Supersymmetry and Homotopy

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Abstract

The homotopical information hidden in a supersymmetric structure is revealed by considering deformations of a configuration manifold. This is in sharp contrast to the usual standpoints such as Connes' programme where a geometrical structure is rigidly fixed. For instance, we can relate supersymmetries of types $N = 2n$ and $N = (n,\ n)$ in spite of their gap due to distinction between $\mathbb{Z}_2$(even-odd)- and integer-gradings.

Our approach goes beyond the theory of real homotopy due to Quillen, Sullivan and Tanré developed, respectively, in the 60's, 70's and 80's, which exhibits real homotopy of a 1-connected space out of its de Rham-Fock complex with supersymmetry. Our main new step is based upon the Taylor (super-)expansion and locality, which links differential geometry with homotopy without the restriction of 1-connectedness. While the homotopy invariants treated so far in relation with supersymmetry are those depending only on $\mathbb{Z}_2$-grading like the index, here we can detect new $\mathbb{N}$-graded homotopy invariants. While our setup adopted here is (graded) commutative, it can be extended also to the non-commutative cases in use of state germs (Haag-Ojima) corresponding to a Taylor expansion.

1 The Algebraic Data of a Physical System

The purpose of this section is to fix notation and our general viewpoint. For simplicity, let us start with a classical system on a finite-dimensional affine space V with its coordinate space V^*. In reference to a chosen probability

measure ω on V, a polynomial algebra $\mathcal{F}^0 = Sym(V^*)$ on V^* generates a sequence of function spaces, $\mathcal{F}^0 \hookrightarrow \mathcal{F}_0 = \mathcal{F}^0.\omega^{1/2} \hookrightarrow L^2(V,\omega) = L^2(V,\omega)^* \hookrightarrow (\mathcal{F}^0)^*$, similar to the Gel'fand triplet, where $\omega^{1/2}$ denotes the half measure [1] corresponding to ω. The inner product of $\mathcal{F}_0$ is given by $\langle f.\omega^{1/2} \mid g.\omega^{1/2}\rangle = \int_V \overline{f(x)}g(x)\omega(dx)$. By $\langle f,\ g.\omega^{1/2}\rangle = \omega(\bar{f}g)$ a duality relation $(\mathcal{F}_0)^* \overset{\omega}{\approx} \mathcal{F}^0$ holds. Hence we have $\mathcal{F}_0 \overset{\omega}{\approx} Sym(V)$.

Then a self-dual algebra P is defined by $P \equiv Sym(V \oplus V^*) \overset{\omega}{\approx} \mathcal{F}_0 \otimes \mathcal{F}^0$ which has a canonical choice of a state determined by the Liouville measure ω. Because of the above duality, a *-involution can be defined on P, through the action of which $\mathcal{F}_0$ and $\mathcal{F}^0$ are interchanged. A linear operator A acting on $\mathcal{F}_0$, $A \in Hom(\mathcal{F}_0,\mathcal{F}_0) \approx \mathcal{F}_0 \otimes \mathcal{F}^0$, and the corresponding one on $\mathcal{F}^0$, $A^* \in Hom(\mathcal{F}^0,\mathcal{F}^0) \approx \mathcal{F}^0 \otimes \mathcal{F}_0$, through ω, have (Noether) charges, $Q_A, Q_{A^*} \in P$. Conversely, when the algebra P is given, the linear spaces $\mathcal{F}_0, \mathcal{F}^0$ are reproduced from it as induced representations from the trivial one on V^* or on V (via ω).

According to the standard procedure, the classical dynamics on $\mathcal{F}_0, \mathcal{F}^0$ is described with the Poisson-Lie bracket on P which is closely related with the quantization $\mathfrak{U}(\hbar eis(V))$ through $P[\hbar] \approx \mathfrak{U}(\hbar eis(V))$. Here $\hbar eis(V) \equiv V \oplus \hbar\mathbb{C} \oplus V^*$ denotes the Heisenberg-Lie algebra equipped with the Lie bracket coming from the evaluation map $V \times V^* \to \mathbb{C}$. Then a dynamics ∂_t on V has a Hamiltonian $H \in P$ as its Noether charge such that $H\}$ defined on P by $(H\})f \equiv \{f,H\}$ implements $\partial_t : \partial_t f = \{f,H\}$. Similarly any symmetry of the dynamics ∂_t constituting a finite-dimensional Lie algebra $\mathfrak{g}$ has a Noether charge in the Poisson-Yang algebra $V \oplus \mathfrak{g}^* \oplus V^*$ which is a generalization of Heisenberg-Lie algebra.

In addition to the bosonic variables belonging to $\mathcal{F}^0$, there are fermionic ones constituting a $\mathcal{F}^0$-module L^0, which are nilpotent $\psi^2 = 0$ for $\psi \in L^0$. Then the above structure survives in an extended form with the evaluation superbracket on $L_0 \oplus \hbar eis(V) \oplus L^0 = L_0 \oplus (V \oplus \hbar\mathbb{C} \oplus V^*) \oplus L^0$ where L^0 and $L_0 = (L^0)^*$ are $\hbar eis(V)$-modules graded by $+1$ and -1, respectively. Again, $\mathcal{F} = \mathcal{F}^0 \otimes \mathcal{F}^{\geq 1}$ is induced from the Poisson-Lie superalgebra $P \otimes (\wedge L^0) \otimes (\wedge L_0)$, where $\mathcal{F}^0 = Sym(V^*)$, $\mathcal{F}^1 = \mathcal{F}^0 \otimes L^0$ and a linear operator $\mathcal{F}^k \to \mathcal{F}^{k+1}$ is implemented by a degree 1 Noether charge $Q \in P \otimes (\wedge L_0) \otimes (\wedge L^0)$.

This simple physical picture can be extended to a general manifold by considering the de Rham complex Ω on it, but not for charges. In fact, its basic algebra $\mathcal{F}^0 = \Omega^0$ consists of differentiable functions, with their commutative pointwise product. The symmetries are given by the (infinite-dimensional) Lie algebra $\mathcal{X} \simeq Der(\Omega^0)$ and differentiable 1-forms $\varphi = \sum f_i dg_i : \mathcal{X} \ni X \longmapsto \sum f_i[X,\ g_i]$ constitute the space Ω^1 of fermionic generators. It is a finitely generated projective Ω^0-module (equivalent to a finite direct sum

of Ω^0), Ω^0-dual to the space Ω_{-1} of the differentiable 1-currents. Namely, a choice of an orientation ω_{-n} and a differentiable oriented volume Φ^n on the orientable double covering of our manifold with dimensionality n determines a differentiable 0-current $\omega = \Phi^n.\omega_{-n}$ dual to Ω^0, and also differentiable (-1)-currents c_X dual to $\Omega^1 : \langle c_X,\ \varphi\rangle_\omega = \omega(\iota_X(\varphi)) = \langle \omega_{-n},\ \Phi^n \wedge \iota_X(\varphi)\rangle$ where ι_X is the evaluation. Then the space Ω_{-1} of differentiable (-1)-currents is identified with $\Omega^{n-1}.\omega_{-n}$. Because of $\Omega^n = \Omega^0\Phi^n$ and of $\Omega^{n-1} = \mathrm{Lin}\{\iota_X(\Phi^n);\ X \in \mathcal{X}\}$, Ω^1 is both Ω^0-dual and $\mathbb{R}$-dual to Ω_{-1}, respectively, by $\langle c_X,\ \varphi\rangle = \iota_X(\varphi)$ and by $\langle c_X,\ \varphi\rangle_\omega = \omega(\iota_X(\varphi))$, while Ω_{-1} is a regular representation of the Lie algebra $\mathcal{X}$. Then we have the Ω^0-Heisenberg algebra $\Omega_{-1} \oplus \hbar\Omega^0 \oplus \Omega^1$, with Ω^0-bilinear evaluation superbracket. It is a deformation of the Poisson superalgebra $P = (\wedge_{\Omega^0}\Omega_{-1}) \otimes_{\Omega^0} (\wedge_{\Omega^0}\Omega^1)$. The latter induces the de Rham complexes Ω_-, Ω^+ of differentiable currents and forms.

Taking X_i as fixed and a p-form φ as variable, $\Omega^p \ni \varphi \longmapsto \langle\varphi(X_1, \cdots, X_p)\rangle_\omega = \omega((\iota_{X_p} \circ \cdots \circ \iota_{X_1})(\varphi)) \equiv \langle c_{X_1\wedge\cdots\wedge X_p},\ \varphi\rangle_\omega$, we obtain a differentiable exterior $(-p)$-current $c_{X_1\wedge\cdots\wedge X_p} \in \Omega_{-p}$ similarly to the above case of $p = 1$. When we choose a suitable sequence of $\omega^{(k)}$ of differentiable 0-currents tending to a Dirac measure δ_z at $z \in Z$, $\omega^{(k)} \underset{k\to\infty}{\to} \delta_z$, the limiting formula is given by $\langle\varphi(X_1, \cdots, X_p)\rangle_\omega = \delta_z(\varphi(X_1, \cdots, X_p)) = \sum\limits_{i_1\cdots i_p} \varphi_{i_1\cdots i_p}(z) X_1^{i_1}(z) \cdots X_p^{i_p}(z)$. Comparing this with $\langle c_{-p}, \varphi\rangle \equiv \int_{c_{-p}}\varphi = \int_{c_{-p}} \sum\limits_{i_1\cdots i_p} \varphi_{i_1\cdots i_p} dz^{i_1}\cdots dz^{i_p}$, we can regard $X_1(z) \wedge \cdots \wedge X_p(z)$ as an infinitesimal version of (an integral over) a cycle c_{-p} at a point $z \in Z$, which generalizes a Dirac measure δ_z to a current $c_{X_1\wedge\cdots\wedge X_p} \in \Omega_{-p}$. The point-like nature of a current $c_{X_1\wedge\cdots\wedge X_p} \in \Omega_{-p}$ and the Ω^0-linearity of the form φ and of $\underset{\Omega^0}{\wedge}$ can be recovered in this form, $\varphi(fX_1 \underset{\Omega^0}{\wedge} \cdots \underset{\Omega^0}{\wedge} X_p) = \varphi(X_1 \underset{\Omega^0}{\wedge} fX_2 \underset{\Omega^0}{\wedge} \cdots \underset{\Omega^0}{\wedge} X_p) = \cdots = f\varphi(X_1 \underset{\Omega^0}{\wedge} \cdots \underset{\Omega^0}{\wedge} X_p)$ for $f \in \Omega^0 : c(\varphi) = \int\langle c_z, \varphi_z\rangle\omega$ with $z \longmapsto \langle c_z, \varphi_z\rangle \in \Omega^0$.

As the symmetries $\in \mathcal{X}$ are not Ω^0-linear but Ω^0-Leibniz, they have no Noether charge in $P \otimes \mathcal{X}^*$ (Poisson-Yang). In the case of invariant de Rham complex on a Lie group G with Lie algebra $\mathfrak{g}$, the Heisenberg algebra becomes $\mathfrak{g}_{-1} \oplus \hbar\mathbb{C} \oplus \mathfrak{g}^1$ and $d : \wedge^k\mathfrak{g} \to \wedge^{k+1}\mathfrak{g}$ has a Noether charge. In general, this problem will be solved by Taylor superexpansions: They provide a Heisenberg algebra $L_- \oplus \hbar\mathbb{C} \oplus L^+$, where L_- is associated with a Lie-algebraic cochain complex $L = \{L_{-k}\}_{k\in\mathbb{N}_0}$ ($\mathbb{N}_0 \equiv \mathbb{N}\cup\{0\}$) (see Definition 3 in Sect.2) augmented by $L_0 \to V$:

$$L(V) :\ \cdots \to L_{-2} \to L_{-1} \to L_0 \to V \to 0, \tag{1}$$

with degree shift by -1, and $L^+ = L_-^*$. L_- provides a regular representation of $L(V)$. Then a symmetry given by a Lie algebra $\mathfrak{g}$ will have a Noether charge

in $P \otimes \mathfrak{g}^*$.

In the general setting with the implementer $H \in P$ of a derivation δ generating a dynamics, an ordinary symmetry is described by an element $X \in \mathfrak{g}^*$ whose charge Q_X (of degree 0) satisfies $[Q_X, H] = 0$. A supersymmetry is given by an operator of degree 1 acting on $\mathcal{F}^+$ or $\mathcal{F}_-$ defined by $\mathcal{F}^+ \equiv \mathcal{A}_c(\overline{L^+})$ and $\mathcal{F}_- \equiv (\mathcal{F}^+)^*$ (see Sect.2) whose charge Q (of degree 1) also satisfies $[Q, H] = 0$. These (super)symmetries form a canonical Lie superalgebra, either with $\mathbb{Z}$ (or $\mathbb{N}$)-grading or $\mathbb{Z}_2$-grading. One of our main objective here is to examine the possibility of passing from $\mathbb{Z}_2$-grading to $\mathbb{Z}$ (or $\mathbb{N}$)-grading, by asking when (super)symmetries can be realized by nilpotent charges of degree 1.

This paper is organized as follows: The essence of algebraic formulation of homotopy is explained in Sect.2, which is extended in Sect.3 to the equivariant situation involving a principal bundle. Then, the mutual relation between de Rham complexes and homotopy is explained in Sect.4 by using the Taylor superexpansions, a graded extension of Taylor expansions. In Sect.5, the relation between the complexes for the total space and for the base space of the bundle is clarified. The method of algebraic homotopy is generalized in Sect.6 to the non 1-connected cases. By using the developed techniques, the obstruction to transform $\mathbb{Z}_2$-grading to $\mathbb{Z}$ (or $\mathbb{N}$)-grading caused by the presence of torsion is shown to be eliminated by a homotopy covering.

2 Algebraic Homotopy

To facilitate the unified treatment of chain and cochain complexes, we note the following simple observation: A chain complex $C = \{C_k\}_{k \in \mathbb{N}_0}$ over a ground ring K,

$$C : \cdots \stackrel{\partial}{\longrightarrow} C_k \stackrel{\partial}{\longrightarrow} C_{k-1} \stackrel{\partial}{\longrightarrow} \cdots \stackrel{\partial}{\longrightarrow} C_0 \stackrel{\partial}{\longrightarrow} 0, \tag{2}$$

consisting of graded K-modules C_k with K-linear boundaries $\partial = \partial_k : C_k \to C_{k-1}$ (of degree -1 and satisfying $\partial^2 = 0$) can be viewed as a *cochain* complex $\mathcal{F}_- = \{\mathcal{F}_{-k}\}$ of *negatively graded* K-modules, simply by a renaming $\mathcal{F}_{-k} \equiv C_k$ to reverse the sign of degrees which makes $\partial : \mathcal{F}_{-k} \to \mathcal{F}_{-k+1}$ to be coboundaries of degree 1:

$$\mathcal{F}_- : \cdots \stackrel{\partial}{\longrightarrow} \mathcal{F}_{-k} \stackrel{\partial}{\longrightarrow} \mathcal{F}_{-k+1} \stackrel{\partial}{\longrightarrow} \cdots \stackrel{\partial}{\longrightarrow} \mathcal{F}_0 \stackrel{\partial}{\longrightarrow} 0. \tag{3}$$

In spite of its degree $+1$, we keep the name *boundary* for $\partial : \mathcal{F}_{-k} \to \mathcal{F}_{-k+1}$, because it appears only in *negatively graded* complexes without any confusions.

Dual to this is a *positively graded* cochain complex, $\mathcal{F}^+ = \{\mathcal{F}^k\}$, defined by $\mathcal{F}^k \equiv (\mathcal{F}_{-k})^*$ with K-linear coboundaries $d = (-)^\bullet \partial^*$ of degree 1:

$$\mathcal{F}^+ : 0 \stackrel{d}{\longrightarrow} \mathcal{F}^0 \stackrel{d}{\longrightarrow} \cdots \stackrel{d}{\longrightarrow} \mathcal{F}^{k-1} \stackrel{d}{\longrightarrow} \mathcal{F}^k \stackrel{d}{\longrightarrow} \cdots . \tag{4}$$

To formulate the duality between $\mathcal{F}^+ = (\mathcal{F}_-)^*$ and $\mathcal{F}_-$ in a consistent and convenient way, this simple trick plays a crucial role, by which both $\mathcal{F}^+$ and $\mathcal{F}_-$ can be treated as cochain complexes on the same ground. Without this it is difficult, for instance, to construct a (universal Poisson) cochain complex $\mathcal{F}'_- \otimes \mathcal{F}^+$ for cochain maps $\in Hom(\mathcal{F}_-, \mathcal{F}'_-)$ between cochains $\mathcal{F}_- \to \mathcal{F}'_-$. Here the tensor product $\mathcal{F}'_- \otimes \mathcal{F}^+$ of two cochain complexes is a cochain complex equipped with the coboundary $D = \partial' \otimes I + (-1)^\bullet I \otimes d$ and the total $\mathbb{Z}$-grading defined by $(\mathcal{F}'_- \otimes \mathcal{F}^+)_k = \bigoplus_{l+m=k,\ l\leq 0,\ m\geq 0} \mathcal{F}'_l \otimes \mathcal{F}^m$. A cochain map ϕ of degree k is defined by a family $\{\phi_{-l}\}$ of K-linear maps $\phi_{-l} : \mathcal{F}_{-l} \to \mathcal{F}'_{-l+k}$ satisfying the condition

$$[Q, \phi] \equiv \partial' \circ \phi - (-)^k \phi \circ \partial = 0, \tag{5}$$

where Q denotes a universal charge for coboundary. By definition, the element $\tilde{\phi}$ in $P = \mathcal{F}'_- \otimes \mathcal{F}^+$ corresponding to ϕ has non-vanishing components only at degree k in $\bigoplus_{l+m=k,\ l\leq 0,\ m\geq 0} \mathcal{F}'_l \otimes \mathcal{F}^m = \bigoplus_{l\leq 0} Hom(\mathcal{F}_{-l}, \mathcal{F}'_{-l+k})$. When ϕ has an algebraic implementer $\tilde{\phi} = \sum c' \otimes f$ (Noether charge) in $P = \mathcal{F}'_- \otimes \mathcal{F}^+$, the condition (5) on ϕ amounts to $D(\tilde{\phi}) = 0$. Actually, the map $\mathcal{F}'_- \otimes \mathcal{F}^+ \to Hom(\mathcal{F}_-, \mathcal{F}'_-)$ sending $c' \otimes f$ to $c' f(\cdot)$ is a 0-degree cochain map. The above equation $D(\tilde{\phi}) = 0$ can be interpreted as the invariance of the charge $\tilde{\phi}$: When ϕ describes an ordinary infinitesimal symmetry (of degree 0) $\mathcal{F}_{-l} \to \mathcal{F}_{-l}$ on a self dual cochain, the condition $D(\tilde{\phi}) = 0$ is seen to be equivalent to each of $[\phi, \partial] = 0$ and $[\phi, \partial^*] = 0$ and hence, to $[\phi, H] = 0$ where $H = \partial\partial^* + \partial^*\partial$.

To emphasize the importance of integer grading (in contrast to $\mathbb{Z}_2$-grading) we mention the following examples of chain complexes:

1) The chain complex $C(\mathcal{C})$ of a category $\mathcal{C}$, based in each degree $k \geq 0$ on the sequences of k arrows $\to\to\to \cdots$, with $\partial = \sum (-)^i \partial_i$, ∂_i expressing the composition of morphisms, and $\partial^2 = 0$ their associativity. The ∂_i lead to an intermediate topological space $B\mathcal{C}$, the so-called classifying one.

2) The infinitesimal version of the above (with half open arrows) is expressed by de Rham differentiable currents.

3) The corresponding equivariant cases are expressed by the Cartan-de Rham currents $\wedge \mathfrak{g}_{-1}$, where $\mathfrak{g}_{-1}$ is the regular representation space of a Lie algebra $\mathfrak{g}$ with degree -1 assigned. This corresponds to the equivalence between representation space V of $\mathfrak{g}$ and Lie structure on $V \oplus \mathfrak{g}$ with 0 bracket on V.

There are two different boundaries ∂ extending the bracket $[\cdot\ ,\ \cdot] : \mathfrak{g}_{-1} \wedge \mathfrak{g}_{-1} \to \mathfrak{g}_{-1}$, one as derivation for the algebra structure and the other as derivation for the coalgebra structure.

We proceed now to introduce in a new way the notion of algebraic homotopy based upon the general notion of freeness in the categorical formulation. We recall that, for a given functor $F' : \mathcal{C} \to \mathcal{C}'$ from a category $\mathcal{C}$ to another one $\mathcal{C}'$, an object X in $\mathcal{C}$ is called $\mathcal{C}'$-*free* if one can find an object X' in $\mathcal{C}'$ and a natural isomorphim

$$\mathcal{C}(X, Y) \approx \mathcal{C}'(X', F'Y) \quad \text{for } \forall Y, \tag{6}$$

Such an X' is called a *basis* for X, and each morphism in $\mathcal{C}$ [called "$\mathcal{C}$-map" hereafter for short] from X corresponds exactly to a $\mathcal{C}'$-map from the basis X'. In standard situations where $\mathcal{C}$ consists in monoidal objects in a category $\mathcal{C}''$ underlying $\mathcal{C}'$, any object X' serves as basis for a free object X constructed formally. In such cases, we have a functor $F : \mathcal{C}' \to \mathcal{C}$ called a (left) adjoint to F',

$$\mathcal{C} \underset{F}{\overset{F'}{\rightleftarrows}} \mathcal{C}', \tag{7}$$

creating a free object $X = F(X')$ in $\mathcal{C}$ from each basis X' in $\mathcal{C}'$ and a natural isomorphism

$$\mathcal{C}(FX', Y) \approx \mathcal{C}'(X', F'Y) \quad \text{for } \forall Y. \tag{8}$$

Then, the canonical maps, $X' \overset{\eta_{X'}}{\to} F'FX'$, $FF'X \overset{\epsilon_X}{\to} X$ (called unit and counit, respectively), generate resolutions such that $\epsilon_{FX'} F(\eta_{X'}) = I_{FX'}$, $F'(\epsilon_X)\eta_{F'X} = I_{F'X}$. So any object $X \in \mathcal{C}$ has a free presentation $FF'X \overset{\epsilon_X}{\to} X \approx FF'X/RX$ with an equivalence relation RX (which is, in an abelian category, nothing but the equivalence w.r.t. $Ker(\epsilon_X)$). Dually, the adjunction defines co-free objects $F'Y$. Adjunction of a category $\mathcal{C}$ with a point category, consisting of one object with its identity, plays an important role. A corresponding free (resp., cofree) object is initial (resp., final) in $\mathcal{C}$. The arrows linking them to other objects correspond bijectively to these objects. In a simplex category, all objects are initial (resp., final). Groups give examples of such a simplex category, with elements of the group as objects, and arrows $\mu : \gamma \longmapsto \mu\gamma$. Also Hilbert spaces give such example, with objects the lines generated by unit vectors X and morphism $X \to Y$ the orthogonal projection: $\mathcal{C}(X, Y) \approx \langle X, Y \rangle$. By adding the 0 object, partial isometries provide adjoint functors. The existence of an adjoint functor (F in the above) to a given one (F' in the above) is the most general form of uniform continuity (also of inversion of F'), since it necessarily

commutes with (category) limits. This is one of the main key in analysis, where it comes not only as integration by parts, but also in spectral theory: It converts the monoid $End(X)$ into a $\mathcal{C}'$-function space on the spectrum X'.

After these generalities worth mentioning, let us examine the case of the forgetting functor $\Phi : Ch_ \to Mod_$ from the category $Ch_$ of cochain complexes $\mathcal{F}_$ (as above) to the category $Mod_$ of negatively graded modules: The functor Φ simply forgets boundary operators absent in $Mod_$, or equivalently sets them equal to 0. To attain a unified treatment of the notion of *contractibility* for complexes with various kinds of algebraic structures, we need the following

Definition 1: A cochain complex $\mathcal{F}_$ (as above) is *contractible* if and only if it is $Mod_$-free or -cofree. We denote such by $\tilde{0}$: $\mathcal{F} = \tilde{0}$.

The term *contractible* is slightly ambiguous because in the usual chain theory it means homotopic to 0 (cofree), while in the space theory it means homotopic to a point (free and cofree), and in the algebraic cochain theory it means homotopic to scalar (free). But these various meanings can be treated in quite a coherent manner as free or cofree object.

Before explaining our use of the definition, let us mention useful intermediate cases, like being $Set_$-free in $Mod_$, or forgetting ∂ beyond some fixed degree. The former case associates to each subchain complex $\mathcal{F}' \subset \mathcal{F}$ a semi-direct decomposition $\mathcal{F} \approx \mathcal{F}' + \mathcal{F}''$, which is actually isomorphic to a direct one if one factor is $\tilde{0}$ (whose proof is easy). The second case describes acylicity above the chosen degree.

Definition 2: Two cochain complexes $\mathcal{F}_$, $\mathcal{F}'_$ are homotopy equivalent if and only if $\mathcal{F}_ \oplus \tilde{0} \approx \mathcal{F}'_ \oplus \tilde{0}'$.

To make more explicit the meaning of this notion, and to see its implication at the chain map level, recall the mapping cylinder $cyl(\phi)$ of a cochain map $\phi : \mathcal{F}_ \to \mathcal{F}'_$ of degree 0. This chain construction mimics the graph of a map, as embedding into a simplex with vertices in either domain or range, and

arrows $x \longrightarrow \phi(x)$:

$$
cyl(\phi) \quad : \quad
\begin{array}{ccccccccc}
 & \downarrow & & & & & & & \\
cyl(\phi)_{-k} & = & \mathcal{F}_{-k} & & \oplus & & \mathcal{F}_{-k+1} & \oplus & \mathcal{F}'_{-k} \\
 & & \downarrow{\scriptstyle \partial_{-k}} & \searrow{\scriptstyle (-1)^{k-1}} & & & \downarrow{\scriptstyle \partial_{-k+1}} & \searrow{\scriptstyle \phi_{-k+1}} & \downarrow{\scriptstyle \partial'_{-k}} \\
cyl(\phi)_{-k+1} & = & \mathcal{F}_{-k+1} & & \oplus & & \mathcal{F}_{-k+2} & \oplus & \mathcal{F}'_{-k+1} \\
 & \downarrow & & & & & & &
\end{array}
\quad . \tag{9}
$$

The subcomplexes, $\mathcal{F}_-$ and $\mathcal{F}'_-$, in $cyl(\phi)$ have their quotients $cone(\phi)$ and $cone(id_{\mathcal{F}}) = cone(\mathcal{F}_-)$, respectively:

$$
cone(\phi) \quad : \quad
\begin{array}{ccccccc}
 & \downarrow & & & & & \\
cone(\phi)_{-k} & = & \mathcal{F}_{-k+1} & & \oplus & & \mathcal{F}'_{-k} \\
 & & \downarrow{\scriptstyle \partial_{-k+1}} & \searrow{\scriptstyle \phi_{-k+1}} & & & \downarrow{\scriptstyle \partial'_{-k}} \\
cone(\phi)_{-k+1} & = & \mathcal{F}_{-k+2} & & \oplus & & \mathcal{F}'_{-k+1} \\
 & \downarrow & & & & &
\end{array}
\tag{10}
$$

$$
cone(id_{\mathcal{F}}) = cone(\mathcal{F}) \quad : \quad
\begin{array}{ccccccc}
 & \downarrow & & & & & \\
cone(\mathcal{F})_{-k} & = & \mathcal{F}_{-k} & & \oplus & & \mathcal{F}_{-k+1} \\
 & & \downarrow{\scriptstyle \partial_{-k}} & \searrow{\scriptstyle (-1)^{k-1}} & & & \downarrow{\scriptstyle \partial_{-k+1}} \\
cone(\mathcal{F})_{-k+1} & = & \mathcal{F}_{-k+1} & & \oplus & & \mathcal{F}_{-k+2} \\
 & \downarrow & & & & &
\end{array}
\quad . \tag{11}
$$

Lemma: The forgetting functor $\Phi : Ch_- \rightarrow Mod_-$ has a left adjoint functor $\Phi^l : Mod_- \rightarrow Ch_-$ and a right adjoint functor $\Phi^r : Mod_- \rightarrow Ch_-$.

Proof: From any given complex $M_- = \{M_k\}$ of negatively graded modules (i.e., $M_k = 0$ for $k > 0$), we can construct a cochain complex $\Phi^l(M_-)$ belonging to Ch_- by defining $\Phi^l(M)_{-k} := M_{-k-1} \oplus M_{-k}$ together with the boundaries $\Phi^l(M_-)_{-k} = M_{-k-1} \oplus M_{-k} \stackrel{\begin{bmatrix} 0 & (-)^k I \\ 0 & 0 \end{bmatrix}}{\longrightarrow} M_{-k} \oplus M_{-k+1} = \Phi^l(M_-)_{-k+1}$, which

terminates with $\Phi^l(M_-)_0 = M_{-1} \oplus M_0$ (not with M_0!) at degree 0. By chasing the commutativity in the diagram (starting from $k = 1$):

$$\begin{array}{ccccccc} \Phi^l(M) & : \rightarrow & M_{-k-1} \oplus M_{-k} & \overset{\left[\begin{smallmatrix} 0 & (-)^k I \\ 0 & 0 \end{smallmatrix}\right]}{\rightarrow} & M_{-k} \oplus M_{-k+1} & \rightarrow & \\ \downarrow & & {\scriptstyle \alpha_{-k}} \searrow \swarrow {\scriptstyle \beta_{-k}} & & {\scriptstyle \alpha_{-k+1}} \searrow \swarrow {\scriptstyle \beta_{-k+1}} & & , \\ \mathcal{F}' & : \rightarrow & \mathcal{F}'_{-k} & \overset{\partial'_{-k}}{\rightarrow} & \mathcal{F}'_{-k+1} & \rightarrow & \end{array} \qquad (12)$$

each chain map $\Phi^l(M_-) \rightarrow \mathcal{F}'_-$ is seen to correspond bijectively to an arbitray choice of module maps $\beta_{-k} : M_{-k} \rightarrow \mathcal{F}'_{-k}$ for $k > 0$ through the relations

$$\partial'_{-k}\beta_{-k} = (-)^k \alpha_{-k+1} \text{ for } k > 0$$

which determine the α_{-k}'s. The cochain $\Phi^l(M_-)$ is related to the preceeding cone on a cochain by means of the shift functor in $Mod_- : (sM_-)_{-k} \equiv M_{-k+1}$: With the 0 coboundary on M_-, $\Phi^l(sM_-) = cone(M_-)$. For the right adjoint, just define $\Phi^r(M_-) = cone(M_-)$. ■

Remark: Taking $\beta_{-k} = I = id$ with $M_- = \Phi(\mathcal{F})$ and $\mathcal{F}'_- = \mathcal{F}_-$ in the above, we see that the counit $\epsilon_{\mathcal{F}_-} : (\Phi^l \circ \Phi)(\mathcal{F}_-) \rightarrow \mathcal{F}_-$ is given by the projection cochain map $(\epsilon_{\mathcal{F}})_{-k} = ((-)^{k+1}\partial_{-k-1},\ I) : \mathcal{F}_{-k-1} \oplus \mathcal{F}_{-k} \longrightarrow \mathcal{F}_{-k}$. As for the unit $\mathcal{F}_- \rightarrow (\Phi^r \circ \Phi)(\mathcal{F}_-)$, it is the cochain inclusion $\mathcal{F}_{-k} \rightarrow \mathcal{F}_{-k} \oplus \mathcal{F}_{-k+1}$.

The importance of the shift functor s is well known in topology to express the topological suspension operator as a quotient of the cylinder $M \times [0,1]$ with each basis collapsed into one point (or, both together with one edge $m_0 \times [0,1]$ into the same point). The suspension can also be an open cone, when the half open interval is used instead. However, its crucial role here will be to express the regular representation of a (Lie) algebra. It appears as the second component in the above $cone(\mathcal{F}_-)$. To stress the shift from a 0-dimensional object to a 1-dimensional arrow we shall also use the famous bar notation $\overline{M}$ for sM and $\overline{Mod}_-$ for the image category of strictly negatively graded modules. Let s, $cone'$, $cone : Ch_- \rightarrow Ch_-$ be the canonical extension of $s, \Phi^l, \Phi^r : Mod_- \rightarrow Ch_-$. Then the relation $\Phi^l(\overline{M}) = \Phi^r(M)$ extends to $cone'(\overline{\mathcal{F}_-}) = cone(\mathcal{F}_-)$.

Theorem 1: $\Phi^r\Phi \approx cone$ by the cochain map ϕ corresponding to the Mod_--map $\Phi(\mathcal{F}_-) \rightarrow \Phi(cone'(\mathcal{F}_-))$. Similarly, $\Phi^l\Phi \approx cone'$ by the cochain map corresponding to the Mod_--map $\Phi(cone(\mathcal{F}_-)) \rightarrow \Phi(\mathcal{F}_-)$.

Proof : Defining ϕ_{-k} for each k by

$$\begin{array}{rcll} \phi_{-k} : \Phi^l\Phi(\overline{\mathcal{F}})_{-k} = & \Phi^r\Phi(\mathcal{F})_{-k} & = \overline{\mathcal{F}}_{-k-1} \oplus \overline{\mathcal{F}}_{-k} = \mathcal{F}_{-k} \oplus \mathcal{F}_{-k+1} & \\ & \longrightarrow & \mathcal{F}_{-k} \oplus \mathcal{F}_{-k+1} = cone(\mathcal{F}_-)_{-k} & (13) \\ & \left[\begin{smallmatrix} I & 0 \\ \partial_{-k} & (-)^k I \end{smallmatrix}\right] & & \end{array}$$

we see that it satisfies $\phi_{-k+1}\partial_{\Phi^l\Phi(\overline{\mathcal{F}_-})} = \partial_{cone(\mathcal{F}_-)}\phi_{-k}$, where $(\partial_{cone(\mathcal{F}_-)})_{-k} = \left[\begin{smallmatrix} \partial_{-k} & (-)^{-k} I \\ 0 & \partial_{-k+1} \end{smallmatrix}\right]$ and $(\partial_{\Phi^l\Phi(\overline{\mathcal{F}_-})})_{-k} := \left[\begin{smallmatrix} 0 & (-)^{-k} I \\ 0 & 0 \end{smallmatrix}\right]$. Thus we obtain a cochain map $\phi : \Phi^l\Phi(\overline{\mathcal{F}_-}) \to cone(\mathcal{F}_-)$ which provides the alluded isomorphism

$$\begin{array}{ccccccc} \Phi^l\Phi(\overline{\mathcal{F}_-}) : & \to & \mathcal{F}_{-k-1} \oplus \mathcal{F}_{-k} & \xrightarrow{\left[\begin{smallmatrix} 0 & (-)^{k+1} I \\ 0 & 0 \end{smallmatrix}\right]} & \mathcal{F}_{-k} \oplus \mathcal{F}_{-k+1} & \to & \\ \phi \downarrow & & \downarrow \left[\begin{smallmatrix} I & 0 \\ \partial_{-k-1} & (-)^{k+1} I \end{smallmatrix}\right] & & \downarrow \left[\begin{smallmatrix} I & 0 \\ \partial_{-k} & (-)^k I \end{smallmatrix}\right] & & \\ cone(\mathcal{F}_-) : & \to & \mathcal{F}_{-k-1} \oplus \mathcal{F}_{-k} & \xrightarrow[\left[\begin{smallmatrix} \partial_{-k-1} & (-)^{k+1} I \\ 0 & \partial_{-k} \end{smallmatrix}\right]]{} & \mathcal{F}_{-k} \oplus \mathcal{F}_{-k+1} & \to & \end{array} \quad . \quad (14)$$

The other isomorphism results from this one by the above relation $\Phi^l(\overline{M_-}) = \Phi^r(M)$. ■

Note that this natural isomorphism ϕ cannot be extended beyond the linear structure, namely, it will not respect bilinear compositions involved in algebraic structures.

Now we can see that $cyl(\phi) \approx cone(\mathcal{F}) \oplus \mathcal{F}' \approx \mathcal{F}' \oplus \tilde{0}$: The projection $cyl(\phi) \to cone(\mathcal{F}_-)$ has a cochain section by using the freeness of $cone'(\overline{\mathcal{F}_-}) = cone(\mathcal{F}_-)$ and the module map $\overline{\mathcal{F}_-} \to cyl(\phi)$. Moreover ϕ factors canonically through $\mathcal{F}_- \hookrightarrow cyl(\phi) \to \mathcal{F}'_-$, the last map being a homotopy equivalence as defined before. Combined with the other inclusion $\mathcal{F}_- \hookrightarrow cyl(\mathcal{F}_-)$, it generates homotopy of maps: Two cochain maps $\mathcal{F}_- \rightrightarrows \mathcal{F}'_-$ are homotopic iff they factor through the two canonical inclusions $\mathcal{F}_- \rightrightarrows cyl(\mathcal{F}_-) \xrightarrow{h} \mathcal{F}'_-$ by some chain map h. Then it has been shown [2] that the homotopy equivalence of cochain complexes as defined previously is equivalent to the existence of a pair of opposite chain maps, with both compositions being homotopy equivalent to the identity.

From the above existence of free and cofree objects, we see that homotopy is a selfdual relation. Then it applies both to negatively graded cochains $\mathcal{F}_-$ and to their positively graded dual $\mathcal{F}^+$. For instance, $cone(\mathcal{F}^+)$ becomes free, while $cone'(\mathcal{F}^+)$ becomes cofree. Both type of cochains come together in duality (see Sect.4).

The classical graded commutative example is the de Rham complex of differentiable p-forms and differentiable currents on a manifold M (called by

him even and odd forms) corresponding to an orientation ω of the orientable cover. More generally, we can define the space $\mathcal{F}_k \equiv \Omega^{n-k}.\omega$ of differentiable currents w.r.t. any fixed 0-current ω with its support being the whole manifold. It is $\Omega^0 = C^\infty$-(pre-)dual to the de Rham algebra of differential forms $\Omega \equiv \mathcal{F}^+$.

Theorem: A Riemannian metric transforms any p-tangent field into a differentiable current $\in \mathcal{F}_{-p}$, and provides a Ω^0-isomorphism $\mathcal{F}_{-p} \approx \mathcal{F}^p$.

The latter Hodge duality $\mathcal{F}_{-k} \approx \mathcal{F}^k$ equips $\mathcal{F}_-$ and $\mathcal{F}^+$ with both boundary and coboundary $d = \partial, d^* = *\partial *$. So they become mixed cochain complexes. The invariance of the Hamiltonian H by d, d^* implies $H = dd^* + d^*d =$ Laplacian.

In the present work, however, we shall use a Taylor superexpansion of a de Rham complex (see Sect.4). Contrary to the case of de Rham complex, it is of infinite degree but degreewise finite dimensional. It displays linearly the nonlinear part of the de Rham complex.

In this way we come to the point to focus upon a *positively* graded cochain complex $\mathcal{F}^+$ put in a duality relation with a negatively graded one $\mathcal{F}_-$. In this situation the coboundary d on $\mathcal{F}^+$ is seen to be *implemented* by the boundary ∂ acting on currents in $\mathcal{F}_-$ in the following sense: For $\forall f \in \mathcal{F}^+$ and $\forall c \in \mathcal{F}_-$, we have $\partial(f \cdot c) = df \cdot c + (-1)^f f \cdot \partial(c)$, or equivalently, $[\partial,\ f\cdot_{\mathcal{F}_-}]c = df \cdot_{\mathcal{F}_-} c$, which means

$$[\partial,\ f\cdot_{\mathcal{F}_-}] = df \cdot_{\mathcal{F}_-} . \tag{15}$$

Combined with the multiplication $f\cdot_{\mathcal{F}^+}$of $f \in \mathcal{F}^+$ on $\mathcal{F}^+$ (distinguished from the one $f\cdot_{\mathcal{F}_-}$on $\mathcal{F}_-$), this equation also implies the derivation property of d on $\mathcal{F}^+$, $d(fg) = (df)g + (-1)^f f(dg)$, for $f,\ g \in \mathcal{F}^+$. Rewriting the latter into

$$[d,\ f\cdot_{\mathcal{F}^+}] = df\cdot_{\mathcal{F}^+},$$

we see that the multiplication in $\mathcal{F}^+$, $(f,\ g) \mapsto fg = f \cdot_{\mathcal{F}^+} g = \mu(f \otimes g)$, is a cochain map, $\mu : \mathcal{F}^+ \otimes \mathcal{F}^+ \to \mathcal{F}^+$, of degree 0, i.e., $d \circ \mu = \mu \circ (d \otimes I + (-)^\bullet I \otimes d)$, where $d \otimes I + (-)^\bullet I \otimes d$ is the coboundary operator of $\mathcal{F}^+ \otimes \mathcal{F}^+$. Taking the dual of this μ, we find a coproduct structure $\mathcal{F}_- \to \mathcal{F}_- \otimes \mathcal{F}_-$ in the (pre-)dual $\mathcal{F}_-$. Note also that $\mathcal{F}^0$ is closed under the composition, that $\mathcal{F}^{>0}$ is a $\mathcal{F}^0$-bimodule, and that $\mathcal{F}^0 \to \mathcal{F}^{>0}$ is a derivation.

The above map μ makes the cochain complex $\mathcal{F}^+$ an associative algebra which we call an *algebraic cochain complex*. It can equally be replaced by a Lie bracket, which yields a *Lie-algebraic cochain complexes:*

Definition 3: A cochain complex C is called an *algebraic* (resp., *Lie-algebraic*) *cochain complex* under a composition map $\mu : C \otimes C \to C$ if μ is a cochain map satisfying the axioms of associative product (resp., Lie bracket). Together with arrows as cochain maps commuting with the structure map μ, the totality of such complexes constitute a category, $AlgCh$ (resp., $LieAlgCh$), of (Lie-) algebraic objects in the category Ch of cochain complexes.

NB: They should not be confused with cochain complexes having an algebra or Lie algebra at each degree.

We shall soon discuss the homotopy theory of algebraic cochain complexes and of Lie-algebraic ones. While the coboundary of an algebraic cochain complex is a graded derivation (which can be viewed as a first variation by algebra automorphisms), the homotopy uses algebra homomorphisms. Similar situation is found for Lie-algebraic cochain complexes of graded derivations, whose maps can be viewed as variations of algebra homomorphisms by inner actions of algebra automorphisms.

To study the homotopy for a (Lie-)algebraic cochain complex in $\mathcal{C} = (Lie)AlgCh^+$ (or with negative integer grading), we consider the forgetting functors $\Phi : AlgCh^+ \to LieCh^+ \to Ch^+$ and their composition, all denoted generically by the same symbol Φ. In contrast to the shift functor s to be discussed later as a regular representation, these functors do not change scalars. As in the above definition, we adopt here a unified notation $\mu(a \otimes b)$ for both the associative product ab of a, b in $AlgCh^+$ and the Lie bracket $[a, b]$ in $LieCh^+$ (which should be easily judged according to the context).

Definition 4: A cochain complex in $\mathcal{C}$ is $\mathcal{C}$-*contractible* if and only if it is Ch^+-free (or cofree) with a cone cochain complex as its basis: $\mathcal{C}(\Phi^l(cone),\ C) \approx Ch^+(cone,\ \Phi(C))$ or $\mathcal{C}(C,\ \Phi^r(cone')) \approx Ch^+(\Phi(C),\ cone')$ with Φ^l and Φ^r, the left and right adjoint functor of the forgetting one $\Phi : \mathcal{C} \to Ch^+$, respectively.

Theorem 2: There exist functors $Ch^+ \stackrel{\mathcal{L}}{\longrightarrow} LieCh^+ \stackrel{\mathcal{U}}{\longrightarrow} AlgCh^+$ together with their composition $\mathcal{A} \equiv \mathcal{U} \circ \mathcal{L}$, all of which are adjoint to the corresponding forgetting ones.

Proof: Once one obtains them neglecting coboundaries, it suffices to extend the coboundaries on generators as derivations. This reduces the construction to the case $Mod^+ \stackrel{\mathcal{L}}{\longrightarrow} Lie^+ \stackrel{\mathcal{U}}{\longrightarrow} Alg^+$. The functor $\mathcal{U}$ is known as the universal envelopping algebra which generates from a Lie algebra an associative algebra $Tens(M^+)$ modulo the identification of the Lie bracket with the algebraic

commutator. The functor $\mathcal{L}$ is constructed as follows:
$\mathcal{L}(M^+) = Tens(M^+)$ modulo [scalars$= 0$, antisymmetry and Jacobi identity]
$= \mathcal{L}(M^0) \otimes (K \oplus M^1 \oplus (M^2 \oplus \mu(M^1 \otimes M^1)) \oplus$
$\oplus (M^3 \oplus \mu(M^1 \otimes M^2) \oplus (\mu(\mu(M^1 \otimes M^1) \otimes M^1) \oplus \cdots)$ modulo [scalars$= 0$].
It has a faithful representation in the underlying Lie algebra of the algebra $Tens(M^+)$, generated by M^+. Finally, the universal enveloping algebra of a free Lie algebra on M^+ is the free algebra $\mathcal{A}(M^+) = \oplus_{k=0}^{\infty}(\underset{j_1+\cdots+j_r=k}{\oplus} (M^{j_1} \otimes \cdots \otimes M^{j_r})) \equiv Tens(M^+)$ with the multiplication structure defined by the tensor product $\otimes$. ■

These functors create free objects. Note that $\mathcal{U}$ applied to a graded Lie algebra M^+ with bracket 0 yields a Mod^+-free algebra $\mathcal{A}_c(M^+)$ which is graded-commutative in Koszul sense. We note that shift operation s, $(sA)^k = A^{k-1}$ destroys the graded product structure of A^+ in $(Lie)AlgCh^+$ replacing it with the underlying graded module structure:

$$s : (Lie)AlgCh^+ \to \overline{Ch^+}.$$

Such a module structure can be regarded as a regular representation in view of the usual identification of a representation of A^+ on V with the algebra $A \oplus V$ where $\mu(V \otimes V) = 0$. As before, we use here the bar notation $s(A^+) = \overline{A^+}$. The elements of A^+ act on this $\overline{A^+}$ as graded operators $A^+ \times \overline{A^+} \to \overline{A^+}$.

Consider the diagram

$$\begin{array}{ccc} (Lie)AlgCh^+ & \overset{\mathcal{A},\mathcal{L}}{\leftrightarrows} & Ch^+ \\ \downarrow \Phi_{d=0} & & \overset{\overline{d}=0}{} \downarrow\uparrow^{cone} \\ (Lie)Alg^+ & \overset{s}{\rightleftarrows} & \overline{Mod^+} \end{array} .$$

Applying $\mathcal{A}$ to $cone(M) = M^+ \oplus \overline{M^+} = cone'(\overline{M})$ we get an algebraic sum (see the end of this section).

Corollary 3: The functor $\Phi_{d=0} : (Lie)AlgCh^+ \to (Lie)Alg^+$ forgetting the coboundary $d = 0$ has a left adjoint functor $\Phi^l_{d=0}$ producing $(Lie)AlgCh^+$-cones.

Proof: For $Alg^+ \to AlgCh^+$ we define it by $\Phi^l_{d=0} := \mathcal{A} \circ cone' \circ s = \mathcal{A} \circ cone \circ \Phi$ modulo the relations $a \otimes b = ab$ in the given graded algebra A. This gives an algebraic sum $A \otimes \mathcal{A}(\overline{A})$ (ibid.), which has the required property. For $Lie^+ \to LieCh^+$, the relation are $a \otimes b - (-)^{ab} b \otimes a = [a, b]$. Then the algebraic sum is $Lie - cone(L) = L \otimes \mathcal{L}(\overline{L})$ by virtue of graded-commutativity. ■

We note, however, that de Rham differential cochain complexes are characterized by Ω^0-freeness over the regular $(Lie)Alg$ representation of the space of vector fields on the space Ω_{-1} of differentiable (-1)-currents. It means to introduce furthermore the relation $a \otimes \overline{b} = \overline{ab}$ or $[a, \overline{b}] = \overline{[a, b]}$ between an algebra A^+ or L^+ and its regular representation $\overline{A^+}$ or $\overline{L^+}$. This gives de Rham cone $A^+ \otimes^n \mathcal{A}(\overline{A})$ and $L^+ \otimes \mathcal{L}(\overline{L})$. As will be discussed in Sect.4, the Taylor expansion homotopy model replaces Ω^0 by the ground scalars when $V = 0$. In that case, a cone $cone(\overline{A^+}) = \overline{A^+} \oplus (s\overline{A^+})$ will be the Weil algebra [3].

Now we relate the above discussion to the bar construction. As in the charge formula, the principle is to transform (derivative) operations into interior mutiplications. Abstractly, this is the induction process to extend K-scalars to larger U-scalars: Given a K-algebra U with rank one representation $\epsilon : U \to K$, the induction $U \otimes M$ provides a U-module with the action of U as an internal operation. These U-modules are the Mod-free ones (w.r.t. the operation-forgetting functor). The corresponding resolution $U \otimes M \to M$ has a kernel, coinciding with the image of $\overline{U} \otimes M$ with $\overline{U} = Ker\epsilon$. Starting again with M replaced by the latter and so on, gives the bar resolution $B(U, M)$ by a Mod-free chain augmented by M in degree -1, equal to $U \otimes (\otimes^n \overline{U}) \otimes M$ in degree n. It has even a canonical linear homotopy $I \sim 0$. Then the representation $\overline{U}$ can be replaced by the full regular one, as sU. Now $B(U, U)$ can be interpreted as de Rham current complex of a classifying space for U-action, and so any birepresentation of U on a vector space S defines the de Rham cochain of the base space with values in S as $B(U, U)^* \otimes S \approx Hom_U(B(U, U), S)$. Its cohomology is called cohomology of U with coefficient S. The explicit form of its coboundary is found in [4].

Definition 5: Two algebraic cochain complexes in a category $\mathcal{C}$ are $\mathcal{C}$-homotopy equivalent iff they become isomorphic by adding $\mathcal{C}$-acyclic algebraic cochain complexes $\tilde{0}$.

Here the adding operation requires explanation. It refers to a category sum, whose definition is to be given by a coherent bijection $\mathcal{C}(X \underset{\mathcal{C}}{+} Y, Z) \approx \mathcal{C}(X, Z) \times \mathcal{C}(Y, Z)$. However, such a sum does not always exist; it may exist for certain objects but not all of them. For instance, free objects FX' w.r.t. a functor $F' : \mathcal{C} \to \mathcal{C}'$ to a category $\mathcal{C}'$ with sum, have a $\mathcal{C}$ sum:

$$\begin{aligned} \mathcal{C}(F(X' \underset{\mathcal{C}'}{+} Y'),\ Z) &\approx \mathcal{C}'(X' \underset{\mathcal{C}'}{+} Y',\ F'Z) \approx \mathcal{C}'(X', F'Z) \times \mathcal{C}'(Y', F'Z) \\ &\approx \mathcal{C}(FX', Z) \times \mathcal{C}(FY', Z). \end{aligned} \tag{16}$$

Any object $X \in \mathcal{C}$ has the free presentation $FF'X \overset{\varepsilon_X}{\to} X$, with relation RX. In

algebraic cases, this gives the sum $X \underset{\mathcal{C}}{+} Y \approx FF'X \underset{\mathcal{C}}{+} FF'Y$, modulo RX, RY. Of course $F'(X \underset{\mathcal{C}}{+} Y) \approx F'X \underset{\mathcal{C}'}{+} F'Y$.

In the case $\mathcal{C} = Alg^+ \to Mod^+ = \mathcal{C}'$, we have the partially defined sum $Tens(M^+) \underset{Alg^+}{+} Tens(N^+) = Tens(M^+ \oplus N^+)$ in Alg^+. For the subcategory $gcAlg^+$ of graded-commutative algebras, it becomes a totally defined sum $\wedge(M^+) \underset{gcAlg^+}{+} \wedge(N^+) = \wedge(M^+ \oplus N^+) \approx (\wedge(M^+)) \otimes (\wedge(N^+))$. In the corresponding infinitesimal case $\mathcal{C} = Lie^+$ of Lie-bracket composition, we have the partially defined sum $\mathcal{L}(M^+) \underset{Lie^+}{+} \mathcal{L}(N^+) \approx \mathcal{L}(M^+ \oplus N^+)$. In the first and third cases, the partially defined sum extends totally by resolution technique.

Now we can use the above definition to find $\mathcal{C}$-cylinder with $\mathcal{C}$ among these algebraic categories. Namely, $cyl(X) \approx X \underset{\mathcal{C}}{+} cone(X)$, with $\mathcal{C}$-cone, and the two inclusions of X in the first and second component. This will define $\mathcal{C}$-homotopy as we said, as well as $\mathcal{C}$-homotopy equivalence. It will be used to deform a given physical supersymmetry data (a priori only even-odd graded) to the one which is integer graded. Then the new data carries the real homotopy type of the configuration space.

3 Principal Algebraic Cochain Complexes

At each step of an expansion we will encounter the appearance of some symmetry groups, which requires us to treat a principal bundle P with a symmetry group G as its structure group. The data of P amounts to a G-equivariant map $\epsilon : P \to P_G$ to a principal G-bundle P_G with a contractible total space, called a *universal* bundle. We start by describing the abstract algebraic structure of the de Rham cochain algebra $\mathcal{F}^+ \equiv \Omega^+(P)$ along the line of [5]. Dually we could use the de Rham coalgebra $\Omega_-(P)$.

1) $\mathcal{F}^+$ is an algebraic cochain complex, either $\mathbb{Z}_2$ (even-odd)- or $\mathbb{N}$-graded,

2) $\mathfrak{g}$ is a (classical) Lie algebra acting on the Alg-cochain complex $(\mathcal{F}^+, d)$ by a 0-derivation θ commuting with d defined by $\theta_X = \partial_s \exp(sX)|_{s=0}$,

3) the vertical action of $\exp X \in G$ comes with a cochain algebraic homotopy $dh_X + h_X d = I - \exp(sX)$ where h_X integrates along the path $\exp(tX), 0 \leq t \leq 1$. Hence h_X commutes with θ_X. The infinitesimal vertical homotopy $\iota_X = \partial_s h_{sX}|_{s=0}$ is a (-1)-derivation of $\mathcal{F}^+$. By differentiation of the above homotopy relation, we have the Cartan identity

$$[d, \iota_X] = d\iota_X + \iota_X d = \theta_X. \tag{17}$$

It is not an algebraic cochain homotopy, but only a linear homotopy, since θ_X does not preserve the algebraic structure. These three maps $\iota_X,\ \theta_X,\ d$ are now derivations of degree $-1,\ 0,\ +1$. In the $\mathbb{Z}_2$-graded case, ± 1 reduces to $odd \equiv +1 \equiv -1$ (mod.2). Since ι_X is constant along the flow generated by X, we have also

$$(\iota_X)^2 = 0, \tag{18}$$

and since $\exp[X,Y]$ is the commutator of $\exp X$ and $\exp Y$, we have

$$[\theta_X\,, \iota_Y] = \iota_{[X,Y]}. \tag{19}$$

(However, any h commuting with θ would give such an operation by taking $\iota = hdh$).

Proposition: The infinitesimal homotopy ι integrates to one for $\exp X$.

Proof : The exponential $\exp(t\theta_X)$ of the 0-derivation $t\theta_X$ is an algebraic automorphism. Because of its spectral decomposition into two self annihilating endomorphisms $td\iota_X + t\iota_X d$, we have

$$\exp(t\theta_X) = \exp(td\iota_X)\exp(t\iota_X d) = \exp(td\iota_X) + \exp(t\iota_X d) - I.$$

Taking the variation between $t = 0,\ 1$, this gives linearly

$$I - \exp(\theta_X) = dh_X + h_X d$$

with h an algebraic cochain homotopy in the categorical sense defined before. ■

Now a G-connection A on P provides another homotopy equivalent cochain subcomplex (one of the Brown models) compatible with θ and ι, consisting of the G-invariant (scalar) p-forms. It is the model $\Omega(P)_{\iota=0} \otimes \wedge\mathfrak{g}^1 \hookrightarrow \mathcal{F}$ in the algebraic form. Neither factor of $\Omega(P)_{\iota=0}$, $\wedge\mathfrak{g}^1$ is a subcomplex, and d has a component $A : \mathfrak{g}^* \to \mathcal{F}^1$. The base algebraic subcomplex is $\Omega(Z) \approx \mathcal{F}_{\iota=0,\theta=0}$. The commutation with ι of the associated algebra inclusion $\wedge\mathfrak{g}^1 \hookrightarrow \mathcal{F}$ yields $\iota_X(A(X^*)) = \iota_X(X^*) = \langle X,\ X^*\rangle$ at lowest degrees 1 and 0. This decomposition $\Omega(P)_{\iota=0} \otimes \wedge\mathfrak{g}^1$ is the degree 1 analog to the degree 0 induction $\Omega^0(Z) \subset \Omega^0(P) \approx \Omega^0(Z) \otimes Sym\mathfrak{g}_o$, which gives the Taylor expansion of the commutative algebra $C(G;\Omega^0(Z))$ for the trivial action equipped with a pointwise product. The Poisson algebra consisting of Noether charges is now augmented in degree 0 by $\mathfrak{g}$. Its quantization is the Heisenberg-Yang-Lie superalgebra $L_- \oplus (\hbar\mathbb{C} \oplus \mathfrak{g})_0 \oplus L^+$.

A fixed connection A provides a classifying object in the sense that $\mathcal{F}^+ = \Omega^+(P)$ has a classifying algebraic cochain map $\epsilon_A : W_{\mathfrak{g}} \to \mathcal{F}^+$ from the Weil algebra $W_{\mathfrak{g}}$. To explain this fundamental notion is worth a short digression back to the categorical adjunction: A category Cat has an adjunction with the point category (on one object with its identity) iff one of its objects I (the free one) satisfies the condition that each object X correponds bijectively to a morphism $\epsilon_X : I \to X$ such that $f\epsilon_X = \epsilon_Y$ for $\forall f : X \to Y$. Then, Cat is said to be contractible, and it looks like a cone. Any category $\mathcal{C}$ gives rise to contractible ones in the following way: For an object $A \in Ob(\mathcal{C})$ fixed, the category $\hat{\mathcal{C}}$ [$= (A \downarrow \mathcal{C})$: the category of objects under A] is contractible when it is defined by $Ob(\hat{\mathcal{C}}) := \{h \in Cat(A,\ X);\ X \in Ob(\mathcal{C})\}$ and by $\hat{\mathcal{C}}(h,k) = \{u_* \equiv u \circ (-)\ ;\ u \in Cat(X,\ Y)$ s.t. $k = u \circ h\}$ as the set of morphisms from $h \in Cat(A,\ X)$ to $k \in Cat(A, Y)$. The reason is because each object $h \in Ob(\hat{\mathcal{C}})$ corresponds bijectively to a morphism h_* in $\hat{\mathcal{C}}$ from I_A(=the identity of A) to $h : h = h_*(I_A) = h \circ I_A$. This is just the essence of the so-called Yoneda's song. But it is nothing but a cone construction identifying objects with arrows, which naturally involves a dimension shift by 1.

Returning to our data, the inclusion of $\mathfrak{g}$-algebra $\wedge\mathfrak{g}^1 \hookrightarrow \mathcal{F}^+$ given by the above connection corresponds to an $AlgCh$-map $\epsilon_A : Alg - cone(\wedge\mathfrak{g}^1) \to \mathcal{F}^+$. This is the classifying map saying that the category of principal algebraic cochain complexes is adjoint to the point category. But in the category of graded-commutative algebras, this cone $Alg - cone(\wedge\mathfrak{g}^1)$ is nothing but the $\mathbb{N}$-graded Weil algebra $W_{\mathfrak{g}}^+ = Sym(\mathfrak{g}^2) \otimes \wedge\mathfrak{g}^1$, which has been seen with the notation $\overline{L} = \mathfrak{g}^1,\ s\overline{L} = \mathfrak{g}^2$ and with the assignment of a 0 superbracket to $\mathfrak{g}^1 \oplus \mathfrak{g}^2$. This approach has the advantage of showing the a priori freeness, in particular the Cat-contractibility. This is usually expressed only as acyclicity, but in our context, the length operator is homotopic to 0, because the extension of the base cochain linear homotopy $dk + kd = I$ as odd-derivations $\overline{d}, \overline{k}$ gives a derivation $[\overline{d}, \overline{k}]$ equal to the identity on generators. But the extension $\overline{I}$ is length 1. Now the classifying map $\epsilon_A : W_{\mathfrak{g}}^+ \to \mathcal{F}^+$ expresses A as push out, $A = \epsilon_A(W)$, of the unique (non-flat) Weil connection $W : \mathfrak{g}^* \to W_{\mathfrak{g}}^1 = \mathfrak{g}^*$.

Our new step is to introduce a principal Lie chain basis for $\Omega(P)$:

$$L(P) :\ \cdots \to L_{-2}(P) \to L_{-1}(P) \to L_0(P) \to P_{+1} \to 0, \tag{20}$$

where $P_{+1} = Z_{+1} \oplus \mathfrak{g}_{+1}$. It projects onto $L(Z)$ with kernel $L(G)$ augmented by $\mathfrak{g}_1$. The homology exact sequence of $0 \to L(G) \to L(P) \to L(Z) \to 0$ will be (Hurewicz-)isomorphic to the homotopy exact sequence of the fibration. The bracket $[\mathfrak{g}_1,\ L_{-p}] \subset L_{-p+1}$ generates the derivation ι on $\Omega(P)$. The map $L_0(P) \to Z_{+1} \oplus \mathfrak{g}_{+1}$ consists of two components taking values in Z_{+1} and in $\mathfrak{g}_{+1}$, each of which can be identified, respectively, with the first order

soldering 1-form and the first order *connection* 1-form. Similarly, the composition $L_{-1}(P) \to L_0(P) \to Z_{+1} \oplus \mathfrak{g}_{+1}$ has the corresponding *torsion* and *curvature* as its two components, both of which are 0. In the next section, we construct these Lie chain bases.

4 Taylor Superexpansion of de Rham Complex

In this section we show how graded Taylor coefficients $\partial_{i_0} \cdots \partial_{i_q} f_{\mu_1 \cdots \mu_p}\, dz^{\mu_1} \wedge \cdots \wedge dz^{\mu_p} (\in J^q \Omega^p)$ of p-forms $\sum_{\mu_1 \cdots \mu_p} f_{\mu_1 \cdots \mu_p}\, dz^{\mu_1} \wedge \cdots \wedge dz^{\mu_p} \in \Omega^p$ are related to the homotopy group $\pi_{q+1}(Z)$ of $\alpha : \mathbb{R}^{q+1} \cup \{\infty\} \to Z$ which sends ∞ to a fixed base point of a manifold Z under consideration.

A first observation is the duality between the space of forms and the differential algebra acting on the former by taking the Taylor coefficients of $\alpha^* f$ at the origin $\in \mathbb{R}^{q+1}$. We shall use the Weil bicomplex expanding the de Rham complex over an open cover [6] to produce a new bicomplex model exhibiting the Hurewicz map $\pi_{k+1}(Z) \to H_{-k-1}(Z)$. Here Z is assumed to be a real analytic manifold, the typical examples of which are given by Lie groups and their homogeneous spaces. On Z let us choose a cover $\mathcal{R} = \{U_i\}$ by open domains of analytic charts, such that each non empty intersection $U_{i_0 \cdots i_q} \equiv U_{i_0} \cap \cdots \cap U_{i_q}$ has a base point named as $i_0 \cdots i_q$ not belonging to a higher order intersection. Choose also a partition of unity $\{\varphi_i\}$, $\sum \varphi_i = 1$ subordinate to the cover $\{U_i\}$: Namely, each φ_i is a real analytic function (> 0) (typically of such a Gaussian shape as constant$\times \exp(\frac{-\varepsilon^2}{\varepsilon^2 - r^2})$, see, e.g., [7] for details) in the interior of $U_i (\simeq \mathbb{R}^n)$ which extends smoothly (but not analytically, of course) as 0 to the outside of U_i. Now let $\Omega^0_{\mathcal{R}}$ be the algebra of functions of the form $f = \sum_{i_0 \cdots i_q} \varphi_{i_0} \cdots \varphi_{i_q} a_{i_0 \cdots i_q}$, where $a_{i_0 \cdots i_q}$ are analytic functions on $U_{i_0 \cdots i_q} \subset Z$ and each term in the sum is differentiably extended by 0 on Z. By the choice of the base points, $a_{i_0 \cdots i_q}$ is determined by the Taylor expansion of f at $i_0 \cdots i_q$. Hence we have a linear decomposition $\Omega^0_{\mathcal{R}} \approx \bigoplus_{i_0 \cdots i_q} \Omega^0(U_{i_0 \cdots i_q})$, where each summand denotes the space of analytic functions. On the manifold Z, $\Omega^0_{\mathcal{R}}$ generates the corresponding $\Omega^0_{\mathcal{R}}$-modules of tangent vector fields $\mathcal{X}_{\mathcal{R}} = Der(\Omega^0_{\mathcal{R}})$, of differentiable (-1)-currents $\Omega^{\mathcal{R}}_{-1} \approx \mathcal{X}_{\mathcal{R}}$, and of cotangent fields $\Omega^1_{\mathcal{R}} = \Gamma_{\mathcal{R}}(T^*(Z))$ as the $\Omega^0_{\mathcal{R}}$-dual of $\Omega^{\mathcal{R}}_{-1}$, and algebraic cochain complexes, $\Omega^{\mathcal{R}}_{-} = \bigoplus_{p \in \mathbb{N}} \bigoplus_{i_0 \cdots i_q} \Omega_{-p}(U_{i_0 \cdots i_q})$, $\Omega^{+}_{\mathcal{R}} = \bigoplus_{p \in \mathbb{N}} \bigoplus_{i_0 \cdots i_q} \Omega^p(U_{i_0 \cdots i_q})$, of Čech type in the indices $i_0 \cdots i_q$ and of de Rham type in p, in a similar way to the de Rham one, as explained in Sect.1: $\Omega_{-p}(U_{i_0 .. i_q}) \equiv \bigwedge^p_{\Omega_{\mathcal{R}}} \Omega_{-1}(U_{i_0 \cdots i_q})$, $\Omega^p(U_{i_0 \cdots i_q}) \equiv \bigwedge^p_{\Omega_{\mathcal{R}}} \Omega^1(U_{i_0 \cdots i_q})$.

Theorem 4: If $\mathcal{R}$ is convex, the complex $\Omega^+_{\mathcal{R}}$ is weakly homotopy equivalent (i.e., as linear cochains) to the de Rham one.

Proof: Here we take $\mathcal{R}$ to be a de Rham convex cover [8, 6] and use the Poincaré lemma on a convex open set, and the (bi)complex technique, in particular, the fact that a short exact sequence $0 \to C' \to C \to C'' \to 0$ is split when $C' \sim 0$ or $C'' \sim 0$ [2]. Let $\partial/\partial t$ be the radial field on the pointed convex set $U_{i_0 \cdots i_q}$. The canonical cone operator defined by $h(f^{(k)})(X_{(k-1)}) = \int_0^1 f^{(k)}(t\ X_{(k-1)} \wedge \partial/\partial t)dt$ for $(k-1)$-vector fields $X_{(k-1)}$ and k-forms $f^{(k)}$ satisfies $hd = I - dh$. It respects analyticity, hence is a homotopy $I \sim 0$ on $\Omega^+(U_{i_0 \cdots i_q})$. ■

The new analytic model $\Omega^+_{\mathcal{R}}$ for the de Rham complex contains much deeper information in view of the familiar formula describing the action of the coboundary $d = d' + d''$ on a form $\varphi \in \Omega^+_{\mathcal{R}}$ in terms of vector fields X_i:

$$\begin{aligned}
(d'\varphi)(X_1, \cdots, X_{p+1}) &= \sum_i (-)^{i+1} [X_i, \varphi(X_1, \cdots, \overset{\vee}{X_i}, \cdots, X_{p+1})], \\
(d''\varphi)(X_1, \cdots, X_{p+1}) &= \sum_{i<j} (-)^{i+j} \varphi([X_i, X_j], X_1, \cdots, \overset{\vee}{X_i}, \cdots, \overset{\vee}{X_j}, \cdots X_{p+1}),
\end{aligned}$$

where $[X,\ f] = \partial_X f = Xf$ for functions f.

While the space $\Omega_{-p}(Z) = \overset{p}{\underset{\Omega^0}{\wedge}} \Omega_{-1}(Z)$ of currents, in Ω^0-duality to p-forms $\Omega^p(Z)$, is the p-th power of $\Omega_{-1}(Z)$ w.r.t. Ω^0-linear exterior product $\underset{\Omega^0}{\wedge}$ over Ω^0, we now consider $\wedge^p \Omega_{-1}(Z)$ taking the $\mathbb{R}$-linear exterior product $\wedge \equiv \underset{\mathbb{R}}{\wedge}$ over $\mathbb{R}$. In contrast to the usual current $c_{X_1 \wedge \cdots \wedge X_p} \in \Omega_{-p}(Z)$ with point-like localization, an element $c_{X_1} \wedge \cdots \wedge c_{X_p} \in \wedge^p \Omega_{-1}(Z)$ is not localized at a single point in the base manifold Z, but it involves p tangent vectors $X_i \in T_{z_i}(Z)$ at p different points $z_i \in Z$. In view of the canonical isomorphism $T_{(z_1, \cdots, z_p)}(Z^p) \approx \overset{p}{\underset{i=1}{\oplus}} T_{z_i}(Z)$ and of the canonical embedding $\wedge^p(\overset{p}{\underset{i=1}{\oplus}} T_{z_i}(Z)) \subset \wedge^p T_{(z_1, \cdots, z_p)}(Z^p)$, we have an isomorphism $\wedge^p \Omega_{-1}(Z) \approx \Omega_{-p}(Z^p - Diag_p(Z))^{sym}$ based upon $c_{X_1} \wedge \cdots \wedge c_{X_p} = (c_{X_1} \wedge \cdots \wedge c_{X_p})(z_1, \cdots, z_p) = \sum_{\sigma = (\sigma_1, \cdots, \sigma_p) \in \mathfrak{S}_p} X_{\sigma_1}(z_1) \wedge \cdots \wedge X_{\sigma_p}(z_p)$, since the last expression vanishes for $(z_1, z_2, \cdots) \in Diag_p(Z) \equiv \{(x_1, x_2, \cdots) \in Z^p; x_i = x_j \text{ for some } i \neq j\} \subset Z^p = \overbrace{Z \times \cdots \times Z}^{p}$. In comparison, we have $(c_{X_1} \wedge \cdots \wedge c_{X_p})(z, \cdots, z) = X_1(z) \wedge \cdots \wedge X_p(z) = c_{X_1 \wedge \cdots \wedge X_p}(z)$. A map $\alpha : \mathbb{R}^p \to Z$ gives $\wedge^p \Omega^1(Z) \to \wedge^p \Omega^1(\mathbb{R}) \approx \Omega^p(\mathbb{R}^p - Diag_p(\mathbb{R}))^{sym} \overset{\cong}{\to} \mathbb{R}$ and

hence an element $c_\alpha \in \wedge^p\Omega_{-1}$. It can be visualized as a smoothing homotopy of $\alpha : \mathbb{R}^p \to Diag_p(Z)$ pushing it away from $Diag_p(Z)$ (i.e. 0 on it; in general, α itself cannot be pushed away from $Diag_p(Z)$). This element c_α behaves as a local current on Z^p under a deformation of α supported around a point. This is the relevance of $\wedge^p\Omega_{-1}(Z)$ to homotopy, even with its bracket $[\alpha, \beta]$ viewed in $Z \vee Z \subset (Z \times Z - Diag_2(Z)) \cup \{\text{base point}\}$, while $\Omega_{-p}(Z)$ is relevant to the homology. Likewise, the jets on $\Omega_{-1}(Z)$ can be defined similarly to the ordinary currents with a Lie derivative ∂_Y and with $J^r = \partial_{Y_1} \cdots \partial_{Y_r}$ as

$$\begin{aligned} (\partial_Y c_X)(\varphi) &\equiv -c_X(\partial_Y \varphi) = -\langle \iota_X \partial_Y(\varphi) \rangle_\omega, \\ J^r c_X &= (-)^r c_X J^r, \end{aligned}$$

but their meaning on $\wedge^p\Omega_{-1}(Z)$ is spread as symmetric currents over $Z^p - Diag_p(Z)$, not localized at a point on Z as ordinary currents $c_{X_1 \wedge \cdots \wedge X_p}$, the latter of which vanish for $p > n$. Keeping this in mind, we write an r-jets in $J^r(\wedge^p\Omega_{-1}(Z))$ as $\partial_{Y_1} \cdots \partial_{Y_r} c_{X_1} \wedge \cdots \wedge c_{X_p}$.

By the above formula $d = d' + d''$ (which is equivalent on $\Omega^1(Z)$ to $[\partial_X,\ \iota_Y] = \iota_{[X,\ Y]}$ and $\partial_X = d\iota_X + \iota_X d$), we get a boundary $\partial : L_{-1} \to L_0$ as the $\mathbb{R}$-dual of $d : \Omega^1(Z) \to \Omega^2(Z)$ by

$$\partial(c_X \wedge c_Y) = -\partial_X c_Y + \partial_Y c_X + c_{[X,Y]}, \tag{21}$$

where $L_0 \equiv \Omega_{-1}(Z)$ and $L_{-1} \equiv \wedge^2\Omega_{-1}(Z)$. The last term $\delta : L_{-1} \ni c_X \wedge c_Y \longmapsto [c_X,\ c_Y] \equiv c_{[X,Y]} \in L_0$, $\mathbb{R}$-dual to d'', defines a Lie bracket $[L_0, L_0] \subset L_0$.

Since the derivative ∂_X taking 1-jets is linear only in $\mathbb{R}$ but not in Ω^0, the canonical quotient map $\wedge^p\Omega_{-1}(Z) \to \Omega_{-p}(Z) = \overset{p}{\underset{\Omega^0}{\wedge}} \Omega_{-1}(Z)$ is not compatible with the 1-jet operation, so the right-hand side of the above formula is not defined over the quotient space $\Omega_{-p}(Z)$. The above ∂ respects the r-jet filtration: $\partial(J^r\Omega_{-1}) \subset J^{r+1}\Omega_{-1}$ and $[J^r\Omega_{-1}, J^s\Omega_{-1}] \subset J^{r+s}\Omega_{-1}$. The role of this jet filtration is just to make the spaces L_0, L_{-1} jet-degreewise finite-dimensional by taking Taylor expansions at a chosen discrete set of base points in $\mathcal{R}$. This ensures a reflexive duality between Ω^1 and Ω_{-1}.

The above structure generates a Lie algebraic chain complex $L \equiv \{L_{-p}\}_{p \in \mathbb{N}_0}$ with

$$L_{-p} \equiv \wedge^{p+1}\Omega_{-1}, \tag{22}$$

which is augmented by the evaluation $L_0 \to L_1 = J^0\Omega_{-1} \approx$ tangent space at a base point. On this complex a mapping $d_L : L_{-p} \longrightarrow L_{-p+1}$ is defined by

$$d_L(c_{X_1} \wedge \cdots \wedge c_{X_{p+1}}) \equiv \sum_i (-)^{i+1} \partial_{X_i} c_{X_1} \wedge \cdots \wedge \overset{\vee}{c_{X_i}} \wedge \cdots \wedge c_{X_{p+1}}, \tag{23}$$

which is $\mathbb{R}$-dual to the first component d' in d in the limit of coincident points:

$$(d_L(c_{X_1} \wedge \cdots \wedge c_{X_{p+1}}))(\varphi) = \sum_i (-)^{i+1} \langle \partial_{X_i} \varphi (X_1 \wedge \cdots \wedge \overset{\vee}{X_i} \wedge \cdots \wedge X_{p+1}) \rangle_\omega$$
$$= \langle (d'\varphi)(X_1 \wedge \cdots \wedge X_{p+1}) \rangle_\omega = \langle c_{X_1 \wedge \cdots \wedge X_{p+1}} (d'\varphi) \rangle_\omega .$$

Similarly, a Lie superbracket $[L_{-p}, L_{-q}] \subset L_{-p-q}$ is defined as a natural extension of the above $\delta : L_{-1} \to L_0$ by

$$[c_{X_1} \wedge \cdots \wedge c_{X_{p+1}}, c_{Y_1} \wedge \cdots \wedge c_{Y_{q+1}}]$$
$$\equiv \sum_{i,j} (-)^{i+j+p+1} c_{[X_i, Y_j]} \wedge c_{X_1} \wedge \cdots \wedge \overset{\vee}{c_{X_i}} \wedge \cdots \wedge c_{X_{p+1}} \wedge$$
$$\wedge c_{Y_1} \wedge \cdots \wedge \overset{\vee}{c_{Y_j}} \wedge \cdots \wedge c_{Y_{q+1}}, \qquad (24)$$

whose coincidence point limit is $\mathbb{R}$-dual to d''. By derivation rule we also have $[J^r L_{-p}, J^s L_{-q}] \subset J^{r+s} L_{-p-q}$ as

$$[J^r c_{X_1} \wedge \cdots \wedge c_{X_{p+1}}, J^s c_{Y_1} \wedge \cdots \wedge c_{Y_{q+1}}]$$
$$\subset J^{r+s} \sum_{i,j} (-)^{i+j+p+1} c_{[X_i, Y_j]} \wedge c_{X_1} \wedge \cdots \wedge \overset{\vee}{c_{X_i}} \wedge \cdots \wedge c_{X_{p+1}} \wedge$$
$$\wedge c_{Y_1} \wedge \cdots \wedge \overset{\vee}{c_{Y_j}} \wedge \cdots \wedge c_{Y_{q+1}} . \qquad (25)$$

Finally, to recapture d'' from L, we take the construction $\mathcal{A}_c(L)$ seen in Sect.2, which gives a free presentation of the original cochain algebra Ω. This uses the reflexivity of the duality mentionned above, so we have to use L_{-p} over the cover $\mathcal{R}$ in such a form as a double chain complex with the total chain complex

$$L_k \equiv \underset{p+q=k}{\oplus} L_{-p}(U_{i_o \cdots i_q}) \qquad (26)$$

filtered by finite-dimensional chain complexes

$$L_{k,l} \equiv \underset{p+q=k}{\oplus} \underset{q+r=l}{\oplus} J^r(\wedge^p \Omega_{-1}(U_{i_0 \cdots i_q})), \qquad (27)$$

where the jets taken at the base point $i_0 \cdots i_q$ constitute a finite-dimensional chain subcomplex. Because of the constraint $p - r = k - l = \text{const}$ (following from $p + q = k$ and $q + r = l$ in Eq.(27)), the gradation of L_k in p with $L_{-p}(Z) \approx \oplus L_{-p}(U_{i_o}) / L_{-p}(U_{i_o i_1})$ can be reinterpreted as a filtration in the jet degree r arising from the Taylor superexpansion of $\Omega_-(Z)$ (in anticommuting variables, each of finite dimensions).

Now $\Omega_{\mathcal{R}}$ has a resolution of the form $\mathcal{F}_{\mathcal{R}} = \mathcal{A}_c(\overline{L(Z)^+})$ seen in Sect.2:

$$\begin{aligned} \mathcal{F}_{\mathcal{R}} \quad &: \quad \mathcal{F}^0 = Sym(Z) \otimes K \longrightarrow \mathcal{F}^1 = Sym(Z) \otimes L^0 \\ \stackrel{d}{\longrightarrow} \mathcal{F}^2 \quad &= \quad Sym(Z) \otimes (L^1 \oplus (L^0 \wedge L^0)) \\ \stackrel{d}{\longrightarrow} \mathcal{F}^3 \quad &= \quad Sym(Z) \otimes (L^2 \oplus (L^0 \wedge L^1) \oplus (\wedge^3 L^0)) \stackrel{d}{\longrightarrow} \cdots . \end{aligned}$$

where $L^p \equiv L^*_{-p}$. One can interpret $L^p(U_{i_0 \cdots i_q})$ as coordinates of a supermanifold Z^L with a body manifold Z (i.e., the 0 degree part). Then $\mathcal{F}_{\mathcal{R}}$ is the de Rham supercomplex. This means that the new model does not lose the geometry [9].

The former geometrico-analytic construction of L based on the locality of Ω realizes a direct categorical construction in the 1-connected case, but it has more profound essence applicable to such general situations as non-simply connected cases or $\mathbb{Z}_2$-graded ones. So let us consider a degreewise finite-dimensional coalgebraic cochain complex of the form $\mathcal{F}_- = \mathcal{A}_c(\overline{L_-(Z)})$ with a Lie superalgebraic cochain complex $L_-(Z)$ augmented by a module Z, together with its dual version, a free algebraic cochain complex over a Lie super-coalgebra $\mathcal{L}^+ = \mathcal{A}_c(\overline{L^+})$ equipped with a cobracket $L^+ \to L^+ \oplus (L^+ \otimes L^+)$. In the case of reflexive duality (e.g., when the dimensions are degreewise finite), the correspondence between L and such $\mathcal{F}^+$ is obviously a Cat isomorphism. Then d on $\mathcal{A}_c(\overline{L_-})$ has a Noether charge in $P = L_- \otimes L^+$ by linearity of the bracket $[\cdot\ ,\ \cdot]$.

Now we want to show that any 1-connected $\mathcal{F}^+$ has such a basis L up to weak homotopy equivalence. The functors $\mathcal{A}_c, \mathcal{L}$ met in Sect.2 allow one to define $\mathcal{L}s^{-1} : coAlg_cCh_- \to LieAlgCh_-$ giving the free Lie superalgebra cochain complex on the shifted cochains, and $co\mathcal{A}_{cs} : LieAlgCh_- \to coAlg_cCh_-$ where $co\mathcal{A}_c = \mathcal{A}_c^*$ giving the free coalgebra on the shifted cochain.

Theorem 5: In the case of degreewise finite dimensions, these functors define an adjunction

$$\phi : LieAlgCh_-(\mathcal{L}s^{-1}(A_-), L_-) \approx coAlg_cCh_-(A_-, co\mathcal{A}_{cs}(L_-)) \qquad (28)$$

for 0-connected A_- (reduced to scalar in degree 0).

Proof: This is due to the following series of adjunctions:

$$\begin{aligned} & LieAlgCh_-(\mathcal{L}s^{-1}(A_-),\ L_-) \approx Ch_-(s^{-1}(A_-), \Phi(L_-)) \approx Ch^+(sL^+, \Phi(A^+)) \\ \approx\ & Alg_cCh^+(\mathcal{A}_{cs}(L^+), A^+) \approx coAlg_cCh_-(A_-, co\mathcal{A}_{cs}(L_-)). \end{aligned} \qquad (29)$$

■

Since adjoint functors carry a free object to a free one, we also have

Corollary 6: Both functors $\mathcal{A}_c$ and $\mathcal{L}$ respect $\mathcal{C}$-cones with $\mathcal{C}$ being Alg_cCh^+ or $coAlg_cCh_-$.

Next, we have

Theorem 7: The unit η and counit ε of the adjunction ϕ are weak homotopy equivalence (i.e., homotopy equivalence in Ch_-).

Proof : The adjunction Φ of both categories $LieAlgCh$ and $(co)AlgCh$ with Ch seen in Sect.2 provides Ch-free objects, which are respected by the adjunction $\Phi, \mathcal{A}, \mathcal{L}$. So one first checks that ε is a homology isomorphism on Ch-free objects, then use Ch-free resolutions [10]. For Ch-free objects, $H(\Phi^*(C))$ is determined by a spectral sequence in $H(C)$. So both $H(\Phi^*(C))$ and $H(L\mathcal{A}_c\Phi^*(C))$ are determined by a spectral sequence in $H(\mathcal{A}_c\Phi^*(C))$. As ε comes from the identity on the latter, it induces an isomorphism on homology. Finally, a chain map inducing a homology isomorphism is a Ch-homotopy equivalence, because its cone chain is acyclic, hence ~ 0 [2]. ■

Now the Lie superalgebra of homology $H_{-k}(L_-(Z))$ is related to that of homotopy $\pi_{k+1}(Z)$ by the Hurewicz map $\pi_{k+1}(Z) \to H_{-k}(L_-(Z))$, which is just the analytic continuation of the constant function 1 along a map $S^{k+1} \to Z$. In fact, by decent in $\Omega^0(U_{i_0 \cdots i_q})$ in the bicomplex any element $\in H_{-k}(L_-(Z))$ represents a constant analytic continuation along a singular closed $(k+1)$-manifold. The discussion on the Hurewicz map leads to the case of G-principal bundle $P \to Z$. The equivariant construction of $L(P)$ is straightforward from the preceeding one. In the case of a G-principal space ($Z =$ point), the invariant de Rham complex is built on $\mathfrak{g}_0$. But the minimal L (i.e. with 0 coboundary) is the odd graded module $L(G) = Prim(G) \approx \pi(G)$ (where $Prim$ denotes the primitive space in the Hopf homology graded algebra $H(G)$).

For a general P, we have an exact sequence of Lie superalgebraic cochain complexes $0 \to L(G) \to L(P) \to L(Z) \to 0$, where $L(Z) = L(P)_{\theta=0,\iota=0}$. The best case is given by a homogeneous space $Z = G/H$. Because the Hurewicz map sends the homotopy exact sequence for principal fibrations to the homology exact sequence for L above, it is an isomorphism on P iff it is so on Z.

Definition 6: We call the Lie superalgebra $H_{-k}(L_-(Z))$ the real homotopy $\pi_{k+1}(Z)$ of Z if Z is simple (i.e., the canonical action of $\pi_1(Z)$ on all $\pi_k(Z)$ is trivial). Otherwise it gives the homotopy with this action trivialized.

To justify this definition, we briefly indicate in the case of a 1-connected space Z how $H_{-k}(L_-) \approx \pi_k(Z)$ is verified, together with an algorithm to compute explicitly π_k from the (co)homology data of $\Omega(Z)$. The basic principle is as follows: The (co)homology of a Lie group is algebraically free with the homotopy as its basis, which is displayed in various odd degrees. Next, this extends to principal bundles by means of the Koszul-Brown homotopy model for the de Rham (co)chain complex of the total space of a principal fibration with a Lie group as its structure group. Finally, the Postnikov sequence of any 1-connected space Z factorizes Z through functional principal fibrations with disjoint homotopy. [Incidentally it is interesting to note the parallelism between the construction of this sequence and a problem in mathematical physics appearing in extending the Doplicher-Roberts superselection theory [11] to the situations with spontaneous symmetry breakdown (SSB): Starting from an algebra acted upon by a compact group H of unbroken symmetry together with some data concerning a homogeneous space G/H with an *unknown* G, one must find a solution to an unknown larger algebra acted upon by this unknown compact group G satisfying the criterion of SSB.] In the topological context, this tower describes the homotopy type of the space. Algebraically, the sequence of principal fibrations is a sequence of extensions $\mathcal{A} \otimes \mathcal{A}_c(L^k)$ with differential $d + d_L$, where $d_L(L) \subset \mathcal{A}$, on which $B \cdots B(\pi_{k+1})$ acts . This is the so-called Hirsch extensions [12]. Then the total L has 0 boundary and they are Taylor expansion (of degree $k \geq 1$) of a "Galois superextension". (Galois extension would occur at $k = 0$). They provide a minimal model with $\partial = 0$, whose basis exhibits the homotopy.

Any 1-connected cochain algebra is a sequence of Hirsch extensions, with an explicit algorithm from the knowledge of cohomology. In terms of Lie-algebraic cochain complexes Hirsch extensions correspond to Quillen extensions [10].

5 Non-Simply Connected Case

Let $Z' \to Z$ be a covering map (with discrete fiber), with classifying map $Z \to B(W)$ where W is the Weyl group of $\pi_1' \equiv \pi_1(Z') \subset \pi_1(Z)$ (defined as the normalizer of $\pi_1' \bmod \pi_1'$). To simplify the discussion let us assume that π_1' is normal, i.e., $W = \pi_1/\pi_1'$.

The (convex) cover $\mathcal{R} = \{U_i\}$ of Z lifts to that of Z', $\mathcal{R}' = \mathcal{R} \times W$. The bicomplex for $\mathcal{R}'$ is $\Omega_{\mathcal{R}'} \approx \Omega_{\mathcal{R}} \times W$, except for the Čech coboundary $\oplus i_0 \to \oplus i_0 i_1$ having a W'-component, which we denote by a "translation" twist $\Omega_{\mathcal{R}} \underset{t}{\times} W$. Similarly $L(Z') \simeq L(Z) \underset{t}{\times} W$. But we need to solve this equation in $L(Z)$. For that purpose, the homotopically exact sequence $Z' \to$

$Z \to B(W)$ gives $\Omega_{\mathcal{R}} \sim \Omega_{\mathcal{R}'} \underset{t}{\otimes} \Omega(B(W))$ and the homotopically exact sequence $W \to P_W \to B(W)$ for the unversal W-principal bundle P_W ($\sim$ point) gives $0 \sim W \underset{t}{\otimes} \Omega(B(W)) \sim cone(W)$. So we obtain $L(Z) \simeq L(Z') \underset{t}{\times} \overline{W}$, where the bar indicates degree 0 (shift of W in degree 1).

The same argument with the universal covering $\tilde{Z} \to Z'$ instead of $Z' \to Z$ gives $L(Z) \sim L(\tilde{Z}) \underset{t}{\times} L_0(\pi_1, \pi_1')$, where $L_0(\pi_1, \pi_1') = W \times \pi_1'$. When both W and π_1' are abelian, we get a linear expression $L(Z) \sim L(\tilde{Z}) \underset{t}{\oplus} L_0(\pi_1, \pi_1')$, with $L_0(\pi_1, \pi_1') = \overline{\mathbb{R}W} \oplus \overline{\mathbb{R}\pi_1'}$. We note that finite order elements in W and π_1' are then eliminated. The central series of π_1 gives a canonical expansion $L(Z) \sim L(\tilde{Z}) \underset{t}{\oplus} L_0(\pi_1(Z))$, where $L_0(\pi) = \underset{k}{\oplus} \mathbb{R}(\pi_1^{k+1}/\pi_1^k)$ which is easily shown to be a canonical graded Lie algebra.

As $\pi_1(Z) \approx H_0(L(Z))$, we have $H_0(L(\tilde{Z})) = 0$, and hence, $H_0(L(Z)) \approx L_0(\pi_1(Z))$. By applying this to the equivariant case considered in Sect.3, we get the interpretation of $L_0(P) \to P_1 \approx Z_1 \oplus \mathfrak{g}_1$ as the first order soldering and connection 1-forms: They depend only on $\pi_1(P)$ (related to $\pi_1(Z)$ by the exact sequence $\pi_1(G) \to \pi_1(P) \to \pi_1(Z) \to 0$, where $\pi_1(G)$ is abelian).

6 Superinduction and FP Ghosts

Let $R = \mathcal{A}_c(\overline{L(P)})$ be a Taylor-de Rham complex for a G-principal bundle $P \to Z$ (Sect.3): The action of the Lie algebra $\mathfrak{g}$ on $L(P)$ can be incorporated by augmenting $L(P)$ with $\mathfrak{g}_0 = \mathfrak{g}$ at degree 0, which produces a new Lie algebraic chain complex $L(P)^\theta$ with bracket $[\mathfrak{g},\ L_{-k}] \subset L_{-k}$, $[\mathfrak{g}_0,\ \mathfrak{g}_0] \subset \mathfrak{g}_0$ and augmentation $L(P)^\theta$ generates the algebraic cochain complex $\mathcal{U}(L(P)^\theta) \approx \mathfrak{U}(\mathfrak{g}_0) \otimes \mathcal{A}_c(\overline{L(P)})$ (partially algebraic isomorphism). If S is a linear representation of $\mathfrak{g}$, the usual induction from the fiber G to P gives a (Fock) space $\mathcal{F} = (R \underset{\mathfrak{U}(G)}{\otimes} S)_{\iota=0} = (R \otimes S)_{\theta=0,\ \iota=0}$ on Z. Actually if R is the usual de Rham complex $\Omega(P)$, $\mathcal{F}$ is the de Rham complex $\Omega(Z) \otimes S$ of S-valued p-forms on Z. If S is a superrepresentation of $\mathfrak{g}$, namely, it is also a $\wedge\mathfrak{g}_1$-module, or rather a $W_{\mathfrak{g}}$-module, the the superinduction is $R \underset{W_{\mathfrak{g}}}{\otimes} S$.

But the (non-commutative) Lie algebraic chain complex $L(P)^\theta$ is weakly homotopy equivalent to another such complex $L(P) + \mathfrak{g}_0$ with two successive augmentations [2], $\cdots \to L_{-1}(P) \to L_0(P) \to P_1 (= Z_1 \oplus \mathfrak{g}_1) \to \mathfrak{g}_2$. The bracket $[L_{-p},\ \mathfrak{g}_1] \subset L_{-p+1}$ generates ι in $\Omega(P)$ while $[\mathfrak{g}_2,\ \cdot] = 0$. The operator $\theta = [\mathfrak{g}_0,\ \cdot]$ is recovered from the identity $\theta = d\iota + \iota d = [\iota, d]$ where $d = \partial^* + \sum_k [L_k,\ \cdot]^*$. The algebraic cochain complex $\mathcal{A}_c(\overline{L(P) + \mathfrak{g}_2})$ is isomor-

phic to $\mathcal{A}_c(\overline{L(P)}) \otimes (\wedge\mathfrak{g}_1)$. For instance, for the universal principal bundle where $L(=\mathfrak{g}_1=\mathfrak{g}_2)$ is generated by FP ghost, we have $\mathcal{A}_c(\overline{L(P)}) = W_\mathfrak{g}$, $\mathcal{A}_c(\overline{L(P)+\mathfrak{g}_2}) \sim \wedge\mathfrak{g}_1$, and so the (co)homology is $H^*(\mathfrak{g})$ [13]. Actually, for any principal G-bundle P over Z, we get by superinduction $\Omega(Z) \sim (\mathcal{A}_c(\overline{L(P)+\mathfrak{g}_2}) \underset{\wedge\mathfrak{g}_1}{\otimes} W_\mathfrak{g})_{\theta=0,\ \iota=0}$. This means that FP ghosts give the homotopy type of the base.

Physically such p-forms (super-expansion) are quite natural as Radon-Nikodym derivatives of a vector $x \in X$ w.r.t. another (e.g., space-time) variable $y \in Y$: The existence of such correlation derivative $\partial_y x \in \Omega(Z) \otimes S$ means that X is fibered over Y locally trivially with fibre Z. The continuity w.r.t. the parameter space Y means that there is a decoupling $X \approx P_Y \underset{G}{\times} Z$ with a G-principal bundle P_Y over Y, where G acts on Z, and a (nonlinear) G-connection on P_Y. The bundle $\Omega(Z_y) \otimes S$ may not be flat. But the cohomology bundle $H(Z_y) \otimes S$ is flat over Y. In the noncommutative (quantized) case, the difference is that instead of fields (infinitesimal arrows) on Z we have arrows and connection of arrows. Then $\Omega(Z)$ becomes a noncommutative de Rham complex [14]. This will be developed in subsequent work.

7 Hodge Star and Spectral Decomposition

Since our Lie-algebraic cochain complexes L are degreewise finite-dimensional, we can safely take over to the present context the theory of Lie bialgebras [15], the details of which are skipped here. In particular, there is a cobracket $\delta_- : L_- \to L_- \otimes L_-$ for which $\partial : L_{-p} \to L_{-p+1}$ is also a derivation, and $\delta^+ \equiv \delta_-^* : L^+ \otimes L^+ \to L^+$ defines a Lie algebraic cochain complex. A self-duality $L \approx L^*$ gives a Hodge star $\partial^\dagger : L^p \to L^{p-1}$ which is a derivation for δ^+. On $\mathcal{F}^+ = \mathcal{A}_c(\overline{L^+})$ equipped with the derivation d generated by $\partial^* + \delta^+$, and the associated self-duality $\mathcal{A}_c(\overline{L^+}) \approx (\mathcal{A}_c(\overline{L_-}))^*$, we have the Hodge star $d^\dagger$ generated by $\partial^\dagger + \delta_-^\dagger$. As a sum of degree $-1$and degree $+1$operators, this is an odd graded. Let us consider the restriction $H_L : L^+ \to \mathcal{F}^+$ of the Hamiltonian $H = dd^\dagger + d^\dagger d$ ($\in$ Poisson algebra) on $\mathcal{F}^+$. By the derivation property of ∂, the mixed compositions in $(\partial^* + \delta^+)(\partial^\dagger + \delta_-^\dagger) + (\partial^\dagger + \delta_-^\dagger)(\partial^* + \delta^+)$ are shown to cancel out: $\delta^+\partial^\dagger + \partial^\dagger\delta^+ = 0 = \delta_-^\dagger\partial^* + \partial^*\delta_-^\dagger$. By the Poisson relations

$$\delta^+\delta_-^\dagger - (-)^{pq}\delta_-^\dagger\delta^+ = \delta_{pq} \tag{30}$$

we have $H_L = H' + N$, where $H' = \partial^*\partial^\dagger + \partial^\dagger\partial^*$ is the Hamiltonian on L and N is the degree-counting operator. The classical harmonic oscillator model shows that $\delta_-^\dagger$ increases the degree by 1 if p is odd, by 2 if p is even, i.e., it unwinds the degree as $\mathbb{Z} \to \mathbb{Z}_2$. So $[L^p,\ L^q] \subset L^{p+q}$. In the principal case where a Lie

algebra $\mathfrak{g}$ acts on L and commutes with H, we have also the action of $\mathfrak{g} = \mathfrak{g}_0$ on each L_{-q}.

Finally, we apply our theory to the problem of a $\mathbb{Z}_2$-graded supersymmetry $\Omega^{(0)} \underset{d}{\overset{d}{\rightleftarrows}} \Omega^{(1)}$ mentioned in [16], as to which kind of topological information can be obtained from such a supersymmetry. There the vanishing of torsion was required, but here with the help of homotopy model L we show that such a restriction can be replaced by the above spectral decomposition as follows. The first step is to associate a $\mathbb{Z}_2$-graded Lie-algebraic cochain complex $L_{(0)} \underset{\partial}{\overset{\partial}{\rightleftarrows}} L_{(1)}$: By our local expansion theory on cover $\mathcal{R}$, it will be obtained under the condition of locality for Ω. This means that it can be displayed on $\mathcal{R} = \{U_i\}$ with local derivatives ∂_i. This is quite a natural a priori condition. The second step is to have a spectral decomposition w.r.t. some Hilbert structure on Ω coming from one on L. It means $H_L = H' + N$ as above, which provides L with a $\mathbb{N}$-graded Lie algebraic cochain complex structure. So we have reached the

Theorem 8: A $\mathbb{Z}_2$-graded supersymmetry on a local anticommutative algebra Ω with Hilbert-norm completion $\overline{\Omega}$ determines a homotopy type π_k, k: integer ≥ 0.

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