \tag{2.5.2}$$

using the graph ζ of g together with an isomorphism $\mathbb{P}^1\backslash\{1\} \to \Delta^1$ sending $\{0,\infty\}$ to S^1. The condition $\mathrm{div}\, g = 0$ means that ζ belongs to $\mathrm{Ker}\,\partial$ in the notation of (2.1), and we get an element of $\bigcap \mathrm{Ker}\,\partial_i$ by adding vertical cycles (i.e. of the form $Z \times \Delta^1$ for a divisor Z of X) to ζ. We have $cl \circ \iota = \mathrm{id}$ by restricting to the smooth part of Y, because the isomorphism in (1.3) is obtained by calculating the period of mixed Hodge structure.

So it remains to show that (2.5.1) is injective, and we may assume that Y smooth by restricting to the smooth part of Y and using the exact sequence in [3, (3.1)]. Then $\mathrm{CH}^1(Y,1) = \Gamma(Y, \mathcal{O}_Y^*)$ by (6.1) in loc. cit, and the assertion follows.

2.6. Corollary of (0.5). *Let X be a variety of pure dimension n, and Y a closed subvariety of X of pure dimension m. Put $U = X\backslash Y$. Let $\xi \in H^{\mathcal{D}}_{2m-1}(X, \mathbb{Q}(m-1))'$ such that its restriction to U is zero. Then there exists $\zeta \in \mathrm{CH}^{n-m+1}(X,1)_{\mathbb{Q}}$ such that $cl(\zeta) = \xi$.*

Proof. We have an exact sequence

$$H^{\mathcal{D}}_{2m-1}(Y, \mathbb{Q}(m-1))' \to H^{\mathcal{D}}_{2m-1}(X, \mathbb{Q}(m-1))' \to H^{\mathcal{D}}_{2m-1}(U, \mathbb{Q}(m-1))'$$

induced by the distinguished triangle

$$\to \mathbb{D}^H_Y \to \mathbb{D}^H_X \to j_*\mathbb{D}^H_U \to$$

(together with the adjoint relation for the direct image and the pull-back), where $j : U \to X$ denotes the inclusion morphism. So the assertion follows from (0.5) together with the compatibility of the cycle map with the direct image (2.4).

2.7. *Proof of the equivalence of* (0.2) *and* (0.3). The implication (0.2) $\Rightarrow$ (0.3) is clear, because $\dim |\zeta| = 1$ for $\zeta \in \mathrm{CH}^2(U,1)$. The converse follows from (2.6).

3. Proof of the equivalence of (0.1) and (0.3)

3.1. *Weight spectral sequence* [6]. Let X be a proper smooth variety of pure dimension 2, and D a divisor with normal crossings on X such that the irreducible components $D_i\,(1 \le i \le r)$ are smooth. Let $D^{(1)}$ be the disjoint union of D_i, and $D^{(2)} = \bigcup_{i<j} D_{i,j}$ with $D_{i,j} = D_i \cap D_j$. Put $U = X\backslash D, D^{(0)} = X$, and $D^{(k)} = \emptyset$ for $k < 0$ or $k > 3$. By [6], we have the weight spectral sequence

$$E_1^{-k,j+k} = H^{j-k}(D^{(k)}, \mathbb{Q}(-k)) \Rightarrow H^j(U, \mathbb{Q}) \tag{3.1.1}$$

such that $d_1 : H^{j-k}(D^{(k)}, \mathbb{Q}(-k)) \to H^{j-k+2}(D^{(k-1)}, \mathbb{Q}(1-k))$ is the Cech-Gysin morphism. In particular, we have

$$\begin{aligned} \mathrm{Gr}_2^W H^2(U, \mathbb{Q}) &= \mathrm{Coker}(\oplus_i H^0(D_i, \mathbb{Q})(-1) \to H^2(X, \mathbb{Q})) \\ \mathrm{Gr}_3^W H^2(U, \mathbb{Q}) &= \mathrm{Ker}(\oplus_i H^1(D_i, \mathbb{Q})(-1) \to H^3(X, \mathbb{Q})) \\ \mathrm{Gr}_4^W H^2(U, \mathbb{Q}) &= \mathrm{Ker}(H^0(D^{(2)}, \mathbb{Q})(-2) \to H^2(D^{(1)}, \mathbb{Q})(-1)). \end{aligned} \tag{3.1.2}$$

If U is affine, then $H^3(Y, \mathbb{Q}) = 0$, and (1.2.5) implies

$$H^3_{\mathcal{D}}(U, \mathbb{Q}(2)) = \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}, H^2(U, \mathbb{Q})(2)). \tag{3.1.3}$$

If furthermore $p_g(X) = 0$ and U is sufficiently small so that

$$\mathrm{Gr}_2^W H^2(U, \mathbb{Q}) = 0 \tag{3.1.4}$$

by Lefschetz, then the last term of (3.1.3) is isomorphic to

$$\mathrm{Coker}(\mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Q}, \mathrm{Gr}_4^W H^2(U, \mathbb{Q})(2)) \to \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}, \mathrm{Gr}_3^W H^2(U, \mathbb{Q})(2))) \tag{3.1.5}$$

by the long exact sequence associated with

$$0 \to \mathrm{Gr}_3^W H^2(U, \mathbb{Q}) \to H^2(U, \mathbb{Q}) \to \mathrm{Gr}_4^W H^2(U, \mathbb{Q}) \to 0. \tag{3.1.6}$$

The extension class associated with (3.1.6) will be denoted by

$$e \in \mathrm{Ext}^1_{\mathrm{MHS}}(\mathrm{Gr}_4^W H^2(U, \mathbb{Q}), \mathrm{Gr}_3^W H^2(U, \mathbb{Q})). \tag{3.1.7}$$

Let p_i be the composition of

$$\mathrm{Gr}_3^W H^2(U, \mathbb{Q}) \to \oplus_i H^1(D_i, \mathbb{Q})(-1) \to H^1(D_i, \mathbb{Q})(-1) \tag{3.1.8}$$

where the first morphism is the inclusion induced by (3.1.2) and the last is the projection.

Let $D_{i,j} = \bigcup_m P_{i,j,m}$ with $P_{i,j.m} \in X$. Then

$$u \in \mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Q}, (\mathrm{Gr}_4^W H^2(U, \mathbb{Q}))(2))$$

is identified with $u(1) = (a_{i,j,m})$ by (3.1.2), where

$$a_{i,j,m} \in \mathbb{Q} = H^0(P_{i,j,m}, \mathbb{Q}). \tag{3.1.9}$$

3.2. Lemma. *With the above notation,*

$$p_i \circ e \circ u \in \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}, H^1(D_i, \mathbb{Q})(1))$$

coincides with the image of the zero cycle with rational coefficients

$$\sum_{j<i} \sum_m a_{j,i,m} P_{j,i,m} - \sum_{j>i} \sum_m a_{i,j,m} P_{i,j,m} \tag{3.2.1}$$

by the Abel-Jacobi map for each i.

Proof. Let $D' = \bigcup_{j \neq i} D_j, U' = X \backslash D'$ and $Y = D_i$ with the inclusion morphisms $j' : U' \to X, j : U \to X, \overline{j} : Y \backslash D' \to Y$. Then we have an exact sequence of mixed Hodge Modules

$$0 \to j'_* \mathbb{Q}^H_{U'}[2] \to j_* \mathbb{Q}^H_U[2] \to \overline{j}_* \mathbb{Q}^H_{Y \backslash D'}(-1)[1] \to 0, \tag{3.2.2}$$

where the direct image by $Y \to X$ is omitted to simplify the notation. The last morphism of (3.2.2) induces a morphism of mixed Hodge structures

$$H^j(U, \mathbb{Q}) \to H^{j-1}(Y \backslash D', \mathbb{Q})(-1) \tag{3.2.3}$$

together with that of weight spectral sequences. The induced morphisms of $E_1^{-1,j+1}$ $(j = 1, 2, 3)$ and $E_1^{-2,4}$ are natural morphisms

$$H^{j-1}(D^{(1)}, \mathbb{Q})(-1) \to H^{j-1}(D_i, \mathbb{Q})(-1),$$
$$H^0(D^{(2)}, \mathbb{Q})(-2) \to H^0(D_i \cap D', \mathbb{Q})(-2),$$

up to a sign coming from the Cech-Gysin morphism. So the assertion follows.

3.3. *Proof of* (0.3) $\Rightarrow$ (0.1). By Roitman's result [9], it is enough to show the injectivity of the Albanese map tensored with $\mathbb{Q}$:

$$\mathrm{CH}^2(X)^0_{\mathbb{Q}} \to \mathrm{Alb}(X)_{\mathbb{Q}}. \tag{3.3.1}$$

Since these groups are birational invariant of smooth proper varieties, we may assume that X is smooth projective by Chow's lemma.

Let $\zeta = \sum_i m_i [P_i] \in \mathrm{CH}^2(X)^0$ belonging to the kernel of (3.3.1). Let Y be a smooth hypersurface of X passing through all the P_i. Let $U = X \backslash Y$. By (3.1.2) (applied to X, Y), ζ determines an element ξ of

$$\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}, (\mathrm{Gr}^W_3 H^2(U, \mathbb{Q}))(2)) \, (\subset \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}, H^1(Y, \mathbb{Q})(1)))$$

using the compatibility of the cycle map (or the Abel-Jacobi map) with the direct image. It can be lifted to

$$\tilde{\xi} \in \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}, H^2(U, \mathbb{Q})(2)),$$

by the right exactness of Ext^1. (Here $\mathrm{Gr}_4^W H^2(U,\mathbb{Q}) = 0$.)

Assuming (0.3), there is a sufficiently small affine open subvariety U' of U such that the image of $\tilde{\xi}$ in $\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}, H^2(U',\mathbb{Q})(2))$ vanishes. Since $p_g(X) = 0$, we may assume that (3.1.4) holds for U' (shrinking U' if necessary).

Let $\pi : X' \to X$ be a composition of blowing-ups with point centers such that it induces an isomorphism over U' and the pull-back D' of $X \backslash U'$ is a divisor with normal crossings whose irreducible components D'_i are smooth. Here D'_1 is the proper transform of Y (which is isomorphic to Y), and $D'_i (2 \le i \le r)$ are exceptional divisors (which are isomorphic to $\mathbb{P}^1$). Let

$$\xi' \in \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}, (\mathrm{Gr}_3^W H^2(U',\mathbb{Q}))(2))$$

denote the image of $\tilde{\xi}$. Since the image of ξ' in $\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}, H^2(U',\mathbb{Q})(2))$ vanishes, we have

$$\xi' = e \circ u \tag{3.3.2}$$

for some $u \in \mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Q}, (\mathrm{Gr}_4^W H^2(U,\mathbb{Q}))(2))$ by (3.1.5), where e is as in (3.1.7).

We have a morphism of mixed Hodge structures

$$H^j(U,\mathbb{Q}) \to H^j(U',\mathbb{Q}) \tag{3.3.3}$$

together with that of weight spectral sequences. The induced morphisms of $E_1^{-1,3}$ is the canonical inclusion

$$H^1(Y,\mathbb{Q})(-1) \to \oplus_i H^1(D'_i,\mathbb{Q})(-1).$$

So $p_1 \circ \xi'$ is identified with ξ which is the image of ζ in the Jacobian of Y tensored with $\mathbb{Q}$, and $p_i \circ \xi'$ vanishes for $i \ge 2$, where p_i are as in (3.1.8)

Then by (3.2) and (3.3.2), there exist rational functions g_i on D'_i together with a nonzero integer a such that

$$\sum_{j<i} \sum_m a_{j,i,m} P_{j,i,m} - \sum_{j>i} \sum_m a_{i,j,m} P_{i,j,m} = a^{-1} \mathrm{div}\, g_i \tag{3.3.4}$$

for each $i \ge 2$, where $a_{j,i,m}$ are associated with u as in (3.1.9). Adding (3.3.4) for $i \ge 2$, we get

$$\sum_{i>1} \sum_m a_{1,i,m} P_{1,i,m} = a^{-1} \sum_{i \ge 2} \mathrm{div}\, g_i.$$

Furthermore, the left-hand side is rationally equivalent to ζ on Y, because their images in the Jacobian of Y (tensored with $\mathbb{Q}$) are the same by (3.2) and $p_i \circ \xi' = \xi$. So the assertion follows.

3.4. *Proof of* (0.1) $\Rightarrow$ (0.3). The argument is essentially as above. Let X, D, U be as in (3.1), and assume that $p_g(X) = 0$ and U is affine and sufficiently small so that

$$H^3(U,\mathbb{Q}) = \mathrm{Gr}_2^W H^2(U,\mathbb{Q}) = 0$$

by (3.1.4). Let $\xi \in H^3_{\mathcal{D}}(U, \mathbb{Q}(2))$. This belongs to

$$\text{(3.4.1)} \qquad \mathrm{Coker}(\mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Q}, \mathrm{Gr}_4^W H^2(U, \mathbb{Q})(2)) \to \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}, \oplus_i H^1(D_i, \mathbb{Q})(1)))$$

by (3.1.5), because $\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}, \mathrm{Gr}_3^W H^2(U, \mathbb{Q})(2))$ is isomorphic to

$$\text{(3.4.2)} \qquad \mathrm{Ker}(\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}, \oplus_i H^1(D_i, \mathbb{Q})(1)) \to \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}, H^3(X, \mathbb{Q})(2)))$$

by (3.1.2) together with the semisimplicity of polarizable Hodge structures [6]. We have to show that ξ vanishes by restricting to a smaller open subvariety if (0.1) is true.

Let $\tilde{\xi}$ be a lift of ξ to $\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}, \oplus_i H^1(D_i, \mathbb{Q})(1))$. It is the direct sum of

$$\text{(3.4.3)} \qquad \xi_i \in \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}, H^1(D_i, \mathbb{Q})(1)).$$

Each ξ_i is the image by the Abel-Jacobi map of a zero cycle ζ_i of degree zero on D_i with rational coefficients. Here we may assume that $\mathrm{supp}\, \zeta_i$ is disjoint from D_j $(j \neq i)$.

By the compatibility of the cycle map with the direct image,

$$\zeta := \sum_i \zeta_i \in \mathrm{CH}^2(X)^0_{\mathbb{Q}}$$

belongs to the kernel of the Albanese map, because ξ belongs to (3.4.2). Assuming (0.1), ζ is then rationally equivalent to zero, and there exist rational functions g_i on D_i $(1 \leq i \leq r)$ and rational functions g_i on integral curves C_i $(r < i \leq s)$ together with a nonzero integer a such that $C_i \neq D_j$ for any i, j, and

$$a\zeta + \sum_j \mathrm{div}\, g_j = 0 \quad \text{in } z^2(X, 0).$$

Let $\pi : X' \to X$ be a composition of blowing-ups with point centers such that it induces an isomorphism over the complement U' of $C \cup D$ and $D' := \pi^{-1}(C \cup D)$ is a divisor with normal crossings whose irreducible components D'_i $(1 \leq i \leq t)$ are smooth. Here D'_i is the proper transform of D_i for $1 \leq i \leq r$, and of C_i for $r < i \leq s$, and is an exceptional divisor (isomorphic to $\mathbb{P}^1$) for $s < i \leq t$.

Let ζ'_i be the pull-back of ζ_i by the isomorphism $D'_i \to D_i$ so that $\pi_* \zeta'_i = \zeta_i$ for $1 \leq i \leq r$. Put $\zeta'_i = 0$ for $i > r$. Let g'_i be the pull-back of g_i to D'_i for $1 \leq i \leq s$. Let

$$\text{(3.4.4)} \qquad \eta'_i = \zeta'_i + a^{-1} \mathrm{div}\, g'_i$$

so that $\pi_* \eta'_i = \zeta_i + a^{-1} \mathrm{div}\, g_i$ for $1 \leq i \leq s$. Then $\sum_{1 \leq i \leq s} \pi_* \eta'_i = 0$, and there exist zero cycles η'_i of degree zero on D'_i for $i > s$ such that

$$\text{(3.4.5)} \qquad \sum_{1 \leq i \leq t} \eta'_i = 0 \quad \text{in } z^2(X', 0).$$

Indeed, $\eta'_i\,(i>s)$ is constructed by using the chain of exceptional divisors connecting the proper transforms of analytic-local irreducible components of D_i, C_j.

By (3.4.5), the supports of η'_i are contained in the singular points of D' (i.e. $D'^{(2)}$ with the notation of (3.1)). So there exist $a_{i,j,m}\in\mathbb{Q}$ for $i<j$ such that

$$\eta'_i=\sum_{j<i}\sum_m a_{j,i,m}P_{j,i,m}-\sum_{j>i}\sum_m a_{i,j,m}P_{i,j,m}\quad\text{for each } i, \tag{3.4.6}$$

with the notation of (3.1). Since $\deg\eta'_i=0$, the $a_{i,j,m}$ determine by (3.1.9)

$$u\in\mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Q},(\mathrm{Gr}_4^W H^2(U',\mathbb{Q}))(2)).$$

Furthermore, (3.2) asserts that

$$\xi'_i:=p_i\circ e\circ u\in\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q},H^1(D'_i,\mathbb{Q})(1)) \tag{3.4.7}$$

is identified with the image of η'_i by the Abel-Jacobi map for any i. So ξ'_i is the pull-back of ξ_i for $1\le i\le r$ by (3.4.4). On the other-hand, $\xi'_i=0$ for $i>r$, because $\zeta'_i=0$ for $r<i\le s$ and $D'_i=\mathbb{P}^1$ for $i>s$. This shows that the pull-back ξ' of ξ to $H^3_{\mathcal{D}}(U',\mathbb{Q}(2))$ is represented by (ξ'_i) in the expression similar to (3.4.1) (using the morphism of the weight spectral sequences as in (3.3)). Then ξ' is zero because (ξ'_i) is the image of u due to (3.4.7). This completes the proof of (0.4).

References

[1] A. Beilinson, Notes on absolute Hodge cohomology, Contemporary Math. 55 (1986) 35–68.

[2] S. Bloch, Lectures on algebraic cycles, Duke University Mathematical series 4, Durham, 1980.

[3] ———, Algebraic cycles and higher K-theory, Advances in Math., 61 (1986), 267–304.

[4] S. Bloch, A. Kas and D. Lieberman, Zero cycles on surfaces with $p_g=0$, Compos. Math. 33 (1976), 135–145.

[5] J. Carlson, Extensions of mixed Hodge structures, in Journées de Géométrie Algébrique d'Angers 1979, Sijthoff -Noordhoff Alphen a/d Rijn, 1980, pp. 107–128.

[6] P. Deligne, Théorie de Hodge I, Actes Congrès Intern. Math., 1970, vol. 1, 425–430 : II, Publ. Math. IHES, 40 (1971), 5–57; III ibid., 44 (1974), 5–77.

[7] F. El Zein and S. Zucker, Extendability of normal functions associated to algebraic cycles, in Topics in transcendental algebraic geometry, Ann. Math. Stud., 106, Princeton Univ. Press, Princeton, N.J., 1984, pp. 269–288.

[8] D. Mumford, Rational equivalence of 0-cycles on surfaces, J. Math. Kyoto Univ. 9 (1969), 195–204.

[9] A. Roitman, The torsion in the group of zero cycles modulo rational equivalence, Ann. Math. 111 (1980), 553–569.

[10] M. Saito, Mixed Hodge Modules, Publ. RIMS, Kyoto Univ., 26 (1990), 221–333.

[11] ———, On the injectivity of cycle maps, Pub. RIMS, Kyoto Univ., 28 (1992), 99–127.

[12] ———, On the formalism of mixed sheaves, RIMS-preprint 784, Aug. 1991.

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