A Simple Derivation of Josephson Formulae in Superconductivity

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Abstract

A simple and general derivation of Josephson formulae for the tunneling currents is presented on the basis of Sewell's general formulation of superconductivity in use of off-diagonal long range order (ODLRO).

1 Introduction

According to the pioneering paper [1] by Haag on the BCS theory of superconductivity [2], the general meaning of Cooper pair condensates can be understood as a "variable at infinity" [3] which is based on the cluster property valid in any thermodynamic *pure phases* described by factor states[1]. Along this line, Sewell [4] takes, as a general characterization of the superconducting BCS states $\langle\cdot\rangle_{BCS}$ with condensed Cooper pairs, the off-diagonal long range order (ODLRO):

$$|\langle\psi(X_1+\frac{\xi_1}{2})\psi(X_1-\frac{\xi_1}{2})\psi^\dagger(X_2-\frac{\xi_2}{2})\psi^\dagger(X_2+\frac{\xi_2}{2})\rangle_{BCS} - \Psi(X_1,\xi_1)\Psi^*(X_2,\xi_2)| \underset{|\vec{X}_1-\vec{X}_2|\to\infty}{\to} 0. \tag{1}$$

Here $\psi(x)$ denotes the second-quantized non-relativistic electron field obeying the canonical anticommutation relation

$$\{\psi(x),\ \psi^\dagger(y)\} = \delta^3(x-y), \tag{2}$$

[1] A *factor* state means a state such that the (GNS) representation of the algebra of physical quantities corresponding to it has a trivial centre.

and the so-called "macroscopic wave function" $\Psi(X,\xi) = \langle\psi(X+\frac{\xi}{2})\psi(X-\frac{\xi}{2})\rangle_{BCS}$ of a Cooper pair should be non-vanishing in the limit of spatial infinity $|\vec{X}| \to \infty$. According to this formulation, he succeeded in giving a general proof of the validity of Meissner effect ($\vec{B} = 0$ inside of superconductor) [4].

Inspired by these attempts to understand the essential features of superconductivity in a model-independent way, I try here to present a simple and general derivation of Josephson formulae[2] [5] which describe the tunnelling currents (*dc* and/or *ac*) caused by the phase differences between two superconductors separated by a thin barrier of the insulator. In contrast to the traditional derivations based on the tunneling Hamiltonian [5], we see that they are just a simple and direct consequence of the above ODLRO and of the fact that the energy contained in one side of the junction gives a non-trivial response to the global gauge transformation caused by the presence of phase difference.

2 Simple Derivation of Josephson Formulae

Although a completely model-independent approach is desirable, the present discussion requires the postulate of the standard BCS Hamiltonian [2, 1] arising effectively from the electron-phonon coupling:

$$H_{BCS}(\Lambda;w) = \int_\Lambda dx[\frac{1}{2m}\vec{\nabla}\psi^\dagger(x)\cdot\vec{\nabla}\psi(x) - \mu\psi^\dagger(x)\psi(x)]$$
$$+\frac{1}{|\Lambda|}\int_\Lambda dx\int_\Lambda dy\int_\Lambda dz\int_\Lambda du\ w(x,y,z,u)\psi^\dagger(x)\psi^\dagger(y)\psi(u)\psi(z), \qquad (3)$$

where m is the mass of an electron, μ the chemical potential of electrons in the bulk superconductor occupying a macroscopically extended spatial region Λ with volume $|\Lambda|$. To exhibit the essence, we use here a simple idealized picture of weakly coupled superconductors placed in two spatial regions Λ_1, Λ_2 ($\subset \Lambda$) separated by a Josephson junction regarded ideally as a *phase boundary* $W \equiv \partial\Lambda_1 = \partial\Lambda_2$. In view of the wide applicability of BCS model (at least in non high T_c cases), the possible differences in the properties of superconductors in Λ_1 and in Λ_2 are supposed to be absorbed in the different choices of the potential functions $w_{\Lambda_1}(x,y,z,u) \equiv w(x,y,z,u)$ for $x,y,z,u \in \Lambda_1$ and $w_{\Lambda_2}(x,y,z,u) \equiv w(x,y,z,u)$ for $x,y,z,u \in \Lambda_2$. Aside from this freedom, we understand that the dynamics of superconductors are described universally by the above H_{BCS} and that the differences of the

[2] While this issue is treated in [4], the arguments there are not clear or explicit enough.

realized thermodynamic phases are all reduced to those in the choice of states $\langle\cdot\rangle$.

Here a remark need be added on a subtle point: To give a precise meaning to the term "thermodynamic *phases*", one should consider the theory in the *thermodynamic limit* with volume tending to infinity. Namely, the sizes of regions Λ_1, Λ_2 (of course, finite on the macroscopic scale) should be treated as "infinitely large" according to the scale of microscopic interactions, and the location of the junction should be supposed to be at "spatial infinity" far away from (the centre of) Λ_1 in this scale. Although such expressions as this may sound quite naive and vague lacking in the mathematical rigour, it is possible to give a mathematically precise meaning to it in a non-standard analytic framework as will be discussed briefly in the next section.

The next essential ingredient is the very definition of the tunneling electric current flowing through the barrier. While our system does not have a *locally conserved* electric current owing to the presence of non-local coupling in Eq.(3), the conservation of electric current is still meaningful in the following sense: We define the electric charge $Q(\Lambda)$ in a spatial region Λ by

$$Q(\Lambda) = -|e| \int_{\Lambda} dx\ \psi^{\dagger}(x)\psi(x), \tag{4}$$

which is conserved in the sense of

$$[H_{BCS}(\Lambda; w),\ Q(\Lambda)] = 0, \tag{5}$$

and which generates the electric global $U(1)$-gauge transformation:

$$[iQ(\Lambda),\ \psi(x)] = i|e|\psi(x), \quad [iQ(\Lambda),\ \psi^{\dagger}(x)] = -i|e|\psi^{\dagger}(x), \tag{6}$$

where $e = -|e|$ is the unit of electric charge. In the situation with $\Lambda = \Lambda_1 \cup \Lambda_2$ $(\cup W)$, the tunneling current J between the two regions Λ_1 and Λ_2 is defined by

$$J = \frac{d}{dt}\langle Q(\Lambda_1)\rangle_{BCS} = \langle [iH_{BCS}(\Lambda),\ Q(\Lambda_1)]\rangle_{BCS}. \tag{7}$$

Note that J cannot be non-vanishing without the presence of the region Λ_2 outside of Λ_1 in view of the simple equality Eq.(5) applied to $\Lambda = \Lambda_1$. (In a sense Eq.(7) can be viewed as the basis of heuristic expression $dN/dt = \partial H/\partial\theta$ discussed in the number-phase picture of Ginzburg-Landau theory.)

In view of the local (anti-)commutativity following from Eq.(2), the right-hand side of (7) can be reduced to

$$J = \langle [iH_{12},\ Q(\Lambda_1)]\rangle_{BCS} = -\langle [iQ(\Lambda_1),\ H_{12}]\rangle_{BCS}. \tag{8}$$

Here, H_{12} appears in the following decomposition of $H_{BCS}(\Lambda; w)$ corresponding to that of the spatial region Λ into Λ_1 and Λ_2 (separated by W):

$$H_{BCS}(\Lambda; w) = H_{BCS}(\Lambda_1; \frac{|\Lambda_1|}{|\Lambda|}w) + H_{BCS}(\Lambda_2; \frac{|\Lambda_2|}{|\Lambda|}w) + H_{12}, \tag{9}$$

according to which it can be written explicitly as

$$H_{12} = \frac{1}{|\Lambda|}\sum_{\{i,j,k,l\}=\{1,2\}}\int_{\Lambda_i}dx\int_{\Lambda_j}dy\int_{\Lambda_k}dz\int_{\Lambda_l}du\ w(x,y,z,u)\psi^\dagger(x)\psi^\dagger(y)\psi(u)\psi(z). \tag{10}$$

Then the commutator in Eq.(8) is calculated as

$$-\langle[iQ(\Lambda_1),\ H_{12}]\rangle_{BCS} = i|e|\frac{1}{|\Lambda|}\sum_{\{i,j,k,l\}=\{1,2\}}\int_{\Lambda_i}dx\int_{\Lambda_j}dy\int_{\Lambda_k}dz\int_{\Lambda_l}du$$
$$\times\ w(x,y,z,u)(-\delta_{i1}-\delta_{j1}+\delta_{k1}+\delta_{l1})\langle\psi^\dagger(x)\psi^\dagger(y)\psi(u)\psi(z)\rangle_{BCS}. \tag{11}$$

To give the precise meaning of the "macroscopic wave function" $\Psi(X,\xi)$ in ODLRO (??), we should now consider the thermodynamic limit of $\Lambda, \Lambda_1, \Lambda_2 \longrightarrow \infty$, in which we assume the ratio $|\Lambda_1|/|\Lambda_2|$ is kept fixed. When we evaluate the right-hand side of (8) in this situation, the contributions of terms coming from the region Λ_2 can be replaced by their expectation value in the BCS state $\langle\cdot\rangle_{BCS}$ because of the cluster property justified by the understanding that the boundary W is infinitely far away in this limit, $\Lambda_1, \Lambda_2 \longrightarrow \infty$ with $|\Lambda_1|/|\Lambda_2|$ kept fixed:

$$\begin{aligned}&|\langle\psi^\#(x_1^{(1)})\cdots\psi^\#(x_{k_1}^{(1)})\psi^\#(x_1^{(2)})\cdots\psi^\#(x_{k_2}^{(2)})\rangle_{BCS}\\ &-\langle\psi^\#(x_1^{(1)})\cdots\psi^\#(x_{k_1}^{(1)})\rangle_{BCS}\langle\psi^\#(x_1^{(2)})\cdots\psi^\#(x_{k_2}^{(2)})\rangle_{BCS}|\\ &\underset{\Lambda_1,\Lambda_2\longrightarrow\infty,\ |\Lambda_1|/|\Lambda_2|:fixed}{\longrightarrow} 0,\end{aligned} \tag{12}$$

where $x_1^{(1)},\cdots,x_{k_1}^{(1)} \in \Lambda_1$, $x_1^{(2)},\cdots,x_{k_2}^{(2)} \in \Lambda_2$ and $\psi^\# = \psi$ or $\psi^\dagger$. Since the expectation values of odd powers of fermionic operators ψ, $\psi^\dagger$ vanishes[3], the only contributions surviving this limit to the above integrand $\chi_{\Lambda_i}(x)\chi_{\Lambda_j}(y)\chi_{\Lambda_k}(z)\chi_{\Lambda_l}(u)(-\delta_{i1}-\delta_{j1}+\delta_{k1}+\delta_{l1})\langle\psi^\dagger(x)\psi^\dagger(y)\psi(u)\psi(z)\rangle_{BCS}$ come from the cases with $i=j=1, k=l=2$ or $i=j=2, k=l=1$:

$$\begin{aligned}&\sum_{\{i,j,k,l\}=\{1,2\}}\chi(x,y,z,u)(-\delta_{i1}-\delta_{j1}+\delta_{k1}+\delta_{l1})\langle\psi^\dagger(x)\psi^\dagger(y)\psi(u)\psi(z)\rangle_{BCS}\\ &\underset{\Lambda_1,\Lambda_2\longrightarrow\infty,\ |\Lambda_1|/|\Lambda_2|:fixed}{\longrightarrow}\quad\sum_{i=j\neq k=l}\chi(x,y,z,u)(-\delta_{i1}-\delta_{j1}+\delta_{k1}+\delta_{l1})\\ &\qquad\times\langle\psi^\dagger(x)\psi^\dagger(y)\rangle_{BCS}\langle\psi(u)\psi(z)\rangle_{BCS}\ ,\end{aligned} \tag{13}$$

[3]This holds under the moderate assumption of spatial homogeneity at infinity [6]

where $\chi(x,y,z,u) \equiv \chi_{\Lambda_i}(x)\ \chi_{\Lambda_j}(y)\chi_{\Lambda_k}(z)\chi_{\Lambda_l}(u)$ with χ_Λ being the indicator function of a set Λ. If we assume the *almost* spatial homogeneity[4] in each of regions Λ_1, Λ_2 and if the potential $w(x,y,z,u)$ between electron pairs can be assumed to be a real quantity, we can put

$$\langle\psi(x)\psi(y)\rangle_{BCS} \simeq \begin{cases} |\Psi_1|e^{i\theta_1}\ (x,y \in \Lambda_1), \\ |\Psi_2|e^{i\theta_2}\ (x,y \in \Lambda_2), \end{cases} \tag{14}$$

from which we obtain the desired result for the *dc*-Josephson current by picking out the phase factors from the integrands:

$$J \simeq \text{constant} \times (e^{i(\theta_1-\theta_2)} - e^{-i(\theta_1-\theta_2)}) \propto \sin(\Delta\theta), \tag{15}$$

where $\Delta\theta \equiv \theta_1 - \theta_2$ is the phase difference of Cooper pairs across the junction.

Once this is obtained, it is straightforward to derive the formula for *ac*-Josephson current applied to the situation with voltage gap V across the junction by replacing the above $\Delta\theta$ with $\Delta\theta + 2eVt$. While the postulated BCS Hamiltonian (3) with non-local interactions is not fully compatible with a local gauge invariant coupling of the system with spatially varying external electromagnetic field, it still allows a coupling with V in a way invariant under local gauge transformations with only temporal dependence. As argued in [4], this gauge freedom allows us to treat the situation with voltage gap V simply by performing a time-dependent local gauge transformation:

$$A^\mu = (\phi,\ \vec{A} = 0) \longrightarrow (\phi + \frac{\partial}{\partial t}(Vt) = V,\ \vec{A} = 0); \tag{16}$$

$$\psi(x,t) \longrightarrow e^{ieVt}\psi(x,t), \tag{17}$$

which causes the above shift $\Delta\theta \longrightarrow \Delta\theta + 2eVt$ of the phase difference. Thus we have the formula $J_{ac} \propto \sin(\Delta\theta + 2eVt)$ for *ac*-Josephson current.

In deriving Eq.(15) we need the spatial homogeneity to extract the factor of phase difference. Although the above kind of qualitative discussion does not allow us to determine the precise coefficient, we can extract the contribution to the energy density of the phase difference at the boundary located infinitely far away in the similar way to the above, which gives us

$$\frac{\langle H_{12}\rangle_{BCS}}{|\Lambda|} \underset{\Lambda_1,\Lambda_2 \longrightarrow \infty,\ |\Lambda_1|/|\Lambda_2|:fixed}{\longrightarrow} \text{constant} \times \cos(\Delta\theta). \tag{18}$$

[4]This can be formulated without difficulty in the non-standard analytic framework mentioned in Sect.3 in such a way that suitable combinations of quantities involving the deviations from the homogeneity are infinitesimal.

If the coefficient of $\cos(\Delta\theta)$ is assured to be of negative sign, this guarantees the self-consistency of the postulate that the phase of Cooper pair condensates in a superconductor should be spatially homogeneous in favour of $\Delta\theta = 0$ in the absence of such a constraint to maintain the phase difference as the barrier. To verify this consistency problem in a more satisfactory way, it seems necessary to confront a challenging problem of how one can justify the notion of a *point-like order parameter* $\Psi(x)$ of Cooper pairs, which appears in the Ginzburg-Landau phenomenological approach [7], and which is crucial for discussing the Type II superconductivity involving in an essential way the spatial inhomogeneity and the local gauge invariance problem.[5]

3 Discussion

Although we refrain from a systematic explanation, we comment here how the method of non-standard analysis [8] can be useful in describing the situation with infinitely large regions Λ_1, Λ_2 separated by a boundary W at infinity. What is important is that it allows us to treat both the finite volume theory and the infinite volume one simultaneously *without disconnecting* the two approaches. Before introducing the distinctions among finite, infinity and infinitesimal (at the syntactic level of *internal objects*), everything looks as if we were in the finite volume theory, but, once such distinctions are introduced (by the interpretation in a *non-standard model*) by regarding $|\Lambda_1|$ and $|\Lambda_2|$ as *infinite numbers* (whose ratio is kept finite), the infinite volume theory is seen to be contained in the former, being extracted through the procedure of taking *finite parts* of quantities which throws away all the infinitesimals such as $1/|\Lambda_1|$. In the situations with only one thermodynamic phase, this kind of treament does not make much difference from the usual one, because the procedure of extracting finite parts is more or less equivalent to taking the thermodynamic limit. In the present situation with two infinitely large regions Λ_1 and Λ_2, however, we have still "another world" in Λ_2 *beyond the infinitely distant* boundary W of infinitely extended Λ_1. Whereas this situation seems difficult to be accommodated in the usual formulation, it can be described without difficulty in the framework mentioned here, where all the infinities and infinitesimals are fully legitimate quantities. Moreover, all such limiting or approximate relations as Eqs.(??), (12), (13), (14), (15), (18) are replaced by simple algebraic equivalence relations modulo infinitesimals, in which one of its conceptual advantages can be found.

[5] This sort of situations may be the very places where the conceptual advantages of non-standard analytic formulation should be exhibited.

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