On Unstable Principal Bundles over Elliptic Curves

Stefan Helmke and Peter Slodowy

April 5, 2000

Contents

Introduction

This article is an outgrowth of a more comprehensive project to provide a detailed account for the results announced in [13], [14]. In these reports a relation between simple elliptic singularities in the sense of Saito [22] and certain holomorphic Kac-Moody loop groups $\hat{\mathcal{L}}G$ is established which generalizes a well known theorem of Brieskorn [6], cf. also [26], relating simple singularities of type A_ℓ, D_ℓ, E_ℓ and the corresponding simple algebraic groups G.

An important ingredient in the derivation of Brieskorn's result is the orbital geometry of the unipotent variety $\mathfrak{U}(G)$ of G, especially in the neighborhood of a subregular unipotent orbit.

In our generalization, the unipotent variety $\mathfrak{U}(G)$ is replaced by an 'unstable' variety $\mathfrak{U}(\hat{\mathcal{L}}G)$ in $\hat{\mathcal{L}}G$ whose orbital geometry can be described, essentially due to an observation by E. Looijenga, in terms of unstable principal G-bundles over elliptic curves.

The purpose of this article is a classification of the unstable bundles which enter into our analogue of Brieskorn's constructions, i.e. so-called regular and subregular ones. Whereas our primary interest lies in the the case of bundles with simply connected structure group G, in particular those of types D_5, E_6, E_7, E_8, we have here also investigated the case of non-simply connected simple structure groups G. They provide a large variety of orbital geometries mainly related to the non-isolated singularities of type A_∞ and D_∞ discussed in [13], [14].

Apart from our envisaged geometrical applications, the results of this paper may be of independent interest. For example, regular unstable G-bundles are of importance in the work of Friedman-Morgan-Witten on moduli spaces of semistable G-bundles,

[10], [9]. Our classification also gives an idea of the complexities involved in the approach of Atiyah-Bott to the cohomology of moduli spaces of G-bundles over arbitrary Riemann surfaces [2]. Finally, our generalization of Brieskorn's Theorem is based on the orbital geometry of holomorphic Kac-Moody loop groups. By a kind of Floquet-theory, this geometry is the same as that for co-adjoint orbits of 2-toroidal or elliptic Lie algebras ([7], [8], [23]). Independently of that device, our results on G-bundles apply to both contexts.

Let us give an overview of the contents of this paper. In Section 1 we recapitulate fundamental results on principal G-bundles over elliptic curves. They are related to the canonical or Harder-Narasimhan reduction of unstable bundles and the moduli spaces of semistable bundles. We mainly refer to previous work by Atiyah-Bott [2], Friedman-Morgan [9], and Schweigert [24].

In Section 2 we investigate the automorphism groups of G-bundles ξ. For the case of semistable bundles, we can refer to recent work of Friedman-Morgan [9]. In the unstable case, we employ the canonical reduction ξ_L of the structure group G of ξ to a Levi subgroup $L \subset G$, to split $\mathrm{Aut}_G(\xi)$ as a semidirect product of two subgroups $\mathrm{Aut}_L(\xi_L)$ and $\mathrm{Aut}_G(\xi)^+$, where the last group is unipotent and transparently determined by the Atiyah-Bott type $\mu(\xi)$ of ξ. We are able to effectively determine $\mu(\xi)$ in case L is a proper maximal Levi subgroup of G.

Smaller Levi subgroups can be dealt with by means of degenerations (Section 3). The computation of $\mathrm{Aut}_L(\xi_L)$ still relies on a precise description of the structure of L. This is given in Section 4, in particular with respect to the occuring fundamental groups.

In Section 5, we assume that G is simple and simply connected, and we tabulate the minimal dimensions of the automorphism groups of unstable G-bundles associated to maximal Levi subgroups. As a result, we show the existence of regular and subregular unstable G-bundles in all cases.

Section 6 addresses the analogous problems for non-simply connected groups. Here, the methods and results are quite involved, and we have to resort to a somewhat tedious case-by-case description. Also here, we exhibit regular and subregular unstable bundles in all cases.

Work on these as well as on the the more far-reaching questions discussed in [13], [14] was started in the the Spring of 1998, when the second named author was a guest at RIMS, Kyoto University. It was continued at the occasion of further encounters at RIMS, and its completion benefitted greatly from a one-year-guest-professorship of the second author at the Kitami Institute of Technology, Hokkaido, 1999-2000. We would like to thank both institutions, and especially K. Saito, Kyoto, and H. Yamada, Kitami, for their continuous moral and material support.

1. General Results

The purpose of this section is to review basic properties of holomorphic principal G-bundles over a complex elliptic curve E where G is a complex, connected, reductive group.

1.1. Topology

Before studying the finer analytic aspects of bundles over E, let us quickly recall some basic facts related to the topology of these bundles. Any holomorphic G-bundle ξ may be regarded as a topological G-bundle. Such bundles are easily classified by an element $\gamma(\xi)$ in $H^2\big(E, \pi_1(G)\big) \cong \pi_1(G)$ where $\pi_1(G)$ is the fundamental group of G, (cf. [2, Section 6], or [20]). We shall call $\gamma(\xi)$ the *topological type* or *topological class* of ξ. It is functorial with respect to extensions $G \longrightarrow G'$ of the structure group.

Let us have a closer look at $\pi_1(G)$. For details on the structure theory of reductive groups we refer, here and in the sequel, to the textbooks [3], [15], [27]. Let $H = (G, G)$ denote the semisimple derived subgroup of G. The corresponding factor commutator group G/H is then a torus A, say of dimension s. Since $\pi_2(A)$ is trivial, we obtain an exact sequence

$$0 \longrightarrow \pi_1(H) \longrightarrow \pi_1(G) \longrightarrow \pi_1(A) \longrightarrow 0$$

and thus $\pi_1(G) \cong \pi_1(H) \times \pi_1(A) = F \times \mathbb{Z}^s$, where $F = \pi_1(H)$ is finite. Henceforth, we shall identify $\pi_1(A)$ with the lattice $X_*(A) = \mathrm{Hom}_{\mathrm{alg\,grp}}(\mathbb{C}^*, A)$ of co-characters (or, multiplicative one-parameter subgroups) of A.

To any G-bundle ξ we may associate the torus bundle $\xi \times^G A$ whose topological type is the image of $\gamma(\xi)$ in $\pi_1(A) = X_*(A)$. We will denote that by $c(\xi)$ and call it the *Chern class* of ξ. For any character χ in the dual lattice $X^*(A) = \mathrm{Hom}_{\mathrm{alg\,grp}}(A, \mathbb{C}^*)$, the integer $\langle \chi, c(\xi)\rangle$ will then be the degree (first Chern class) of the associated line bundle $\xi \times^G \mathbb{C}_\chi$.

Later, we shall have to view the Chern class $c(\xi)$ as an element of the Lie algebra $\mathfrak{g}$ of G. This comes about as follows. Let $C := C(G)^\circ$ denote the identity component of the center of G. It is a torus, of dimension $s = \dim A$ as well, and the natural map $C \longrightarrow A$ is an isogeny of tori with kernel $C \cap H$. Thus, we may naturally identify $X_*(A) \otimes \mathbb{Q}$ with $X_*(C) \otimes \mathbb{Q}$ which, itself, may be regarded as a $\mathbb{Q}$-subspace of $\mathfrak{g}$. Viewing thus $c(\xi)$ as an element of $\mathfrak{g}$, we have the following elementary observation.

(1.1.1) **Lemma.** *Let $\rho : G \longrightarrow GL(F)$ be a representation of G on an n-dimensional vector space F, and let $d\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(F)$ be the corresponding representation of its Lie algebra. Assume that $c(\xi) \in \mathfrak{g}$ acts by a scalar, i.e. $d\rho\big(c(\xi)\big) = s \cdot \mathrm{id}_F$ for some $s \in \mathbb{C}$. Then $s = d/n$, where d is the degree of the associated vector bundle $\xi \times^G F$.*

(1.1.2) Proof. The representation of G on $\det(F) = \bigwedge^n(F)$ is given by a character χ in $X^*(G) = X^*(A)$. As remarked above, the integer $\langle \chi, c(\xi)\rangle$ equals the degree d of the line bundle $\xi \times^G \mathbb{C}_\chi = \xi \times^G \det(F)$. But this number also agrees with the value of the infinitesimal representation of $c(\xi)$ on $\det(F)$, which is $n \cdot s \in \mathbb{C} = \mathfrak{gl}\big(\det(F)\big)$. □

1.2. Stability

Let us now turn to analytic invariants. Recall that the *slope* $\mu(V)$ for a holomorphic vector bundle V over E is defined by $\mu(V) = \deg(V)/\operatorname{rank}(V)$. Such a bundle is called *stable* (resp. *semistable*) if for every non-trivial holomorphic subbundle W of V the inequality

$$\mu(W) \;<\; \mu(V) \qquad \big(\text{resp.}\quad \mu(W) \;\leqslant\; \mu(V)\big)$$

holds. It is called *unstable* if it is not semistable, i.e. if there is a proper subbundle $W \subset V$ such that $\mu(W) > \mu(V)$.

The existence of a subbundle W of rank k inside V, of rank n, means that the structure group $GL_n(\mathbb{C})$ of V can be reduced to a parabolic subgroup P corresponding to upper diagonal block matrices with step size k and $n-k$. The character group $X^*(P) = \mathrm{Hom}(P, \mathbb{C}^*)$ is freely spanned by the determinants, χ_1 and χ_2 , on the two blocks, the global determinant det being their product, $\det = \chi_1 + \chi_2$ (when written additively). The characters vanishing on the center, i.e. the scalars of $GL_n(\mathbb{C})$ are proportional to the character $\chi = (n-k)\chi_1 - k\chi_2 = n\chi_1 - k\det$. Now, we have $\mu(W) < \mu(V)$ exactly when the associated line bundle $\chi(V)$ is of negative degree.

Motivated by this example, Ramanathan gave the following definition [20] for a general reductive group G. Call a character on a parabolic subgroup P of G *dominant* if it vanishes on the 'connected center' $C^\circ(G)$ of G and is dominant with respect to some Borel subgroup contained in P.

(1.2.1) **Definition.** Let G be a connected reductive group and ξ a holomorphic principal G-bundle over E. Then ξ is called *stable* (resp. *semistable*) if, for any reduction ξ_P of ξ to a parabolic subgroup P of G and any dominant character χ of P, the associated line bundle $\xi_P \times^P \mathbb{C}_\chi$ is of negative (resp. non-positive) degree. It is called *unstable* if it is not semistable.

It turns out that, on an elliptic curve, there are only very few stable bundles. In particular, from the viewpoint of moduli spaces, it is reasonable to concentrate directly on the class of semistable bundles. Here are some equivalent characterizations, formulated in [9, Theorem 2.2] and based on [2], [20], [21]. We let $\mathfrak{g}$ denote the Lie algebra and G^{ad} the adjoint group $G/C(G)$ of G. The group G^{ad} is semisimple or trivial (in case G is a torus).

(1.2.2) **Theorem.** *The following conditions on a holomorphic principal G-bundle ξ over E are equivalent:*

(*a*) *The bundle ξ is semistable.*

(*b*) *The associated principal G^{ad}-bundle $\xi \times^G G/C(G)$ (extension of structure group) is semistable.*

(*c*) *The associated adjoint vector bundle* $\mathrm{ad}(\xi) = \xi \times^G \mathfrak{g}$ *is semistable.*

(*d*) *For every finite dimensional irreducible representation $\rho : G \longrightarrow GL(F)$ the associated vector bundle $\xi \times^G F$ is semistable.*

1.3. Canonical Reduction

Let V be an arbitrary holomorphic vector bundle over E. According to a well known procedure (valid over arbitrary curves) by Harder and Narasimhan ([12], cf. e.g. [17, Ch. V]) there is a canonical filtration of V by unique subbundles,

$$0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_r = V$$

such that all successive quotients V_i/V_{i-1} are semistable of slopes $\mu_i = \mu(V_i/V_{i-1})$ with strictly decreasing values

$$\mu_1 > \mu_2 > \mu_3 > \cdots > \mu_r.$$

Of course, V is semistable exactly when $r = 1$. Since E is an elliptic curve, this filtration even splits, i.e. V is isomorphic to its (canonically) associated graded bundle $gr(V) = \bigoplus_i V_i/V_{i-1}$. This is a standard argument using the vanishing of the higher cohmology of indecomposable vector bundels of positive degree over an elleptic curve and the additivity of the slope for tensor products of vector bundles (see e.g. [9, proof of Prop. 2.6], [2, Lemma 10.1]).

In terms of reductions, the filtration corresponds to a reduction of the structure group $GL_n(\mathbb{C})$ of V to a parabolic subgroup P, and the splitting corresponds to a further reduction to a Levi subgroup L of P.

Applying this to the adjoint bundle $\mathrm{ad}(\xi) = \xi \times^G \mathfrak{g}$ of a given G-bundle ξ, we get a decomposition of $\mathrm{ad}(\xi)$ into a direct sum of semistable bundles γ_i. Let M denote the additive subgroup of $\mathbb{Q}$ generated by the slopes μ_i of the γ_i and reindex γ_i as $\gamma(\mu_i)$. Then we may write

$$\mathrm{ad}(\xi) \;\cong\; \bigoplus_{\nu \in M} \gamma(\nu).$$

Invoking once more the additivity of the slope for tensor products, we easily see that $\mathrm{ad}(\xi)$ has become a bundle of M-graded Lie algebras. The induced grading on a special fibre, $\mathfrak{g} \cong \bigoplus_{\nu \in M} \mathfrak{g}(\nu)$, corresponds to the choice of an infinitesimal semisimple automorphism μ of the Lie algebra $\mathfrak{g}$, i.e. an element of $\mathrm{Lie}(G^{ad}) = [\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}$. With this extra datum, the structure group G^{ad} of $\mathrm{ad}(\xi)$ reduces to the centralizer L' of μ in G^{ad}, a (connected) Levi subgroup [3], [27, 6.4.7]. Let L denote the preimage of L' in G. Then L is a Levi subgroup of G reducing the structure group of ξ to L (note that the bundles $\xi/L = \xi \times^G G/L$ and $(\xi \times^G G^{ad}) \times^{G^{ad}} G^{ad}/L'$ are the same and that respective reductions are given by sections of these bundles, cf. e.g. [11] or [21]).

As a result, the adjoint bundle $\mathrm{ad}(\xi)$ is associated to a principal L-bundle ξ_L by means of the restricted adjoint representation $\mathrm{ad}_L : L \longrightarrow \mathrm{Aut}(\mathfrak{g})$ of L. Using the action of L on the graded components $\mathfrak{g}(\nu)$, it may also be viewed as the direct sum of semistable associated bundles

$$\mathrm{ad}(\xi) \;\cong\; \bigoplus_{\nu \in M} \gamma(\nu) \;\cong\; \bigoplus_{\nu \in M} \xi_L \otimes^L \mathfrak{g}(\nu).$$

Note that $\mathfrak{g}(0)$ is the Lie algebra of L and that $\xi_L \otimes^L \mathfrak{g}(0) = \mathrm{ad}(\xi_L)$ is a maximal semistable subbundle of subalgebras in ξ.

We have thus arrived, on a slightly different way, at the following basic result of Friedman and Morgan, [9, Theorem 2.7].

(1.3.1) **Theorem.** *Let ξ be an unstable holomorphic G-bundle over E. There is a proper Levi subgroup L of G such that*

(a) ξ reduces to a semistable principal L-bundle ξ_L,

(b) L is maximal among Levi subgroups with this property.

The group L is well defined up to G-conjugacy, and the bundle ξ_L is determined up to isomorphism of principal bundles.

(1.3.2) Remark. Let $\mathfrak{p} := \bigoplus_{\nu \in M_{\geqslant 0}} \mathfrak{g}(\nu)$. Then $\mathfrak{p}$ is the Lie algebra of a parabolic subgroup P of G, of which L is a Levi subgroup. Using the fact that the normalizer of $\mathfrak{p}$ in G equals P (see e.g. [27, Theorem 6.4.7]), one obtains the more familiar but weaker reduction of ξ to a P-bundle ξ_P already by only exploiting the inclusion $\bigoplus_{\nu \in M_{\geqslant 0}} \gamma(\nu) \subset \xi$ of Lie algebra bundles, cf. [2, Section 10], and [9, Theorem 2.7]. This procedure works over Riemann surfaces of arbitrary genus. The pair (P, ξ_P) is usually called the canonical or Harder-Narasimhan reduction of ξ. However, we shall stick to the following definition, special to genus 1.

(1.3.3) **Definition.** The pair $(L,\, \xi_L)$, for ξ unstable, (resp. (G, ξ), if ξ is semistable) is called the *canonical reduction* of ξ.

1.4. Atiyah-Bott Type

Since the canonically reduced bundle ξ_L of a G-bundle ξ is an invariant of the latter, all further invariants of ξ_L will also be invariants of ξ. In particular, we can look at the Chern class $c(\xi_L)$ of ξ_L which we may view as an element of the Lie algebra $\mathfrak{g}$ via the inclusions

$$X_*\big(L/(L,L)\big) \;\subset\; \mathrm{Lie}(L) \;\subset\; \mathrm{Lie}(G) \;=\; \mathfrak{g}.$$

In view of Theorem 1.3.1, the element $\mu(\xi) := c(\xi_L) \in \mathfrak{g}$ is well defined up to conjugacy by elements of G. Thus, we should view it as a semisimple conjugacy class in $\mathfrak{g}$ or, alternatively, as an element in the fundamental Weyl chamber $K \subset X_*(T) \otimes \mathbb{Q}$ for a fixed pair (T, B) consisting of a maximal torus T and a Borel subgroup $B \supset T$ of G. Because of its rôle in the work of Atiyah and Bott [2, Section 10], we define:

(1.4.1) **Definition.** Let ξ be a G-principal bundle and ξ_L its canonical reduction. Then, the Chern class $\mu(\xi) = c(\xi_L)$ is called the *Atiyah-Bott-type* of ξ.

(1.4.2) Remark. The semsisimple element $\mu \in \mathfrak{g}$ used in the reduction process of ξ is, by Lemma 1.1.1, just the image $\mu = \mathrm{ad}\big(\mu(\xi)\big)$ under the adjoint representation $\mathrm{ad} : \mathfrak{g} \longrightarrow [\mathfrak{g}, \mathfrak{g}]$. If ξ is semistable, then, of course, $\mu(\xi) \in \mathfrak{c}$, the center of $\mathfrak{g}$.

(1.4.3) Remark. The Atiyah-Bott-type $\mu(\xi)$ of ξ is a discrete invariant of ξ since the denominators occuring in a reduced rational linear combination of elements from $X_*(T)$ are bounded by the exponents of the finite abelian groups $C(L)^\circ \cap (L, L)$.

(1.4.4) Remark. The element $\mu(\xi)$ determines the 'reducing' groups $L(\xi)$ and $P(\xi)$. Independently of the choice of a pair (T, B), they may be described in the following way: $L(\xi) = C_G\big(\mu(\xi)\big)$ and $P(\xi) = P\big([\mu(\xi)]\big)$ where $P\big([\mu(\xi)]\big)$ is the 'limit parabolic' attached to a one-parameter group $[\mu(\xi)]$ on the positive ray through $\mu(\xi)$, cf. [19, Ch. 2 §2], or [27, Prop. 8.4.5].

If we fix $\mu(\xi)$ as an element of the fundamental Weyl chamber $K \subset V := X_*(T) \otimes \mathbb{Q}$ associated to a pair (T, B), the groups $L(\xi)$ and $P(\xi)$ will be of 'standard type'. In that respect, recall the geometric structure of K. Let $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ denote a system of simple roots in $X^*(T)$ relative to (T, B). Then $K = \big\{\, v \in V \;\big|\; \alpha(v) \geqslant 0 \text{ for all } \alpha \in \Delta \,\big\}$ is a convex cone with faces $K_\Theta = \big\{\, v \in K \;\big|\; \alpha(v) = 0 \text{ for all } \alpha \in \Theta \,\big\}$ attached to arbitrary subsets $\Theta \subset \Delta$. It is the disjoint union of its relative interiors

$$K \;=\; \dot{\bigcup}_{\Theta \subset \Delta} \dot{K}_\Theta \quad \text{where} \quad \dot{K}_\Theta \;=\; \big\{\, v \in K_\Theta \;\big|\; \alpha(v) > 0 \text{ for all } \alpha \in \Delta \setminus \Theta \,\big\}.$$

If $\mu(\xi) \in \dot{K}_\Theta$, then $L(\xi) = L_\Theta$ and $P(\xi) = P_\Theta$ where L_Θ is generated by T and the unipotent root subgroups U_α $\alpha \in \Theta$, of G and where P_Θ is generated by L_Θ and B. The connected center $C^\circ(L_\Theta)$ of L_Θ is then given by the subtorus

$$S_\Theta \;=\; \big\{\, t \in T \;\big|\; \alpha(t) = 1 \text{ for all } \alpha \in \Theta \,\big\}^\circ,$$

and the vector space $X_*(S_\Theta) \otimes \mathbb{Q} =: V_\Theta \subset V$ is the supporting subspace for the fundamental face K_Θ, i.e. $K_\Theta = V_\Theta \cap K$ and $\dim V_\Theta = \dim K_\Theta$.

1.5. Moduli Spaces

The canonical semistable reduction ξ_L of a G-bundle ξ also has continuous invariants, furnished by the corresponding point in the moduli space of semistable bundles. To these moduli spaces, we turn now.

(1.5.1) **Definition.** Let $\mathcal{B}$ denote a set of G-bundles over E, and let ξ be a single G-bundle over E. Then ξ is called *adjacent to* $\mathcal{B}$, or a *degeneration of* $\mathcal{B}$, if there is a pointed complex space (S, s) and a holomorphic G-bundle Ξ over $E \times S$ such that, for all $t \neq s$, the restricted bundle $\Xi_t = \Xi|_{E\times t}$ is isomorphic to an element of $\mathcal{B}$, and Ξ_s is isomorphic to ξ. A pair $\big((S, s), \Xi\big)$ as above will be called a *degenerating family* (from $\mathcal{B}$ to ξ).

It may happen that $\mathcal{B}$ consists of a single element η. Then we will speak of a *degeneration from η to ξ* or say that ξ is *adjacent to η*. The equivalence relation generated by the adjacency relation between G-bundles is called *S-equivalence*.

(1.5.2) **Theorem** (Ramanathan). *There exists a coarse moduli space $\mathcal{M}(G)$ of S-equivalence classes of semistable G-bundles over E. The connected components of $\mathcal{M}(G)$ are normal projective varieties $\mathcal{M}(G, \gamma)$ corresponding bijectively to the elements γ in the fundamental group $\pi_1(G)$ of G or, alternatively, to the topological isomorphism classes of G-bundles over E.*

Note that, in particular, any topological G-bundle carries a holomorphic structure, and that the space $\mathcal{M}(G)$ is connected if G is simply connected. A more explicit description is due to Friedman, Morgan, Witten and Schweigert ([10], [9, Theorems 5.19, 5.22, 5.23], and [24], [4]). Its origin goes back to Looijenga's paper [18], which had been motivated by the deformation theory of simple elliptic singularities. Here are the most important cases.

(1.5.3) **Theorem.** *Let G be simple, simply connected and of rank ℓ. Let T be a maximal torus with Weyl group W and co-character lattice $\Lambda = X_*(T)$. Then $\mathcal{M}(G)$ is isomorphic to the natural quotient space $(E \otimes_{\mathbb{Z}} \Lambda)/W$, a weighted projective space of dimension ℓ.*

Now, assume that G is of the form $G = \tilde{G}/\langle c\rangle$ where $\tilde{G}$ is simply connected and simple, and where c is an element of the center $C(\tilde{G})$ of $\tilde{G}$. Since $\langle c\rangle$ may be identified with the fundamental group $\pi_1(G)$, there is a component $\mathcal{M}(G, c)$ of the moduli space $\mathcal{M}(G)$ consisting of bundles with topological type c. Let $\tilde{T}$ and $T = \tilde{T}/C(G)$ be corresponding maximal tori. Consider, with [5], the action of the affine Weyl groups $\hat{W} = W \ltimes X_*(\tilde{T})$ and $\check{W} = W \ltimes X_*(T)$ on $X_*(T) \otimes \mathbb{Q}$. The stabilizer $\check{W}_A$ of a fundamental alcove A for the $\hat{W}$-action may then be identified with the quotient $\check{W}/\hat{W} \cong X_*(T)/X_*(\tilde{T}) \cong \langle c\rangle$ as well as with a subgroup $\langle w_c\rangle$ of $W = \check{W}/X_*(T)$. This way, $\langle c\rangle$ acts as a symmetry group of A and the extended Dynkin diagram of G (for the explicit form, cf. the tables in [5]). Let $\ell_c{+}1$ denote the number of $\langle c\rangle$-orbits of vertices of that diagram. We shall call ℓ_c the *reduced rank* with respect to c. Then the torus $T_c = T/(\mathrm{id} - w_c)T$ is of dimension ℓ_c and acted upon by the centralizer $W_c = Z_W(w_c)$. Let Λ_c denote its co-character lattice $X_*(T_c)$.

(1.5.4) **Theorem.** *Let G be simple of rank ℓ and of the form $G = \tilde{G}/\langle c\rangle$ as described above. Then $\mathcal{M}(G, c)$ is isomorphic to $(E \otimes_{\mathbb{Z}} \Lambda_c)/W_c$, a weighted projective space of dimension ℓ_c.*

Finally, let G be reductive of the form $G = \tilde{G}/\langle c\rangle$, where $\tilde{G} = \prod_{j=0}^{r} \tilde{G}_j$ is a product of a central torus $\tilde{G}_0$ of dimension s with r simple and simply connected factors $\tilde{G}_j$, and where the element c is the product of central elements $c_j \in \tilde{G}_j$. Put $G_j := \tilde{G}_j/\langle c_j\rangle$.

(1.5.5) **Theorem.** *The moduli space $\mathcal{M}(G,c)$ is an étale fiber bundle over the product $\prod_{j=1}^{r} \mathcal{M}(G_j, c_j)$ with fiber an abelian variety isogeneous to E^s.*

(1.5.6) Remark. The weights occuring in the definition of the weighted projective spaces mentioned in 1.5.3 and 1.5.4 are essentially given by the coefficients of a minimal isotropic coroot (central element) of the involved affine root systems. For more details, cf. [4] as well as [16], [18], [24].

(1.5.7) Remark. One may also consider the spaces $\mathcal{M}(G,c)$ for arbitrary reductive groups G. They are just finite quotients of the spaces described in Theorem 1.5.5, cf. [9, Cor. 5.24].

(1.5.8) Example. Let us look at the 'classical case' studied by Atiyah [1], i.e. vector bundles of rank n, corresponding to principal GL_n-Bundles ξ over E. The exact sequence

$$1 \longrightarrow SL_n \longrightarrow GL_n \xrightarrow{\det} \mathbb{C}^* \longrightarrow 1$$

induces an isomorphism $\pi_1(GL_n) \cong \mathbb{Z}$ which identifies the topological type of ξ with the degree of the associated vector bundle $\xi \times^{GL_n} \mathbb{C}^n$. In the case that the determinant bundle $\xi \times^{GL_n} \bigwedge^n \mathbb{C}^n$ is trivial, the group of the bundle reduces to SL_n and, according to Theorem 1.5.3 and the respective references, we have

$$\mathcal{M}(GL_n, 0) \;=\; \mathcal{M}(SL_n) \;\cong\; (E^{n-1})/S_n \;\cong\; \mathbb{P}^{n-1}(\mathbb{C}).$$

Next, consider bundles with structure group $PGL_n = SL_n/\mu_n$. Then

$$\pi_1(PGL_n) \;=\; \mu_n \;=\; \left\{ e^{2\pi i m/n} \;\middle|\; m = 0, 1, \ldots, n-1 \right\}.$$

Let $c = e^{2\pi i m/n} \in \mu_n = C(SL_n)$ and $g = \gcd(m,n)$. The extended Dynkin diagram $\tilde{\Delta}$ may be identified with the unit circle $S^1 \subset \mathbb{C}$, the nodes being given by the subset μ_n. Then c acts simply by multiplication, i.e. through rotation by $2\pi m/n$, yielding g orbits of nodes for the group $\langle c\rangle$. According to Theorem 1.5.4, resp. the relevant references, we now have

$$\mathcal{M}(PGL_n, c) \;\cong\; (E^{g-1})/S_g \;\cong\; \mathbb{P}^{g-1}(\mathbb{C}).$$

Finally, consider GL_n-bundles ξ of arbitrary degree $m \in \mathbb{Z}$, and put $g = gcd(m,n)$ again. From the sequence

$$1 \longrightarrow \mathbb{C}^* \longrightarrow GL_n \longrightarrow PGL_n \longrightarrow 1,$$

and the 'anti-diagonal' quotient realisation $GL_n = (\mathbb{C}^* \times SL_n)/\mu_n$ we see that the topological type of the extended PGL_n-bundle $\bar{\xi} = \xi \times^{GL_n} PGL_n$ is $c = e^{-2\pi i m/n}$. Thus, by Theorem 1.5.5, we have a fibering

$$\mathcal{M}(GL_n, m) \longrightarrow \mathcal{M}\big(PGL_n, e^{-2\pi i m/n}\big) \;\cong\; \mathbb{P}^{g-1}(\mathbb{C}),$$

whose fibre may, in fact, be identified with $E/{}_fE \cong E$ (here $n = f.g$, and ${}_fE$ denotes the f-torsion points of E).

2. Automorphism Groups

Our final interest will be in the orbital geometry of 'big' or 'almost generic' bundles. These are principal G-bundles over E which, in any of their local deformations $\big((S,s),\,\Xi\big)$ with $\Xi_s \cong \xi$, are found over a space of small codimension in S. This corresponds to a small infinitesimal deformation space $H^1\big(E,\,\mathrm{ad}(\xi)\big)$, or, using Serre and Killing duality, to a small automorphism group, $\mathrm{Lie}\big(\mathrm{Aut}(\xi)\big) \cong H^0\big(E,\,\mathrm{ad}(\xi)\big)$. We shall therefore investigate the automorphism groups of G-bundles in this section.

As is easily seen, there are unstable bundles with automorphism groups of arbitrarily large dimension. In contrast, as shown below, 2.2, all semistable bundles have an automorphism group embeddable into the original group G. In that sense, they all may be considered to be big.

(2.1) **Definition.** Let G be reductive. A semistable G-bundle is called *regular* (resp. *semisimple*) if its automorphism group has minimal dimension among all semistable G-bundles (resp. maximal dimension among all members in its semistable S-equivalence class).

For simple groups $G = \tilde{G}/\langle c\rangle$ of the type considered in Theorems 1.5.3 and 1.5.4, we have the following result which is a reformulation, from the angle of [26, 3.10], of results and constructions contained in [9, especially Theorem 4.1, Corollaries 6.2 and 6.3].

(2.2) **Theorem.** *Each S-equivalence class $[\xi]$ of semistable bundles in $\mathcal{M}(G,c)$ consists of finitely many isomorphism classes which are in a bijective, adjacency-preserving correspondence with the the unipotent conjugacy classes in a reductive subgroup $Z\big([\xi]\big)$ of rank ℓ_c in G. Under this correspondence, the automorphism group $\mathrm{Aut}_G(\xi)$ is identified with the stabilizer of the corresponding unipotent element in $Z\big([\xi]\big)$. Thus, for all $\xi \in [\xi]$ we have*

$$\ell_c \;\leqslant\; \dim \mathrm{Aut}_G(\xi) \;\leqslant\; \dim Z\big([\xi]\big) \quad \text{and} \quad \dim \mathrm{Aut}_G(\xi) \;\equiv\; \ell_c \pmod 2.$$

There are unique isomorphism classes of regular and of semisimple bundles in $[\xi]$. If ξ is a regular bundle, then $\mathrm{Aut}_G(\xi)^\circ$ is an abelian subgroup of $Z\big([\xi]\big)$ of dimension ℓ_c. If ξ is a semisimple bundle, then $\mathrm{Aut}_G(\xi) = Z\big([\xi]\big)$. Furthermore, there is an open dense subset $\mathcal{M}(G,c)_{rs} \subset \mathcal{M}(G,c)$ whose members $[\xi]$ consist of a single, regular and semisimple, isomorphism class.

Call a semistable G-bundle ξ with $[\xi] \in \mathcal{M}(G,c)$, *subregular* if $\dim \mathrm{Aut}_G(\xi) = \ell{+}2$. Then we can add the following to (2.2), using [26, 5.4].

(2.3) **Addition.** *An S-equivalence class $[\xi]$ of semistable bundles in $\mathcal{M}(G,c)$ contains subregular bundles if and only if $Z\big([\xi]\big)^\circ$ is not a torus, i.e. of dimension $\geqslant \ell{+}2$. In that case, the isomorphism classes of subregular bundles are in bijection with the subregular unipotent classes in $Z\big([\xi]\big)$ which are, themselves, in bijection with the orbits of the action of the component group $Z\big([\xi]\big)/Z\big([\xi]\big)^\circ$ on the simple (quasi-)factors of $Z\big([\xi]\big)^\circ$.*

Let now ξ be an unstable G-bundle. We can get useful insight into its automorphism group $\mathrm{Aut}_G(\xi)$ by exploiting its canonical reduction $(L,\,\xi_L)$ and its Atiyah-Bott type $\mu(\xi)$. Let $P = P(\xi) \supset L$ be the associated parabolic. Recall that the Lie algebra of

$\mathrm{Aut}_G(\xi)$ can be written as the space of global sections of the adjoint bundle. Using the decomposition $\mathfrak{g} = \bigoplus_{\nu\in M} \mathfrak{g}(\nu)$ of $\mathfrak{g}$ with respect to $\mu(\xi)$, we get

$$\begin{aligned}\mathrm{Lie}\big(\mathrm{Aut}_G(\xi)\big) &= H^0\big(E, \mathrm{ad}(\xi)\big) = H^0(E, \xi_L \times^L \mathfrak{g}) = \\ &= H^0\big(E, \mathrm{ad}(\xi_L)\big) \oplus \bigoplus_{\nu\in M_{>0}} H^0\big(E, \xi_L \times^L \mathfrak{g}(\nu)\big) = H^0\big(E, \mathrm{ad}(\xi_P)\big)\end{aligned}$$

(since only non-negative degrees contribute). The dimension can be computed in an elegant way. Namely, $\mathfrak{g}(\nu)$ is spanned by the root spaces $\mathfrak{g}_\alpha$ for all roots α with $\alpha\big(\mu(\xi)\big) = \nu$ (and $\mathfrak{h}$, if ν is zero). Since ν is the slope of $\xi_L \times^L \mathfrak{g}(\nu)$, the degree of $\xi_L \times^L \mathfrak{g}(\nu)$, and hence the dimension of $H^0\big(E, \xi_L \times^L \mathfrak{g}(\nu)\big)$ with $\nu > 0$, is given by $\nu \cdot \dim \mathfrak{g}(\nu)$. Thus

$$\dim\Big(\bigoplus_{\nu\in M_{>0}} H^0\big(E, \xi_L \otimes^L \mathfrak{g}(\nu)\big)\Big) = \sum_{\alpha\in\Sigma+} \alpha\big(\mu(\xi)\big).$$

As usual ([5]), denote the half sum of the positive roots by $\rho \in X^*(T) \otimes \mathbb{Q}$, and let $\mathrm{Aut}_G(\xi)^+$ be the connected subgroup of $\mathrm{Aut}_G(\xi)$ corresponding to the subalgebra $\bigoplus_{\nu\in M_{>0}} H^0\big(E, \xi_L \times^L \mathfrak{g}(\nu)\big)$ of $\mathrm{Lie}\big(\mathrm{Aut}_G(\xi)\big)$. Collecting the above information, we obtain the following statement (cf. also [2, 10.7] and [9, Theorem 2.7]).

(2.4) **Proposition.** *Let ξ be an unstable G-bundle, (L, ξ_L) its canonical reduction and $\mu(\xi)$ its Atiyah-Bott type. Let $P = P(\xi) \supset L$ be the associated parabolic.*

- *(a) We have equality for the identity components,* $\mathrm{Aut}_G(\xi)^\circ = \mathrm{Aut}_P(\xi_P)^\circ$.
- *(b) The group* $\mathrm{Aut}_G(\xi)^+$ *is a unipotent normal subgroup of* $\mathrm{Aut}_G(\xi)$ *of dimension* $\langle 2\rho, \mu(\xi)\rangle$*, and* $\mathrm{Aut}_G(\xi)^\circ$ *is a semidirect product* $\mathrm{Aut}_L(\xi_L)^\circ \ltimes \mathrm{Aut}_G(\xi)^+$.
- *(c)* $\dim \mathrm{Aut}_G(\xi) = \dim \mathrm{Aut}_L(\xi_L) + \langle 2\rho, \mu(\xi)\rangle$.

According to this result, the problem of determining the dimension of $\mathrm{Aut}_G(\xi)$ is now broken into

- (i) the computation of $\dim \mathrm{Aut}_L(\xi_L)$, and
- (ii) the evaluation of $\langle 2\rho, \mu(\xi)\rangle$.

Concerning (i), we have Theorem 2.2 which, however, has to be completed by a precise description of the structure of the Levi subgroups of G. Concerning (ii), we have to determine the possible Atiyah-Bott types $\mu(\xi)$. In the following, we will address both problems in the simplest cases, i.e. for those bundles ξ whose parabolic type is a maximal parabolic subgroup P of G.

Let us fix some notations. Following [5], a system of simple roots in $X^*(T)$ will be denoted by $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$. Then there are well-defined fundamental co-weights $\varpi_k^\vee \in X_*\big(T \cap (G, G)\big) \otimes \mathbb{Q}$ $k = 1, \ldots, \ell$, i.e. $\langle \alpha_i, \varpi_k^\vee\rangle = \delta_{ik}$ with respect to the natural pairing. If G is semisimple, we also fix fundamental dominant weights $\varpi_k \in X^*(T) \otimes_Z \mathbb{Q}$. Let Λ be any integer lattice and $\omega \in \Lambda \otimes \mathbb{Q}$. We shall write $[\omega]$ for the smallest positive integer multiple of ω which lies in Λ.

(2.5) **Definition.** Let γ be an element in $\pi_1(G)$ and P a parabolic subgroup of G. An unstable bundle ξ of topological type γ is called *γ-P-maximal* if its associated parabolic subgroup is P and if its automorphism group $\mathrm{Aut}_G(\xi)$ has minimal dimension among all bundles of topological type γ with associated parabolic P. If a bundle ξ is *γ-P-maximal* for all $\gamma \in \pi_1(G)$ we will call it *P-maximal.*

In the following, we shall view $\mu(\xi)$ as an element of a fixed fundamental chamber. The associated parabolics are then 'standard'. We put $P_k := P_{\Delta\setminus\{\alpha_k\}}$ for the maximal standard parabolic subgroup of G attached to a simple root $\alpha_k \in \Delta$.

(2.6) **Theorem.** *Let G be semisimple and ξ an unstable G-bundle of type P_k. If ξ is P_k-maximal, we have*

$$\begin{aligned} \mu(\xi) &= \varpi_k^\vee/\langle[\varpi_k],\, \varpi_k^\vee\rangle, \quad i.e. \\ \dim \mathrm{Aut}_G(\xi)^+ &= \langle 2\rho,\, \varpi_k^\vee\rangle/\langle[\varpi_k],\, \varpi_k^\vee\rangle. \end{aligned}$$

If ξ is not necessarily P_k-maximal, its Atiyah-Bott type is a positive integral multiple of the element $\varpi_k^\vee/\langle[\varpi_k],\, \varpi_k^\vee\rangle$.

(2.7) Proof. Put $P = P_k$, $L = L_{\Delta\setminus\{\alpha_k\}}$, and $S = S_{\Delta\setminus\{\alpha_k\}}$. Consider the composition of group homomorphisms

$$\phi : \mathbb{C}^* \cong S \longrightarrow L \longrightarrow A \;=\; L/(L,L) \;=\; P/(P,P) \cong \mathbb{C}^*.$$

On the level of co-characters, the map $\phi_* : \mathbb{Z} = X_*(S) \longrightarrow X_*(A) = \mathbb{Z}$ is given by multiplication with the degree $\deg(\phi)$. Note that $X_*(S) = \mathbb{Z}[\varpi_k^\vee]$, and that ϕ is given by the character $[\varpi_k]$ restricted to $S \subset T$. Thus, $\deg(\phi) = \langle[\varpi_k],\, [\varpi_k^\vee]\rangle$, and $X_*(A)$ may be identified with the sublattice $\mathbb{Z}[\varpi_k^\vee]/\langle[\varpi_k],\, [\varpi_k^\vee]\rangle \subset X_*(S) \otimes \mathbb{Q}$. A minimal Chern class $\mu(\xi) \in X_*(T) \otimes \mathbb{Q}$ for a bundle ξ with associated parabolic P_k is thus given by

$$[\varpi_k^\vee]/\langle[\varpi_k],\, [\varpi_k^\vee]\rangle \;=\; \varpi_k^\vee/\langle[\varpi_k],\, \varpi_k^\vee\rangle. \qquad \square$$

(2.8) Example. Let us look at simple groups of type A_ℓ, i.e. $G = SL_n/\mu_g$, where g is a divisor of $n = \ell+1$. Then, using the tables of [5] and their notation, one can show

$$\begin{aligned} [\varpi_k] \;=\; \kappa.\varpi_k \quad \text{with} \quad \kappa \;=\; g/(g,k) \quad \text{and thus} \\ \langle 2\rho, \varpi_k^\vee\rangle/\langle[\varpi_k], \varpi_k^\vee\rangle \;=\; (g,k)\cdot(n/g). \end{aligned}$$

Here (a, b) denotes the greatest common divisor of the integers a and b. In particular, for SL_n, $(g = 1)$, the simply connected case, we get the value n for all k, and for PGL_n (where $g = n$), we get $\langle 2\rho,\, \varpi_k^\vee\rangle/\langle[\varpi_k],\, \varpi_k^\vee\rangle = (n, k)$.

Due to the complications, related to the embedding of the lattices $X_*(S) \subset X_*(A)$ and their intersection with the fundamental Weyl chamber, it seems practically impossible, apart from individual cases, to develop similar formulae or, at least, reasonable estimates for arbitrary parabolic subgroups. For our purposes, some general geometric properties will do. They may also be of independent interest.

3. Degenerations

Let G be reductive, again, with pair (T,B) fixing a fundamental Weyl chamber K in $X_*(T) \otimes \mathbb{Q}$. On $X_*(T) \otimes \mathbb{Q}$ we fix some invariant scalar product, which we shall also denote by $\langle\ ,\ \rangle$, thus identifying $X_*(T) \otimes \mathbb{Q}$ with ist dual $X^*(T) \otimes \mathbb{Q}$. Note that the central part $X_*\big(C(G)\big) \otimes \mathbb{Q}$ will split off as an orthogonal summand. Let $L = L_\Theta$ be a standard Levi subgroup with connected center $S \subset T$ and commutative quotient $A = L/(L, L)$.

(3.1) **Lemma.** *Under the identification of $X_*(A) \otimes \mathbb{Q}$ with $X_*(S) \otimes \mathbb{Q}$ induced by the isogeny $S \longrightarrow A$, the projection $\pi : X_*(T) \otimes \mathbb{Q} \longrightarrow X_*(A) \otimes \mathbb{Q}$ coincides with the orthogonal projection along the subspace $X_*\big(T \cap (L, L)\big) \otimes \mathbb{Q}$.*

(3.2) Proof. Since $X_*\big(T \cap (L, L)\big) \otimes \mathbb{Q}$ is the kernel of π, we only have to show that this space is orthogonal to $X_*(S) \otimes \mathbb{Q}$. But these spaces are generated by the simple co-roots $\alpha^\vee, \alpha \in \theta$, resp. the fundamental co-weights $\varpi_\beta^\vee$ with $\beta \in \Delta \setminus \Theta$ and the center $X_*\big(C(G)\big) \otimes \mathbb{Q}$, which are orthogonal to each other. □

Recall the notion of adjacency, 1.5.1.

(3.3) **Proposition.** *Let ξ be an unstable G-bundle of Atiyah-Bott type $\mu(\xi)$ lying in $X_*\big(T/(T \cap (P, P))\big)$ with associated parabolic P. Let Q be a parabolic containing P. Then ξ is adjacent to some unstable bundle of Atiyah-Bott type $\pi\big(\mu(\xi)\big)$ lying in $X_*\big(T/(T \cap (Q, Q))\big)$ with associated parabolic Q. In particular, any unstable G-bundle is adjacent to a semistable one.*

(3.4) Proof. Let us first reduce the general case to the special one ($Q = G$). From the canonical reduction (L, ξ_L) of ξ to the (standard) Levi subgroup $L \subset P$ we obtain another, intermediate reduction $\xi_M = \xi_L \times^L M$ of ξ by extension to the (standard) Levi subgroup $M \supset L$ of Q. By the construction of the canonical reduction, ξ_M will be unstable and, granting the special case of our claim for the group M, there will be a degenerating family $\big((S, s),\ \Xi_M\big)$ from a semistable M-bundle η_M to ξ_M. This family extends to a family of G-bundles $\big((S, s),\ \Xi\big)$, where $\Xi = \Xi_M \times^M G$, degenerating $\eta = \eta_M \times^M G$ into ξ. We now have to determine the Atiyah-Bott type $\mu(\eta)$ of η.

Note that the natural projection $\pi : X_*\big(T/(T \cap (L, L)\big) \longrightarrow X_*\big(T/(T \cap (M, M)\big)$ maps the Chern class $c(\xi_L)$ $(= \mu(\xi))$ of ξ_L to $c(\xi_M)$. Since Chern classes are topological invariants, they do not change in flat families, thus $c(\eta_M) = c(\xi_M)$. Using Lemma 3.1 and the acuteness of the dihedral angles of the fundamental Weyl chamber K (i.e. $\langle\alpha,\ \beta\rangle \leqslant 0$ for all $\alpha \neq \beta$ in Δ), we see that π maps the interior of the face $K \cap X_*\big(T/(T \cap (L, L)\big) \otimes \mathbb{Q}$ to the interior of the face $K \cap X_*\big(T/(T \cap (M, M)\big) \otimes \mathbb{Q}$. Thus, the element $\pi\big(\mu(\xi)\big)$ coincides with the Atiyah-Bott type $\mu(\eta)$.

We finally have to prove the special case ($Q = G$). Here, we need some insights from the deformation theory of ξ, which should be well-known. They are consequences of the parametrization of G-bundles in terms of $(0, 1)$-connections, [2], or of loop groups (for some facts, cf. [2, Section 15]). Because of $\dim_{\mathbb{C}}(E) = 1$, the deformation theory of ξ is unobstructed due to $H^2\big(E,\ \mathrm{ad}(\xi)\big) = 0$. We thus have a semi-universal deformation $\big((\mathfrak{U}, 0),\ \Xi\big)$ of ξ with smooth base space $(\mathfrak{U}, 0)$,

$$T_0\mathfrak{U} \;=\; H^1\big(E,\ \mathrm{ad}(\xi)\big) \;=\; H^0\big(E,\ \mathrm{ad}(\xi)\big)^\vee,$$

on which the automorphism group $\mathrm{Aut}_G(\xi)$ acts. There is then a multiplicative subgroup $M \cong \mathbb{C}^* \subset \mathrm{Aut}_L(\xi_L)^\circ \subset \mathrm{Aut}_G(\xi)$ given by an integer multiple of $\mu(\xi)$ inside $X_*(T)$ which acts on $\mathfrak{U}$ with non-positive weights. Let $\mathfrak{U}_-$ be the smooth subspace of $\mathfrak{U}$ which is contracted to the base point 0 by the action of M (when going to ∞). The bundle Ξ_u for a generic $u \in \mathfrak{U}_-$ will be semistable. Now, let S be the closure of the M-orbit of u inside $\mathfrak{U}_-$. The restricted family $\big((S,0), \Xi|_S\big)$ is a degeneration as requested. $\square$

There is a much more general result first proved for vector bundles by Shatz [25] and then reformulated and generalized to arbitrary structure groups by Atiyah and Bott [2, 7.8, 10.6, 12.8] which, in particular, gives a necessary criterion for adjacency. In the following, the Atiyah-Bott type $\mu(\xi)$ of a G-bundle ξ will be considered as an element in $V = X_*(T) \otimes \mathbb{R}$, and we will denote by $\widehat{W\mu}$ the convex hull of the Weyl group orbit of an element μ in V.

(3.5) **Theorem** (Atiyah-Bott). *Let Ξ be a family of G-bundles over the pointed space (S,s), i.e. a holomorphic G-bundle over the product space $E \times S$. Assume that for all $t \neq s$ the Atiyah-Bott type $\mu(\Xi_t)$ is a constant μ, and let Ξ_s be of Atiyah-Bott type μ_0. Then μ is contained in the convex hull $\widehat{W\mu_0}$ of the W-orbit of μ_0.*

The convex hull of $W\mu_0$ can be conveniently described, cf. [2, Lemma 12.14].

(3.6) **Lemma.** *Let μ and μ_0 be elements of the fundamental chamber $C \subset V$. Then $\mu \in \widehat{W\mu_0} \iff$ for all $\omega \in C$ we have $\langle \omega, \mu \rangle \leqslant \langle \omega, \mu_0 \rangle$.*

(3.7) Example. In Proposition (3.3), we have a degeneration of Atiyah-Bott type $\mu = \pi(\mu_0)$ into Atiyah-Bott type $\mu_0 = \mu(\xi)$, where π is the orthogonal projection onto a face C' of C. A quick way to see that $\mu \in \widehat{W\mu_0}$ is to write μ as a weighted sum over the stabilizer W' of C' in W:

$$\mu \;=\; \pi(\mu_0) \;=\; \frac{1}{|W'|} \sum_{w \in W'} w(\mu_0).$$

(3.8) Example. Let $G = Spin_{10}$ of type D_5. For any $k = 1, \ldots, 5$, there are P_k-maximal unstable G-bundles of Atiyah-Bott type $\mu_k = \varpi_k / \langle \varpi_k, \varpi_k \rangle$ (we are identifying weights and co-weights). In a later context, we are interested in the question whether there exist families of G-bundles, whose generic member is of type μ_2 and whose special member has type μ_5. For that, we check the inequality of Lemma 3.6 on the element $\omega = 2\mu_2 \in C$. In the notations of [5], we have:

$$\mu_2 \;=\; (\varepsilon_1 + \varepsilon_2)/2, \quad \mu_5 \;=\; 2(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5)/5,$$
$$\langle \omega, \mu_2 \rangle \;=\; 1 \quad \text{and} \quad \langle \omega, \mu_5 \rangle \;=\; 4/5.$$

Thus, there are no such families.

The following rounds off our geometric discussion.

(3.9) **Proposition.** *Let ξ and η be G-bundles. If ξ is adjacent to η, then*

$$\dim \mathrm{Aut}(\xi) \;>\; \dim \mathrm{Aut}(\eta).$$

(3.10) PROOF. Let $\big((S,s),\,\Xi\big)$ be a degeneration of η into ξ. Locally at s, this family is induced by a morphism ϕ from the germ (S,s) to the base space $\mathfrak{U}$ of the semi-universal deformation of ξ. Because of the non-triviality of the family Ξ, the image of ϕ has to have positive dimension. Openness of versality inside $\mathfrak{U}$ now implies $\dim H^1\big(E,\,\mathrm{ad}(\eta)\big) < \dim H^1\big(E,\,\mathrm{ad}(\xi)\big)$. Serre and Killing duality give the corresponding inequality for the Lie algebras of the automorphism groups. □

4. On the Structure of Levi Subgroups

We will now turn to the structure of the maximal Levi subgroups of G. Let us start with the following general recollections.

(4.1) **Lemma.** *Let G be semisimple and $L \subset G$ a Levi subgroup $L = L_\Theta \subset P_\Theta$ associated with a subset $\Theta \subset \Delta$. Denote the semisimple derived subgroup (L,L) of L by H and put $S = C(L)^\circ$.*

- *(a) The group L is an almost direct product of H and S, i.e. $L = S.H$ and $S \cap H$ is finite. H itself is an almost direct product $H = H_1H_2\cdots H_r$ of simple groups H_j corresponding to the connected components of Θ.*
- *(b) If L is maximal, i.e $L = L_k = L_\Theta$ where $\Theta = \Delta \setminus \{\alpha_k\}$, then $S \cap H$ is a cyclic subgroup of $S \cong \mathbb{C}^*$ of order $d = \langle [\varpi_k],\,[\varpi_k^\vee]\rangle$.*
- *(c) If G is simply connected, the groups H, H_j are as well, and H is a direct product $H = \prod_{j=1}^r H_j$.*

(4.2) PROOF. These facts are well known (cf. e.g. [27, Chap. 8, esp. 8.4.6.(6)]). If L is maximal, we only have to add that the subtorus $S \cong \mathbb{C}^*$ is generated by $[\varpi_k^\vee] \in X_*(T)$, and that the intersection $S \cap H$ equals the kernel of the character $[\varpi_k] : L \longrightarrow \mathbb{C}^*$, restricted to S. □

The following lemma provides some topological information.

(4.3) **Lemma.** *Let G be semisimple and $L \subset G$ a Levi subgroup $L = L_\Theta \subset P_\Theta$ associated with a subset $\Theta \subset \Delta$. Denote the semisimple derived subgroup (L,L) of L by H and the connected centre $C(L)^\circ$ by S. Put $\bar{S} = L/H = S/S\cap H$ and $\bar{H} = L/S = H/S \cap H$. Then the long exact homotopy sequences associated with the fibrations*

$$H \longrightarrow L \longrightarrow L/H, \quad S \longrightarrow L \longrightarrow L/S, \quad L \longrightarrow G \longrightarrow G/L, \quad H \longrightarrow G \longrightarrow G/H$$

induce short exact sequences

$$\begin{aligned}
0 &\longrightarrow \pi_1(H) \longrightarrow \pi_1(L) \longrightarrow \pi_1(\bar{S}) \longrightarrow 0,\\
0 &\longrightarrow \pi_1(S) \longrightarrow \pi_1(L) \longrightarrow \pi_1(\bar{H}) \longrightarrow 0,\\
0 &\longrightarrow \pi_2(G/L) \longrightarrow \pi_1(L) \longrightarrow \pi_1(G) \longrightarrow 0,\\
0 &\longrightarrow \pi_1(H) \longrightarrow \pi_1(G) \longrightarrow \pi_1(G/H) \longrightarrow 0.
\end{aligned}$$

(4.4) PROOF. These sequences are easy consequences of the vanishing $\pi_2(\bar{S}) = 0$, $\pi_2(\bar{H}) = 0$, $\pi_1(G/L) = \pi_1(G/P) = 0$, $\pi_2(G) = 0$ and $\pi_2(G/H) = 0$. For the last equality, observe that G/H is finitely covered by $\tilde{G}/\tilde{H}$, where $\tilde{G}$ and $\tilde{H}$ are the simply connected coverings of G and H, and that $\pi_2(\tilde{G}/\tilde{H}) = 0$ (use $\pi_1(\tilde{H}) = \pi_1(\tilde{G}) = 0$, Lemma 4.1). □

(4.5) REMARK. Henceforth, we shall call the map $\pi_1(L) \longrightarrow \pi_1(\bar{S})$ from the first sequence the *degree map*. If $L = L_k$ is maximal and η an L-bundle of topological type $\gamma = \gamma(\eta)$ in $\pi_1(L)$, then the image of γ in $\pi_1(\bar{S}) = \mathbb{Z}$ is an integer, the degree of the line bundle $\eta \times^L \mathbb{C}$ associated with the character $[\varpi_k]$.

(4.6) REMARK. For computational questions, especially like those to come in the non-simply connected case, it may be helpful to interpret the exact sequences of Lemma 4.3 in terms of lattices related to $X_*(T) = \pi_1(T)$. For any simple root α, let $\alpha^\vee \in X_*(T)$ denote the corresponding co-root and $\varpi_\alpha \in X^*(T) \otimes \mathbb{Q}$ the corresponding fundamental dominant weight. Let T' be the subtorus $T \cap H$ and $\bar{T}$ the quotient torus $T/S = T \cap H/S \cap H$. We have

$$\begin{aligned} X_*(S) &= \{\gamma \in X_*(T) \mid \langle \alpha, \gamma \rangle = 0 \text{ for all } \alpha \in \Theta\} \quad \text{and} \\ X_*(T') &= \{\gamma \in X_*(T) \mid \langle \varpi_\beta, \gamma \rangle = 0 \text{ for all } \beta \in \Delta \setminus \Theta\}. \end{aligned}$$

Let $Q^\vee$, resp. $Q'^\vee$, denote the sublattice of $X_*(T)$ generated by all dual roots, resp. by those dual to Θ. Then the above four sequences can be written in the following form

$$\begin{array}{l} 0 \longrightarrow X_*(T')/Q'^\vee \longrightarrow X_*(T)/Q'^\vee \longrightarrow X_*(T)/X_*(T') \longrightarrow 0 \\ 0 \longrightarrow X_*(S) \longrightarrow X_*(T)/Q'^\vee \longrightarrow X_*(\bar{T})/Q'^\vee \longrightarrow 0 \\ 0 \longrightarrow Q^\vee/Q'^\vee \longrightarrow X_*(T)/Q'^\vee \longrightarrow X_*(T)/Q^\vee \longrightarrow 0 \\ 0 \longrightarrow X_*(T')/Q'^\vee \longrightarrow X_*(T)/Q^\vee \longrightarrow X_*(T)/\big(X_*(T') + Q^\vee\big) \longrightarrow 0 \end{array}$$

In the third sequence, we have the interpretation

$$\pi_2(G/L) = \pi_2(G/P) = H_2(G/P, \mathbb{Z}) = Q^\vee/Q'^\vee = \bigoplus_{\beta \in \Delta \setminus \Theta} \mathbb{Z}\beta^\vee.$$

(4.7) REMARK. The first two of the sequences in 4.3, together with the covering group sequences,

$$\begin{array}{l} 0 \longrightarrow \pi_1(S) \longrightarrow \pi_1(\bar{S}) \longrightarrow \pi_0(S \cap H) = S \cap H \longrightarrow 0, \\ 0 \longrightarrow \pi_1(L) \longrightarrow \pi_1(\bar{H}) \longrightarrow \pi_0(S \cap H) = S \cap H \longrightarrow 0, \end{array}$$

give rise to an exact sequence of the following form.

$$0 \longrightarrow \pi_1(L) \longrightarrow \pi_1(\bar{S}) \times \pi_1(\bar{H}) \xrightarrow{+} \pi_0(S \cap H) = S \cap H \longrightarrow 0,$$

which embeds $\pi_1(L)$ into the product $\pi_1(\bar{S}) \times \pi_1(\bar{H})$ as the subset of all pairs (α, β) such that $\pi(\alpha) = -\pi(\beta)$. Here π denotes the respective projection to $S \cap H$. This is a consequence of general covering theoretic facts, cf. also [2, Ch. 6].

Here is a result in the simply connected case.

(4.8) **Corollary.** *Assume that G is simply connected and that L is a maximal Levi subgroup $L = L_k$. Let $d = \langle \varpi_k, [\varpi_k^\vee] \rangle$. Then the sequence*

$$0 \longrightarrow \pi_1(S) \longrightarrow \pi_1(L) \longrightarrow \pi_1(\bar{H}) \longrightarrow 0$$

is isomorphic to the sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{d\cdot} \mathbb{Z} \longrightarrow \mathbb{Z}/d\mathbb{Z} \longrightarrow 0.$$

The situation for non-simply connected groups turns out to be quite involved, from the algebra as well as from the topology. Since our main interest lies in the case where G is simple and simply connected, we shall separate the two cases and treat the simply connected case first.

5. Simply Connected Simple Groups

Let G be simple and simply connected, and let $L = L_k$ be a maximal Levi subgroup G. According to Lemma 4.1, the group L may be described as the quotient $\mathbb{C}^* \times^{\mu_d, \varphi} H$ of $\mathbb{C}^* \times H$ by a central cyclic subgroup $C_d = \left\{ (t, \varphi(t)^{-1}) \in \mathbb{C}^* \times H \;\middle|\; t \in \mu_d \right\}$ associated with an embedding ('identification') $\varphi : \mu_d \longrightarrow C(H)$ of μ_d into the center of H. Note that, in the product $C(H) = \prod_{j=1}^r C(H_j)$, all factors $C(H_j)$ are cyclic subgroups μ_{d_j}, except in case D_{even}, where we have $\mu_2 \times \mu_2$. Thus, any component φ_j of φ can be written in the form $t \longmapsto t^{m_j}$ (resp. $t \longmapsto (t^{m_j}, t^{n_j})$, if H_j is of type D_{even}). Here, the m_j, (n_j), $j = 1, \ldots, r$, are integers satisfying $d_j\, m_j \equiv d_j\, n_j \equiv 0$ modulo d. They are uniquely determined modulo d and up to outer automorphism of the factors H_j. Conversely, they determine φ and thus L, up to isomorphism.

In the following discussion, we will refer to these data by the expression

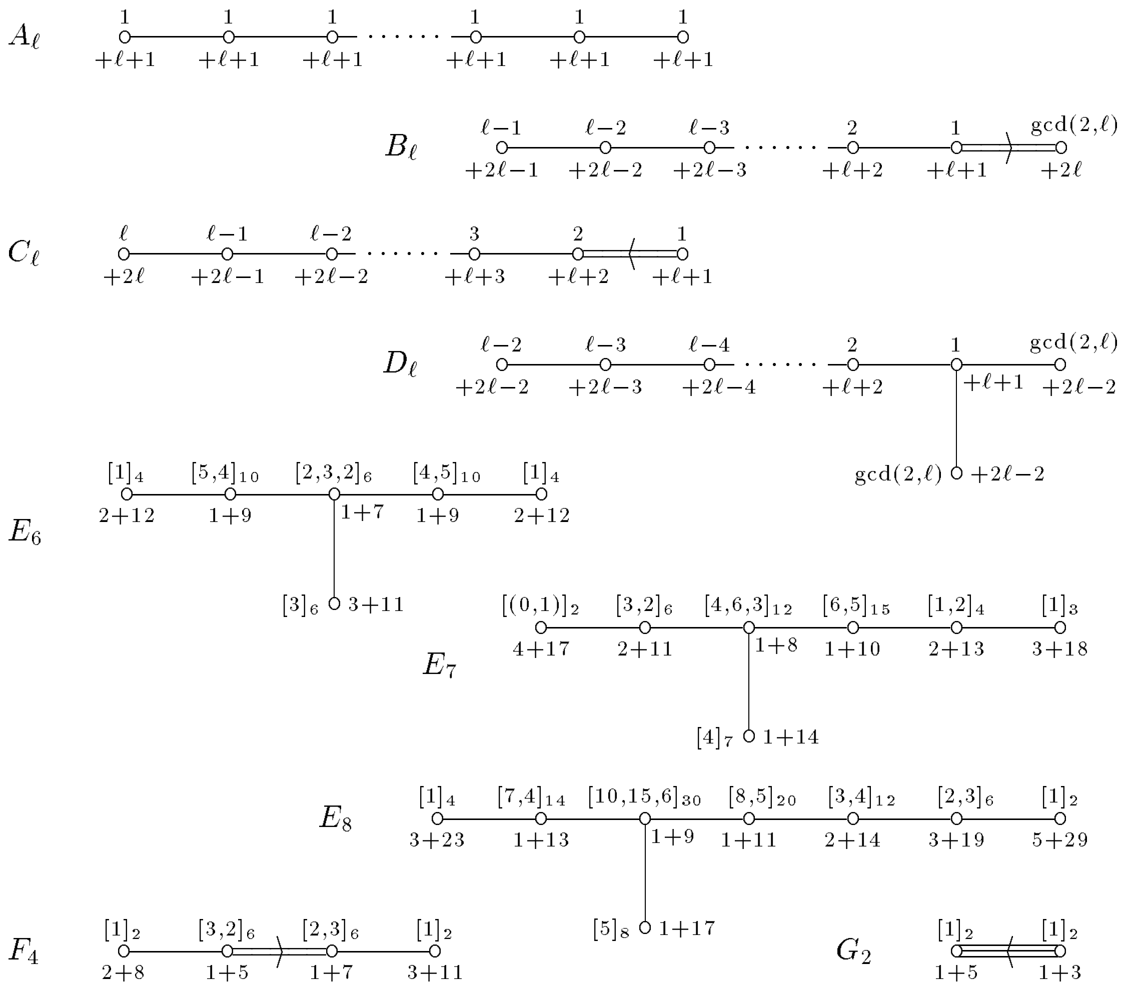
$$[m_1, \ldots, m_r]_d, \qquad \text{resp.} \quad [m_1, \ldots, (m_j, n_j), \ldots, m_r]_d,$$

which we shall call the *symbol* of the Levi subgroup L. Assuming the Dynkin diagrams in the form and with the numbering as in [5] or the tables below, the types of the simple factors H_j of L and their order in the symbol can be simply read off from their position in the Dynkin diagram Δ, i.e. the left factor will correspond to the left component of $\Delta \setminus \{\alpha_k\}$, etc.

Finally note that, for any positive integer κ, the symbol $[\kappa m_1, \ldots, \kappa m_r]_{\kappa d}$ gives rise, in a similar way, to an abstract algebraic group $\mathbb{C}^* \times^{\mu_{\kappa d}, \varphi \circ \kappa} H$ isomorphic to L. Whereas the reconstruction of the groups L as subgroups inside G leads only to 'reduced' symbols (since μ_d embeds into H), we shall nonetheless make use of this fact for the purpose of notational simplification and uniformization.

In the table of the following theorem, we shall list the dimensions of the P-maximal G-bundles ξ for all simple and simply connected G and for all maximal $P \subset G$. This is done slightly differently for classical and exceptional types. In the classical case, one finds, above or in front of every node $\alpha_k \in \Delta$, the dimension $\dim \mathrm{Aut}_{L_k}(\xi_{L_k})$ of the automorphism group of the canonical semistable reduction ξ_{L_k} of ξ. Below or behind the node, one finds added (with an extra $+$-sign) the dimension $\dim \mathrm{Aut}_G(\xi)^+$ as given in Theorem 2.6. In the exceptional cases, these two numbers are always put together, below or behind the relevant node. Above or in front of the node, we find the symbol of the corresponding Levi subgroup L_k from which we will compute $\dim \mathrm{Aut}_{L_k}(\xi_{L_k})$ by means of Theorem 2.2.

(5.1) **Theorem.** *Let G be simple and simply connected, α_k a simple root and P_k the associated parabolic subgroup of G. Let ξ be a P_k-maximal G-bundle. Then the dimension of its automorphism group $\mathrm{Aut}_G(\xi)$ is given in the following table.*



(5.2) Proof. Let ξ be a P_k-maximal G-bundle with canonical semistable reduction $(L,\,\xi_L)$, $L = L_k$. The formula $\dim \mathrm{Aut}_G(\xi)^+ = \langle 2\rho,\, \varpi_k^\vee\rangle / \langle \varpi_k,\, \varpi_k^\vee\rangle$ provided by Theorem 2.6 can be easily evaluated with the help of the tables in [5].

(5.2.1) The computation of $\dim \mathrm{Aut}_L(\xi_L)$ will be reduced to Theorem 2.2 as follows. Let us adhere to the previous notation as in 4.1, 4.3, 4.8, and the subsequent discussion. Then we have an exact sequence

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(S) & \longrightarrow & \pi_1(L) & \longrightarrow & \pi_1(\bar H) & \longrightarrow & 1\\
 & & \| & & \| & & \| & & \\
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\;d\cdot\;} & \mathbb{Z} & \longrightarrow & \mathbb{Z}/d\mathbb{Z} & \longrightarrow & 0.
\end{array}$$

As fundamental group, we may identify $\mathbb{Z}/d\mathbb{Z}$ with the cyclic subgroup $S\cap H = \varphi(\mu_d)$ in the center of H. Because of the realisation $L = \mathbb{C}^* \times^{\mu_d,\varphi} H$, the generator $\bar 1$ in $\mathbb{Z}/d\mathbb{Z}$ is then identified with the inverse of the generator $c = \varphi(e^{2\pi i/d})$.

By extension of structure group, $L \longrightarrow \bar H$, any semistable L-bundle η of topological type $\gamma(\eta) = m \in \mathbb{Z}$ induces a semistable $\bar H$-bundle $\bar\eta = \eta \times^L \bar H$ of topological type

$\gamma(\bar\eta) = \bar m \in \mathbb{Z}/d\mathbb{Z}$. Conversely, since $H^2(E, \underline{\mathbb{C}}^*_E) = 0$, every semistable $\bar H$-bundle $\bar\eta$ reduces to a semistable L-bundle η, unique up to an action of $H^1(E, \underline{\mathbb{C}}^*_E) \cong E \times \mathbb{Z}$ (cf. e.g. [11]). A comparison of the adjoint bundles $\mathrm{ad}(\eta)$ and $\mathrm{ad}(\bar\eta)$ immediately shows

$$\dim \mathrm{Aut}_L(\eta) \;=\; \dim \mathrm{Aut}_{\bar H}(\bar\eta) + 1.$$

Assume that η, like ξ_L, is of minimal Chern class, i.e. $\gamma(\eta) = 1$. Then the topological class of $\bar\eta$ is $\bar 1 \in \mathbb{Z}/d\mathbb{Z}$. Thus, the isomorphism class of $\bar\eta$ belongs to

$$\mathcal{M}\big(\bar H,\, c^{-1}\big) \;=\; \mathcal{M}\big(H/\langle c\rangle,\, c^{-1}\big) \;=\; \prod_{j=1}^{r} \mathcal{M}\big(H_j/\langle c_j\rangle,\, c_j^{-1}\big),$$

where we have expressed c as a product $c = \prod_{j=1}^r c_j$ in $H = \prod_{j=1}^r H_j$. The dimension of $\mathrm{Aut}_{\bar L}(\bar\eta)$ is now described in Theorem 2.2. In particular, for regular η, like ξ_L, we get the minimal value

$$\dim \mathrm{Aut}_L(\eta) \;=\; \sum_{j=1}^{r} \ell_{c_j} \;+\; 1,$$

where $\ell_{c_j} + 1$ is the number of $\langle c_j\rangle$-orbits on the extended Dynkin diagram of H_j, cf. Theorem 1.5.4 and its preceding explanations.

In the classical cases, the structure of the maximal Levi subgroups L can be determined by direct inspection.

(5.2.2) CASE A_ℓ. Let $n = \ell + 1$. The k-th Levi subgroup L_k may be written as $S(GL_k \times GL_{n-k}) = \big\{\, (g,h) \in GL_k \times GL_{n-k} \;\big|\; \det(g).\det(h) = 1 \,\big\}$. Its (possibly non-reduced) symbol with respect to the product $H_k = SL_k \times SL_{n-k}$ is $[n-k, k]_{k(n-k)}$, yielding the generating element $c = (e^{2\pi i/k}, e^{2\pi i/n-k})$ of $S \cap H$ inside the center $C(H_k) = \mu_k \times \mu_{n-k}$. Let η be a semistable L_k-bundle of minimal topological type $\gamma(\eta) = 1$. Using Example 1.5.8, we get $0+0+1 = 1$ for the dimension of its automorphim group $\mathrm{Aut}_{L_k}(\eta)$.

(5.2.3) CASE B_ℓ. Let $n = 2\ell + 1$, and let us first look at the extremal case of L_ℓ. The corresponding Levi subgroup OL_ℓ of the orthogonal group SO_n has the simple form $OL_\ell = GL_\ell$. The preimage $L_\ell \subset Spin_n$ is given as the double ('spin') cover ${}_sGL_\ell = \big\{\, (g,s) \in GL_\ell \times \mathbb{C}^* \;\big|\; \det(g) = s^2 \,\big\}$, which may be described by the symbol $[2]_\ell$. This corresponds to the central element $c = e^{4\pi i/\ell}$ in $\mu_\ell = C(H_\ell)$. The dimension of $\mathrm{Aut}(\eta)$ for a regular semistable ${}_sGL_\ell$-bundle η with $\gamma(\eta) = 1$ is thus $0+1$ or $1+1$, according to whether ℓ is odd or even. Let us now look at the remaining cases, $k < \ell$. There, the Levi subgroup $OL_k \subset SO_n$ admits a product decomposition $OL_k = GL_k \times SO_{n-2k}$ of type $A_{k-1} \times B_{\ell-k}$. This is not preserved under pull-back into $Spin_n$. However, like in the case of spin groups for direct sums, the group L_k is now a quotient $L_k \cong ({}_sGL_k \times Spin_{n-2k})/\mu_2$ of the individual spin covers (μ_2 diagonally embedded)). It can thus be described by the symbol $[2,k]_{2k}$, corresponding to the central element $c = (e^{2\pi i/k}, -1) \in \mu_k \times \mu_2 = C(H_k)$. This gives the dimension $0 + (\ell - k - 1) + 1 = \ell - k$ for the group $\mathrm{Aut}_{L_k}(\eta)$ of a regular semistable L_k-bundle η with $\gamma(\eta) = 1$.

(5.2.4) CASE C_ℓ. Here, the Levi subgroup L_k is a product $GL_k \times Sp_{2(\ell-k)}$ of type $A_{k-1} \times C_{\ell-k}$. Its symbol with respect to the product $H_k = SL_k \times Sp_{2(\ell-k)}$ is given by $[1,0]_k$, corresponding to the central element $c = (e^{2\pi i/k}, 1) \in \mu_k \times \mu_2 = C(H_k)$. The dimension of $\mathrm{Aut}_{L_k}(\eta)$ for a regular L_k-bundle is thus $0 + (\ell - k) + 1 = \ell - k + 1$.

(5.2.5) CASE D_ℓ. This case is very similar to that of B_ℓ. For the extremal indices $k = \ell - 1, \ell$, the corresponding Levi subgroup L_k is again ${}_sGL_\ell$, leading to minimal dimensions 1 or 2, according to the parity (odd/even) of ℓ. In the remaining cases, $k < \ell - 1$, the Levi subgroups L_k are of semisimple type $A_{k-1} \times D_{\ell-k}$ and realized as quotients $L_k \cong ({}_sGL_k \times Spin_{2(\ell-k)})/\mu_2$ (μ_2 diagonally embedded, also into the center $\mu_2 \times \mu_2 \subset Spin_{2(\ell-k)}$, if $\ell - k$ is even). They are described by the symbol $[2,k]_{2k}$ ($\ell - k$ odd) resp. $[2,(k,k)]_{2k}$ ($\ell - k$ even) , corresponding to central elements $c = (e^{2\pi i/k}, -1) \in \mu_k \times \mu_4$ resp. $c = (e^{2\pi i/k}, -1, -1) \in \mu_k \times \mu_2 \times \mu_2$ of H_k. This yields the minimal dimension $0 + (\ell - k - 2) + 1 = \ell - k - 1$.

(5.2.6) To investigate the structure of L_k in the exceptional cases, we employ a general method, which may also be used in the classical cases (it leads, however, to reduced symbols which, due to the underlying arithmetic, involve a greater number of case specifications). To determine the symbol of L_k, we look at irreducible representations W_j of L_k generated by the highest weight vectors v_j in certain irreducible representations V_j of G with highest weight ω_j. The action of the central torus $S_k \cong \mathbb{C}^*$ of L_k on W_j is then given by the weight $m = \langle \omega_j, [\varpi_k^\vee] \rangle \in \mathbb{Z}$. Comparing this with the action of the center of H_k on W_j, will allow an identification of the elements of $\mu_d \subset S_k$ with those in the center of H_k, provided we have looked at sufficiently many such representations W_j. This will be the case as soon as the direct sum $\bigoplus_j W_j$ is a faithful H_k-module. In most cases, choosing among the extremal fundamental dominant weights $\varpi_1, \varpi_2, \varpi_{\ell-1}, \varpi_\ell$ for G will do (notation as in the tables of [5]). Using different weights or, alternatively, outer automorphisms, we have occasionally been able to obtain smaller exponents m inside the symbol.

Let us demonstrate this procedure in three examples.

(5.2.7) Let G be of type E_8 with Levi subgroup L_4 of semisimple type $A_2 \times A_1 \times A_4$. In this case, may identify roots and coroots, and we immediately get $[\varpi_4^\vee] = \varpi_4^\vee = \varpi_4$ in $X_*(T)$. From the tables in [5], we read off $d = \langle \varpi_4, \varpi_4 \rangle = 30$. The fundamental G-representations with dominant weights ϖ_1, ϖ_2 and ϖ_8 will, on their respective top-level, induce the natural representations of the factors of $H_4 \cong SL_3 \times SL_2 \times SL_5$. Thus their sum is faithful. The central subgroup $S_4 \cong \mathbb{C}^* \subset L_4$ will act on these modules by means of the weights $\langle \varpi_1, \varpi_4 \rangle = 10$, $\langle \varpi_2, \varpi_4 \rangle = 15$, and $\langle \varpi_8, \varpi_4 \rangle = 6$, yielding the symbol $[10,15,6]_{30}$ and the central element $c = (e^{2\pi i/3}, e^{\pi i}, e^{2\pi i/5})$ in the center $\mu_3 \times \mu_2 \times \mu_5$ of H_4. The minimal dimension of $\mathrm{Aut}(\eta)$ for a semistable L_4-bundle η with $\gamma(\eta) = 1$ is thus $0 + 0 + 0 + 1 = 1$.

(5.2.8) Next, consider G of type E_7 with Levi subgroup L_1 of semisimple type D_6. Again, we may identify roots and coroots, and we see from the tables in [5] that $[\varpi_1^\vee] = \varpi_1^\vee = \varpi_1 \in X_*(T)$. Furthermore, $\langle \varpi_1, \varpi_1 \rangle = 2$. The fundamental G-representations with dominant weights ϖ_2 and ϖ_3 will, on their top-levels, induce the two half-spin representations Δ_+ and Δ_- on $H_1 \cong Spin_{12}$, whose sum, $\Delta_+ \oplus \Delta_-$, realizes a faithful representation of $Spin_{12}$. Restricted to the center $\mu_2 \times \mu_2$ of $Spin_{12}$, the two representations $\Delta_{+/-}$ realize the two projections. The central torus $S_1 \cong \mathbb{C}^*$ of L_1 acts on these modules by the weights $\langle \varpi_2, \varpi_1 \rangle = 2$ and $\langle \varpi_3, \varpi_1 \rangle = 3$. Restriction to the subgroup $\mu_2 \subset S_1$ now yields the symbol $[(0,1)]_2$ and the corresponding central element in $c = (1,-1) \in \mu_2 \times \mu_2 = C(H_1)$. The minimal dimension of $\mathrm{Aut}(\eta)$ for a semistable L_1-bundle η with $\gamma(\eta) = 1$ is thus $3+1=4$ (the extended Dynkin diagram of type D_6 is just folded once over its midpoint by the automorphism corresponding to the element c).

(5.2.9) Finally, let us look at G of type E_6 with Levi subgroup L_5 of semisimple type $A_4 \times A_1$. Again, we identify roots and coroots. This time, we find from the tables of [5] that $[\varpi_5^\vee] = 3\varpi_5^\vee = 3\varpi_5$ in the 'simply connected' coroot lattice $X_*(T)$ and that $d = \langle \varpi_5, [\varpi_5^\vee]\rangle = 10$. The fundamental G-representations with dominant weights ϖ_1 and ϖ_6 will, on their top-levels, induce the natural representations of the factors of $H_5 \cong SL_5 \times SL_2$. Thus their sum is faithful. The central subgroup $S_5 \cong \mathbb{C}^* \subset L_5$ will act on these modules by means of the weights $\langle \varpi_1, [\varpi_5^\vee]\rangle = 4$ and $\langle \varpi_6, [\varpi_5]^\vee\rangle = 5$, yielding the symbol $[4,5]_{10}$ and the central element $c = (e^{4\pi i/5}, -1)$ in the center $\mu_5 \times \mu_2 = C(H_5)$. The minimal dimension of $\mathrm{Aut}(\eta)$ for a semistable L_5-bundle η with $\gamma(\eta) = 1$ is now $0+0+1=1$. □

(5.3) **Definition.** An unstable G-bundle ξ of topological type $\gamma \in \pi_1(G)$ is called *regular* if its automorphism group $\mathrm{Aut}_G(\xi)$ has minimal dimension among all unstable G-bundles of topological type γ.

(5.4) Remark. This definition is similar to the definition for semistable bundles 2.1. However, as seen below, there is a discrepancy in the dimensions of $\mathrm{Aut}_G(\xi)$ for regular semistable and regular unstable bundles. To 'correct' this fact, one has to regard G-bundles as parametrized by extended loop groups with conjugacy as equivalence, as we will do in subsequent work, cf. [13], [14] for an outline. Then the elements representing regular, resp. subregular, bundles ξ will acquire the same stabilizer dimension, irrespective of their stability nature.

The following observations are already related to this loop group context. Note that the elliptic curve E acts on the set of all bundles by pull back via translations in the base E. Let ξ be an unstable G-bundle with canonical reduction (L, ξ_L). Due to the unicity (up to isomorphism) of the canonical reduction, the association $\xi \mapsto \xi_L$ is compatible with translation. In particular, since ξ_L has non-zero degree, ξ can be isomorphic to only finitely many of its E-translates.

(5.5) **Definition.** Two G-bundles ξ and ζ are called *extended isomorphic* if they become isomorphic as G-bundles after possible translation in the base E.

(5.6) **Theorem.** *Let G be simple and simply connected.*

(*a*) *An unstable G-bundle ξ is regular exactly when* $\dim \mathrm{Aut}_G(\xi) = \ell + 2$.

(*b*) *For $G = SL_n$ there are exactly $n-1$ distinct extended isomorphism classes of regular unstable bundles ξ. They are in bijection with the $n-1$ distinct conjugacy classes of maximal Levi subgroups L_k of G by means of their canonical reduction (L_k, ξ_{L_k}).*

(*c*) *If G is not isomorphic to SL_n, then, up to extended isomorphism, there exists a unique regular unstable G-bundle.*

(5.7) PROOF. The description in 5.1 gives a list of the minimal dimensions of the automorphism groups of unstable bundles ξ whose parabolic type P_k is maximal. This leaves only one candidate for k, except in case A_ℓ. In all cases, the degeneration results, Propositions 3.3 and 3.9, imply then that the Levi subgroup of the canonical reduction of a regular unstable bundle cannot have a semisimple rank less than $\ell - 1$. The (relative) unicity of regular unstable G-bundles ξ is seen as follows. The dimension $\dim \mathrm{Aut}_L(\xi_L) = 1$ is also the dimension of the moduli space $\mathcal{M}\big(L, \gamma(\xi_L)\big)$ of all canonical reductions of unstable bundles of the same type $\mu(\xi)$ as ξ. Since, by translation, the compact group E acts non-trivially on that compact and connected space, it must consist of a single orbit. □

(5.8) REMARK. An announcement of this result was given first in [10].

(5.9) EXAMPLES. In [10], one also finds 'explicit' constructions for the regular bundles. In case of the classical groups, they are realized in terms of the vector bundles associated to the natural representations.

(5.9.1) Here is the case $G = SL_n$. Let $0 < k < n$ be an integer and U a stable vector bundle of rank k and degree $\deg(U) = 1$. Then there is a unique stable vector bundle U' of rank $n-k$ and degree -1 whose determinant line bundle $\bigwedge^{n-k} U'$ is the inverse of that of U (cf. e.g. [1]). The principal G-bundle ξ derived from $U \oplus U'$ is then a regular unstable bundle with canonical Levi subgroup $L_k = S(GL_k \times GL_{n-k})$.

(5.9.2) In the case of an exceptional group, we can reformulate the constructions of [10] in a somewhat more systematic way which, suitably modified, works in all corank-1-cases. We shall restrict ourselves to the three cases where the Dynkin diagram has three branches, D_ℓ, E_6, E_7, E_8. Then the derived subgroup H of the canonical Levi subgroup L is a product $H = SL_p \times SL_q \times SL_r$, p, q, r denoting the lengths of the branches. By exploiting the (explicit) symbol of L, one easily sees that the group L may be realized as the subgroup of a product of three general linear groups

$$L \;=\; \big\{\, (A, B, C) \in GL_p \times GL_q \times GL_r \;\big|\; \det(A) = \det(B) = \det(C) \,\big\}.$$

From the long exact sequence of cohomology sets attached to the exact sequence of group sheaves over E

$$1 \longrightarrow \underline{L} \longrightarrow \underline{GL}_p \times \underline{GL}_q \times \underline{GL}_r \longrightarrow \underline{\mathbb{C}}^* \times \underline{\mathbb{C}}^* \longrightarrow 1$$

we see that any L-bundle η is given by a unique $GL_p \times GL_q \times GL_r$-bundle (η_p, η_q, η_r) such that $\det(\eta_p) = \det(\eta_q) = \det(\eta_r)$. In particular, a bundle of minimal degree has to satisfy $\deg(\eta_p) = \deg(\eta_q) = \deg(\eta_r) = 1$. If we choose the direct sum of the three natural representations $V = \mathbb{C}^p \oplus \mathbb{C}^q \oplus \mathbb{C}^r$ as a faithful module for L, then the associated vector bundle $\eta \times^L V$ is a direct sum $U_p \oplus U_q \oplus U_r$ of stable vector bundles U_s with $\mathrm{rank}(U_s) = s$ and with all determinant line bundles equal, of degree 1. This realisation illuminates the unicity statement. To get a G-bundle, however, one still has to extend the bundle η to G. The adjoint bundle $\mathrm{ad}(\xi)$ can then be obtained from $\eta \times^L V$ by applying suitable tensor functors to V. For example, the representation of L on $\mathfrak{g}(1)$ (the lowest positively graded piece of $\mathfrak{g}$ with respect to $\mu(\xi)$) is given by $V(1) = \mathbb{C}^p \otimes \mathbb{C}^q \otimes (\mathbb{C}^r)^*$, where $r = 2$.

Let us now turn to the more involved case of subregular bundles.

(5.10) **Definition.** An unstable G-bundle ξ of topological type $\gamma \in \pi_1(G)$ is called *subregular* if its automorphism group $\mathrm{Aut}_G(\xi)$ has dimension $\dim \mathrm{Aut}_G(\xi_r) + 2$ where ξ_r is a regular unstable G-bundles of topological type γ.

(5.11) **Theorem.** *Let G be simple and simply connected. There are no unstable G-bundles ξ with $\dim \mathrm{Aut}_G(\xi) = \ell+3$, and we have $\dim \mathrm{Aut}_G(\xi) = \ell+4$ for all subregular unstable G-bundles ξ. Up to extended isomorphism, there are the following subregular unstable G-bundles.*

(*a*) *If G is of type A_1, E_7, E_8, F_4 or G_2, there is a unique subregular bundle.*

(*b*) *If G is of type E_6, there are exactly two subregular unstable bundles permuted by the outer automorphism.*

(*c*) *If $G = SL_n$, $n > 2$: There are $n-2$ disjoint families of subregular unstable bundles, each parametrized by the elliptic curve E. They are in bijection with the $n-2$ subsets $\{k, k+1\} \subset \{1, \dots, n-1\}$ by means of their canonical reduction (L, ξ_L), in which L is a Levi subgroup of corank 2 conjugate to $L_{\Delta \setminus \{\alpha_k, \alpha_{k+1}\}}$.*

(*d*) *If G is of type B_ℓ with $\ell \neq 3$ or of type C_ℓ or of type D_ℓ with $\ell > 5$, there is a family of subregular unstable bundles parametrized by the projective line $\mathbb{P}^1$.*

(*e*) *If G is of type D_4, there are three disjoint families of subregular unstable bundles, each parametrized by the projective line $\mathbb{P}^1$.*

(*f*) *If G is of type B_3, resp. D_5, there is a family of subregular unstable bundles parametrized by the projective line $\mathbb{P}^1$. In addition, there are one, resp. two more subregular bundles which are not adjacent to the first mentioned family.*

(5.12) Proof. The strategy of proof is the same as in the regular case. Except for the case of SL_n, one finds from 5.1 the correct Levi subgroups, unicity or type of families for unstable bundles of minimal degree 1. Using Propositions 3.3 and 3.9, one excludes non-maximal Levi subgroups. The statement about the non-adjacency in the D_5-case already appeared in the example to Theorem 3.5. The case B_3 is settled similarly.

It remains to treat the case of SL_n. Here, the group SL_2 is a special case by itself. In fact, since for an unstable G-bundle $\dim \mathrm{Aut}_G(\xi)^+$ is larger than a positive multiple of $\ell + 1$, this is the only case, where an L-bundle of degree 2 can lead to a subregular bundle. The associated rank-2 vector bundle is then $U \oplus U^*$, where U is a line bundle of degree 2. For $n > 2$ we have to use corank-2 Levi subgroups L of SL_n. The vector bundles U associated naturally to corresponding maximal unstable SL_n-bundles ξ decompose into a direct sum of 3 summands, say

$$U = U_1 \oplus U_2 \oplus U_3, \ \ \mathrm{rank}(U_i) = n_i, \ \ \deg(U_i) = d_i, \ \ \sum_i n_i = n, \ \ \sum_i d_i = 0.$$

Assuming a decreasing sequence of slopes d_i/n_i and $d_2 \geqslant 0$ (otherwise, use the dual bundle), one computes the values $\dim \mathrm{Aut}_{SL_n}(\xi)^+ = (n + n_2)d_1 + (n - n_1)d_2$ and $\dim \mathrm{Aut}_L(\xi_L)) \geqslant 2$. This leaves the only choice: $d_1 = 1$, $d_2 = 0$, $d_3 = -1$ and $n_2 = 1$. From that, our respective assertions follow. □

(5.13) Remark. Whenever one of the families discussed above is parametrized by $\mathbb{P}^1$, this is a consequence of the relevant moduli space $\mathcal{M}(\bar{H}, c^{-1})$ of semistable $\bar{H}$-bundles being a weighted projective space of dimension 1. At first sight, the parametrization is only set-theoretical. However, as shown in [10] in the case of simply connected $\bar{H}$, this family may be realized holomorphically by exploiting a $\mathbb{C}^*$-equivariant 'transversal slice' for a regular unstable $\bar{H}$-bundle. For general semisimple $\bar{H}$, the same procedure can be applied. The existence problem for regular unstable bundles is settled in our next section.

6. Non-Simply Connected Simple Groups

In this section, we want to deal with the general but more involved case of non-simply connected groups G. Let ξ be a P-maximal G-bundle with canonical reduction $(L,\, \xi_L)$, where $P = P_k$ is a maximal parabolic with Levi subgroup $L = L_k$. The strategy for computing the dimension of $\mathrm{Aut}_G(\xi)$ is the same as in the simply connected case. Theorem 2.6 gives a formula for $\dim \mathrm{Aut}_G(\xi)^+$, and $\dim \mathrm{Aut}_L(\xi_L)$ is reduced to Theorem 2.2 by means of the exact sequence

$$1 \longrightarrow \pi_1(S) \longrightarrow \pi_1(L) \longrightarrow \pi_1(\bar{H}) \longrightarrow 1.$$

All semistable L-bundles η extend to semistable $\bar{H}$-bundles $\bar{\eta}$ and, conversely, any semistable $\bar{H}$-bundle $\bar{\eta}$ reduces to a semistable L-bundle η unique up to an action of $E \times \mathbb{Z}$. The topological class $\gamma(\eta) \in \pi_1(L)$ is mapped to $\gamma(\bar{\eta}) \in \pi_1(\bar{H})$ under the above surjection, and $\dim \mathrm{Aut}_L(\eta) = \dim \mathrm{Aut}_{\bar{H}}(\bar{\eta}) + 1$. The problem lies mainly in the determination of the relation between the topological type of ξ and that of $\bar{\eta}$, which is now complicated by the interference of the fundamental groups of G and H.

$$\begin{array}{ccccc} \pi_1(\bar{H}) & \longleftarrow & \pi_1(L) & \longrightarrow & \pi_1(G) \\ \gamma(\bar{\eta}) & \longleftarrow & \gamma(\eta) & \longrightarrow & \gamma(\xi) \end{array}$$

(6.1) In principle, the structure of L could still be described by a symbol like in the simply connected case. However, the center $C(H)$ of H may not split any more according to the simple factors of H. On the other hand, for adjoint G, the group $C(H)$ itself is cyclic and isomorphic to $S \cap H$. Because of the more involved relations between the various fundamental groups, we will essentially proceed by a case-by-case analysis. Some general observations may nonetheless precede the individual computations.

(6.1.1) We may restrict our considerations to groups of adjoint type. Let G be a semisimple group with projection $\pi : G \longrightarrow G^{ad}$ onto its adjoint group G^{ad}. Via π, any G-bundle ξ extends automatically to a G^{ad}-bundle $\bar{\xi}$, preserving the dimensions $\dim \mathrm{Aut}_G(\xi) = \dim \mathrm{Aut}_{G^{ad}}(\bar{\xi})$. Conversely, a G^{ad}-bundle $\bar{\xi}$ reduces to a G-bundle ξ exactly when its topological class $\gamma(\bar{\xi})$ lies in the subgroup $\pi_*\big(\pi_1(G)\big) \subset \pi_1(G^{ad})$.

(6.1.2) Assume that G is simple of adjoint type with Lie algebra $\mathfrak{g}$. The action of $S = S_k \cong \mathbb{C}^*$ provides $\mathfrak{g}$ with a $\mathbb{Z}$-grading, $\mathfrak{g} = \bigoplus_{m \in \mathbb{Z}} \mathfrak{g}(m)$. Then $L = L_k$ acts on the graded components $\mathfrak{g}(m)$ by the restricted adjoint action, and $\mathfrak{g}(0) = \mathrm{Lie}(L)$. It is easily seen, (cf. e.g. [16, Prop. 8.6]), that $\mathfrak{g}(-1)$ is an irreducible L-module with heighest weight $-\alpha_k$. Moreover, it is a faithful L-module since, together with $\mathfrak{g}(0)$ and its contragredient $\mathfrak{g}(1)$, it generates $\mathfrak{g}$. This provides an identication of $S \cap H \cong \mu_d \subset \mathbb{C}^*$ with the full center $C(H)$ of H. In particular, the quotient group $\bar{H} = H/S \cap H$ is of adjoint type.

From the topological point of view, there are three different situations which we will encounter; recall that the fourth sequence in 4.3 gives an embedding of $\pi_1(H)$ into $\pi_1(G)$:

$$\text{(I)} \quad \pi_1(H) = 0, \quad \text{(II)} \quad \pi_1(H) = \pi_1(G), \quad \text{(III)} \quad 0 \neq \pi_1(H) \neq \pi_1(G).$$

The third situation (III) can only arise if the order of $\pi_1(G)$ is not a prime. For simple G, this can happen only if G is of type A or D, and we refrain from developing any generalities. In the diagrams, the nodes of type (I), (II), (III), will be marked by a $\circ$, a $\circledast$, or a $\circledcirc$, respectively.

(6.1.3) $\pi_1(H) = 0$ (I). This is closest to the simply connected case. According to Lemma 4.1, the intersection $S \cap H$ is a cyclic group of order $d = \langle [\varpi_k], [\varpi_k^\vee] \rangle$, and because of $\pi_1(H) = 0$, the degree map $\pi_1(L) \longrightarrow \pi_1(\bar{S})$ induces now an isomorphism $\mathbb{Z} \longrightarrow \mathbb{Z}$, and the second exact sequence from 4.3 reads

$$0 \longrightarrow \mathbb{Z} \xrightarrow{d} \mathbb{Z} \longrightarrow \mathbb{Z}/d\mathbb{Z} = \pi_1(\bar{H}) \longrightarrow 0.$$

Thus, if η is an L-bundle of degree m, the topological class $\gamma(\bar{\eta})$ in $\pi_1(\bar{H}) = \mathbb{Z}/d\mathbb{Z}$ is $\bar{m}$. Let ξ be the G-extension of η. The third sequence of 4.3 tells us already that the topological class of ξ is the residue class of m in $\pi_1(G) \cong \mathbb{Z}/e\mathbb{Z}$, where $e = \langle [\varpi_k], \alpha_k^\vee \rangle$. In many cases, i.e. if $\pi_1(G)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, this will suffice for the determination of $\gamma(\xi)$. In the other cases, however, we have to relate this to a fixed isomorphism, i.e. $\pi_1(G) = X_*(T)/Q^\vee$. For that, we have to compute the residue class of $[\varpi_k^\vee]/d \in X_*(T)/Q'^\vee$ in $X_*(T)/Q^\vee$. The topological class of ξ is then the m-fold of this element.

Because of $\pi_1(L) = 0$, one may also describe L and the resulting topological invariants somewhat more precisely by means of a symbol as in 5 and 5.1 (however, since the element $c = \varphi(e^{2\pi i/d})$ will always be a generator of the cyclic group $C(H)$, we effectively don't need that precision).

(6.1.4) $\pi_1(H) = \pi_1(G)$ (II). This condition implies that the kernel $C \cong \pi_1(G)$ of the universal covering $\tilde{G} \longrightarrow G$ is contained in the analogous subgroup $\tilde{H} \subset \tilde{G}$ which, in turn, implies $[\varpi_k] = \varpi_k$. The degree map, now given by ϖ_k, induces an isomorphism of $\mathbb{Z}\alpha_k^\vee \subset \pi_1(L)$ with $\mathbb{Z} = \pi_1(\bar{S})$ and thus a natural splitting

$$\pi_1(L) \;=\; \mathbb{Z}\alpha_k^\vee \times \pi_1(G),$$

such that the natural map $\pi_1(L) \longrightarrow \pi_1(G)$ identifies with the second projection. In particular, the topological type of the G-extension ξ of an L-bundle η does not depend on the degree of η. The degree of η enters, however, into the topological type of the $\bar{H}$-extension η according to the the surjection

$$\pi_1(L) \;=\; \mathbb{Z}\alpha_k^\vee \times \pi_1(H) \longrightarrow \pi_1(\bar{H}).$$

The image of $\pi_1(H)$ in $\pi_1(\bar{H})$ is usually easy to describe. To determine the image of $\alpha_k^\vee$, we have to evaluate its class in $\pi_1(\bar{H}) = X_*(T)/\big(Q'^\vee + \mathbb{Z}[\varpi_k^\vee]\big)$. For example, if $X_*(T) = P^\vee$, the class of $\alpha_k^\vee$ is a negative sum of 'neighbouring' fundamental coweights $\varpi_j^\vee$, $\langle \alpha_j, \alpha_k^\vee \rangle \neq 0$.

Let us now investigate the individual cases. Since the behaviour of regular and subregular unstable bundles varies strongly between different cases, we have not collected that in a separate statement. Instead, in each case, we will directly record the relevant properties.

(6.2) THE CASE A_ℓ, $G = PGL_n$, $n = \ell + 1$. This is the adjoint group of type A_ℓ with fundamental group $\mathbb{Z}/\ell\mathbb{Z}$. We shall identify co-weights and weights and $\mathbb{Z}/\ell\mathbb{Z}$ with P/Q, in such a way, that the element $\bar{k}$ corresponds to the class of the $(n-k)$-th fundamental weight ϖ_{n-k}. The reduced rank $\ell_{\bar{k}}$ is $g-1$, where $g = (k,n) = (n-k,n)$. For the fundamental dominant weights, we have

$$[\varpi_k] \;=\; (n/g)\,\varpi_k \quad \text{and} \quad d_k \;=\; \langle [\varpi_k],\, \varpi_k^\vee \rangle \;=\; k(n-k)/g,$$

cf. also the Example following Theorem 2.6. The Levi subgroup L_k is the quotient of $S(GL_k \times GL_{n-k})$ by the full center μ_n, and its semisimple subgroup H_k is the quotient of $SL_k \times SL_{n-k}$ by $\mu_g \subset \mu_n$. Thus, we have $\pi_1(H_k) \cong \mathbb{Z}/g\mathbb{Z}$, and we are in Situation I exactly when $g = (k,n) = 1$. Otherwise, we meet Situation III. The following explicit forms of the fundamental groups and their relating maps will be useful.

The fundamental group $\pi_1(L_k) = P/Q' \cong \mathbb{Z} \times \mathbb{Z}/g\mathbb{Z}$ of L_k may be written in lattice form

$$P/Q' \;=\; (\mathbb{Z}\varpi_1 \oplus \mathbb{Z}\varpi_\ell)/\langle k\varpi_1 - (n-k)\varpi_\ell\rangle \;=\; (\mathbb{Z}\alpha_k \oplus \mathbb{Z}\varpi_\ell)/\langle k\alpha_k - n\varpi_\ell\rangle$$

(the equivalence of these two realisations follows from the identity (of classes in P/Q') $\alpha_k = \varpi_1 + \varpi_\ell$). Note that under the degree map to $\mathbb{Z} = \pi_1(\bar{S})$, the elements α_k and ϖ_ℓ are mapped to n/g and k/g. Under the surjection $\pi_1(L_k) \longrightarrow \pi_1(G)$, the element ϖ_ℓ is mapped to $\bar{1} \in \mathbb{Z}/n\mathbb{Z}$, and, of course, α_k goes to $\bar{0}$. We shall make the identification $\pi_1(\bar{H}_k) = P/Q' + \mathbb{Z}\varpi_k = \mathbb{Z}/k\mathbb{Z} \times \mathbb{Z}/(n-k)\mathbb{Z}$ such that the class of ϖ_1 corresponds to $(\bar{1},\bar{0})$ and that of ϖ_ℓ to $(\bar{0},\bar{1})$. Then, the class of the element α_k corresponds to $(\bar{1},\bar{1})$.

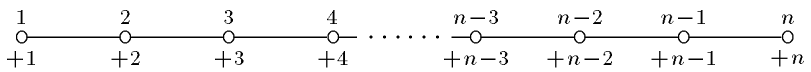
Assume that ξ is an unstable G-bundle of topological type $\gamma(\xi) = \bar{m}$ in the fundamental group $\pi_1(G) = \mathbb{Z}/n\mathbb{Z}$ with canonical L_k-reduction η. Then the topological type of η has to be of the form $\gamma(\eta) = u\alpha_k + m\varpi_\ell$, where u is an arbitrary integer and $m \in \mathbb{Z}$ is a lift of $\bar{m}$. This gives $\deg(\eta) = (mk + un)/g$ and, as a consequence of Theorem 2.6, $\dim \mathrm{Aut}_G(\xi)^+ = mk + un$. For fixed k and m, the minimal such dimension is given as the positive residue of km modulo n or n, if that residue is 0. For the computation of $\dim \mathrm{Aut}_{L_k}(\eta)$, we have to identify the image of $\gamma(\eta)$ in $\pi_1(\bar{H}_k)$. Writing $\gamma(\eta)$ in the above form, we obtain $(\bar{u}, \bar{u} + \bar{m})$ in $\mathbb{Z}/k\mathbb{Z} \times \mathbb{Z}/(n-k)\mathbb{Z}$.

Here is a list of the dimensions of the automorphism groups of maximal unstable bundles in some (essentially low rank) cases. Note that tables for $\gamma(\xi) = -\bar{k}$ are obtained from those for $\gamma(\xi) = \bar{k}$ by applying the diagram symmetry.

If the topological class of the unstable G-bundle ξ is $\bar{m} = \bar{1}$, we can make the following general observation. For a reducing L_k-bundle η of minimal degree, we get $\gamma(\eta) = 0\alpha_k + 1\varpi_\ell$ and, accordingly, $\dim \mathrm{Aut}_G(\xi)^+ = \deg(\eta) = k$. The type of $\gamma(\bar{\eta})$ in $\mathbb{Z}/k\mathbb{Z} \times \mathbb{Z}/(n-k)\mathbb{Z}$ will be $(\bar{0},\bar{1})$. Thus, we also get $\dim \mathrm{Aut}_{L_k}(\eta) = k$.

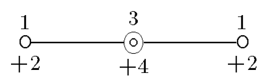
$A_\ell,\ PGL_n,\ n = \ell + 1$

$\gamma(\xi) = \bar{1}$



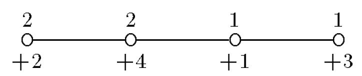
$A_3,\ PGL_4$

$\gamma(\xi) = \bar{2}$



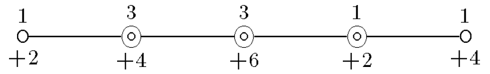
$A_4,\ PGL_5$

$\gamma(\xi) = \bar{2}$

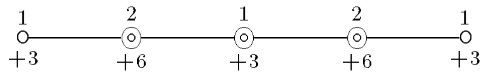


A_5, PGL_6

$\gamma(\xi) = \bar{2}$

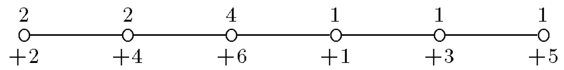


$\gamma(\xi) = \bar{3}$

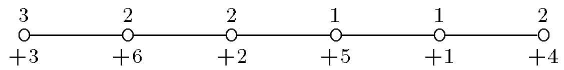


A_6, PGL_7

$\gamma(\xi) = \bar{2}$

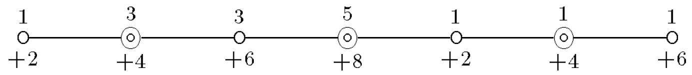


$\gamma(\xi) = \bar{3}$

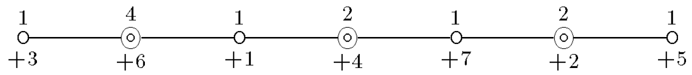


A_7, PGL_8

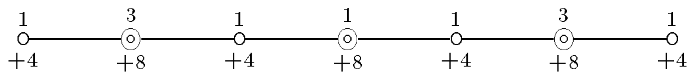
$\gamma(\xi) = \bar{2}$



$\gamma(\xi) = \bar{3}$



$\gamma(\xi) = \bar{4}$



The behaviour of regular and subregular unstable G-bundles ξ is most involved for this group, $G = PGL_n$. In each individual case, the sought for information can be extracted from our explicit formulae. We shall only give some relevant indications. Let $\gamma(\xi) = \bar{m}$ for some $0 < m < n$ and $g = (m, n)$.

The regular case is still easy to overlook. Note that there are integers $u, k \in \mathbb{Z}$ such that $un + km = g$, and where k is well determined up to multiples of n/g. Thus there are g choices for k satisfying $0 < k < n$ and, correspondingly, g distinct Levi subgroups L_k with semistable bundles η_k satisfying $\dim \mathrm{Aut}_G(\eta_k \times^{L_k} G) = g$ (the minimal positive value attainable for G-bundles ξ of type $\bar{m}$). We claim that also $\dim \mathrm{Aut}_{L_k}(\eta_k)$ attains the minimal value, 1. This follows from $(u, k) = 1$ and $(u + m, n - k) = 1$. The first equality is a consequence of $un + km = g$ and $(m, n) = g$, and the second is an easy consequence of the first (rewrite $un + km = g$ as $u(n - k) + k(m + u) = g$, then $g' = (u + m, n - k)$ has to divide g, thus m and n, thus u and k, which gives $g' = 1$). As a consequence, there are exactly g regular unstable bundles up to extended isomorphism.

Let us turn to subregular bundles. If $g = 1$, there is always a $\mathbb{P}^1$-family of subregular bundles, up to extended isomorphism. This follows from an argument similar to that in the regular case. Namely, there is exactly one integer k, $0 < k < n$, and an integer u satisfying $un + km = 2$. Similar to the arguments in the regular case, one checks either $(u, k) = 2$ and $(u + m, n - k) = 1$, or $(u, k) = 1$ and $(u + m, n - k) = 2$. These last conditions guarantee $\dim \mathrm{Aut}_{L_k}(\eta_k) = 2$ (in obvious notation) from which our assertions follow. This family may be accompanied by an additional single subregular unstable bundle (up to extended isomorphism). This single bundle is then associated to a Levi subgroup L_{3k-n}, where the number k satisfies $0 < k < n$, $un + km = 1$ and the inequality $n < 3k < 2n$.

Assume $g \geqslant 2$. Then there are $g - 1$ distinct families of subregular unstable bundles parametrized by the elliptic curve E. This follows from an analysis of the unstable bundles associated with corank-2 Levi subgroups L. One can show that the

Levi subgroups of type $L = L_{k,k+n/g}$, where k satisfies $un + km = g$, and only those, give rise to unstable bundles ξ with $\dim \mathrm{Aut}_G(\xi)^+ = g + 1$. They are most easily described in terms of the associated projective space bundles, i.e. by using a fractional line bundle

$$\xi \quad \longleftrightarrow \quad U_0^{m/n} \otimes \big(U_1 \oplus U_2 \oplus U_3\big).$$

Here, U_0 is a line bundle of degree 1, and U_i is a stable vector bundle of rank n_i and degree d_i for $i = 1, 2, 3$, where

$$(n_1, d_1) \;=\; (k, u), \quad (n_2, d_2) \;=\; (n/g, -m/g), \quad (n_3, d_3) \;=\; (k', u')$$

with $k' = n - (n/g) - k$ and $un + km = g$, $u'n + k'm = -g$. One easily verifies that these bundles satisfy $\dim \mathrm{Aut}_L(\xi_L) = 2$.

If $g = 2$ and $n > 4$, then there is, up to extended isomorphism, an additional single subregular unstable bundle associated to one of the two Levi subgroups L_k whose index k satisfies $un + km = 4$.

The existence of isolated subregular unstable bundles can be interpreted in terms of orbit closures inside the extended loop group $\hat{\mathcal{L}}G$ (cf. [13], [14]). In the above cases, they lead to simple elliptic singularities of type $\tilde{E}_6$, $g = 1$, and $\tilde{D}_5$, $g = 2$. The families of subregular unstable bundles are related to singularities of type A_∞ and D_∞ depending on the parametrization by either E or $\mathbb{P}^1$.

(6.3) THE CASE B_ℓ. The adjoint group of type B_ℓ is isomorphic to the special orthogonal group $SO_{2\ell+1}$. Its fundamental group is $\mathbb{Z}/2\mathbb{Z}$, and the reduced rank is $\ell_{\bar{1}} = \ell - 1$. For the fundamental dominant weights, we have the following relations

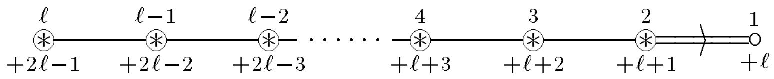
$$[\varpi_k] \;=\; \varpi_k \quad \text{for} \quad k < \ell \quad \text{and} \quad [\varpi_\ell] \;=\; 2\varpi_\ell,$$

leading, by Theorem 2.6, to the dimensions $\dim \mathrm{Aut}_G(\xi)^+$ for maximal bundles of type P_k as tabulated below. The structure of the Levi subgroups L_k has already been discussed before in relation to $Spin_{2\ell+1}$, i.e. $L_k = GL_k \times SO_{2(\ell-k)+1}$. Let us first consider the extremal case $k = \ell$, in which $L_\ell = GL_\ell$. Then $H_\ell = SL_\ell$ is simply connected (Situation I). The exact sequence

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \pi_2(G/L_\ell) & \longrightarrow & \pi_1(L_\ell) & \longrightarrow & \pi_1(G) & \longrightarrow & 0 \\
 & & \| & & \| & & \| & & \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 0
\end{array}$$

shows that GL_ℓ-bundles η of odd degree extend to $SO_{2\ell+1}$-bundles ξ of topological type $\gamma(\xi) = \bar{1}$. For bundles η of degree 1 we get $\dim \mathrm{Aut}_{L_\ell}(\eta) = 1$. Assume now $k < \ell$. Then $\pi_1(H_k) = \mathbb{Z}/2\mathbb{Z}$ maps isomorphically to $\pi_1(G)$ (Situation II). For L_k-bundles η of degree 1, we have two choices according to the topological type of the $SO_{2(\ell-k)+1}$-component η_2. If η_2 is a proper $SO_{2(\ell-k)+1}$-bundle, then, as we have seen in the discussion of $Spin_{2\ell+1}$, the bundle η itself reduces to the Spin-Levi subgroup $\tilde{L}_k \subset Spin_{2\ell+1}$. Thus η_2 has to be a topologically trivial bundle, now, reducing to $Spin_{2(\ell-k)+1}$. Then $\dim \mathrm{Aut}_{SO_{2(\ell-k)+1}}(\eta_2) = \ell - k$ and $\dim \mathrm{Aut}_{L_k}(\eta) = \ell + 1 - k$.

(In more geometric terms, the reason behind the above argument is that the Stiefel-Whitney class of the extended bundle η is the sum of the individual Stiefel-Whitney classes of η_1 and η_2 -in obvious notation-).



Up to extended isomorphism, we find one regular unstable bundle associated with the Levi subgroup L_ℓ and a $\mathbb{P}^1$-family of subregular unstable bundles associated with $L_{\ell-1}$.

(6.4) THE CASE C_ℓ. The adjoint group of this type is $G = PSp_{2\ell} = Sp_{2\ell}/\langle -1\rangle$ with fundamental group $\pi_1 = \mathbb{Z}/2\mathbb{Z}$ and reduced rank $\ell_{\bar{1}} = m$, where either $\ell = 2m$ or $\ell = 2m+1$. For the fundamental dominant (co-)weights, the following relations hold

$$[\varpi_k] \;=\; 2\varpi_k,\ \text{for } k \text{ odd},\ \ [\varpi_k] \;=\; \varpi_k,\ \text{for } k \text{ even and } [\varpi_k^\vee] \;=\; \varpi_k^\vee,\ \text{for all } k,$$

which, together with Theorem 2.6, gives the minimal values for $\dim \operatorname{Aut}_G(\xi)^+$ of G-bundles ξ with $\gamma(\xi) = \bar{1}$.

Concerning the structure of the Levi subgroups and semistable bundles of minimal degree, the parities of the involved numbers ℓ, k, and $k/2$, if k is even, will play a role.

Let us first have a look at the Levi subgroups L_k for odd k, in which case we have $\langle [\varpi_k], \varpi_k^\vee\rangle = 2k$. The group L_k is the quotient of $GL_k \times Sp_{2(\ell-k)}$ by the diagonal subgroup μ_2 with simply connected semisimple part $H_k = SL_k \times Sp_{2(\ell-k)}$ (Situation I). Let η be an L_k-bundle of minimal degree 1. Then the topological class of its $\bar{H}_k$-extension $\bar{\eta}$ is a generator of $\pi_1(\bar{H}_k) = \mathbb{Z}/k\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. From that, we conclude $\dim \operatorname{Aut}_{L_k}(\eta) = 0 + \ell' + 1 = m + 1 - (k-1)/2$, if ℓ is odd, and $\dim \operatorname{Aut}_{L_k}(\eta) = 0 + \ell' + 1 = m - (k-1)/2$, if ℓ is even.

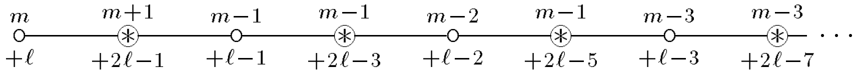
We now turn to the case $k = 2s$ even. Again L_k is the quotient of $GL_k \times Sp_{2(\ell-k)}$ by the diagonal subgroup μ_2. However, this time, the semisimple part H_k is not any more simply connected but isomorphic to the diagonal quotient $(SL_k \times Sp_{2(\ell-k)})/\mu_2$ with fundamental group $\pi_1(H_k) = \mathbb{Z}/2\mathbb{Z}$ (Situation II). By the general arguments in 6.1, we can identify $\pi_1(L) \longrightarrow \pi_1(G)$ with the second projection

$$\mathbb{Z}\varpi_k^\vee \times \pi_1(G) \longrightarrow \pi_1(G).$$

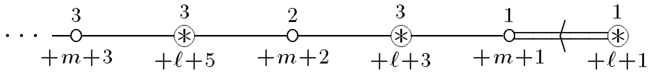
A semistable L_k-bundle η of minimal degree 1 whose G-extension has non-trivial type in $\pi_1(G)$ has topological type $\gamma(\eta) = (\alpha_k^\vee, \bar{1})$ in $\mathbb{Z}\alpha_k^\vee \times \mathbb{Z}/2\mathbb{Z}$. Let $\bar{\eta}$ be the semistable $\bar{H}$-bundle extension of η. We have to compute its topological class $\gamma(\bar{\eta})$ in $\pi_1(\bar{H}) = \pi_1(PGL_k) \times \pi_1\big(PSp_{2(\ell-k)}\big) = \mathbb{Z}/k\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Observe that under the projection $H \longrightarrow \bar{H}$, the group $\pi_1(H)$ is embedded diagonally into $\pi_1(\bar{H}) = \mathbb{Z}/k\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and the element $\alpha_k^\vee \in \pi_1(L)$ is mapped to the element $(\bar{1}, \bar{0})$ in $\mathbb{Z}/k\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. (This last fact is checked by a computation in the coweight lattice of type C_ℓ or, dually, in the weight lattice of type B_ℓ.) Thus, the topological type $\gamma(\bar{\eta})$ is $(\bar{1} + \bar{s}, \bar{1})$, $s = k/2$, yielding $\dim \operatorname{Aut}_L(\eta) = 1 + (m-s) + 1 = m + 2 - s$, if s is odd, resp. $\dim_L \operatorname{Aut}(\eta) = 0 + (m-s) + 1 = m + 1 - s$, if s is even. Note that this reasoning covers also the case that $k = \ell$.

Here are diagrams representing that information graphically. Their left ends depend on the parity of ℓ, and their right ends on the residue of ℓ modulo 4.

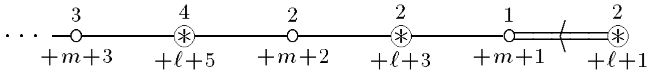
$C_\ell,\ \ell = 2m$



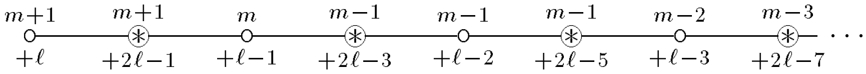
For m even, the right end looks like:



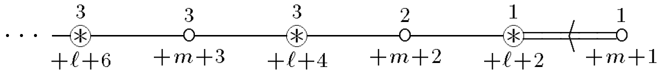
And for m odd, we have:



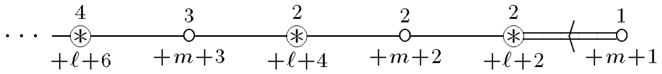
$C_\ell,\ \ell = 2m+1$



The right end for m even:

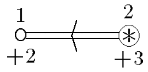


The right end for m odd:

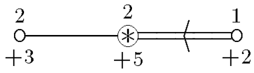


Here are some diagrams of low rank.

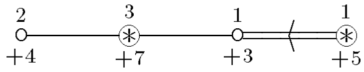
C_2, PSp_4



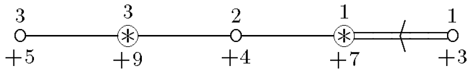
C_3, PSp_6



C_4, PSp_8

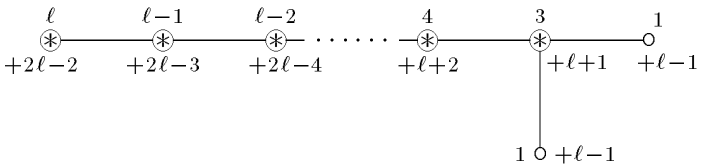


C_5, PSp_{10}



Up to extended isomorphism, we find one regular unstable bundle and a $\mathbb{P}^1$-family of subregular unstable bundles. They are associated with the Levi subgroups $L_{\ell-1}$ and $L_{\ell-3}$ if ℓ is even, resp. L_ℓ and $L_{\ell-2}$ if ℓ is odd.

(6.5) THE CASE D_ℓ, $G = SO_{2\ell}$. This is one of the non simply connected groups of type D_ℓ with fundamental group $\mathbb{Z}/2\mathbb{Z}$ and reduced rank $\ell_{\bar{1}} = \ell - 2$. It can be treated exactly as the odd orthogonal case. However, one has to take care of two extremal cases now, $k = \ell - 1$ and $k = \ell$. The case $\ell = 3$ which corresponds to $SO_6 \cong SL_4/\mu_2$ can also be included.



Up to extended isomorphism, we find two regular unstable bundle associated with the Levi subgroups $L_{\ell-1}$ and L_ℓ. There is also a family of subregular unstable bundles ξ parametrized by the elliptic curve E and associated with the corank-2 Levi subgroup

$L_{\ell-1,\ell}$. The orthogonal vector bundle associated to ξ and the natural representation of $SO_{2\ell}$ has the form

$$U_{\ell-1} \oplus (U_{\ell-1})^* \oplus U_1 \oplus (U_1)^*,$$

where $U_{\ell-1}$ is stable of rank $\ell-1$ and degree 1, and where U_1 is a line bundle of degree 0. In case $\ell = 3$, this gives a family of subregular unstable SL_4/μ_2-bundles of topological class $\bar{2}$.

(6.6) THE CASE D_ℓ, $\ell = 2m+1$, $G = PSO_{2\ell}$. This is the other non simply connected, i.e. adjoint group of this type D_ℓ with fundamental group $\mathbb{Z}/4\mathbb{Z} = P/Q$. As a generator $\bar{1}$ of P/Q we fix the class of ϖ_ℓ. Then $\bar{2}$ corresponds to the class of ϖ_1 and $\bar{3}$ to that of $\varpi_{\ell-1}$. Bundles of type $\gamma(\xi) = \bar{0}$ or $\bar{2}$ reduce to $Spin_{2\ell}$ or $SO_{2\ell}$ and have been dealt with before. Moreover, the outer automorphism exchanges the two types $\bar{1}$ and $\bar{3}$. We may therefore restrict to bundles of type $\bar{1}$. The reduced rank is $\ell_{\bar{1}} = m-1$.

Since G is adjoint, all fundamental coweights $\varpi_k^\vee$ lie in $X_*(T)$. For the fundamental weights, we have

$$[\varpi_k] \;=\; 2\varpi_k, \text{ for } k < \ell-1, \text{ odd}, \qquad [\varpi_k] \;=\; \varpi_k, \text{ for } k < \ell-1, \text{ even}$$

$$\text{and} \quad [\varpi_k] \;=\; 4\varpi_k, \text{ for } k = \ell-1 \text{ and } k = \ell.$$

For $k < \ell-1$, the k-th Levi subgroup L_k is the quotient of $GL_k \times SO_{2(\ell-k)}$ by the diagonal subgroup μ_2, and $L_{\ell-1} \cong L_\ell \cong GL_\ell/\mu_2$. The semisimple part H_k has a fundamental group $\pi_1(H_k)$ of order $2, 4, 1$ in the respective cases. In particular, if $k < \ell-1$ is odd, we have $\pi_1(H_k) = \mathbb{Z}/2\mathbb{Z}$, canonically embedded into $\pi_1(G) = \mathbb{Z}/4\mathbb{Z}$ (Situation III). For even $k < \ell-1$, we get an isomorphism $\pi_1(H_k) = \pi_1(G) = \mathbb{Z}/4\mathbb{Z}$ (Situation II), and for $k = \ell-1, \ell$, we get $\pi_1(H_k) = 0$ (Situation I).

Let us begin with the extremal case $k = \ell-1, \ell$ (Situation I). From the lattice interpretation of the surjective map

$$\pi_1(L_k) \;=\; \mathbb{Z} \longrightarrow \mathbb{Z}/4\mathbb{Z} \;=\; \pi_1(G),$$

we see that, for maximal G-bundles ξ of type $\bar{1}$, we have to consider L_ℓ-bundles η of degree 1 or $L_{\ell-1}$-bundles η of degree 3, in case $\ell \equiv 1 \bmod 4$. If $\ell \equiv 3 \bmod 4$, the rôles of $\ell-1$ and ℓ are interchanged. According to the surjection

$$\pi_1(L_k) \;=\; \mathbb{Z} \longrightarrow \mathbb{Z}/\ell\mathbb{Z} \;\cong\; \pi_1(\bar{H}_k) \;\cong\; \pi_1(PGL_\ell),$$

the topological type of the $\bar{H}_k$-extension $\bar{\eta}$ will be $\bar{1}$ resp. $\bar{3}$. This leads to the value $\dim \mathrm{Aut}_{L_k}(\eta) = (3, \ell)$, which is 1 or 3.

We now attack the next, still familiar, case $k = 2s < \ell-1$, (Situation II). Then H_k is the diagonal quotient of $SL_k \times SO_{2(\ell-k)}$ by the diagonal μ_2 with fundamental group $\pi_1(H_k) = \pi_1(G) = \mathbb{Z}/4\mathbb{Z}$. Thus, we may restrict to L-bundles η of degree 1. Up to isomorphism of the factors, the surjection

$$\pi_1(L) \;=\; \mathbb{Z}\alpha_k^\vee \times \mathbb{Z}/4\mathbb{Z} \longrightarrow \mathbb{Z}/k\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \;\cong\; \pi_1(PGL_k \times PSO_{2(\ell-k)})$$

maps $\alpha_k^\vee$ to $(\bar{1}, \bar{2})$ and $(0, \bar{1})$ to $(\bar{s}, \bar{1})$. Thus, for $\bar{\eta}$ of topological type $\bar{1}$ or $\bar{3}$, we obtain $\dim \mathrm{Aut}_L(\eta) = \big((s+1, 2s) - 1\big) + (m-s-1) + 1 = m-s-1+(s+1, 2s)$. Note that the value of $(s+1, 2s)$ is 1 or 2 according to s being even or odd.

Finally, assume that k is odd and $k < \ell - 1$ (Situation III). Then we have $H_k \cong SL_k \times SO_{2(\ell-k)}$ with fundamental group $\pi_1(H_k) = \mathbb{Z}/2\mathbb{Z}$ injecting into the group $\pi_1(G) = \mathbb{Z}/4\mathbb{Z}$. From the surjective map

$$\pi_1(L_k) \;=\; \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}/4\mathbb{Z} \;=\; \pi_1(G),$$

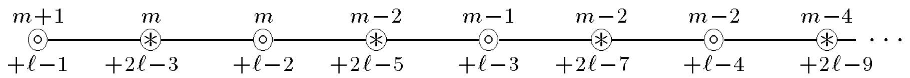
we see that, for maximal G-bundles ξ of type $\bar{1}$ or $\bar{3}$, it is sufficient to consider L_k-bundles η of degree 1. The homomorphism

$$\pi_1(L) \;=\; \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}/k\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \;\times \mathbb{Z}/2\mathbb{Z} \;\cong\; \pi_1\big(PGL_k \times PSO_{2(\ell-k)}\big)$$

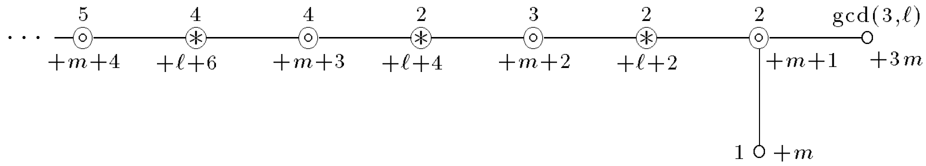
which maps the element $(0,\bar{1})$ to $(\bar{0},\bar{1},\bar{1})$, can be surjective only, if it maps $(1,\bar{0})$ to $(\bar{1},\bar{1},\bar{0})$ (up to possible isomorphism of the target). Thus, for $\deg(\eta) = 1$ or 3 we get $\dim \mathrm{Aut}_{L_k}(\eta) = 0 + (\ell-k)/2 + 1 = m + 1 - (k-1)/2$.

The diagrams representing the dimension values start the same on their left end.

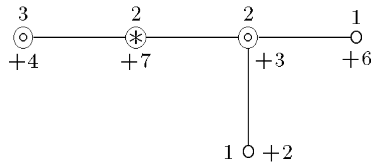
$PSO_{2\ell}$, $\ell = 2m+1$, $\gamma(\xi) = \bar{1}$.



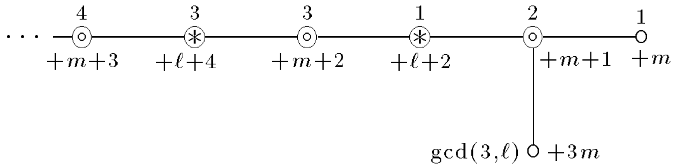
Diagrams for $\gamma(\xi) = \bar{3}$ are obtained by applying the diagram automorphism. The right end depends on the parity of m. Here is m even.



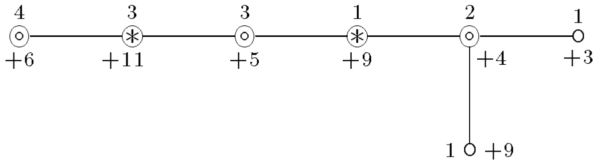
An example of smallest rank, D_5, $m = 2$.



Here is m odd.



An example of smallest rank, D_7, $m = 3$.



Up to extended isomorphism, we find one regular unstable bundle and a $\mathbb{P}^1$-family of subregular unstable bundles. They are associated with the Levi subgroups L_ℓ and $L_{\ell-2}$ if m is even, resp. $L_{\ell-1}$ and $L_{\ell-2}$ if m is odd.

(6.7) THE CASE D_ℓ, $\ell = 2m$, $G = PSO_{2\ell}$. This is the adjoint group of type D_ℓ with fundamental group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} = P/Q$, generated by $(\bar{1}, \bar{0}) = \varpi_{\ell-1}$ and $(\bar{0}, \bar{1}) = \varpi_\ell$. The element $(\bar{1}, \bar{1})$ corresponds to the class of ϖ_1 in P/Q. Bundles of type $\gamma(\xi) = (\bar{0}, \bar{0})$ or $(\bar{1}, \bar{1})$ reduce to $Spin_{2\ell}$ or $SO_{2\ell}$ and have been dealt with before. The other bundles reduce to the half-spin groups $Spin_{2\ell}^{+/-}$. However, this reduction has no effect on the dimension of their automorphism groups, and we therefore stay with the more accessible adjoint group. The outer automorphism exchanges the two types $(\bar{1}, \bar{0})$ and $(\bar{0}, \bar{1})$. We may therefore restrict to bundles of one of these types. The reduced rank is $\ell_{(\bar{1},\bar{0})} = m$. Since G is adjoint, all fundamental coweights $\varpi_k^\vee$ lie in $X_*(T)$. For the fundamental weights, we have

$$[\varpi_k] \;=\; \varpi_k, \text{ for } k \text{ even, } k \neq \ell, \qquad [\varpi_k] \;=\; 2\varpi_k, \text{ for } k \text{ odd, or } k = \ell.$$

We start with the somewhat familiar Situation II, i.e. $k = 2s \neq \ell$. Then L_k is the quotient of $GL_k \times SO_{2(\ell-k)}$ by the diagonal subgroup μ_2, and $H_k = SL_k \times SO_{2(\ell-k)}/\mu_2$ has fundamental group $\pi_1(H_k) = \pi_1(G) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Accordingly, we may restrict our attention to L_k-bundles η of degree 1. The surjection

$$\begin{array}{ccc} \pi_1(L) & \longrightarrow\!\!\!\!\rightarrow & \pi_1(PGL_k \times PSO_{2(\ell-k)}) \\ \| & & \| \\ \mathbb{Z}\alpha_k^\vee \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} & \longrightarrow & \mathbb{Z}/k\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \end{array}$$

maps $\alpha_k^\vee$ to $(\bar{1}, \bar{1}, \bar{1})$ and, up to automorphism, $(0, \bar{1}, \bar{0})$ to $(\bar{s}, \bar{1}, \bar{0})$ and $(0, \bar{0}, \bar{1})$ to $(\bar{s}, \bar{0}, \bar{1})$. Assuming $\bar{\eta}$ to be of topological type $(\bar{1}, \bar{0})$ or $(\bar{0}, \bar{1})$, we obtain $\dim \operatorname{Aut}_L(\eta) = \big((s+1, 2s) - 1\big) + (\ell - k)/2 + 1 = m - s + (s+1, 2s)$. Note that $(s+1, 2s)$ is 1 or 2 according to s being even or odd.

Next, assume that $k = 2s+1$ is odd and $k < \ell - 1$ (Situation III). Then L_k is a diagonal quotient $GL_k \times SO_{2(\ell-k)}/\mu_2$ with semisimple part $H_k \cong SL_k \times SO_{2(\ell-k)}$. The fundamental group $\pi_1(H_k) = \mathbb{Z}/2\mathbb{Z}$ injects diagonally into $\pi_1(G) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. From the surjective map

$$\pi_1(L_k) \;=\; \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \;=\; \pi_1(G),$$

we see that, for maximal unstable G-bundles ξ of type $(\bar{1}, \bar{0})$ or $(\bar{0}, \bar{1})$, it is sufficient to consider L_k-bundles η of degree 1. The homomorphism

$$\pi_1(L) \;=\; \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}/k\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \;\cong\; \pi_1(PGL_k \times PSO_{2(\ell-k)})$$

maps the element $(0, \bar{1})$ to $(\bar{0}, \bar{2})$. Because of surjectivity, it must map the element $(1, \bar{0})$ to $(\bar{1}, \bar{1})$, up to isomorphism of the target. Thus, for the relevant L_k-bundles η we get $\dim \operatorname{Aut}_{L_k}(\eta) = 0 + \big((\ell - k - 1)/2 - 1\big) + 1 = m - 1 - s$.

Finally, let us treat the extremal cases $k = \ell - 1$, ℓ (Situation III) (by symmetry, it is sufficient to look at $k = \ell$, but then we have to allow both relevant types of the bundle ξ). In the lattice interpretation of the surjective map

$$\pi_1(L_\ell) \;=\; \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \;=\; \pi_1(G),$$

the class of ϖ_1 (chosen generator of $\mathbb{Z}$) is mapped to $(\bar{1}, \bar{1})$, and the torsion element $(0, \bar{1})$, i.e. the class of $(\ell - 2)/2\varpi_1 - \varpi_{\ell-1}$, is mapped to $(\bar{1}, \bar{0})$, if $m = \ell/2$ is odd, and to $(\bar{0}, \bar{1})$, if $m = \ell/2$ is even. An L_ℓ-bundle η of minimal positive degree will thus extend to a G-bundle ξ of type $\gamma(\xi) = (\bar{1}, \bar{0})$ exactly when its class in $\pi_1(L_\ell)$ is $(1, \bar{1})$, m even, or $(2, \bar{1})$, m odd. We will have $\gamma(\xi) = (\bar{0}, \bar{1})$ exactly when $\gamma(\eta) = (1, \bar{1})$, m odd, or $(2, \bar{1})$, m even.

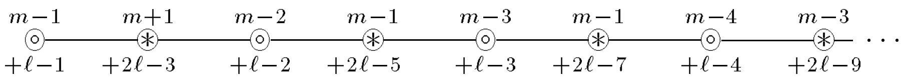
With respect to the same generators of $\pi_1(L_\ell)$ and the generating class ϖ_1 of $\pi_1(\bar{H})$, the map

$$\pi_1(L_\ell) \;=\; \mathbb{Z} \longrightarrow \mathbb{Z}/\ell\mathbb{Z} \;\cong\; \pi_1(\bar{H}) \;\cong\; \pi_1(PGL_\ell)$$

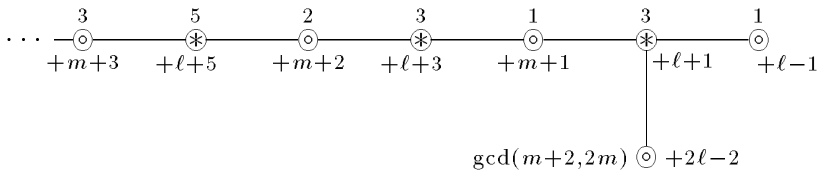
sends ϖ_1 to $\bar{1}$ and $(0, \bar{1})$ to $\bar{\ell}/2$. In particular, the elements $(1, \bar{1}), (2, \bar{1})$, are mapped to $\bar{m} + \bar{1}$, resp. $\bar{m} + \bar{2}$. The corresponding dimensions $\dim \mathrm{Aut}_L(\eta)$ are $(m+1, 2m)$, resp. $(m+2, 2m)$. If m is odd, these are 2, resp. 1. If $m = 2s$ is even, they are 1 resp. 2 or 4, depending on s being even or odd.

The diagrams representing the dimension values start the same on their left end.

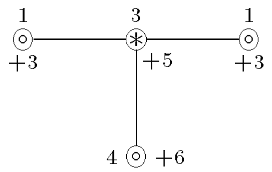
$PSO_{2\ell}$, $\ell = 2m$, $\gamma(\xi) = (\bar{1}, \bar{0})$.



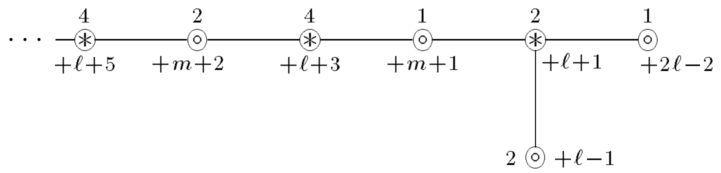
The right end depends on the parity of m. Here is m even.



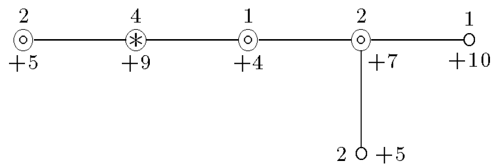
An example of smallest rank, D_4, $m = 2$. By triality, this is equivalent to the case of SO_8.



Here is m odd.



An example of smallest rank, D_6, $m = 3$.



In the following, we shall exclude the case $m = 2$, i.e. D_4 which has been dealt with already (SO_8). Up to extended isomorphism, we find one regular unstable bundle and a P^1-family of subregular unstable bundles. They are associated with the Levi subgroups $L_{\ell-3}$ and $L_{\ell-5}$.

(6.8) THE CASE E_6. The adjoint group $G = E_6^{ad}$ of type E_6 has fundamental group $\pi_1 = \mathbb{Z}/3\mathbb{Z}$ and reduced rank $\ell_{\bar{1}} = \ell_{\bar{2}} = 2$. The set of indices $k = 1, \dots, 6$ of Δ decomposes into two parts, $I_I = \{1, 3, 5, 6\}$, which contains those k with $[\varpi_k] = 3\varpi_k$, and $I_{II} = \{2, 4\}$, which contains those k with $[\varpi_k] = \varpi_k$. These cases will also correspond to the two situations $\pi_1(H_k) = 0$ resp. $\pi_1(H_k) = \pi_1(G) = \mathbb{Z}/3\mathbb{Z}$. The outer automorphism of G of order 2 interchanges the two non trivial center elements of $\tilde{G}$ and accordingly G-bundles of topological types $\bar{1}$ and $\bar{2}$. It is therefore sufficient to look at bundles of type $\bar{1}$. We shall fix the isomorphism $\pi_1(G) = P/Q$ and the element $\bar{1}$ as the class of the weight $\varpi_1 \in P/Q$ (here we have identified the weight and coweight lattices, which we shall also do below).

Let us first study the Situation I which essentially means $k = 1, 3$, the cases $k = 5, 6$ being symmetric to those by the outer automorphism. We have $H_1 = Spin_{10}$ and $H_3 = SL_2 \times SL_5$ with respective modules $\mathfrak{g}(-1)$ isomorphic to Δ_+, one of the half-spin representations, and $\mathbb{C}^2 \otimes \bigwedge^3 \mathbb{C}^5$. This leads to the symbol $[3]_4$ for L_1 resp. $[5, 8]_{10}$ for L_3. If η is an L-bundle of degree 1 this gives the value 2 resp. 1 for $\dim \mathrm{Aut}_L(\eta)$. If $\deg(\eta) = 2$ we get the value 4 resp. 2. The classes of ϖ_k/d, $k = 1, 3, 5, 6$ in P/Q are ϖ_1, $2\varpi_1$, ϖ_1, $2\varpi_1$ (note that $d \equiv 1 \pmod 3$ in all cases). Thus, extending η to a G-bundle ξ gives topological type $\gamma(\xi) = \bar{1}$ if $\deg(\eta) = 1, 2, 1, 2$ for $k = 1, 3, 5, 6$.

The cases $k = 2, 4$ belong to Situation II. Let $k = 2$. We have $H_2 = SL_6/\mu_3$ with fundamental group $\mathbb{Z}/3\mathbb{Z}$, acting on $\mathfrak{g}(-1)$ by means of $\bigwedge^3 \mathbb{C}^6$. The value of $d = \langle \varpi_2, \varpi_2 \rangle$ is 2, and the map

$$\pi_1(L_2) \;=\; \mathbb{Z}\alpha_2 \times \pi_1(H_2) \longrightarrow \pi_1(\bar{H}_2) \;=\; \pi_1(PGL_6)$$

can be written explicitly as follows

$$\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \longrightarrow \mathbb{Z}/6\mathbb{Z}, \quad (m, \bar{n}) \longmapsto (3\bar{m} + 2\bar{n}).$$

As a consequence, for any L_2-bundle of degree 1 with topologically non-trivial extension to G, we get $\dim \mathrm{Aut}_{L_2}(\eta) = 1$.

Let $k = 4$. Here, $H_4 = (Sl_3 \times SL_2 \times SL_3)/\mu_3$ with fundamental group $\mathbb{Z}/3\mathbb{Z}$. It acts on $\mathfrak{g}(-1)$ by means of the representation resp. $\mathbb{C}^3 \otimes \mathbb{C}^2 \otimes \mathbb{C}^3$. We have $d = \langle \varpi_4, \varpi_4 \rangle = 6$, and the map

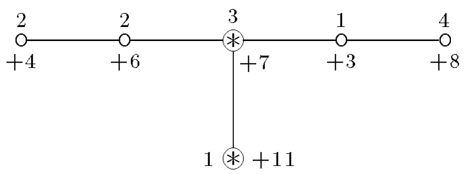
$$\pi_1(L_4) \;=\; \mathbb{Z}\alpha_4 \times \pi_1(H_4) \longrightarrow \pi_1(\bar{H}_4) \;=\; \pi_1(PGL_3) \times \pi_1(PGL_2) \times \pi_1(PGL_3)$$

can be written explicitly as follows

$$\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \longrightarrow \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}, \quad (m, \bar{n}) \longmapsto (\bar{m} + \bar{n}, \bar{m}, \bar{m} + 2\bar{n}).$$

For all L_4-bundles of degree 1 with topologically non-trivial extension to G, this gives $\dim \mathrm{Aut}_{L_4}(\eta) = 3$.

Here is a diagram collecting the numerical values for all P_k-maximal unstable G-bundles ξ of topological type $\gamma(\xi) = \bar{1}$. The diagram for topological type $\gamma(\xi) = \bar{2}$ is obtained by applying the diagram symmetry.



Up to extended isomorphism, we find one regular unstable bundle and a P^1-family of subregular unstable bundles. They are associated with the Levi subgroups L_5 and L_1. Of course, when changing from $\gamma(\xi) = \bar{1}$ to $\gamma(\xi) = \bar{2}$, these Levi subgroups are replaced by L_3 and L_6.

(6.9) THE CASE E_7. The adjoint group $G = E_7^{ad}$ of type E_7 has fundamental group $\pi_1 = \mathbb{Z}/2\mathbb{Z}$ and reduced rank $\ell_{\bar{1}} = 4$. Again, the set of indices $k = 1, \ldots, 7$ decomposes into two parts, $I_I = \{2, 5, 7\}$, which contains those k with $[\varpi_k] = 2\varpi_k$ and $\pi_1(H_k) = 0$, and $I_{II} = \{1, 3, 4, 6\}$, which contains those k with $[\varpi_k] = \varpi_k$ and $\pi_1(H_k) = \pi_1(G) = \mathbb{Z}/2\mathbb{Z}$.

The Situation I arises with $k = 2, 5, 7$. We have $H_2 = SL_7$, $H_5 = SL_5 \times SL_2$ and $H_7 = E_6^{sc}$ acting on $\mathfrak{g}(-1)$ by a representation isomorphic to $\bigwedge^3 \mathbb{C}^7$ resp. $\bigwedge^3 \mathbb{C}^5 \otimes (\mathbb{C}^3)^*$ resp. $\mathbb{C}^{27}$. This leads to the symbols $[2]_7$, resp. $[3, 10]_{15}$, resp. $[2]_3$. If η is a semistable L-bundle of degree 1, this gives the value 1, resp. 1, resp. 3 for $\dim \mathrm{Aut}_L(\eta)$.

Let us now treat $k \in I_{II} = \{1, 3, 4, 6\}$. The group H_1 is isomorphic to the half-spin group $Spin_{12}^+$ with fundamental group $\mathbb{Z}/2\mathbb{Z}$. It acts faithfully on $\mathfrak{g}(-1)$ by means of its half-spin representation. We have $d = \langle \varpi_1, \varpi_1 \rangle = 2$, $\bar{H}_1 = PSO_{12}$, and the map

$$\pi_1(L_1) \;=\; \mathbb{Z}\alpha_1 \times \pi_1(H_1) \longrightarrow \pi_1(\bar{H}_1) \;=\; \pi_1(PSO_{12})$$

can be written explicitly as follows

$$\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \quad (m, \bar{n}) \longmapsto (\bar{m}, \bar{n}).$$

For an L_1-bundle η of degree 1 with non-trivial extension to G, this gives the dimension 5 for $\mathrm{Aut}_{L_1}(\eta)$.

The group H_3 is isomorphic to $SL_2 \times SL_6/\mu_2$ with fundamental group $\mathbb{Z}/2\mathbb{Z}$. It acts faithfully on $\mathfrak{g}(-1)$ by means of the representation $\mathbb{C}^2 \otimes \bigwedge^2 \mathbb{C}^6$. The order of intersection with S_3 is $d = \langle \varpi_3, \varpi_3 \rangle = 6$, $\bar{H}_3 = PGL_2 \times PGL_6$, and the map

$$\pi_1(L_3) \;=\; \mathbb{Z}\alpha_3 \times \pi_1(H_3) \longrightarrow \pi_1(\bar{H}_3) \;=\; \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$$

can be written explicitly as follows

$$\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}, \quad (m, \bar{n}) \longmapsto (\bar{m}, 2\bar{m} + 3\bar{n}).$$

For an L_3-bundle η of degree 1 with non-trivial extension to G, this gives the dimension 1 for $\mathrm{Aut}_{L_3}(\eta)$.

The group H_4 is isomorphic to $SL_3 \times (SL_2 \times SL_4)/\mu_2$ with fundamental group $\mathbb{Z}/2\mathbb{Z}$. It acts faithfully on $\mathfrak{g}(-1)$ by means of the representation $\mathbb{C}^3 \otimes \mathbb{C}^2 \otimes \mathbb{C}^4$. The order of intersection with S_4 is $d = \langle \varpi_4, \varpi_4 \rangle = 12$, $\bar{H}_4 = PGL_3 \times PGL_2 \times PGL_4$, and the map

$$\pi_1(L_4) \;=\; \mathbb{Z}\alpha_4 \times \pi_1(H_4) \longrightarrow \pi_1(\bar{H}_4) \;=\; \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$$

can be written explicitly as follows

$$\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}, \quad (m, \bar{n}) \longmapsto (\bar{m}, \bar{m} + \bar{n}, \bar{m} + 2\bar{n}).$$

For an L_4-bundle η of degree 1 with non-trivial extension to G, this gives the dimension 2 for $\mathrm{Aut}_{L_4}(\eta)$.

Finally, the group H_6 is isomorphic to $(Spin_{10} \times SL_2)/\mu_2$ with fundamental group $\mathbb{Z}/2\mathbb{Z}$. It acts faithfully on $\mathfrak{g}(-1)$ by means of the tensor product representation $\Delta^+ \otimes \mathbb{C}^2$, where Δ^+ is a half-spin module. The order of intersection with S_6 is $d = \langle \varpi_6, \varpi_6 \rangle = 4$, $\bar{H}_6 = PSO_{10} \times PGL_2$, and the map

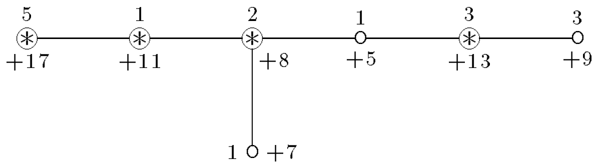
$$\pi_1(L_6) \;=\; \mathbb{Z}\alpha_6 \times \pi_1(H_6) \longrightarrow \pi_1(\bar{H}_6) \;=\; \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$

can be written explicitly as follows

$$\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \quad (m, \bar{n}) \longmapsto (\bar{m} + 2\bar{n}, \bar{m} + \bar{n}).$$

For an L_6-bundle η of degree 1 with non-trivial extension to G, this gives the dimension 3 for $\mathrm{Aut}_{L_6}(\eta)$.

The following diagram collects the dimension values for all P_k-maximal unstable E_7^{ad}-bundles ξ of topological type $\gamma(\xi) = \bar{1}$.



Up to extended isomorphism, we find one regular and one subregular unstable bundle. They are associated with the Levi subgroups L_5 and L_2.

(6.10) REMARK. One can investigate the orbit closure singularity arising in this situation as described in [13], [14]. One finds a simply elliptic singularity of type $\tilde{E}_6$ which is unfolded semiuniversally with respect to a natural $\mathbb{Z}_2$-symmetry. This deformation has been studied before by Yano [28]. It is a free deformation of type $F_4^\vee$ in his terminology.

References

[1] M. F. ATIYAH, *Vector bundles over an elliptic curve,* Proc. London Math. Soc. **7** (1957), 414–452.

[2] M. F. ATIYAH and R. BOTT, *The Yang-Mills equations over Riemann surfaces* Phil. Transact. Royal Soc. London Ser. A **308** (1982), 523–615.

[3] A. BOREL, *Linear algebraic groups, 2nd ed.* Graduate Texts in Math. **126**, Springer, Berlin, Heidelberg, New York, 1996.

[4] A. BOREL, R. FRIEDMAN and J. MORGAN, *Almost commuting elements in compact Lie groups,* eprint math.GR/9907007.

[5] N. BOURBAKI, *Groupes et algèbres de Lie, Chap. IV, V, VI,* Hermann, Paris, 1968.

[6] E. BRIESKORN, *Singular elements of semisimple algebraic groups,* in Actes Congr. Int. Math., Nice, tome **2** (1970), 279–284.

[7] P. ETINGOF and I. B. FRENKEL, *Central extensions of current groups in two dimensions,* Commun. Math. Phys. **165** (1994), 429–444.

[8] P. ETINGOF and B. A. KHESIN, *Affine Gelfand-Dickey brackets and holomorphic vector bundles,* Geom. Funct. Anal. **4** (1994), 399–423.

[9] R. FRIEDMAN and J. MORGAN, *Holomorphic principal bundles over elliptic curves,* Duke eprint math.AG/9811130.

[10] R. FRIEDMAN, J. MORGAN and E. WITTEN, *Vector bundles and F-theory,* Commun. Math. Phys. **187** (1997), 679–743.

[11] A. GROTHENDIECK, *A general theory of fiber spaces with structure sheaf,* University of Kansas Report Number **4** (1955).

[12] G. HARDER and M. S. NARASIMHAN, *On the cohomology groups of moduli spaces of vector bundles over curves,* Math. Ann. **212** (1975), 215–248.

[13] S. HELMKE, *Loop groups and elliptic singularities,* in Proc. Kinosaki Conf. on Algebraic Geometry, Nov. 1998, 98–105.

[14] S. HELMKE and P. SLODOWY, *Loop groups, principal bundles over elliptic curves and elliptic singularities,* in Annual Meeting of the Math. Soc. of Japan, Hiroshima, Sept. 1999, Abstracts, Section Infinite-dimensional Analysis, 67–77.

[15] J. HUMPHREYS, *Linear algebraic groups,* Graduate Texts in Math. **21**, Springer, Berlin, Heidelberg, New York, 1975.

[16] V. G. KAC, *Infinite dimensional Lie algebras,* Cambridge University Press, 1990.

[17] S. KOBAYASHI, *Differential geometry of complex vector bundles,* Publications of the Mathematical Society of Japan **15**, Iwanami Shoten and Princeton University Press, Tokyo, Princeton, 1987.

[18] E. J. LOOIJENGA, *Root systems and elliptic curves,* Invent. Math. **38** (1976), 17–32.

[19] D. MUMFORD, J. FOGARTY and F. KIRWAN, *Geometric invariant theory 3rd ed.,* Grundlehren der mathematischen Wissenschaft **333**, Springer, Berlin, Heidelberg, New York, 1994.

[20] A. RAMANATHAN, *Stable principal bundles on a compact Riemann surface,* Math. Annalen **213** (1975), 129–152.

[21] S. RAMANAN and A. RAMANATHAN, *Some remarks on the instability flag,* Tohoku Math. Journ. **36** (1984), 269–291.

[22] K. SAITO, *Einfach elliptische Singularitäten,* Invent. Math. **23** (1974), 289–325.

[23] K. SAITO and D. YOSHII, *Extended affine root systems IV (Simply-laced elliptic Lie algebras),* Publ. RIMS, to appear.

[24] C. SCHWEIGERT, *On moduli spaces of flat onnections with non-simply connected structure group,* Nucl. Phys. B **492** (1997), 743-755.

[25] S. S. SHATZ, *The decomposition and specialization of algebraic families of vector bundles,* Compositio Math. **35** (1977), 163–187.

[26] P. SLODOWY, *Simple Singularities and Simple Algebraic Groups,* Springer Lecture Notes in Math. **815**, Springer, 1980.

[27] T. A. SPRINGER, *Linear algebraic groups, 2nd ed.* Progress in Math. **9**, Birkhäuser, Basel, Boston 1998.

[28] T. YANO, *Free deformations of simply elliptic singularities,* Science Reports of the Saitama University **9** (1980), 71–79.

RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES
Kyoto University, Kyoto 606–8502, Japan

MATHEMATISCHES SEMINAR DER UNIVERSITÄT HAMBURG
Universität Hamburg, D-20146 Hamburg, Germany

EMAIL ADDRESSES: helmke@kurims.kyoto-u.ac.jp
slodowy@math.uni-hamburg.de