MONODROMY AT INFINITY AND THE WEIGHTS OF COHOMOLOGY

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Abstract. We show that the size of the Jordan blocks with eigenvalue one of the monodromy at infinity is estimated in terms of the weights of the cohomology of the total space and a general fiber.

Let $f : X \to S$ be a morphism of complex algebraic varieties with relative dimension n. Assume S is a smooth curve. Let U be a dense open subvariety of S such that the $H^j(X_s, \mathbb{Q})$ for $s \in U$ form a local system (which is naturally isomorphic to $R^j f_* \mathbb{Q}_X|_U$), where $X_s = f^{-1}(s)$.

It is known that the size of the Jordan blocks of the monodromy of $H^j(X_s, \mathbb{Q})$ around any point of $S \setminus U$ (with arbitrary eigenvalues) does not exceed $\min\{j+1, 2n-j+1\}$. This is a part of the monodromy theorem in the case the generic fiber is proper smooth. See e.g. [12], [18]. More precisely, the size is estimated in terms of the nonzero *Hodge numbers* of a general fiber. This is due to [17] when X is smooth, and the general case (with a slightly sharper bound) follows from the theory of limit mixed Hodge structure ([24], [26], [8], [13], etc.) combined with [4, (8.2.4)]. See (2.4) below.

However, it is often observed that the size of the Jordan blocks with eigenvalue 1 of the monodromy at infinity (i.e. around a point of $\bar{S} \setminus S$ with $\bar{S}$ the smooth compactification of S) is strictly smaller than the bound mentioned above even though the cohomology of a general fiber has enough nonzero Hodge numbers, e.g. if $f : \mathbb{A}^{n+1} \to \mathbb{A}^1$ is a polynomial map. See [6] in the case $n = 1$.

In this paper, we show that this phenomenon is closely related to the vanishing of the cohomology of the total space, or more precisely, to the upper bound of the *weights* of the cohomology of a general fiber and the total space.

0.1. Theorem. *Let $f : X \to S$ be a morphism of complex algebraic varieties such that $\dim X = n+1$ and $\dim S = 1$. Let j be a positive integer, and r, r' be integers such that $r' < r$. Then the Jordan blocks of the monodromies at infinity of $H^j(X_s, \mathbb{Q})$ with eigenvalue 1 have size $\le r - r'$ if $H^{j+1}(X, \mathbb{Q})$ has weights $\le j + r$ and $H^j(X_s, \mathbb{Q})$ for a general $s \in S$ has weights in $[j + r', j + r]$.*

0.2. Corollary. *Assume $X = \mathbb{A}^{n+1}$ and $S = \mathbb{A}^1$ so that f is a polynomial map. Let j be a positive integer. Then the Jordan blocks with eigenvalue 1 of the monodromy at infinity of $H^j(X_s, \mathbb{Q})$ have size $\le j$.*

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Theorem mean that the restriction on the weights of the cohomology of the total space and a general fiber implies a certain restriction on the monodromy at infinity. By [4] we may assume $r' = 0$ if X is smooth. The converse of Theorem is true in a weak sense if the generic fiber is smooth proper so that $H^j(X_s, \mathbb{Q})$ is pure of weight j for a general s. See Remark after (2.3). Note that the weights in the hypothesis are relevant to the mixed Hodge structure in [4, III], and not to the limit mixed Hodge structure. For instance, the assumption on $H^j(X_s, \mathbb{Q})$ in the case X smooth is satisfied if X_s has a smooth compactification $\bar{X}_s$ such that $\bar{X}_s \setminus X_s$ is a divisor with normal crossings and the intersection of any $r+1$ local irreducible components is empty. (Similarly for $H^{j+1}(X, \mathbb{Q})$ with r replaced by $r-1$.) See loc. cit. (The condition on $H^{j+1}(X, \mathbb{Q})$ can be replaced by that on $H^0(S, {}^pR^{j+1}f_*\mathbb{Q}_X)$, see (2.3).)

Corollary was proved in the case $n = 1$ by [6], and follows from [21], Cor. 10, if f is cohomologically tame in the sense of loc. cit. See also [10], [11]. For $n = j = 2$, there is an example such that the size of the Jordan block is j (see [7], Example (5.3.2): $f(x, y, z) = x + y + z + x^2y^2z^2$). Note that Corollary does not hold for the monodromy around a point of $S \setminus U$ (consider e.g. $f(x, y) = y^2 + x^3 - 3x$), although the local analogue is true (0.4).

We can prove also

0.3. Proposition. *Let f be as in (0.1). Assume the monodromy at infinity of $\mathrm{Gr}^W_{j+k}H^j(X_s, \mathbb{Q})$ has a Jordan block of size $r\,(>0)$ with eigenvalue 1, and $H^{j+1}(X, \mathbb{Q})$ has weights $\le j+k+r$. Then, either the local system $\{\mathrm{Gr}^W_{j+k+r+1}H^j(X_s, \mathbb{Q})\}_{s\in U}$ has a nonzero global section with $r \le j-k-1$, or $\{\mathrm{Gr}^W_{j+k}H^j(X_s, \mathbb{Q})\}_{s\in U}$ has a nonzero global section with $r = 1$. We have furthermore $k > r'$ in the second case if $H^j(X, \mathbb{Q})$ has weights $> j + r'$ and $H^j(X_s, \mathbb{Q})$ has weights $\ge j + r'$.*

This follows from [6] when f is a polynomial map as in (0.2) with $n = 1$ (because $\mathrm{Gr}^W_1H^1(X_s, \mathbb{Q})$ coincides with the cohomology of the smooth compactification). The assumption on the weights of the cohomology of X is trivially satisfied if $X = \mathbb{A}^{n+1}$. Note that $\{H^j(X, \mathbb{Q})\}_{s\in U}$ form a variation of mixed Hodge structure, and in particular, $\{W_kH^j(X, \mathbb{Q})\}_{s\in U}$ form a local system.

If f is a polynomial map and a general fiber X_s admits a smooth compactification such that the divisor at infinity is smooth (e.g. if the hypersurface in $\mathbb{P}^n$ defined by the highest degree part of f is reduced and smooth), then $H^j(X_s, \mathbb{Q})$ has weights in $[j, j+1]$, and Proposition implies that 1 is not an eigenvalue of the monodromy at infinity of $\mathrm{Gr}^W_jH^j(X_s, \mathbb{Q})$ and the size of the Jordan block with eigenvalue 1 for $\mathrm{Gr}^W_{j+1}H^j(X_s, \mathbb{Q})$ is ≤ 1. In particular, the last assertion holds also for $H^j(X_s, \mathbb{Q})$. (Note that it follows also from Theorem.)

For the proofs of (0.1–3), we use the fact that the action of the nilpotent part N of the monodromy on the nearby cycles at infinity (endowed with the limit mixed Hodge structure) is a morphism of mixed Hodge structures of type $(-1, -1)$ (see [4]) so that the assertion is reduced to the estimate of the weights of the nearby cycles at infinity. We first consider the spectral sequence (2.1.1) which relates the hypercohomology over S of the (perverse) direct image of the constant sheaf by f to that of the graded pure pieces of the direct image. For each graded piece, the weight filtration on its nearby cycles is given by

the monodromy filtration up to a shift. The weights on the nearby cycles can be estimated by using its higher direct image by the inclusion to the compactification of S (1.4.4), and then using its hypercohomology over S (1.5.1).

The corresponding local assertion is more or less well-known. For a holomorphic function f on a complex manifold X, we have $N^{n+1} = 0$ on the nearby cycle sheaf $\psi_f \mathbb{Q}_X[n]$ (by reducing to the normal crossing case). See e.g. [20]. Consequently, Jordan blocks of the monodromy on the jth cohomology of the Milnor fiber at any point of X have size $\leq j+1$ (by restricting to a generic hyperplane and using the vanishing of certain relative cohomology [19]). See also [9]. We have furthermore

0.4. Proposition. *The support of the perverse sheaves $N^j \psi_f \mathbb{Q}_X[n]$, $N^{j-1} \varphi_{f,1} \mathbb{Q}_X[n]$ have dimension $\leq n-j$. See (2.5).*

This is actually equivalent to $\dim \operatorname{supp} \operatorname{Gr}^W_{n-j} \psi_f \mathbb{Q}[n] \leq n-j$ by (1.4.2) and (1.4.5), and implies $\dim \operatorname{supp} \operatorname{Gr}^W_{n-j} \mathbb{Q}_{X_0}[n] \leq n-j$ by (1.4.5), where W is the weight filtration of the mixed Hodge Module [23]. In particular, $N^n = 0$ on the vanishing cycle sheaf with unipotent monodromy $\varphi_{f,1} \mathbb{Q}_X[n]$, and Jordan blocks with eigenvalue 1 of the monodromy on the jth reduced cohomology of the Milnor fiber have size $\leq j$ (see [25] in the isolated singularity case).

In Sect. 1 we review some basic facts from the theory of mixed Hodge Modules [22], [23], and prove (0.1–4) in Sect. 2.

1. Mixed Hodge Modules

1.1. For a smooth complex algebraic variety X, let $\mathrm{MHM}(X)$ denote the category of mixed Hodge Modules on X. See [23, 4.2]. An object $\mathcal{M}$ of $\mathrm{MHM}(X)$ consists of $((M,F,W),(K,W),\alpha)$ where (M,F) is a filtered $\mathcal{D}_X$-Module with the filtration W, (K,W) is a filtered perverse sheaf with rational coefficients on X, and α is an isomorphism of perverse sheaves $\mathrm{DR}(M) \simeq K \otimes_{\mathbb{Q}} \mathbb{C}$ compatible with W. They satisfy several good conditions. The filtrations F and W are called respectively the Hodge and weight filtrations. The category $\mathrm{MHM}(X)$ is an abelian categories, and every morphisms is strictly compatible with the two filtrations (F,W) in the strong sense [22].

For morphisms f of smooth complex algebraic varieties, we can construct canonical functors f_*, $f_!$, f^*, $f^!$ between the bounded derived categories of mixed Hodge Modules $D^b\mathrm{MHM}(X)$ [23, 4.3-4]. We will denote by $H^i : D^b\mathrm{MHM}(X) \to \mathrm{MHM}(X)$ the natural cohomology functor.

Remark. Let $f : X \to Y$ be a morphism of smooth complex algebraic varieties, and $\mathcal{M}$ a bounded complex of mixed Hodge Module on X with a finite decreasing filtration G. Then we have a spectral sequence in the category of mixed Hodge Modules

$$E_1^{j,i-j} = H^i f_* \operatorname{Gr}^j_G \mathcal{M} \Rightarrow H^i f_* \mathcal{M}. \tag{1.1.1}$$

Indeed, the direct image $f_*\mathcal{M}$ is represented by a complex of mixed Hodge Modules endowed with a filtration induced by the filtration G on $\mathcal{M}$ by the definition of direct

image [23, 4.3]. In particular, Gr_G^i commutes with the direct image, and the spectral sequence follows (see e.g. [4, (1.3.1)]).

Applying this to the truncation τ on $f_*\mathcal{M}$, we get the Leray spectral sequence

$$E_2^{p,q} = H^p g_* H^q f_* \mathcal{M} \Rightarrow H^{p+q}(gf)_* \mathcal{M}, \tag{1.1.2}$$

as in [4, (1.4.8)] for morphisms $f : X \to Y$ and $g : Y \to Z$.

1.2. We say that $\mathcal{M}$ is pure of weight r if $\mathrm{Gr}_k^W \mathcal{M} = 0$ for $k \neq r$. A pure Hodge Module is also called a polarizable Hodge Module. It admits a decomposition by strict support

$$\mathcal{M} = \oplus_Z \mathcal{M}_Z, \tag{1.2.1}$$

where the direct sum is taken over irreducible closed subvarieties Z of X, and $\mathcal{M}_Z$ has support Z or $\emptyset$, but has no nontrivial subobject or quotient object with strictly smaller support. The underlying perverse sheaf K_Z of $\mathcal{M}_Z$ is an intersection complex with local system coefficients, i.e. there exist a dense open smooth affine subvariety U of Z with the inclusion $j : U \to Z$, and a local system L_Z on U such that K_Z is the intermediate direct image

$$j_{!*}(L_Z[\dim Z]) := \mathrm{Im}(j_!(L_Z[\dim Z]) \to j_*(L_Z[\dim Z])) \tag{1.2.2}$$

in the sense of [1]. If Z is a curve, then $K_Z[-1]$ is a sheaf in the usual sense, and is isomorphic to j_*L_Z (where the direct image is in the usual sense).

Remark. Let f be a proper morphism of smooth complex algebraic varieties, and $\mathcal{M}$ a pure Hodge Modules of weight n on X. Then the cohomological direct image $H^j f_* \mathcal{M}$ is pure of weight $n+j$. See [23, 4.5]. A pure Hodge Module is also stable by the intermediate direct image (1.2.2) due to loc. cit.

1.3. For $X = pt$, we have naturally an equivalence of categories

$$\mathrm{MHM}(pt) = \mathrm{MHS}^p, \tag{1.3.1}$$

where the right-hand side denotes the category of graded-polarizable mixed Hodge structures with rational coefficients [4] (and $F^p = F_{-p}$). See [23, (4.2.12)]. So mixed Hodge Modules on pt will be identified with graded-polarizable mixed Hodge structures.

More generally, a mixed Hodge Module $((M, F, W), (K, W); \alpha)$ such that $K[-d]$ is a local system can be naturally identified with an admissible variation of mixed Hodge structure [16], [26] by replacing K with $K[-d]$, and W with $W[-d]$, where $d = \dim X$ and $(W[-d])_k = W_{k+d}$. See [23, 3.27]. In particular, a polarizable Hodge Module of weight w such that the underlying perverse sheaf is a local system (up to a shift) can be identified with a polarizable variation of Hodge structure of weight $w - d$. We will denote by $\mathbb{Q}_X^H[d] \in \mathrm{MHM}(X)$ the pure Hodge Module of weight d corresponding to the constant variation of Hodge structure of type $(0, 0)$.

1.4. We have the nearby and vanishing cycle functors ψ_g and φ_g for a function g on X. They are exact functors of mixed Hodge Modules on X, and correspond to $\psi_g[-1]$, $\varphi_g[-1]$

on the underlying perverse sheaves [3]. Let $Y = g^{-1}(0)$. If Y is smooth, the functors will be viewed as a functor to $\mathrm{MHM}(Y)$ due to [23, (4.2.4)]. (Actually we may assume Y smooth by using the direct image by the inclusion of the graph of g as in [22, 3.2.1] if necessary).

The semisimple part T_s of the monodromy acts naturally on $\psi_g\mathcal{M}$, $\varphi_g\mathcal{M}$, and $\mathrm{Ker}(T_s-1)$ is denoted by $\psi_{g,1}\mathcal{M}$, $\varphi_{g,1}\mathcal{M}$. Let $N = (2\pi i)^{-1}\log T_u$, where T_u is the unipotent part of the monodromy. Then N gives morphisms of mixed Hodge Modules

$$N : \psi_g\mathcal{M} \to \psi_g\mathcal{M}(-1), \quad N : \varphi_g\mathcal{M} \to \varphi_g\mathcal{M}(-1), \tag{1.4.1}$$

where (-1) is the Tate twist as in [4] (i.e. the Hodge filtration is shifted by -1 and the weight filtration by 2).

The weight filtration is given by the relative monodromy filtration in the sense of [5, (1.6.13)] (see also [26]). In particular, if $\mathcal{M}$ is pure of weight k, then we have isomorphisms

$$\begin{aligned} N^i : \mathrm{Gr}^W_{k-1+i}\psi_g\mathcal{M} &\xrightarrow{\sim} \mathrm{Gr}^W_{k-1-i}\psi_g\mathcal{M}(-i), \\ N^i : \mathrm{Gr}^W_{k+i}\varphi_{g,1}\mathcal{M} &\xrightarrow{\sim} \mathrm{Gr}^W_{k-i}\varphi_{g,1}\mathcal{M}(-i). \end{aligned} \tag{1.4.2}$$

Let $U = X \setminus Y$ with the inclusion morphisms $i : Y \to X$, $j : U \to X$. Since $\psi_g\mathcal{M}$ depends only on $\mathcal{M}|_U$, $\psi_g j_*\mathcal{M}$ for $\mathcal{M} \in \mathrm{MHM}(U)$ will be denoted by $\psi_g\mathcal{M}$. We have a canonical isomorphism

$$i^*j_*\mathcal{M} = \mathrm{Cone}(N : \psi_{g,1}\mathcal{M} \to \psi_{g,1}\mathcal{M}(-1)) \tag{1.4.3}$$

by [23, 2.24]. (Here Y is assumed to be smooth).

Indeed, $\mathrm{Var} : \varphi_{g,1}j_*\mathcal{M} \to \psi_{g,1}\mathcal{M}(1)$ is an isomorphism (because Var corresponds to the action of t on the underlying $\mathcal{D}$-Module, see e.g. [22, 3.4.12]) and $\mathrm{can} : \psi_{g,1}\mathcal{M} \to \varphi_{g,1}j_*\mathcal{M}$ is identified with N. (This is related to [26] when X is a curve and the local monodromies are unipotent.)

In particular, we have

$$\mathrm{Gr}^W_{k+1+i}H^0i^*j_*\mathcal{M} = (P_N\mathrm{Gr}^W_{k-1+i}\psi_{g,1}\mathcal{M})(-1), \tag{1.4.4}$$

where the right-hand side denotes the primitive part by the action of N. So the dimension of $H^0i^*j_*\mathcal{M}$ coincides with the number of Jordan blocks with eigenvalue 1 in the case $\dim X = 1$ and g is a local coordinate.

Remarks. (i) Let $\mathbb{Q}^H_Y[n]$ denote the mixed Hodge Module such that the underlying perverse sheaf is $\mathbb{Q}_Y[n]$ where $n = \dim Y$. Then we have a short exact sequence of mixed Hodge Modules

$$0 \to \mathbb{Q}^H_Y[n] \to \psi_{g,1}\mathbb{Q}^H_X[n+1] \xrightarrow{\mathrm{can}} \varphi_{g,1}\mathbb{Q}^H_X[n+1] \to 0.$$

See [23, (2.4.3)]. Combined with [22, 5.1.7], this implies

$$\begin{aligned} \mathbb{Q}^H_Y[n] &= \mathrm{Ker}(N : \psi_{g,1}\mathbb{Q}^H_X[n+1] \to \psi_{g,1}\mathbb{Q}^H_X[n+1](-1)), \\ \varphi_{g,1}\mathbb{Q}^H_X[n+1] &= \mathrm{Coim}(N : \psi_{g,1}\mathbb{Q}^H_X[n+1] \to \psi_{g,1}\mathbb{Q}^H_X[n+1](-1)). \end{aligned} \tag{1.4.5}$$

(ii) Assume X is a smooth curve, and let $0 \in X$ with a local coordinate t such that $\{0\} = t^{-1}(0)$. Let $\mathcal{M}$ be an admissible variation of mixed Hodge structure on U, which is identified with a mixed Hodge Module on U. Assume the monodromy around 0 is unipotent. Then the nearby cycles $\psi_t\mathcal{M}$ can be defined as in [24], [26] by extending the Hodge bundles to Deligne's canonical extension [2] as subbundles, and then restricting them to the fiber at 0. (In general, we have to use the filtration V ([15], [20]) indexed by rational numbers as in [22].)

In particular, the set of integers p such that $\mathrm{Gr}_F^p H^j(X_s, \mathbb{Q}) \neq 0$ do not change by passing to the limit Hodge filtration. This is used for the estimate of the size of Jordan blocks stated at the beginning of the introduction. See (2.4).

1.5. Lemma. *Let S be a smooth affine curve with the structure morphism $a_S : S \to pt$, and $\bar{S}$ be the smooth compactification of S with the inclusions $i : \bar{S} \setminus S \to \bar{S}$, $j : S \to \bar{S}$. Let $\mathcal{M}$ be a pure Hodge Module of weight k on S, and let $j_{!*}\mathcal{M} = \mathrm{Im}(j_!\mathcal{M} \to j_*\mathcal{M})$, the intermediate direct image. Then we have an exact sequence of mixed Hodge structures*

$$0 \to H^0(a_{\bar{S}})_* j_{!*}\mathcal{M} \to H^0(a_S)_*\mathcal{M} \to (a_{\bar{S}\setminus S})_* H^0 i^* j_*\mathcal{M} \to H^1(a_{\bar{S}})_* j_{!*}\mathcal{M} \to 0, \tag{1.5.1}$$

and an isomorphism

$$H^{-1}(a_{\bar{S}})_* j_{!*}\mathcal{M} = H^{-1}(a_S)_*\mathcal{M}. \tag{1.5.2}$$

In particular, $H^{-1}(a_S)_*\mathcal{M}$ is pure of weight $k-1$, and is isomorphic to the dual of $H^1(a_{\bar{S}})_* j_{!*}\mathcal{M}$ up to a Tate twist.

Proof. Since S is affine, it is enough to show the short exact sequence of mixed Hodge Modules

$$0 \to j_{!*}\mathcal{M} \to j_*\mathcal{M} \to i_* H^0 i^* j_*\mathcal{M} \to 0.$$

Indeed, $H^i(a_{\bar{S}})_* j_{!*}\mathcal{M}$ is pure of weight $k+i$ (see Remark after (1.2)), and $\mathcal{M}$ is self-dual up to a Tate twist due to the polarization.

Let K be the underlying perverse sheaf of $\mathcal{M}$. Replacing $\bar{S}$ with an open neighborhood of $\bar{S} \setminus S$, we may assume that $\mathcal{M}$ is a variation of mixed Hodge structure, i.e. $K[-1]$ is a local system. Then the underlying $\mathbb{Q}$-complex of $(j_{!*}\mathcal{M})[-1]$ is $j_*(K[-1])$ where j_* is the direct image in the usual sense. So the assertion follows from the distinguished triangle

$$\to j_*(K[-1]) \to \mathbf{R}j_*(K[-1]) \to R^1 j_*(K[-1]) \to,$$

which gives the exact sequence of the underlying perverse sheaves after shifting the complexes by 1.

Remark. If every local monodromy of $K[-1]|_U$ is unipotent, we can prove the assertion by using [26].

2. Proof of Theorem

2.1. Proposition. *Let S be a smooth affine curve, and $\mathcal{M}$ a mixed Hodge Module on S with the weight filtration W. Let U be a dense open subvariety of S on which $\mathcal{M}$ is a variation of mixed Hodge structure. Assume $H^0(a_S)_*\mathcal{M}$ has weights $\le m$. Then $H^0(a_S)_*\mathrm{Gr}_k^W\mathcal{M}$ has weights $\le m$ if $\mathcal{M}|_U$ has weights $\le m+1$. Moreover, $H^{-1}(a_S)_*\mathrm{Gr}_{i+1}^W\mathcal{M} \ne 0$ if $\mathrm{Gr}_i^W H^0(a_S)_*\mathrm{Gr}_k^W\mathcal{M} \ne 0$ with $i > m$.*

Proof. By (1.1.1) we have a spectral sequence of mixed Hodge structures

$$E_1^{-k,k+i} = H^i(a_S)_*\mathrm{Gr}_k^W\mathcal{M} \Rightarrow H^i(a_S)_*\mathcal{M}. \tag{2.1.1}$$

Since S is affine, $E_1^{p,q} = 0$ if $p+q < -1$ or $p+q > 0$. By (1.5), $E_1^{-k,k-1}$ is pure of weight $k-1$, and $E_1^{-k,k-1} = 0$ for $k-1 > m$ using the decomposition (1.2.1) applied to $\mathrm{Gr}_k^W\mathcal{M}$, because the direct factor of $\mathrm{Gr}_k^W\mathcal{M}$ with discrete support does not contribute to $H^{-1}(a_S)_*\mathrm{Gr}_k^W\mathcal{M}$. So the assertion is clear.

2.2. Proposition. *With the above notation, assume $\mathcal{M}$ is a pure Hodge Module of weight k on S. For $s \in \bar{S} \setminus S$, let t be a local coordinate at s. If $H^0(a_S)_*\mathcal{M}$ has weights $\le m$, then $\psi_{t,1}\mathcal{M}$ has weights between $\min\{k-1, 2k-m\}$ and $\max\{k-1, m-2\}$. Conversely, if the monodromy of $\psi_{t,1}\mathcal{M}$ has a Jordan block of size r, then either $\mathrm{Gr}_{k+r}^W H^0(a_S)_*\mathcal{M} \ne 0$, or $\mathrm{Gr}_{k+r}^W H^1(a_{\bar{S}})_* j_{!*}\mathcal{M} \ne 0$ with $r = 1$.*

Proof. We have the symmetry of the weights of nearby cycles by (1.4.2). So it is enough to estimate the maximal weight for the first assertion, and it is verified by using the exact sequence (1.5.1) and taking H^0 of (1.4.3). For the last assertion, the existence of the Jordan block of size r corresponds to the nonvanishing of the primitive part of $\mathrm{Gr}_{k+r}^W(a_{\bar{S}\setminus S})_* H^0 i^* j_*\mathcal{M}$ by (1.4.4). Then it corresponds by (1.5.1) to that of $\mathrm{Gr}_{k+r}^W H^0(a_S)_*\mathcal{M}$ or $\mathrm{Gr}_{k+r}^W H^1(a_{\bar{S}})_* j_{!*}\mathcal{M}$. In the second case, we have $r = 1$ because $H^1(a_{\bar{S}})_* j_{!*}\mathcal{M}$ is pure of weight $k+1$. So the assertion follows.

2.3. *Proofs of* (0.1–3). We may assume S connected and then affine (because otherwise $\bar{S} = S$). Let $\mathcal{M} = H^{j+1}f_*\mathbb{Q}_X^H$ so that ${}^pR^{j+1}f_*\mathbb{Q}_X$ is the underlying perverse sheaf of $\mathcal{M}$ (in particular, its restriction to U is $(R^j f_*\mathbb{Q}_X|_U)[1]$). Here ${}^pR^{j+1}f_*$ means ${}^pH^{j+1}\mathbf{R}f_*$. See [1]. By hypothesis $\mathcal{M}|_U$ has weights in $[j+r'+1, j+r+1]$. See (1.3) for the shift of weight. Furthermore $H^0(a_S)_*\mathcal{M}$ has weights $\le j+r$ by the exact sequence of mixed Hodge structures

$$0 \to H^0(a_S)_* H^{j+1}f_*\mathbb{Q}_X^H \to H^{j+1}(X,\mathbb{Q}) \to H^{-1}(a_S)_* H^{j+2}f_*\mathbb{Q}_X^H \to 0. \tag{2.3.1}$$

(This follows from (1.1.2), because S is affine so that $E_2^{p,q} = 0$ except for $p = -1, 0$.)

Then $H^0(a_S)_*\mathrm{Gr}_k^W\mathcal{M}$ has weights $\le j+r$ by (2.1). Using (2.2), this implies that the $\psi_{t,1}\mathrm{Gr}_k^W\mathcal{M}$ for $j+r'+1 \le k \le j+r+1$ (and hence $\psi_{t,1}\mathcal{M}$) have weights in $[j-r+2r'+1, j+r]$. This completes the proof of (0.1), and (0.2) follows by using Remark (2.4) below.

The proof of (0.3) is similar. Indeed, the existence of the Jordan block of size r corresponds by (2.2) (with k replaced by $j+k+1$) to the nonvanishing of

$$\mathrm{Gr}_{j+k+r+1}^W H^0(a_S)_*\mathrm{Gr}_{j+k+1}^W\mathcal{M} \quad \text{or} \quad \mathrm{Gr}_{j+k+r+1}^W H^1(a_{\bar{S}})_* j_{!*}\mathrm{Gr}_{j+k+1}^W\mathcal{M}.$$

In the first case, the assertion follows from the last assertion of (2.1), where $j+k+r+1 \le 2j$ by [4, (8.2.4)] (see also (2.4) below). In the second case, we have $r = 1$ by (2.2), and $k > r'$ if $H^j(X, \mathbb{Q})$ has weights $> j + r'$, because it implies that $\{\mathrm{Gr}^W_{j+r'} H^j(X_s, \mathbb{Q})\}_{s \in U}$ has no nonzero global section (using (2.1.1)). This completes the proof of (0.1–3).

Remark. Assume X is smooth and f is proper. Then $H^{j+1}(X, \mathbb{Q})$ has weights $\le j + r + 1$ if the Jordan blocks with eigenvalue 1 of the local monodromies at any points of $\bar{S} \setminus S$ have size $\le r$. This follows from (1.4.3) and (1.5.1).

2.4. *Remark.* Let Y be a complex algebraic variety of dimension n. Let $h^{j,p,q}(Y) = \dim \mathrm{Gr}_F^p \mathrm{Gr}^W_{p+q} H^j(Y, \mathbb{C})$. Then by [4, (8.2.4)], $h^{j,p,q}(Y) = 0$ except when $(p, q) \in [0, j] \times [0, j]$ and $j \le n$, or $(p, q) \in [j - n, n] \times [j - n, n]$ and $j \ge n$. Combined with Remark (ii) after (1.4), this gives the upper bound of the size of the Jordan blocks of the local monodromies mentioned in the introduction, because the $H^j(X_s, \mathbb{Q})$ for $s \in U$ form an admissible variation of mixed Hodge structure ([8], [13], [26], etc.) See also [18], [12] in the case the generic fiber is proper smooth. More precisely, the above argument shows that the size of the Jordan blocks does not exceed the maximal length of successive numbers p such that $\mathrm{Gr}_F^p H^j(X_s, \mathbb{C}) \ne 0$. This was first shown in [24] when the generic fiber is proper smooth. See also [17] for a slightly weaker but more general assertion.

If Y is smooth, we have furthermore $h^{j,p,q}(Y) = 0$ for $p + q < j$. In particular, $H^j(Y, \mathbb{Q})$ have weights in $[j, 2j]$ for $j \le n$, and in $[j, 2n]$ otherwise.

2.5. *Proof of* (0.4). It is well-known that $N^{n+1} = 0$ on the nearby cycle sheaf $\psi_f \mathbb{Q}_X[n]$. See e.g. [20]. This implies $N^n = 0$ on the vanishing cycles with unipotent monodromy $\varphi_{f,1} \mathbb{Q}_X[n]$ by (1.4.5). Now we take a Whitney stratification of X satisfying Thom's A_f-condition, which exists at least locally by [14]. (Here $f^{-1}(0)$ is assumed to be a union of strata.) For each stratum S_α in $f^{-1}(0)$, let X_α be a transversal space which is a locally closed complex submanifold of X. Applying the above argument to the restriction of f to X_α, we get the assertion on the dimension on the support of $\mathrm{Im}\, N^j$.

Remarks. (i) We can also prove (0.4), or rather the equivalent assertion after (0.4), by reducing to the normal crossing case and then using the calculation of nearby cycle sheaf as in [25] or [22, (3.6.10)]. Here we can also show that if $f^{-1}(0)$ is a divisor with normal crossings, we have $\mathrm{Gr}^W_k \psi_f \mathbb{Q}[n] = 0$ for $|k - n| \ge r$ if the intersection of any $r + 1$ local irreducible components of $f^{-1}(0)$ is empty. (See also [18], [9].)

(ii) The statement on the monodromy on the reduced cohomology of the Milnor fiber after (0.4) follows from (0.4) because a perverse sheaf K on an analytic space Y satisfies $\mathcal{H}^i K = 0$ for $i < -\dim Y$. This can be verified by induction on $\dim Y$ using the transversal space to each stratum with positive dimension of a Whitney stratification of Y and also the long exact sequence associated to local cohomology.

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