# The $q$-twisted Cohomology and The $q$-hypergeometric Function at $|q| = 1$

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**Abstract.** We construct the $q$-twisted cohomology associated with the $q$-multiplicative function of Jordan-Pochhammer type at $|q| = 1$. In this framework, we prove the Heine's relations and a connection formula for the $q$-hypergeometric function of the Barnes type. We also prove an orthogonality relation of the $q$-little Jacobi polynomials at $|q| = 1$.

## 1 Introduction

In this paper we construct the $q$-twisted cohomology at $|q| = 1$ in Jordan-Pochhammer case and prove some properties of the $q$-hypergeometric function at $|q| = 1$ defined in [NU].

The basic hypergeometric function with $0 < |q| < 1$ [GR] is represented in terms of a Jackson integral. In [A1, AK], a formulation of Jackson integrals is given. Namely, for a $q$-multiplicative function defined by means of $q$-version of Sato's $b$-functions [Sa], the $q$-twisted cohomology is defined. In this approach, Jackson integrals can be regarded as a pairing between this cohomology and $q$-cycles.

We consider the case that $|q| = 1$ and $q$ is not a root of unity. Then the structure of $b$-functions is the same as in the case of $0 < |q| < 1$ and associated $q$-multiplicative function can be constructed in terms of the double sine function [B]. The problem is to define a suitable integral which is a certain generalization of Jackson integrals to the case of $|q| = 1$.

In [MT], a family of solutions to the quantum Knizhnik-Zamolodchikov ($q$KZ) equation at $|q| = 1$ was constructed. The solution is represented in terms of a pairing between two functional spaces, which is called the hypergeometric pairing. The hypergeometric pairing was defined by Tarasov and Varchenko in the study of the rational $q$KZ [TV1] and the trigonometric $q$KZ for $0 < |q| < 1$ [TV2]. In the trigonometric case, the hypergeometric pairing is a pairing between a space of trigonometric functions and that of elliptic functions. It is given by an integral over a closed contour with a kernel function defined by a product of the infinite product with step $q$. This kernel function is the $q$-multiplicative funtion mentioned above. By taking residues, we can represent this integral in terms of a Jackson integral.

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In the case of $|q| = 1$, the hypergeometric pairing is a pairing between two spaces of trigonometric functions which depend on two respective values of a deformation parameter

$$q = e^{2\pi i\omega} \quad \text{and} \quad Q = e^{\frac{2\pi i}{\omega}}.$$

Hence the pairing induces certain duality of two functional spaces at $|q| = 1$. This type of duality has appeared in mathmatical physics: for example, modular double of quantum group [F] and matrix elements in quantum Toda chain [Sm].

In this paper we define an integral associated with the $q$-multiplicative function of Jordan-Pochhammer type at $|q| = 1$ in a similar way to [MT]. This integral is regarded as a pairing between two functional spaces which depend on $q$ and $Q$, respectively. Then we can define a cohomology of these spaces associated with this integral. In this way we get the $q$-twisted cohomology at $|q| = 1$. From this point of view, we can prove some relations satisfied by the $q$-hypergeometric function of the Barnes type at $|q| = 1$.

The plan of the paper is as follows. In Section 2, we recall the result of $q$-analogue of $b$-funcions following [A2] and define a $q$-multiplicative function at $|q| = 1$. In Section 3, we construct the $q$-twisted cohomology associated with the $q$-multiplicative function of Jordan-Pochhammer type. In this case we can find a basis of the cohomology. In order to prove linear independence, we write down the formula for a determinant, see (3.23). This formula gives us the $q$-Beta integral formula at $|q| = 1$ in a special case. In Section 4, we prove two properties of the $q$-hypergeometric function at $|q| = 1$: Heine's relations and a connection formula. Section 5 is additional one. We discuss the $q$-little Jacobi polynomials and their orthogonality with respect to the kernel of the $q$-Beta integral at $|q| = 1$ given in Section 3.

# 2 The $q$-multiplicative function at $|q| = 1$

Let $q$ be a nonzero complex number. In this paper, we consider the case that $|q| = 1$ and $q$ is not a root of unity. We put $q = e^{2\pi i\omega}$ $(\omega > 0, \omega \notin \mathbb{Q})$.

Let $L$ be an $l$ dimensional integer lattice in $\mathbb{C}^l$:

$$L := \{\chi = (\chi_1, \cdots, \chi_l) | \chi_j \in \mathbb{Z}, j = 1, \cdots, l\} \in \mathbb{C}^l. \tag{2.1}$$

For a set of nonzero rational functions $\{b_\chi(t)\}_{\chi \in L}$, where

$$b_\chi(t) = b_\chi(t_1, \cdots, t_l) \in \mathbb{C}(t_1, \cdots, t_l)^\times, \tag{2.2}$$

we consider the following system of difference equations:

$$\Phi(z + \chi) = b_\chi(t)\Phi(z), \quad (\chi \in L), \tag{2.3}$$

where $z = (z_1, \cdots, z_l) \in \mathbb{C}^l$ and

$$t = (t_1, \cdots, t_l) := (e^{2\pi i\omega z_1}, \cdots, e^{2\pi i\omega z_l}). \tag{2.4}$$

The compatibility condition of (2.3) implies

$$b_0(t) = 1, \tag{2.5}$$

$$b_{\chi+\chi'}(t) = b_\chi(t) b_{\chi'}(q^\chi \cdot t) \quad \text{for any } \chi, \chi' \in L, \tag{2.6}$$

where $q^{\chi}\cdot t = (q^{\chi_1}t_1, \cdots, q^{\chi_l}t_l)$. The conditions (2.5) and (2.6) mean that the set $\{b_\chi(t)\}_{\chi\in L}$ defines a *1-cocycle*. A set $\{b_\chi(t)\}_{\chi\in L}$ is said to be a *1-coboundary* if and only if there exists a nonzero rational funcion $\varphi(t)$ such that

$$b_\chi(t) = \frac{\varphi(q^\chi \cdot t)}{\varphi(t)} \quad \text{for any } \chi \in L. \tag{2.7}$$

Let us consider the quotient $H^1 :=\{\text{1-cocycles}\}/\{\text{1-coboundaries}\}$. $H^1$ has a mutiplicative group structure. For $\mu \in L^* := \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$, we put

$$\mu(m) = \mu(0, \cdots, \overset{m-\text{th}}{1}, \cdots, 0) \in \mathbb{Z} \tag{2.8}$$

and

$$t^\mu = t_1^{\mu(1)} \cdots t_l^{\mu(l)}. \tag{2.9}$$

Note that $\mu(\chi) = \sum_{m=1}^{l} \mu(m)\chi_m$ for $\chi = (\chi_1, \cdots, \chi_l) \in L$.

Then the following result holds.

**Proposition 2.1** *$H^1$ is represented by cocycles of the following form:*

$$b_\chi(t) = a_\chi \prod_{\nu=0}^{\mu_0(\chi)-1} (q^\nu t^{\mu_0}) \frac{\prod_{j=1}^{k} (q^{\gamma_j} t^{\mu_j}; q)_{\mu_j(\chi)}}{\prod_{j=1}^{k'} (q^{\gamma'_j} t^{\mu'_j}; q)_{\mu'_j(\chi)}} \tag{2.10}$$

*for $\mu_0, \mu_j, \mu'_j \in L^*$ and $\gamma_j, \gamma'_j \in \mathbb{C}$. Here $\{a_\chi\}_{\chi\in L}$ is a set of nonzero constants satisfying $a_{\chi+\chi'} = a_\chi a_{\chi'}$ for any $\chi, \chi' \in L$, and*

$$(x;q)_n := \begin{cases} \prod_{j=1}^{n-1} (1 - xq^j), & \text{for } n \geqslant 0, \\ \prod_{j=1}^{-n} (1 - xq^{-j})^{-1}, & \text{for } n < 0. \end{cases} \tag{2.11}$$

*The expression (2.10) is not unique.*

This result is a $q$-analogue of Sato's result [Sa] and was stated by Aomoto in [A2]. In [A2], Aomoto stated this result in the case of $0 < |q| < 1$. However, it can be checked that the proposition holds unless $q$ is a root of unity.

Let us find a solution $\Phi(z)$ to (2.3) for the 1-cocycle $\{b_\chi(t)\}_{\chi\in L}$ given by (2.10). Following [N], we set

$$\langle x \rangle := \exp\left(\frac{\pi i}{2}\left((1+\omega)x - \omega x^2\right)\right) S_2(x|1, \frac{1}{\omega}), \tag{2.12}$$

where $S_2(x)$ is the double sine function. We refer the reader to [JM] for the double sine function. Moreover, we define a funcion $\sigma(x)$ by

$$\sigma(x) := \exp\left(\pi i \left((1+\omega)x - \omega x^2\right)\right) = \langle x \rangle \langle 1 + \frac{1}{\omega} - x \rangle. \tag{2.13}$$

These functions satisfy

$$\frac{\langle x+1\rangle}{\langle x\rangle} = \frac{1}{1-e^{2\pi i\omega x}}, \quad \frac{\sigma(x+1)}{\sigma(x)} = -e^{-2\pi i\omega x}. \tag{2.14}$$

For $\mu \in L^*$, we set

$$\mu(z) := \sum_{m=1}^{l} \mu(m) z_m. \tag{2.15}$$

Then we have

$$\frac{\langle \mu(z+\chi)+\gamma\rangle}{\langle \mu(z)+\gamma\rangle} = \frac{1}{(q^\gamma t^\mu;q)_{\mu(\chi)}}, \quad \frac{\sigma(\mu(z+\chi))}{\sigma(\mu(z))} = (-1)^{\mu(\chi)} \prod_{\nu=0}^{\mu(\chi)-1} (q^\nu t^\mu)^{-1}. \tag{2.16}$$

Hence, we can get a solution to (2.3) in the following form:

$$\Phi(z) = t_1^{\alpha_1}\cdots t_l^{\alpha_l} \frac{\prod_{j=1}^{n'} \langle \mu'_j(z)+\gamma'_j\rangle}{\prod_{j=1}^{n} \langle \mu_j(z)+\gamma_j\rangle}, \tag{2.17}$$

where $\alpha_m, \gamma_j, \gamma'_j \in \mathbb{C}$, $\mu_j, \mu'_j \in L^*$.

A function $\Phi(z)$ of type (2.17) is called a *q-multiplicative function* at $|q|=1$.

# 3 The $q$-twisted cohomology in Jordan-Pochhammer case

Let us consider the $q$-multiplicative function of Jordan-Pochhammer type given by

$$\Phi(z) = t^\alpha \prod_{j=1}^{n} \frac{\langle z+\gamma'_j\rangle}{\langle z+\gamma_j\rangle}, \tag{3.1}$$

where $z \in \mathbb{C}$, $t = e^{2\pi i\omega z} = q^z$ and $\alpha, \gamma_j, \gamma'_j \in \mathbb{C}$. We assume $\gamma_j \neq \gamma'_k$ for any $j,k = 1,\cdots,n$.

We denote by $D, D_j$ and $D'_j$ the difference operators corresponding to the displacements $\alpha \mapsto \alpha+1, \gamma_j \mapsto \gamma_j+1$ and $\gamma'_j \mapsto \gamma'_j+1$, respectively. Let $\mathcal{A}$ be the commutative algebra generated by $D^{\pm1}, D_j^{\pm1}$ and $D_j'^{\pm1}$ $(j=1,\cdots,n)$ over $\mathbb{C}$. We define a subspace $Z$ of $\mathbb{C}(t)$ by

$$Z := \{(\kappa\Phi)/\Phi \,|\, \kappa \in \mathcal{A}\}. \tag{3.2}$$

It is easy to see that

$$Z = \left\{ \frac{f(t)}{\prod_{j=1}^{n} (c_j q^{-\ell_j} t;q)_{\ell_j} (c'_j t;q)_{\ell'_j}} \,\middle|\, f(t) \in \mathbb{C}[t,t^{-1}] \text{and } \ell_j, \ell'_j \in \mathbb{Z}_{\geqslant 0} \right\}, \tag{3.3}$$

where $c_j = e^{2\pi i\omega\gamma_j}$ and $c'_j = e^{2\pi i\omega\gamma'_j}$.

Next we define another space of rational functions. An important point is that the function $\langle x \rangle$ satisfies also the following functional relation:

$$\frac{\langle x+\frac{1}{\omega}\rangle}{\langle x\rangle} = \frac{1}{1-e^{2\pi i x}}. \tag{3.4}$$

We denote by $\widetilde{D}, \widetilde{D}_j$ and $\widetilde{D}'_j$ the difference operators corresponding to the displacements $\alpha \mapsto \alpha + \frac{1}{\omega}, \gamma_j \mapsto \gamma_j + \frac{1}{\omega}$ and $\gamma'_j \mapsto \gamma'_j + \frac{1}{\omega}$, respectively. In the same way as before, we consider the commutative algebra $\tilde{\mathcal{A}}$ generated by $\widetilde{D}^{\pm 1}, \widetilde{D}_j{}^{\pm 1}$ and $\widetilde{D}'_j{}^{\pm 1}$ $(j = 1, \cdots, n)$ over $\mathbb{C}$ and set

$$\tilde{Z} := \{(\tilde{\kappa}\Phi)/\Phi | \tilde{\kappa} \in \tilde{\mathcal{A}}\}. \tag{3.5}$$

Then we have

$$\tilde{Z} = \left\{ \frac{\tilde{f}(T)}{\prod_{j=1}^n (C_j Q^{-\widetilde{\ell}_j} T; Q)_{\widetilde{\ell}_j} (C'_j T; Q)_{\widetilde{\ell}'_j}} \middle| \tilde{f}(T) \in \mathbb{C}[T, T^{-1}] \text{ and } \widetilde{\ell}_j, \widetilde{\ell}'_j \in \mathbb{Z}_{\geqslant 0} \right\}, \tag{3.6}$$

where $Q = e^{\frac{2\pi i}{\omega}}, T = e^{2\pi i z}, C_j = e^{2\pi i \gamma_j}$ and $C'_j = e^{2\pi i \gamma'_j}$.

Now we define a pairing between $Z$ and $\tilde{Z}$. For

$$\varphi(t) = \frac{t^m}{\prod_{j=1}^n (c_j q^{-\ell_j} t; q)_{\ell_j} (c'_j t; q)_{\ell'_j}} \in Z \tag{3.7}$$

and

$$\tilde{\varphi}(T) = \frac{T^{\widetilde{m}}}{\prod_{j=1}^n (C_j Q^{-\widetilde{\ell}_j} T; Q)_{\widetilde{\ell}_j} (C'_j T; Q)_{\widetilde{\ell}'_j}} \in \tilde{Z}, \tag{3.8}$$

we set

$$I(\varphi, \tilde{\varphi}) := \int_C dz \Phi(z) \varphi(t) \tilde{\varphi}(T). \tag{3.9}$$

Here the contour $C$ is taken to be the imaginary axis $(-i\infty, i\infty)$ except that the poles at

$$-\gamma_j + \ell_j + \frac{\widetilde{\ell}_j}{\omega} + \mathbb{Z}_{\leqslant 0} + \frac{1}{\omega}\mathbb{Z}_{\leqslant 0} \quad (j = 1, \cdots, n) \tag{3.10}$$

are on the left of $C$ and the poles at

$$-\gamma'_j - \ell'_j - \frac{\widetilde{\ell}'_j}{\omega} + \mathbb{Z}_{\geqslant 1} + \frac{1}{\omega}\mathbb{Z}_{\geqslant 1} \quad (j = 1, \cdots, n) \tag{3.11}$$

are on the right of $C$. Then the integral (3.9) is absolutely convergent if

$$0 < \mathrm{Re}\alpha + m + \frac{\widetilde{m}}{\omega} < \sum_{j=1}^n (\mathrm{Re}\gamma_j - \mathrm{Re}\gamma'_j) + \sum_{j=1}^n (\ell_j + \ell'_j) + \frac{1}{\omega}\sum_{j=1}^n (\widetilde{\ell}_j + \widetilde{\ell}'_j). \tag{3.12}$$

Under the condition (3.12) the integrand of (3.9) decreases exponentially as $z \to \pm i\infty$.

Let us consider cohomologies of $Z$ and $\widetilde{Z}$ associated with the integral (3.9). We set

$$B := \mathrm{span}_{\mathbb{C}} \left\{ \psi(t) - b_\chi(t)\psi(q^\chi t) | \psi(t) \in Z, \chi \in \mathbb{Z} \right\}, \tag{3.13}$$

where $b_\chi(t) = \Phi(z+\chi)/\Phi(z)$. Note that for any $\psi(t) \in Z, \widetilde{\varphi}(T) \in \widetilde{Z}$ and $\chi \in \mathbb{Z}$ we can deform the contour $C$ so that there are no poles of the function $\Phi(z)\psi(t)\widetilde{\varphi}(T)$ between $C$ and $C+\chi$. Thus we have

$$\int_C dz\Phi(z)\{\psi(t) - b_\chi(t)\psi(q^\chi t)\}\widetilde{\varphi}(T) = \left( \int_C - \int_{C+\chi} \right) dz\Phi(z)\psi(t)\widetilde{\varphi}(T) = 0, \tag{3.14}$$

if all the integrals are convergent. Here we used the fact that $T = e^{2\pi i z}$ is invariant under the change $z \to z - \chi$. Hence, we find

$$I(\varphi_0, \widetilde{\varphi}) = 0 \quad \text{for } \varphi_0 \in B \text{ and } \widetilde{\varphi} \in \widetilde{Z}. \tag{3.15}$$

Similarly, we set

$$\widetilde{B} := \mathrm{span}_{\mathbb{C}} \left\{ \widetilde{\psi}(T) - \widetilde{b_\chi}(T)\widetilde{\psi}(Q^\chi T) | \widetilde{\psi}(T) \in \widetilde{Z}, \chi \in \mathbb{Z} \right\}, \tag{3.16}$$

where $\widetilde{b_\chi}(T) := \Phi(z + \frac{\chi}{\omega})/\Phi(z)$. Then we have

$$I(\varphi, \widetilde{\varphi_0}) = 0 \quad \text{for } \varphi \in Z \text{ and } \widetilde{\varphi_0} \in \widetilde{B}. \tag{3.17}$$

From these relations, we define the *q-twisted cohomology* $H$ and $\widetilde{H}$ by

$$H := Z/B \quad \text{and} \quad \widetilde{H} := \widetilde{Z}/\widetilde{B}. \tag{3.18}$$

We note that the structure of the cohomology $H$ is determined by the parameters $q, q^\alpha, c_j$ and $c'_j\,(j = 1, \cdots, n)$. We write down this dependence explicitly as

$$H = \mathrm{H}(q|q^\alpha; c_1, \cdots, c_n; c'_1, \cdots, c'_n). \tag{3.19}$$

Then $\widetilde{H}$ is written as

$$\widetilde{H} = \mathrm{H}(Q|Q^{\omega\alpha}; C_1, \cdots, C_n; C'_1, \cdots, C'_n). \tag{3.20}$$

It is easy to see that the following proposition holds.

**Proposition 3.1** *The cohomology* $\mathrm{H}(q|q^\alpha; c_1, \cdots, c_n; c'_1, \cdots, c'_n)$ *is generated by*

$$\left\{ \frac{1}{1 - c'_j t} \Big| j = 1, \cdots, n \right\}, \tag{3.21}$$

*if the parameters* $q, q^\alpha, c_j$ *and* $c'_j\,(j = 1, \cdots, n)$ *are generic.*

Moreover, we see that the set $\{\frac{1}{1-c'_j t}\}_{j=1,\cdots,n}$ is a basis of $H$ from the following determinant formula.

**Proposition 3.2** *Set*

$$\varphi_j(t) = \frac{1}{1-c'_j t}, \quad \widetilde{\varphi}_j(T) = \frac{1}{1-C'_j T}, \quad (j = 1, \cdots, n). \tag{3.22}$$

*Then*

$$\begin{aligned} \det\left(I(\varphi_j, \widetilde{\varphi}_k)\right)_{j,k=1,\cdots,n} &= \langle 1\rangle^n \exp\left(-2\pi i\omega\alpha \sum_{j=1}^n \gamma'_j\right) \\ &\times \frac{\langle \alpha + \sum_{j=1}^n \gamma_j - \sum_{j=1}^n \gamma'_j\rangle}{\langle\alpha\rangle \prod_{j,k=1}^n \langle \gamma_j - \gamma'_k\rangle} \\ &\times \prod_{1\leqslant j<k\leqslant n} \frac{(1-e^{2\pi i\omega(\gamma'_j-\gamma'_k)})(1-e^{2\pi i(\gamma'_j-\gamma'_k)})}{\sigma(\gamma'_j-\gamma'_k)}. \end{aligned} \tag{3.23}$$

*Proof.* First we set

$$\psi_k(t) = \frac{1}{1-c'_k t}\prod_{j=1}^{k-1}\frac{1-c_j t}{1-c'_j t}, \quad \widetilde{\psi}_k(T) = \frac{1}{1-C'_k T}\prod_{j=1}^{k-1}\frac{1-C_j T}{1-C'_j T}, \quad (j = 1, \cdots, n). \tag{3.24}$$

Then we have

$$\psi_k(t) = \sum_{p=1}^{k-1}\frac{1-c_p/c'_p}{1-c'_k/c'_p}\prod_{\substack{j=1\\ j\neq p}}^{k-1}\frac{1-c_j/c'_p}{1-c'_j/c'_p}\varphi_p(t) + \prod_{j=1}^{k-1}\frac{1-c_j/c'_k}{1-c'_j/c'_k}\varphi_k(t) \tag{3.25}$$

and a similar formula for $\widetilde{\psi}_k(T)$.

Hence we find

$$\det\left(I(\varphi_j, \widetilde{\varphi}_k)\right) = \prod_{1\leqslant j<k\leqslant n}\left(\frac{1-c'_j/c'_k}{1-c_j/c'_k}\frac{1-C'_j/C'_k}{1-C_j/C'_k}\right)\det\left(I(\psi_j, \widetilde{\psi}_k)\right). \tag{3.26}$$

The determinant in the right hand side of (3.26) is a special case of the determinant discussed in [MT]. Combining the result in [MT] and (3.26), we get the formula (3.23). □

*Remark.* In the case of $n = 1$ and $\gamma'_1 = 0$, the formula (3.23) is represented as follows:

$$\int_C q^{\alpha z}\frac{\langle z\rangle}{\langle z+\beta\rangle}\frac{1}{1-t}\frac{1}{1-T}dz = \int_C q^{\alpha z}\frac{\langle z+1+\frac{1}{\omega}\rangle}{\langle z+\beta\rangle}dz = \frac{\langle 1\rangle\langle\alpha+\beta\rangle}{\langle\alpha\rangle\langle\beta\rangle}. \tag{3.27}$$

We may call (3.27) the *q-Beta integral formula* at $|q| = 1$.

To finish this section we find a system of difference equation in $\alpha$ satisfied by the function

$$\Psi(\alpha) = \Psi(\alpha|\widetilde{\varphi}) := \int_C dz\Phi(z)\widetilde{\varphi}(T) \quad \text{for}\, \widetilde{\varphi}(T) \in \widetilde{H} \tag{3.28}$$

in a similar manner to [AK].

For the $q$-multiplicative function (3.1), we can represent the funcion $b_\chi(t) = \frac{\Phi(z+\chi)}{\Phi(z)}$ $(\chi \in \mathbb{Z})$ as follows:

$$b_\chi(t) = q^{\chi\alpha} \frac{b_\chi^+(t)}{b_\chi^-(t)}, \tag{3.29}$$

where $b_\chi^+(t)$ and $b_\chi^-(t)$ are polynomials in $t$ and have no common factor. For example, if $\chi = 1$ we have

$$b_1^+(t) = \prod_{j=1}^{n}(1 - c_j t), \quad b_1^-(t) = \prod_{j=1}^{n}(1 - c_j' t). \tag{3.30}$$

By setting $\psi(t) = b_\chi^-(q^{-\chi}t)$ in (3.13), we find

$$b_\chi^-(q^{-\chi}t) - q^{\chi\alpha} b_\chi^+(t) \in B. \tag{3.31}$$

Note that $t\,\Phi = D\Phi$, where $D$ is the difference operator defined by $\alpha \mapsto \alpha + 1$. Therefore, we get

$$\left\{ b_\chi^-(q^{-\chi}D) - q^{\chi\alpha} b_\chi^+(D) \right\} \Psi = 0 \tag{3.32}$$

for $\chi \in \mathbb{Z}$ such that $b_\chi^-(q^{-\chi}D)\Psi$ and $b_\chi^+(D)\Psi$ are defined. These equations are $q$-analogues of Mellin-Sato hypergeometric equations [AK] at $|q| = 1$. In the case of $\chi = 1$, the equation (3.32) is given by

$$\left\{ \prod_{j=1}^{n}(1 - q^{-1}c_j' D) - q^\alpha \prod_{j=1}^{n}(1 - c_j D) \right\} \Psi = 0. \tag{3.33}$$

# 4 Application to the $q$-hypergeometric function

## 4.1 Preliminaries

Following [GR], we recall some properties of the basic hypergeometric series with $0 < |q| < 1$ given by

$$\phi(a, b, c; t) := \sum_{k=0}^{\infty} \frac{(a;q)_k (b;q)_k}{(q;q)_k (c;q)_k} t^k \tag{4.1}$$

for $|t| < 1$.

This function satisfies the Heine's relations:

$$\phi(a, b, q^{-1}c) - \phi(a, b, c) = tc \frac{(1-a)(1-b)}{(q-c)(1-c)} \phi(qa, qb, qc), \tag{4.2}$$

$$\phi(qa, b, c) - \phi(a, b, c) = ta \frac{1-b}{1-c} \phi(qa, qb, qc), \tag{4.3}$$

$$\phi(qa, q^{-1}b, c) - \phi(a, b, c) = q^{-1} t \frac{aq - b}{1-c} \phi(qa, b, qc). \tag{4.4}$$

Here we abbreviated $\phi(a,b,c;x)$ to $\phi(a,b,c)$.

It also satisfies a connection formula:

$$\begin{aligned}\phi(a,b,c;t) &= \frac{(b)_\infty (c/a)_\infty}{(c)_\infty (b/a)_\infty}\frac{\Theta(at)}{\Theta(t)}\phi(a,qa/c,qa/b;qc/abt)\\ &+ \frac{(a)_\infty (c/b)_\infty}{(c)_\infty (a/b)_\infty}\frac{\Theta(bt)}{\Theta(t)}\phi(b,qb/c,qb/a;qc/abt),\end{aligned} \tag{4.5}$$

where $(a)_\infty := \prod_{j=1}^{\infty}(1-q^{j-1}a)$ and $\Theta(x) := (q)_\infty (x)_\infty (q/x)_\infty$.

Now we consider the $q$-hypergeometric function of the Barnes type at $|q|=1$ [NU], which is defined as follows in our notation:

$$\Psi(\alpha,\beta,\gamma;x) := \frac{\langle\alpha\rangle\langle\beta\rangle}{\langle 1\rangle\langle\gamma\rangle}\left(-\frac{1}{2\pi i}\right)\int_{C_0}\frac{\langle z+1\rangle\langle z+\gamma\rangle}{\langle z+\alpha\rangle\langle z+\beta\rangle}\frac{\pi(-q^x)^z}{\sin\pi z}dz, \tag{4.6}$$

where $-q^x = e^{2\pi i\omega x+\pi i}$ and the contour $C_0$ is the imaginary axis $(-i\infty, i\infty)$ except that the poles at

$$-\alpha+\mathbb{Z}_{\leqslant 0}+\frac{1}{\omega}\mathbb{Z}_{\leqslant 0}, \quad -\beta+\mathbb{Z}_{\leqslant 0}+\frac{1}{\omega}\mathbb{Z}_{\leqslant 0} \tag{4.7}$$

are on the left of $C_0$ and the poles at

$$\mathbb{Z}_{\geqslant 0}+\frac{1}{\omega}\mathbb{Z}_{\geqslant 0}, \quad -\gamma+\mathbb{Z}_{\geqslant 1}+\frac{1}{\omega}\mathbb{Z}_{\geqslant 1} \tag{4.8}$$

are on the right of $C_0$.

By using

$$\left(-\frac{1}{2\pi i}\right)\frac{\pi(-q^x)^z}{\sin\pi z} = q^{xz}\frac{1}{1-e^{2\pi i z}}, \tag{4.9}$$

we can rewrite (4.6) as follows:

$$\Psi(\alpha,\beta,\gamma;x) = \frac{\langle\alpha\rangle\langle\beta\rangle}{\langle 1\rangle\langle\gamma\rangle}\int_{C_0} q^{xz}\frac{\langle z+1+\frac{1}{\omega}\rangle\langle z+\gamma\rangle}{\langle z+\alpha\rangle\langle z+\beta\rangle}dz. \tag{4.10}$$

Now we denote by $\Phi(z)$ the integrand of (4.10):

$$\Phi(z) = q^{xz}\frac{\langle z+1+\frac{1}{\omega}\rangle\langle z+\gamma\rangle}{\langle z+\alpha\rangle\langle z+\beta\rangle}. \tag{4.11}$$

This function $\Phi(z)$ is the $q$-multiplicative function of Jordan-Pochhammer type. From (3.12), the integral (4.10) is absolutely convergent if

$$0 < \mathrm{Re}x < 1+\frac{1}{\omega}+\mathrm{Re}\gamma-\mathrm{Re}\alpha-\mathrm{Re}\beta. \tag{4.12}$$

In this case, the equation (3.33) is nothing but the hypergeometric difference equation at $|q|=1$:

$$\left\{(1-D)(1-q^{\gamma-1}D)-q^x(1-q^\alpha D)(1-q^\beta D)\right\}\Psi = 0, \tag{4.13}$$

where $D$ is the difference operator defined by $x \mapsto x+1$.

For the $q$-multiplicative function (4.10), we define two functional spaces $Z$ and $\widetilde{Z}$ as in the previous section, and set

$$\Psi(\alpha,\beta,\gamma;x|\widetilde{\varphi}) := \frac{\langle\alpha\rangle\langle\beta\rangle}{\langle 1\rangle\langle\gamma\rangle}\int_{C_0}\Phi(z)\widetilde{\varphi}(T)dz \qquad \text{for} \quad \widetilde{\varphi}\in\widetilde{Z}. \tag{4.14}$$

Note that $\Psi(\alpha,\beta,\gamma;x)=\Psi(\alpha,\beta,\gamma;x|1)$. Then the function (4.14) also satisfies (4.13).

For simplicity's sake, hereafter we use the following notation:

$$t=e^{2\pi i\omega z},\ a=e^{2\pi i\omega\alpha},\ b=e^{2\pi i\omega\beta},\ c=e^{2\pi i\omega\gamma} \tag{4.15}$$

and

$$T=e^{2\pi iz},\ A=e^{2\pi i\alpha},\ B=e^{2\pi i\beta},\ C=e^{2\pi i\gamma}. \tag{4.16}$$

## 4.2 Heine's relations

**Proposition 4.1** *We abbreviate $\Psi(\alpha,\beta,\gamma;x)$ to $\Psi(\alpha,\beta,\gamma)$. Then the following equalities hold:*

$$\Psi(\alpha,\beta,\gamma-1)-\Psi(\alpha,\beta,\gamma)=q^x c\frac{(1-a)(1-b)}{(q-c)(1-c)}\Psi(\alpha+1,\beta+1,\gamma+1), \tag{4.17}$$

$$\Psi(\alpha+1,\beta,\gamma)-\Psi(\alpha,\beta,\gamma)=q^x a\frac{1-b}{1-c}\Psi(\alpha+1,\beta+1,\gamma+1), \tag{4.18}$$

$$\Psi(\alpha+1,\beta-1,\gamma)-\Psi(\alpha,\beta,\gamma)=q^{x-1}\frac{aq-b}{1-c}\Psi(\alpha+1,\beta,\gamma+1). \tag{4.19}$$

*Proof.* For $f(t)\in Z$, we set

$$[f] := \frac{\langle\alpha\rangle\langle\beta\rangle}{\langle 1\rangle\langle\gamma\rangle}\int_{C_0}\Phi(z)f(t)dz. \tag{4.20}$$

First we prove (4.17). It is easy to see that

$$\Psi(\alpha,\beta,\gamma-1)=\left[\frac{1-q^{-1}ct}{1-q^{-1}c}\right],\quad \Psi(\alpha,\beta,\gamma)=[1]. \tag{4.21}$$

Hence, we have

$$\Psi(\alpha,\beta,\gamma-1)-\Psi(\alpha,\beta,\gamma)=\left[\frac{c(1-t)}{q-c}\right]. \tag{4.22}$$

On the other hand, by changing the variable $z\to z-1$, we find

$$\Psi(\alpha+1,\beta+1,\gamma+1)=\left[q^{-x}\frac{(1-c)(1-t)}{(1-a)(1-b)}\right]. \tag{4.23}$$

From (4.22) and (4.23), we get (4.17). We can prove (4.18) in the same way as above.

Next we prove (4.19). By changing the variable $z \to z+1$, we have

$$\Psi(\alpha+1,\beta-1,\gamma) = \left[ q^{x-1} \frac{(q-b)(1-at)(1-qat)}{(1-a)(1-qt)(1-ct)} \right]. \tag{4.24}$$

It is easy to see that

$$\Psi(\alpha,\beta,\gamma) = [1], \quad \Psi(\alpha+1,\beta,\gamma+1) = \left[ \frac{(1-c)(1-at)}{(1-a)(1-ct)} \right]. \tag{4.25}$$

By using this, we can find the following:

$$\begin{aligned} &\Psi(\alpha+1,\beta-1,\gamma) - \Psi(\alpha,\beta,\gamma) - q^{x-1}\frac{aq-b}{1-c}\Psi(\alpha+1,\beta,\gamma+1) \\ &\qquad = \left[ -1 + q^x \frac{(1-at)(1-bt)}{(1-qt)(1-ct)} \right]. \end{aligned} \tag{4.26}$$

Note that

$$-1 + q^x \frac{(1-at)(1-bt)}{(1-qt)(1-ct)} = -\{1 - b_1(t)\cdot 1\} \in B. \tag{4.27}$$

Therefore, (4.26) equals to 0. This completes the proof of (4.19). $\square$

In the proof above, we see that Heine's relations come from some relations in $H$. Hence, we find that the funcion $\Psi(\alpha,\beta,\gamma;x|\widetilde{\varphi})$ (4.14) also satisfies Heine's relations.

## 4.3 Connection formula

**Proposition 4.2**

$$\begin{aligned} &\Psi(\alpha,\beta,\gamma;x) \\ &= \frac{\langle\beta\rangle\langle\gamma-\alpha\rangle}{\langle\gamma\rangle\langle\beta-\alpha\rangle}\frac{\sigma(x+\alpha)}{\sigma(x)}\Psi(\alpha,1+\alpha-\gamma,1+\alpha-\beta;1+\frac{1}{\omega}+\gamma-\alpha-\beta-x) \\ &+ \frac{\langle\alpha\rangle\langle\gamma-\beta\rangle}{\langle\gamma\rangle\langle\alpha-\beta\rangle}\frac{\sigma(x+\beta)}{\sigma(x)}\Psi(\beta,1+\beta-\gamma,1+\beta-\alpha;1+\frac{1}{\omega}+\gamma-\alpha-\beta-x). \end{aligned} \tag{4.28}$$

*Proof.* We rewrite the integral

$$\begin{aligned} &\Psi(\alpha,1+\alpha-\gamma,1+\alpha-\beta;1+\frac{1}{\omega}+\gamma-\alpha-\beta-x) \\ &\qquad = \frac{\langle\alpha\rangle}{\langle 1\rangle}\int_{C_0'} q^{(1+\frac{1}{\omega}+\gamma-\alpha-\beta-x)z}\frac{\langle 1+\alpha-\gamma\rangle}{\langle 1+\alpha-\beta\rangle}\frac{\langle z+1+\frac{1}{\omega}\rangle\langle z+1+\alpha-\beta\rangle}{\langle z+\alpha\rangle\langle z+1+\alpha-\gamma\rangle}dz, \end{aligned} \tag{4.29}$$

where $C_0'$ is the contour associated with the set of parameters $(\alpha, 1+\alpha-\gamma, 1+\alpha-\beta)$.

By changing the variable $z \to -z-\alpha$, we have

$$(4.30) = \frac{\langle\alpha\rangle}{\langle 1\rangle}\int_{C_0} q^{(1+\frac{1}{\omega}+\gamma-\alpha-\beta-x)(-z-\alpha)}\frac{\langle 1+\alpha-\gamma\rangle}{\langle 1+\alpha-\beta\rangle}\frac{\langle -z-\alpha+1+\frac{1}{\omega}\rangle\langle -z+1-\beta\rangle}{\langle -z\rangle\langle -z+1-\gamma\rangle}dz, \tag{4.30}$$

where $C_0$ is the contour defined in (4.6). By using (2.13), we have

$$
\begin{aligned}
(4.31)\qquad &\text{the integrand of (4.30)}\\
&= q^{(1+\frac{1}{\omega}+\gamma-\alpha-\beta-x)(-z-\alpha)}\frac{\sigma(1+\alpha-\gamma)}{\sigma(1+\alpha-\beta)}\frac{\sigma(-z-\alpha+1+\frac{1}{\omega})\sigma(-z+1-\beta)}{\sigma(-z)\sigma(-z+1-\gamma)}\\
&\times\frac{\langle\frac{1}{\omega}+\beta-\alpha\rangle}{\langle\frac{1}{\omega}+\gamma-\alpha\rangle}\frac{\langle z+1+\frac{1}{\omega}\rangle\langle z+\gamma+\frac{1}{\omega}\rangle}{\langle z+\alpha\rangle\langle z+\beta+\frac{1}{\omega}\rangle}.
\end{aligned}
$$

It can be shown that

$$
\begin{aligned}
(4.32)\qquad & q^{(1+\frac{1}{\omega}+\gamma-\alpha-\beta-x)(-z-\alpha)}\frac{\sigma(1+\alpha-\gamma)}{\sigma(1+\alpha-\beta)}\frac{\sigma(-z-\alpha+1+\frac{1}{\omega})\sigma(-z+1-\beta)}{\sigma(-z)\sigma(-z+1-\gamma)}\\
&= q^{xz}\frac{\sigma(x)}{\sigma(x+\alpha)}.
\end{aligned}
$$

From (3.4), we have

$$
\begin{aligned}
(4.33)\qquad &\frac{\langle\frac{1}{\omega}+\beta-\alpha\rangle}{\langle\frac{1}{\omega}+\gamma-\alpha\rangle}\frac{\langle z+1+\frac{1}{\omega}\rangle\langle z+\gamma+\frac{1}{\omega}\rangle}{\langle z+\alpha\rangle\langle z+\beta+\frac{1}{\omega}\rangle}\\
&= \frac{\langle\beta-\alpha\rangle}{\langle\gamma-\alpha\rangle}\frac{\langle z+1+\frac{1}{\omega}\rangle\langle z+\gamma\rangle}{\langle z+\alpha\rangle\langle z+\beta\rangle}\frac{(A-C)(1-BT)}{(A-B)(1-CT)}.
\end{aligned}
$$

Combining (4.32) and (4.33), we get

$$
\begin{aligned}
(4.34)\qquad &\Psi(\alpha,1+\alpha-\gamma,1+\alpha-\beta;1+\frac{1}{\omega}+\gamma-\alpha-\beta-x)\\
&= \frac{\langle\alpha\rangle\langle\beta-\alpha\rangle}{\langle1\rangle\langle\gamma-\alpha\rangle}\frac{\sigma(x)}{\sigma(x+\alpha)}\int_{C_0}\Phi(z)\frac{(A-C)(1-BT)}{(A-B)(1-CT)}dz.
\end{aligned}
$$

By exchanging $\alpha$ and $\beta$, we find

$$
\begin{aligned}
(4.35)\qquad &\Psi(\beta,1+\beta-\gamma,1+\beta-\alpha;1+\frac{1}{\omega}+\gamma-\alpha-\beta-x)\\
&= \frac{\langle\beta\rangle\langle\alpha-\beta\rangle}{\langle1\rangle\langle\gamma-\beta\rangle}\frac{\sigma(x)}{\sigma(x+\beta)}\int_{C_0}\Phi(z)\frac{(B-C)(1-AT)}{(B-A)(1-CT)}dz.
\end{aligned}
$$

Therefore, we get

$$
\begin{aligned}
(4.36)\ \text{the rhs of (4.28)} &= \frac{\langle\alpha\rangle\langle\beta\rangle}{\langle1\rangle\langle\gamma\rangle}\int_{C_0}\Phi(z)\left\{\frac{(A-C)(1-BT)}{(A-B)(1-CT)}+\frac{(B-C)(1-AT)}{(B-A)(1-CT)}\right\}dz\\
&= \frac{\langle\alpha\rangle\langle\beta\rangle}{\langle1\rangle\langle\gamma\rangle}\int_{C_0}\Phi(z)\cdot1dz=\Psi(\alpha,\beta,\gamma;x).
\end{aligned}
$$

□

In the proof above, we see that the formula (4.28) comes from the following simple relation in $\widetilde{H}$:

$$
(4.37)\qquad \frac{(A-C)(1-BT)}{(A-B)(1-CT)}+\frac{(B-C)(1-AT)}{(B-A)(1-CT)}=1.
$$

If we consider $H$ as a cohomology and $\widetilde{H}$ as its dual, that is a homology, then the relation (4.37) is a relation among some homologies, and the formula (4.28) is a linear relation among the integrals associated with different homologies.

# 5 The $q$-little Jacobi polynomials at $|q| = 1$

First we recall the definition of the $q$-little Jacobi polynomials in the case of $0 < |q| < 1$ [GR]:

$$p_n^{(\alpha,\beta)}(t) := \phi(q^{-n}, q^{\alpha+\beta+n+1}, q^{\alpha+1}; qt), \quad (n = 0, 1, \cdots). \tag{5.1}$$

The following orthogonality relation holds [AA, GR]:

$$\int_0^1 t^{\alpha-1} \frac{(tq)_\infty}{(tq^\beta)_\infty} p_m^{(\alpha-1,\beta-1)}(t) p_n^{(\alpha-1,\beta-1)}(t) d_q t = \delta_{m,n} c_n, \tag{5.2}$$

where

$$c_n = (1-q) \frac{(q)_\infty (q^{\alpha+\beta})_\infty}{(q^\alpha)_\infty (q^\beta)_\infty} \frac{1-q^{\alpha+\beta-1}}{1-q^{\alpha+\beta+2n-1}} \frac{(q)_n (q^\beta)_n}{(q^{\alpha+\beta-1})_n (q^\alpha)_n} q^{n\alpha}. \tag{5.3}$$

In (5.2), the integral is a Jackson integral defined by

$$\int_0^1 f(t) d_q t := (1-q) \sum_{k=0}^{\infty} f(q^n) q^n, \tag{5.4}$$

and $(a)_n = (a;q)_n$.

The formula (5.2) means that the $q$-little Jacobi polynomials are orthogonal polynomials with respect to the kernel of the $q$-Beta integral given by

$$\int_0^1 t^{\alpha-1} \frac{(tq)_\infty}{(tq^\beta)_\infty} d_q t = (1-q) \frac{(q)_\infty (q^{\alpha+\beta})_\infty}{(q^\alpha)_\infty (q^\beta)_\infty}. \tag{5.5}$$

Let us consider the case of $|q| = 1$. We can get the $q$-little Jacobi polynomials at $|q| = 1$ from the $q$-hypergeometric function (4.10) as follows.

**Proposition 5.1** *For $n \in \mathbb{Z}_{\geqslant 0}$, we have*

$$\lim_{\alpha \to -n} \Psi(\alpha, \beta, \gamma; x) = \phi(q^{-n}, q^\beta, q^\gamma; q^x). \tag{5.6}$$

*Note that the right hand side of (5.6) is a polynomial in $q^x$ and so makes sense at $|q| = 1$.*

*Proof.* Recall the definition of $\Psi(\alpha, \beta, \gamma; x)$:

$$\Psi(\alpha, \beta, \gamma; x) := \frac{\langle \alpha \rangle \langle \beta \rangle}{\langle 1 \rangle \langle \gamma \rangle} \int_{C_0} \Phi(z) dz. \tag{5.7}$$

At $\alpha=-n$, the coefficient $\langle\alpha\rangle$ has a zero and the integral has a pole because of pinches of the contour $C_0$ by poles at $z=0,1,\cdots,n$ and $z=-\alpha-n,-\alpha-n+1,\cdots,-\alpha$, respectively. In order to avoid these pinches, we take the residues at $z=-\alpha-n,\cdots,-\alpha$. Then we get

$$\begin{aligned}(5.8)\qquad \Psi(\alpha,\beta,\gamma;x) &= \frac{\langle\alpha\rangle\langle\beta\rangle}{\langle 1\rangle\langle\gamma\rangle}\sum_{k=0}^{n}2\pi i\mathrm{res}_{z=-\alpha-k}\Phi(z)dz\\ &+ \frac{\langle\alpha\rangle\langle\beta\rangle}{\langle 1\rangle\langle\gamma\rangle}\times(\text{regular at }\alpha=-n).\end{aligned}$$

The second term of the rhs of (5.8) equals zero at $\alpha=-n$. Hence it suffices to calculate the limit of the first term.

By using (2.14), we have

$$\begin{aligned}(5.9)\qquad & 2\pi i\mathrm{res}_{z=-\alpha-(n-k)}\Phi(z)dz\\ &= 2\pi i\mathrm{res}_{z=-\alpha}\Phi(z)\prod_{j=1}^{n-k}\frac{(1-q^{-j+1}t)(1-q^{-j}ct)}{(1-q^{-j}at)(1-q^{-j}bt)}q^{-(n-k)x}dz\\ &= \langle 1\rangle\langle 1+\frac{1}{\omega}-\alpha\rangle\frac{\langle\gamma-\alpha\rangle}{\langle\beta-\alpha\rangle}\prod_{j=1}^{n-k}\frac{(1-q^{-j+1}/a)(1-q^{-j}c/a)}{(1-q^{-j})(1-q^{-j}b/a)}q^{-(\alpha+n-k)x}.\end{aligned}$$

Here we used the notation (4.15) and

$$(5.10)\qquad 2\pi i\mathrm{res}_{z=0}\frac{dz}{\langle z\rangle}=\frac{i}{\sqrt{\omega}}=\langle 1\rangle.$$

Therefore, we get

$$\begin{aligned}(5.11)\quad & \lim_{\alpha\to -n}\Psi(\alpha,\beta,\gamma;x)\\ &= \lim_{\alpha\to -n}\langle\alpha\rangle\langle 1+\frac{1}{\omega}-\alpha\rangle\frac{\langle\beta\rangle}{\langle\beta-\alpha\rangle}\frac{\langle\gamma-\alpha\rangle}{\langle\gamma\rangle}\sum_{k=0}^{n}q^{-(\alpha+n-k)x}\prod_{j=1}^{n-k}\frac{(1-q^{-j+1}/a)(1-q^{-j}c/a)}{(1-q^{-j})(1-q^{-j}b/a)}\\ &= \sigma(-n)\frac{\langle\beta\rangle}{\langle\beta+n\rangle}\frac{\langle\gamma+n\rangle}{\langle\gamma\rangle}\sum_{k=0}^{n}q^{kx}\prod_{j=1}^{n-k}\frac{(1-q^{n+1-j})(1-q^{n-j}c)}{(1-q^{-j})(1-q^{n-j}b)}\\ &= (-1)^n q^{-\frac{n(n+1)}{2}}\prod_{j=1}^{n}\frac{1-q^j}{1-q^{-j}}\sum_{k=0}^{n}q^{kx}\prod_{j=0}^{k-1}\frac{(1-q^{-n+j})(1-q^jb)}{(1-q^j)(1-q^jc)}\\ &= \phi(q^{-n},q^\beta,q^\gamma;q^x).\end{aligned}$$

□

From this proposition, we get

$$(5.12)\qquad p_n^{(\alpha,\beta)}(q^x)=\Psi(-n,\alpha+\beta+n+1,\alpha+1;x+1),\quad (n=0,1,\cdots).$$

Then we find that the $q$-little Jacobi polynomials (5.12) satisfy the orthogonal relation associated with the $q$-Beta integral at $|q|=1$ (3.27).

**Proposition 5.2**

$$\int_C q^{\alpha z}\frac{\langle z+1+\frac{1}{\omega}\rangle}{\langle z+\beta\rangle}p_m^{(\alpha-1,\beta-1)}(t)p_n^{(\alpha-1,\beta-1)}(t)dz=\delta_{m,n}c_n, \tag{5.13}$$

*where* $t=q^z=e^{2\pi i\omega z}$ *and*

$$c_n=\frac{\langle 1\rangle\langle\alpha+\beta\rangle}{\langle\alpha\rangle\langle\beta\rangle}\frac{1-q^{\alpha+\beta-1}}{1-q^{\alpha+\beta+2n-1}}\frac{(q)_n(q^\beta)_n}{(q^{\alpha+\beta-1})_n(q^\alpha)_n}q^{n\alpha}. \tag{5.14}$$

*In the left hand side of (5.13), the contour* $C$ *is the imaginary axis* $(-i\infty,i\infty)$ *except that the poles at* $\mathbb{Z}_{\geqslant 0}+\frac{1}{\omega}\mathbb{Z}_{\geqslant 0}$ *are on the right of* $C$ *and the poles at* $-\beta+\mathbb{Z}_{\leqslant 0}+\frac{1}{\omega}\mathbb{Z}_{\leqslant 0}$ *are on the left of* $C$*.*

*Proof.* First we rewrite the orthogonality relation with $0<|q|<1$ (5.2) as follows. We expand the product

$$p_m^{(\alpha-1,\beta-1)}(t)p_n^{(\alpha-1,\beta-1)}(t)=\sum_{k=0}^{m+n}A_k^{m,n}t^k,\quad A_k^{m,n}\in\mathbb{C}\,. \tag{5.15}$$

By using (5.5), we have

$$\begin{aligned}&\int_0^1 t^{\alpha-1}\frac{(tq)_\infty}{(tq^\beta)_\infty}p_m^{(\alpha-1,\beta-1)}(t)p_n^{(\alpha-1,\beta-1)}(t)d_qt\\&=\sum_{k=0}^{m+n}A_k^{m,n}\int_0^1 t^{\alpha+k-1}\frac{(tq)_\infty}{(tq^\beta)_\infty}d_qt\\&=\sum_{k=0}^{m+n}A_k^{m,n}(1-q)\frac{(q)_\infty(q^{\alpha+\beta+k})_\infty}{(q^{\alpha+k-1})_\infty(q^\beta)_\infty}\\&=(1-q)\frac{(q)_\infty(q^{\alpha+\beta})_\infty}{(q^\alpha)_\infty(q^\beta)_\infty}\sum_{k=0}^{m+n}A_k^{m,n}\prod_{j=0}^{k-1}\frac{1-q^{\alpha+j}}{1-q^{\alpha+\beta+j}}.\end{aligned} \tag{5.16}$$

Hence the relation (5.2) is equivalent to

$$\sum_{k=0}^{m+n}A_k^{m,n}\prod_{j=0}^{k-1}\frac{1-q^{\alpha+j}}{1-q^{\alpha+\beta+j}}=\delta_{m,n}\frac{1-q^{\alpha+\beta-1}}{1-q^{\alpha+\beta+2n-1}}\frac{(q)_n(q^\beta)_n}{(q^{\alpha+\beta-1})_n(q^\alpha)_n}q^{n\alpha}. \tag{5.17}$$

Note that (5.17) is an algebraic equality and holds also in the case of $|q|=1$.

On the other hand, we find the following from (3.27) in the same way as (5.16):

$$\int_C q^{\alpha z}\frac{\langle z+1+\frac{1}{\omega}\rangle}{\langle z+\beta\rangle}p_m^{(\alpha-1,\beta-1)}(t)p_n^{(\alpha-1,\beta-1)}(t)dz=\frac{\langle 1\rangle\langle\alpha+\beta\rangle}{\langle\alpha\rangle\langle\beta\rangle}\sum_{k=0}^{m+n}A_k^{m,n}\prod_{j=0}^{k-1}\frac{1-q^{\alpha+j}}{1-q^{\alpha+\beta+j}}. \tag{5.18}$$

From (5.17) and (5.18), we get (5.13). □

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