DUALITY FOR THE $G_{r,s}$ QUANTUM GROUP[1]

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Abstract

The two parameter quantum group $G_{r,s}$ is generated by five elements, four of which form a Hopf subalgebra isomorphic to $GL_q(2)$, while the fifth generator relates $G_{r,s}$ to $GL_{p,q}(2)$. We construct explicitly the dual algebra of $G_{r,s}$ and show that it is isomorphic to the single parameter deformation of $gl(2) \oplus gl(1)$, with the second parameter appearing in the costructure. We also formulate a differential calculus on $G_{r,s}$ which provides a realisation of the calculus on $GL_{p,q}(2)$.

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I Introduction

The q-deformation of the universal enveloping algebra is commonly referred to in recent literature as quantum universal enveloping algebras or simply quantum algebras. There are some approaches [1-5] to quantum groups in which the objects may be called *quantum matrix groups* which are Hopf algebras in duality to the quantum algebras. In particular this means the existence of a doubly nondegenerate bilinear form between the two Hopf algebras. It was Sudbery [6] who first gave a formalism for such a duality motivated by the fact that, at the classical level, an element of the Lie algebra corresponding to a Lie group is a tangent vector at the identity of the Lie group. The q-analogue of tangent vector at the identity would then be obtained by first differentiating the elements of the given Hopf algebra H and then taking the counit operation. The elements thus obtained would belong to the dual Hopf algebra H*. Following this simple procedure, Sudbery obtained $U_q(sl(2)) \otimes U(u(1))$ as the algebra of tangent vectors at the identity of $GL_q(2)$. This technique has been successfully applied to the case of multiparametric $GL_{p,q}(2)$ [7] and quantum $GL(n)$ [8] groups. The quantised universal enveloping algebra (i.e. the dual) corresponding to a quantum group can also be obtained using the R-matrix formalism [1].

Recently, there has been a considerable development in the study of multiparametric quantum groups, both from algebraic and differential geometric aspects. We wish to focus our attention on a new quantum group [9] $G_{r,s}$, depending on two deformation parameters and five generators. The first four generators of this Hopf algebra form a Hopf subalgebra, which coincides exactly with the single parameter dependent $GL_q(2)$ quantum group when $q = r^{-1}$. However, the two parameter dependent $GL_{p,q}(2)$ can also be realised through the generators of this $G_{r,s}$ Hopf algebra, provided the sets of deformation parameters p, q and r, s are related to each other in a particular fashion. This new algebra

can, therefore, be used to realise both $GL_q(2)$ and $GL_{p,q}(2)$ quantum groups. Alternatively, this $G_{r,s}$ structure can be considered as a two parameter quantisation of the classical $GL(2) \times GL(1)$ group. The first four generators of $G_{r,s}$, i.e. a, b, c, d correspond to $GL(2)$ group at the classical level and the remaining generator f is related to $GL(1)$ group. Given the rich structure of this simple quantum group it is surprising that no further studies have so far been done. A starting point in the further investigation would be the explicit description of the dual algebra to $G_{r,s}$ and to look at its differential structure.

This is the problem that we address in this paper. We obtain the dual algebra to the two-parameter matrix quantum group $G_{r,s}$. We show that the Hopf algebra dual to $G_{r,s}$ may be realised using the method described by Sudbery [6] and exhibit its Hopf Structure. The dual algebra is also explicitly constructed using the R-matrix approach [1]. We then employ Jurco's constructive procedure [10] for a description of the differential calculus on $G_{r,s}$.

In Section II, we describe the essential features of $G_{r,s}$. The dual algebra for $G_{r,s}$ is then constructed in Section III within the framework of Sudbery's approach wherein we also detail the commutation relations and Hopf structure of the generators of the dual. Section IV deals with the R-matrix formulation of $G_{r,s}$ while Section V is devoted to the construction of a bicovariant differential calculus on $G_{r,s}$. Concluding remarks are made in Section VI.

II The quantum group $G_{r,s}$

The Hopf algebra $G_{r,s}$ is generated by elements a, b, c, d, and f satisfying the relations

$$\begin{aligned} ab &= r^{-1}ba, \quad & db &= rbd \\ ac &= r^{-1}ca, \quad & dc &= rcd \\ bc &= cb, \quad & [a,d] &= (r^{-1}-r)bc \end{aligned}$$

and

$$af = fa, \qquad cf = sfc$$
$$bf = s^{-1}fb, \quad df = fd$$

where r and s are two deformation parameters with arbitrary nonzero complex numbers. Elements a, b, c, d satisfying the first set of commutation relations form a subalgebra which coincides exactly with $GL_q(2)$ when $q = r^{-1}$. The generators are arranged in a 3×3 matrix as

$$T = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & f \end{pmatrix}$$

The Hopf structure is given as

$$\Delta \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & f \end{pmatrix} = \begin{pmatrix} a \otimes a + b \otimes c & a \otimes b + b \otimes d & 0 \\ c \otimes a + d \otimes c & c \otimes b + d \otimes d & 0 \\ 0 & 0 & f \otimes f \end{pmatrix}$$

$$\varepsilon \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & f \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The Casimir operator is defined as $\mathbf{D} = ad - r^{-1}bc$. The inverse is assumed to exist and satisfies $\Delta(\mathbf{D}^{-1}) = \mathbf{D}^{-1} \otimes \mathbf{D}^{-1}$, $\varepsilon(\mathbf{D}^{-1}) = 1$, $S(\mathbf{D}^{-1}) = \mathbf{D}$, which enables one to determine the antipode matrix $S(T)$, as

$$S \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & f \end{pmatrix} = \mathbf{D}^{-1} \begin{pmatrix} d & -rc & 0 \\ -r^{-1}c & a & 0 \\ 0 & 0 & \mathbf{D}f \end{pmatrix}$$

The elements a, b, c and d of $G_{r,s}$ evidently form a Hopf subalgebra, which coincides with $GL_q(2)$ quantum group. In order to investigate the connection

of $G_{r,s}$ with $GL_{p,q}(2)$, a simple realisation of $GL_{p,q}(2)$ generators through the elements of $G_{r,s}$ has been proposed as

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = f^N \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where N is any fixed nonzero integer. For this to give a realisation of $GL_{p,q}(2)$, the deformation parameters (p,q) and (r,s) are related via

$$p = r^{-1}s^N \quad \text{and} \quad q = r^{-1}s^{-N}$$

For $N \neq 0$ and any given value of (p,q), one can find out the corresponding values of the deformation parameters (r,s). The full Hopf algebra structure related to $GL_{p,q}(2)$ can in fact be reproduced through the above realisation. The mapping from (r,s)-plane to (p,q)-plane depends on the choice of N $(\neq 0)$. Taking different values of N, a single point on the (r,s)-plane can be mapped over infinite number of discrete points which satisfy

$$p_N q_N = r^{-2}, \quad \frac{p_{N+1}}{p_N} = s, \quad \frac{q_{N+1}}{q_N} = s^{-1},$$

where

$$p_N = r^{-1}s^N \quad \text{and} \quad q_N = r^{-1}s^{-N}$$

Here (p_N, q_N) denotes a point on the (p,q)-plane corresponding to the N-th mapping. One finds that from the representation of $G_{r,s}$ algebra for a particular value of deformation parameters (r,s), one can build up the representations of $GL_{p,q}(2)$ quantum group for infinitely many discrete values of (p,q) parameters. The choice of (9×9) R-matrix is given by

$$R = \sum_i e_{ii} \otimes e_{ii} + \sum_{i \neq j} f_{ii} \cdot e_{ii} \otimes e_{jj} + (r - r^{-1}) \sum_{i<j} e_{ij} \otimes e_{ji}$$

where $f_{12} = f_{23} = 1$, $f_{13} = s$ and $f_{ij} = f_{ji}^{-1}$ and $i,j \in [1,3]$. This satisfies the quantum Yang-Baxter equation which when recast in the form of the RTT-relations yields the commutators of the elements $G_{r,s}$. In fact, the most general

Hopf algebra generated by this R-matrix is $GL_q(3)$ which can easily reproduce the $G_{r,s}$ structure. Therefore, $G_{r,s}$ might be interpreted as a quotient of multi-parameter deformed $GL(3)$ group.

III The Dual algebra for $G_{r,s}$

Two bialgebras U and A are in duality if there exists a doubly non-degenerate bilinear form

$$\langle , \rangle : U \otimes A \rightarrow C; \quad \langle , \rangle : (u, a) \rightarrow \langle u, a \rangle; \quad \forall u \in U, a \in A$$

such that for $u, v \in U$ and $a, b \in A$, have

$$\langle u, ab \rangle = \langle \Delta_U(u), a \otimes b \rangle$$
$$\langle uv, a \rangle = \langle u \otimes v, \Delta_A(a)$$

$$\langle \mathbf{1}_U, a \rangle = \varepsilon_A(a)$$
$$\langle u, \mathbf{1}_A \rangle = \varepsilon_U(u)$$

For the two bialgebras to be in duality as Hopf algebras, U and A further satisfy

$$\langle S_U(u), a \rangle = \langle u, S_A(a) \rangle$$

It is enough to define the pairing between the generating elements of the two algebras. Pairing for any other elements of U and A follows from the above relations and the bilinear form inherited by the tensor product. For example, for $\Delta(u) = \sum_i u'_i \otimes u''_i$, we have

$$\begin{aligned} \langle u, ab \rangle = \langle \Delta_U(u), a \otimes b \rangle &= \sum_i \langle u'_i \otimes u''_i, a \otimes b \rangle \\ &= \sum_i \langle u'_i, a \rangle \langle u''_i, b \rangle \end{aligned}$$

Sudbery's idea for duality was motivated by the fact that, at the classical level, "an element of Lie Algebra corresponding to the Lie Group is a tangent vector

of the identity of the Lie Group". Let H be a given Hopf algebra generated by non-commuting elements a, b, c, d. Then q-analogue of tangent vector of identity is obtained by differentiating elements of H (polynomials in a, b, c, d) and then putting $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ later on (i.e. taking the co-unit operation). The elements thus obtained belong to the dual H^*. For the sake of convenience differentiation is taken from the right. He obtained $U_q(sl(2)) \otimes U(u(1))$ as the algebra of tangent vectors at the identity of $GL_q(2,C)$.

As a Hopf algebra, $G_{r,s}$ is generated by elements a, b, c, d, f and a basis is given by all monomials of the form

$$g = g_{kltmn} = a^k d^l f^t b^m c^n$$

where $k,l,t,m,n \in Z_+$, and δ_{00000} is the unit of the algebra $\mathbf{1}_A$. We use the so-called normal ordering i.e. first put the diagonal elements from the T-matrix then use the lexicographic order for the others. Let $U_{r,s}$ be the algebra generated by tangent vectors at the identity of $G_{r,s}$. Then $U_{r,s}$ is dually paired with $G_{r,s}$. The pairing is defined through the tangent vectors as follows

$$\langle Y, g\rangle = \frac{\partial g}{\partial y}\Big|_{\left(\begin{smallmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & f \end{smallmatrix}\right)=\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)} \quad ; \quad y \in \{a,b,c,d,f\}$$

or more consistently

$$\langle Y, g\rangle = \varepsilon(\frac{\partial g}{\partial y}) \quad ; \quad y \in \{a,b,c,d,f\}$$

where $Y \in \{A,B,C,D,F\}$ are the generating elements of the dual algebra (which has unit $\mathbf{1}_U$). Such a generalised approach in which initial pairings are postulated to be equal to the classical undeformed results was given by Dobrev

in [7]. Writing explicitly the 5 possible cases, we have

$$\begin{aligned}
\langle A, g\rangle &= \varepsilon(\frac{\partial g}{\partial a}) = k\delta_{m0}\delta_{n0} \\
\langle B, g\rangle &= \varepsilon(\frac{\partial g}{\partial b}) = \delta_{m1}\delta_{n0} \\
\langle C, g\rangle &= \varepsilon(\frac{\partial g}{\partial c}) = \delta_{m0}\delta_{n1} \\
\langle D, g\rangle &= \varepsilon(\frac{\partial g}{\partial d}) = l\delta_{m0}\delta_{n0} \\
\langle F, g\rangle &= \varepsilon(\frac{\partial g}{\partial f}) = t\delta_{m0}\delta_{n0}
\end{aligned}$$

where differentiation is from the right as this is most suitable for differentiation in this basis. Variables b, c commute and their differentiation is classical. For example, $\langle B, g\rangle$ is obtained by first calculating

$$(a^k d^l f^t b^m c^n)\frac{\partial}{\partial b} = (a^k d^l f^t c^n b^m)\frac{\partial}{\partial b} = a^k d^l f^t c^n m b^{m-1}$$

(because of differentiation from right), and then applying the counit operation

$$\varepsilon(a^k d^l f^t c^n m b^{m-1}) = \varepsilon(a)^k \varepsilon(d)^l \varepsilon(f)^t \varepsilon(c)^n m \varepsilon(b)^{m-1} = \delta_{n0} m \delta_{m-1,0} = \delta_{n0}\delta_{m1}$$

Other pairings follow in a similar way. As a consequence of the above pairings, the following relations hold

$$\langle A, \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & f \end{pmatrix}\rangle = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\langle D, \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & f \end{pmatrix}\rangle = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\langle F, \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & f \end{pmatrix}\rangle = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\langle B, \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & f \end{pmatrix}\rangle = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\langle C, \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & f \end{pmatrix}\rangle = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\langle Y, \mathbf{1}_A\rangle = 0 \quad Y = A, B, C, D, F$$

$$\langle \mathbf{1}_u, a^k d^l f^t b^m c^n \rangle = \delta_{m0}\delta_{n0}$$

$$\left[\because LHS = \varepsilon_A(a^k d^l f^t b^m c^n) = (\varepsilon_A(a))^k (\varepsilon_A(d))^l (\varepsilon_A(f))^t (\varepsilon_A(b))^m (\varepsilon_A(c))^n = \delta_{m0}\delta_{n0}\right]$$

Commutation Relations

The action of the monomials in $U_{r,s}$ on $g = a^k d^l f^t b^m c^n$ is given by the following:

$$\begin{aligned}
\langle BC, g\rangle &= \delta_{m0}\delta_{n0}\sum_{j=0}^{k-1} r^{2(j-l)} + r^{-1}\delta_{m1}\delta_{n1} \\
\langle CB, g\rangle &= \delta_{m0}\delta_{n0}\sum_{j=0}^{k-1} r^{-2j} + r\delta_{m1}\delta_{n1} \\
\langle AB, g\rangle &= (k+1)\delta_{m1}\delta_{n0} = (k+1)\langle B, g\rangle \\
\langle BA, g\rangle &= k\delta_{m1}\delta_{n0} = k\langle B, g\rangle \\
\langle AC, g\rangle &= k\delta_{m0}\delta_{n1} = k\langle C, g\rangle \\
\langle CA, g\rangle &= (k+1)\delta_{m0}\delta_{n1} = (k+1)\langle C, g\rangle \\
\langle DB, g\rangle &= l\delta_{m1}\delta_{n0} = l\langle B, g\rangle \\
\langle BD, g\rangle &= (l+1)\delta_{m1}\delta_{n0} = (l+1)\langle B, g\rangle \\
\langle DC, g\rangle &= (l+1)\delta_{m0}\delta_{n1} = (l+1)\langle C, g\rangle \\
\langle CD, g\rangle &= l\delta_{m0}\delta_{n1} = l\langle C, g\rangle \\
\langle AD, g\rangle &= \langle DA, g\rangle = kl\delta_{m0}\delta_{n0} = kl\langle \mathbf{1}_U, g\rangle \\
\langle AF, g\rangle &= \langle FA, g\rangle = kt\delta_{m0}\delta_{n0} = kt\langle \mathbf{1}_U, g\rangle \\
\langle DF, g\rangle &= \langle FD, g\rangle = lt\delta_{m0}\delta_{n0} = lt\langle \mathbf{1}_U, g\rangle \\
\langle BF, g\rangle &= (t+1)\delta_{m1}\delta_{n0} = (t+1)\langle B, g\rangle \\
\langle FB, g\rangle &= t\delta_{m1}\delta_{n0} = t\langle B, g\rangle \\
\langle CF, g\rangle &= t\delta_{m0}\delta_{n1} = t\langle C, g\rangle \\
\langle FC, g\rangle &= (t+1)\delta_{m0}\delta_{n1} = (t+1)\langle C, g\rangle
\end{aligned}$$

As an illustration, $\langle BC, g\rangle$ is obtained by using the assumed duality,

$$
\begin{aligned}
\langle BC, \quad & a^k d^l f^t b^m c^n\rangle \\
= \quad & \langle B\otimes C, \Delta_A(a^k d^l f^t b^m c^n)\rangle \\
= \quad & \langle B\otimes C, (\Delta_A(a))^k(\Delta_A(d))^l(\Delta_A(f))^t(\Delta_A(b))^m(\Delta_A(c))^n\rangle \\
= \quad & \langle B\otimes C, (a\otimes a + b\otimes c)^k(c\otimes b + d\otimes d)^l(f\otimes f)^t(a\otimes b + b\otimes d)^m(c\otimes a + d\otimes c)^n\rangle \\
= \quad & \langle B\otimes C, (a^k\otimes a^k + \sum_{j=0}^{k-1} a^{k-1-j}ba^j\otimes a^{k-1-j}ca^j)(d^l\otimes d^l + \sum_{j=0}^{l-1} d^{l-1-j}cd^j\otimes d^{l-1-j}bd^j) \\
& (f^t\otimes f^t)(\delta_{m0} + (a\otimes b + b\otimes d)\delta_{m1})(\delta_{n0} + (c\otimes a + d\otimes c)\delta_{n1})\rangle \\
& \text{(keeping terms involving } b,c \text{ at most of degree 1)} \\
= \quad & \langle B\otimes C, (\sum_{j=0}^{k-1} a^{k-1-j}ba^j d^l f^t\otimes a^{k-1-j}ca^j d^l f^t)\delta_{m0}\delta_{n0} + (a^k d^l f^t bd\otimes a^k d^l f^t dc)\delta_{m1}\delta_{n1} \\
& \text{(keeping only contributing terms involving exactly one factor } 'b' \text{ to the left of } \otimes \text{ and} \\
& \text{one factor } 'c' \text{ to the right of } \otimes)
\end{aligned}
$$

Next, use $G_{r,s}$ commutation relations to reorder the elements to the basis order $(adfbc)$. Then, we have

$$
\begin{aligned}
\langle BC, \quad & a^k d^l f^t b^m c^n\rangle \\
= \quad & \langle B\otimes C, (\sum_{j=0}^{k-1} a^{k-1} d^l f^t b r^{j-l} s^{-t}\otimes a^{k-1} d^l f^t c r^{j-l} s^t)\delta_{m0}\delta_{n0} \\
& + (a^k d^{l+1} f^t b r^{-1}\otimes a^k d^{l+1} f^t c)\delta_{m1}\delta_{n1}\rangle \\
& \text{(Moving the elements to the right (left) brings in the powers of } r \text{ and } s)
\end{aligned}
$$

Now, applying the pairings proves $\langle BC, g\rangle$. Similarly for $\langle CB, g\rangle$,we have

$$
\begin{aligned}
&\langle CB, \quad a^k d^l f^t b^m c^n \rangle \\
&= \quad \langle C \otimes B, (a^k \otimes a^k)(d^l \otimes d^l + \sum_{j=0}^{l-1} d^{l-1-j} c d^j \otimes d^{l-1-j} b d^j)(f^t \otimes f^t) \\
&\qquad (\delta_{m0} + (a \otimes b)\delta_{m1})(\delta_{n0} + (c \otimes a)\delta_{n1}) \rangle \\
&= \quad \langle C \otimes B, (\sum_{j=0}^{l-1} a^k d^{l-1-j} c d^j f^t \otimes a^k d^{l-1-j} b d^j f^t)\delta_{m0}\delta_{n0} + (a^k d^l f^t ac \otimes a^k d^l f^t ba)\delta_{m1}\delta_{n1} \rangle \\
&= \quad \langle C \otimes B, (\sum_{j=0}^{l-1} a^k d^{l-1} f^t c r^{-j} s^t \otimes a^k d^{l-1} f^t b r^{-j} s^{-t})\delta_{m0}\delta_{n0} + ((a^{k+1} d^l f^t - a^k f^t b c^2 (r^{-l} - r^l)) \\
&\qquad \otimes (a^{k+1} d^l f^t br - a^k f^t b^2 cr(r^{-l} - r^l))\delta_{m1}\delta_{n1} \rangle
\end{aligned}
$$

Again, applying the pairings $\langle B, g\rangle$ and $\langle C, g\rangle$ proves the result $\langle CB, g\rangle$. Now,

$$r\langle BC, g\rangle - r^{-1}\langle CB, g\rangle = (r\sum_{j=0}^{k-1} r^{2(j-l)} - r^{-1}\sum_{j=0}^{l-1} r^{-2j})\delta_{m0}\delta_{n0}$$

Using the formula

$$\sum_{j=0}^{k-1} x^j = \frac{1-x^k}{1-x}$$

we obtain

$$r\langle BC, g\rangle - r^{-1}\langle CB, g\rangle = \frac{r^{2(k-l)} - 1}{r - r^{-1}}\delta_{m0}\delta_{n0}$$

Also obtained as a corollary from the action on $g = a^k d^l f^t b^m c^n$ of the monomials in $U_{r,s}$are the following relations

$$
\begin{aligned}
&\langle [A, B], g\rangle = \langle B, g\rangle, \qquad \langle [A, C], g\rangle = -\langle C, g\rangle \\
&\langle [D, B], g\rangle = -\langle B, g\rangle, \quad \langle [D, C], g\rangle = \langle C, g\rangle \\
&\langle [A, D], g\rangle = 0, \qquad \langle [A, F], g\rangle = 0, \qquad \langle [D, F], g\rangle = 0, \\
&\langle [B, F], g\rangle = \langle B, g\rangle, \qquad \langle [C, F], g\rangle = -\langle C, g\rangle
\end{aligned}
$$

From this one obtains the commutation relations in the algebra $U_{r,s}$ dual to $A_{r,s}$ as

$$rBC - r^{-1}CB = \frac{r^{2(A-D)} - \mathbf{1}_U}{r - r^{-1}}$$

$$\begin{aligned}
&[A,B] = B, \quad [A,C] = -C, \quad [D,B] = -B \\
&[D,C] = C, \quad [A,D] = 0 \\
&[A,F] = 0, \quad [D,F] = 0 \\
&[B,F] = B, \quad [C,F] = -C
\end{aligned}$$

The Hopf structure of the dual algebra is obtained and the coproduct of the elements of the dual is given by

$$\begin{aligned}
\Delta(A) &= A \otimes \mathbf{1} + \mathbf{1} \otimes A \\
\Delta(B) &= B \otimes r^{A-D} s^{-F} + \mathbf{1} \otimes B \\
\Delta(C) &= C \otimes r^{A-D} s^{F} + \mathbf{1} \otimes C \\
\Delta(D) &= D \otimes \mathbf{1} + \mathbf{1} \otimes D \\
\Delta(F) &= F \otimes \mathbf{1} + \mathbf{1} \otimes F
\end{aligned}$$

The counit $\varepsilon(Y) = 0$; where $Y = A, B, C, D, F$ and the antipode is given as

$$\begin{aligned}
S(A) &= -A \\
S(B) &= -B r^{-(A-D)} s^{F} \\
S(C) &= -C r^{-(A-D)} s^{-F} \\
S(D) &= -D \\
S(F) &= -F
\end{aligned}$$

IV The R-matrix approach

Here we obtain the dual algebra employing the R-matrix for $G_{r,s}$. We work with a different 9×9 R-matrix by labelling the index 3 as 0 and transposing

the R-matrix given in [9]. This reads

$$R = \begin{pmatrix} r & 0 & 0 & 0 \\ 0 & S^{-1} & 0 & 0 \\ 0 & \Lambda & S & 0 \\ 0 & 0 & 0 & R_r \end{pmatrix}$$

in block form i.e. in the order (00), (01), (02), (10), (20), (11), (12), (21), (22) where

$$R_r = \begin{pmatrix} r & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & 0 & r \end{pmatrix} \quad S = \begin{pmatrix} s & 0 \\ 0 & 1 \end{pmatrix}; \Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}; \lambda = r - r^{-1}$$

Also, the the matrix of generators is transposed as

$$\mathcal{T} = \begin{pmatrix} f & 0 \\ 0 & T \end{pmatrix} \quad \text{with} \quad T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

The linear functionals $(L^{\pm})^a_b$ are defined by their value on the elements of the matrix of generators T

$$\langle (L^{\pm})^a_b, T^c_d \rangle = (R^{\pm})^{ac}_{bd}$$

where

$$\begin{aligned} (R^{+})^{ac}_{bd} &= c^{+}(R)^{ca}_{db} \\ (R^{-})^{ac}_{bd} &= c^{-}(R^{-1})^{ac}_{bd} \end{aligned}$$

and c^{+} , c^{-} are free parameters. Matrices $(L^{\pm})^a_b$ satisfy

$$\langle (L^{\pm})^a_b, uv \rangle = \langle (L^{\pm})^a_c \otimes (L^{\pm})^c_d, u \otimes v \rangle = (L^{\pm})^a_c(u)(L^{\pm})^c_d(v)$$

$$\text{i.e.} \quad \Delta(L^{\pm})^a_b = (L^{\pm})^a_c \otimes (L^{\pm})^c_d$$

For $G_{r,s}$, the (R^+) and (R^-) matrices read

$$(R^+) = c^+ \begin{pmatrix} r & 0 & 0 & 0 \\ 0 & S^{-1} & \Lambda & 0 \\ 0 & 0 & S & 0 \\ 0 & 0 & 0 & R_r^T \end{pmatrix} ; \quad (R^-) = c^- \begin{pmatrix} r & 0 & 0 & 0 \\ 0 & S & 0 & 0 \\ 0 & -\Lambda & S^{-1} & 0 \\ 0 & 0 & 0 & R_r^{-1} \end{pmatrix}$$

Note that $R_r^{-1} = R_{r^{-1}}$. The $(\mathcal{L}^{\pm})_b^a$ functionals are dual to the elements $\mathcal{T}_b^a$ in the fundamental representation. From the above, it is to be noted that the $\mathcal{L}^{\pm}$ matrices are obtained by evaluating their action on the generators of the quantum group $G_{r,s}$. Using the duality pairings, we get

$$\mathcal{L}^+ = c^+ r \begin{pmatrix} s^{\frac{1}{2}(\tilde{F}-H_2-1)} r^{\frac{1}{2}(\tilde{F}-H_1-1)} & 0 & 0 \\ 0 & s^{\frac{1}{2}(\tilde{F}-H_1+1)} r^{\frac{1}{2}(-\tilde{F}+H_2-1)} & r^{-1}\lambda\tilde{C} \\ 0 & 0 & s^{-\frac{1}{2}(\tilde{F}+H_1-1)} r^{\frac{1}{2}(-\tilde{F}-H_2-1)} \end{pmatrix}$$

$$\mathcal{L}^- = c^- r^{-1} \begin{pmatrix} s^{-\frac{1}{2}(\tilde{F}-H_2-1)} r^{-\frac{1}{2}(\tilde{F}-H_1-1)} & 0 & 0 \\ 0 & s^{-\frac{1}{2}(\tilde{F}-H_1+1)} r^{-\frac{1}{2}(-\tilde{F}+H_2-1)} & 0 \\ 0 & -r\lambda\tilde{B} & s^{\frac{1}{2}(\tilde{F}+H_1-1)} r^{-\frac{1}{2}(-\tilde{F}-H_2-1)} \end{pmatrix}$$

where $H_1 = \tilde{A} + \tilde{D}$, $H_2 = \tilde{A} - \tilde{D}$ and $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}, \tilde{F}$ are elements of the algebra dual to $G_{r,s}$. More conveniently, one can write

$$\mathcal{L}^+ = \begin{pmatrix} J & 0 & 0 \\ 0 & M & P \\ 0 & 0 & N \end{pmatrix} \quad \text{and} \quad \mathcal{L}^- = \begin{pmatrix} J^{-1} & 0 & 0 \\ 0 & M^{-1} & 0 \\ 0 & Q & N^{-1} \end{pmatrix}$$

where

$$J = s^{\frac{1}{2}(\tilde{F}-H_2-1)} r^{\frac{1}{2}(\tilde{F}-H_1+1)}$$
$$M = s^{\frac{1}{2}(\tilde{F}-H_1+1)} r^{\frac{1}{2}(-\tilde{F}+H_2+1)}$$
$$N = s^{-\frac{1}{2}(\tilde{F}+H_1-1)} r^{\frac{1}{2}(-\tilde{F}-H_2+1)}$$
$$P = \lambda \tilde{C}$$
$$Q = -\lambda \tilde{B}$$

These can also be arranged in terms of smaller L^+ and L^- matrices

$$\mathcal{L}^+ = c^+ \begin{pmatrix} J & 0 \\ 0 & L^+ \end{pmatrix} \qquad \text{where} \qquad L^+ = \begin{pmatrix} M & P \\ 0 & N \end{pmatrix}$$
$$\mathcal{L}^- = c^- \begin{pmatrix} J^{-1} & 0 \\ 0 & L^- \end{pmatrix} \qquad \text{where} \qquad L^- = \begin{pmatrix} M^{-1} & 0 \\ Q & N^{-1} \end{pmatrix}$$

Commuation relations of the dual

The algebra dual to $G_{r,s}$ generated by functionals (matrices) $\mathcal{L}^{\pm}$ satisfy the q-commuation relations (the so-called $R\mathcal{L}\mathcal{L}$- relations)

$$R_{12}\mathcal{L}_2^{\pm}\mathcal{L}_1^{\pm} = \mathcal{L}_1^{\pm}\mathcal{L}_2^{\pm}R_{12}$$
$$R_{12}\mathcal{L}_2^{+}\mathcal{L}_1^{-} = \mathcal{L}_1^{-}\mathcal{L}_2^{+}R_{12}$$

where $\mathcal{L}_1^{\pm} = \mathcal{L}^{\pm} \otimes \mathbf{1}$ and $\mathcal{L}_2^{\pm} = \mathbf{1} \otimes \mathcal{L}^{\pm}$. The dual algebra for $G_{r,s}$ can be constructed using the theory developed in [1]. Since $G_{r,s}$ is a quotient Hopf algebra, the R-matrix for the $R\mathcal{L}\mathcal{L}$ - relations is different from the one used in the RTT - relations for the $G_{r,s}$ algebra. Instead, it is necessary to amend the R-matrix to remove relations which are inconsistent with the quotient structure. For $G_{r,s}$, this means that the $R\mathcal{L}\mathcal{L}$ - relations are constructed with the R-matrix

$$R_{12} = c^- \langle \mathcal{L}^-, \mathcal{T} \rangle^{-1} = \begin{pmatrix} r & & & \\ & S^{-1} & & \\ & & S & \\ & & & R_r \end{pmatrix}$$

Evaluating $\mathcal{L}_1^{\pm}$, $\mathcal{L}_2^{\pm}$ matrices and substituting in the above $R\mathcal{L}\mathcal{L}$- relations yields the dual algebra commutation relations. From $R_{12}\mathcal{L}_2^-\mathcal{L}_1^- = \mathcal{L}_1^-\mathcal{L}_2^- R_{12}$ and $R_{12}\mathcal{L}_2^+\mathcal{L}_1^+ = \mathcal{L}_1^+\mathcal{L}_2^+ R_{12}$ we obtain

$$\begin{aligned} R_r L_2^- L_1^- &= L_1^- L_2^- R_r \\ R_r L_2^+ L_1^+ &= L_1^+ L_2^+ R_r \\ MJ &= JM \\ NJ &= JN \\ JQ &= s^{-1}QJ \\ JP &= sPJ \end{aligned}$$

where

$$\begin{aligned} R_r L_2^- L_1^- = L_1^- L_2^- R_r \quad &\Longrightarrow \quad MQ = rQM, \quad QN = rNQ \quad \text{and} \quad NM = MN \\ R_r L_2^+ L_1^+ = L_1^+ L_2^+ R_r \quad &\Longrightarrow \quad PM = rMP, \quad NP = rPN \quad \text{and} \quad NM = MN \end{aligned}$$

In addition, the cross relation $R_{12}\mathcal{L}_2^+\mathcal{L}_1^- = \mathcal{L}_1^-\mathcal{L}_2^+ R_{12}$ yields $R_r L_2^+ L_1^- = L_1^- L_2^+ R_r$ which further implies

$$QP - PQ = -\lambda(N^{-1}M - NM^{-1})$$

Simplifying the above, we get the following commutation relations

$$\begin{aligned} [\tilde{A}, \tilde{B}] &= \tilde{B}, & [\tilde{A}, \tilde{C}] &= -\tilde{C} \\ [\tilde{D}, \tilde{B}] &= -\tilde{B}, & [\tilde{D}, \tilde{C}] &= \tilde{C} \\ [\tilde{A}, \tilde{D}] &= 0, & [\tilde{F}, \bullet] &= 0 \end{aligned}$$

and

$$[\tilde{B},\tilde{C}] = \frac{\left(r^{\tilde{A}-\tilde{D}}s^{\tilde{F}}\right) - \left(r^{\tilde{A}-\tilde{D}}s^{\tilde{F}}\right)^{-1}}{r-r^{-1}} = \frac{r^{\tilde{A}-\tilde{D}+\gamma\tilde{F}} - r^{-(\tilde{A}-\tilde{D}+\gamma\tilde{F})}}{r-r^{-1}} \quad \text{where} \quad \gamma = \frac{\ln s}{\ln r}$$

So, defining $H = \tilde{A} - \tilde{D} + \gamma\tilde{F}$, $X_+ = \tilde{B}$ and $X_- = \tilde{C}$ we obtain

$$[H, X_\pm] = 2X_\pm; \quad [X_+, X_-] = [H]; \quad [\tilde{F}, \bullet] = 0$$

i.e. the Drinfeld-Jimbo form of a single parameter deformation of $gl(2) \oplus gl(1)$.

V Differential Calculus on $G_{r,s}$

We now proceed towards the construction of a differential calculus on the $G_{r,s}$ quantum group. We use Jurco's constructive procedure [10] based on the R-matrix formulation and using the $\mathcal{T}$- and the $\mathcal{L}^\pm$ matrices of the previous section.

Quantum one-forms

Define ω to be the basis of all left-invariant quantum one-forms. So, we have

$$\Delta_L(\omega) = \mathbf{1} \otimes \omega$$

This defines the left action on the bimodule Γ (space of quantum one-forms). The bimodule Γ is further characterised by the commutation relations between ω and $a \in A$ (A being the $G_{r,s}$ Hopf algebra),

$$\omega a = (f * a)\omega$$

The left convolution product is defined

$$f * a = (\mathbf{1} \otimes f)\Delta(a)$$

where $f \in A'(= Hom(A, \mathcal{C}))$ i.e. belongs to the dual. This means

$$\omega a = (\mathbf{1} \otimes f)\Delta(a)\omega$$

Now, the linear functional f is defined in terms of the $\mathcal{L}^{\pm}$ matrices as

$$f = S(\mathcal{L}^{+})\mathcal{L}^{-}$$

Thus, we have

$$\omega a = [(\mathbf{1} \otimes S(\mathcal{L}^{+})\mathcal{L}^{-})\Delta(a)]\omega$$

In terms of components, one can write

$$\omega_{ij} a = [(\mathbf{1} \otimes S(l^{+}_{ki})l^{-}_{jl})\Delta(a)]\omega_{kl}$$

using the expressions $\mathcal{L}^{\pm} = l^{\pm}_{ij}$ and $\omega = \omega_{ij}$ where $i, j = 1..3$.
For Γ to be a bicovariant bimodule, the right coaction is given as

$$\Delta_R(\omega) = \omega \otimes M$$

where functionals M are defined in terms of the matrix of generators T,

$$M = TS(T)$$

Again, in component form, one can write

$$\Delta_R(\omega_{ij}) = \omega_{kl} \otimes t_{ki} S(t_{jl})$$

Using the above formulae, we obtain the commutation relations of all the left-

invariant one forms with the elements of the $G_{r,s}$ quantum group as follows

$$\omega^0 a = r^2 s^2 a\omega^0 \qquad \omega^0 b = r^2 b\omega^0$$
$$\omega^1 a = r^{-2} a\omega^1 \qquad \omega^1 b = b\omega^1$$
$$\omega^+ a = r^{-1} a\omega^+ \qquad \omega^+ b = r^{-1} b\omega^+ - \lambda r^{-1} a\omega^1$$
$$\omega^- a = r^{-1} a\omega^- - \lambda r^{-1} b\omega^1 \qquad \omega^- b = r^{-1} b\omega^-$$
$$\omega^2 a = a\omega^2 - \lambda b\omega^+ \qquad \omega^2 b = r^{-2} b\omega^2 - \lambda r^{-1} a\omega^- + \lambda^2 b\omega^1$$
$$\omega^0 c = r^2 s^2 c\omega^0 \qquad \omega^0 d = r^2 d\omega^0$$
$$\omega^1 c = r^{-2} c\omega^1 \qquad \omega^1 d = d\omega^1$$
$$\omega^+ c = r^{-1} c\omega^+ \qquad \omega^+ d = r^{-1} d\omega^+ - \lambda r^{-1} c\omega^1$$
$$\omega^- c = r^{-1} c\omega^- - \lambda r^{-1} d\omega^1 \qquad \omega^- d = r^{-1} d\omega^-$$
$$\omega^2 c = c\omega^2 - \lambda d\omega^+ \qquad \omega^2 d = r^{-2} d\omega^2 - \lambda r^{-1} c\omega^- + \lambda^2 d\omega^1$$

$$\omega^0 f = f\omega^0$$
$$\omega^1 f = s^{-2} f\omega^1$$
$$\omega^+ f = s^{-1} f\omega^+$$
$$\omega^- f = s^{-1} f\omega^-$$
$$\omega^2 f = f\omega^2$$

where $\omega^0 = \omega_{11}, \omega^1 = \omega_{22}, \omega^+ = \omega_{23}, \omega^- = \omega_{32}, \omega^2 = \omega_{33}$ and the components $\omega_{12}, \omega_{13}, \omega_{21}, \omega_{31}$ have null contribution given the structure of the $\mathcal{T}$- matrix (i.e. $t_{12} = t_{13} = t_{21} = t_{31} = 0$).

Vector Fields

The linear space Γ (space of all left invariant one-forms) contains a bi-invariant element $\tau = \sum_i \omega_{ii}$ which can be used to define a derivative on A (the $G_{r,s}$ Hopf algebra). For $a \in A$, one sets

$$\mathbf{d}a = \tau a - a\tau$$

$$\text{Now} \qquad \omega_{ii}a = [(\mathbf{1}\otimes S(l^+_{ki})l^-_{il})\Delta(a)]\omega_{kl}$$

$$\text{So} \qquad \mathbf{d}a = [(\mathbf{1}\otimes \chi_{kl})\Delta(a)]\omega_{kl}$$

where $\chi_{kl} = S(l^+_{ki})l^-_{il} - \delta_{kl}\varepsilon$, ε being the counit. Denote

$$\chi_{ij} = S(l^+_{ik})l^-_{kj} - \delta_{ij}\,\varepsilon$$

or more compactly

$$\chi = S(\mathcal{L}^+)\mathcal{L}^- - \mathbf{1}\varepsilon$$

the matrix of left-invariant vector fields χ_{ij} on A. On elements of $G_{r,s}$ the vector fields act as

$$\begin{aligned}
\chi_{ij}\,a &= (S(l^+_{ik})l^-_{kj} - \delta_{ij}\,\varepsilon)a \\
\chi_{ij}\,a &= \langle S(l^+_{ik})l^-_{kj}, a\rangle - \delta_{ij}\,\varepsilon(a); \qquad\qquad a \in G_{r,s}
\end{aligned}$$

We obtain explicitly the following

$$\begin{aligned}
&\chi_0(a) = r^2s^2 - 1 && \chi_0(b) = 0 \\
&\chi_1(a) = r^{-2} - 1 && \chi_1(b) = 0 \\
&\chi_+(a) = 0 && \chi_+(b) = 0 \\
&\chi_-(a) = 0 && \chi_-(b) = -(r - r^{-1}) \\
&\chi_2(a) = 0 && \chi_2(b) = 0
\end{aligned}$$

$$\begin{aligned}
&\chi_0(c) = 0 && \chi_0(d) = r^2 - 1 \\
&\chi_1(c) = 0 && \chi_1(d) = (r - r^{-1})^2 \\
&\chi_+(c) = -(r - r^{-1}) && \chi_+(d) = 0 \\
&\chi_-(c) = 0 && \chi_-(d) = 0 \\
&\chi_2(c) = 0 && \chi_2(d) = r^{-2} - 1
\end{aligned}$$

$$\chi_0(f) = 0$$

$$\chi_1(f) = s^{-2} - 1$$

$$\chi_+(f) = 0$$

$$\chi_-(f) = 0$$

$$\chi_2(f) = 0$$

where $\chi_0 = \chi_{11}, \chi_1 = \chi_{22}, \chi_+ = \chi_{23}, \chi_- = \chi_{32}, \chi_2 = \chi_{33}$ and again (by previous argument) the components $\chi_{12}, \chi_{13}, \chi_{21}, \chi_{31}$ have null contribution. The left convolution products are given as

$$\chi_0 * a = ((rs)^2 - 1)a \qquad \chi_0 * b = (r^2 - 1)b$$

$$\chi_1 * a = (r^{-2} - 1)a \qquad \chi_1 * b = ((r - r^{-1})^2)b$$

$$\chi_+ * a = -(r - r^{-1})b \qquad \chi_+ * b = 0$$

$$\chi_- * a = 0 \qquad \chi_- * b = -(r - r^{-1})a$$

$$\chi_2 * a = 0 \qquad \chi_2 * b = (r^{-2} - 1)b$$

$$\chi_0 * c = ((rs)^2 - 1)c \qquad \chi_0 * d = (r^2 - 1)d$$

$$\chi_1 * c = (r^{-2} - 1)c \qquad \chi_1 * d = ((r - r^{-1})^2)d$$

$$\chi_+ * c = -(r - r^{-1})d \qquad \chi_+ * d = 0$$

$$\chi_- * c = 0 \qquad \chi_- * d = -(r - r^{-1})c$$

$$\chi_2 * c = 0 \qquad \chi_2 * d = (r^{-2} - 1)d$$

$$\chi_0 * f = 0$$

$$\chi_1 * f = (s^{-2} - 1)f$$

$$\chi_+ * f = 0$$

$$\chi_- * f = 0$$

$$\chi_2 * f = 0$$

Exterior Derivatives

Using the formula $\mathbf{d}a = \sum_i (\chi_i * a)\omega^i$ for $a \in A$, we obtain the action of the exterior derivative on the generating elements of $G_{r,s}$

$$\begin{aligned}
\mathbf{d}a &= ((rs)^2 - 1)a\omega^0 + (r^{-2} - 1)a\omega^1 - \lambda b\omega^+ \\
\mathbf{d}b &= (r^2 - 1)b\omega^0 + \lambda^2 b\omega^1 - \lambda a\omega_- + (r^{-2} - 1)b\omega^2 \\
\mathbf{d}c &= ((rs)^2 - 1)c\omega^0 + (r^{-2} - 1)c\omega^1 - \lambda d\omega^+ \\
\mathbf{d}d &= (r^2 - 1)d\omega^0 + \lambda^2 d\omega^1 - \lambda c\omega_- + (r^{-2} - 1)d\omega^2 \\
\mathbf{d}f &= (s^{-2} - 1)f\omega^1
\end{aligned}$$

where $\lambda = r - r^{-1}$. The exterior derivative $\mathbf{d} : A \longrightarrow \Gamma$ satisfies the Leibnitz rule and $\mathbf{d}A$ generates Γ as a left A-module. This then defines a first order differential calclulus $(\Gamma, \mathbf{d})$ on $G_{r,s}$. Furthermore, the calculus is bicovariant due to the coexistence of the left and the right actions

$$\Delta_L : \Gamma \longrightarrow A \otimes \Gamma$$

$$\Delta_R : \Gamma \longrightarrow \Gamma \otimes A$$

since $\mathbf{d}$ has the invariance property

$$\Delta_L \mathbf{d} = (\mathbf{1} \otimes \mathbf{d})\Delta$$

$$\Delta_R \mathbf{d} = (\mathbf{d} \otimes \mathbf{1})\Delta$$

Motivated by the observation that $G_{r,s}$ provides a realisation of the two parameter quantum group $GL_{p,q}(2)$, we wish to investigate the relationship between the differential calculus on $G_{r,s}$ and that of $GL_{p,q}(2)$. Again, if we denote the primed generators as those belonging to $GL_{p,q}(2)$ while the unprimed ones as the generators of $G_{r,s}$, then the relation

$$\mathbf{d}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \mathbf{d}\left(f\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$$

provides a relisation of the differential calculus on $GL_{p,q}(2)$ [13], with the defining relations between the two sets of deformation parameters (p,q) and (r,s) as before. In establishing the above homomorphism at the differential calculus level, one has to ignore the term involving ω^0 as this is the one-form related to the fifth element f, which is nonexistent in the case of $GL_{p,q}(2)$. A large part of the differential calculus depends on the deformation parameter r with the second parameter s being related to the element f and corrsponding one-forms and vector fields.

We note that the calculus on $G_{r,s}$ contains the calculus on $GL_q(2)$ and our results match with those given in [11]. We also expect that the calculus on $G_{r,s}$ could be obtained by projection from the calculus on multiparameter q-deformed $GL(3)$ [12]. The physical interest in studing $G_{r,s}$ lies in the observation that when endowed with a $*$- structure, this quantum group would specialise to a two parameter quantum deformation of $SU(2)\otimes U(1)$ which is precisely the gauge group for the theory of electroweak interactions. Since gauge theories have an obvious differential geometric description, the above study of differential calculus could provide insights in constructing a q-gauge theory based on $G_{r,s}$. The homomorphism between $G_{r,s}$ and $GL_{p,q}(2)$ at the level of differential calculus is then, also expected to play a significant role.

VI Conclusions

In this work, we have explicitly constructed the dual algebra to the quantum group $G_{r,s}$ and exhibited its Hopf algebraic structure. This has been obtained using the method described by Sudbery [6] as well as the R-matrix approach [1]. It is to be noted that the algebra sector of the dual to $G_{r,s}$ is a single parameter deformation of $gl(2) \oplus gl(1)$ whereas the second parameter only appears in the coalgebra. This can be clearly understood as an application of the twisting procedure. As a next stage of this analysis, we have constructed a bicovariant differential calculus on the quantum group $G_{r,s}$, which would be of significance in constructing a corresponding gauge theory. Furthermore, our analysis shows that the quantum group homomorphism between $G_{r,s}$ and $GL_{p,q}(2)$ has a natural extention at the level of their differential calculii.

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