ARITHMETIC MIXED SHEAVES

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We give a formalism of arithmetic mixed sheaves, including the case of arithmetic mixed Hodge structures which are recently studied by P. Griffiths, M. Green [16], [17] and M. Asakura [1]. This notion became necessary for us to describe the image of Griffiths' Abel-Jacobi map for a generic hypersurface (inspired by previous work of M. Green [15] and C. Voisin [38], [39]), and also to prove the following variant of results of D. Mumford [24] and A. Roitman [27] suggested (and proved in the case $\dim X = 2$) by S. Bloch in Exercise of Appendix to Lecture 1 of [6]:

Let X be a smooth proper complex algebraic variety. If there is a morphism of complex varieties $S \to X$ inducing a surjective morphism $\mathrm{CH}_0(S)_{\mathbb{Q}} \to \mathrm{CH}_0(X)_{\mathbb{Q}}$, then $\Gamma(X, \Omega_X^j) = 0$ for $j > \dim S$.

Actually, it turns out that the usual Hodge theory [9] is enough for this. See Prop. 1.4 of Lect. 6 in [39]. But the attempt led us to the following formulation.

For a subfield A of $\mathbb{R}$, and a subfield k of $\mathbb{C}$ with finite transcendence degree, the category of k-finite mixed A-Hodge structures $\mathrm{MHS}(A)_{\langle k\rangle}$ is defined to be the inductive limit of the category of admissible variations of mixed A-Hodge structures on $S = \operatorname{Spec} R$ over $k_R := \overline{k} \cap R$, with R running over finitely generated smooth k-subalgebras of $\mathbb{C}$. (Here an admissible variation on S means an admissible variation of mixed A-Hodge structures on $S_{\mathbb{C}} := S \otimes_{k_R} \mathbb{C}$ in the sense of [21], [37] such that the Hodge filtration, the connection and the polarizations are defined on S/k_R.) More generally, for a complex algebraic variety X, there is a k-subalgebra R of $\mathbb{C}$ as above such that X is defined over R, and the category $\mathrm{MHM}(X, A)_{\langle k\rangle}$ of k-finite mixed Hodge Modules on X is defined similarly. See (2.1). These can be extended to the mixed sheaves in the sense of [30] where l-adic sheaves can also be included, and the main theorems of this paper hold in the generalized situation. (In fact, they apply even to the subcategory consisting of the objects of geometric origin.)

A similar category was defined in [30,(1.9)] for a subfield K of $\mathbb{C}$ finitely generated over k, by assuming that the fractional field of R is K in the above definition. (This was inspired by [6], p. 1.20.) Then it is enough to take further the inductive limit over K in order to get the above category of arithmetic mixed Hodge Modules (or sheaves, more generally). See also [5, 6.1.7] and 1.1 of Lect. 6 in [39]. The origin of the idea can be found in p. 86 and pp. 98–99 of [34].

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The category of arithmetic mixed Hodge structures is closer to a conjectural category of mixed motives of Beilinson [3] than the category of graded-polarizable mixed A-Hodge structures $\mathrm{MHS}(A)$ in the usual sense [9], but is different from both (at least if we do not restrict to the subcategory consisting of the objects of geometric origin). There is a natural functor $\iota : \mathrm{MHS}(X,A)_{\langle k\rangle} \to \mathrm{MHS}(X,A)$, but this is not fully faithful. See (2.4) (ii). We have a constant object $A_{\langle k\rangle}$ in $\mathrm{MHS}(A)_{\langle k\rangle}$ represented by a constant variation of Hodge structure of type $(0,0)$ on S. For a complex variety X, we can define the cohomology $H^j(X, A_{\langle k\rangle})$ in $\mathrm{MHS}(A)_{\langle k\rangle}$ in a compatible way with the functor ι, and show the following:

0.1. Theorem. *Let X be a smooth proper complex variety of dimension n. If $\Gamma(X,\Omega_X^j) \neq 0$, then $\mathrm{Ext}^j(A_{\langle k\rangle}, H^{2n-j}(X, A_{\langle k\rangle}(n)))$ is an infinite dimensional vector space over $A \cap \overline{k}$, where Ext^j is taken in $\mathrm{MHS}(A)_{\langle k\rangle}$. See (4.2).*

So we get a positive answer to the problem of showing the nonvanishing of the higher extension groups in a variant of the category of mixed Hodge structures. See also [1], [16], [17], [40], etc. The proof uses an argument similar to the passage from a normal function which is an element of an extension group, to the corresponding Hodge cycle as in [31], [32]. Note that (0.1) and the other main theorems hold for the arithmetic mixed sheaves and also for the objects of geometric origin, where A is assumed to be $\mathbb{Q}$.

For the relation with the cycle map, we define an analogue of Deligne cohomology by

$$H^i_{\mathcal{D}}(X, A_{\langle k\rangle}(j)) = \mathrm{Ext}^i(A_{\langle k\rangle}, (a_X)_* a_X^* A_{\langle k\rangle}(j))\,(= \mathrm{Ext}^i(a_X^* A_{\langle k\rangle}, a_X^* A_{\langle k\rangle}(j))),$$

where $a_X : X \to \mathrm{Spec}\,\mathbb{C}$ denotes the structure morphism, and the extension group is taken in the derived category of $\mathrm{MHS}(A)_{\langle k\rangle}$ or $\mathrm{MHM}(X,A)_{\langle k\rangle}$. By the decomposition theorem, we have the Leray filtration L on $H^i_{\mathcal{D}}(X, A_{\langle k\rangle}(j))$ (associated with the Leray spectral sequence) such that

$$\mathrm{Gr}^r_L H^i_{\mathcal{D}}(X, A_{\langle k\rangle}(j)) = \mathrm{Ext}^r(A_{\langle k\rangle}, H^{i-r}(X, A_{\langle k\rangle}(j)).$$

See (3.3). For $A = \mathbb{Q}$, we have a cycle map

$$cl : \mathrm{CH}^p(X)_{\mathbb{Q}} \to H^{2p}_{\mathcal{D}}(X, \mathbb{Q}_{\langle k\rangle}(p)),$$

and the Chow group $\mathrm{CH}^p(X)_{\mathbb{Q}}$ has the induced filtration L. (This can be generalized to Bloch's higher Chow groups [7] due to [30, (8.3)].) By definition, $L^1\mathrm{CH}^p(X)_{\mathbb{Q}}$ consists of homologically equivalent to zero cycles, and $L^2\mathrm{CH}^p(X)_{\mathbb{Q}}$ is contained in the kernel of Griffiths' Abel-Jacobi map, but it is not clear if the last two coincide in general. The graded piece of the cycle map

$$\mathrm{Gr}^r_L cl : \mathrm{Gr}^r_L \mathrm{CH}^p(X)_{\mathbb{Q}} \to \mathrm{Ext}^r(A_{\langle k\rangle}, H^{2p-r}(X, \mathbb{Q}_{\langle k\rangle}(p))$$

for $r \geq 2$ is sometimes called the higher Abel-Jacobi map. See also [1], [16], [17], [40], etc. By the same argument as in [33, II, (3.3)] we can verify that the filtration L is preserved by the action of a correspondence, and the action on the graded pieces depends only on the cohomology class of the correspondence. See (3.5). Using Murre's result [25], [26], we can prove

0.2. Theorem. *Let X be a smooth proper variety of dimension n. Then the image of the second Abel-Jacobi map $\mathrm{Gr}_L^2 cl$ for $p = n$ is an infinite dimensional $\mathbb{Q}$-vector space if $\Gamma(X, \Omega_X^2) \neq 0$. See (4.4).*

In the surface case this is analogous to a recent result of C. Voisin [40], where she uses an explicit description of a second Abel-Jacobi map due to M. Green [16]. For a zero cycle ζ on a surface S, it is given by a composition of the extension classes associated with the inclusions $|\zeta| \to C$ and $C \to S$, where C is a curve containing $|\zeta|$. But it is a difficult problem how to eliminate the cohomology of C in the composition of the two extensions, and the relation between the two second Abel-Jacobi maps is not trivial. (See also [1] for the nonvanishing of the second Abel-Jacobi map in the case X is the self-product of a curve defined over $\overline{\mathbb{Q}}$ such that its genus is > 1 and the rank of the Neron-Severi group of the product over $\overline{\mathbb{Q}}$ is three.)

The nonvanishing of $\mathrm{Gr}_L^r cl$ for $p = n$ would hold under the assumption $\Gamma(X, \Omega_X^r) \neq 0$, if the strong Künneth decomposition in the sense of Murre [26] holds. Note that the relation with Murre's result was suggested in [33, II, (3.4)]. In the case X is a smooth hypersurface of $\mathbb{P}^{n+1}$ defined over k, we can show that the image of $\mathrm{Gr}_L^n cl$ is infinite dimensional if $\Gamma(X, \Omega_X^n) \neq 0$ (i.e. if $\deg X \geq n+2$). Similar arguments apply to cycles of lower codimensions. See Remark after (4.4). We can prove an analogue of (0.2) for the higher Chow groups. See (5.2).

As to the image of Griffiths' Abel-Jacobi map, we have a natural morphism of $H_{\mathcal{D}}^{2p}(X, \mathbb{Q}_{\langle k \rangle}(p))$ to the usual Deligne cohomology $H_{\mathcal{D}}^{2p}(X, \mathbb{Q}(p))$, and its image is called the k-finite part of the Deligne cohomology, and is denoted by $H_{\mathcal{D}}^{2p}(X, \mathbb{Q}(p))^{\langle k \rangle}$. The k-finite part $J^p(X)_{\mathbb{Q}}^{\langle k \rangle}$ of Griffiths' intermediate Jacobian tensored with $\mathbb{Q}$ is defined by intersecting it with $H_{\mathcal{D}}^{2p}(X, \mathbb{Q}(p))^{\langle k \rangle}$. The Abel-Jacobi map induces a morphism of $L^1\mathrm{CH}^p(X)_{\mathbb{Q}}$ to $J^p(X)_{\mathbb{Q}}^{\langle k \rangle}$. By an argument similar to [31], [32], we can show (see (4.1)) :

0.3. Proposition. *Let X and k be as in (0.1). If $H^{2p-1}(X, \mathbb{Q}_{\langle k \rangle})$ is global section-free (see (2.3)), and the Hodge conjecture for codimension p cycles holds for any smooth projective complex varieties defined over k, then the cycle map $cl : \mathrm{CH}^p(X)_{\mathbb{Q}} \to H_{\mathcal{D}}^{2p}(X, \mathbb{Q}(p))^{\langle k \rangle}$ and the Abel-Jacobi map $L^1\mathrm{CH}^p(X)_{\mathbb{Q}} \to J^p(X)_{\mathbb{Q}}^{\langle k \rangle}$ are surjective.*

The first assumption is satisfied if X is a generic hypersurface, or more generally, if X is a generic hypersurface of a smooth projective variety Y defined over a subfield of k such that $H^{2p-1}(Y, \mathbb{Q}) = 0$. If the first assumption of (0.3) is not satisfied, the cycle map $cl : \mathrm{CH}^p(X)_{\mathbb{Q}} \to H_{\mathcal{D}}^{2p}(X, \mathbb{Q}(p))^{\langle k \rangle}$ and the Abel-Jacobi map $L^1\mathrm{CH}^p(X)_{\mathbb{Q}} \to J^p(X)_{\mathbb{Q}}^{\langle k \rangle}$ cannot be surjective in general, because Ext^1 in $\mathrm{MHS}(k, A)$ is too big. See Remark (ii) after (4.1). (One possibility of getting smaller extension groups is to restrict to objects of geometric origin. See [33].)

For the injectivity of the cycle map, let Y be a smooth variety over an algebraic number field k, where k is assumed to be algebraically closed in the function field of Y. We consider the Abel-Jacobi map over k:

$$L^1\mathrm{CH}^p(Y)_{\mathbb{Q}} \to J^p(Y/k)_{\mathbb{Q}} := \mathrm{Ext}^1(\mathbb{Q}_k, H^{2p-1}(Y/k, \mathbb{Q}(p)))$$

where $L^1\mathrm{CH}^p(Y)_{\mathbb{Q}}$ is the subgroup of cohomologically equivalent to zero cycles (with respect to $H^\bullet(X\otimes_k\overline{k},\mathbb{Q}_l)$ or $H^\bullet(X(\mathbb{C}),\mathbb{Q})$), and $\mathbb{Q}_k, H^{2p-1}(Y/k,\mathbb{Q}(p))$ are defined in $\mathrm{MHS}(k,A)$, the category of graded-polarizable mixed $\mathbb{Q}$-Hodge structures whose $\mathbb{C}$-part is defined over k. See (1.3). (In the case $k=\mathbb{C}$ and $p=1$, we can show that this Abel-Jacobi map is bijective. See [33, I, (3.4)].)

0.4. Theorem. *Let X,k be as in (0.1), and assume k is a number field. Then the cycle map $\mathrm{CH}^p(X)_{\mathbb{Q}}\to H^{2p}_{\mathcal{D}}(X,\mathbb{Q}_{\langle k\rangle}(p))$ is injective if the above Abel-Jacobi map over k is injective for any smooth varieties Y over number fields. In the case $p=2$, the last assumption is reduced to the same injectivity for any smooth projective variety over a number field having a morphism to another variety such that the base change of its generic fiber is isomorphic to X. See (4.6).*

This shows that the second hypothesis in a theorem of [1] is not necessary. It is expected that the above Abel-Jacobi map for smooth projective varieties over number fields would be injective if the target is replaced by the motivic extension group. See also [3], [19]. However, it is not clear whether the morphism of the motivic extension group to the above extension group should be injective, because it is closely related to the fully faithfulness of the forgetful functor from the category of mixed motives to that of mixed Hodge structures, or of systems of realizations. See [33, I]. So it is better to use the systems of realizations here.

If the conclusion of (0.4) is true, and the Künneth decomposition in the Chow group holds for the smooth proper variety X, then we can verify that L coincides with the filtration of Murre [26], using the same argument as in [33, II, (3.3)]. See (4.9) below. We can also show that Bloch's conjecture in [6] on the converse of Mumford 's result [24] can be reduced to the last hypothesis of (0.4) (although the latter may be more difficult than the former). See Remark (i) after (4.8). We can prove an analogue of (0.4) for the higher Chow groups. See (5.3).

In §1, we introduce the notion of mixed Hodge structure whose $\mathbb{C}$-part is defined over k, and, more generally, that of mixed sheaf. In §2, we study k-finite mixed Hodge Modules (and also mixed sheaves), and define the cycle map in §3. The proofs of the above assertions in the generalized situation are given in §4. We show analogues of (0.2) and (0.4) for the higher Chow groups in §5.

1. Mixed Sheaves Defined over a Subfield

1.1. Let A be a subfield of $\mathbb{R}$, and k a subfield of $\mathbb{C}$. For varieties X over k, let $\mathrm{MHM}(X/k,A)$ denote the categories of mixed Hodge Modules [29] on $X_{\mathbb{C}}:=X\otimes_k\mathbb{C}$ such that the underlying filtered $\mathcal{D}$-Modules $(M_{\mathbb{C}},F;W)$ on $X_{\mathbb{C}}$ are defined over k (i.e., a filtered $\mathcal{D}$-Module $(M_k,F;W)$ on X/k together with an isomorphism $(M_k,F;W)\otimes_k\mathbb{C}=(M_{\mathbb{C}},F;W)$ is given). We also assume that polarizations on the graded pieces of the weight filtration are defined over k. See (1.8, ii) in [30].

More generally, it is possible to consider the categories of mixed sheaves $\mathcal{M}(X/k,A)$ for varieties X over k which satisfy the axioms of mixed sheaves. See loc. cit. Here we have

to choose an embedding $k \to \mathbb{C}$ for the later argument. We assume furthermore that for $k \subset k' \subset \mathbb{C}$ and for a k-variety X, we have canonically the base change functor

$$\mathcal{M}(X/k, A) \to \mathcal{M}(X \otimes_k k'/k', A), \tag{1.1.1}$$

which is exact and compatible with dual, pull-back, direct image, external product, etc. We also assume that there exist forgetful functors

$$\mathcal{M}(X/k, A) \to \mathrm{MHM}(X_{\mathbb{C}}, A) \tag{1.1.2}$$

compatible with the forgetful functors to $\mathrm{Perv}(X_{\mathbb{C}}, A)$.

For example, we can consider the category consisting of $((M_k; F, W), (K, W), (K_l, W))$ where (M_k, F) is a regular holonomic filtered $\mathcal{D}$-Module on X/k endowed with a finite filtration W, (K, W) is a filtered pervers sheaf on $X_{\mathbb{C}}^{\mathrm{an}}$ with $\mathbb{Q}$-coefficients, and (K_l, W) are filtered perverse l-adic sheaves on $X_{\overline{k}} := X \otimes_k \overline{k}$ with $\mathbb{Q}_l$-coefficients which are endowed with a continuous action of the Galois group of $\bar{k}/k$ as in [32] (i.e. the action is lifted to perverse sheaves with $\mathbb{Z}_l$-coefficients). Furthermore, they are given comparison isomorphisms

$$\mathrm{DR}((M_k, W) \otimes_k \mathbb{C}) = (K, W) \otimes_{\mathbb{Q}} \mathbb{C}, \quad \epsilon^* i_{\overline{k}}^*(K_l, W) = (K, W) \otimes_{\mathbb{Q}} \mathbb{Q}_l. \tag{1.1.3}$$

Here $A = \mathbb{Q}$, $\overline{k}$ is the algebraic closure of k in $\mathbb{C}$, and $i_{\overline{k}}$: $X_{\mathbb{C}} \to X_{\overline{k}}$ is the canonical morphism. (See [5] for ϵ^*.) It is also possible to consider the perverse sheaves K_σ on $(X \otimes_{k,\sigma} \mathbb{C})^{\mathrm{an}}$ for any embeddings $\sigma : k \to \mathbb{C}$ so that we get the systems of realizations as in [30, (1.8)]. See [11], [12], [19], etc. for the case $X = \mathrm{Spec}\, k$.

Remark. If X is a smooth complex variety, a mixed A-Hodge Module consists of $((M, F; W), (K, W), \alpha)$ where (M, F) is a filtered regular holonomic $\mathcal{D}_X$-Module, K is a perverse sheaf [5] with A-coefficients on X^{an}, W is a finite increasing filtration on M, K (called the weight filtration), and $\alpha : \mathrm{DR}(M, W) = (K, W) \otimes_A \mathbb{C}$ is an isomorphism of filtered perverse sheaves with $\mathbb{C}$-coefficients on X^{an}. They satisfy several good conditions. See [29]. Note that the condition of mixed Hodge Module on $\mathcal{M} = ((M, F; W), (K, W), \alpha)$ is Zariski-local, and we have locally the following: If g is a function on X such that the restriction of K to the complement U of $Y := g^{-1}(0)_{\mathrm{red}}$ is a local system up to a shift of complex, then $\mathcal{M}$ is a mixed Hodge Module if and only if the following four conditions are satisfied :
(a) The restriction of $\mathcal{M}$ to $X \setminus Y$ is an admissible variation of mixed Hodge structure in the sense of [21], [37].
(b) The three filtrations F, V, W on $(i_g)_* M$ (the direct image as a $\mathcal{D}$-Module) are compatible, where i_g is the closed embedding by the graph of g.
(c) The relative monodromy filtration exists on $\psi_g(K, W)$ and $\varphi_{g,1}(K, W)$.
(d) $\psi_g \mathcal{M}$ and $\varphi_{g,1} \mathcal{M}$ are mixed Hodge Modules.
See [27] for the detail.

If X is a singular complex variety, the underlying filtered $\mathcal{D}$-Module of a mixed A-Hodge Module is defined by using closed embeddings $U \to V$ where U are open subvariety of X and V are smooth. See for example [28, 2.1.20].

1.2. Theorem. *With the above notation, the category $\mathcal{M}(X/k,A)$ is an $(A\cap k)$-linear abelian category, and every morphism is strictly compatible with (F,W) on M_k. The bounded derived categories $D^b\mathcal{M}(X/k,A)$ are stable by standard functors like $f_*, f_!, f^*, f^!$, etc. in a compatible way with (1.1.2).*

Proof. This follows from [30]. In the case $\mathcal{M}(X/k,A)=\mathrm{MHM}(X_{\mathbb{C}},A)$, the first assertion follows from [28, 5.1.14], and the last from 4.3 and 4.4 of loc. cit.

Remark. For the construction of the direct images $f_*, f_!$, it is enough to construct the cohomological direct images $H^if_*, H^if_!$ (thanks to the strictness of (F,W)). Indeed, if X is quasi-projective, $f_*, f_!$ are defined by taking two sets of open coverings of X associated with general hyperplane sections, and using the combination of co-Cech and Cech complexes together with Artin's vanishing theorem for the (perverse) cohomological direct images by an affine morphism. See [4], [5].

For the pull-back by a closed embedding $i: X\to Y$, it is enough to show an equivalence of categories

$$i_*: D^b\mathrm{MHM}(X/k,A)\to D^b_X\mathrm{MHM}(Y/k,A),$$

where the target is the full subcategory of $D^b\mathrm{MHM}(Y/k,A)$ defined by the condition : $\mathrm{supp}\,H^jM\subset X$ for any j. This is reduced to the case where X is a divisor defined by a function g. Then the assertion is verified by using the functor ξ_g in [29, 2.22]. This is inspired by [4].

1.3. Let $\mathrm{MHS}(k,A)=\mathrm{MHM}(\mathrm{Spec}\,k/k,A)$ (where we put $F^p=F_{-p}$). This is the category of graded-polarizable mixed A-Hodge structures [9] whose $\mathbb{C}$-part is defined over k (and polarizations are assumed to be defined over k as above). An object H of $\mathrm{MHS}(k,A)$ consists of $((H_k,F;W),(H_A,W),(H_{\mathbb{C}},W);\alpha_k,\alpha_A)$, where H_Λ is a filtered (or bifiltered) Λ-vector spaces for $\Lambda=k,A,\mathbb{C}$ with comparison isomorphisms $\alpha_k: H_k\otimes_k\mathbb{C}\to H_{\mathbb{C}}$, $\alpha_A: H_A\otimes_A\mathbb{C}\to H_{\mathbb{C}}$ compatible with the filtration W.

For $j\in\mathbb{Z}$, we define $A_k(j)\in\mathrm{MHS}(k,A)$ by $H_k=k$ and $H_A=(2\pi i)^jA\subset H_{\mathbb{C}}=\mathbb{C}$, where α_k and α_A are natural isomorphisms, and $\mathrm{Gr}_F^i=0$ for $i\ne -j$ and $\mathrm{Gr}_i^W=0$ for $i\ne -2j$. Let $A_k=A_k(0)$. We have also $A_k(j)\in\mathcal{M}(k,A)$ in general. See [30].

For a k-variety X with structure morphism $a_{X/k}: X\to\mathrm{Spec}\,k$, let

$$A_{X/k}(j)=a_{X/k}^*A_k(j),\quad H^j(X/k,A(j))=H^j(a_{X/k})_*A_{X/k}(j). \tag{1.3.1}$$

For a smooth k-variety X, we have a cycle map

$$cl_k:\mathrm{CH}^p(X)_{\mathbb{Q}}\to\mathrm{Ext}^{2p}(A_{X/k},A_{X/k}(p))=\mathrm{Ext}^{2p}(A_k,(a_{X/k})_*A_{X/k}(p)), \tag{1.3.2}$$

where the extension groups are taken in the derived categories of $\mathcal{M}(X/k,A)$ or $\mathcal{M}(k,A)$. If a cycle ζ is represented by an irreducible closed subvariety Z, then $cl(\zeta)$ is represented by the composition of

$$A_{X/k}\to A_{Z/k}\to\mathrm{IC}_{Z/k}A[-d_{Z/k}]$$

with its dual, by using the dualities

$$\mathbb{D}(A_{X/k})=A_{X/k}(d_{X/k})[2d_{X/k}],\quad \mathbb{D}(\mathrm{IC}_{Z/k}A)=\mathrm{IC}_{Z/k}A(d_{Z/k}),$$

where $\mathrm{IC}_{Z/k}A$ is the intersection complex, and $d_{X/k} = \dim X/k$. See [29, (4.5.15)].

This cycle map is compatible with the pushdown and the pull-back of cycles (where we assume that a morphism is proper for the pushdown.) See [33, II]. We define

$$L^1\mathrm{CH}^p(X)_{\mathbb{Q}} = \mathrm{Ker}(\mathrm{CH}^p(X)_{\mathbb{Q}} \to \mathrm{Hom}(\mathbb{Q}_k, H^{2p}(X/k, \mathbb{Q}(p)))) \tag{1.3.3}$$

where the last morphism is the composition of the cycle map with the natural morphism to $\mathrm{Hom}(\mathbb{Q}_k, H^{2p}(X/k, \mathbb{Q}(p)))$. Then the cycle map induces the Abel-Jacobi map over k

$$L^1\mathrm{CH}^p(X)_{\mathbb{Q}} \to J^p(X/k)_{\mathbb{Q}} := \mathrm{Ext}^1(\mathbb{Q}_k, H^{2p-1}(X/k, \mathbb{Q}(p))) \tag{1.3.4}$$

by using the Leray spectral sequence

$$E_2^{i,j} = \mathrm{Ext}^i(\mathbb{Q}_k, H^j(X/k, \mathbb{Q}(p))) \Rightarrow \mathrm{Ext}^{i+j}(\mathbb{Q}_{X/k}, \mathbb{Q}_{X/k}),$$

because the vanishing of $E_2^{i,j}$ for $i < 0$ implies the injective morphisms

$$E_\infty^{i,j} \to E_2^{i,j} \ \ (i \le 1). \tag{1.3.5}$$

Similar assertions hold also for $\mathcal{M}(X/k, A)$. See [30].

1.4. Lemma. *For $H_i \in \mathrm{MHS}(k, A)(i = 1, 2)$, let $H = \mathcal{H}om(H_1, H_2) \in \mathrm{MHS}(k, A)$, and*

$$C(H)^0 = \mathrm{Ker}(F^0W_0H_k \oplus W_0H_A \to \mathrm{Gr}_0^W H_{\mathbb{C}}), \quad C(H)^1 = W_{-1}H_{\mathbb{C}},$$

so that we get a complex $C(H) = [C(H)^0 \to C(H)^1]$. Then we have natural isomorphisms

$$H^0(C(H)) = \mathrm{Hom}(H_1, H_2), \ \ H^1(C(H)) = \mathrm{Ext}^1(H_1, H_2),$$

where Hom *and* Ext^1 *are taken in* $\mathrm{MHS}(k, A)$.

(The proof is left to the reader. The argument is similar to [8], taking account of the semisimplicity of $\mathrm{Gr}_0^W H$.)

1.5. Corollary. *For $i > 1$, $\mathrm{Ext}^i(H_1, H_2) = 0$.*

2. Limit of Mixed Sheaves

2.1. *k-finite Mixed Sheaves.* Let A, k be as in (1.1). For a complex variety X, let R be a finitely generated smooth k-subalgebra of $\mathbb{C}$ such that X is defined over R, i.e., there is an R-scheme X_R with an isomorphism $X_R \otimes_R \mathbb{C} = X$ (and $\mathrm{Spec}\, R$ is smooth over k). Then, for a finitely generated smooth k-subalgebra R' of $\mathbb{C}$ containing R, we put $X_{R'} = X_R \otimes_R R', S' = \mathrm{Spec}\, R', d_{R'} = \dim_k S'$, and $k_{R'} = \overline{k} \cap R'$, where $\overline{k}$ is the algebraic closure of k in $\mathbb{C}$. Note that $k_{R'} = \overline{k} \cap k(S')$ with $k(S')$ the function field of S' (because S' is normal) so that $S/k_{R'}$ is geometrically irreducible and $S'_{\mathbb{C}} := S' \otimes_{k_{R'}} \mathbb{C}$ is connected.

Let R', R'' be as above such that $R' \subset R''$ and $S'' := \operatorname{Spec} R''$ is smooth over S'. Let $\mathcal{M}(X/k, A)$ be as in (1.1) (e.g. $\mathcal{M}(X/k, A) = \mathrm{MHM}(X/k, A)$). Then we have an exact functor by the the composition of pull-backs

(2.1.1)

$$\mathcal{M}(X_{R'}/k_{R'}, A)[-d_{R'}] \to \mathcal{M}(X_{R'} \otimes_{k_{R'}} k_{R''}/k_{R''}, A)[-d_{R'}] \to \mathcal{M}(X_{R''}/k_{R''}, A)[-d_{R''}]$$

where $\mathcal{M}(X_{R'}/k_{R'}, A)[-d_{R'}]$ denotes the category of mixed sheaves $\mathcal{M}(X_{R'}/k_{R'}, A)$ shifted by $-d_{R'}$ in the derived category. Since X is identified with the (general) closed fiber of $X_{R'} \otimes_{k_{R'}} \mathbb{C}$ over the closed point of $S'_{\mathbb{C}}$ corresponding to the inclusion $R' \subset \mathbb{C}$, we have an exact functor

$$\mathcal{M}(X_{R'}/k_{R'}, A)[-d_{R'}] \to \mathrm{MHM}(X, A) \tag{2.1.2}$$

compatible with the above functor by using (1.1.2) and restricting to the fiber. Let I be the ordered set consisting of finitely generated smooth k-subalgebras R' of $\mathbb{C}$ containing R, where $R' < R''$ if and only if $R' \subset R''$ and R'' is smooth over R'. We define

$$\mathcal{M}(X, A)_{\langle k \rangle} = \operatorname{ind}\lim_I \mathcal{M}(X_{R'}/k_{R'}, A)[-d_{R'}]. \tag{2.1.3}$$

This is independent of the choice of R and X_R at the beginning of this section. An object of this category is represented by an object of $\mathcal{M}(X_{R'}/k, A)[-d_{R'}]$ for some R', and is called a k-finite mixed A-Hodge Module. By (2.1.2) we have a natural functor

$$\iota : \mathcal{M}(X, A)_{\langle k \rangle} \to \mathrm{MHM}(X, A). \tag{2.1.4}$$

We have (shifted) cohomological functors H^i to $\mathcal{M}(X_{R'}/k_{R'}, A)[-d_{R'}]$, $\mathcal{M}(X, A)_{\langle k \rangle}$, $\mathrm{MHM}(X, A)$ from their derived categories such that the restriction of H^0 to $\mathcal{M}(X_{R'}/k_{R'}, A)[-d_{R'}]$, etc. is the identity, and they are compatible with the functors (2.1.1), (2.1.2) and (2.1.4).

2.2. *Mixed Sheaves over a function field.* Let K be a subfield of $\mathbb{C}$ which contains k and is finitely generated over k. For a K-variety X_K, there exist a finitely generated smooth k-subalgebra R of K whose fractional field is K, and an R-scheme X_R with an isomorphism $X_R \otimes_R K = X_K$. Let k' be the algebraic closure of k in K, and $d_{K/k}$ the transcendence degree of K over k. Then we can define

$$\mathcal{M}(X_K/K, A)_{\langle k \rangle} = \operatorname{ind}\lim_{R'} \mathcal{M}(X_{R'}/k', A)[-d_{K/k}], \tag{2.2.1}$$

where R' runs over localizations of R by one element, and $X_{R'} = X_R \otimes_R R'$. (This is considered in [30, (1.9)], see also [34, 2.12].)

For a complex variety X, there exist a subfield K of $\mathbb{C}$ which contains k and is finitely generated over k, and a K-variety X_K with an isomorphism $X_K \otimes_K \mathbb{C} = X$. Then we have by definition

$$\mathcal{M}(X, A)_{\langle k \rangle} = \operatorname{ind}\lim_{K'} \mathcal{M}(X_{K'}/K', A)_{\langle k \rangle}[-d_{K'/k}], \tag{2.2.2}$$

where K' runs over subfields of $\mathbb{C}$ which contains K and is finitely generated over k, and $X_{K'} = X_K \otimes_K K'$.

We can define the category $\mathcal{M}(X/\Omega,A)_{\langle k\rangle}$ similarly for any algebraically closed subfield Ω of $\mathcal{C}$ with infinite transcendence degree.

2.3. *k-finite Mixed Hodge structures.* With the notation of (1.1) and (2.1), let

$$\mathcal{M}(A)_{\langle k\rangle}=\mathcal{M}(\operatorname{Spec}\mathbb{C},A)_{\langle k\rangle}. \tag{2.3.1}$$

If the mixed sheaves in (1.1) are mixed Hodge Modules, this is called the category of k-finite mixed A-Hodge structures $\mathrm{MHS}(A)_{\langle k\rangle}$. In general, we have a natural functor

$$\iota:\mathcal{M}(A)_{\langle k\rangle}\to\mathrm{MHS}(A), \tag{2.3.2}$$

where $\mathrm{MHS}(A)$ is the category of graded-polarizable mixed A-Hodge structures in the usual sense [9].

For a finitely generated smooth k-algebra R of $\mathbb{C}$, put $S=\operatorname{Spec}R$ and $S_{\mathbb{C}}=\operatorname{Spec}R\otimes_{k_R}\mathbb{C}$. Let $\mathrm{VMHS}(S/k_R,A)^{\mathrm{ad}}$ be the category of admissible variations of mixed A-Hodge structures on S/k_R as in the introduction. Then we have by [29, 3.27]

$$\mathrm{VMHS}(\operatorname{Spec}R/k_R,A)^{\mathrm{ad}}=\mathrm{MHM}(\operatorname{Spec}R/k_R,A)^{\mathrm{sm}}[-d_R], \tag{2.3.3}$$

where the right-hand side is the category of mixed Hodge A-Modules on $\operatorname{Spec}R/k_R$ whose underlying perverse sheaves are local systems. So we get

$$\mathrm{MHS}(A)_{\langle k\rangle}=\operatorname{ind}\lim_R\mathrm{VMHS}(\operatorname{Spec}R/k_R,A)^{\mathrm{ad}}, \tag{2.3.4}$$

where R runs over finitely generated smooth k-subalgebras of $\mathbb{C}$.

For K as in (2.2), we define

$$\mathrm{MHS}(K,A)_{\langle k\rangle}=\operatorname{ind}\lim_R\mathrm{VMHS}(\operatorname{Spec}R/k_R,A)^{\mathrm{ad}},$$

with R running over finitely generated smooth k-subalgebras of K whose fractional field is K. Then

$$\mathrm{MHS}(A)_{\langle k\rangle}=\operatorname{ind}\lim_K\mathrm{MHS}(K,A)_{\langle k\rangle}.$$

In general, we can define $\mathcal{M}(A)_{\langle k\rangle},\mathcal{M}(K,A)_{\langle k\rangle}$ similarly by using the subcategory $\mathcal{M}(\operatorname{Spec}R/k_R,A)^{\mathrm{sm}}[-d_R]$ of $\mathcal{M}(\operatorname{Spec}R/k_R,A)[-d_R]$ consisting of the objects whose underlying $\mathbb{Q}$-complexes are local systems.

For $j\in\mathbb{Z}$, we have $A_{\langle k\rangle}(j)\in\mathrm{MHS}(A)_{\langle k\rangle}$ which is represented by a constant variation of Hodge structure $A_{S/k_R}(j)$ of type $(-j,-j)$ on $S=\operatorname{Spec}R$. We have also $A_{K,\langle k\rangle}(j)\in\mathrm{MHS}(K,A)_{\langle k\rangle}$ as the limit of $A_{S/k_R}(j)$.

Similarly, we have $A_{K,\langle k\rangle}(j)\in\mathcal{M}(K,A)_{\langle k\rangle},A_{\langle k\rangle}(j)\in\mathcal{M}(A)_{\langle k\rangle}$ in general.

Definition. We say that $H\in\mathrm{MHS}(A)_{\langle k\rangle}$ (or $\mathcal{M}(A)_{\langle k\rangle}$) is global section-free, if for any finitely generated smooth k-subalgebra R of $\mathbb{C}$ such that H is represented by an admissible variation of mixed Hodge structure (or a smooth mixed sheaf) on $S:=\operatorname{Spec}R$, the underlying local system on $S_{\mathbb{C}}$ has no nonzero global section.

2.4. *Remarks.* (i) When $k = \overline{\mathbb{Q}}$, $\mathrm{MHS}(A)_{\langle k\rangle}$ is called the category of arithmetic mixed Hodge structures according to M. Asakura [1], and is closely related to recent work of P. Griffiths and M. Green [16], [17] (see also [40]).

If $H \in \mathrm{MHS}(A)$ is the image of $\mathcal{M} \in \mathrm{MHS}(A)_{\langle k\rangle}$ by the functor ι in (2.3.2), then the underlying $\mathbb{C}$-part $H_{\mathbb{C}}$ has naturally an integrable connection $\nabla : H_{\mathbb{C}} \to H_{\mathbb{C}} \otimes_{\mathbb{C}} \Omega^1_{\mathbb{C}/\overline{k}}$ which is the limit of the connection $M_R \to M_R \otimes_R \Omega^1_{R/k_R}$, where $M_R = \Gamma(S, M_S)$ with M_S the underlying $\mathcal{D}_S$-Modules of the representatives of $\mathcal{M}$.

So we can also consider the category $\mathrm{MHS}(A)_{\nabla/\overline{k}}$ of mixed Hodge structures whose $\mathbb{C}$-part has an integrable connection ∇ over $\overline{k}$. Then we can show that $\mathrm{MHS}(A)_{\langle k\rangle}$ is a full subcategory of $\mathrm{MHS}(A)_{\nabla/\overline{k}}$. Indeed, a morphism $A \to \iota(\mathcal{M})$ in $\mathrm{MHS}(A)_{\nabla/\overline{k}}$ defines a section of $\mathrm{Ker}\,\nabla$ in M_R for R sufficiently large, and the corresponding global section of $H_{\mathbb{C}}$ on $S_{\mathbb{C}}$ belongs to the A-local system H_A at the point corresponding to the inclusion $R \to \mathbb{C}$. See also [1], [16], [17].

(ii) The canonical functor $\mathcal{M}(A)_{\langle k\rangle} \to \mathrm{MHS}(A)$ is not fully faithful, because the morphism of extension classes is not injective. For example, let $S = \mathrm{Spec}\,R$ with $R = k[t, t^{-1}]$, and consider an admissible variation of mixed Hodge structure on S which is an extension of $\mathbb{Q}_S$ by $\mathbb{Q}_S(1)$, and which has an $\mathcal{O}_S$-basis e_0, e_1 such that $t\partial_t e_0 = e_1, t\partial_t e_1 = 0$, F^0 is generated by e_0 over $\mathcal{O}_S$, and the $\mathbb{Q}$-lattice of the fiber at t is generated by e_1 and $e_0 + (2\pi i)^{-1}\log(t/c)e_1$, where $c \in \mathbb{C}^*$ corresponds to the point of $S_{\mathbb{C}}$ determined by the embedding $R \to \mathbb{C}$.

For another example, we can use the noninjectivity of $\mathrm{Ext}^1(A_k, H) \to \mathrm{Ext}^1(A, a^*_{\mathbb{C}/k}H)$ for $H \in \mathrm{MHS}(A)_k$ (see (1.4)), where $a^*_{\mathbb{C}/k}H$ is the underlying graded-polarizable mixed A-Hodge structure of H. Indeed, the morphism is factorized by the injective morphism $\mathrm{Ext}^1(A_k, H) \to \mathrm{Ext}^1(A_{S/k}, a^*_{S/k}H)$.

(iii) If the objects of $\mathcal{M}(X/k, A)$ contain l-adic sheaves as in (1.1), then an objects of $\mathcal{M}(A)_{\langle k\rangle}$ contains the inductive limit of l-adic sheaves which has a Galois action in the weak sense. (This is however enough to define the weight filtration.) Indeed, the l-adic part of an object of $\mathcal{M}(K, A)_{\langle k\rangle}$ is identified with a $\mathbb{Q}_l$-vector space having an action of $\mathrm{Gal}(\overline{K}/K)$ (unramified over $\mathrm{Spec}\,R$ for some finitely generated smooth k-subalgebra R of K whose fractional field is K). Furthermore the pull-back by $K' \supset K$ is given by the composition of natural morphisms $\mathrm{Gal}(\overline{K'}/K') \to \mathrm{Gal}(\overline{K}K'/K') = \mathrm{Gal}(\overline{K}/\overline{K} \cap K') \to \mathrm{Gal}(\overline{K}/K)$.

(iv) Let $\mathrm{MHM}(X, A)^{\mathrm{go}}$, $\mathrm{MHS}(A)^{\mathrm{go}}$ be the categories of mixed Hodge Modules or structures of geometric origin. See [33]. We can define similarly $\mathrm{MHM}(X, A)^{\mathrm{go}}_{\langle k\rangle}$, $\mathrm{MHS}(A)^{\mathrm{go}}_{\langle k\rangle}$ as full subcategories of $\mathrm{MHM}(X, A)_{\langle k\rangle}$, $\mathrm{MHS}(A)_{\langle k\rangle}$ using morphisms of complex varieties and the direct image and pull-back functors between $\mathrm{MHM}(X, A)_{\langle k\rangle}$. Then we have a natural functor

$$\mathrm{MHM}(X, A)^{\mathrm{go}}_{\langle k\rangle} \to \mathrm{MHM}(X, A)^{\mathrm{go}}. \quad \mathrm{MHS}(A)^{\mathrm{go}}_{\langle k\rangle} \to \mathrm{MHS}(A)^{\mathrm{go}}. \tag{2.4.1}$$

These are essentially surjective. But it is not clear whether they are fully faithful. It would be true for morphisms between pure objects, if the Hodge conjecture is true. But the mixed case is not clear.

In (2.1) we can restrict to the subcategory consisting of objects of geometric origin,

because $\mathcal{M}(X,A)^{\mathrm{go}}_{\langle k\rangle} = \operatorname{ind\,lim} \mathcal{M}(X_{R'}/k_{R'},A)^{\mathrm{go}}[-d_{R'}]$. In this case the surjectivity of the cycle map is equivalent to the Hodge type conjecture. See [30], [33].

2.5. Proposition. *Let K, K' be as in (2.2) such that $K \subset K'$ and $\overline{k} \cap K = \overline{k} \cap K'$. Let $\pi' : \operatorname{Spec} K' \to \operatorname{Spec} K$ denote the canonical morphism. Then for a pure object M in $\mathrm{MHS}(K,A)_{\langle k\rangle}$, the canonical morphism*

$$\operatorname{Ext}^i(A_{K,\langle k\rangle}, M) \to \operatorname{Ext}^i(A_{K',\langle k\rangle}, \pi'^* M) \tag{2.5.1}$$

is injective.

Proof. Let $k' = \overline{k} \cap K$. Let u be an element of the source, and assume its image in the target is zero. Then there exist finitely generated smooth k'-subalgebras $R \subset R'$ of $\mathbb{C}$ such that K, K' are the fractional fields of R, R', and u is defined over R. We may assume that there is a multivalued section of $S' \to S$ such that its image S'' is finite etale over S. Let $\pi'' : S'' \to S$ denote the canonical morphism. Then the image of u in

$$\operatorname{Ext}^i(A_{S''/k'}, \pi''^* M) = \operatorname{Ext}^i(A_{S/k'}, \pi''_* \pi''^* M)$$

is zero, and the injection $M \to \pi''_* \pi''^* M$ splits by the semisimplicity of pure objects. So u is zero, and the assertion follows.

2.6. Proposition. *We have equivalences of categories*

$$\mathrm{MHM}(X,A)_{\langle k\rangle} \to \mathrm{MHM}(X,A)_{\langle \overline{k}\rangle}, \quad \mathrm{MHS}(A)_{\langle k\rangle} \to \mathrm{MHS}(A)_{\langle \overline{k}\rangle}.$$

induced by $\mathrm{MHM}(X_{R'}/k_{R'}, A) \to \mathrm{MHM}(X_{R'} \otimes_{k_{R'}} \overline{k}/\overline{k}, A)$.

Proof. It is enough to verify that a filtered $\mathcal{D}$-Module on $X_{R'} \otimes_{k_{R'}} \overline{k}/\overline{k}$ is defined on $X_{R'} \otimes_{k_{R'}} k'/k'$ for a sufficiently large algebraic extension k' of k in $\overline{k}$ (and a similar assertion holds for morphisms).

2.7. Theorem. *The category $\mathcal{M}(X,A)_{\langle k\rangle}$ is an $(A \cap \overline{k})$-linear abelian category (where $A = \mathbb{Q}$ unless $\mathcal{M}(X,A)_{\langle k\rangle} = \mathrm{MHM}(X,A)_{\langle k\rangle}$), and the bounded derived categories $D^b\mathcal{M}(X,A)_{\langle k\rangle}$ are stable by standard functors like $f_*, f_!, f^*, f^!$, etc. in a compatible way with the functor ι.*

Proof. This follows from (1.2) using an analogue of the generic base change theorem in [10].

Remark. Any object of $\mathcal{M}(X,A)_{\langle k\rangle}$ is represented by an object of $\mathcal{M}(X_{R'/k_{R'}}, A)[-d_{R'}]$ whose underlying perverse sheaf has a Whitney stratification of $\overline{X}_{R'/k_{R'}}$ such that the strata are smooth over R', where $\overline{X}_{R'/k_{R'}}$ is a smooth compactification of $X_{R'/k_{R'}}$ over R'.

3. Cycle Maps

3.1. For a complex variety X with structure morphism $a_X : X \to \mathbb{C}$, we define

$$A_{X,\langle k\rangle}(j) = a_X^* A_{\langle k\rangle}(j) \in D^b\mathcal{M}(X,A)_{\langle k\rangle},$$
$$H^i(X, A_{\langle k\rangle}(j)) = H^i(a_X)_* A_{X,\langle k\rangle}(j) \in \mathcal{M}(A)_{\langle k\rangle}.$$

These are compatible with the functor ι. By definition, $H^i(X, A_{\langle k\rangle}(j))$ is the limit of $H^i\pi_* A_{X_R/k_R}(j)$, where $\pi : X_R \to S = \operatorname{Spec} R$ denotes the structure morphism.

For X_K as in (2.2) with structure morphism $a_{X_K/K} : X_K \to \operatorname{Spec} K$, we define similarly

$$A_{X_K/K,\langle k\rangle}(j) = a_{X_K/K}^* A_{K,\langle k\rangle}(j) \in D^b\mathcal{M}(X_K/K, A)_{\langle k\rangle},$$
$$H^i(X_K/K, A_{\langle k\rangle}(j)) = H^i(a_{X_K/K})_* A_{X_K/K,\langle k\rangle}(j) \in \mathcal{M}(K,A)_{\langle k\rangle}.$$

If X is smooth over $\mathbb{C}$ and $A = \mathbb{Q}$, let

$$\operatorname{Hdg}^p_{\mathcal{M}}(X,\mathbb{Q})^{\langle k\rangle} = \operatorname{Hom}(\mathbb{Q}_{\langle k\rangle}, H^{2p}(X, \mathbb{Q}_{\langle k\rangle}(p))).$$

If the mixed sheaves in (1.1) are mixed Hodge Modules, this is called the group of k-finite (or k-horizontal) Hodge cycles of codimension p (and the index $\mathcal{M}$ will be omitted).

3.2. *Remark.* (i) Assume the mixed sheaves in (1.1) are mixed Hodge Modules. By Remark (i) of (2.4), $\operatorname{Hdg}^p(X,\mathbb{Q})^{\langle k\rangle}$ is identified with the space of $\mathbb{Q}$-Hodge cycles (in the usual sense) which are annihilated by the connection over $\overline{k}$. See also [16], [17].

When $A = \mathbb{Q}$, the Hodge conjecture implies that every $\mathbb{Q}$-Hodge cycle in the usual sense is a $\overline{\mathbb{Q}}$-finite (or $\overline{\mathbb{Q}}$-horizontal) Hodge cycle. It may be more natural to restrict the Hodge conjecture to $\overline{\mathbb{Q}}$-finite (or $\overline{\mathbb{Q}}$-horizontal) Hodge cycles.

3.3. *Deligne cohomology.* Let X be a proper smooth complex variety, and $H^i_{\mathcal{D}}(X,\mathbb{Q}(j))$ the $\mathbb{Q}$-Deligne cohomology in the usual sense [13]. By [2] it is isomorphic to

$$\operatorname{Ext}^i(\mathbb{Q}^H, (a_X)_*\mathbb{Q}^H_X(j)) = \operatorname{Ext}^i(\mathbb{Q}^H_X, \mathbb{Q}^H_X(j)),$$

where $\mathbb{Q}^H$ and $\mathbb{Q}^H_X$ belong to $D^b\mathrm{MHS}(\mathbb{Q})$ and $D^b\mathrm{MHM}(X,\mathbb{Q})$ respectively (see [29]), and the last isomorphism follows from the adjoint relation between the direct image and the pull-back. We have by [8]

$$J^p(X)_{\mathbb{Q}} = \operatorname{Ext}^1(\mathbb{Q}^H, H^{2p-1}(X,\mathbb{Q}(p))), \tag{3.3.1}$$

where $J^p(X)_{\mathbb{Q}}$ is Griffiths' intermediate Jacobian [18] tensored with $\mathbb{Q}$.

We define $H^i_{\mathcal{D}}(X, \mathbb{Q}_{\langle k\rangle}(j))$ as

$$\operatorname{Ext}^i(\mathbb{Q}_{\langle k\rangle}, (a_X)_*\mathbb{Q}_{X,\langle k\rangle}(j)) = \operatorname{Ext}^i(\mathbb{Q}_{X,\langle k\rangle}, \mathbb{Q}_{X,\langle k\rangle}(j)),$$

where the extension group is taken in the derived category of $\mathcal{M}(A)_{\langle k\rangle}$ or $\mathcal{M}(X,A)_{\langle k\rangle}$. This is the limit of $H^i_{\mathcal{D}}(X_R/k_R,\mathbb{Q}_k(j))$ which is defined as

$$\mathrm{Ext}^i(\mathbb{Q}_{k_R},(a_{X_R/k_R})_*\mathbb{Q}_{X_R/k_R}(j))=\mathrm{Ext}^i(\mathbb{Q}_{X_R/k_R},\mathbb{Q}_{X_R/k_R}(j)). \tag{3.3.2}$$

See (1.3). By (2.1.4) we have a natural morphism $H^i_{\mathcal{D}}(X,\mathbb{Q}_{\langle k\rangle}(j))\to H^i_{\mathcal{D}}(X,\mathbb{Q}(j))$. Its image is denoted by $H^i_{\mathcal{D}}(X,\mathbb{Q}(j))^{\langle k\rangle}$, and is called the k-finite part of the Deligne cohomology associated with the mixed sheaves $\mathcal{M}$. Since we have a noncanonical isomorphism

$$(a_X)_*a_X^*\mathbb{Q}_{\langle k\rangle}=\oplus_j H^j(X,\mathbb{Q}_{\langle k\rangle})[-j]$$

in the derived category of $\mathrm{MHS}(A)_{\langle k\rangle}$ (see [30, (8.1.7)]), there exists a decreasing filtration L on $H^i_{\mathcal{D}}(X,\mathbb{Q}_{\langle k\rangle}(j))$ induced by the truncation τ on $(a_X)_*\mathbb{Q}_{X,\langle k\rangle}(j)$ such that

$$\mathrm{Gr}^r_L H^i_{\mathcal{D}}(X,\mathbb{Q}_{\langle k\rangle}(j))=\mathrm{Ext}^r(\mathbb{Q}_{\langle k\rangle},H^{i-r}(X,\mathbb{Q}_{\langle k\rangle}(j))). \tag{3.3.3}$$

See [30, (8.1)]. For $(i,j)=(2p,p)$, we have a morphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J^p(X)^{\langle k\rangle}_{\mathbb{Q}} & \longrightarrow & H^{2p}_{\mathcal{D}}(X,\mathbb{Q}(p))^{\langle k\rangle} & \longrightarrow & \mathrm{Hdg}^p_{\mathcal{M}}(X,\mathbb{Q})^{\langle k\rangle} & \longrightarrow & 0\\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & J^p(X)_{\mathbb{Q}} & \longrightarrow & H^{2p}_{\mathcal{D}}(X,\mathbb{Q}(p)) & \longrightarrow & \mathrm{Hdg}^p(X,\mathbb{Q}) & \longrightarrow & 0 \end{array} \tag{3.3.4}$$

where $J^p(X)^{\langle k\rangle}_{\mathbb{Q}}$ is defined as the image of

$$\mathrm{Ext}^1(\mathbb{Q}_{\langle k\rangle},H^{2p-1}(X,\mathbb{Q}_{\langle k\rangle}(p)))\to\mathrm{Ext}^1(\mathbb{Q}^H,H^{2p-1}(X,\mathbb{Q}(p)))=J^p(X)_{\mathbb{Q}}. \tag{3.3.5}$$

Remark. It is not clear whether (3.3.5) is injective. See Remark (ii) of (2.4).

3.4. *Cycle map.* For a smooth proper complex variety X, let $\mathrm{CH}^p(X)_{\mathbb{Q}}$ be the Chow group of codimension p cycles on X tensored with $\mathbb{Q}$. By [29], [30] we have a cycle map

$$cl:\mathrm{CH}^p(X)_{\mathbb{Q}}\to H^{2p}_{\mathcal{D}}(X,\mathbb{Q}_{\langle k\rangle}(p)). \tag{3.4.1}$$

This is the limit of the cycle map of (1.3.2) :

$$cl_R:\mathrm{CH}^p(X_R)_{\mathbb{Q}}\to H^{2p}_{\mathcal{D}}(X_R/k_R,\mathbb{Q}_k(p)). \tag{3.4.2}$$

We have the induced filtration L (see (3.3.3)) on $\mathrm{CH}^p(X)_{\mathbb{Q}}$, and the graded pieces of the cycle map

$$\mathrm{Gr}^r_L cl:\mathrm{Gr}^r_L\mathrm{CH}^p(X)_{\mathbb{Q}}\to\mathrm{Ext}^r(\mathbb{Q}_{\langle k\rangle},H^{2p-r}(X,\mathbb{Q}_{\langle k\rangle}(p)) \tag{3.4.3}$$

is called the higher Abel-Jacobi map. It may be expected that L gives a conjectural filtration of Beilinson and Bloch (see [3], [6] and also [26]) on the Chow group. See for example [1].

For X_K as in (2.2), we have similarly the induced filtration L on $\mathrm{CH}^p(X_K)_{\mathbb{Q}}$ and

$$\mathrm{Gr}_L^r cl_K : \mathrm{Gr}_L^r \mathrm{CH}^p(X_K)_{\mathbb{Q}} \to \mathrm{Ext}^r(\mathbb{Q}_{K,\langle k\rangle}, H^{2p-r}(X_K/K, \mathbb{Q}_{\langle k\rangle}(p))). \tag{3.4.4}$$

This gives (3.4.3) by passing to the limit.

We have a short exact sequence

$$0 \to L^1\mathrm{CH}^p(X)_{\mathbb{Q}} \to \mathrm{CH}^p(X)_{\mathbb{Q}} \to \mathrm{Gr}_L^1\mathrm{CH}^p(X)_{\mathbb{Q}} \to 0, \tag{3.4.5}$$

and the cycle map induces a morphism of this exact sequence to the first row of the commutative diagram (3.3.4). We can verify that the morphism $L^1\mathrm{CH}^p(X)_{\mathbb{Q}} \to J^p(X)_{\mathbb{Q}}$ coincides with Griffiths' Abel-Jacobi map (by using [13]). See [29, (4.5.20)]. Indeed, we can prove that, for $\zeta \in \mathrm{CH}^p(X)_{\mathbb{Q}}$ represented by a closed irreducible subvariety Z of X, $cl(\zeta)$ is represented by the composition

$$\mathbb{Q}^H \to H_Z^{2p}(X, \mathbb{Q})(p) \to (a_X)_*\mathbb{Q}_X^H(p)[2p].$$

(The last has been known to specialists as Deligne's cycle map.)

Remark. We can define a cycle map from Bloch's higher Chow group $\mathrm{CH}^p(X,n)_{\mathbb{Q}}$ [7] to $\mathrm{Ext}^{2p-n}(\mathbb{Q}_{X,\langle k\rangle}, \mathbb{Q}_{X,\langle k\rangle}(p))$, and get the filtration L. See [30, (8.3)].

3.5. *Correspondence.* For smooth proper complex varieties X, Y, we define the group of correspondences by

$$C^i(X,Y)_{\mathbb{Q}} = \mathrm{CH}^{i+\dim X}(X \times Y)_{\mathbb{Q}},$$

where X is assumed to be equidimensional. Then we have the cycle map

$$\begin{aligned} C^i(X,Y)_{\mathbb{Q}} &\to \mathrm{Ext}^{2i+2\dim X}(\mathbb{Q}_{X\times Y,\langle k\rangle}, \mathbb{Q}_{X\times Y,\langle k\rangle}(i+\dim X)) \\ &= \mathrm{Hom}((a_X)_*\mathbb{Q}_{X,\langle k\rangle}, (a_Y)_*\mathbb{Q}_{Y,\langle k\rangle}(i)[2i]). \end{aligned} \tag{3.5.1}$$

This cycle map is compatible with the composition of correspondences. See [33, II, (3.3)]. This implies that the action of a correspondence on the Chow groups corresponds by the cycle map (3.4.1) to the composition of morphisms with the image of (3.5.1), because $C^i(pt, X)_{\mathbb{Q}} = \mathrm{CH}^i(X)_{\mathbb{Q}}$. So we get a commutative diagram

$$\begin{array}{ccc} \mathrm{CH}^p(X)_{\mathbb{Q}} & \xrightarrow{cl} & \mathrm{Hom}(\mathbb{Q}_{\langle k\rangle}, (a_X)_*\mathbb{Q}_{X,\langle k\rangle}(p)[2p]) \\ \big\downarrow{\scriptstyle \Gamma_*} & & \big\downarrow{\scriptstyle \Gamma_*} \\ \mathrm{CH}^{p+i}(Y)_{\mathbb{Q}} & \xrightarrow{cl} & \mathrm{Hom}(\mathbb{Q}_{\langle k\rangle}, (a_Y)_*\mathbb{Q}_{Y,\langle k\rangle}(p+i)[2p+2i]) \end{array} \tag{3.5.2}$$

for a correspondence $\Gamma \in C^i(X,Y)_{\mathbb{Q}}$. Since

$$\Gamma_* : (a_X)_*\mathbb{Q}_{X,\langle k\rangle} \to (a_Y)_*\mathbb{Q}_{Y,\langle k\rangle}(i)[2i]$$

preserves the filtration τ, we see that the action of a correspondence on the Chow groups preserves the filtration L, and the action on the graded pieces depends only on the cohomology class of the correspondence.

For X_K and K as in (2.2), we can define similarly $C^i(X_K, Y_K)_{\mathbb{Q}}$ and the cycle map

$$C^i(X_K, Y_K)_{\mathbb{Q}} \to \mathrm{Hom}((a_{X_K/K})_*\mathbb{Q}_{X_K/K,\langle k\rangle}, (a_{Y_K/K})_*\mathbb{Q}_{Y_K/K,\langle k\rangle}(i)[2i]). \tag{3.5.3}$$

This gives (3.5.1) by passing to the limit. We have also a commutative diagram similar to (3.5.2).

Remark. If $\zeta \in C^i(X,Y)_{\mathbb{Q}}$ is represented by an irreducible closed subvariety Z, let $Z' \to Z$ be a resolution of singularities, and p_i denote the composition with the projection to each direct factor. Then by [loc. cit] the image of ζ by (3.5.1) is the composition of the restriction and Gysin morphisms :

$$p_1^* : (a_X)_*\mathbb{Q}_{X,\langle k\rangle} \to (a_{Z'})_*\mathbb{Q}_{Z',\langle k\rangle}, \quad p_{2*} : (a_{Z'})_*\mathbb{Q}_{Z',\langle k\rangle} \to (a_Y)_*\mathbb{Q}_{Y,\langle k\rangle}(i)[2i].$$

3.6. Proposition. *Let X_K be a smooth proper K-variety of dimension n, where K is as in (2.2). Then $L^2\mathrm{CH}^n(X_K)_{\mathbb{Q}}$ (see (3.4.4)) coincides with the kernel of the Albanese map, and we get an injection*

$$\mathrm{Alb}_{X_K}(K) \to \mathrm{Ext}^1(A_{K,\langle k\rangle}, H^{2n-1}(X_K/K, \mathbb{Q}_{\langle k\rangle}(p))).$$

Proof. To simplify the notation, we will write X for X_K. Since $\mathrm{CH}^n(X_K)_{\mathbb{Q}}, \mathrm{Alb}_{X_K}(K)$, and $H^{2n-1}(X_K/K, \mathbb{Q}_{\langle k\rangle}(p))$ do not change by a birational morphism of smooth proper varieties (see [14], [23]), the assertion is reduced to the projective case.

It is clear that $L^2\mathrm{CH}^n(X_K)_{\mathbb{Q}}$ is contained in the kernel of the Albanese map. (Indeed, restricting to the fiber over the point corresponding to the inclusion $K \to \mathbb{C}$ and using (1.1.2), the assertion is reduced to the case $K = \mathbb{C}$.) Let $\zeta \in L^1\mathrm{CH}^n(X)_{\mathbb{Q}}$ belonging to the kernel of the Albanese map. Let Γ be the projector in $\mathrm{CH}^n(X_K \times_K X_K)_{\mathbb{Q}}$ corresponding to the Albanese motive in [25, 4.1] (i.e. $\Gamma^2 = \Gamma$ and the action on $H^i(X_K \otimes_K \overline{K}, \mathbb{Q}_l)$ is the identity for $i = 2n-1$, and zero otherwise). Furthermore $\Gamma_*\zeta = 0$ by loc. cit. We have to show the vanishing of

$$cl(\zeta) \in \mathrm{Ext}^1(\mathbb{Q}_{K,\langle k\rangle}, H^{2n-1}(X/K, \mathbb{Q}_{\langle k\rangle})).$$

But this follows from (3.5.2) applied to Γ and ζ, because the action of Γ_* on $H^{2n-1}(X/K, \mathbb{Q}_{\langle k\rangle})$ is the identity. So we get the assertion. ∎

4. Proof of Main Theorems

In this section we prove the main theorems for mixed sheaves in (1.1).

4.1. *Proof of* (0.3). It is enough to show the surjectivity of

$$\mathrm{CH}^p(X)_{\mathbb{Q}} \to \mathrm{Hdg}^p_{\mathcal{M}}(X, \mathbb{Q})^{\langle k\rangle}, \quad L^1\mathrm{CH}^p(X)_{\mathbb{Q}} \to J^p(X)^{\langle k\rangle}_{\mathbb{Q}}.$$

Let R be a finitely generated smooth k-subalgebra of $\mathbb{C}$ with a smooth proper morphism $\pi : X_R \to S := \operatorname{Spec} R$ such that $X_R \otimes_R \mathbb{C} = X$. Put $A = \mathbb{Q}$. Then $H^i(X, \mathbb{Q}_{\langle k\rangle}(j))$ is represented by a variation of Hodge structure $H^i\pi_* A_{X_R/k_R}(j)$ on S. For $i = 0, 1$, we have

$$\operatorname{Ext}^i(A_{S/k_R}, H^{2p-i}\pi_* A_{X_R/k_R}(p)) = \operatorname{Hom}(A_{k_R}, H^i(a_{S/k_R})_* H^{2p-i}\pi_* A_{X_R/k_R}(p)).$$

by using the Leray spectral sequence

$$E_2^{i,j} = \operatorname{Ext}^i(A_{k_R}, H^j(a_{S/k_R})_* M) \Rightarrow \operatorname{Ext}^{i+j}(A_{S/k_R}, M), \tag{4.1.1}$$

where $M = H^{2p}\pi_* A_{X_R/k_R}(p)$ or $H^{2p-1}\pi_* A_{X_R/k_R}(p)$ (because $E_2^{i,0} = 0$ for $i = 1, 2$ in the latter case by assumption). Since we have a noncanonical isomorphism (see [30, (6.10)]) :

$$\pi_* A_{X_R/k_R} = \oplus_i (H^i\pi_* A_{X_R/k_R})[-i] \quad \text{in } D^b\mathrm{MHM}(S/k_R, A), \tag{4.1.2}$$

we can identify $H^i(a_{S/k_R})_* H^{2p-i}\pi_* A_{X_R/k_R}(p)$ noncanonically with a subobject (and canonically with a subquotient) of $H^{2p}(X_R/k_R, A(p))$. So the assertion is reduced to the Hodge conjecture for a smooth compactification of $X_R \otimes_{k_R} \mathbb{C}$. See Remark (i) below. Then it is further reduced to the Hodge conjecture for a smooth projective variety due to Chow's lemma.

Remarks. (i) Let X_k be a smooth k-variety such that $k = \overline{k} \cap k(X_k)$ (i.e. X_k/k is geometrically irreducible). Let $\overline{X}_k$ be a smooth compactification of X_k such that $\overline{X}_k \setminus X_k$ is a divisor with normal crossings. Then the cycle map

$$\mathrm{CH}^p(X)_{\mathbb{Q}} \to \operatorname{Hom}(\mathbb{Q}_k, H^{2p}(X_k/k, \mathbb{Q}_k(p))) \tag{4.1.3}$$

is surjective if the Hodge conjecture for codimension p cycles on $\overline{X}_{\mathbb{C}} := \overline{X}_k \otimes_k \mathbb{C}$ is true.

Indeed, $H^{2p}(X_k/k, \mathbb{Q}_k)$ has weights $\geq 2p$ and $\mathrm{Gr}_{2p}^W H^{2p}(X_k/k, \mathbb{Q}_k)$ is a quotient of $H^{2p}(\overline{X}_k/k, \mathbb{Q}_k)$. See [9]. So the assertion is reduced to the case X_k is smooth proper. Then the assertion is more or less well-known (using the Galois action on l-adic cohomology or de Rham cohomology). See for example [31, (8.5), (iii)].

(ii) For a smooth proper k-variety X_k and a finitely generated smooth k-subalgebra R of $\mathbb{C}$, let $X = X_k \otimes_k \mathbb{C}, X_R = X_k \otimes_k R$, and $S = \operatorname{Spec} R$. Assume $k = \overline{k} \cap k(X_k)$ and the mixed sheaves in (1.1) are mixed Hodge Modules. Then $J^p(X)^{\langle k\rangle} = J^p(X)$, and the Abel-Jacobi map to $J^p(X)^{\langle k\rangle}$ is not surjective if $H^{2p-1}(X)$ has level > 1.

4.2. *Proof of* (0.1). Let R be a finitely generated smooth k-subalgebra of $\mathbb{C}$ with a smooth proper morphism $\pi : X_R \to S := \operatorname{Spec} R$ such that $X_R \otimes_R \mathbb{C} = X$. We may assume X is connected. Put

$$M = H^{2n-j}\pi_* A_{X_R/k_R}(n).$$

Let R' be the affine ring of an affine open subvariety U of X_R (i.e. $U = \operatorname{Spec} R'$). Put $S' = \operatorname{Spec} R'$ with $\pi' : S' \to S$ the canonical morphism. Let $R^{(m)}$ be the m-ple tensor product of R' over R. We choose an embedding $R^{(m)} \to \mathbb{C}$ over R. (Here $R^{(m)} \otimes_{k_R} \overline{k}$ is integral so that $R^{(m)} \cap \overline{k} = k_R$, because π' is smooth, and the generic fiber is geometrically

irreducible.) Put $S^{(m)} = \operatorname{Spec} R^{(m)}$ with $\pi^{(m)} : S^{(m)} \to S$ the canonical morphism. For a nonempty affine open subvariety S'' of $S^{(m)}$, let $\pi'' : S'' \to S$ denote the restriction of $\pi^{(m)}$. It is enough to show that

$$\operatorname{ind}\lim_{S''} \operatorname{Ext}^j(A_{S''/k_R}, \pi''^* M) \tag{4.2.1}$$

with S'' running over nonempty affine open subvarieties of $S^{(m)}$, has at least dimension m, and the morphism induced by the base change associated with the first functor of (2.1.1) has at least rank m, because the effect by the second functor of (2.1.1) is covered by (2.5). (Concerning the first functor of (2.1.1), we can do the same construction as above after taking the base change by a finite extension $k_{R'} \to k_{R''}$.)

By the edge morphism of the Leray spectral sequence, we have a canonical morphism

$$\operatorname{Ext}^j(A_{S''/k_R}, \pi''^* M) \to \operatorname{Hom}(A_{S/k_R}, H^j \pi''_* \pi''^* M)$$

This is compatible with the transition morphism of the inductive limit, and is surjective by the Lemma below. So we may replace (4.2.1) with

$$\operatorname{ind}\lim_{S''} \operatorname{Hom}(A_{S/k_R}, H^j \pi''_* \pi''^* M), \tag{4.2.2}$$

and we have to show that it has at least dimension m, and the morphism induced by the first functor of (2.1.1) has at least rank m.

We first consider the case $m = 1$, where we may assume $S'' = S'$ shrinking U if necessary. Let

$$M' = H^{2n-j} \pi'_! A_{U/k_R}(n), \quad M'^* = H^j \pi'_* A_{U/k_R},$$

where π' is identified with the restriction of π to U. We may assume that the underlying $\mathbb{Q}$-complexes of M' and M'^* are local systems (shrinking S if necessary). Then they are dual of each other (up to a Tate twist and a shift of complex), and

$$H^j \pi'_* \pi'^* M = M'^* \otimes M = \mathcal{H}om(M', M).$$

We have a canonical element ξ in

$$\operatorname{Hom}(A_{S/k_R}, \mathcal{H}om(M', M)) = \operatorname{Hom}(M', M),$$

corresponding to the canonical morphism $M' \to M$. We see that this is nonzero for any $U \subset X_R$ by restricting to the fiber at the geometric generic point of S and using the functor (2.1.4) together with the level of Hodge structure, because $\Gamma(X, \Omega_X^j) \neq 0$. The compatibility with the base change is clear.

Now we consider the case $m > 1$. Since $\pi_*^{(m)} A_{S^{(m)}/k_R}$ is the m-ple tensor product of $\pi'_* A_{S'/k_R}$, we get a canonical injection

$$\overset{m}{\oplus}(M'^* \otimes M) \to H^j \pi_*^{(m)} A_{S^{(m)}/k_R} \otimes M.$$

We can verify that this gives an m-dimensional subspace of

$$\operatorname{Hom}(A_{S/k_R}, H^j \pi_*^{(m)} A_{S^{(m)}/k_R} \otimes M),$$

and of (4.2.2) using (2.1.4) and the level of Hodge structure as above.

4.3. Lemma. *Let X, S be smooth k-schemes of finite type with a smooth morphism $f : X \to S$ such that k is algebraically closed in the function field of S and $H^j f_* A_{X/k}$ is smooth (i.e. its underlying $\mathbb{Q}$-complex is a local system). Let $M \in \mathcal{M}(S/k, A)^{\mathrm{sm}}[-\dim S]$ with pure weight $-j$. Then the canonical morphism*

$$\mathrm{Ext}^j(A_{X/k}, f^*M) = \mathrm{Ext}^j(A_{S/k}, f_* f^* M) \to \mathrm{Hom}(A_{S/k}, H^j f_* f^* M)$$

is surjective by shrinking S if necessary.

Proof. Let $\overline{f} : \overline{X} \to S$ be a smooth compactification of f (i.e., $\overline{f}$ is smooth proper). Then the canonical morphism $H^j \overline{f}_* \overline{f}^* M \to \mathrm{Gr}_0^W H^j f_* f^* M$ is surjective, and $W_{-1} H^j f_* f^* M = 0$. This implies the surjectivity of

$$\mathrm{Hom}(A_{S/k}, H^j \overline{f}_* \overline{f}^* M) \to \mathrm{Hom}(A_{S/k}, H^j f_* f^* M),$$

and the assertion is reduced to the case where f is smooth proper. Then it follows from the decomposition (4.1.2) applied to the direct image of $\overline{f}^* M$ by $\overline{f}$.

4.4. *Proof of* (0.2). We may assume X is connected. It is enough to construct cycles in $L^2\mathrm{CH}^n(X)_{\mathbb{Q}}$ whose images in (4.2.2) coincide with the elements constructed in (4.2). We first consider the case X is projective. With the notation of (4.2), let K be the fractional field of R, and $X_K = X_R \otimes_R K$. Let K' be the function field of X_K, and $X_{K'} = X_K \otimes_K K'$. By Murre's result [25], [26], there is a middle dimensional cycle ζ on $X_K \times_K X_K$ such that $\zeta \circ \zeta = \zeta$ (as correspondences) and the action of its base change $\zeta_{\mathbb{C}} \in \mathrm{CH}^n(X \times X)$ on $H^j(X, \mathbb{Q})$ (via the correspondence) is the identity for $j \le 2n-2$, and zero otherwise. Let ζ' be the restriction of ζ to the generic fiber $X_{K'}$ of the first projection of $X_K \times_K X_K$ to X_K. Let $\zeta_{K'}$ be the pull-back of ζ to $X_{K'} \times_{K'} X_{K'}$. Then $(\zeta_{K'})_* \zeta' = \zeta'$ by $\zeta \circ \zeta = \zeta$ (see Lemma below), and hence $\zeta' \in L^2\mathrm{CH}^n(X_{K'})_{\mathbb{Q}}$ (using a diagram similar to (3.5.2)). Here we may assume that ζ is defined on $X_R \times_R X_R$ replacing R if necessary. We see that the image of ζ in $\mathrm{Hom}(A_{S/k}, H^2 \pi''_* \pi''^* M)$ in the notation of (4.2) coincides with the canonical element constructed in (4.2), using the action of $\zeta_{\mathbb{C}}$ on $H^{2n-2}(X, \mathbb{Q})$ together with(2.1.4). (A similar argument works for $L^r\mathrm{CH}^n(X)_{\mathbb{Q}}$ if the Künneth decomposition in the Chow group [26] holds.)

To show that the image of the second Abel-Jacobi map has dimension $\ge m$, it is enough to take the $(m+1)$-ple fiber product of X_R over R, and consider the pull-back of the above cycle by the projection to the fiber product of the i-th and the $(m+1)$-th factors for $1 \le i \le m$.

In the case X is not necessarily projective, let $\rho : X' \to X$ be a birational morphism such that X' is smooth projective. We may assume that X' is defined over R, and let X'_R, X'_K be as above. Then we have an isomorphism $\rho_* : L^2\mathrm{CH}_0(X')_{\mathbb{Q}} \to L^2\mathrm{CH}_0(X)_{\mathbb{Q}}$ (see [14]) and a decomposition $H^{2n-2}(X', \mathbb{Q}_{\langle k \rangle}) = H^{2n-2}(X, \mathbb{Q}_{\langle k \rangle}) \oplus N$, where N has level zero. So N does not contribute to the m-dimensional vector space constructed in (4.2), and the assertion is reduced to the projective case.

Remark. Let X be a smooth hypersurface of $\mathbb{P}^{n+1}$ defined over k. Then we have $R = K = k$ in the notation of (4.1). Take a closed point $\mathrm{Spec}\, k'$ of X_k, and let

$$\zeta = \Delta - [k' : k]^{-1}(X_k \times_k \mathrm{Spec}\, k') \in \mathrm{CH}^n(X_k \times_k X_k)_{\mathbb{Q}},$$

where Δ is the diagonal. Let K' be the function field of X_k, and ζ' the restriction of ζ to $X_{K'}$ as before. Assume that the mixed sheaves in (1.1) are mixed Hodge Modules. Then

$$\zeta' \in L^n \mathrm{CH}^n(X_{K'})_{\mathbb{Q}}.$$

Indeed, it is enough to show by (1.5)

$$\mathrm{Ext}^1(\mathbb{Q}_k, H^{2i-1}(U/k, \mathbb{Q}(i)) = 0, \quad \mathrm{Hom}(\mathbb{Q}_k, H^{2j}(U/k, \mathbb{Q}(j)) = 0$$

for $0 < i \leq n/2, 0 < j < n/2$, and U a sufficiently small open subvariety of X_k, because $H^r(X/k, \mathbb{Q}) = \mathbb{Q}(-r/2)$ or zero for $r > n$. But the assertion is clear by

$$W_{2i-1}H^{2i-1}(U/k, \mathbb{Q}) = W_{2j}H^{2j}(U/k, \mathbb{Q}) = 0.$$

We can show furthermore that $\mathrm{Gr}_L^n cl_{K'}(\zeta') \neq 0$, and the image of $\mathrm{Gr}_L^n cl$ is infinite dimensional as in (4.2) if $\deg X \geq n+2$.

(ii) Let X, Y be smooth proper complex varieties of dimension n, m respectively, and $\zeta \in C^i(Y, X)$ for a positive integer i such that $i+m \leq n$. Assume they are defined over k so that they come from X_k, etc. Let K be the function field of Y_k/k, and ζ_K the restriction of ζ_k to $X_K := X_k \otimes_k K$. Assume further $\zeta_K \in L^m \mathrm{CH}^{i+m}(X_K)$. Then $\mathrm{Gr}_L^m cl(\zeta_K)$ is nonzero if the image of $\zeta_* : H^m(Y, \mathbb{Q}) \to H^{2i+m}(X, \mathbb{Q})(i)$ has level of Hodge structure m.

4.5. Lemma. *Let X, Y, Z be smooth proper k-varieties, and $\zeta \in C^i(X, Y), \zeta' \in C^j(Y, Z)$. Let $\zeta'' = \zeta \circ \zeta' \in C^{i+j}(X, Z)$. Assume X is irreducible, and let K be the function field of X. Put $Y_K = Y \otimes_k K$, and similarly for Z_K. Let ζ_K, ζ''_K be the restrictions of ζ, ζ'' to Y_K, Z_K, and $\zeta'_K \in C^j(Y_K, Z_K)$ be the pull-back of ζ'. Then*

$$\zeta''_K = (\zeta'_K)_* \zeta_K.$$

Proof. This follows from $\zeta \circ \zeta' = p_* i^*(\zeta \times \zeta')$ where $i = id \times \delta \times id : X \times Y \times Z \to X \times Y \times Y \times Z$ with δ the diagonal, and $p : X \times Y \times Z \to X \times Z$ is the projection.

4.6. *Proof of* (0.4). With the notation of (4.1), we have

$$\mathrm{Ext}^{2p}(\mathbb{Q}_{S/k_R}, \pi_* \mathbb{Q}_{X_R/k_R}(p)) = \mathrm{Ext}^{2p}(\mathbb{Q}_{k_R}, (a_{X_R/k_R})_* \mathbb{Q}_{X_R/k_R}(p)),$$

and $H^{2p}_{\mathcal{D}}(X, \mathbb{Q}_{\langle k \rangle}(p))$ is equal to

$$\mathrm{ind}\lim_R \mathrm{Ext}^{2p}(\mathbb{Q}_{k_R}, (a_{X_R/k_R})_* \mathbb{Q}_{X_R/k_R}(p)).$$

So the first assertion is reduced to the injectivity of the cycle map

$$\mathrm{CH}^p(X_R)_{\mathbb{Q}} \to \mathrm{Ext}^{2p}(\mathbb{Q}_{k_R}, (a_{X_R/k_R})_* \mathbb{Q}_{X_R/k_R}(p)).$$

See (3.4.2). Then it follows from the assumption by using the injective morphisms in (1.3.5). The last assertion is reduced to the following.

4.7. Proposition. *Let X be a smooth proper variety over k such that k is algebraically closed in the function field of X. Let Y be a divisor on X, and $\rho : X' \to X$ be a birational k-morphism such that X' is smooth projective and $Y' := \rho^{-1}(Y)$ is a divisor with normal crossings whose irreducible components Y'_i are smooth. Then the Able-Jacobi map over k for $U := X \setminus Y$ and $p = 2$ is injective if the same injectivity holds for X' and $p = 2$.*

Proof. By the compatibility of the cycle map with the pull-back, the assertion is easily reduced to that for $U' = \rho^{-1}(U)$. So we assume $X' = X$ and Y'_i will be denoted by Y_i.

Let $M = H^3(U/k, \mathbb{Q})(2)$. Then it has weights ≥ -1, and we have an exact sequence

$$\mathrm{Hom}(\mathbb{Q}_k, \mathrm{Gr}_0^W M) \to \mathrm{Ext}^1(\mathbb{Q}_k, \mathrm{Gr}_{-1}^W M) \to \mathrm{Ext}^1(\mathbb{Q}_k, M), \tag{4.7.1}$$

where W is the weight filtration so that

$$\mathrm{Gr}_{-1}^W M = \mathrm{Coker}(\oplus_i H^1(Y_i/k, \mathbb{Q}(1)) \to H^3(X/k, \mathbb{Q}(2)))$$

and $\mathrm{Gr}_0^W M$ is isomorphic to a quotient of

$$\mathrm{Ker}(\oplus_i H^2(Y_i/k, \mathbb{Q}(1)) \to H^4(X/k, \mathbb{Q}(2))).$$

See [9]. The first morphism of (4.7.1) is also induced by $(a_{Y/k})_* i^! \mathbb{Q}_{X/k} \to (a_{X/k})_* \mathbb{Q}_{X/k}$, where $i : Y \to X$ is the inclusion morphism. So the cycle map gives a morphism of an exact sequence

$$(\oplus_i \mathrm{CH}^1(Y_i)_{\mathbb{Q}})^{(0)} \to L^1\mathrm{CH}^2(X)_{\mathbb{Q}} \to L^1\mathrm{CH}^2(U)_{\mathbb{Q}} \tag{4.7.2}$$

to (4.7.1), where $(\oplus_i \mathrm{CH}^1(Y_i)_{\mathbb{Q}})^{(0)} = \mathrm{Ker}((\oplus_i \mathrm{CH}^1(Y_i)_{\mathbb{Q}}) \to H^4(X_{\mathbb{C}}, \mathbb{Q}(p)))$. Since the Hodge conjecture for codimension one cycles on Y_i is true, the morphism of the left terms is surjective. See Remark (i) after (4.1). This implies also the surjectivity of $L^1\mathrm{CH}^2(X)_{\mathbb{Q}} \to L^1\mathrm{CH}^2(U)_{\mathbb{Q}}$. Then the assertion is reduced to the next proposition, because

$$\mathrm{Ext}^1(\mathbb{Q}_k, \mathrm{Gr}_{-1}^W M) = \mathrm{Coker}(\oplus_i J^1(Y_i/k)_{\mathbb{Q}} \to J^2(X/k)_{\mathbb{Q}}).$$

4.8. Proposition. *Let X,Y be smooth projective k-varieties of pure dimension n and $n-1$ respectively. Let $g : Y \to X$ be a morphism of k-varieties. Then*

$$\mathrm{Gr}_L^1 cl(g_* L^1\mathrm{CH}^1(Y)_{\mathbb{Q}}) = \mathrm{Gr}_L^1 cl(L^1\mathrm{CH}^2(X)_{\mathbb{Q}}) \cap \mathrm{Im}(g_* : J^1(Y/k)_{\mathbb{Q}} \to J^2(X/k)_{\mathbb{Q}})$$

Proof. Let l denote the hyperplane section class of X. We will denote also by l the induced morphism

$$l : H^i(X/k, \mathbb{Q}) \to H^{i+2}(X/k, \mathbb{Q})(1).$$

(It is defined over k because the condition for a general hyperplane section is Zariski -open, and k is an infinite field.) By the Lefschetz decomposition, we have

$$H^3(X/k, \mathbb{Q}) = H^3(X/k, \mathbb{Q})^{\mathrm{prim}} \oplus lH^1(X/k, \mathbb{Q}(-1)),$$

where $H^3(X/k,\mathbb{Q})^{\rm prim} = \mathrm{Ker}\, l^{n-2} \subset H^3(X/k,\mathbb{Q})$. This implies

$$J^2(X/k)_{\mathbb{Q}} = J^2(X/k)^{\rm prim}_{\mathbb{Q}} \oplus lJ^1(X/k)_{\mathbb{Q}},$$

where $J^2(X/k)^{\rm prim}_{\mathbb{Q}} = \mathrm{Ext}^1(\mathbb{Q}_k, H^3(X/k,\mathbb{Q}(2))^{\rm prim})$.

Let $e \in J^1(Y/k)_{\mathbb{Q}}$ such that $g_*e \in \mathrm{Gr}^1_L cl(L^1\mathrm{CH}^2(X)_{\mathbb{Q}})$. We have to show $g_*e \in \mathrm{Gr}^1_L cl(g_*L^1\mathrm{CH}^1(Y)_{\mathbb{Q}})$. We first reduce the assertion to the case $g_*e \in J^2(X/k)^{\rm prim}_{\mathbb{Q}}$.

Since l is algebraically defined as a correspondence, we have

$$l^{n-2}g_*e \in \mathrm{Gr}^1_L cl(L^1\mathrm{CH}^n(X)_{\mathbb{Q}}) \subset J^n(X/k)_{\mathbb{Q}}.$$

By (3.6), we can identify $\mathrm{Gr}^1_L cl(L^1\mathrm{CH}^n(X)_{\mathbb{Q}})$ with $\mathrm{Alb}_X(k)_{\mathbb{Q}}$. Let P_Y be the Picard variety of Y. Since $l^{n-2}g_*$ is algebraically defined by a correspondence Γ, we have a commutative diagram

$$\begin{array}{ccc} P_Y(k)_{\mathbb{Q}} & \longrightarrow & J^1(Y/k)_{\mathbb{Q}} \\ \big\downarrow{\scriptstyle \Gamma_*} & & \big\downarrow{\scriptstyle \Gamma_*} \\ \mathrm{Alb}_X(k)_{\mathbb{Q}} & \longrightarrow & J^n(X/k)_{\mathbb{Q}} \end{array}$$

where the horizontal morphisms are injective. We see that the left vertical morphism comes from a morphism of abelian varieties. (Indeed, if D denotes the Poincaré divisor on $P_Y \times_k Y$, the composition $D\circ\Gamma \in \mathrm{CH}^n(P_Y \times_k X)$ as correspondences gives further $(D\circ\Gamma)_{\rm Alb} \in \mathrm{CH}^m(P_Y \times_k \mathrm{Alb}_X)$ which is given by the graph of a morphism of abelian varieties, and induces $\Gamma_* : P_Y(k) \to \mathrm{Alb}_X(k)$, where $m = \dim \mathrm{Alb}_X$.) Then the diagram induces injective morphisms on the kernel and the cokernel of Γ_* (by using the base change by $k \to \mathbb{C}$ together with (1.1.2)). So $l^{n-2}g_*e$ belongs to $\Gamma_*P_Y(k)_{\mathbb{Q}}$, and there exists $\zeta \in L^1\mathrm{CH}^1(Y)_{\mathbb{Q}}$ such that

$$l^{n-2}g_*e = l^{n-2}g_*\mathrm{Gr}^1_L cl(\zeta).$$

Then we may assume $l^{n-2}g_*e = 0$ (i.e. g_*e is primitive) by replacing e with $e - \mathrm{Gr}^1_L cl(\zeta)$.

Now let

$$P_Y(k)^{(0)}_{\mathbb{Q}} = \mathrm{Ker}\,\Gamma_* \subset P_Y(k)_{\mathbb{Q}}, \;\; J^1(Y/k)^{(0)}_{\mathbb{Q}} = \mathrm{Ker}\,\Gamma_* \subset J^1(Y/k)_{\mathbb{Q}},$$

and consider

$$g^*l^{n-3}g_*e \in \mathrm{Alb}_Y(k)_{\mathbb{Q}} \subset J^{n-1}(Y/k)_{\mathbb{Q}}.$$

By a similar argument, there exists $\zeta' \in P_Y(k)^{(0)}_{\mathbb{Q}}$ such that

$$g^*l^{n-3}g_*e = g^*l^{n-3}g_*\mathrm{Gr}^1_L cl(\zeta').$$

So it remains to show that the restriction of g^*l^{n-3} to $\mathrm{Im}\, g_* \cap H^3(X/k,\mathbb{Q})^{\rm prim}$ is injective, or equivalently, the pairing $\langle u, l^{n-3}v\rangle$ is nondegenerate on $\mathrm{Im}\, g_* \cap H^3(X/k,\mathbb{Q})^{\rm prim}$. But this is clear because the paring is a polarization of Hodge structure. This completes the proof of (4.7-8) and (0.4).

Remarks. (i) We can verify that Bloch's conjecture [6] is true if the conclusion of (0.4) holds. Let X be a smooth proper complex variety of pure dimension 2, and $N^1H^2(X,\mathbb{Q}_{\langle k\rangle})$ the space of Hodge (i.e. algebraic) cycles of codimension one, which is a direct sum of $\mathbb{Q}_{\langle k\rangle}(-1)$. Then for $\zeta \in L^2\mathrm{CH}^2(X)_{\mathbb{Q}}$, we can show that $cl(\zeta) = 0$ if $cl(\zeta) \in \mathrm{Ext}^2(\mathbb{Q}_{\langle k\rangle}, N^1H^2(X,\mathbb{Q}_{\langle k\rangle}(2)))$.

Indeed, let C be a (not necessarily connected) smooth proper curve with a morphism $g : C \to X$ such that $N^1H^2(X,\mathbb{Q}_{\langle k\rangle}) = g_*H^0(C,\mathbb{Q}_{\langle k\rangle}(-1))$. Then we have an injection

$$g^* : N^1H^2(X,\mathbb{Q}_{\langle k\rangle}) \to H^2(C,\mathbb{Q}_{\langle k\rangle}),$$

but $g^*cl(\zeta) = cl(g^*\zeta) = 0$. So $cl(\zeta) = 0$. See also [33, II, (4.12)].

In general, it is easy to show that for a smooth projective variety X and $\zeta \in L^r\mathrm{CH}^p(X)_{\mathbb{Q}}$ with $r > p$ we have $\mathrm{Gr}_L^r cl(\zeta) = 0$ and hence $cl(\zeta) = 0$ (using the pull-back to an intersection of generic hyperplanes of dimension $2p - r$). See [33, I].

(ii) The extension groups in $\mathrm{MHS}(\mathbb{Q})_{\langle k\rangle}$ are generally too big. For example, we have

$$\mathrm{Ext}^2(\mathbb{Q}_{\langle k\rangle}, \mathbb{Q}_{\langle k\rangle}(1)) \neq 0$$

by (1.4–5) in the case the mixed sheaves in (1.1) are mixed Hodge Modules. Indeed, for R, S as in (2.3), we have

$$\mathrm{Ext}^1(\mathbb{Q}_k, H^1(S/k_R, \mathbb{Q}(1))) \neq 0,$$

if $\mathrm{Gr}_1^W H^1(S_{\mathbb{C}}, \mathbb{Q}) \neq 0$.

(iii) With the notation of (0.4), we have a commutative diagram

$$\begin{array}{ccc} L^1\mathrm{CH}^p(Y)_{\mathbb{Q}} & \longrightarrow & J^p(Y/k)_{\mathbb{Q}} \\ \downarrow & & \downarrow \\ L^1\mathrm{CH}^p(Y_{\mathbb{C}})_{\mathbb{Q}} & \longrightarrow & J^p(Y_{\mathbb{C}})_{\mathbb{Q}} \end{array}$$

where the horizontal morphisms are the Able-Jacobi maps. If the composition of the horizontal and vertical morphisms is injective, it would imply the injectivity of the Abel-Jacobi map defined over k. But the converse is not clear, because the right vertical morphism is not injective by (1.4) (although so is the left one).

4.9. *Remarks.* (i) Assume that a smooth proper complex variety X admits the Künneth decomposition in the Chow group in the sense of Murre [25] [26] (i.e., if there exist $p_i \in C^0(X,X)$ for $i \in \mathbb{Z}$ such that $p_i = 0$ for $i > 2\dim X$ or $i < 0$, $\sum_i p_i$ is the diagonal of X, $p_ip_j = \delta_{i,j}p_i$, and the action of p_i on $H^j(X,\mathbb{Q})$ is $\delta_{i,j}id$). Let $q_{r,j} = \sum_{0\le i<j} p_{r-i}$ and $q'_{r,j} = \sum_{i<0} p_{r-i} + \sum_{i\ge j} p_{r-i}$. Then, following Murre, we define the filtration F_M on $\mathrm{CH}^p(X)_{\mathbb{Q}}$ by

$$F_M^j\mathrm{CH}^p(X)_{\mathbb{Q}} = (q'_{2p,j})_*\mathrm{CH}^p(X)_{\mathbb{Q}} = \mathrm{Ker}((q_{2p,j})_* : \mathrm{CH}^p(X)_{\mathbb{Q}} \to \mathrm{CH}^p(X)_{\mathbb{Q}}).$$

Let L be as in (3.4.3). Then we can verify

$$F_M \subset L \tag{4.9.1}$$

inductively, using $p_ip_j = \delta_{i,j}p_i$ together with (3.5). Similarly, we see

$$\mathrm{Gr}^j_{F_M}\mathrm{Gr}^i_L = 0 \quad \text{for } i \neq j, \tag{4.9.2}$$

because the action of p_{2p-j} on $\mathrm{Gr}^j_{F_M}$ is the identity. In particular,

$$F_M = L \mod \bigcap_i L^i. \tag{4.9.3}$$

If the cycle map (3.4.1) is injective for X, we have $\bigcap_i L^i = 0$ and the action of p_i on $\mathrm{CH}^p(X)_{\mathbb{Q}}$ for $i > 2p$ or $i < p$ is zero. (The relation with a conjectural filtration of Beilinson and Bloch is explained in [20] assuming the algebraicity of the Künneth components of the diagonal.)

(ii) In [36], Shuji Saito has defined a filtration F_{Sh} on $\mathrm{CH}^p(X)_{\mathbb{Q}}$ for a smooth projective complex variety X by letting $F^0_{\mathrm{Sh}}\mathrm{CH}^p(X)_{\mathbb{Q}} = \mathrm{CH}^p(X)_{\mathbb{Q}}$ and using an inductive formula

$$F^{j+1}_{\mathrm{Sh}}\mathrm{CH}^p(X)_{\mathbb{Q}} = \sum \Gamma_* F^j_{\mathrm{Sh}}\mathrm{CH}^{p-i}(Y)_{\mathbb{Q}}.$$

Here the summation is taken over correspondences $\Gamma \in C^i(Y,X)_{\mathbb{Q}}$ with i integers and Y smooth projective complex varieties such that the image of

$$\Gamma_* : H^{2p-j-2i}(Y,\mathbb{Q})(-i) \to H^{2p-j}(X,\mathbb{Q}) \tag{4.9.4}$$

is contained in $N^{p-j+1}H^{2p-j}(X,\mathbb{Q})$. (Here N denotes the "coniveau" filtration.) It is possible to consider some variants of this definition. We get filtrations $F_\alpha, F_\beta, F_\gamma$ by modifying the condition on Γ respectively as follows.
(α) There exists $\Gamma' \in C^i(Y,X)_{\mathbb{Q}}$ such that the morphisms (4.9.4) for Γ and Γ' coincide and Γ' is supported on $Y \times Z$ with $\dim Z < \dim X - p + j$.
(β) The morphism (4.9.4) is zero.
(γ) The morphism (4.9.4) is zero and $Y = X, i = 0$.

We have clearly $F_{\mathrm{Sh}} \supset F_\alpha \supset F_\beta \supset F_\gamma$ (and they contain F_M if the latter exists). The filtrations F_α, F_β are stable by the action of correspondences (using [14] for F_α). We can show also

$$F_\alpha \subset L. \tag{4.9.5}$$

But it does not seem easy to verify $F_{\mathrm{Sh}} = F_\alpha$ or $F_{\mathrm{Sh}} \subset L$ without assuming the standard conjectures [22]. If the Künneth components of the diagonal are algebraic for X, then

$$\mathrm{Gr}^j_{F_*}\mathrm{Gr}^i_L = 0 \quad \text{for } i \neq j \text{ and } * = \mathrm{Sh}, \alpha, \beta, \gamma. \tag{4.9.6}$$

So we have $F_* = L$ for $* = \alpha, \beta, \gamma$, if the Künneth components of the diagonal are algebraic and the cycle map (3.4.1) is injective for X.

5. Higher Chow Groups

5.1. Let k be a subfield of $\mathbb{C}$ as above. For a k-variety X of pure dimension n, let $\mathrm{CH}^p(X,m)_{\mathbb{Q}}$ denote Bloch's higher Chow group with rational coefficients [7]. Its elements are represented by cycles of codimension p on $X\times_k\mathbb{A}^m$. Let $\mathcal{M}(X/k,\mathbb{Q})$ be the categories of mixed sheaves for k-varieties X as in (1.1). Let $d=n-p$. Then by [30] we can construct a cycle map

$$cl:\mathrm{CH}^p(X,m)_{\mathbb{Q}}\to H^{\mathcal{D}}_{2d+m}(X/k,\mathbb{Q}(d)),\tag{5.1.1}$$

where the target is defined by

$$\mathrm{Ext}^{-2d-m}(a^*_{X/k}\mathbb{Q}_k,a^!_{X/k}\mathbb{Q}_k(-d))(=\mathrm{Ext}^{-2d-m}(\mathbb{Q}_k,(a_{X/k})_*a^!_{X/k}\mathbb{Q}_k(-d))),$$

with Ext taken in $D^b\mathcal{M}(X/k,\mathbb{Q})$ (or $D^b\mathcal{M}(\mathrm{Spec}\,k/k,\mathbb{Q})$). If X is smooth, it becomes

$$\begin{aligned}H^{2p-m}_{\mathcal{D}}(X/k,\mathbb{Q}(p))&:=\mathrm{Ext}^{2p-m}(\mathbb{Q}_{X/k},\mathbb{Q}_{X/k}(p))\\&(=\mathrm{Ext}^{2p-m}(\mathbb{Q}_k,(a_{X/k})_*\mathbb{Q}_{X/k}(p))),\end{aligned}$$

in the notation of (1.3). If furthermore X is smooth proper and $m\ge 1$, then the cycle map induces a generalized Abel-Jacobi map

$$cl':\mathrm{CH}^p(X,m)_{\mathbb{Q}}\to\mathrm{Ext}^1(\mathbb{Q}_k,H^{2p-m-1}(X/k,\mathbb{Q}(p))),\tag{5.1.2}$$

by taking the composition with the canonical morphism

$$H^{2p-m}_{\mathcal{D}}(X/k,\mathbb{Q}(p))\to\mathrm{Ext}^1(\mathbb{Q}_k,H^{2p-m-1}(X/k,\mathbb{Q}(p))),$$

because $H^{2p-m}(X/k,\mathbb{Q}(p))$ is pure of weight $-m$.

Let now X be a complex algebraic variety. Then (5.1.1) induces the cycle map

$$cl:\mathrm{CH}^p(X,m)_{\mathbb{Q}}\to H^{\mathcal{D}}_{2d+m}(X,\mathbb{Q}_{\langle k\rangle}(d)),\tag{5.1.3}$$

where the target is defined in a similar way by replacing $\mathcal{M}(X/k,\mathbb{Q})$ with $\mathcal{M}(X,\mathbb{Q})_{\langle k\rangle}$. If X is smooth, it becomes $H^{2p-m}_{\mathcal{D}}(X,\mathbb{Q}_{\langle k\rangle}(p))$ in (3.3).

If $m=1$, we have a natural map

$$\mathrm{CH}^{p-1}(X)_{\mathbb{Q}}\otimes_{\mathbb{Z}}\mathbb{C}^*\to\mathrm{CH}^p(X,1)_{\mathbb{Q}},\tag{5.1.4}$$

using the isomorphism $\mathrm{CH}^1(pt,1)=\mathbb{C}^*$. See [7]. (Indeed, an element of $\mathrm{CH}^p(X,1)$ is represented by $\sum_i(Z_i,g_i)$ where Z_i is an integral closed subvariety of codimension $p-1$ in X and g_i are nonzero rational functions on Z_i such that $\sum_i\mathrm{div}g_i=0$ in X.) An element of the image of (5.1.4) is called decomposable. Consider the cycle map to the usual Deligne cohomology for a smooth projective variety X over $\mathbb{C}$

$$cl:\mathrm{CH}^p(X,1)_{\mathbb{Q}}\to H^{2p-1}_{\mathcal{D}}(X,\mathbb{Q}(p))=\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q},H^{2p-2}(X,\mathbb{Q}(p))).\tag{5.1.5}$$

The image of the decomposable elements by (5.1.5) coincides with

$$\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q},N^{p-1}H^{2p-2}(X,\mathbb{Q}(p)))=N^{p-1}H^{2p-2}(X,\mathbb{Q}(p-1))\otimes_{\mathbb{Z}}\mathbb{C}^*,$$

where $N^{p-1}H^{2p-2}(X,\mathbb{Q}(p-1))$ is the subspace of algebraic cycle classes (which is contained in the subspace of Hodge cycles). So we cannot detect the image of $\mathrm{CH}^{p-1}_{\mathrm{hom}}(X)_{\mathbb{Q}}\otimes_{\mathbb{Z}}\mathbb{C}^*$ in $\mathrm{CH}^p(X,1)_{\mathbb{Q}}$ by using (5.1.5), where $\mathrm{CH}^{p-1}_{\mathrm{hom}}(X)_{\mathbb{Q}}$ denotes the subgroup of homologically equivalent to zero cycles.

5.2. Theorem. *Let X be a smooth complex projective variety such that $N^{p-2}H^{2p-3}(X,\mathbb{Q}) \neq 0$, where N is the coniveau filtration. Then for any positive integer m, there exist $\zeta_i \in \mathrm{CH}^{p-1}_{\mathrm{hom}}(X)_{\mathbb{Q}}\,(1 \leq i \leq m)$ and a subfield K of $\mathbb{C}$ finitely generated over k such that for any complex numbers $\alpha_1,\dots,\alpha_m$ not algebraic over K, the images of $\zeta_i \otimes \alpha_i\,(1 \leq i \leq m)$ by the composition of (5.1.4) and (5.1.3) are linearly independent over $\mathbb{Q}$.*

Remark. If $p = 2$ or $\dim X + 1$, the assumption on the coniveau filtration is equivalent to the nonvanishing of $H^1(X,\mathcal{O}_X)$. Furthermore, (5.1) in the case $p = \dim X + 1$ holds by replacing that condition with the nonvanishing of $H^2(X,\mathcal{O}_X)$ using an argument similar to (4.4).

Proof of (5.2). By assumption on the coniveau filtration, there exist an irreducible smooth complex projective curve C and $\zeta' \in \mathrm{CH}^{p-1}(C \times X)$ such that the induced morphism $H^i(C,\mathbb{Q}) \to H^{2p-4+i}(X,\mathbb{Q}(p-2))$ is nontrivial for $i = 1$ and vanishes for $i = 2$. (Indeed, for a closed subvariety Y of codimension $p-2$ in X, there is a curve together with a divisor on $C \times \tilde{Y}$ such that the induced morphism $H^i(C,\mathbb{Q}) \to H^i(\tilde{Y},\mathbb{Q})$ is surjective for $i = 1$ and vanishes for $i = 2$, where $\tilde{Y} \to Y$ is a resolution of singularities.)

Let R, S, X_R be as in (2.1). We may assume that C and ζ' are defined over R, i.e. there exist an R-scheme C_R and $\zeta_R \in \mathrm{CH}^{p-1}(C_R \times_R X_R)$ such that their base changes by $R \to \mathbb{C}$ are C and ζ'. Let U be a nonempty affine open subvariety of C_R. We define S' to be the fiber product of m-copies of U over $S = \operatorname{Spec} R$, and $\zeta'_{i,S'} \in \mathrm{CH}^{p-1}(S' \times_R X_R)$ to be the pull-back of ζ'_R by $p_i \times id$ where p_i is the composition of the projection to the m-th factor $S' \to U$ and the inclusion $U \to C_R$. Let R' be the affine ring of S'. Then R' is integral. We choose an embedding of $K = k(S')$ into $\mathbb{C}$. The condition on α_i is relevant to this K (i.e. the α_i are not algebraic over K). Let $\zeta_i \in \mathrm{CH}^{p-1}(X)$ be the base change of $\zeta_{i,S'}$. It is homologically equivalent to zero by the assumption on ζ'.

Let R'' be a finitely generated smooth k-subalgebra of $\mathbb{C}$ such that $R' \subset R''$, $\operatorname{Spec} R''$ is smooth over $\operatorname{Spec} R'$, and $\alpha_i \in R''$ for any i. Let k_R be the algebraic closure of k in R. We may assume that k_R is also algebraically closed in R'' by replacing R, R' with the base change by $k_R \to k_{R''}$ (and k_R with $k_{R''}$). Note that the algebraic closure of K is unchanged. Let $\pi : S'' \to S'$ be a smooth projective compactification of $\operatorname{Spec} R'' \to S'$. Let D_i be the support of the divisor defined by α_i. Put $D = \cup_i D_i$. Note that D_i are nonempty by assumption on α_i.

We have to show that the images of $\zeta_i \otimes \alpha_i$ by the cycle map are linearly independent. Let U'' be a nonempty open subvariety of S''. We assume that $U'' \cap D$ is nonempty and smooth over S', because it is only required that any nonempty open subvariety of $\operatorname{Spec} R''$ contains $U'' \setminus D$ for some U''. We can prove a commutative diagram

$$\begin{array}{ccc} \mathrm{CH}^p((U''\setminus D)\times_R X_R,1)_{\mathbb{Q}} & \longrightarrow & \mathrm{CH}^{p-1}((U''\cap D)\times_R X_R)_{\mathbb{Q}} \\ \downarrow & & \downarrow \\ H^{2p-1}_{\mathcal{D}}((U''\setminus D)\times_R X_R/k_R,\mathbb{Q}(p)) & \longrightarrow & H^{2p-2}_{\mathcal{D}}((U''\cap D)\times_R X_R/k_R,\mathbb{Q}(p-1)) \end{array}$$

See for example [35]. By definition $\zeta_i \otimes \alpha_i$ is represented by the pull-back of $\zeta_{i,S'}$ to $(U'' \setminus D) \times_R X_R$ together with the function α_i coming from $U'' \setminus D$. It is enough to show

that the images of their nontrivial linear combinations in $H^{2p-2}_{\mathcal{D}}((U''\cap D)\times_R X_R,\mathbb{Q}(p))$ are nonzero by the same argument as in (4.2) using (2.5) together with (4.1.2) applied to $(U''\backslash D)\times_R X_R\to U''\backslash D$. Consider the image of the cycle in $\mathrm{CH}^{p-1}((U''\cap D)\times_R X_R)_{\mathbb{Q}}$. It is identified with a linear combination of the pull-back of $\zeta_{i,S'}$ by $(U''\cap D)\times_R X_R\to S'\times_R X_R$ where the coefficients are multiplied by the multiplicity of zero or pole of α_i along the divisor. So the assertion follows from the same argument as in (4.2).

5.3. Theorem. *Let X be a smooth projective complex algebraic variety. Then the cycle map (5.1.3) for $p=2,m=1$ is injective if the generalized Abel-Jacobi map (5.1.2) for the same p,m is injective for any smooth projective variety over a number field having a morphism to another variety such that the base change of its generic fiber is isomorphic to X.*

Proof. Let $\zeta'\in\mathrm{CH}^2(X,1)_{\mathbb{Q}}$. Replacing R if necessary, we may assume that ζ' is defined over R, i.e. there exists a cycle ζ'_R on $X_R\times_R A^1_R$ such that $\zeta'_R\otimes_R\mathbb{C}=\zeta'$. Let $Y=X_R$. Then there exist smooth projective compactifications $\bar Y,\bar S$ of Y,S together with a projective morphism $\bar Y\to\bar S$ extending $X_R\to S$. Let $Z=\bar Y\setminus Y$. We may assume it is a divisor on $\bar Y$. Let k_R be the algebraic closure of k in $k(S)$. Then k_R is also algebraically closed in $k(Y)$ (because the generic fiber of $Y\to S$ is geometrically irreducible.) We may assume that each irreducible components of Z are geometrically irreducible (taking the base change of $\bar Y,S$ by a finite extension of k_R if necessary).

Let $n=\dim Y$. Then the cycle map induces a morphism of the localization sequence in [7] to the corresponding exact sequence (see e.g. [35]):

$$\begin{array}{ccccccc}
\mathrm{CH}^1(Z,1)_{\mathbb{Q}} & \longrightarrow & \mathrm{CH}^2(\bar Y,1)_{\mathbb{Q}} & \longrightarrow & \mathrm{CH}^2(Y,1)_{\mathbb{Q}} & \longrightarrow & \mathrm{CH}^1(Z)_{\mathbb{Q}}\\
\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle(*)}\\
H^{\mathcal{D}}_{2n-3}(Z,n-2) & \longrightarrow & H^3_{\mathcal{D}}(\bar Y,2) & \longrightarrow & H^3_{\mathcal{D}}(Y,2) & \longrightarrow & H^{\mathcal{D}}_{2n-4}(Z,n-2)
\end{array}$$

where $H^{\mathcal{D}}_{2n-3}(Z,n-2)$ is the abbreviation of $H^{\mathcal{D}}_{2n-3}(Z/k_R,\mathbb{Q}(n-2))$, etc.

Let $\zeta=\zeta'_R\in\mathrm{CH}^2(Y,1)_{\mathbb{Q}}$, and assume $cl(\zeta)=0$. Since $(*)$ is injective (after taking the base change by $k_R\to\mathbb{C}$, see loc. cit), we see that ζ is the image of some $\bar\zeta\in\mathrm{CH}^2(\bar Y,1)_{\mathbb{Q}}$. Furthermore, $cl(\bar\zeta)$ comes from $H^{\mathcal{D}}_{2n-3}(Z/k_R,\mathbb{Q}(n-2))$ by the commutativity. Since (5.1.2) is injective by hypothesis, it is enough to show that the image of $\bar\zeta$ in $\mathrm{Ext}^1(\mathbb{Q}_{k_R},H^2(\bar Y/k_R,\mathbb{Q}(2)))$ comes from $\mathrm{CH}^1(Z,1)_{\mathbb{Q}}$. So the assertion is reduced to the following:

5.4. Proposition. *Let Y be a smooth projective k-variety of pure dimension n, and Z a divisor on it. Assume Y and the irreducible components of Z are geometrically irreducible over k. Then*

$$\begin{aligned}cl'(\mathrm{CH}^2(Y,1)_{\mathbb{Q}})\cap\mathrm{Im}(H^{\mathcal{D}}_{2n-3}(Z/k,\mathbb{Q}(n-2))\to\mathrm{Ext}^1(\mathbb{Q}_k,H^2(Y/k,\mathbb{Q}(2))))\\ =\mathrm{Im}(\mathrm{CH}^1(Z,1)_{\mathbb{Q}}\to\mathrm{Ext}^1(\mathbb{Q}_k,H^2(Y/k,\mathbb{Q}(2)))).\end{aligned}$$

Proof. Let $\tilde Z\to Z$ be a resolution of singularities. By Lemma (5.5) below, we may replace Z in the formula by $\tilde Z$. Then $H^{\mathcal{D}}_{2n-3}(Z/k,\mathbb{Q}(n-2))$ becomes $\mathrm{Ext}^1(\mathbb{Q}_k,H^0(\tilde Z/k,\mathbb{Q}(1)))$, and

the morphism of Deligne cohomologies is induced by the Gysin morphism for $g : \tilde{Z} \to Y$. Let $\xi = cl'(\zeta)$ be an element of the left-hand side (with Z replaced by $\tilde{Z}$). We have the decomposition $\xi = \xi_1 + \xi_2$ corresponding to the primitive decomposition as in (4.8), where ξ_1 corresponds to the primitive part.

We first reduced the assertion to the case $\xi_2 = 0$. Let Z_i be the irreducible components of Z. We may assume that the cohomology class of some Z_i is not contained in the primitive part (because $\xi_2 = 0$ otherwise). Let l denote the hyperplane section class as in (4.8). Consider the pushforward of $l^{n-1}\zeta \in \mathrm{CH}^{n+1}(Y,1)_{\mathbb{Q}}$ by $Y \to \operatorname{Spec} k$. Let $\zeta_i \in \mathrm{CH}^1(\tilde{Z}_i,1)_{\mathbb{Q}}$ be its pull-back by $\tilde{Z}_i \to \operatorname{Spec} k$. Then, modifying ξ by a constant multiple of the image of ζ_i, we may assume $\xi_2 = 0$.

Now let V be a maximal dimensional vector subspace of $H^0(\tilde{Z},\mathbb{Q})$ which is sent injectively to a subspace of the primitive part of $H^2(Y,\mathbb{Q}(1))$ by the Gysin morphism for $g : \tilde{Z} \to Y$. Then there exists $\Gamma \in \mathrm{CH}^0(\tilde{Z} \times_k \tilde{Z})_{\mathbb{Q}}$ such that the action of the composition $\Gamma_* \circ g^* \circ l^{n-2} \circ g_*$ on V is the identity. So ξ comes from $(\Gamma_* \circ g^* \circ l^{n-2})\zeta \in \mathrm{CH}^1(\tilde{Z},1)_{\mathbb{Q}}$, and the assertion follows.

5.5. Lemma. *Let X be a purely n-dimensional k-variety, and $\pi : \tilde{X} \to X$ be a resolution of singularities. Then*

$$H^{\mathcal{D}}_{2n-1}(X/k,\mathbb{Q}(n-1)) = cl(\mathrm{CH}^1(X,1)_{\mathbb{Q}}) + \pi_*(H^{\mathcal{D}}_{2n-1}(\tilde{X}/k,\mathbb{Q}(n-1)))$$

Proof. Let U be a dense open subvariety of X such that $\tilde{U} := \pi^{-1}(U) \to U$ is an isomorphism and $Z := X \setminus U$ and $\tilde{Z} := \tilde{X} \setminus \tilde{U}$ are divisors. Let $\xi \in H^{\mathcal{D}}_{2n-1}(X/k,\mathbb{Q}(n-1))$, and ξ' be its restriction to $\tilde{U} = U$ which belongs to $H^1_{\mathcal{D}}(U/k,\mathbb{Q}(1))$. Consider the commutative diagram induced by the cycle map:

$$\begin{array}{ccccccc}
\mathrm{CH}^1(\tilde{X},1)_{\mathbb{Q}} & \longrightarrow & \mathrm{CH}^1(\tilde{U},1)_{\mathbb{Q}} & \longrightarrow & \mathrm{CH}^0(\tilde{Z})_{\mathbb{Q}} & \longrightarrow & \mathrm{CH}^1(\tilde{X})_{\mathbb{Q}} \\
\downarrow & & \downarrow & & \downarrow (*) & & \downarrow (**) \\
H^{\mathcal{D}}_{2n-1}(\tilde{X},n-1) & \longrightarrow & H^1_{\mathcal{D}}(\tilde{U},1) & \longrightarrow & H^{\mathcal{D}}_{2n-2}(\tilde{Z},n-1) & \longrightarrow & H^{\mathcal{D}}_{2n-2}(\tilde{X},n-1)
\end{array}$$

where $H^{\mathcal{D}}_{2n-1}(\tilde{X},n-1)$ means $H^{\mathcal{D}}_{2n-1}(\tilde{X}/k,\mathbb{Q}(n-1))$ as before. By Lemma below, $(*)$ is an isomorphism and $(**)$ is injective (after taking the base change by $k \to \mathbb{C}$). So there exists $\zeta \in \mathrm{CH}^1(\tilde{U},1)_{\mathbb{Q}}$ such that $\xi' - cl(\zeta)$ comes from an element ξ'' of $H^{\mathcal{D}}_{2n-1}(\tilde{X}/k,\mathbb{Q}(n-1))$.

Using a similar diagram with $\tilde{X},\tilde{U},\tilde{Z}$ replaced by X,U,Z and also the morphism between the two diagrams induced by the direct image, we see that the image of ζ in $\mathrm{CH}^0(Z)_{\mathbb{Q}}$ is zero, because the image of ζ' in $H^{\mathcal{D}}_{2n-2}(Z/k,\mathbb{Q}(n-1))$ is zero and $(*)$ remains injective with $\tilde{Z}$ replaced by Z. So ζ comes from an element of $\mathrm{CH}^1(X,1)_{\mathbb{Q}}$, and we get the assertion using the injectivity of $H^1_{\mathcal{D}}(X/k,\mathbb{Q}(1)) \to H^1_{\mathcal{D}}(U/k,\mathbb{Q}(1))$, which follows from the vanishing of $H^{\mathcal{D}}_{2n-1}(Z/k,\mathbb{Q}(n-1))$.

5.6. Lemma. *Let X be a purely n-dimensional k-variety with irreducible components X_i. Then we have a canonical isomorphism*

$$H^{\mathcal{D}}_{2n}(X,\mathbb{Q}(n)) = \oplus_i H^{\mathcal{D}}_{2n}(X_i,\mathbb{Q}(n)) = \oplus_i \mathbb{Q}.$$

Proof. By the localization sequence, we may assume X smooth by deleting a closed subvariety of dimension $< n$ in X. So the first isomorphism is clear and we may assume further X irreducible. Then the assertion becomes

$$\mathrm{Hom}(\mathbb{Q}_{X/k}, \mathbb{Q}_{X/k}) = \mathbb{Q},$$

and it is clear if X is geometrically irreducible. In general, if the left-hand side is not $\mathbb{Q}$, there exists an endomorphism of $\mathbb{Q}_{X/k}$ inducing zero on some irreducible components of $X \otimes_k \mathbb{C}$ and nonzero on some other components (because the endomorphisms on each component are the multiplications by rational numbers). But this is impossible because it comes from an endomorphism of $\mathcal{O}_X$ as $\mathcal{D}_X$-Module. So the assertion follows.

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