A revision of Minty's algorithm for finding a maximum weight stable set of a claw-free graph

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Abstract: The maximum weight/cardinality stable set problem is to find a maximum weight/cardinality stable set of a given graph. It is well known that these problems for general graphs belong to the class of NP-hard. However, for several classes of graphs, e.g., for perfect graphs and claw-free graphs and so on, these problem can be solved in polynomial time. For instance, Minty (1980), Sbihi (1980) and Lovász and Plummer (1986) have proposed polynomial time algorithms finding a maximum cardinality stable set of a claw-free graph. Moreover, it has been believed that Minty's algorithm is the unique algorithm finding a maximum weight stable set of a claw-free graph up to date. Here we show that Minty's algorithm for the weighted version fails for some special cases, and give modifications to overcome it.

1 Introduction

Let $G = (V, E)$ be a graph with vertex-set V and edge-set E. Without loss of generality, we assume that G is simple, that is, each edge $e \in E$ is described by a pair (a, b) of two distinct vertices a and b. If $e = (a, b)$, we said that e is incident to a and b, and that a and b are endpoints of e. Two vertices are said to be *adjacent* if there is an edge incident to the vertices; otherwise to be *nonadjacent*. A subset S of V is called a *stable set* (or an independent set or a vertex packing) if any two elements of S are nonadjacent. A subset M of E is called a *matching* if no two elements of M are incident to the same vertex.

Given a weight function $w : V \to \Re$, a *maximum weight stable set* is a stable set S maximizing the sum of weights of all of its elements, $w(S) = \sum_{v \in S} w(v)$. Similarly, given a weight function $\hat{w} : E \to \Re$, a *maximum weight matching* is a matching M maximizing $\hat{w}(M) = \sum_{e \in M} \hat{w}(e)$. We will deal with the problems of finding a maximum weight stable set/matching, the so-called *the maximum weight stable set/matching problem*. Especially, if $w(v) = 1$ for all $v \in V$ ($\hat{w}(e) = 1$ for all $e \in E$), we call these the *maximum cardinality stable set/matching problem*. The maximum weight matching problem can be easily transformed to the maximum weight stable set problem by using line graphs. The line graph $\ell(G)$

of a given graph $G = (V, E)$ is a graph whose vertex set is E and whose two distinct vertices e and f are adjacent if and only if e and f have an endpoint in common in G. Since the matchings of G correspond to the stable sets of $\ell(G)$, it is easily seen that the maximum weight matching problem is a special case of the maximum weight stable set problem. However, it is well known that there is a big gap between these two problems. The maximum weight stable set problem is NP-hard, even if $w(v) = 1$ for $v \in V$ (see [8]). On the other hand, many polynomial time algorithms for the maximum weight/cardinality matching problem have been proposed, e.g., [11, 9, 5] for bipartite graphs and [2, 3, 15, 6] for general graphs. Moreover, these polynomial time algorithms have been extended to those solving more general problems, for instance, the maximum weight/cardinality stable set problem for claw-free graphs [18, 16, 14], the linear matroid parity problem [12, 13, 7, 17] and the linear delta-matroid parity problem [10].

In this paper, we will deal with the maximum weight stable set problem for claw-free graphs. We call the complete bipartite graph $K_{1,3}$ a *claw*. A graph is said to be *claw-free* if it does not contain a vertex-induced subgraph which is isomorphic to a claw (see [4] for a survey on claw-free graphs). The claw is one of the forbidden subgraphs of line graphs [1]. That is, the line graphs are claw-free, and the maximum weight stable set problem for claw-free graphs is a generalization of the maximum weight matching problem. Up to date, three polynomial time algorithms, one by Minty [16], one by Sbihi [18] and the third by Lovász and Plummer [14] have been proposed for the maximum cardinality stable set problem for claw-free graphs. Furthermore, it has been believed that Minty's algorithm is the only algorithm which can be extended to the weighted problem. This method finds a maximum weight stable set by constructing "semi-optimal" stable sets while increasing their cardinalities. Here we call a stable set S is semi-optimal if there is no stable set T such that $|T| \leq |S|$ and $w(T) > w(S)$. In order to find the next semi-optimal stable set from a current one, Minty's algorithm transforms the problem to the maximum weight matching problem by constructing a graph called the Edmonds' graph. However, the method does not always work well for the weighted problem (it is correct for the cardinality version). We will describe an example for which the algorithm fails. Furthermore, we will show that Minty's algorithm can be revised by modifying the construction of the Edmonds' graph. Hence, the maximum weight stable set problem for claw-free graphs is polynomially solvable.

Section 2 briefly explains Minty's algorithm. Section 3 gives an example for which Minty's algorithm fails and Section 4 proposes our revision.

2 Minty's algorithm

2.1 Basic idea

Here we briefly explain the basic idea of Minty's algorithm. Fix a claw-free graph $G = (V, E)$, a weight function $w : V \to \Re$ and a stable set S of G. We call the elements of S *black* and other vertices *white*. Note that a white vertex is adjacent to at most two black

vertices. A white vertex is said to be *bounded* if it is adjacent to two black vertices, *free* if it is adjacent to exactly one black vertex and otherwise *super free*.

Let x and y be different black vertices, and let W denote the set of all the bounded vertices adjacent to both x and y. If W is not empty, W is called a *wing* adjacent to x (and y).

Lemma 2.1 ([16]) *Let x and y be different black vertices and u be a bounded vertex adjacent to both x and y. If a fourth vertex v is adjacent to u, then v is adjacent to x or y.*

Proof. If the assertion is false, then $\{u, v, x, y\}$ induces a claw. □

A simple path or cycle X is called an *alternating path* or an *alternating cycle* of S, if white and black vertices appear alternately, and its white vertices form a stable set. Note that the symmetric difference of two stable sets of a claw free graph is partitioned into alternating paths and alternating cycles. We define the *weight* of a path (or cycle) by the sum of weights of its white vertices minus the sum of weights of its black vertices and denote it by $\delta(X)$. By Lemma 2.1, one can easily verify the following lemmas.

Lemma 2.2 ([16]) *A simple cycle $v_1, v_2, \ldots, v_{2\ell}, v_{2\ell+1} = v_1$ ($\ell \geq 2$) is an alternating cycle if and only if white and black vertices appear alternately, and each white vertex v_{i-1} is not adjacent to v_{i+1}.*

Lemma 2.3 ([16]) *A simple path $v_1, v_2, \ldots, v_\ell$ ($\ell \geq 1$) is an alternating path if and only if white and black vertices appear alternately, each white vertex v_{i-1} is not adjacent to v_{i+1}, neither v_1 nor v_ℓ is adjacent to two black vertices of the path, and v_1 and v_ℓ are not adjacent to each other if they are white.*

Here we denote the difference and the symmetric difference of two sets by using symbols '$-$' and '$\triangle$'. For an alternating cycle C, $S \triangle C$ is a stable set and $w(S \triangle C) = w(S) + \delta(C)$. If $\delta(C) > 0$, C is called an *augmenting cycle*. For an alternating path P, if no vertex of $S - P$ is adjacent to endpoints of P, then $S \triangle P$ is a stable set and $w(S \triangle P) = w(S) + \delta(P)$. In addition, if $\delta(P) > 0$, P is called an *augmenting path*.

We call an alternating path *white* (or *black*) if both of its endpoints are white (black), and otherwise *white-black*.

A stable set S is said to be *semi-optimal* if there is no stable set T such that $|T| \leq |S|$ and $w(T) > w(S)$. Minty's algorithm finds semi-optimal stable sets while increasing their cardinalities, until it finds a maximum weight stable set. For this purpose, white augmenting paths play an important role. Minty proved the following property.

Lemma 2.4 ([16]) *Let S be a semi-optimal stable set but not a maximum weight stable set. Then, there exists a semi-optimal stable set T such that $|T| = |S| + 1$ and $w(T) > w(S)$. For any semi-optimal stable set T with $|T| = |S| + 1$, $S \triangle T$ contains a white augmenting path having the maximum weight among the white augmenting paths for S. Conversely, if P is a white augmenting path having the maximum weight among the white augmenting paths for S, then $S \triangle P$ is a semi-optimal stable set with one more vertex than S.*

Hence Minty's algorithm can be briefly described as below.

(1) $S \leftarrow \emptyset$;
(2) Find a white augmenting path P having the maximum weight for S ;
(3) If there exists no white augmenting path for S, then stop (S is optimal) ;
(4) $S \leftarrow S \triangle P$ and return to (2).

Obviously, endpoints of a white augmenting path are super free or free. In the former case, the length of the path must be zero. (Here we define the length of a path by the number of edges.) Let us consider the latter case (the two endpoints a and b must be distinct and nonadjacent to each other). Let x_a be the black vertex adjacent to a, and x_b be the black vertex adjacent to b. If $x_a = x_b$, (a, x_a, b) is the unique white alternating path between a and b. So we can easily find all white augmenting paths of length zero or two. In the sequel, we will consider a method for finding a white augmenting path having the maximum weight between two distinct free vertices a and b which are nonadjacent and $x_a \neq x_b$. For convenience, we call such a white augmenting path an *MWWAP* (a maximum weight white augmenting path) between a and b. If one can find an MWWAP in polynomial time, Minty's algorithm is a polynomial time algorithm because the number of iterations is at most $|V|$ and because it is enough to find at most $|V|^2$ MWWAPs in each iteration.

2.2 The Edmonds' graph

We assume that a semi-optimal stable set S and free vertices a and b satisfying the above conditions are given. We first delete all the super free vertices, free vertices except a and b, and all the white vertices that are adjacent to a or b, since they never appear in any white augmenting path between a and b. The reduced graph is called an *RBS* (a reduced basic structure), including the weight function w and the semi-optimal stable set S. In the rest of this paper, we will deal with the RBS instead of a given graph G.

Vertices x_a and x_b are called *regular I*. Other black vertices are classified as follows: a black vertex is called *regular II* if it is adjacent to three or more wings, *irregular* if it is adjacent to exactly two wings, and otherwise *useless*.

Given a vertex v, let $N(v)$ denote the set of all the vertices adjacent to v. We partition $N(x_a)$ into two sets $N^1(x_a) = \{a\}$ and $N^2(x_a) = N(x_a) - \{a\}$, and define $N^1(x_b)$ and $N^2(x_b)$ similarly. Obviously, the unique element a of $N^1(x_a)$ is nonadjacent to any element of $N^2(x_a)$. This property can be extended to regular II vertices as bellow.

Lemma 2.5 ([16]) *For any regular II vertex v, $N(v)$ is uniquely (except exchanging) partitioned into $N^1(v)$ and $N^2(v)$ so that for any x and y in distinct wings adjacent to v, x is not adjacent to y if and only if one of x and y is in $N^1(v)$ and the other is in $N^2(v)$.*

Lemma 2.5 enables us easily to deal with the set of alternating paths of the RBS. Before explaining this, we define a new terminology. Let $(v_0, v_1, v_2, \ldots, v_{2\ell-1}, v_{2\ell})$ be a black alter-

nating path. If $v_2, v_4, \ldots, v_{2\ell-2}$ are all irregular, then the subpath $(v_1, v_2, \ldots, v_{2\ell-1})$ is called an *IWAP* (irregular white alternating path) between v_0 and $v_{2\ell}$.

Lemma 2.6 ([16]) *Let P be a path between a and b with a form:*

$$P = (a, x_a = x_0, P_{01}, x_1, P_{12}, x_2, P_{23}, x_3, \ldots, x_\ell, P_{\ell,\ell+1}, x_{\ell+1} = x_b, b)$$

where $x_1, \ldots, x_\ell$ are distinct regular II vertices and $P_{i,i+1}$ is an IWAP between x_i and x_{i+1}. Suppose that $v^i_{i,i+1}$ and $v^{i+1}_{i,i+1}$ are the endpoints of $P_{i,i+1}$ adjacent to x_i and x_{i+1} respectively, and that $v^i_{i,i+1} \in N^{p_i}(x_i)$ and $v^{i+1}_{i,i+1} \in N^{q_{i+1}}(x_{i+1})$, where $p_i, q_{i+1} \in \{1,2\}$. Then P is an alternating path if and only if $p_i \neq q_i$ for all $i = 1, \ldots, \ell$.

Proof. Directly from Lemmas 2.3 and 2.5. □

Lemma 2.6 guarantees that if $P_{i,i+1}$ is replaced by an arbitrary IWAP between x_i and x_{i+1} such that its endpoints belong to $N^{p_i}(x_i)$ and $N^{q_{i+1}}(x_{i+1})$ respectively, the new path is also an alternating path. Thus, in order to find an MWWAP, we must find a maximum weight IWAP between any two regular vertices x and y such that its endpoints belong to $N^p(x)$ and $N^q(y)$ for $p, q \in \{1,2\}$, if it exists. For fixed $p, q \in \{1,2\}$, the problem can be transformed to the longest path problem of an acyclic directed graph. Hence maximum weight IWAPs between regular vertices can be found in polynomial time. Then, the remaining is to determine a sequence of regular II vertices in some MWWAP between a and b. For this purpose, Minty's algorithm construct the *Edmonds' graph* $Ed(G, S, w)$ defined as below. Let $x_1, \ldots, x_r$ be all the regular (I or II) vertices of the RBS. The Edmonds' graph $Ed(G, S, w)$ has $2r + 2$ vertices labeled by x^1_i, x^2_i $(i = 1, \ldots, r)$ and $\hat{a}, \hat{b}$. Each edge of the graph is painted black or white and assigned a weight $\hat{w}(\cdot)$ in the following manner. For each regular vertex x_i, join x^1_i and x^2_i by a black edge with weight $\hat{w}((x^1_i, x^2_i)) = w(x_i)$. Join $\hat{a}$ and x^1_a by a white edge with $\hat{w}((\hat{a}, x^1_a)) = w(a)$, and join $\hat{b}$ and x^1_b by a white edge with $\hat{w}((\hat{b}, x^1_b)) = w(b)$. These two white edges correspond to white vertices a and b. For each pair of regular vertices x_i and x_j and for $p, q \in \{1,2\}$, if there exists an IWAP between x_i and x_j whose endpoints are in $N^p(x_i)$ and $N^q(x_j)$, join x^p_i and x^q_j by a white edge whose weight is the maximum weight among such IWAPs. Now we finish constructing the Edmonds' graph. We briefly explain the construction of the Edmonds' graph by using an example in Figure 1. The RBS has four regular II vertices $\{x_1, x_2, x_3, x_4\}$. For example, let us consider two black vertices x_3 and x_4. By Lemma 2.5, $N(x_3)$ and $N(x_4)$ are partitioned into $N^1(x_3) = \{g, j\}$ and $N^2(x_3) = \{f, \ell\}$, and $N^1(x_4) = \{h, k\}$ and $N^2(x_4) = \{i, m\}$, respectively. Between x_3 and x_4, there are four IWAPs (j, z_1, k), (j, z_1, m), (ℓ, z_1, k) and (ℓ, z_1, m) which join vertices belonging to distinct pairs of partitions of $N(x_3)$ and $N(x_4)$. Thus, the Edmonds' graph of the RBS has four white edges (x^1_3, x^1_4), (x^1_3, x^2_4), (x^2_3, x^1_4) and (x^2_3, x^2_4) whose weights are 10, 11, 11 and 12, respectively. Figure 2 in which the waved lines represent the black edges is the Edmonds' graph of the RBS in Figure 1.

The black edges of the Edmonds' graph are a matching. Here we recall several terminologies relative to matchings. Let us consider a graph with $2k$ vertices, a weight function

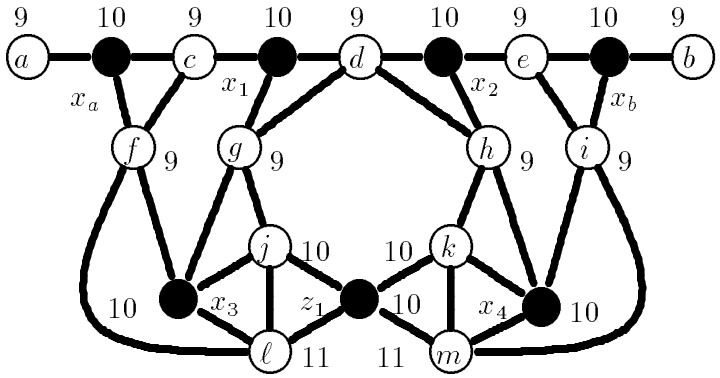

Figure 1: An example of the RBS.

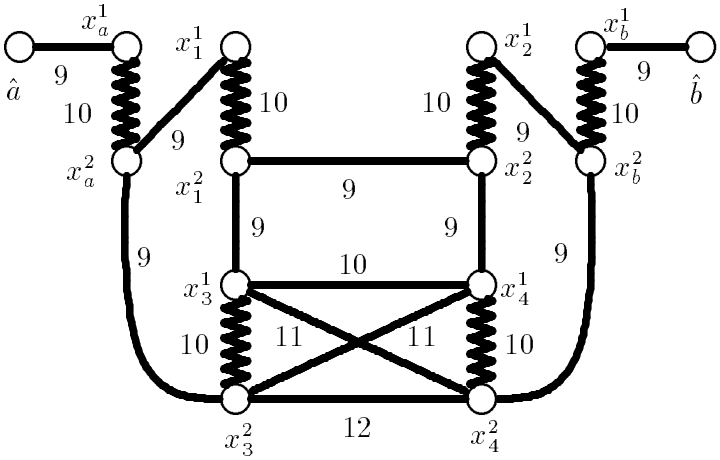

Figure 2: An example of the Edmonds' graph.

$\hat{w}$ on edges and a matching M. We call the edges in M *black* and the other edges *white*. If an edge of M is incident to a vertex v, v is said to be *matched* by M. A simple path (or cycle) is called an *alternating path* (or *alternating cycle*) of M if white and black edges appear alternately. The *weight* of the path (or cycle), denoted by $\hat{\delta}(\cdot)$, is the sum of weights of the white edges minus the sum of weights of the black edges in it. If an alternating cycle C has positive weight ($\hat{\delta}(C) > 0$), C is called an *augmenting cycle*. If an alternating path P has positive weight and no edge of $M - P$ is incident to endpoints of P, then P is called an *augmenting path*. We call an alternating path or augmenting path *white* (or *black*) if it has one more white (black) edge, and otherwise *white-black*.

From the construction of the Edmonds' graph, the following lemma holds.

Lemma 2.7 ([16]) *If there exists no white augmenting path (of edges) in the Edmonds' graph, then there exists no white augmenting path between a and b in the RBS. Moreover, any maximum weight augmenting path between $\hat{a}$ and $\hat{b}$ in the Edmonds' graph corresponds to a certain MWWAP between a and b in the RBS.*

Lemma 2.7 says that the problem finding an MWWAP can be transformed to the problem finding a maximum weight alternating path between $\hat{a}$ and $\hat{b}$ in the Edmonds' graph. In general, given a graph, a matching and a weight function on edges, the problem finding a maximum weight alternating path between two specified unmatched vertices belongs to the class of NP-hard, because the longest path problem for directed graphs is polynomially reduced to the problem. However, in some special cases, this problem can be solved in polynomial time. For instance, we suppose that the given matching M has the maximum weight among the matchings with at most $|M|$ edges. Here we call such a matching *semi-optimal* in the same way as the case of stable sets. By the following property and by using a polynomial time algorithm for the maximum weight matching problem, one can easily solve the above problem.

Lemma 2.8 *Let G be a graph with $2k$ vertices. Suppose that M is a semi-optimal matching of size $k-1$. If M has the maximum weight, then there is no white augmenting path of M. Otherwise, for any maximum weight matching M^* of G, $M \triangle M^*$ contains a maximum weight white augmenting path of M.*

For the cardinality case, i.e., for the case when $w(v) = 1$ for all vertices of a given claw-free graph, all edges of the Edmonds' graph have the same weight 1. That is, the black edges of the Edmonds' graph form a semi-optimal matching. From the above discussion, Minty's algorithm is correct for the cardinality case.

Theorem 2.9 ([16]) *Minty's algorithm find a maximum cardinality stable set of a claw-free graph in polynomial time.*

In [16], Minty claimed that the matching of black edges of the Edmonds' graph is semi-optimal and that an MWWAP in the RBS can be found in polynomial time by using the above method. However, the matching of the Edmonds' graph is not always semi-optimal. In the next section, we describe such an example.

3 An error of Minty's algorithm for the weighted case

If the black edges in the Edmonds' graph is not semi-optimal, then the Edmonds' graph must contain one of the followings:

(1) an augmenting cycle,
(2) a white-black augmenting path whose one endpoint is unmatched,
(3) a black alternating path P and a white alternating path Q such that the endpoints of Q are unmatched, P and Q are vertex-disjoint and $\hat{\delta}(P) + \hat{\delta}(Q) > 0$, where we call them an *augmenting path pair*,
(4) a black augmenting path.

However, the Edmonds' graph contains none of them except one special case.

Lemma 3.1 *The Edmonds' graph contains none of augmenting cycles of length at least* 6*, white-black augmenting paths, augmenting path pairs or black augmenting paths.*

Proof. Assume that there is an augmenting cycle C of length at least 6 in the Edmonds' graph. Each white edge of C corresponds an IWAP in the RBS. Since the length of C is greater than or equal to 6, these IWAPs do not join the same pair of regular vertices (this does not hold for cycles of length 4). By Lemma 2.5 and the construction of the Edmonds' graph, C corresponds to an alternating cycle C' in the RBS. Moreover $\delta(C') = \hat{\delta}(C)$. Hence C' is an augmenting cycle in the RBS, contradicting the assumption that the given stable set is semi-optimal.

Next let us consider the case when an augmenting path pair (P, Q) exists. In the same way as above, P and Q correspond to alternating paths P' and Q' in the RBS, respectively. Since P and Q are vertex-disjoint, P' and Q' are also vertex-disjoint. From the construction of the Edmonds' graph, the white vertices of P' (or Q') are nonadjacent to each other. Assume to the contrary that a white vertex v of Q' is adjacent to a white vertex u of P'. Since P' is a black alternating path, u is adjacent to exactly two black vertices x and y. By Lemma 2.1, v must be adjacent to either x or y. Thus, either x or y must be contained in Q'. However, this contradicts that P' and Q' are vertex-disjoint. Hence the white vertices of P' and Q' are nonadjacent to each other. The symmetric difference of the given stable set S and $(P' \cup Q')$ is a stable set of the RBS, because two endpoints of Q' are a and b and because a white vertex of P' and Q' is adjacent to none of $S - (P' \cup Q')$. Furthermore, $\delta(P') + \delta(Q') = \hat{\delta}(P) + \hat{\delta}(Q)$ holds. This contradicts that S is semi-optimal.

We can similarly show the assertion of the remaining cases. □

Unfortunately, there can exist augmenting cycles of length 4 in the Edmonds' graph, since two white edges of this cycle may correspond to some vertices in a common wing in the RBS. Let us consider the example in Figure 1. We can show the followings: there is no augmenting cycle, and the maximum weights of white-black alternating paths whose white endpoint is a or b, white alternating paths between a and b and black alternating paths are -1, 6 and -8, respectively. Hence the black vertices form the unique semi-optimal stable

set of cardinality 7 and of weight 70, and it must be found after 7 iterations in Minty's algorithm. The Edmonds' graph in Figure 2 contains an augmenting cycle $(x_3^1, x_3^2, x_4^2, x_4^1, x_3^1)$ of length 4 and of weight 2. Minty's algorithm finds a maximum weight matching M^* of the Edmonds' graph, e.g., $M^* = \{(\hat{a}, x_a^1), (x_a^2, x_1^1), (x_1^2, x_2^2), (x_b^2, x_2^1), (\hat{b}, x_b^1), (x_3^1, x_4^1), (x_3^2, x_4^2)\}$ which is not unique. Then the symmetric difference of the set of black edges and M^* consists of a white augmenting path $(\hat{a}, x_a^1, x_a^2, x_1^1, x_1^2, x_2^2, x_2^1, x_b^2, x_b^1, \hat{b})$ of weight 5 and an augmenting cycle $(x_3^1, x_3^2, x_4^2, x_4^1, x_3^1)$ of weight 2. Thus, the algorithm finds a stable set of cardinality 8 and of weight 75 by using this white augmenting path. However, there exists a white augmenting path $(\hat{a}, x_a^1, x_a^2, x_3^2, x_3^1, x_4^1, x_4^2, x_b^2, x_b^1, \hat{b})$ of weight 6, which is an MWWAP (but is not unique). So we fail to find a semi-optimal stable set of cardinality 8 and of weight 76, which is a maximum weight stable set.

4 Revised Edmonds' graphs

We will overcome an error of Minty's algorithm by revising the Edmonds' graph so that the assertions of Lemma 2.7 is preserved and that the black edges form a semi-optimal matching in the revised Edmonds' graph. Then, from Lemma 2.8, the Minty's algorithm using the revised Edmonds' graph finds a maximum stable set in polynomial time. Lemma 3.1 suggests that it can be done by eliminating all augmenting cycles of length 4 from the Edmonds' graph preserving all alternating paths between $\hat{a}$ and $\hat{b}$.

Since the Edmonds' graph has no alternating cycle containing (x_a^1, x_a^2) or (x_b^1, x_b^2), let us fix distinct regular II vertices x_i and x_j. Suppose that P_{pq} denotes the maximum weight IWAP corresponding to the edge (x_i^p, x_j^q) in the Edmonds' graph if the edge exists, for $p, q \in \{1, 2\}$. We consider the following cases when an augmenting cycle of length 4 exists in the Edmonds' graph:

Case A: there exist both P_{11} and P_{22}, and $\hat{\delta}(P_{11}) + \hat{\delta}(P_{22}) > w(x_i) + w(x_j)$, or

Case B: there exist both P_{12} and P_{21}, and $\hat{\delta}(P_{12}) + \hat{\delta}(P_{21}) > w(x_i) + w(x_j)$.

We first discuss an easy situation. A wing W is said to be *irregular reachable* to a regular vertex x if there exist an integer $\ell \geq 1$, distinct irregular vertices $z_1, \ldots, z_{\ell-1}$ and distinct wings $W_1 (= W), W_2, \ldots, W_\ell$ such that W_1 is adjacent to z_1 and W_k is adjacent to z_{k-1} and z_k for $k = 2, \ldots, \ell$, where $z_\ell = x$. Let $W(x_i, x_j)$ denote the union of all the wings that are irregular reachable to both x_i and x_j.

Lemma 4.1 *If $N^1(x_j) \subseteq W(x_i, x_j)$, then any white alternating path between a and b in the RBS passes through neither P_{12} nor P_{22}. That is, we can delete the edges (x_i^1, x_j^2) and (x_i^2, x_j^2) from the Edmonds' graph. Similarly if $N^2(x_j) \subseteq W(x_i, x_j)$, we can delete (x_i^1, x_j^1) and (x_i^2, x_j^1). If $N^1(x_i) \subseteq W(x_i, x_j)$, we can delete (x_i^2, x_j^1) and (x_i^2, x_j^2). If $N^2(x_i) \subseteq W(x_i, x_j)$, we can delete (x_i^1, x_j^1) and (x_i^1, x_j^2).*

Proof. Assume to the contrary that a white alternating path from a to b passes x_i, P_{12} (or P_{22}) and x_j. Before x_j, it passes a vertex in $N^2(x_j)$. Hence it must pass a vertex in $N^1(x_j)$ after x_j. However, since $N^1(x_j) \subseteq W(x_i, x_j)$, it must pass x_i again. □

Under the condition of Lemma 4.1, we can delete an augmenting cycle of length 4.

In the sequel, we suppose that none of $N^1(x_i)$, $N^2(x_i)$, $N^1(x_j)$ and $N^2(x_j)$ belongs to $W(x_i, x_j)$. We now consider Case A. (It is symmetric to consider Case B.)

Lemma 4.2 *Paths P_{11} and P_{22} have the same set of irregular vertices (which may be empty).*

Proof. If P_{11} and P_{22} have no irregular vertex, the assertion clearly holds. Suppose that P_{11} or P_{22} contains a irregular vertex. Assume to the contrary that the assertion does not hold. Since P_{11} and P_{22} has no regular vertex, the sets of irregular vertices of these IWAPs have no intersection. Thus, all white vertices of P_{11} and P_{22} belong to distinct wings. Let us consider the cycle $C = (x_i, P_{11}, x_j, P_{22}, x_i)$. By Lemmas 2.2, 2.5 and the assumption $\delta(C) = \hat{\delta}(P_{11}) + \hat{\delta}(P_{22}) - w(x_i) - w(x_j) > 0$, C is an augmenting cycle in the RBS. However this contradicts the fact that the given stable set is semi-optimal. □

Lemma 4.2 says that P_{11} and P_{22} can be represented as follows:

$$P_{11} = (y^1_1, z_1, y^1_2, z_2, \ldots, y^1_{\ell-1}, z_{\ell-1}, y^1_\ell) \quad \text{and} \quad P_{22} = (y^2_1, z_1, y^2_2, z_2, \ldots, y^2_{\ell-1}, z_{\ell-1}, y^2_\ell).$$

Here $z_1, \ldots, z_{\ell-1}$ are irregular vertices, both y^1_k and y^2_k are in a common wing W_k for $k = 1, \ldots, \ell$, $y^1_1 \in N^1(x_i)$, $y^2_1 \in N^2(x_i)$, $y^1_\ell \in N^1(x_j)$ and $y^2_\ell \in N^2(x_j)$.

Lemma 4.3 *There exists k such that $2 \le k \le \ell - 1$ and $y^1_k = y^2_k$, or there exists k such that $1 \le k \le \ell$ and y^1_k is adjacent to y^2_k.*

Proof. Suppose that $y^1_k \neq y^2_k$ for all k $(2 \le k \le \ell - 1)$. Note that $y^1_1 \neq y^2_1$ and $y^1_\ell \neq y^2_\ell$ since $N^1(x_i) \cap N^2(x_i) = \emptyset = N^1(x_j) \cap N^2(x_j)$. Let $z_0 = x_i$ and $z_\ell = x_j$. Since $0 < \hat{\delta}(P_{11}) + \hat{\delta}(P_{22}) - w(x_i) - w(x_j) = \sum_{k=1}^{\ell}(w(y^1_k) - w(z_{k-1}) + w(y^2_k) - w(z_k))$, we can choose k such that $w(y^1_k) - w(z_{k-1}) + w(y^2_k) - w(z_k) > 0$. If y^1_k and y^2_k are not adjacent, $C = \{y^1_k, z_{k-1}, y^2_k, z_k\}$ is an augmenting cycle in the RBS, contradicting the assumption that the given stable set is semi-optimal. Hence y^1_k and y^2_k must be adjacent. □

We first consider the case when $\ell = 1$.

Lemma 4.4 *Suppose that $\ell = 1$. Then, x_i is adjacent to precisely three wings: W_1, a wing $W^1_i \subseteq N^1(x_i)$ and a wing $W^2_i \subseteq N^2(x_i)$, and x_j is also adjacent to precisely three wings: W_1, $W^1_j \subseteq N^1(x_j)$ and $W^2_j \subseteq N^2(x_j)$. Moreover, W^1_i and W^1_j are adjacent to a common black vertex v_1, and W^2_i and W^2_j are adjacent to a common black vertex v_2.*

Proof. Let us fix a vertex $u \in N^1(x_i) - W_1 (\supseteq N^1(x_i) - W(x_i, x_j) \neq \emptyset)$. Suppose that v_1 is the black vertex adjacent to u other than x_i. Let w be any vertex in $N^1(x_j) - W_1 (\neq \emptyset)$. By Lemma 4.3, y^1_1 is adjacent to y^2_1. Lemma 2.5 says that y^1_1 is adjacent to both u and w, but y^2_1 is adjacent to neither u nor w. Thus w must be adjacent to u, since otherwise $\{y^1_1, y^2_1, u, w\}$ induces a claw. Since $w \notin W_1$, w is not adjacent to x_i. Then, from Lemma 2.1, w must be adjacent to v_1. Now we can conclude that all the vertices in $N^1(x_j) - W_1$ belong to one wing W^1_j which is adjacent to both x_j and v_1. Similarly, one can prove the other assertions. □

Lemma 4.5 *If $\ell = 1$, any white alternating path from a to b passes through neither P_{11} nor P_{22}. Hence we can delete the edges (x_i^1, x_j^1) and (x_i^2, x_j^2) from the Edmonds' graph. Moreover, Case B does not occur for the same pair x_i and x_j.*

Proof. If a white alternating path from a to b passes through x_i, P_{11} (or P_{22}) and x_j, it must pass through v_2 (or v_1) twice, a contradiction.

Assume to the contrary that Case B occurs. By Lemma 4.4, P_{12} and P_{21} pass through neither v_1 nor v_2, even if v_1 and v_2 are irregular. That is, P_{12} and P_{21} belong to the wing W_1. The parallel discussion to Case A concludes the "twisting" version of Lemma 4.4, but this contradicts the assertion of Lemma 4.4 for Case A. □

Finally we consider the case when $\ell \geq 2$.

Lemma 4.6 *If $\ell \geq 2$, the followings hold.*

(1) There exists k such that $2 \leq k \leq \ell - 1$ and $y_k^1 = y_k^2$, or there exists k such that $1 \leq k \leq \ell - 1$, y_k^1 is not adjacent to y_{k+1}^2 and y_k^2 is not adjacent to y_{k+1}^1.

(2) For such k, let

$$P_{11i} = (y_1^1, z_1, y_2^1, \ldots, z_{k-1}, y_k^1), \qquad P_{11j} = (y_{k+1}^1, z_{k+1}, \ldots, y_{\ell-1}^1, z_{\ell-1}, y_\ell^1),$$
$$P_{22i} = (y_1^2, z_1, y_2^2, \ldots, z_{k-1}, y_k^2) \quad \text{and} \quad P_{22j} = (y_{k+1}^2, z_{k+1}, \ldots, y_{\ell-1}^2, z_{\ell-1}, y_\ell^2),$$

and let $P'_{12} = (P_{11i}, z_k, P_{22j})$ and $P'_{21} = (P_{22i}, z_k, P_{11j})$. Then $\hat{\delta}(P'_{12}) + \hat{\delta}(P'_{21}) = \hat{\delta}(P_{11}) + \hat{\delta}(P_{22})$, P'_{12} is an IWAP between $N^1(x_i)$ and $N^2(x_j)$, and P'_{21} is an IWAP between $N^2(x_i)$ and $N^1(x_j)$.

(3) $\hat{\delta}(P_{11}) + \hat{\delta}(P_{22}) = \hat{\delta}(P_{12}) + \hat{\delta}(P_{21})$.

(4) $\hat{\delta}(P'_{12}) = \hat{\delta}(P_{12})$ and $\hat{\delta}(P'_{21}) = \hat{\delta}(P_{21})$.

Proof.

(1): Suppose that $y_k^1 \neq y_k^2$ for all k with $2 \leq k \leq \ell - 1$. Recall that $y_1^1 \neq y_1^2$ and $y_\ell^1 \neq y_\ell^2$.

Suppose that y_1^1 is adjacent to y_1^2. Fix a vertex $u \in N^1(x_i) - W(x_i, x_j)$. Lemma 2.5 says that u is adjacent to y_1^1, but not to y_1^2. Vertices y_1^2 and y_2^2 are not adjacent to each other, since they belong to P_{22}. Vertex u must be nonadjacent to y_2^2, since otherwise, from Lemma 2.1, u must be adjacent to either z_1 or z_2, a contradiction to $u \notin W(x_i, x_j)$. Thus y_1^1 is not adjacent to y_2^2; otherwise $\{y_1^1, y_1^2, y_2^2, u\}$ induces a claw. Similarly, y_1^2 is not adjacent to y_2^1. Hence $k = 1$ satisfies the latter assertion of (1).

If y_ℓ^1 is adjacent to y_ℓ^2, one can analogously show that $k = \ell - 1$ satisfies the latter assertion.

Finally let us consider the other case. Lemma 4.3 guarantees that there is k such that $2 \leq k \leq \ell - 1$ and y_k^1 is adjacent to y_k^2, but y_{k-1}^1 is not adjacent to y_{k-1}^2. Obviously, z_{k-1} is adjacent to y_{k-1}^1, y_{k-1}^2, and y_k^1. Vertices y_{k-1}^1 and y_k^1 are nonadjacent since they belong to P_{11}. Thus y_{k-1}^2 is adjacent to y_k^1, since otherwise $\{z_{k-1}, y_{k-1}^1, y_{k-1}^2, y_k^1\}$ induces a claw. Vertices y_{k-1}^2, y_k^2 and y_{k+1}^2 are mutually nonadjacent since they belong to P_{22}. Hence y_k^1 must be nonadjacent to y_{k+1}^2. Similarly, we can prove that y_k^2 is not adjacent to y_{k+1}^1.

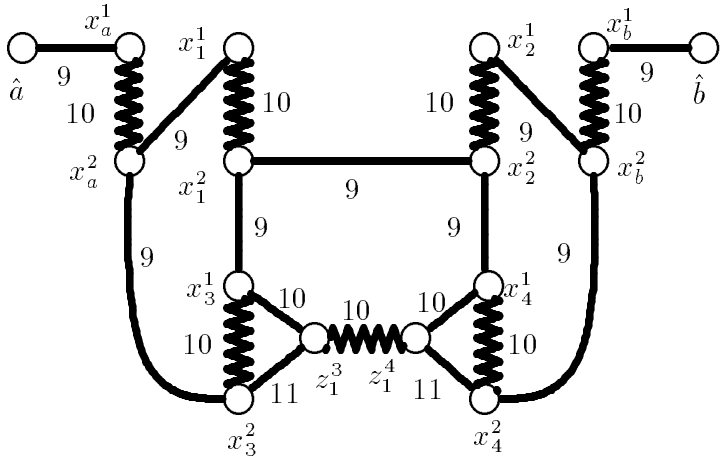

Figure 3: The revised Edmonds' graph of the RBS in Figure 1.

(2): Obviously the assertion holds by (1) and Lemma 2.3.
(3): By the definition of P_{12} and P_{21}, $\hat{\delta}(P'_{12}) \leq \hat{\delta}(P_{12})$ and $\hat{\delta}(P'_{21}) \leq \hat{\delta}(P_{21})$. Hence $\hat{\delta}(P_{11}) + \hat{\delta}(P_{22}) \leq \hat{\delta}(P_{12}) + \hat{\delta}(P_{21})$. Assume to the contrary that $\hat{\delta}(P_{11}) + \hat{\delta}(P_{22}) < \hat{\delta}(P_{12}) + \hat{\delta}(P_{21})$ holds. By the parallel discussion about P_{12} and P_{21}, and by Lemma 4.5, if P_{12} and P_{21} are of length 0, then Case A does not occur, a contradiction. Thus, lengths of P_{12} and P_{21} are greater than or equal to two (i.e., $\ell \geq 2$). From the assertions (1) and (2) of this lemma for P_{12} and P_{21}, there exist P'_{11} and P'_{22} such that $\hat{\delta}(P'_{11}) + \hat{\delta}(P'_{22}) = \hat{\delta}(P_{12}) + \hat{\delta}(P_{21}) > \hat{\delta}(P_{11}) + \hat{\delta}(P_{22})$. However, this implies $\hat{\delta}(P'_{11}) > \hat{\delta}(P_{11})$ or $\hat{\delta}(P'_{22}) > \hat{\delta}(P_{22})$, contradicting the maximality of P_{11} or P_{22}.
(4): It follows from $\hat{\delta}(P'_{12}) \leq \hat{\delta}(P_{12})$, $\hat{\delta}(P'_{21}) \leq \hat{\delta}(P_{21})$ and $\hat{\delta}(P'_{12}) + \hat{\delta}(P'_{21}) = \hat{\delta}(P_{12}) + \hat{\delta}(P_{21})$. □

Summing up the above discussion, we propose a revision of the Edmonds' graph. The above discussion deals with three cases, Lemmas 4.1, 4.5 and 4.6. In the first two cases, elimination of augmenting cycles can be easily done by deleting edges. Lemmas 4.1 and 4.5 guarantee that the alternating paths between $\hat{a}$ and $\hat{b}$ are preserved and no new alternating path is generated. Let us consider the last case. We revise the Edmonds' graph as below (see Figure 3):

- delete four edges (x_i^1, x_j^1), (x_i^2, x_j^2), (x_i^1, x_j^2) and (x_i^2, x_j^1) (Lemma 4.6 guarantees the existence of these four edges),
- add two new vertices z_k^i and z_k^j, join z_k^i and z_k^j by a black edge and assign its weight $\hat{w}((z_k^i, z_k^j))$ to be $w(z_k)$, where k satisfies the conditions of Lemma 4.6,
- add four white edges (x_i^1, z_k^i), (x_i^2, z_k^i), (x_j^1, z_k^j) and (x_j^2, z_k^j), and assign their weights to be $\hat{w}((x_i^1, z_k^i)) = \hat{\delta}(P_{11i})$, $\hat{w}((x_i^2, z_k^i)) = \hat{\delta}(P_{22i})$, $\hat{w}((x_j^1, z_k^j)) = \hat{\delta}(P_{11j})$ and $\hat{w}((x_j^2, z_k^j)) = \hat{\delta}(P_{22j})$.

All alternating paths passing through black edges (x_i^1, x_i^2) and (x_j^1, x_j^2) can be preserved by our revision, because (x_i^p, x_j^q) in the original Edmonds' graph ($p, q \in \{1, 2\}$) is interpreted by

the path $\{x_i^p, z_k^i, z_k^j, x_j^q\}$ in the revised Edmonds' graph. Furthermore, Lemma 4.6 guarantees that weights of these four edges are equal to those of such four paths, respectively.

Lemma 4.7 *For each pair of regular II vertices x_i and x_j, if Case A or Case B occurs, we apply the above revision. Then, the black edges in the revised Edmonds' graph are a semi-optimal matching.*

Proof. Obviously, all the augmenting cycles of length 4 are eliminated and no new augmenting cycle of length 4 is generated. In the same way as the proof of Lemma 3.1, we can prove that the revised Edmonds' graph contain no augmenting cycle of length at least 6, no white-black augmenting path, no augmenting path pair or no black augmenting path. □

Theorem 4.8 *A maximum weight stable set of a claw-free graph can be found in polynomial time in the numbers of vertices and edges.*

Proof. The number of alternating cycles of length 4 is polynomially bounded. For each pair of regular II vertices x_i and x_j, $W(x_i, x_j)$ can be found in polynomial time. Hence the revised Edmonds' graph can be constructed in polynomial time. □

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