LECTURES ON FACTORIZATION OF BIRATIONAL MAPS

Kenji Matsuki

in memory of Prof. Kunihiko Kodaira

This is an expanded version of the notes for the lectures given by the author at RIMS in the summer of 1999 to provide a detailed account of the proof for the (weak) factorization theorem of birational maps by Abramovich-Karu-Matsuki-Włodarczyk [1]. All the main ideas of these lecture notes, therefore, are derived from the collaboration and joint work with D. Abramovich, K. Karu and J. Włodarczyk. The author would like to emphasize that the theory of "Birational Cobordism", which plays the central role in our proof, was developed in Włodarczyk [2] and that its origin may be traced back to the earlier work of Morelli [1][2] in his brilliant solution to the (strong) factorization conjecture for toric birational maps[1]. The author would also like to emphasize that there is another independent proof for the (weak) factorization theorem by Włodarczyk [3], which is NOT discussed in these notes. It is noteworthy that not only the ideas of but also some essential steps of the two proofs rely on the work of Morelli: our proof in Abramovich-Karu-Matsuki-Włodarczyk [1] reduces the factorization of general birational maps to that of toroidal birational maps and then uses the combinatorial algorithm by Morelli to factor toroidal ones. The proof in Włodarczyk [3] uses the process of "π-desingularization", which is the most difficult and subtle part of the algorithm by Morelli. In fact, all that we cover in these notes is the above mentioned reduction step "from general to toroidal", taking the combinatorial algorithm for the factorization of toroidal birational maps as a black box. We refer the reader to Abramovich-Matsuki-Rashid [1] for what is inside of the black box, including the details of the process of π-desingularization and clarifications of some discrepancies contained in the original arguments of Morelli [1][2][2].

The deepest gratitude of the author goes to Prof. S. Mori for his generosity to provide an opportunity to deliver these lectures at RIMS and to Profs. Y. Miyaoka, N. Nakayama and the participants of the seminar, who listened to and suggested innumerable improvements to the presentation and contents of the lectures. In the talks subsequently given at Tokyo University, another set of invaluable comments was given to the author by Profs. S. Iitaka, Y. Kawamata and others. The present notes are made more comprehensible due to their constructive criticism than the

[1,2] Unfortunately, K. Karu pointed out that the algorithm of Morelli for the strong factorization described in Abramovich-Matsuki-Rashid [1] as well as the original one in Morelli [1] does not work, as some example of a cobordism representing a toric birational map in dimension 3 indicates. Thus despite their claims in Abramovich-Matsuki-Rashid [1] Morelli [1] the strong factorization conjecture remains open even for toric birational maps, as of May 11, 2000. See Matsuki [2], which is an erratum to Abramovich-Matsuki-Rashid [1], for the details.

original manuscripts. The author hopes that our sweat in the midst of the humidity and heat of "TSUYU" (the rainy season in Japan) served some purpose.

CONTENTS

Chapter 0. Introduction

§0-1. Outline of the strategy
§0-2. Some historical remarks

Chapter 1. Preliminaries

§1-1. Brief review of Geometric Invariant Theory and Toric Geometry
§1-2. "Locally Toric" structures vs. "Toroidal" structures
§1-3. K^*-action on locally toric and toroidal structures
§1-4. Elimination of points of indeterminacy

Chapter 2. Birational Cobordism

§2-1. Definition of a birational cobordism and the toric main example
§2-2. Construction of a (collapsible) birational cobordism
§2-3. Interpretation by Geometric Invariant Theory
§2-4. Factorization into locally toric birational maps

Chapter 3. Torification

§3-1. Construction of the torific ideal
§3-2. The torifying property of the torific ideal

Chapter 4. Recovery of Nonsingularity

§4-1. Application of the canonical resolution of singularities
§4-2. Conclusion of the proof for the weak factorization theorem

Chapter 5. Generalizations

§5-1. Factorization of bimeromorphic maps
§5-2. Equivariant factorization under group action
§5-3. Factorization over a non-algebraically closed field
§5-4. Factorization in the logarithmic category

Chapter 6. Problems

§6-1. Effectiveness of the construction
§6-2. Strong Factorization Conjecture
§6-3. Toroidalization Conjecture

CHAPTER 0. INTRODUCTION

The purpose of these lecture notes is to provide a detailed account of the proof given by Abramovich-Karu-Matsuki-Włodarczyk [1] for the following (weak) factorization conjecture for birational maps. (We note that there is another proof independently given by Włodarczyk [3], which we will NOT discuss in these notes.)

Weak Factorizatioin Theorem of Birational Maps. *Let $\phi : X_1 \dashrightarrow X_2$ be a birational map between complete nonsingular varieties over an algebraically closed field K of characteristic zero. Let $X_1 \supset U \subset X_2$ be a common open subset over which ϕ is an isomorphism. Then ϕ can be factored into a sequence of blowups and blowdowns with smooth irreducible centers disjoint from U. That is to say, there exists a sequence of birational maps between complete nonsingular varieties*

$$X_1 = V_1 \overset{\psi_1}{\dashrightarrow} V_2 \overset{\psi_2}{\dashrightarrow} \cdots \overset{\psi_{i-1}}{\dashrightarrow} V_i \overset{\psi_i}{\to} V_{i+1} \overset{\psi_{i+1}}{\to} \cdots \overset{\psi_{l-2}}{\dashrightarrow} V_{l-1} \overset{\psi_{l-1}}{\to} V_l = X_2$$

where

(i) $\phi = \psi_{l-1} \circ \psi_{l-2} \circ \cdots \circ \psi_2 \circ \psi_1$,

(ii) ψ_i are isomorphisms over U, and

(iii) either $\psi_i : V_i \dashrightarrow V_{i+1}$ or $\psi_i^{-1} : V_{i+1} \dashrightarrow V_i$ is a morphism obtained by blowing up a smooth irreducible center on V_{i+1} or V_i, respectively, disjoint from U.

Furthermore, if both X_1 and X_2 are projective, then we can choose a factorization so that all the intermediate varieties V_i are projective. More precisely, in the factorization we construct, there exists an index i_o such that the birational maps $V_i \dashrightarrow X_1$ are projective birational morphisms for $i \le i_o$ and that the birational maps $V_i \dashrightarrow X_2$ are projective birational morphisms for $i_o \le i$.

It is called the WEAK factorization theorem as we allow the sequence to consist of blowups and blowdowns in any order. If we insist that the sequence in the factorization should consist only of blowups immediately followed by blowdowns, then we obtain the following STRONG factorization conjecture, which remains open.

Strong Factorization Conjecture. *Let $\phi : X_1 \dashrightarrow X_2$ be as above. Then a sequence as above can be chosen with the extra condition that there exists an index i_o so that ψ_i^{-1} for $1 \le i \le i_o$ are blowups while ψ_i for $i_o \le i \le l-1$ are blowdowns.*

We note that, if both X_1 and X_2 are projective, in a sequence asserted in the strong factorization conjecture all the intermediate varieties are automatically projective, while in an arbitrary sequence in the weak factorization it may have some complete but nonprojective intermediate varieties and that the "Furthermore" part of the weak factorization theorem asserts the existence of a sequence preserving the projectivity.

Several generalizations of the theorem will be discussed in Chapter 5.

§0-1. Outline of the strategy

First, we simply state the strategy of our proof in a symbolic way with a brief description of each of the following five main steps:

Strategy of the Proof

Step 1: Reduction to the case where ϕ is a projective birational morphism

Elimination of points of indeterminacy

Step 2: Factorization into locally toric birational maps

Construction of a birational cobordism

Step 3: Factorization into toroidal birational maps

Torification : Blowing up the torific ideal

Step 4: Recovery of nonsingularity

Canonical resolution of singularities

Step 5. Factorization of toroidal birational maps among nonsingular toroidal embeddings

Morelli's combinatorial algorithm.

Now we will explain each step more in detail and describe the contents of the chapters of these notes accordingly.

Let $\phi : X_1 \dashrightarrow X_2$ be a birational map between complete nonsingular varieties over an algebraically closed field K of characteristic zero.

Step 1: Reduction to the case where ϕ is a projective birational morphism

Hironaka's elimination of points of indeterminacy asserts that there exist sequences of blowups with smooth centers $X_1' \to X_1$ and $X_2' \to X_2$ so that the induced map $\phi' : X_1' \to X_2'$ is a projective birational morphism. By replacing the original ϕ with ϕ' we may assume in the factorization problem that ϕ is a projective birational morphism. This process is discussed in §1-4 of Chapter 1.

Step 2: Factorization into locally toric birational maps

This is the key step of our proof, discussed in Chapter 2, which allows us to bring in a "Morse Theoretic" view point to approach the factorization problem via the theory of "Birational Cobordism".

Let us briefly recall what the Morse theory tells us: Consider the situation where we have a usual cobordism $C = C(M_1, M_2)$ between differentiable manifolds M_1 and M_2 equipped with a Morse function $f : C(M_1, M_2) \to \mathbb{R}$. (By extending the cobordism appropriately, we may assume that f is surjective.)

Picture 0-1-1.

By looking at the vector field $grad(f)$ (with respect to a Riemannian metric on $C(X_1, X_2)$), we can introduce the action of the "time" $t \in \mathbb{R}$ on the cobordism, namely, we have a map from the time $\mathbb{R}$ to the group $Diff(C)$ of diffeomorphisms of C denoted by

$$(t \in \mathbb{R}) \mapsto (\varphi_t : C \to C \in Diff(C))$$

where $\varphi_t(p)$ is the position of a point $p \in C$ after time t along the flow given by integrating the vector field $grad(f)$. Taking the exponential $exp(t) \in \mathbb{R}_{>0}$ of the time $t \in \mathbb{R}$, we see that the multiplicative group $(\mathbb{R}_{>0}, \times) = exp(\mathbb{R}, +)$ acts as well. The crucial information in the Morse Theory can be read off in terms of this action of the multiplicative group:

(C-o) the multiplicative group $\mathbb{R}_{>0}$ acts on a cobordism C,

(C-i) M_2 and M_1 being on the top and at the bottom of the cobordism C, respectively, we can describe them as the quotients

$$M_2 \cong C_+/\mathbb{R}_{>0}$$
$$M_1 \cong C_-/\mathbb{R}_{>0}$$

where the sets C_+ and C_- are defined to be

$$C_+ := \{x \in C; \lim_{t \to \infty} t(x) = \lim_{exp(t) \to \infty} exp(t)(x) \text{ does NOT exist in } C\}$$
$$C_- := \{x \in C; \lim_{t \to -\infty} t(x) = \lim_{exp(t) \to 0} exp(t)(x) \text{ does NOT exist in } C\},$$

(C-ii) the critical points of f are precisely the fixed points of the action and the homotopy types of fibers of f change as we pass through these fixed points.

This interpretation of the Morse Theory in terms of the action of the multiplicative group led Włodarczyk [2] naturally to the notion of a birational cobordism $B = B_\phi(X_1, X_2)$ for a birational map ϕ:

(B-o) the multiplicative group $t \in K^*$ acts on a (nonsingular or normal) variety B of $\dim B = \dim X_1 + 1 = \dim X_2 + 1$,

(B-i) we can describe X_2 and X_1 as the quotients

$$X_2 \cong B_+/K^*$$
$$X_1 \cong B_-/K^*$$

where the sets B_+ and B_- are defined to be

$$B_+ := \{x \in B; \lim_{t\to\infty} t(x) \text{ does NOT exist in } B\}$$
$$B_- := \{x \in B; \lim_{t\to 0} t(x) \text{ does NOT exist in } B\}$$

and the birational map induced by the natural inclusions

$$X_1 = B_-/K^* \supset (B_- \cap B_+)/K^* \subset B_+/K^* = X_2$$

coincides with ϕ,

(B-ii) "passing through" the fixed points of the action induces the birational transformations.

The precise definition of the birational cobordism is found in §2-1. We give the construction of a birational cobordism (only for a projective birational morphism, after the reduction Step 1, and leaving the construction for a general birational map to the original Włodarczyk [2]) in §2-2, together with the discussion of the "collapsibility" in order to line up the fixed points nicely in the above algebraic setting where we lack a Morse function f, which, in the usual differentiable setting, would assign the levels to the critical points and hence line them up nicely. We also provide an interpretation of the notion of the birational cobordism in §2-3 from the view point of Geometric Invariant Theory (reviewed briefly in §1-1 together with Toric Geometry), in connection with the work of Thaddeus and others. This view point allows us to preserve the projectivity of the intermediate varieties in our factorization if both X_1 and X_2 are projective.

We would like to take a closer look at the birational transformations as we pass through the fixed points (when the cobordism B is nonsingular).

Let $p \in B^{K^*}$ be a fixed point of the birational cobordism B. Analytically locally, the action of K^* on B at p is equivalent to the action of K^* on the tangent space $T_{B,p}$, which is a vector space V_p of $\dim V_p = n$. As the multiplicative group K^* is reductive, the vector space V_p splits into the eigenspaces according to the characters of the action. By choosing a basis $\{z_1, \cdots, z_n\}$ from the eigenspaces, we can describe the action of $t \in K^*$ on $T_{B,p} = V_p$ as

$$t(z_1, \cdots, z_n) = (t^{\alpha_1} z_1, \cdots, t^{\alpha_n} z_n)$$

where the $\alpha_i \in \mathbb{Z}$ are the characters for the z_i. The crucial sets $(V_p)_+$ and $(V_p)_-$ for the local behavior of the cobordism, defined similarly as B_+ and B_-, have simple descriptions:

$$(V_p)_+ = \{z = (z_1, \cdots, z_n) \in V_p; z_i \neq 0 \text{ for some } i \text{ with } \alpha_i > 0\}$$
$$(V_p)_- = \{z = (z_1, \cdots, z_n) \in V_p; z_i \neq 0 \text{ for some } i \text{ with } \alpha_i < 0\}.$$

If we identify the vector space V_p with the affine n-space $\mathbb{A}^n = X(N, \sigma)$, regarded as the toric variety associated to the standard cone $\sigma = \langle v_1, \cdots, v_n \rangle \subset N_{\mathbb{R}} = N \otimes \mathbb{R}$ generated by the standard $\mathbb{Z}$-basis of the lattice $N \cong \mathbb{Z}^n$, and identify the action of K^* with the one-parameter subgroup corresponding to the point $a = (\alpha_1, \cdots, \alpha_n) \in N$, we have a simple local description of the birational transformation, namely the birational transformation between $(V_p)_+/K^*$ and $(V_p)_-/K^*$ in terms of toric geometry associatde to the projection $\pi : N_{\mathbb{R}} \to N_{\mathbb{R}}/\mathbb{R} \cdot a$;

$$(V_p)_+/K^* = X(\pi(N), \pi(\partial_+\sigma))$$
$$(V_p)_-/K^* = X(\pi(N), \pi(\partial_-\sigma))$$

where the upper boundary face and lower boundary face of σ are defined to be

$$\partial_+\sigma := \{v \in \sigma; v + \epsilon \cdot (-a) \notin \sigma \text{ for } \epsilon > 0\}$$
$$\partial_-\sigma := \{v \in \sigma; v + \epsilon \cdot a \notin \sigma \text{ for } \epsilon > 0\}.$$

Therefore, analytically locally, the birational transformation induced by passing through the fixed point $p \in B$ is equivalent to a toric birational transformation between toric varieties corresponding to (the projections of) the upper boundary face and lower boundary face of the standard cone σ (with respect to the one-parameter subgroup $a \in N$), which is a typical example of the "polyhedral cobordism" used by Morelli to solve the factorization probelm for toric birational maps. Thus via the birational cobordism we succeed in factoring ϕ into locally toric birational transformations as we pass through the fixed points (at the several levels ordered nicely by the collapsibility). This is the content of §2-4.

Picture 0-1-2.

One might naively expect at this point that since ϕ is now factored into "locally toric" birational maps and since "toroidal" birational maps can be factored by Morelli's combinatorial algorithm, we had already completed a proof for the weak factorization theorem, especially when the notion of "locally toric" and that of "toroidal" are mixed and sometimes confused in the existing literatures. But there is a huge GAP between locally toric birational maps and toroidal birational maps. Actually the distinction between the two notions is one of the key issues in our argument and the main bodies §1-2 and §1-3 of Chapter 1 are devoted to the detailed discussion and clarification of them.

Step 3: Factorization into toroidal birational maps

The birational cobordism B has a locally toric structure, as we observed above, each point $p \in B$ having an analytic neighborhood isomorphic to a toric variety V_p, and the action of K^* is translated into that of a one-parameter subgroup in this locally toric chart. But the choice of coordinates (or equivalently the choice of coordinate divisors) is not canonical and hence the embedded tori may vary from chart to chart as we move from point to point. In particular, these locally toric coordinates do NOT patch together. On the other hand, we require a toroidal structure (U_X, X) to have a fixed open subset $U_X \subset X$ so that $X - U_X$ provides the global coordinate divisors. Thus in order to give B (or its modification) a toroidal structure we have to create such global divisors. "Torification" achieves exactly this goal by blowing up the "torific" ideal and hence creating these desired global coordinate divisors defined by the pull-back of the torific ideal. The torific ideal is canonically defined in terms of the action of K^* only.

Here arises one technical but essential difficulty. The torific ideal is only well-defined over smaller pieces of the birational cobordism, called the quasi-elementary pieces of B, but not on the entire birational cobordism B. (The quasi-elementary pieces correspond to the semi-stable loci of different linearizations of the action of K^*, when the birational cobordism is interpreted via Geometric Invariant Theory.) Thus we can only introduce toroidal structures, after blowing up the torific ideals, to the quasi-elementary pieces and these toroidal structures may not be compatible with each other. As a consequence, though we succeed in factoring ϕ into toroidal birational maps by torification, we have to pay the price: these toroidal birational maps are only enough to provide weak factorization but not strong factorization, due to the incompatibility of the toroidal structures (even if we succeed in proving the strong factorization for toroidal birational maps).

The precise definition of the torific ideal, together with the subtlety why it is only well-defined over the quasi-elementary pieces, is given in §3-1. The core of "torification" is discussed in §3-2, i.e., the torifying property of the torific ideal asserting that blowing it up induces a toroidal structure so that the action of K^* is that of a one-parameter subgroup in a toroidal chart of each point.

Step 4: Recovery of nonsingularity

Now that we succeeded in factoring ϕ into toroidal birational maps after Step 3, are we done with the proof ? Not yet !!

Though the birational maps are toroidal, the toroidal embeddings we have con-

structed through Steps 1-3 may be, and in most of the cases they actually are, SINGULAR. (In fact, even at the stage of factoring ϕ into locally toric birational maps the intermediate varieties may be singular.) Since we seek to factor ϕ into blowups and blowdowns with smooth centers on NONSINGULAR varieties, we have to recover nonsingularity. This is done via the application of the canonical resolution of singularities (and the canonical principalization of some ideals) in §4-1. The canonicity of the resolution allows us to preserve the toroidal structures through the process of desingularization.

Step 5. Factorization of toroidal birational maps among nonsingular toroidal embeddings

Finally, applying Morelli's combinatorial algorithm to toroidal birational maps between nonsingular toroidal embeddings, we complete the proof of the weak factorization in §4-2.

In Chapter 5, we discuss several generalizations of the weak factorization theorem, modifying the arguments discussed above according to the different purposes. §5-1 accomplishes the factorization of bimeromorphic maps in the analytic category. In §5-2, we consider the case where there is a group acting and establish the factorization equivariantly. (We remark that we may have to blow up several smooth centers simultaneously to maintain the equivariance.) As an application, we show in §5-3 that we can remove the assumption of the base field K being algebraically closed, by considering the action of the Galois group. It is noteworthy that we can NOT remove the assumption of the characteristic of K being zero for the time being, as the canonical resolution of singularities sits in the center of our method, that it is the only place where we use the characteristic assumption and hence that any future development of the method of the canonical resolution of singularities in positive characteristics would allow us to remove the assumption. §5-4 discusses the factorization in the logarithmic category, which has an application to the study of the behavior of the Hodge structures under birational transformations.

In Chapter 6, we mention some problems related to our proof of the weak factorization conjecture. Though the proof, providing a specific procedure to factor a birational map, is constructive, it falls short of being effective at several places, of which §6-1 makes a list. In §6-2, we discuss briefly the strong factorization conjecture, which remains open even for toric birational maps. In §6-3, we discuss the toroidalization conjecture and gives a proof in dimension 2. The toroidalization conjecture seems to lie at the heart of the whole circle of ideas, inspired by the work of De Jong [1] and others, involving the factorization problem, resolution of singularities, semi-stable reductions, the log category of Kato [1] and beyond.

§0-2. Some historical remarks

The following is the auhtor's personal and prejudiced view of how the factorization problem evolved over years. It is by NO means meant to be a complete account of the subject and the author apologizes in advance for the omission of any references or topics which should be included.

Early origins of the problem

A birational map $\phi : X_1 \dashrightarrow X_2$ between algebraic varieties is by definition a rational map which induces an isomorphism of the function fields $\phi^* : K(X_2) \xrightarrow{\sim} K(X_1)$ or equivalently a rational map which induces an isomorphism over a common dense open subset $X_1 \supset U \subset X_2$.

If we restrict our attention to complete nonsingular varieties, a birational map is nothing but an isomorphism for curves, i.e., varieties of dimension 1. In dimension 2 or higher, however, the picture changes drastically. There are many birational maps which are not isomorphisms, typical examples of which are BLOWUPS with smooth centers, their inverses BLOWDOWNS. A natural and fundamental question arises then if these exhaust all the possible birational maps, i.e., if a given birational map is a composite of blowups and blowdowns with smooth centers, which came to be known as the factorization problem of birational maps.

The history of the factorization problem of birational maps could be traced back to the Italian school of algebraic geometers, who already knew that the operation of blowing up points on surfaces is a fundamental source of richness for surface geometry. The importance of the strong factorization theorem in dimension 2 by Zariski cannot be overestimated in the analysis of the birational geometry of algebraic surfaces. The strong factorization problem was stated in the form of a question as "Question (F')" in Hironaka [2], who recognized the connection with the problem of resolution of singularities and gave a partial answer in the form of Elimination of Points of Indeterminacy. Though the question of the weak factorization was also raised in Oda [1], the problem remained largely open in higher dimensions despite the efforts and interesting results of many, e.g., Crauder [1] Kulikov [1] Schaps [1] and others. These were summarized in Pinkham [1], where the weak factorization conjecture is explicitly stated in the form presented here.

Toric Case

For toric birational maps, the equivariant version of the factorization problem under the torus action was posed in Oda [1] and came to be known as Oda's weak and strong conjectures. Though the toric geometry can be interpreted in terms of the geometry of convex cones and hence sometimes is considered to be a "baby" or easier version of the real case, the factorization problem presented a substantial challenge and difficulty in combinatorics. (Actually the fact that all the known examples (cf. Hironaka [1] Sally [1] Shannon [1]) demonstrating the difficulties in higher dimensions were toric made some of us even suspect that it is not a coincidence but an indication that all the combinatorial complications in general are somehow concentrated in the toric case.) In dimension 3, Danilov [2]'s proof for the weak factorization was later supplemented by Ewald [1]. Oda's weak conjecture was solved in arbitrary dimension by Włodarczyk [1]. A big breakthrough was brought by Morelli [1][2], who not only solved Oda's strong conjecture in arbitrary dimension[3] but also introduced the notion of a "cobordism" between fans into the analysis. This theory of the polyhedral cobordism made the combinatorial approach to the problem conceptually transparent, so much so that it made us eager to hope

[3] See the footnotes 1 and 2.

for the extension of the theory to be applied to general birational maps. But because of the apparent disguise as a purely combinatorial object with little hope of extension, we had to wait for Włodarczyk [2] to reveal its true meaning as the algebraic version of the Morse Theory.

Minimal Model Program

It is worthwhile to note the relation of the factorization problem to the development of the Mori program. Hironaka [1] used the cone of effective curves to study the properties of birational morphisms. This direction was further developed and given a decisive impact by Mori [2], who, motivated by Kleiman's criterion for ampleness and inspired by the solution to a conjecture of Hartshorne (cf. Mori [1]), introduced the notion of extremal rays and systematically used it in an attempt to construct minimal models in higher dimensions, called the minimal model program. Danilov [2] introduced the notion of canonical and terminal singularities in conjunction with the factorization problem. This was developed by Reid [1][2][4] into a general theory of these singularities, which appear in an essential way in the minimal model program. The minimal model program is so far proven up to dimension 3 (See Mori [3] Kawamata [1][2][3] Kollár [1] Shokurov [1].) and for toric varieties in arbitrary dimension (See Reid [3]). In the steps of the minimal model program one is only allowed to contract a divisor into a variety with terminal singularities or to perform a flip, modifying some codimension 2 loci. This allows a factorization of a given projective birational morphism into such "elementary operations". An algorithm to factor birational maps among uniruled varieties, known as "Sarkisov's program", has also been developed and established in dimension 3 in general (See Sarkisov [1] Reid [5] Corti [1].) and in arbitrary dimension for toric varieties (See Matsuki [1]). It was hoped that the detailed understanding of such "elementary operations" would yield a solution, at least in dimension 3, to the classical factorization problem. Still, we do not know of a way to carry out this approach.

Resurgence of Toroidal Geometry

It was a shock when De Jong [1] showed that the resolution of singularities in positive characteristics, though up to finite alterations, can be achieved and reduced to the toroidal case by an elegant application of the theory of the moduli space of (marked) stable curves. Resolution of singularities (easy but weak) in characteristic zero, without a finite alteration, was also achieved shortly after by Abramovich-DeJong [1] and Bogomolov-Pantev [1] (See also Paranjape [1].), again reducing the general case to the toroidal case. When we saw Abramovich-Karu [1] solve the problem of semi-stable reduction over higher dimensional base as an application of the method above, extending the original theorem by Kempf-Knudsen-Mumford-SaintDonat [1], the inspiration was developing into a belief that toric (toroidal) geometry is not just a baby version testing the real case but reflects the geometry in general and that the factorization problem could be approached in a similar manner by some reduction step yet to be found.

Local Version

There is a local version of the factorization problem, formulated and proved in dimension 2 by Abhyankar [1]. Chritensen [1] posed the problem in general and solved it for some special cases in dimension 3. Here the varieties X_1 and X_2 for a birational map $\phi : X_1 \dashrightarrow X_2$ are replaced by appropriate birational local rings dominated by a fixed valuation, and blowups are replaced by monoidal transforms subordinate to the valuation. This local conjecture was recently solved by S.D. Cutkosky in its strongest form[4] in a series of papers (cf. Cutkosky [1][2][3]). The main ingredient of his proof is the "monomialization" of a given birational map, i.e., the local version of the reduction step to the toroidal case. Thus the results of Cutkosky, though we do not use any of his method in our proof, not only gave us a psycological support that there is no local obstruction to solving the global factorization conjecture but also presented another evidence supporting the belief mentioned above.

Birational Cobordism and Connection with G.I.T.

Włodarczyk [2] provides the long-sought-after and key ingredient to reduce the factorization problem of GENERAL birational maps to that of LOCALLY TORIC (ultimately TOROIDAL after torification) birational maps by revealing the true nature of Morelli's cobordism and hence opening the way to applying his combinatorial analysis to the general case via the theory of "birational cobordism", which can be regarded as an algebraic version of the Morse Theory. His theory can also be interpreted in the frame work of Geometric Invariant Theory and has a direct connection with the subject of the change of G.I.T. quotients associated to the change of linearizations, which has been intensively studied by Thaddeus [1][2] and others.

Canonical Resolution of Singularities

There has been much progress after Hironaka's original work toward the understanding and simplification of the algorithms of resolution of singularities, notably by Bierstone-Milman [1] Villamayor [1] Encinas-Villamayor [1] among others. Since the elimination of points of indeterminacy, it has been a common consensus that the problem of resolution of singularities is in close connection with the factorization problem. In fact, the canonicity of these algorithms for resolution of singularities plays an indispensible role in our proof for the weak factorization.

Solutions

The theory of birational cobordism by Włodarczyk [2] almost immediately factors a given birational map into locally toric birational transformations, the factorization which is a consequence of a local analysis in nature. But in order to apply Morelli's combinatorial algorithm, which is a global analysis in nature, we still have to bridge

[4] After monomialization, Cutkosky [3] also relies on the algorithm of Abramovich-Matsuki-Rashid [1] Morelli [1] for the strong factorization, which has been found to fail to work. Thus the local factorization is also only established in the weak form but not in the strong from at the moment.

the gap between "local" and "global".

Włodarczyk [3] provides this bridge by constructing a global combinatorial object, much like the conical complexes associated to toroidal embeddings in Kempf-Knudsen-Mumford-SaintDonat [1], and hence a solution for the weak factroization problem. Abramovich-Karu-Matsuki-Włodarczyk [1], meanwhile, provides this bridge by transforming the locally toric structure into a (global) toroidal structure by the process of torification. It is obtained by blowing up the "torific" ideal. The idea originates from the above-mentioned work of Abramovich-DeJong [1] for the weak resolution of singularities.

Both proofs share essentially the same difficulty toward the strong factorization conjecture (even assuming a solution to the strong factorization for toric or toroidal birational maps), which remains as an open question.

Today and Future

Though based upon the same belief, our proof presented here is different from what we had originally conceived as a possible approach to the factorization problem. King, in a paper jointly with Abkulut, essentially presented what is now known as the toroidalization conjecture, our original approach, stated explicitly in Abramovich-Karu [1] Abramovich-Matsuki-Rashid [1].

Toroidalization Conjecture. *Let $\phi : X_1 \dashrightarrow X_2$ be as above. Then there exist sequences of blowups and blowdowns with smooth centers $X_1' \to X_1$ and $X_2' \to X_2$ such that the induced morphism $\phi' : X_1' \to X_2'$ becomes toroidal (with respect to some suitably chosen toroidal structures on X_1' and X_2').*

The conjecture has been known to hold in diemsnion 2 by the original works of Abkulut-King [1], Abramovich-Karu [1] Karu [1] and that of Abramovich presented in Abramovich-Karu-Matsuki-Włodarczyk [1], as well as by a recent local algorithmic proof by Cutkosky-Piltant [1]. The toroidalization conjecture makes sense not only for a birational morphism but also for a morphism $f : X \to Y$ between varieties of different dimensions, the flexibility which makes us hopeful for the inductional structure in a possible proof. (Cutkosky announced a proof for the case $\dim X = 3$ and $\dim Y = 2$ in the summer of 1999.)

An affirmative solution for the toroidalization conjecture will, once we have established the strong factorization for toroidal birational maps, automatically provide one for the strong factorization conjecture.

Also by resolution of singularities, it is easily reduced to the following case of morphisms between toroidal embeddings: Let $f : (U_X, X) \to (U_Y, Y)$ be a morphism between complete nonsingular toroidal embeddings with $f|_{U_X} : U_X = f^{-1}(U_Y) \to U_Y$ being smooth. Then there should exist sequences of blowups and blowdowns $\sigma_X : (U_{X'}, X') \to (U_X, X)$ and $\sigma_Y : (U_{Y'}, Y') \to (U_Y, Y)$ with smooth centers sitting inside the boundaries so that the induced morphism $f' : (U_{X'}, X') \to (U_{Y'}, Y')$ is "log smooth", or equivalently, toroidal. The notion of a morphism being log smooth was introduced by Kato [1] together with that of the log category. (See also the log category and logarithmic ramification formula of Iitaka [1].) It gives us an intrinsic justification of the toroidalization conjecture, which otherwise might stand as a mere technical speculation. Interpreted in this way, the toroidalization conjecture may be regarded as the problem of resolution of singularities of

morphisms (maps) in the log-category, similar to the problem of resolution of singularities of varieties in the usual category. The pursuit of a canonical algorithm for toroidalization is the future topic which invokes intensive research of today.

CHAPTER 1. PRELIMINARIES

In the first part of this chapter, we recall some basic definitions and facts from Geometric Invariant Theory (G.I.T. for short, cf. Kirwan-Fogarty-Mumford [1]) and Toric Geometry (cf. Danilov [1] Fulton [1] Kempf-Knudsen-Mumford-SaintDonat [1] Oda [1]), mainly to fix the notation for these lecture notes.

The second part is devoted to the discussion of "**locally toric**" and "**toroidal**" structures. Since one of the critical issues in our argument is the distinction between the two notions and since in the existing literatures they appear in some mixed terminologies, the understanding of this part is crucial as well as basic.

In the third part of this chapter, we discuss the actions of the multiplicative group K^* of the base field on the locally and/or toroidal structures. In the course of studying birational transformations via birational cobordisms, we would like to introduce locally toric and/or toroidal structures not only on the birational cobordisms but also on their quotients by the actions of K^*. This naturally leads us to the (well-known) notion of a "strongly étale" K^*-equivariant morphism, utilized in the form of **Luna's Fundamental Lemma** and **Luna's Étale Slice Theorem**.

In these lecture notes, all the varieties are assumed to be irreducible and reduced schemes of finite type over an algebraically closed field K of characteristic zero, EXCEPT for Chapter 5 where we specifically try to loosen some of the restrictions.

§1-1. Brief Review of G.I.T. and Toric Geometry

Review 1-1-1 (Geometric Invariant Theory).

Suppose a reductive group G acts on an algebraic variety X. We denote by X/G what we call the geometric quotient, i.e., the space of orbits, and by $X//G$ the space of equivalence classes of orbits, where the equivalence relation is generated by the condition that two orbits are equivalent if their closures intersect.

These spaces are endowed with the structures (of algebraic varieties) which satisfy the usual universal property (if they exist in the category of algebraic varieties).

In most of our situations,

- G is taken to be the multiplicative group K^* or a finite group,
- when we take the quotient $X//G$, a variety X is such that the closure of any orbit contains a unique closed orbit, i.e., contains a unique fixed point in case $G = K^*$ (See the definition of a quasi-elementary cobordism in Chapter 2.),
- if a variety X is normal, then the quotient X/G or $X//G$ is also a normal variety (if it exists in the category of algebraic varieties) .

If X is endowed with an ample line bundle $\mathcal{L}$ with a linear G-action compatible with the G-action on X (called a linearization of the action), then we denote by

$X^{ss}_{\mathcal{L}}$ the set of semi-stable points with respect to this linearization

$$X^{ss}_{\mathcal{L}} = \{x \in X; \exists s \in H^0(X, \mathcal{L}^{\otimes n})^G \text{ for some } n \in \mathbb{N} \text{ s.t. } s(x) \neq 0\}$$

and by $X^{ss}_{\mathcal{L}}//G$ the so-called G.I.T. quotient, which has the structure of a projective algebraic variety

$$X^{ss}_{\mathcal{L}}//G = \text{Proj} \oplus_{n \geq 0} H^0(X, \mathcal{L}^{\otimes n})^G.$$

We discuss more on G.I.T. related to the work of Thaddeus and others, i.e., the subject of the change of the G.I.T. quotients associated to the change of linearizations in Chapter 2.

Review 1-1-2 (Toric Geometry).

Let $N \cong \mathbb{Z}^n$ be a lattice and $\sigma \subset N_{\mathbb{R}} = N \otimes \mathbb{R}$ be a strictly convex rational polyhedral cone. We denote the dual lattice by $M = Hom_{\mathbb{Z}}(N, \mathbb{Z})$ and the dual cone by $\sigma^\vee \subset M_{\mathbb{R}} = M \otimes \mathbb{R}$. The affine toric variety X associated to σ is defined to be

$$X = X(N, \sigma) = \text{Spec } K[M \cap \sigma^\vee],$$

containing the torus

$$X \supset T_X = \text{Spec } K[M].$$

The lattice N is identified with the group of one-parameter subgroups of T_X and the dual lattice with their characters. As we only consider toric varieties associated to the saturated lattices $M \cap \sigma^\vee$ as above, they are all normal by definition.

More generally the toric variety associated to a fan Σ in $N_{\mathbb{R}}$ is denoted by $X(N, \Sigma)$.

If $X_1 = X(N, \Sigma_1)$ and $X_2 = X(N, \Sigma_2)$ are toric varieties having the same lattice N, the embeddings of the torus $T_{X_1} = T_{X_2} = \text{Spec } K[M]$ define an equivariant birational map $\phi : X_1 \dashrightarrow X_2$, called a toric birational map. This map ϕ is a morphism if and only if every cone in Σ_1 is contained in a cone in Σ_2 and proper if and only if the support of Σ_1 coincides with that of Σ_2.

Suppose that K^* acts on an affine toric variety $X = X(N, \sigma)$ as a one-parameter subgroup of the torus T_X, corresponding to a lattice point $a \in N$. If $t \in K^*$ and $m \in M$, the action on the monomial z^m is given by

$$t^*(z^m) = t^{(a,m)} z^m,$$

where $(\cdot, \cdot)$ is the natural pairing on $N \times M$. (For $m \in M$, we denote its image in the semi-group algebra $K[M \cap \sigma^\vee]$ by z^m.)

The K^*-invariant monomials correspond to the lattice points $M \cap a^\perp$, hence

$$X//K^* = \text{Spec } K[M \cap \sigma^\vee \cap a^\perp].$$

Let $\pi : N_{\mathbb{R}} \to N_{\mathbb{R}}/a \cdot \mathbb{R}$ be the projection. Then the lattice $M \cap a^\perp$ is dual to the lattice $\pi(N)$ and

$$M \cap \sigma^\vee \cap a^\perp = Hom_{\mathbb{Z}}(\pi(N), \mathbb{Z}) \cap \pi(\sigma)^\vee.$$

Therefore, $X//K^* = X(\pi(N), \pi(\sigma))$ is again an affine toric variety defined by the lattice $\pi(N)$ and the cone $\pi(\sigma)$ (though $\pi(\sigma)$ may no longer be strictly convex). This quotient is a geometric quotient precisely when $\pi : \sigma \to \pi(\sigma)$ is a linear isomorphism (while the map between the lattices $\pi : N \cap \sigma \to \pi(N) \cap \pi(\sigma)$ may not be an isomorphism).

§1-2. "Locally Toric" Structures vs. "Toroidal" Structures

There is some confusion in the existing literatures among the terminologies expressing the notion of toroidal embeddings and toroidal maps (cf. Kempf-Knudsen-Mumford-SaintDonat [1] Abramovich-Karu [2]) and that of toroidal varieties (cf. Danilov [1] Włodarczyk [2]), the latter of which we prefer to call locally toric varieties (and locally toric maps). Since one of the critical issues in our argument is the distinction of the two notions "locally toric" and "toroidal", we clarify our usage of these two terminologies below.

Definition 1-2-1 (Locally Toric & Toroidal Structures).
(i) A variety W is locally toric if for every closed point $p \in W$ there exists a closed point $x \in X$ in an affine toric variety such that we have an isomorphism of complete local rings over K

$$\widehat{\mathcal{O}_{W,p}} \quad \tilde{\to} \quad \widehat{\mathcal{O}_{X,x}}.$$

(ii) A pair (U_W, W) consisting of a dense open subset $U_W \subset W$ contained in a variety W is called a toroidal embedding (or alternatively we say (U_W, W) has a toroidal structure) if for every closed point $p \in W$ there exists a closed point $x \in X$ in an affine toric variety such that we have an isomorphism of complete local rings over K

$$\widehat{\mathcal{O}_{W,p}} \quad \overset{\eta^*}{\tilde{\to}} \quad \widehat{\mathcal{O}_{X,x}}$$

which induces an isomorphism of ideals defining the boundaries

$$(\widehat{\mathcal{I}_{W-U_W}})_p = \mathcal{I}_{W-U_W} \otimes \widehat{\mathcal{O}_{W,p}} \quad \overset{\eta^*}{\tilde{\to}} \quad (\widehat{\mathcal{I}_{X-T_X}})_x = \mathcal{I}_{X-T_X} \otimes \widehat{\mathcal{O}_{X,x}}$$

where $\mathcal{I}_{W-U_W}$ (resp. $\mathcal{I}_{X-U_X}$) is the ideal defining the reduced structure on the boundary $W - U_W$ (resp. $X - T_X$).

If all the irreducible components of the boundary $W - U_W$ is normal, then (U_W, W) is called a toroidal embedding **without self-intersection**.

Definition 1-2-2 (Locally Toric & Toroidal Birational Morphisms).
(i) A proper birational morphism $f : W_1 \to W_2$ between locally toric varieties is locally toric if for every closed point $p_1 \in W_1$ mapping to $p_2 = f(p_1) \in W_2$, there exists a toric morphism $\varphi : X_1 \to X_2$ mapping a closed point $x_1 \in X_1$ to $x_2 = \varphi(p_1) \in X_2$ such that we have a commutative diagram of homomorphisms between complete local rings

$$\begin{array}{ccc} \widehat{\mathcal{O}_{W_1,p_1}} & \overset{\eta_1^*}{\tilde{\to}} & \widehat{\mathcal{O}_{X_1,x_1}} \\ {\scriptstyle f^*}\uparrow & & \uparrow{\scriptstyle \varphi^*} \\ \widehat{\mathcal{O}_{W_2,p_2}} & \overset{\eta_2^*}{\tilde{\to}} & \widehat{\mathcal{O}_{X_2,x_2}}. \end{array}$$

(ii) A proper birational morphism $f : (U_{W_1}, W_1) \to (U_{W_2}, W_2)$ between toroidal embeddings is toroidal if for every closed point $p_1 \in W_1$ mapping to $p_2 = f(p_1) \in W_2$, there exists a toric morphism $\varphi : X_1 \to X_2$ mapping a closed point $x_1 \in X_1$ to $x_2 = \varphi(p_1) \in X_2$ such that we have a commutative diagram of homomorphisms between complete local rings

$$\begin{array}{ccc} \widehat{\mathcal{O}_{W_1,p_1}} & \overset{\eta_1^*}{\leftarrow} & \widehat{\mathcal{O}_{X_1,x_1}} \\ {\scriptstyle f^*}\uparrow & & \uparrow{\scriptstyle \varphi^*} \\ \widehat{\mathcal{O}_{W_2,p_2}} & \overset{\eta_2^*}{\leftarrow} & \widehat{\mathcal{O}_{X_2,x_2}} \end{array}$$

which induces a commutative diagram of ideals defining the boundaries

$$\begin{array}{ccc} \widehat{(\mathcal{I}_{W_1-U_{W_1}})_{p_1}} & \overset{\eta_1^*}{\leftarrow} & \widehat{(\mathcal{I}_{X_1-U_{X_1}})_{x_1}} \\ {\scriptstyle f^*}\uparrow & & \uparrow{\scriptstyle \varphi^*} \\ \widehat{(\mathcal{I}_{W_1-U_{W_1}})_{p_2}} & \overset{\eta_2^*}{\leftarrow} & \widehat{(\mathcal{I}_{X_1-U_{X_1}})_{x_2}}. \end{array}$$

Recall that a birational map $f : W_1 \dashrightarrow W_2$ is proper (cf. Iitaka [1]) if both projections $p_1 : \Gamma_f \to W_1$ and $p_2 : \Gamma_f \to W_2$ are proper morphisms in the usual sense, where $\Gamma_f \subset W_1 \times W_2$ is the graph of f. (If f is well-defined as a morphism over a dense open subset $U \subset W_1$, then Γ_f is the closure in $W_1 \times W_2$ of the graph of the morphism $f|_U : U \to W_2$. This closure is independent of the choice of the open subset U.)

Definition 1-2-3 (Locally Toric & Toroidal Birational Maps).
(i) A proper birational map $f : W_1 \dashrightarrow W_2$ between locally toric varieties is locally toric if there exists another locally toric variety Z which dominates both W_1 and W_2 by proper birational morphisms

$$W_1 \overset{f_1}{\leftarrow} Z \overset{f_2}{\rightarrow} W_2$$

where we require f_1 and f_2 to be locally toric as in Definition 1-2-2 (i).

(ii) A proper birational map $f : (U_{W_1}, W_1) \dashrightarrow (U_{W_2}, W_2)$ between toroidal embeddings is toroidal if there exists another toroidal embedding (U_Z, Z) which dominates both (U_{W_1}, W_1) and (U_{W_2}, W_2) by proper birational morphisms

$$(U_{W_1}, W_1) \overset{f_1}{\leftarrow} (U_Z, Z) \overset{f_2}{\rightarrow} (U_{W_2}, W_2)$$

where we require f_1 and f_2 to be toroidal as in Definition 1-2-2 (ii).

Remark 1-2-4.

(i) Though we presented the notions of "locally toric" and "toroidal" as if they were parallel and of equal importance, the collection of locally toric varieties

with locally toric birational morphisms and/or birational maps does NOT form a category, in clear contrast to the fact that the collection of toroidal embeddings with toroidal birational morphisms and/or toroidal birational maps does form a category as shown in Proposition 1-2-5.

A composition of locally toric birational morphisms or maps is NOT locally toric in general (cf. Proposition 1-2-5 (i) (ii) for the toroidal case): We take $g = f_2 \circ f_1$ to be a composite of the blowup $f_2 : W_2 \to W_3$ of a point on a smooth threefold W_3 and the blowup $f_1 : W_1 \to W_2$ of a smooth curve on W_2 which is not transversal to the exceptional divisor for f_2. Then while f_1 and f_2 are locally toric, the composition g is not.

It is NOT clear and probably not true (cf. Proposition 1-2-5 (iii) for the toroidal case) that the definition of being locally toric for a birational morphism in Definition 1-2-2 (i) is compatible with the definition of being locally toric for a birational map in Definition 1-2-3 (i), though the author does not have a specific example showing incompatibility.

(ii) There are examples where a locally toric birational morphism $f : W_1 \to W_2$ cannot be toroidal no matter how we introduce toroidal structures on W_1 and W_2: Let W_2 be a smooth surface. First we blow up a point on W_2 and then blow up 3 or more distinct points on the exceptional divisor to obtain $f : W_1 \to W_2$.

Proposition 1-2-5.

(i) A composition of two proper toroidal birational morphisms is again proper toroidal.

(ii) A composition of two proper toroidal birational maps is again proper toroidal.

(iii) A proper birational morphism between toroidal embeddings is toroidal in the sense of Definition 1-2-3 (ii) if and only if it is toroidal in the sense of Definition 1-2-2 (ii).

In particular, the collection of toroidal embeddings with proper toroidal birational morphisms and/or proper toroidal birational maps forms a category.

Proof.

The assertion (i) is proved in Abramovich-Karu [1] Karu [1] for a composition of general (not necessarily birational) toroidal morphisms.

The assertions (ii) and (iii) can be proved as an application of the following lemmas, after reducing the case of general toroidal embeddings to that of toroidal embeddings without self-intersection via some appropriate toroidal blowups and base changes.

Lemma 1-2-6. *Let (U_W, W) be a toroidal embedding. Then there exists a sequence $(U_{\tilde{W}}, \tilde{W}) \to (U_W, W)$ of toroidal blowups with smooth centers (i.e., the centers analytically locally correspond to the smooth closures of orbits) such that the resulting toroidal embedding $(U_{\tilde{W}}, \tilde{W})$ is nonsingular and without self-intersection.*

Proof.

First take the canonical resolution of singularities $r : W^{res} \to W$ and set $U_{W^{res}} = r^{-1}(U_W)$. Then by the property ($\spadesuit^{res} - 1$) of the canonical resolution (We refer the reader to Remark 4-1-1 for the details of the properties of the canonical

resolution of singularities.) we conclude that $r : (U_{W^{res}}, W^{res}) \to (U_W, W)$ is a toroidal morphism (between toroidal embeddings) obtained as a sequence of toroidal blowups with smooth centers. Now starting from blowing up the 0-dimensional strata of the boundary components of $(U_{W^{res}}, W^{res})$, we consecutively blow up (the closures of) the strict transforms of the original k-dimensional strata of the boundary of the original toroidal structure $(U_{W^{res}}, W^{res})$ for $k = 1, \cdots, \dim W - 1$ to obtain $s : (U_{\tilde{W}} = s^{-1}(U_{W^{res}}), \tilde{W}) \to (U_{W^{res}}, W^{res})$. (Note that the closures of the strata of the boundary components are the closures of the generic points which, analytically locally, correspond to the intersections of the boundary divisors of the toroidal charts.) This is a sequence of toroidal blowups with smooth centers so that the resulting toroidal embedding $(U_{\tilde{W}}, \tilde{W})$ is nonsingular and without self-intersection.

Lemma 1-2-7. *Let $f : (U_{W_1}, W_1) \to (U_{W_2}, W_2)$ be a proper toroidal birational morphism between toroidal embeddings without self-intersection. Then f is allowable (in the sense of Kempf-Knudsen-Mumford-SaintDonat [1]). In particular, such proper toroidal birational morphisms over (U_{W_2}, W_2) are in one-to-one correspondence with the subdivisions of the conical complex associated to the toroidal embedding (U_{W_2}, W_2) without self-intersection.*

Proof.

For a proof, we refer the reader to Abramovich-Karu [1] Karu [1] and Kempf-Knudsen-Mumford-SaintDonat [1].

Now we go back to the proof of Proposition 1-2-5.

In order to see the assertion (ii), suppose

$$f : (U_{W_1}, W_1) \dashrightarrow (U_{W_2}, W_2) \text{ and } g : (U_{W_2}, W_2) \dashrightarrow (U_{W_3}, W_3)$$

are proper toroidal birational maps among toroidal embeddings. Let

$$(U_{W_1}, W_1) \xleftarrow{f_1} (U_Z, Z) \xrightarrow{f_2} (U_{W_2}, W_2) \text{ and } (U_{W_2}, W_2) \xleftarrow{g_2} (U_{Z'}, Z') \xrightarrow{g_3} (U_{W_3}, W_3)$$

be some toroidal embeddings dominating $(U_{W_1}, W_1), (U_{W_2}, W_2)$ and (U_{W_3}, W_3) by proper toroidal birational morphisms as required in Definition 1-2-3 (ii). By taking a sequence of toroidal blowups with smooth centers of (U_{W_2}, W_2) using Lemma 1-2-6, we can make it without self-intersection. Since f_2 and g_2 are toroidal, we can "pull-back" these toroidal blowups to the toroidal blowups on (U_Z, Z) and on $(U_{Z'}, Z')$. Again using Lemma 1-2-6 and the assertion (i), we may assume that $(U_Z, Z), (U_{W_2}, W_2)$ and $(U_{Z'}, Z')$ are all without self-intersection. By Lemma 1-2-7, the morphisms f_2 and g_2 are associated with the subdivisions Δ_Z and $\Delta_{Z'}$ of the conical complex Δ_{W_2}. Let $(U_{\hat{Z}}, \hat{Z}) \to (U_{W_2}, W_2)$ be the toroidal embedding associated to the common refinement $\Delta_{\hat{Z}}$ of Δ_Z and $\Delta_{Z'}$. Then by construction $(U_{\hat{Z}}, \hat{Z})$ dominates (U_Z, Z) and $(U_{Z'}, Z')$ by proper toroidal birational morphisms

$$(U_{W_1}, W_1) \xleftarrow{f_1} (U_Z, Z) \xleftarrow{\hat{f}_1} (U_{\hat{Z}}, \hat{Z}) \xrightarrow{\hat{g}_3} (U_{Z'}, Z') \xrightarrow{g_3} (U_{W_3}, W_3).$$

Since the compositions $f_1 \circ \hat{f}_1$ and $g_3 \circ \hat{g}_3$ are both toroidal by Lemma 1-2-6, we conclude that the composition $g \circ f$ is also a toroidal birational map by Definition 1-2-3 (ii).

In order to see the assertion (iii), let $f : (U_{W_1}, W_1) \to (U_{W_2}, W_2)$ be a proper birational morphism between toroidal embeddings. If f is toroidal as a birational morphism according to Definition 1-2-2 (ii), then it is obviously toroidal as a birational map according to Definition 1-2-3 (ii) by taking the diagram

$$(U_{W_1}, W_1) \overset{identity}{\leftarrow} (U_{W_1}, W_1) \overset{f}{\to} (U_{W_2}, W_2).$$

Thus we only have to show the converse, i.e., if f is toroidal as a birational map as in Definition 1-2-3 (ii), then it is toroidal as a birational morphism as in Definition 1-2-2 (ii).

According to Definition 1-2-3 (ii), there exists another toroidal embedding (U_Z, Z) which dominates both (U_{W_1}, W_1) and (U_{W_2}, W_2)

$$(U_{W_1}, W_1) \overset{f_1}{\leftarrow} (U_Z, Z) \overset{f_2}{\to} (U_{W_2}, W_2)$$

where f_1 and f_2 are proper toroidal birational morphisms according to Definition 1-2-2 (ii). By Lemma 1-2-6 and Proposition 1-2-5 (i), we may assume that (U_Z, Z) is a nonsingular toroidal embedding without self-intersection.

Let $p_1 \in W_1$ be a closed point mapping to $p_2 = f(x_1) \in W_2$.

Via the analytic isomorphisms as in the definition of the toroidal structures

$$\widehat{\mathcal{O}_{W_1,p_1}} \overset{\sim}{\to} \widehat{\mathcal{O}_{X_1,x_1}}$$
$$\widehat{\mathcal{O}_{W_2,p_2}} \overset{\sim}{\to} \widehat{\mathcal{O}_{X_2,x_2}}$$

we pull back the standard coordinates for the tori

$$\{z_1, \cdots, z_n\} \text{ for the torus } T_{X_1} = \text{Spec } K[z_1^{\pm 1}, \cdots, z_n^{\pm 1}]$$
$$\{w_1, \cdots, w_n\} \text{ for the torus } T_{X_2} = \text{Spec } K[w_1^{\pm 1}, \cdots, w_n^{\pm 1}],$$

which we denote by the same letters by abuse of notation.

We only have to show the claim that there exists a unimodular $n \times n$ matrix $A = \{a_{ij}\}$ and units $u_i \in \widehat{\mathcal{O}_{W_1,p_1}}$ for $i, j = 1, \cdots, n$ such that

$$f^* w_i = u_i \prod z_j^{a_{ij}} \text{ in the field of fractions of } \widehat{\mathcal{O}_{W_1,p_1}}.$$

Remark that

$$\hat{f}_1 : (U_Z, Z) \times \text{Spec } \widehat{\mathcal{O}_{W_1,p_1}} \to (U_{W_1}, W_1) \times \text{Spec } \widehat{\mathcal{O}_{W_1,p_1}}$$
$$\hat{f}_2 : (U_Z, Z) \times \text{Spec } \widehat{\mathcal{O}_{W_2,p_2}} \to (U_{W_2}, W_2) \times \text{Spec } \widehat{\mathcal{O}_{W_2,p_2}}$$

are both proper birational toroidal morphisms between toroidal embeddings WITHOUT self-intersection and hence allowable and that they correspond to the subdivisions of the conical complexes by Lemma 1-2-7, which we denote by

$$\Delta_{\hat{f}_1} : \Delta_{Z,p_1} \to \Delta_{W_1,p_1}$$
$$\Delta_{\hat{f}_2} : \Delta_{Z,p_2} \to \Delta_{W_2,p_2}.$$

Remark also that the natural map $\Delta_{Z,p_1} \to \Delta_{Z,p_2}$ (induced by $(U_Z, Z) \times \operatorname{Spec} \widehat{\mathcal{O}_{W_1,p_1}} \to (U_Z, Z) \times \operatorname{Spec} \widehat{\mathcal{O}_{W_2,p_2}}$) is an immersion, since $f_1^{-1}(p_1) \subset f_2^{-1}(p_2)$ and since the completion of the local ring at $p \in f_1^{-1}(p_1) \subset f_2^{-1}(p_2)$ is the same no matter which ambient space we choose.

For the sake of simplicity of the presentation, we only deal with the case where $\dim \Delta_{Z,p_1} = \dim \Delta_{Z,p_2} = n = \dim W_1 = \dim W_2$, leaving the general case of $\dim \Delta_{Z,p_1} \leq \dim \Delta_{Z,p_2} \leq n = \dim W_1 = \dim W_2$ to the reader via the use of some arguments in Abramovich-Karu [1] Karu [1].

In this case, Δ_{Z,p_1} is a conical complex of full dimension immersed in another conical complex Δ_{Z,p_2} with the compatible integral structures. Since Δ_{Z,p_1} is a subdivision of Δ_{W_1,p_1} and since Δ_{Z,p_2} is a subdivision of Δ_{W_2,p_2}, the pull-backs of $\{z_1, \cdots, z_n\}$ and $\{w_1, \cdots, w_n\}$ form the dual $\mathbb{Z}$-bases of these compatible integral structures. Since f_2 factors as $f_2 = f_1 \circ f$, this implies that $\{f^* w_1, \cdots, f^* w_n\}$ as well as $\{z_1, \cdots, z_n\}$ forms a dual basis of the integral structure of Δ_{W_1,p_1}. Therefore, we have the claim.

This completes the proof of Proposition 1-2-5.

Remark 1-2-8.

(i) As we have seen in Remark 1-2-4 (i), a composition of two locally toric birational morphisms is not necessarily locally toric, since the locally toric structures for the two locally toric morphisms may not be compatible in the middle. Even worse, the locally toric structures for the morphisms $W_1 \stackrel{f_1}{\leftarrow} Z$ and $Z \stackrel{f_2}{\to} W_2$ in Definition 1-2-3 (i) of a locally toric birational map may not be compatible on Z. Thus it is nontrivial to prove the assertions (i), (ii) and (iii) in Proposition 1-2-5, settleing a rather subtle issue of the compatibility of toroidal structures for several different toroidal morphisms.

(ii) We define a stronger version of the notion of locally toric and toroidal birational maps, namely the notion of V-locally toric and V-toroidal birational maps below. These turn out to be the only maps we have to consider in our notes, the fact which enables us to avoid dealing with the subtleties as above.

Therefore, except for Lemma 1-2-6 and Lemma 1-2-7, we will NOT use the results of Proposition 1-2-5 in the later argument of our proof.

Actually the only place where Lemma 1-2-7 appears in our notes is when we use the allowablity in the proof of Theorem 4-2-1. However, in the case of V-toroidal birational maps the allowablity (after reducing the factorization to that of a birationa map between toroidal embeddings without self-intersection) in Theorem 4-2-1 is automatic and hence our argument does not even use Lemma 1-2-7 in an essential way (though it appears in Theorem 4-2-1 dealing with the factorization of general toroidal birational maps).

Remark 1-2-9 (Comparison with the definitions in Abramovich-Karu-Matsuki-Włodarczyk [1]).

In Abramovich-Karu-Matsuki-Włodarczyk [1], they give more restrictive definitions for locally toric and toroidal structures as well as for locally toric and

toroidal birational morphisms/maps via the use of (allowable) Zariski locally toric and toroidal charts.

Definition of locally toric and toroidal structures in Abramovich-Karu-Matsuki-Włodarczyk [1]:

(i) A variety W is locally toric if for every closed point $p \in W$ there exists a Zariski open neighborhood $V_p \subset W$ of p and an étale morphism $\eta_p : V_p \to X_p$ to a toric variety X_p.

(ii) A pair (U_W, W) consisting of a dense open subset $U_W \subset W$ in a variety W is called a toroidal embedding if for every closed point $p \in W$ there exists a Zariski open neighborhood V_p and an étale morphism $\eta_p : V_p \to X_p$ to a toric variety such that $\eta_p^{-1}(T_{X_p}) = U_W$ where T_{X_p} is the torus for X_p.

Definition of locally toric and toroidal morphisms in Abramovich-Karu-Matsuki-Włodarczyk [1]:

(i) A proper birational morphism $f : W_1 \to W_2$ between locally toric varieties is called locally toric if for every closed point $p_2 \in W_2$ there exists a Zariski open neighborhood V_{p_2} and an étale morphism $\eta_{p_2} : V_{p_2} \to X_{p_2}$ to a toric variety, together with a toric proper birational morphism $Y_{p_2} \to X_{p_2}$ such that we have a fiber square

$$\begin{array}{ccc} f^{-1}(V_{p_2}) = V_{p_2} \times_{X_{p_2}} Y_{p_2} & \to & Y_{p_2} \\ \downarrow & \square & \downarrow \\ V_{p_2} & \to & X_{p_2} \end{array}$$

(ii) A proper birational morphism $f : (U_{W_1}, W_1) \to (U_{W_2}, W_2)$ between toroidal embeddings is called toroidal if for every closed point $p_2 \in W_2$ there exists a Zariski open neighborhood V_{p_2} and an étale morphism $\eta_{p_2} : V_{p_2} \to X_{p_2}$ to a toric variety, together with a toric proper birational morphism $Y_{p_2} \to X_{p_2}$ such that we have a fiber square of toroidal embeddings

$$\begin{array}{ccc} (U_{W_1} \cap f^{-1}(V_{p_2}), f^{-1}(V_{p_2})) = V_{p_2} \times_{X_{p_2}} (T_{Y_{p_2}}, Y_{p_2}) & \to & (T_{Y_{p_2}}, Y_{p_2}) \\ \downarrow & \square & \downarrow \\ (U_{W_2} \cap V_{p_2}, V_{p_2}) = V_{p_2} \times_{X_{p_2}} (T_{X_{p_2}}, X_{p_2}) & \to & (T_{X_{p_2}}, X_{p_2}) \end{array}$$

Definition of locally toric and toroidal birational maps in Abramovich-Karu-Matsuki-Włodarczyk [1]:

(i) A proper birational map $f : W_1 \dashrightarrow W_2$ between locally toric varieties is locally toric if there exists another locally toric variety Z which dominates both W_1 and W_2 by proper birational morphisms

$$W_1 \stackrel{f_1}{\leftarrow} Z \stackrel{f_2}{\to} W_2$$

where we require f_1 and f_2 to be locally toric as in (i) above.

(ii) A proper birational map $f : (U_{W_1}, W_1) \dashrightarrow (U_{W_2}, W_2)$ between toroidal embeddings is toroidal if there exists another toroidal embedding (U_Z, Z) which dominates both (U_{W_1}, W_1) and (U_{W_2}, W_2) by proper birational morphisms

$$(U_{W_1}, W_1) \stackrel{f_1}{\leftarrow} (U_Z, Z) \stackrel{f_2}{\to} (U_{W_2}, W_2)$$

where we require f_1 and f_2 to be toroidal as in (ii) above.

The above definitions differ from ours in several subtle ways. Here are a couple of differences:

∘ The toroidal embeddings in their sense are automatically without self-intersection, while those in our sense may not. For example, a pair (U_S, S) consisting of a nonsingular surface S with an irreducible nodal curve as a boundary component $S - U_S$ is a toroidal embedding in our sense but not in their sense.

∘ Locally toric birational morphisms in our sense may not be locally toric in their sense. For example, as in Remark 1-2-4, if we blowup a point on a smooth surface W_2 and further blowup three or more points on the exceptional divisor to obtain W_1, then the morphism $f : W_1 \to W_2$ is locally toric in our sense but not in their sense.

On the other hand, we would like to emphasize that for the category of toroidal embeddings WITHOUT self-intersection and proper birational morphisms and maps, their definitions and our definitions coincide.

In fact, we have the following:

∘ The toroidal embedding without self-intersection in our sense always has a Zariski toroidal chart as required in their definition.

Let $p \in (U_W, W)$ be a closed point in a toroidal embedding without self-intersection in our sense and let $x \in X$ be a closed point in an affine toric variety as described in Definition 1-2-1 (ii). The Cartier divisors in Spec $\widehat{\mathcal{O}_{W,p}}$ = Spec $\widehat{\mathcal{O}_{X,x}}$ supported on the support of $\widehat{\mathcal{O}_{W,p}}/\widehat{I_{W-U_W}} = \widehat{\mathcal{O}_{X,x}}/\widehat{I_{X-T_X}}$ give rise to a lattice in a cone of effective Cartier divisors (the dual cone sits in the conical complex associated to (U_W, W), which is well-defined as (U_W, W) is without self-intersection). We only provide a proof in case the dual cone has the full dimension (being equal to $n = \dim W$), leaving the general case as an exercise to the reader. Let $z_1, \cdots, z_n$ be the standard coordinates for the torus T_X = Spec $K[z_1^{\pm}, \cdots, z_n^{\pm}]$. Since $z_1, \cdots, z_n$ define Cartier divisors which form a $\mathbb{Z}$-basis of the lattice and since (U_W, W) is without self-intersection, we can find rational functions $r_1, \cdots, r_n \in K(W)$ for W such that the Cartier divisors associated to them coincide, analytically locally at p, with those associated to $z_1, \cdots, z_n$. Now let $m_1, \cdots, m_l$ be the monomials in $z_1, \cdots, z_n$ generating the affine coordinate ring for X and hence the maximal ideal corresponding to the point $x \in X$, and let $m_1', \cdots, m_l'$ be the same monomials but in $r_1, \cdots, r_n$. In a suitable affine Zariski open neighborhood V_p, we may assume all the monomials $m_1', \cdots, m_l'$ are regular over V_p. It is easy to see $r_1, \cdots, r_n$ are transcendental over K, as well as $z_1, \cdots, z_n$ are over K. Therefore, we have a well-defined morphism

$$\eta_p : V_p \to X_p = X$$

defined by sending $m_1, \cdots, m_l$ to $m_1', \cdots, m_l'$, respectively. Now by construction we see that the homomorphism

$$\eta_p{}^* : \widehat{\mathcal{O}_{X,x}} \to \widehat{\mathcal{O}_{V,p}}$$

is surjective and that it is also injective as so is $\eta_p{}^* : A(X) \to A(V_p)$, hence an isomorphism. (Note that we are in the situation where we DO have the preservation

of injectivity at the completion level, unlike the counter-examples in Gabrielov [1][2] to a conjecture of Grothendieck [1]. See, e.g., Hübl [1].) Therefore, by shrinking V_p, we may assume η_p is étale. By shrinking V_p further, we may also assume that all the irreducible divisors in $V_p \cap (W - U_W)$ contain p and hence $\eta_p^{-1}(T_X) = U_W \cap V_p$. Thus $\eta_p : V_p \to X_p$ is a desired Zariski toroidal chart.

∘ The toroidal birational morphisms in our sense between toroidal embeddings without self-intersection are toroidal in their sense.

Let $f : (U_{W_1}, W_1) \to (U_{W_2}, W_2)$ be a proper birational morphism which is toroidal in our sense according to Definition 1-2-2 (ii). For a point $p_2 \in W_2$ we take a Zariski toroidal chart $\eta_p : V_{p_2} \to X_{p_2}$ constructed as above. By shrinking V_{p_2} and X_{p_2} we may assume that the Cartier divisors in V_{p_2} supported on $V_{p_2} \cap (W - U_W)$ are in one-to-one correspondence with Cartier divisors in X_{p_2} supported on $X_{p_2} - T_{X_{p_2}}$ and hence that the conical complexes $\Delta_{V_{p_2}}$ associated to $V_{p_2} \cap (W - U_W)$ and $\Delta_{X_{p_2}}$ associated to X_{p_2} coincide. Now by the results of Abramovich-Karu [1] Karu [1], f is allowable and hence corresponds to a subdivision of $\Delta_{V_{p_2}}$ over V_{p_2}. By taking $Y_{p_2} \to X_{p_2}$ to be the toric birational morphism corresponding to the same subdivision of $\Delta_{X_{p_2}}$, we obtain the commutative diagram of fiber squares

$$\begin{array}{ccc} (U_{W_1} \cap f^{-1}(V_{p_2}), f^{-1}(V_{p_2})) = V_{p_2} \times_{X_{p_2}} (T_{Y_{p_2}}, Y_{p_2}) & \to & (T_{Y_{p_2}}, Y_{p_2}) \\ \downarrow & \square & \downarrow \\ (U_{W_2} \cap V_{p_2}, V_{p_2}) = V_{p_2} \times_{X_{p_2}} (T_{X_{p_2}}, X_{p_2}) & \to & (T_{X_{p_2}}, X_{p_2}). \end{array}$$

Therefore, f is also toroidal in their sense.

Now that we see that their definitions and our definitons coincide for the category of toroidal embeddings without self-intersection, the statements of Proposition 1-2-5 (i), (ii) and (iii) also hold using their definitions in the category of toroidal embeddings without self-intersection.

Definition 1-2-10 (V-Locally Toric & V-Toroidal Birational Maps).

(i) A proper birational map $f : W_1 \dashrightarrow W_2$ between locally toric varieties is called V-locally toric if there exists another locally toric variety Y which both W_1 and W_2 dominate by proper birational morphisms

$$W_1 \to Y \leftarrow W_2$$

such that for every closed point $y \in Y$ we can find étale morphisms $i : V \to Y$ with $y \in i(V)$ and $\eta : V \to X_Y$ to an affine toric variety X_Y which some toric varieties X_1 and X_2 dominate by proper toric birational morphisms

$$X_1 \to X_Y \leftarrow X_2$$

with the property that when we take the fiber-products with V they coincide, i.e., we have a commutative diagram of the form

$$\begin{array}{ccccc}
W_1 & \to & Y & \leftarrow & W_2 \\
\uparrow & \square & \uparrow & \square & \uparrow \\
W_1 \times_Y V & \to & V & \leftarrow & W_2 \times_Y V \\
\| & & \| & & \| \\
X_1 \times_{X_Y} V & \to & V & \leftarrow & X_2 \times_{X_Y} V \\
\downarrow & \square & \downarrow & \square & \downarrow \\
X_1 & \to & X_Y & \leftarrow & X_2.
\end{array}$$

(Note that X_Y and hence X_1 & X_2 are allowed to vary depending on the points $y \in Y$.)

(ii) A proper birational map $f : (U_{W_1}, W_1) \dashrightarrow (U_{W_2}, W_2)$ between toroidal embeddings is called V-toroidal if there exists another toroidal embedding (U_Y, Y) which both (U_{W_1}, W_1) and (U_{W_2}, W_2) dominate by proper birational morphisms

$$(U_{W_1}, W_1) \to (U_Y, Y) \leftarrow (U_{W_2}, W_2)$$

such that for every closed point $y \in Y$ we can find étale morphisms $i : V \to Y$ with $y \in i(V)$ and $\eta : V \to X_Y$ to an affine toric variety X_Y which some toric varieties X_1 and X_2 dominate by proper toric birational morphisms

$$X_1 \to X_Y \leftarrow X_2$$

with the property that when we take the fiber-products with V they coincide, i.e., we have a commutative diagram of the form

$$\begin{array}{ccccc}
(U_{W_1}, W_1) & \to & (U_Y, Y) & \leftarrow & (U_{W_2}, W_2) \\
\uparrow & \square & \uparrow & \square & \uparrow \\
(U_{W_1}, W_1) \times_Y V & \to & (U_Y, Y) \times_Y V & \leftarrow & (U_{W_2}, W_2) \times_Y V \\
\| & & \| & & \| \\
(T_{X_1}, X_1) \times_{X_Y} V & \to & (T_{X_Y}, X_Y) \times_Y V & \leftarrow & (U_{X_2}, X_2) \times_{X_Y} V \\
\downarrow & \square & \downarrow & \square & \downarrow \\
(T_{X_1}, X_1) & \to & (T_{X_Y}, X_Y) & \leftarrow & (T_{X_2}, X_2).
\end{array}$$

(Note again that X_Y and hence X_1 & X_2 are allowed to vary depending on the points $y \in Y$.)

§1-3. K^*-action on locally toric and toroidal structures

Definition 1-3-1 (Strongly Étale Morphisms). *Let V and X be affine varieties with K^*-actions and let*

$$\eta : V \to X$$

be a K^*-equivariant étale morphism. Then η is said to be **strongly étale** *if*

(i) the quotient map

$$\eta//K^* : V//K^* \to X//K^*$$

is étale, and

(ii) the natural map

$$V \to X \times_{X//K^*} V//K^*$$

is an isomorphism.

Remark 1-3-2.

(i) The statement below does NOT hold in general:
For a K^*-equivariant étale morphism $\eta : V \to X$ between affine varieties with K^*-actions, the quotient map $\eta//K^* : V//K^* \to X//K^*$ is also étale.

For example, take

$$V = \text{Spec } K[u, u^{-1}, y] \text{ with the } K^*\text{-action given by } t(u, y) = (tu, t^{-1}y)$$

and

$$X = \text{Spec } K[w, w^{-1}, y] \text{ with the } K^*\text{-action given by } t(w, y) = (t^2 w, t^{-1}y),$$

while a K^*-equivariant étale morphism $\eta : V \to X$ is associated to the ring homomorphism $\eta^* : K[w, y] \to K[u, y]$ defined by $\eta^*(w) = u^2$. Then the quotient map is given by

$$\eta//K^* : V//K^* = \text{Spec } K[uy] \to X//K^* = \text{Spec } K[wy^2]$$

which ramifies over the origin as $wy^2 = (uy)^2$.

The "reason" why $\eta//K^*$ fails to be étale is that in general the stabilizer $Stab(v) \subset K^*$ of a point $v \in V$ is "smaller" than the stabilizer $Stab(x) \subset K^*$ of the image $x = \eta(v)$. (The inclusion $Stab(v) \subset Stab(x) \subset K^*$ is obvious.) Thus, roughly speaking, the quotient $X//K^*$ is divided more than $V//K^*$ by the difference $Stab(x)/Stab(v)$. In the above example, if a point $p \in V$ has the coordinate $y = 0$ the stabilizer is $Stab(v) = \{1\} \subset K^*$, while the image $x = \eta(v)$ has the stabilizer $Stab(x) = \{\pm 1\} \subset K^*$.

It is not difficult even to come up with an example where $V//K^*$ is nonsingular while $X//K^*$ is singular, and hence $\eta//K^* : V//K^* \to X//K^*$ obviously fails to be étale.

(ii) On the other hand, if $\eta : V \to X$ is strongly étale between affine varieties with K^*-actions, it follows from the condition (ii) that

$$Stab(v) = Stab(x) \quad \forall v \in V \text{ where } x = \eta(v).$$

(iii) Luna's Fundamental Lemma stated below (cf. Lemma 1-3-6) can be considered as a statement in the above remark (i) with the extra condition that $Stab(v) = Stab(x)$ for all $v \in V$ where $x = \eta(v)$ is the image, when properly interpreted. Therefore, in a very rough sense, a K^*-equivariant étale morphism $\eta : V \to X$ is strongly étale if and only if the stabilizers are preserved by η.

Definition 1-3-3 (Locally Toric & Toroidal K^*-actions).

(i) Let W be a locally toric variety with a K^*-action. We say that the action is locally toric if for every closed point $p \in W$ we can find a K^*-invariant affine neighborhood $p \in U_p \subset W$, an affine variety V_p with a K^*-action and an affine toric variety X_p with K^* acting as a one-parameter subgroup such that we have K^*-equivariant and strongly étale morphisms η_p and i_p

$$X_p \xleftarrow{\eta_p} V_p \xrightarrow{i_p} U_p.$$

We call such a set of stronly étale morphisms **Luna's locally toric chart at p** *for the locally toric K^*-action.*

(ii) Let (U_W, W) be a toroidal embedding with a K^*-action. We say that the action is toroidal if for every closed point $p \in W$ we can find a K^*-invariant affine neighborhood $p \in U_p \subset W$, an affine toroidal embedding (U_{V_p}, V_p) with a K^*-action and an affine toric variety (T_{X_p}, X_p) with K^* acting as a one-parameter subgroup such that we have K^*-equivariant and strongly étale morphisms η_p and i_p

$$(T_{X_p}, X_p) \xleftarrow{\eta_p} (U_{V_p}, V_p) \xrightarrow{i_p} (U_W \cap U_p, U_p),$$

where in the condition (ii) of η_p and i_p being strongly étale as stated in Definition 1-3-1 we require that the natural morphisms

$$(U_{V_p}, V_p) \to (T_{X_p}, X_p) \times_{X_p//K^*} V_p//K^*$$
$$(U_{V_p}, V_p) \to (U_W \cap U_p, U_p) \times_{U_p//K^*} V_p//K^*$$

to be the isomorphisms of the toroidal embeddings and hence η_p and i_p coincide with the obvious étale toroidal morphisms

$$\begin{array}{ccc}
(T_{X_p}, X_p) & \xleftarrow{\eta_p} & (U_{V_p}, V_p) \\
\| & & \| \\
(T_{X_p}, X_p) \times_{X_p//K^*} X_p//K^* & \longleftarrow & (T_{X_p}, X_p) \times_{X_p//K^*} V_p//K^* \\
(U_{V_p}, V_p) & \xrightarrow{i_p} & (U_W \cap U_p, U_p) \\
\| & & \| \\
(U_W \cap U_p, U_p) \times_{U_p//K^*} V_p//K^* & \longrightarrow & (U_W \cap U_p, U_p) \times_{U_p//K^*} U_p//K^*.
\end{array}$$

We call such a set of strongly étale (toroidal) morphisms **Luna's toroidal chart at p** *for the toroidal K^*-action.*

Proposition 1-3-4. *Any K^*-action on a nonsingular variety W is locally toric, i.e., for every closed point $p \in W$ we can find Luna's locally toric chart at p.*

Remark 1-3-5.

We give two alternative proofs for Proposition 1-3-4:

A) One which uses the canonical resolution of singularities and Luna's Fundamental Lemma. In this proof, we show that Luna's locally toric chart

$$X_p \xleftarrow{\eta_p} V_p \xrightarrow{i_p} U_p$$

can be taken so that i_p is an isomorphism, i.e., $V_p = U_p$ is a K^*-invariant Zariski open neighborhood of p and that X_p is nonsingular. Moreover, we show that in case $p \in W$ is not a fixed point Luna's locally toric chart can be chosen so that none of $U_p = V_p$ or X_p have any fixed points.

B) One which uses Luna's Étale Slice Theorem. In this proof, we show that Luna's locally toric chart can be taken so that i_p is surjective in general and X_p is nonsingular and that i_p is an isomorphism if $p \in W$ is a fixed point of the K^*-action. Moreover, we show that in case $p \in W$ is not a fixed point Luna's locally toric chart can be chosen so that none of U_p, V_p or X_p has any fixed points.

While Luna's Fundamental Lemma is characteristic free, the canonical resolution of singularities (for the moment) and Luna's Étale Slice Theorem are only valid in characteristic 0.

Proof.

A) By Sumihiro's equivariant completion theorem (cf. Sumihiro [1][2]), we can embed K^*-equivariantly W into a complete variety $\overline{W}$ with a K^*-action. By taking the canonical resolution of singularities, which is a sequence of blowups with centers outside of W (cf. the condition $(\spadesuit^{res} - 0)$ of the canonical resolution in Remark 4-1-1), we may assume $\overline{W}$ is nonsingular as well. Note that the K^*-action lifts to the canonical resolution by the condition $(\spadesuit^{res} - 1)$.

Let $p \in W$ be a closed point. Since $\overline{W}$ is complete, there exists a fixed point $q \in \overline{O(p)}$ in the closure of the orbit of p. By another theorem of Sumihiro, there exists a K^*-invariant affine open neighborhood $q \in V_q = \text{Spec } A(V_q)$. Since the maximal ideal $m_q \subset A(V_q)$ associated to the fixed point q is K^*-invariant, it splits into a direct sum of the eigenspaces

$$m_q = \oplus_{\alpha \in \mathbb{Z}} m_{q,\alpha}$$

where $m_{q,\alpha}$ consists of all the eigenfunctions (in m_q) of K^*-character α. Therefore, we can choose eigenfunctions

$$f_1, \cdots, f_n \in m_q \ (n = \dim W)$$

with K^*-characters $\alpha_1, \cdots, \alpha_n$, i.e.,

$$t^*(f_j) = t^{\alpha_j} f_j \text{ for } t \in K^*,$$

such that they form a basis of m_q/m_q^2 and hence that they generate the maximal ideal over $\mathcal{O}_{W,q}$. Consider the morphism

$$\eta : V_q = \text{Spec } A(V_q) \to X = \mathbb{A}^n = \text{Spec } [z_1, \cdots, z_n]$$

defined by

$$\eta^*(z_1) = f_1, \cdots, \eta^*(z_n) = f_n.$$

Letting $t \in K^*$ act on $\mathbb{A}^n$ by $t^*(z_j) = t^{\alpha_j} \cdot z_j$, i.e., letting K^* act as the one parameter subgroup $a = (\alpha_1, \cdots, \alpha_n) \in N$ on $\mathbb{A}^n$ regarded as a standard toric variety, and shrinking V_q if necessary, we see that η is a K^*-equivariant étale morphism.

Lemma 1-3-6 (Luna's Fundamental Lemma). *Let G be a reductive group acting on affine varieties V and X. Let $\eta : V \to X$ be a G-equivariant morphism. Let T be a closed orbit of G in V such that*

(i) η is étale at some point of T,

(ii) $\eta(T)$ is closed in X,

(iii) η is injective on T, and

(iv) V is normal along T.

Then there are G-invariant affine open subsets $V' \subset V$ and $X' \subset X$, with $T \subset V'$, such that $\eta|_{V'} : V' \to X'$ is a strongly étale G-equivariant morphism of V' onto X'.

First we apply Luna's Fundamental Lemma to $\eta : V \to X$ with $T = \{q\}$ and $G = K^*$ to observe that for the choice of K^*-invariant open subsets $V' \subset V$ and $X' \subset X$ as above

$$V' \to X' \times_{X'//K^*} V'//K^*$$

is an isomorphism.

Since $O(p) \subset V'$, we see that η is injective on $O(p)$. The morphism η is étale at all points of $O(p)$. Let $J = \{j; f_j(p) \neq 0\} \subset \{1, \cdots, n\}$. By replacing V and X with

$$\{v \in V; \prod_{j \in J} f_j(v) \neq 0\}$$

$$\{x \in X; \prod_{j \in J} z_j(x) \neq 0\},$$

respectively, we may assume $\eta(O(p))$ is closed in X, which is still an affine nonsingular toric variety. By assumption, $V \subset W$ is nonsingular. With these conditions satisfied, we apply Luna's Fundamental Lemma secondly to $\eta : V \to X$ with $T = O(p)$ and $G = K^*$ to have

$$\eta|_{V'} : V' \to X' (\subset X)$$

being strongly étale and hence

$$V' \to X' \times_{X'//K^*} V'//K^*$$

being an isomorphism.

Case: $p \in W$ is not a fixed point.

In this case, by the first application of Luna's Fundamental Lemma we conclude that there exists $j \in J$ with $f_j(p) \neq 0$ and $\alpha_j \neq 0$. By the choice of X, we see that X has no fixed points. Thus we see that $X'//K^* \subset X//K^*$ is an open subset.

Therefore, we conclude

(i) $\eta_p = \eta|_{V'} : V_p = V' \to X_p = X$ is étale, and

(ii) $V_p = V' \to X' \times_{X'//K^*} V'//K^* = X_p \times_{X_p//K^*} V_p//K^*$ is an isomorphism.

Thus

$$X_p \overset{\eta_p}{\leftarrow} V_p \overset{i_p}{=} U_p$$

is Luna's locally toric chart at p.

Case: p is a fixed point.

In this case, $\eta(p)$ is also a fixed point. Let F_X be the set of fixed points in X with respect to the K^*-action. Let

$$\pi_X : X \to X//K^*$$

be the quotient map. Since $F_X - X'$ is a K^*-invariant closed subset of X, we conclude that $\pi_X(F_X - X')$ is a closed subset of $X//K^*$ and hence that $\pi_X^{-1}(\pi_X(F_X - X'))$ is a closed subset of X. Therefore, we conclude that the subsets defined below

$$C := F_X \cap X'$$
$$D := \pi_X^{-1}(\pi_X(F_X - X')) \cap X'$$

are disjoint K^*-invariant closed subsets of X'. Therefore, by Corollary 1.2 in Mumford-Fogarty-Kirwan [1], there exists a K^*-invariant function $f \in A(X')^{K^*}$ such that

$$f \equiv 1 \text{ on } C \quad \& \quad f \equiv 0 \text{ on } D.$$

Then by construction, setting

$$X'_f = \{x \in X'; f(x) \neq 0\},$$

we see that $\eta(p) \in X'_f$ and that $X'_f//K^* \subset X//K^*$ is an open subset. Set $V'_f = \eta^{-1}(X'_f)$. Then since

$$V' \to X' \times_{X'//K^*} V'//K^*$$

is an isomorphism, so is

$$V'_f \to X'_f \times_{X'_f//K^*} V'_f//K^*.$$

Therefore, we conclude

(i) $\eta_p = \eta|_{V_p} : V_p = V'_f \to X_p = X$ is étale, and

(ii) $V_p \to X'_f \times_{X'_f//K^*} V'_f//K^* = X_p \times_{X_p//K^*} V_p//K^*$ is an isomorphism.

Thus

$$X_p \overset{\eta_p}{\leftarrow} V_p \overset{i_p}{=} U_p$$

is Luna's locally toric chart at p.

This completes the proof A) for Proposition 1-3-4 using the canonical resolution of singularities and Luna's Fundamental Lemma.

B) We recall Luna's Étale Slice Theorem.

Theorem 1-3-7 (Luna's Étale Slice Theorem). *Let G be a reductive group. Let U be an affine variety with a G-action and let T be a closed orbit of G in U along which U is normal. If $p \in T$, then there exists a locally closed G_p-invariant (G_p is the stabilizer of p in G) affine subvariety Z of U with $p \in Z$ such that $U_p = G \cdot Z$ is affine open in U and the G-equivariant morphism*

$$\iota : G \times_{G_p} Z \to U_p$$

is strongly étale. (Note that $G \times_{G_p} Z$ is by definition the quotient of $G \times Z$ by the action of G_p given by

$$g \cdot (t, z) = (tg^{-1}, g \cdot w) \text{ for } g \in G_p \text{ and } (t, w) \in G \times Z.)$$

Moreover, if U is nonsingular, then Z can be taken to be also nonsingular and we have the following commutative diagram of fiber squares

$$\begin{array}{ccccc} G \times_{G_p} N_p & \overset{\text{étale}}{\leftarrow} & G \times_{G_p} Z & \overset{\text{étale surjective}}{\rightarrow} & U_p \\ \downarrow & \square & \downarrow & \square & \downarrow \\ N_p//G_p & \overset{\text{étale}}{\leftarrow} & Z//G_p & \overset{\text{étale surjective}}{\rightarrow} & U_p//G, \end{array}$$

where N_p is the normal vector space to Z at p.

Let $p \in W$ be a closed point. By a theorem of Sumihiro, there exists a K^*-invariant affine open neighborhood U of p such that U is also contained in $W - \{\overline{O(p)} - O(p)\}$. Now $T = O(p)$ is a closed orbit of $G = K^*$ in U, which is nonsingular by assumption. Then applying Luna's Étale Slice Theorem, we obtain strongly étale morphisms

$$\begin{aligned} i_p &: V_p = G \times_{G_p} Z \to U_p = G \cdot Z \quad \text{(surjective)} \\ \eta_p &: V_p = G \times_{G_p} Z \to X_p = G \times_{G_p} N_p, \end{aligned}$$

where $X_p = G \times_{G_p} N_p = K^* \times_{(K^*)_p} N_p$ is easily seen to be a nonsingular affine toric variety with K^* acting as a one-parameter subgroup, as it acts on the first factor of $K^* \times_{(K^*)_p} N_p$ by the usual multiplication.

Thus

$$X_p \overset{\eta_p}{\leftarrow} V_p \overset{i_p}{\rightarrow} U_p$$

is Luna's locally toric chart at p.

Moreover, if $p \in W$ is a fixed point, then i_p as above is an isomorphism, i.e., $V_p = U_p$ is a K^*-invariant affine neighborhood of p. If $p \in W$ is not a fixed point, then by taking U to be contained in $W - F_W$ where F_W is the fixed point set in W, we see that none of X_p, V_p or U_p has any fixed points by construction.

This completes the proofs A) and B) of Proposition 1-3-4.

§1-4. Elimination of points of indeterminacy

For the weak factorization problem, we start with a general birational map $\phi : X_1 \dashrightarrow X_2$ between complete nonsingular varieties. In this section, we show, by the method of elimination of points of indeterminacy, that we only have to deal with the case where ϕ is a projective birational morphism.

Lemma 1-4-1. *There is a commutative diagram*

$$\begin{array}{ccc} X_1' & \stackrel{\phi'}{\to} & X_2' \\ {\scriptstyle g_1}\downarrow & & \downarrow{\scriptstyle g_2} \\ X_1 & \stackrel{\phi}{\dashrightarrow} & X_2 \end{array}$$

such that g_1 and g_2 are sequences of blowups with smooth centers disjoint from U and that ϕ' is a projective birational morphism.

Proof.

By Hironaka's theorem on elimination of points of indeterminacy, there is a sequence of blowups $g_2 : X_2' \to X_2$ with smooth centers disjoint from U such that the birational map $h := \phi^{-1} \circ g_2 : X_2' \to X_1$ is a morphism. By the same theorem, there is a sequence of blowups $g_1 : X_1' \to X_1$ with smooth centers disjoint from U such that the birational map $\phi' := h^{-1} \circ g_1 : X_1' \to X_2$ is a morphism.

Since the composite $h \circ \phi'$ is projective, we conclude that ϕ' is also projective.

We only have to replace $\phi : X_1 \dashrightarrow X_2$ by $\phi' : X_1' \to X_2'$ as above to see the reduction Step 1 in the strategy for the proof described in Chapter 0. Introduction.

Remark 1-4-2 (Elimination of Points of Indeterminacy).

We should make a couple of remarks about Hironaka's method of elimination of points of indeterminacy for a birational map $\phi : X_1 \dashrightarrow X_2$.

First we may assume that ϕ^{-1} is a morphism. Otherwise, replace X_2 by the graph of ϕ. (If one wants to stay in the nonsingular category, take a resolution of singularities of the graph.)

Secondly, if ϕ^{-1} is a projective morphism then we can find an ideal I, trivial over U, such that ϕ^{-1} is the blowup of X_1 along I. (See, for example, the proof of Theorem 2-2-2.) In general, including the case where ϕ^{-1} is not a projective morphism, Hironaka's version of Chow's Lemma (cf. Hironaka [3]) asserts that there exists an ideal I on X_1 such that the blowup of X_1 along I factors through X_2. (Remark that although it is not explicitly stated in Hironaka [3], the ideal I can be taken so that it is trivial over U. Remark also that although Hironaka [3] works specifically over the field $\mathbb{C}$ of complex numbers, the canonicity of his method and Lefschetz principle imply that it is actually valid over any field of characteristic zero (cf. Chapter 4 and Chapter 5). See also Section 5 in Abramovich-Karu-Matsuki-Włodarczyk [1].) We only have to take $g_1 : X_1' \to X_1$ to be a principalization of the ideal I by a sequence of blowups with smooth centers (lying over the support of $\mathcal{O}_{X_2}/I$) in order to obtain a morphism $h = \phi \circ g_1 : X_1' \to X_2$.

Remark-Example 1-4-3.

In dimension 2, Step 1 leads immediately to the end of the proof as we know that any projective (proper) birational morphism $\phi : X_1 \to X_2$ between nonsingular surfaces is a sequence of blowups of points. In dimension 3 or higher, this is no longer the case as the following examples of Hironaka and Shannon, both of which are toric, demonstrate.

(An example by Hironaka)

Let $\phi : X_1 = X(N, \tilde{\sigma}) \to X_2 = \mathbb{A}^3 = X(N, \sigma)$ be a toric (projective birational) morphism where $\sigma = \langle v_1, v_2, v_3 \rangle$ is the regular cone in $N_{\mathbb{R}} = N \otimes \mathbb{R} \cong \mathbb{Z}^3 \otimes \mathbb{R}$ generated by the standard $\mathbb{Z}$-basis and where $\tilde{\sigma}$ is the subdivision of σ as below whose 1-dimensional rays are generated by $v_1, v_2, v_3, v_1 + v_2, v_2 + v_3, v_3 + v_1$ and $v_1 + v_2 + v_3$.

Picture 1-4-4.

There are only 4 subdivisions of σ which correspond to toric blowups with smooth centers. As the map of fans $\phi_\Delta : \tilde{\sigma} \to \sigma$ factors through none of these subdivisions, we conclude that ϕ cannot be factored into a sequence of TORIC blowups with smooth centers. Since these 4 blowups of $\mathbb{A}^3$ are the only blowups whose centers lie in the image of the exceptional locus for ϕ and which give rise to the valuations corresponding to the exceptional divisors for ϕ, we also conclude that ϕ cannot be factored into a sequence of ANY blowups with smooth centers.

On the other hand, it is easy to see that for any point $p_1 \in X_1$ and its image $p_2 = \phi(p_1) \in X_2$ the associated homomorphism of local rings $\mathcal{O}_{X_2,p_2} \to \mathcal{O}_{X_1,p_1}$ can be factored into a sequence of ring homomorphisms associated to a sequence of blowups with smooth centers followed by localization. (Note that the sequence may vary from point to point.) In this sense, Hironaka's example is only a GLOBAL (counter-)example (to the statement that any projective birational morphism between nonsingular varieties is a sequence of blowups with smooth centers) but not a LOCAL one.

(An example by Shannon)

This time we let $\psi : X_1 = X(N, \hat{\sigma}) \to X_2 = \mathbb{A}^3 = X(N, \sigma)$ be a toric (projective birational) morphism where $\sigma = \langle v_1, v_2, v_3 \rangle$ is the regular cone in $N_{\mathbb{R}} = N \otimes \mathbb{R} \cong \mathbb{Z}^3 \otimes \mathbb{R}$ as before, but $\hat{\sigma}$ is a nonsingular projective subdivision of σ which contains the following cone

$$\tau = \langle 2v_1 + v_3, v_1 + 2v_2 + v_3, 3v_2 + v_3 \rangle.$$

(We leave it to the reader as an easy exercise to show the existence of such a subdivision.)

Picture 1-4-5.

Now let $p_1 \in X_1$ be the closed point corresponding to the cone τ and let $p_2 = \psi(p_1) \in X_2$ be its image. Then it is easy to see that the associated homomorphism of local rings $\mathcal{O}_{X_2,p_2} \to \mathcal{O}_{X_1,p_1}$ can NOT be factored into a sequence of ring homomorphisms associated to a sequence of blowups with smooth centers followed by localization, as τ is not contained in any cone in the only 4 subdivisions of σ mentioned in Hironaka's example. Thus this example by Shannon gives a LOCAL (counter-)example (and hence also is a GLOBAL one).

CHAPTER 2. BIRATIONAL COBORDISM

In this chapter, we study the theory of birational cobordisms by Włodarczyk [2] (See also Morelli [1].), which is the main tool for our construction of the factorization of a given birational map $\phi : X_1 \dashrightarrow X_2$ between nonsingular complete varieties. The power of the theory lies in the fact that it allows us to analyze the structure of birational transformations much like the usual Morse theory allows us to study the structure of homotopy transformations via the usual cobordisms. The theory may be considered an algebraic version of the Morse theory in terms of the action of the multiplicative group K^* through this analogy, which also justifies the definition we give below. (See the introduction in Chapter 0.)

It may be worthwhile to note that the connection between the Morse theory and the action of K^* on an algebraic variety has been known and may even be considered classical, as was observed by many authors (e.g. Brion-Processi [1] Dolgachev-Hu [1] Frankel [1] Kirwan [1] Mumford-Fogarty-Kirwan [1] Thaddeus [1] [2]). When $K = \mathbb{C}$ is the field of complex numbers, it is also studied in the realm of symplectic geometry. Actually we take advantage of the interpretation of the birational cobordisms in terms of Geometric Invariant Theory to show that, if both X_1 and X_2 are projective, then we can choose our factorization so that all the intermediate varieties are also projective. The subject of the change of the G.I.T. quotients associated to the change of linearizations has been found to have a close connection with the factorization problem of certain birational maps, as studied by Thaddeus and others.

The novelty of the ideas by Włodarczyk, therefore, is in connecting the long-standing factorization problem of GENERAL birational maps to these classical ideas by constructing birational cobordisms and hence providing a view point from the Morse theory to approach the problem.

We cannot probably emphasize too much the importance of the original work of Morelli [1], who introduced the notion of combinatorial polyhedral cobordisms to solve the factorization problem of toric birational maps. It is the algebro-geometric interpretation of Morelli's combinatorial objects that ultimately led Włodarczyk to the theory of birational cobordisms for general birational maps.

§2-1. Definition of a birational cobordism and the toric main example

Definition 2-1-1 (Birational Cobordism). *Let $\phi : X_1 \dashrightarrow X_2$ be a proper birational map between normal varieties defined over K, isomorphic over a common open subset $X_1 \supset U \subset X_2$. A normal variety B is called a* **birational cobordism** *for ϕ and denoted by $B_\phi(X_1, X_2)$ if it satisfies the following conditions:*

(i) the multiplcative group K^* acts on $B = B_\phi(X_1, X_2)$, (We denote the action of $t \in K^*$ on $x \in B$ by $t(x)$.)

(ii) the sets

$$B_+ := \{x \in B; \lim_{t\to\infty} t(x) \text{ does NOT exist in } B\}$$
$$B_- := \{x \in B; \lim_{t\to 0} t(x) \text{ does NOT exist in } B\}$$

are nonempty Zariski open subsets of B, and

(iii) there are isomorphisms

$$B_+/K^* \xrightarrow{\sim} X_2$$
$$B_-/K^* \xrightarrow{\sim} X_1$$

so that the birational map induced by the inclusions $B_- \supset B_- \cap B_+ \subset B_+$ and the isomorphisms above

$$X_1 = B_-/K^* \supset (B_- \cap B_+)/K^* \subset B_+/K^* = X_2$$

coincides with ϕ.

We say that **B respects the open subset U** *if U is contained in $(B_- \cap B_+)/K^*$.*

We discuss the following fundamental example of a birational cobordism in the toric setting, as was first observed by Morelli [1].

Main Toric Example 2-1-2.

Let $B = \mathbb{A}^n = \text{Spec } K[z_1, \cdots, z_n]$ with a K^*-action given by

$$t(z_1, \cdots, z_j, \cdots, z_n) = (t^{\alpha_1} z_1, \cdots, t^{\alpha_j} z_j, \cdots, t^{\alpha_n} z_n).$$

We regard $\mathbb{A}^n = X(N, \sigma)$ as a toric variety defined by a lattice $N \cong \mathbb{Z}^n$ and the regular cone $\sigma \subset N_{\mathbb{R}}$ generated by the standard $\mathbb{Z}$-basis of N

$$\sigma = \langle v_1, \cdots, v_j, \cdots, v_n \rangle.$$

The dual cone $\sigma^\vee$ is generated by the dual $\mathbb{Z}$-basis

$$\sigma^\vee = \langle v_1^*, \cdots, v_j^*, \cdots, v_n^* \rangle$$

and we identify

$$z_j = z^{v_j^*}.$$

The K^*-action then corresponds to the one parameter subgroup

$$a = (\alpha_1, \cdots, \alpha_j, \cdots, \alpha_n) \in N.$$

We have the obvious description of the sets B_+ and B_-

$$B_+ = \{(z_1, \cdots, z_n); z_j \neq 0 \text{ for some } j \text{ with } \alpha_j = (v_j^*, a) > 0\}$$
$$B_- = \{(z_1, \cdots, z_n); z_j \neq 0 \text{ for some } j \text{ with } \alpha_j = (v_j^*, a) < 0\}.$$

We define the upper boundary and lower boundary of σ (with respect $a \in N$) to be

$$\partial_+\sigma = \{x \in \sigma; x + \epsilon \cdot (-a) \notin \sigma \text{ for } \epsilon > 0\}$$
$$\partial_-\sigma = \{x \in \sigma; x + \epsilon \cdot a \notin \sigma \text{ for } \epsilon > 0\}.$$

Then we obtain the description of B_+, B_- and $B_+ \cap B_-$ as the toric varieties corresponding to the fans $\partial_+\sigma$, $\partial_-\sigma$ and $\partial_+\sigma \cap \partial_-\sigma$, i.e.,

$$B_+ = X(N, \partial_+\sigma)$$
$$B_- = X(N, \partial_-\sigma).$$
$$B_+ \cap B_- = X(N, \partial_+\sigma \cap \partial_-\sigma)$$

Accordingly, if we denote by $\pi : N_{\mathbb{R}} \to N_{\mathbb{R}}/\mathbb{R} \cdot a$ the projection onto $N_{\mathbb{R}}/\mathbb{R} \cdot a$ with the lattice $\pi(N)$, then we have the description of the quotients as the toric varieties

$$B_+/K^* = X(\pi(N), \pi(\partial_+\sigma))$$
$$B_-/K^* = X(\pi(N), \pi(\partial_-\sigma))$$
$$(B_+ \cap B_-)/K^* = X(\pi(N), \pi(\partial_+\sigma \cap \partial_-\sigma))$$
$$B//K^* = X(\pi(N), \pi(\sigma))$$

where $\pi(\partial_+\sigma), \pi(\partial_-\sigma), \pi(\partial_+\sigma \cap \partial_-\sigma)$ and $\pi(\sigma)$ are the fans in $N_{\mathbb{R}}/\mathbb{R} \cdot a$ defined by

$$\pi(\partial_+\sigma) = \{\pi(\tau); \tau \in \partial_+\sigma\}$$
$$\pi(\partial_-\sigma) = \{\pi(\tau); \tau \in \partial_-\sigma\}$$
$$\pi(\partial_+\sigma \cap \partial_-\sigma) = \{\pi(\tau); \tau \in \partial_+\sigma \cap \partial_-\sigma\} = \pi(\partial_+\sigma) \cap \pi(\partial_-\sigma)$$
$$\pi(\sigma) = \{\pi(\sigma) \text{ and its proper faces in } N_{\mathbb{R}}/\mathbb{R} \cdot a\},$$

respectively. Since $\pi(\partial_+\sigma)$ and $\pi(\partial_-\sigma)$ are subdivisions of $\pi(\sigma)$, we obtain a diagram of toric birational maps

$$\begin{array}{ccccc}
B_-/K^* & & \overset{\phi}{\dashrightarrow} & & B_+/K^* \\
& \searrow & & \swarrow & \\
\| & & B//K^* & & \| \\
X(\pi(N), \pi(\partial_-\sigma)) & & \dashrightarrow & & X(\pi(N), \pi(\partial_+\sigma)) \\
& \searrow & \| & \swarrow & \\
& & X(\pi(N), \pi(\sigma)) & &
\end{array}$$

where ϕ coincides with the birational map induced by

$$\begin{array}{ccccc}
B_-/K^* & \supset & (B_- \cap B_+)/K^* & \subset & B_+/K^* \\
\| & & \| & & \| \\
X(\pi(N), \pi(\partial_-\sigma)) & \supset & X(\pi(N), \pi(\partial_-\sigma \cap \partial_+\sigma)) & \subset & X(\pi(N), \pi(\partial_+\sigma)).
\end{array}$$

Therefore, we conclude that B is a birational cobordism for ϕ.

More generally, one can prove (See Morelli [1] Abramovich-Matsuki-Rashid [1].) that if Σ is a subdivision of a convex polyhedral cone in $N_{\mathbb{R}}$ with the lower boundary $\partial_-\Sigma$ and upper boundary $\partial_+\Sigma$ with respect to a one-parameter subgroup $a \in N$, then the toric variety $X(N,\Sigma)$ corresponding to Σ with the K^*-action given by the one-parameter subgroup $a \in N$, is a birational cobordism between the two toric varieties $X(\pi(N), \pi(\partial_-\Sigma))$ and $X(\pi(N), \pi(\partial_-\Sigma))$.

Next in the pursuit of analogy to the usual Morse theory where the critical points are "lined up nicely" according to the levels given by the Morse function, we would like to have the fixed points of a birational cobordism "ordered nicely" so that we can study the birational transformations as we go through the fixed points "from the bottom to the top", though we may not see a Morse function explicitly. This requirement naturally leads us to the notion of "**collapsibility**" introduced by Morelli [1]. In the toric setting that Morelli [1] studied, the fixed points of the cobordism, associated to a fan Σ, correspond to the cones σ of the maximal dimension (bubbles) in Σ. Factorization then corresponds to going from $\partial_-\Sigma$ to $\partial_+\Sigma$, by replacing $\partial_-\sigma$ with $\partial_+\sigma$ one at a time (collapsing of the bubble σ). (See the main toric example 2-1-2 and Morelli [1] Abramovich-Matsuki-Rashid [1].) Thus the question of whether or not we can order the fixed points nicely corresponds to that of whether or not we can collapse these bubbles in a nicely ordered manner, thus giving rise to the name "collapsibility".

First we introduce the notations for some specific subsets in B associated to the fixed point set.

Notation 2-1-3. *Let $B = B_\phi(X_1, X_2)$ be a birational cobordism, and let $F \subset F_B = B^{K^*}$ be a subset of the fixed point set with respect to the K^*-action. We define*

$$\begin{aligned} F^+ &:= \{x \in B; \lim_{t\to 0} t(x) \in F\} \\ F^- &:= \{x \in B; \lim_{t\to \infty} t(x) \in F\} \\ F^\pm &:= F^+ \cup F^- \\ F^* &:= F^\pm - F. \end{aligned}$$

Definition 2-1-4. *Let $B = B_\phi(X_1, X_2)$ be a birational cobordism. We define a relation $\prec$ among the connected components of the fixed point set B^{K^*} as follows: let $F_1, F_2 \subset B^{K^*}$ be two (not necessarily distinct) connected components, and set $F_1 \prec F_2$ if there exists a point $p \in B$ with $p \notin B^{K^*}$ such that*

$$\lim_{t\to 0} t(p) \in F_1 \text{ and } \lim_{t\to \infty} t(p) \in F_2.$$

That is to say, in terms of Notation 2-1-3, we have the relation $F_1 \prec F_2$ if and only if $(F_1^+ - F_1) \cap (F_2^- - F_2) \neq \emptyset$.

Definition 2-1-5 (Collapsibility). *We say that a birational cobordism $B = B_\phi(X_1, X_2)$ is collapsible if the relation $\prec$ is a strict preorder, namely, there is no directed cycle of the connected components of the fixed point set*

$$F_1 \prec F_2 \prec \cdots \prec F_m \prec F_1.$$

Note that collapsibility excludes the possibility of a self-loop $F_1 \prec F_1$ for any connected component $F_1 \subset B^{K^*}$.

In Chapter 3, we will consider the torific ideals on a birational cobordism B where the existence of a point $p \in B$ with $p \notin B^{K^*}$ such that both limits $\lim_{t\to 0} t(p)$ and $\lim_{t\to\infty} t(p)$ exist within B would cause a problem in order to have such ideals well-defined. The notion of a quasi-elemetary birational cobordism below is introduced exactly to avoid this problem.

Definition 2-1-6 (Quasi-Elementary Birational Cobordism). *A birational cobordism B is said to be quasi-elementary if there does not exist any point $p \in B$ with $p \notin B^{K^*}$ such that both limits $\lim_{t\to 0} t(p)$ and $\lim_{t\to\infty} t(p)$ exist within B. In terms of the relation $\prec$, this is equivalent to saying that for any two (not necessarily distinct) connected components $F_1, F_2 \subset B^{K^*}$ neither $F_1 \prec F_2$ nor $F_2 \prec F_1$ holds. That is to say, in terms of Notation 2-1-3, B is quasi-elementary if and only if $(F^+ - F) \cap (F^- - F) = \emptyset$ where $F = B^{K^*}$ is the entire fixed point set in B.*

Definition 2-1-7 (Elementary Birational Cobordism). *A quasi-elementary birational cobordism is said to be elementary if the fixed point set B^{K^*} is connected.*

We will observe that the birational transformation represented by an elementary nonsingular birational cobordism corresponds to a (weighted) blowup of a connected center, which is étale locally equivalent to a toric birational transformation as in Main Toric Example 2-1-2. (See §2-4 for the details.) One might want to call such a birational transformation "elementary", as Włodarczyk [2] does, and hence name the corresponding birational cobordism also "elementary".

§2-2. Construction of a (collapsible) birational cobordism

After defining a birational cobordism, the natural and important issue is its existence, which Włodarczyk [2] shows for any proper birational map between (normal) varieties. Here we present a simple construction of a collapsible birational cobordism for a projective birational morphism $\phi : X_1 \to X_2$ between complete nonsingular varieties, which suffices for our purposes after reducing the factorization problem of an arbitrary birational map to that of a projetive morphism via Hironaka's theorem on elimination of points of indeterminacy. (See §1-4 for the details of the reduction step.)

Theorem 2-2-2 (Construction of Birational Cobordism). *Let $\phi : X_1 \to X_2$ be a projective (in the sense of Grothendieck and not as defined in Hartshorne [1]) birational morphism between complete nonsingular varieties (defined over K), which is an isomorphism over a common open subset U. Then there exists a complete nonsingular variety $\overline{B}$ with an effective K^*-action, (Note that a K^*-action*

is called **effective** *if the action is not induced from that of the nontrivial quotient of K^*, that is to say, if $\cap_{p\in\overline{B}} Stab(p) = \{1\}$.) satisfying the following properties:*

(i) there exist closed embeddings

$$i_1 : X_1 \hookrightarrow \overline{B}$$
$$i_2 : X_2 \hookrightarrow \overline{B}$$

with disjoint images in $\overline{B}^{K^*}$,

(ii) there is a coherent sheaf $\mathcal{E}$ on X_2, with a K^*-action (which is compatible with the trivial action of K^* on X_2), and a K^*-equivariant closed embedding

$$\overline{B} \hookrightarrow \mathbb{P}(\mathcal{E}) := Proj_{X_2} \oplus_{m\geq 0} Sym^m \mathcal{E},$$

(iii) the open subvariety $B := \overline{B} - (i_1(X_1) \cup i_2(X_2))$ is a collapsible birational cobordism for ϕ.

We call such a variety $\overline{B}$ a **compactified birational cobordism** *projective over X_2.*

Remark 2-2-3.

Suppose that a projective birational morphism between complete nonsingular varieties $\phi : X_1 \to X_2$ is a sequence of blowups with smooth centers. Then we can construct a compatified birational cobordism $\overline{B}$ projective over X_2 in the following simple manner: We start with the product $W_0 = X_2 \times \mathbb{P}^1$ where K^* acts on the second factor as the multiplication on $K^* = \mathbb{P}^1 - \{0, \infty\}$. We take the sequence of blowups $W \to X_2 \times \mathbb{P}^1$ with the centers identified with those for ϕ but considered as lying in (the strict transforms of) the 0-section $X_2 \times \{0\}$ instead of lying in (the subsequent blowups of) X_2. Now $\overline{B} = W$ is a complete nonsingular variety projective over X_2, with an effective K^*-action, satisfying the properties:

(i) we have two closed embeddings

$$i_1 : X_1 \xrightarrow{\sim} (\text{the strict transform of } X_2 \times \{0\}) \subset \overline{B}$$
$$i_2 : X_2 \xrightarrow{\sim} X_2 \times \{\infty\} \subset \overline{B}$$

with disjoint images $i_1(X_1), i_2(X_2) \subset \overline{B}^{K^*}$

(ii) the centers of blowups lie in the fixed point set of the K^*-action and hence the morphism $\overline{B} \to X_2 \times \mathbb{P}^1 \to X_2$ is obviously K^*-equivariant, which easily implies the existence of such a coherent sheaf $\mathcal{E}$ as above,

(iii) $B = \overline{B} - (i_1(X_1) \cup i_2(X_2))$ is a collapsible birational cobordism for ϕ. Remark that the factorization induced by B, as will be discussed in §2-4, coincides with the given sequence of blowups with smooth centers.

The above birational cobordism can be considered as the standard birational cobordism associated to a sequence of blowups with smooth centers factoring a projective birational morphism ϕ (if such a sequence exists at all for ϕ). The construction below for the proof of Theorem 2-2-2 may be considered as modeled on that of the standard birational cobordism discussed above.

Proof of Theorem 2-2-2.

We start with the product $W_0 = X_2 \times \mathbb{P}^1$ where K^* acts on the second factor as the multiplication on $K^* = \mathbb{P}^1 - \{0, \infty\}$.

Since ϕ is a projective birational morphism which is an isomorphism over U, there exists an ideal sheaf $J \subset \mathcal{O}_{X_2}$ such that $\phi : X_1 \to X_2$ is the blowup morphism of X_2 along J and that the support of $\mathcal{O}_{X_2}/J$ is disjoint from U. (This can be verified as follows: Take a divisor D on X_1 which is ϕ-ample. Then we see $D - \phi^*\phi_* D$ is also ϕ-ample and that it is of the form $\Sigma(-a_i)E_i$, where $f_* D$ is the cycle-theoretic image of D under ϕ as a divisor and where the E_i are exceptional divisors for ϕ with $a_i > 0$. We only have to take $J = \phi_* \mathcal{O}_{X_1}(l \cdot \Sigma(-a_i)E_i)$ for some sufficiently large $l \in \mathbb{N}$. Remark also that in our case the existence of such a ϕ-ample divisor $\Sigma(-a_i)E_i$ follows from the construction. (cf. §5-1.) In our construction of the factorization the projective morphism ϕ is obtained as $\phi' : X_1' \to X_2'$ starting from the original birational map $\phi : X_1 \dashrightarrow X_2$ via elimination of points of indeterminacy. Thus in the notation of Lemma 1-4-1 we have a g_1-ample divisor of the form $\Sigma(-b_j)F_j$, where F_j are g_1-exceptional divisors whose centers on X_1 lie outside of U. The divisor $\Sigma(-b_j)F_j$ is hence ϕ-ample. We can take $J = \phi_* \mathcal{O}_{X_1}(l \cdot \Sigma(-b_j)F_j)$ for some sufficiently large $l \in \mathbb{N}$.) Let I_0 be the ideal of the origin $0 \in \mathbb{P}^1$. We set

$$I = (p_1^{-1} J + p_2^{-1} I_0) \cdot \mathcal{O}_{W_0}$$

where

$$p_1 : W_0 = X_2 \times \mathbb{P}^1 \to X_1$$
$$p_2 : W_0 = X_2 \times \mathbb{P}^1 \to \mathbb{P}^1$$

are the projections.

Let W be the blowup of W_0 along I.

We claim that X_1 and X_2 are embedded in the nonsingular locus of W, as the strict transform of $X_2 \times \{0\}$ and $X_2 \times \{\infty\}$, respectively. For $i_2 : X_2 \xrightarrow{\sim} X_2 \times \{\infty\} \subset W$ this is clear, as the centers of blowup for $X_2 \times \mathbb{P}^1$ only lie over $X_2 \times \{0\}$. For $i_2 : X_1 \xrightarrow{\sim}$ (the strict transform of $X_2 \times \{0\}$), in order to prove that X_1 is in the nonsingular locus of W, it suffices to show that X_1 is a Cartier divisor as X_1 itself is nonsingular. We look at local coordinates. Let $A(V)$ be the affine coordinate ring for an affine open subset $V \subset X_2$ and let $y_1, \cdots, y_m$ be a set of generators for J on V. Let $K[x]$ be the affine coordinate ring for $\mathbb{A}^1 = \mathbb{P}^1 - \{\infty\}$. Then the affine open subset $V \times \mathbb{A}^1 \subset X_2 \times \mathbb{P}^1$ has the affine coordiante ring $A(V) \otimes_K K[x]$, where the ideal I is generated by $y_1, \cdots, y_m, x$. The charts of the blowup containing the strict transform of $V \times \{0\} = \{x = 0\}$ are of the form

$$\text{Spec } A[\frac{y_1}{y_i}, \cdots, \frac{y_m}{y_i}, \frac{x}{y_i}] = \text{Spec } A[\frac{y_1}{y_i}, \cdots, \frac{y_m}{y_i}] \times \text{Spec } K[\frac{x}{y_i}],$$

where K^* acts on the second factor by the multiplication on x. The strict transform of $\{x = 0\}$ is defined by $\frac{x}{y_i}$ and hence is Cartier.

Let $\overline{B} \to W$ be the canonical resolution of singularities. Since the centers of blowups of the canonical resolution of singularities are taken over the singular locus

(cf. the condition ($\spadesuit^{res}-0$)) and since the (effective) K^*-action lifts to $\overline{B}$ (cf. the condition ($\spadesuit^{res}-1$)), the above analysis of W immediately implies the properties (i) and (iii), except for the collapsibility of B.

Remark that the morphism $\tau : \overline{B} \to X_2$ is K^*-equivariant as well as projective. Therefore, there exists a relatively ample line bundle $\mathcal{L}$ on $\overline{B}$ over X_2, equipped with a K^*-action (compatible with the K^*-action on $\overline{B}$). (Remark that in our case the existence of such a relatively ample line bundle follows directly from the construction. See also the discussion in §5-1.) In order to see the property (ii), we only have to set

$$\mathcal{E} = \tau_*(\mathcal{L}^{\otimes l}) \text{ for sufficiently large } l \in \mathbb{N}.$$

Finally we show that the birational cobordism B constructed as above is collapsible.

Let $\mathcal{C}$ be the set of the connected components of B^{K^*}. It suffices to show that there exists a strictly increasing function $\chi : \mathcal{C} \to \mathbb{Z}$, i.e., a function such that

$$F_1 \prec F_2 \Longrightarrow \chi(F_1) < \chi(F_2).$$

Since K^* acts trivially on X_2 and since K^* is reductive, there exists a direct sum decomposition

$$\mathcal{E} = \oplus_{b \in \mathbb{Z}} \mathcal{E}_b$$

where $\mathcal{E}_b$ is the subsheaf on which the K^*-action is given by the character $t \mapsto t^b$. Denote by

$$b_0 < b_1 < \cdots < b_{h-1} < b_h$$

the characters which appear in this presentation. Note that there are disjoint embeddings $\mathbb{P}(\mathcal{E}_b) \subset \mathbb{P}(\mathcal{E})$.

Let $p \in B$ be a point lying in the fiber $\mathbb{P}(E_q)$ over $q \in X_2$, where $E_q = \mathcal{E} \otimes \mathcal{O}_{X_2,q}/m_{X_2,q}$. We choose a basis of E_q, which are then considered to be homogeneous coordinates of $\mathbb{P}(E_q)$

$$(x_{b_0,1}, \cdots, x_{b_0,d_0}, \cdots, x_{b_h,1}, \cdots, x_{b_h,d_h})$$

where $x_{b_j,k} \in E_{b_j} = \mathcal{E}_{b_j} \otimes \mathcal{O}_{X_2,q}/m_{X_2,q}$.

It is straightforward to observe that $p \in B$ is a fixed point if and only if

$$\exists b_{j_p} \text{ such that } x_{b_j,k}(p) = 0 \text{ whenever } b_j \neq b_{j_p}.$$

Therefore, if $F \subset B^{K^*}$ is a connected component, then F is contained in $\mathbb{P}(\mathcal{E}_{b_j})$ for some b_j. We define

$$\chi(F) := b_j.$$

Suppose $F_1 \prec F_2$, i.e., there exists a point $p \in B$ but $p \notin B^{K^*}$ with

$$\lim_{t \to 0} t(p) \in F_1 \text{ and } \lim_{t \to \infty} t(p) \in F_2.$$

It is easy to see that

$$\lim_{t\to 0} t(p) \in \mathbb{P}(E_{b_{min}})$$
$$\lim_{t\to \infty} t(p) \in \mathbb{P}(E_{b_{max}}),$$

where

$$b_{min} = \min\{b_j ; x_{b_j,k}(p) \neq 0 \text{ for some } k\}$$
$$b_{max} = \max\{b_j ; x_{b_j,k}(p) \neq 0 \text{ for some } k\}.$$

Observe also that

$$b_{min} < b_{max}$$

since $p \in B$ is not a fixed point. Thus we conclude that

$$\chi(F_1) = b_{min} < b_{max} = \chi(F_2),$$

showing χ is a strictly increasing function.

Therefore, the birational cobordism B is collapsible.

This completes the proof of Theorem 2-2-2.

§2-3. Interpretation by Geometric Invariant Theory

In this section, we apply Geometric Invariant Theory to the compactified birational cobordism $\overline{B}$ projective over X_2 and see that the birational cobordism B decomposes into the quasi-elementary pieces B_{a_i} (See below for the precise definition), which are the semi-stable loci of the different linearizations of the K^*-action on $\overline{B}$. Thus we can analyze the birational transformations of $B_{a_i}//K^*$ as the change of the G.I.T. quotients when we vary the linearizations, a subject which is extensively studied by Thaddeus [1][2] and others.

We continue using the notation of Theorem 2-2-2.

Let $\mathcal{E}$ be a coherent sheaf on X_2, with a K^*-action (which is compatible with the trivial action of K^* on X_2), and a K^*-equivariant closed embedding

$$\overline{B} \hookrightarrow \mathbb{P}(\mathcal{E}) := Proj_{X_2} \oplus_{m\geq 0} Sym^m \mathcal{E}$$

and let

$$\mathcal{E} = \oplus_{b\in\mathbb{Z}} \mathcal{E}_b$$

be its decomposition into the eigen-subsheaves according to their characters. Let

$$a_1 < a_2 < \cdots < a_{s-2} < a_{s-1}$$

be the values of $\chi : \mathcal{C} \to \mathbb{Z}$, i.e., $\{a_i\}$ is the subset of $\{b_j\}$, which are the values associated to the connected components of the fixed point set B^{K^*}. By using the

Veronese embedding $\mathbb{P}(\mathcal{E}) \hookrightarrow \mathbb{P}(Sym^2\mathcal{E})$ and replacing $\mathcal{E}$ with $Sym^2\mathcal{E}$, we may assume that all the a_i are even and that in particular

$$a_{i-1} < a_i - 1 < a_i < a_i + 1 < a_{i+1} \text{ for } i = 2, \cdots, s-2.$$

(This is a technical condition which avoids the rational "twists" and allows us only to deal with the integral ones.)

Denote by

$$\rho_0(t)$$

the original K^*-action on $\mathcal{E}$. For any $r \in \mathbb{Z}$ we consider the "twisted" action of K^* on $\mathcal{E}$, denoted by

$$\rho_r(t) = t^{-r} \cdot \rho_0(t),$$

so that in the decomposition $\mathcal{E} = \oplus_{b\in\mathbb{Z}}\mathcal{E}_b$ the $\rho_r(t)$ acts on $\mathcal{E}_b$ by the multiplication t^{b-r}. Note that the "twists" do not change the induced action of K^* on $\mathbb{P}(\mathcal{E})$ but only change the linearizations on $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$.

We can now apply Geometric Invariant Theory in its relative form (cf. Hu [2] Pandripande [1]) to the situation.

We denote by

$$(\mathbb{P}(\mathcal{E}), \rho_r)^{ss}$$

the semi-stable locus of $\mathbb{P}(\mathcal{E})$ with respect to the linearization of $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ induced by the twisted action ρ_r of K^* on $\mathcal{E}$. Recall that a point $p \in \mathbb{P}(\mathcal{E})$ is semi-stable, i.e., $p \in (\mathbb{P}(\mathcal{E}), \rho_r)^{ss}$ if there exists a local section $s \in Sym^n\mathcal{E}$ for some positive $n \in \mathbb{N}$, invariant under the twisted action ρ_r of K^* on $Sym^n\mathcal{E}$, such that $s(p) \neq 0$. We denote by

$$(\overline{B}, \rho_r)^{ss}$$

the semi-stable locus of $\overline{B}$ with respect to the linearization of $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)|_{\overline{B}}$ coming from that of $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ induced by the twisted action ρ_r of K^*. Then we have

$$(\overline{B}, \rho_r)^{ss} = \overline{B} \cap (\mathbb{P}(\mathcal{E}), \rho_r)^{ss}$$

by Theorem 1.19 in Kirwan-Fogarty-Mumford [1].

Now Thaddeus and others tell us that we should observe the change of the semi-stable loci when the twists pass through critical values and that we should also observe the birational transformations of the G.I.T. quotients.

In our situtaion, these critical values (aside from a_0 where $i(X_1) \subset \mathbb{P}(\mathcal{E}_{a_0})$ and a_s where $i(X_2) \subset \mathbb{P}(\mathcal{E}_{a_s})$) turn out to be just

$$a_1 < a_2 < \cdots < a_{s-2} < a_{s-1}$$

and the change of the semi-stable loci provides the decomposition of the birational cobordism B into the quasi-elementary pieces.

Proposition 2-3-1. *Let the situation be as above. Then*

(o) the critical values for the change of the semi-stable loci (aside from a_0 where $i(X_1) \subset \mathbb{P}(\mathcal{E}_{a_0})$ and a_s where $i(X_2) \subset \mathbb{P}(\mathcal{E}_{a_s})$) are

$$a_1 < a_2 < \cdots < a_{s-2} < a_{s-1},$$

(i) at each critical value a_i the semi-stable locus $(\overline{B}, \rho_{a_i})^{ss} := B_{a_i}$ is a quasi-elementary birational cobordism and we have set-theoretic descriptions

$$\begin{aligned} B_{a_i} &= (\overline{B}, \rho_{a_i})^{ss} \\ &= B - \{(\cup_{F \in \mathcal{C}, \chi(F) < a_i} F^-) \cup (\cup_{F \in \mathcal{C}, \chi(F) > a_i} F^+)\} \\ (B_{a_i})_+ &= (\overline{B}, \rho_{a_i+1})^{ss} \\ (B_{a_i})_- &= (\overline{B}, \rho_{a_i-1})^{ss}, \end{aligned}$$

(ii) we have the commutative diagarm of (proper) birational morphsims and maps between varieties PROJECTIVE over X_2

$$\begin{array}{ccccc} (B_{a_i})_-/K^* & & \overset{\varphi_i}{\dashrightarrow} & & (B_{a_i})_+/K^* \\ & \searrow & & \swarrow & \\ \| & & B_{a_i}//K^* & & \| \\ (\overline{B}, \rho_{a_i-1})^{ss}/K^* & & \dashrightarrow & & (\overline{B}, \rho_{a_i+1})^{ss}/K^* \\ & \searrow & \| & \swarrow & \\ & & (\overline{B}, \rho_{a_i})^{ss}//K^* & & \end{array}$$

(iii) we also have

$$\begin{aligned} (B_{a_1})_- &= B_- \text{ and } (B_{a_1})_-/K^* = B_-/K^* = X_1 \\ (B_{a_{s-1}})_+ &= B_+ \text{ and } (B_{a_{s-1}})_+/K^* = B_+/K^* = X_2. \end{aligned}$$

Proof.

The proof is an easy consequence of the (relative) G.I.T. and the definition of the semi-stable locus as above and left to the reader as an exercise.

§2-4. Factorization into locally toric birational maps

As a summary of the results discussed in the previous sections §2-1, §2-2 and §2-3 and the use of elimination of points of indeterminacy by Hironaka, we obtain the following factorization theorem of a birational map into locally toric transformations by Włodarczyk [2].

Theorem 2-4-1 (Factorization into Locally Toric Birational Maps). *Let $\phi : X_1 \to X_2$ be a birational map between complete nonsingular varieties over K, which is an isomorphism over a common open subset $X_1 \supset U \subset X_2$. Then there exists a sequence of birational maps*

$$X_1 = W_1 \overset{\varphi_1}{\dashrightarrow} W_2 \overset{\varphi_2}{\dashrightarrow} \cdots \overset{\varphi_{i-1}}{\dashrightarrow} W_i \overset{\varphi_i}{\dashrightarrow} W_i \overset{\varphi_{i+1}}{\dashrightarrow} \cdots \overset{\varphi_{s-1}}{\dashrightarrow} W_s = X_2$$

such that

(i) $\phi = \varphi_{s-1} \circ \varphi_{s-2} \circ \cdots \varphi_2 \circ \varphi_1$,

(ii) φ_i is V-locally toric and an isomorphism over U, and that

(iii) there exists an index i_o with the property that for all $i \leq i_o$ the maps $W_i \dashrightarrow X_1$ are projective birational morphisms and for all $i \geq i_o$ the maps $W_i \dashrightarrow X_2$ are projective birational morphisms, and hence if X_1 and X_2 are projective then all the W_i are projective.

Proof.

By the reduction step as discussed in §1-4, we may assume that $\phi : X_1 \to X_2$ is a projective birational morphism. Thus we are in a situation to apply Theorem 2-2-2 to obtain a compactified birational cobordism $\overline{B} \subset \mathbb{P}(\mathcal{E})$ projective over X_2, where $\mathcal{E}$ is a coherent sheaf on X_2 with a K^*-action. The associated birational cobordism $B = \overline{B} - (i_1(X_1) \cup i_2(X_2))$ is collapsible and the connected components F of the fixed point set are ordered by the values of a strictly increasing function $\chi : \mathcal{C} \to \mathbb{Z}$

$$a_1 < a_2 < \cdots < a_{s-2} < a_{s-1}.$$

By looking at the semi-stable loci with respect to the linearizations induced by the "twisted" actions of K^* on $\mathcal{E}$ and at the corresponding G.I.T. quotients, we obtain a factorization

$$\begin{array}{ccccc}
W_i = (B_{a_i})_-/K^* & & \overset{\varphi_i}{\dashrightarrow} & & W_{i+1} = (B_{a_i})_+/K^* \\
& \searrow & & \swarrow & \\
\| & & B_{a_i}//K^* & & \| \\
(\overline{B}, \rho_{a_i-1})^{ss}/K^* & & \dashrightarrow & & (\overline{B}, \rho_{a_i+1})^{ss}/K^* \\
& \searrow & \| & \swarrow & \\
& & (\overline{B}, \rho_{a_i})^{ss}//K^* & &
\end{array}$$

starting with $X_1 = B_-/K^* = (B_{a_1})_-/K^* = W_1$ and ending with $W_s = (B_{a_{s-1}})_+/K^* = B_+/K^* = X_2$, as given in Proposition 2-3-1. By construction, each W_i is projective over X_2 and φ_i is an isomorphism over U. The existence of the index i_o as claimed in (iii) is also clear, once one takes the reduction in Step 0 into consideration.

Thus it remains to show that each φ_i is a V-locally toric birational map.

Lemma 2-4-4. *Let B_{a_i} be a quasi-elementary cobordism. Then for every closed point $p \in B_{a_i} - F^*_{B_{a_i}}$, where $F_{B_{a_i}} = B_{a_i}^{K^*}$ and $F^*_{B_{a_i}}$ is the set as defined in Notation 2-1-3, there exists Luna's locally toric chart*

$$X_p \xleftarrow{\eta_p} V_p \xrightarrow{i_p} U_p \subset B_{a_i}$$

such that the following condition $(\star)$ is satisfied:

$(\star)$ If a K^*-orbit $O(q)$ lies in U_p, then its closure $\overline{O(q)}$ in B_{a_i} also lies in U_p.

Proof.

We take Luna's locally toric chart

$$X_p \xleftarrow{\eta_p} V_p \xrightarrow{i_p} U_p \subset B_{a_i}$$

as constructed in Proposition 1-3-4, which may not satisfy the condition $(\star)$. Let $\pi : B_{a_i} \to B_{a_i}//K^*$ be the quotient map.

Case: $p \in B_{a_i} - F^*_{B_{a_i}}$ is not a fixed point.

In this case, $p \in B_{a_i} - \pi^{-1}(\pi(F_{B_{a_i}}))$. Thus by shrinking U_p (and accordingly V_p) if necessary, we may assume $U_p \subset B_{a_i} - \pi^{-1}(\pi(F_{B_{a_i}}))$, in which case the condition $(\star)$ is automatically satisfied.

Case: $p \in B_{a_i} - F^*_{B_{a_i}}$ is a fixed point.

In this case, set

$$\begin{aligned} C &:= F_{B_{a_i}} \cap U_p \\ D &:= \pi^{-1}(\pi(F_{B_{a_i}} - U_p)) \cap U_p = (F_{B_{a_i}} - U_p)^{\pm} \cap U_p. \end{aligned}$$

Since C and D are disjoint K^*-invariant closed subsets of an affine K^*-invariant variety U_p, Corollary 1.2 in Kirwan-Fogarty-Mumford [1] implies that there exists a K^*-invariant function $f \in A(U_p)^{K^*}$ such that

$$f \equiv 1 \text{ on } C \quad \& \quad f \equiv 0 \text{ on } D.$$

We only have to replace the oiginal U_p and V_p with

$$\begin{aligned} (U_p)_f &= \{q \in U_p ; f(q) \neq 0\} \text{ and} \\ (V_p)_f &= \{v \in V_p ; i_p^* f(v) \neq 0\}, \end{aligned}$$

respectively, in order for Luna's locally toric chart to satisfy the condition $(\star)$.

We resume the proof for φ_i being V-locally toric.

Remark that $\{U_p\}_{p \in B_{a_i} - F^*_{B_{a_i}}}$, where we have Luna's locally toric chart

$$X_p \xleftarrow{\eta_p} V_p \xrightarrow{i_p} U_p \subset B_{a_i}$$

satisfying the condition $(\star)$, is an open covering of B_{a_i}. By the condition $(\star)$ the morphism $V_p//K^* \to B_{a_i}//K^*$ is étale, as it is a composite of an étale morphism $V_p//K^* \to U_p//K^*$ with an inclusion $U_p//K^* \subset B_{a_i}//K^*$. We also have the commutative diagram

$$
\begin{array}{ccccc}
(B_{a_i})_-/K^* & \to & B_{a_i}//K^* & \leftarrow & (B_{a_i})_+/K^* \\
\uparrow & \square & \uparrow & \square & \uparrow \\
(U_p)_-/K^* \times_{U_p//K^*} V_p//K^* & \to & V_p//K^* & \leftarrow & (U_p)_+/K^* \times_{U_p//K^*} V_p//K^* \\
\| & & \| & & \| \\
(V_p)_-/K^* & \to & V_p//K^* & \leftarrow & (V_p)_+/K^* \\
\| & & \| & & \| \\
(X_p)_-/K^* \times_{X_p//K^*} V_p//K^* & \to & V_p//K^* & \leftarrow & (X_p)_+/K^* \times_{X_p//K^*} V_p//K^* \\
\downarrow & \square & \downarrow & \square & \downarrow \\
(X_p)_-/K^* & \to & X_p//K^* & \leftarrow & (X_p)_+/K^*.
\end{array}
$$

Therefore, according to Definition 1-2-10 (i), the proper birational map φ_i is V-locally toric.

This completes the proof of Theorem 2-4-1.

CHAPTER 3. TORIFICATION

In the previous chapter, we observed that the K^*-action on the entire cobordism B and hence on the quasi-elementary cobordisms B_{a_i} is locally toric and that this gives rise to the V-locally toric birational map

$$\begin{array}{ccc} W_i & \overset{\varphi_i}{\dashrightarrow} & W_{i+1} \\ \| & & \| \\ (B_{a_i})_-/K^* & & (B_{a_i})_+/K^* \\ & \searrow \quad \swarrow & \\ & (B_{a_i})/K^* & \end{array}$$

The reason why $\varphi_i : W_i = (B_{a_i})_-/K^* \dashrightarrow W_{i+1} = (B_{a_i})_+/K^*$ is merely locally toric and not toroidal is that the choices of the local coordinate divisors (for Luna's locally toric charts for B_{a_i}) are not canonical and hence that they do NOT patch together to provide a global boundary divisor $B_{a_i} - U_{B_{a_i}}$ necessary for a desired toroidal structure $(U_{B_{a_i}}, B_{a_i})$.

In order to create such a global boundary divisor, we blow up B_{a_i} along an ideal I, called the "**torific**" ideal on B_{a_i}, canonically defined in terms of the locally toric K^*-action on B_{a_i}. We show that the divisor D^{tor} defined by the principal ideal $\mu^{-1}(I) \cdot B_{a_i}^{tor}$, where

$$\mu : B_{a_i}^{tor} \to B_{a_i}$$

is the normalization of the blowup of B_{a_i} along I, provides the toroidal structure $(U_{B_{a_i}^{tor}} := B_{a_i}^{tor} - D^{tor}, B_{a_i}^{tor})$ with respect to which the induced K^*-action on $B_{a_i}^{tor}$ is toroidal. We call such a procedure "**torification**".

The basic idea that if one blows up an ideal then the divisors defined by the principalized ideal provide the resulting variety with some useful extra structure, can be traced back at least to Hironaka. The torific ideal we construct is closely related to the one used by Abramovich-DeJong [1] for their proof of Weak Resolution of Singularities.

After torification, $B_{a_i}^{tor}$ turns out to be a quasi-elementary cobordism and hence we obtain a V-toroidal birational map

$$\begin{array}{ccc} (U_{W_{i-}^{tor}}, W_{i-}^{tor}) & \overset{\varphi_i^{tor}}{\dashrightarrow} & (U_{W_{i+}^{tor}}, W_{i+}^{tor}) \\ \| & & \| \\ (U_{B_{a_i}^{tor}}, B_{a_i}^{tor})_-/K^* & & (U_{B_{a_i}^{tor}}, B_{a_i}^{tor})_+/K^* \\ & \searrow \quad \swarrow & \\ & (U_{B_{a_i}^{tor}}, B_{a_i}^{tor})/K^* & \end{array}$$

Remark in general that $W_i, W_{i\pm}^{tor}$ may be all singular and that the induced morphisms $f_{i\pm} : W_{i\pm}^{tor} \to W_i$ are not sequences of blowups with smooth centers on nonsingular varieties. Thus it seems that through the birational cobordism we decomposed ϕ into a sequence of locally toric birational maps, which were made then toroidal via torification, but at the fatal cost of introducing singularities. In Chapter 4, we remedy this situation by applying canonical resolution of singularities to $W_i, W_{i\pm}^{tor}$ (and canonical principalization of some ideals on them) so that we can apply the weak factorization theorem of toroidal birational maps between nonsingular toroidal embeddings by Morelli [1] (cf. Włodarczyk [1] Abramovich-Matsuki-Rashid [1] Abramovich-Karu-Matsuki-Włodarczyk [2]) to finish the proof of the Weak Factorization Theorem.

The ideal situation would have been that we could define the torific ideal on the entire cobordism B, the blowup of the torific ideal would then torify the K^*-action and hence we could reduce the factorization of ϕ to that of ONE toroidal birational map. But at the moment, for a small but essential reason to be discussed below, we can only define the torific ideal on each quasi-elementary cobordism B_{a_i} separately but not on the entire cobordism B. Thus we are forced to break up ϕ into SEVEARL toroidal birational maps and cannot make their toroidal structures compatible with each other. This is the main reason why we can only prove the WEAK Factorization Theorem but not the STRONG one (even assuming the strong factorization for toroidal birational maps for the moment).

In this chapter, $B = B_{a_i}$ always refers to a nonsingular quasi-elementary cobordism by abuse of notation with K^* acting effectively. We also omit the subscript a_i from the corresponding notations, e.g., $F = F_{a_i}, F^* = F_{a_i}^*$ etc. This convention is restricted to this chapter only.

§3-1. Construction of the torific ideal

Definition 3-1-1. *Let $p \in B$ be a closed point with $G_p \subset K^*$ being the stabilizer of p. Fix an integer $\alpha \in \mathbb{Z}$. Then we define*

$$J_{\alpha,p} \subset \mathcal{O}_{B,p}$$

to be the ideal generated by all the semi-invariant functions $f \in \mathcal{O}_{B,p}$ of G_p-character α, that is,

$$t^*(f) = t^\alpha f \text{ for } t \in G_p \subset K^*.$$

Theorem 3-1-2. *There exists a unique coherent K^*-equivariant ideal sheaf $I_\alpha \subset \mathcal{O}_B$ such that*

$$(I_\alpha)_p = J_{\alpha,p} \quad \forall p \in B - F^*.$$

(We note that a K^*-equivariant sheaf is nothing but a K^*-sheaf in the sense of Mumford [1], where he considers the notion of a G-sheaf when a group G acts on a variety. In our case, $\mathcal{O}_B$ has the natural structure of a K^*-sheaf induced by the K^*-action on B and we require that the ideal I_α has the K^*-subsheaf structure.)

Definition 3-1-3. *The coherent sheaf I_α as above is called the α-torific ideal sheaf with respect to the K^*-action on B.*

Remark 3-1-4.

(i) It is absolutely necessary to exclude the points in F^* from the ones on which we require the condition $(I_\alpha)_p = J_{\alpha,p}$. The collection of ideals $J_{\alpha,p}$ for all the points $p \in B$ does not define a coherent sheaf in general. As an example, let $B = \mathbb{A}^2$ and let K^* act on it by

$$t(x,y) = (tx, t^{-1}y) \text{ for } t \in K^*.$$

We fix $\alpha = 1$. Then at $p = (0,0)$ the stabilizer is $G_p = K^*$ and $J_{1,p} = (x)$. On the other hand, at any other point $q \in B - \{p\}$, the stabilizer is trivial $G_q = \{1\} \subset K^*$ and hence $J_{1,q} = \mathcal{O}_{B,q}$. While $J_{1,q} = \mathcal{O}_{B,q}$ for $q = (x,y)$ with $x \neq 0$ and $J_{1,p} = (x)$ patch together, $J_{1,q} = \mathcal{O}_{B,q}$ for $q = (x,y)$ with $x = 0$ and $J_{1,p} = (x)$ do NOT.

(ii) For a point $q \in F^*$, the stalk $(I_\alpha)_q$ is determined by the K^*-equivariance of I_α and by the fact that the closure $\overline{O(q)}$ of the orbit of q has a unique fixed point $p \in F \subset B - F^*$, since B is quasi-elementary. (See the proof for the uniqueness of the α-torific ideal.) On the other hand, if B is not quasi-elementary, the closure $\overline{O(q)}$ may have two distinct fixed points p_1 and p_2, and the possible stalks at q induced from those at p_1 and p_2, respectively, by the K^*-equivariance may not coincide. This is why we have trouble defining the torific ideal if B is not quasi-elementary.

Proof of Theorem 3-1-2.

(**Uniqueness**) Suppose we have two coherent ideal sheaves I_α and I'_α satisfying the condition of Theorem 3-1-2. Then

$$(I_\alpha)_p = (I'_\alpha)_p = J_{\alpha,p} \text{ for } p \in B - F^*.$$

We have to check

$$(I_\alpha)_q = (I'_\alpha)_q \text{ for } q \in F^*.$$

Since B is quasi-elementary, the closure $\overline{O(q)}$ contains only one fixed point $r \in \overline{O(q)}$. Since $r \in F \subset B - F^*$, we have

$$(I_\alpha)_r = (I'_\alpha)_r = J_{\alpha,r}.$$

Since both I_α and I'_α are coherent, there exists an open neighborhood $r \in U$ such that

$$I_\alpha|_U = I'_\alpha|_U.$$

Take a point $q_o = t_o(q) \in O(q) \cap U$ for some $t_o \in K$. Then by the K^*-equivariance we have

$$(I_\alpha)_q = t_o^*\{(I_\alpha)_{q_o}\} = t_o^*\{(I'_\alpha)_{q_o}\} = (I'_\alpha)_q,$$

proving the uniqueness.

(**Existence**)

Step 1. First observe that it is enough to prove the following.

Claim 3-1-5. *For each point $p \in B - F^*$, there exist a K^*-invariant open neighborhood $p \in U_p$ satisfying the condition $(\star)$:*

$(*)$ *If a K^*-orbit $O(q)$ lies in U_p, then its closure $\overline{O(q)}$ also lies in U_p.*

and the α-torific ideal over U_p, i.e., a K^*-equivariant coherent ideal sheaf I_{α,U_p} over U_p such that

$$(I_{\alpha,U_p})_q = J_{\alpha,q} \quad \forall q \in U_p \cap (B - F^*).$$

In fact, the collection $\{U_p\}_{p \in B-F^*}$ is an open covering of B and the collection of ideals $\{I_{\alpha,U_p}\}_{p \in B-F^*}$ patches together by the uniqueness argument, using the condition $(\star)$, to provide the global α-torific ideal over B as required.

Step 2. We take Luna's locally toric chart for $p \in B - F^*$

$$X \xleftarrow{\eta} V \xrightarrow{i} Y = U_p \subset B$$

where η and i are strongly étale with i being surjective, X_p is a nonsingular affine toric variety and where $Y = U_p$ satisfies the condition $(\star)$.

Claim 3-1-6. *In order to show the existence of the α-torific ideal I_{α,U_p}, it suffices to show the existence of the α-torific ideal $I_{\alpha,X}$ over the toric variety X.*

We spend the rest of Step 2 for the verification of this claim.

Lemma 3-1-7. *Let $p \in B$ and $J_{\alpha,p} \subset \mathcal{O}_{B,p}$ be the ideal generated by all the semi-invarinat functions in $\mathcal{O}_{B,p}$ of G_p-character α as in Definition 3-1-1. We lift the G_p-action on $\mathcal{O}_{B,p}$ to the one on the completion $\hat{\mathcal{O}}_{B,p}$. Then the completion $\widehat{J_{\alpha,p}}$ of the ideal $J_{\alpha,p}$*

$$\widehat{J_{\alpha,p}} = J_{\alpha,p} \otimes \widehat{\mathcal{O}_{B,p}} \subset \mathcal{O}_{B,p} \otimes \widehat{\mathcal{O}_{B,p}} = \widehat{\mathcal{O}_{B,p}}$$

is characterized as the ideal generated by all the semi-invariant functions in $\widehat{\mathcal{O}_{B,p}}$ of G_p-character α with respect to the induced G_p-action on $\widehat{\mathcal{O}_{B,p}}$.

Proof.

The affine coordinate ring for $Y = \mathrm{Spec}A$ splits into the direct sum

$$A = \oplus_{\beta \in \mathbb{Z}} A_\beta$$

where the A_β consist of all the semi-invariant functions of K^*-character β. This splitting induces the splitting of the maximal ideal m_p for the point p

$$m_p = \oplus_{\beta' \in \mathbb{Z}/(b_p)} m_{p,\beta'}$$

where $m_{p,\beta'}$ are the elements of m_p with G_p-character β'. (The stabilizer $G_p \subset K^*$ is identified with the group of b_p-th roots of unity. We set $b_p = 0$ when $G_p = K^*$.) In fact, it is easy to see that any direct summand A_β must be contained in m_p if $\beta (\mathrm{mod}\ b_p) \neq 0$. Thus taking $m_{p,0} = m_p \cap \oplus_{\beta (\mathrm{mod}\ b_p)=0} A_\beta$ and $m_{p,\beta'} = \oplus_{\beta (\mathrm{mod}\ b_p)=\beta'} A_\beta$ for $\beta' \neq 0$ we obtain the above splitting. Therefore, we can choose semi-invariant functions $z_1, \cdots, z_n$ (with respect to the G_p-action) which form a basis of the vector space m_p/m_p^2 and hence generates m_p as an $\mathcal{O}_{B,p}$-module by Nakayama's Lemma. We have a G_p-equivariant isomorphism

$$\widehat{\mathcal{O}_{B,p}} \cong K[[z_1, \cdots, z_n]]$$

where the action of G_p on the formal power series is induced by the G_p-characters on $z_1, \cdots, z_n$.

Since the G_p-action respects the natural grading on $K[[z_1, \cdots, z_n]]$ given by the degree and since $K[[z_1, \cdots, z_n]]$ is noetherian, the ideal K_α generated by all the semi-invariant functions in $\widehat{\mathcal{O}_{B,p}}$ of G_p-character α is generated by a finite number of monomials $m_1, \cdots, m_l$ of G_p-character α in $z_1, \cdots, z_n$. Since obviously $m_1, \cdots, m_l \in J_{\alpha,p}$, we have

$$K_\alpha = (m_1, \cdots, m_l) \subset \widehat{J_{\alpha,p}}.$$

Since $J_{\alpha,p} \subset K_\alpha$ by definition and hence $\widehat{J_{\alpha,p}} \subset K_\alpha$, we conclude

$$K_\alpha = \widehat{J_{\alpha,p}}$$

as required.

We resume the verification of Claim 3-1-6.

For any point $q \in Y$, let $q_V \in i^{-1}(q)$ be a point in the inverse image of q by i and $q_X = \eta(q_V)$. (We note that q_V and hence q_X may not be determined uniquely only by specifying the point q.) Observe by the condition $(\star)$ that we have

$$q \in B - F^* \Longleftrightarrow q_X \in X - F_X^*$$

and in particular

$$p \in B - F^* \Longrightarrow p_X \in X - F_X^*.$$

Let $\mathcal{H}_\alpha \subset \mathcal{O}_Y$ be the ideal sheaf over Y generated by the monomials $m_1, \cdots, m_l$ as taken in the proof of Lemma 3-1-7. Then we have

$$(\eta^* \widehat{I_{\alpha,X}})_{p_V} = \eta^*((\widehat{I_{\alpha,X}})_{p_X}) \overset{\eta^*}{\underset{\sim}{\longleftarrow}} (\widehat{I_{\alpha,X}})_{p_X}$$

and

$$\widehat{(i^*\mathcal{H}_\alpha)_{p_V}} = i^*(\widehat{(\mathcal{H}_\alpha)_p}) \xleftarrow{i^*} \widehat{(\mathcal{H}_\alpha)_p} = \widehat{J_{\alpha,p}}.$$

Now since $I_{\alpha,X}$ is the α-torific ideal over X and since $p_X \in X - F_X^*$, Lemma 3-1-7 characterizes both $\widehat{(\eta^* I_{\alpha,X})_{p_V}}$ and $\widehat{(i^*\mathcal{H})_{p_V}}$ as the ideal generated by all the semi-invariant functions in $\widehat{\mathcal{O}_{V,p_V}}$ of G_{p_V}-character α. (Remark that $G_{p_V} = G_{p_X} = G_p$ since η and i are both strongly étale.) Therefore, we have

$$\widehat{(\eta^* I_{\alpha,X})_{p_V}} = \widehat{(i^*\mathcal{H}_\alpha)_{p_V}},$$

which implies (See, e.g., Matsumura [1] Theorem 8.15 (3).)

$$(\eta^* I_{\alpha,X})_{p_V} = (i^*\mathcal{H}_\alpha)_{p_V}.$$

Therefore, noting that the equality above holds for all the points $p_V \in i^{-1}(p)$ and by shrinking $U_p = Y$ and accordingly V if necessary, we may assume

$$\eta^* I_{\alpha,X} = i^*\mathcal{H}_\alpha.$$

Now for any point $q \in Y - F^*$ we have $q_X \in X - F_X^*$ and

$$i^*(\widehat{J_{\alpha,q}}) = \eta^*(\widehat{(I_{\alpha,X})_{q_X}}),$$

both sides being characterized as the ideal generated by all the semi-invariant functions of $G_{q_V}(= G_q = G_{q_X})$-character α in $\widehat{\mathcal{O}_{V,q_V}}(\cong \widehat{\mathcal{O}_{B,q}} \cong \widehat{\mathcal{O}_{X,q_X}})$. On the other hand, we have

$$\begin{aligned}\eta^*(\widehat{(I_{\alpha,X})_{q_X}}) &= \widehat{(\eta^* I_{\alpha,X})_{q_V}} \\ &= \widehat{(i^*\mathcal{H}_\alpha)_{q_V}} \\ &= i^*(\widehat{(\mathcal{H}_\alpha)_q}).\end{aligned}$$

Since i is étale, we conclude

$$\widehat{J_{\alpha,q}} = \widehat{(\mathcal{H}_\alpha)_q},$$

which implies

$$J_{\alpha,q} = (\mathcal{H}_\alpha)_q.$$

Since η and i are K^*-equivariant and since $I_{\alpha,X}$ is K^*-equivariant as it is the α-torific ideal over X, the equality $\eta^* I_{\alpha,X} = i^*\mathcal{H}_\alpha$ implies that $\mathcal{H}_\alpha$ is also K^*-equivariant.

Therefore, we conclude that we can take $\mathcal{H}_\alpha$ to be the α-torific ideal I_{α,U_p} over $Y = U_p$.

Step 3. We show the existence of the α-torific ideal on an affine nonsingular toric variety X where K^*-acts as a one-parameter subgroup.

Proposition 3-1-8. *Let $X = X(N, \sigma)$ be an affine nonsingular toric variety where K^* acts as a one-parameter subgroup. Then the ideal generated by all the monomials of K^*-character α is the α-torific ideal over X.*

Proof.

We take a $\mathbb{Z}$-basis $\{v_1, \cdots, v_n\}$ of the lattice N so that

$$\sigma = \langle v_1, \cdots, v_l \rangle$$
$$\sigma^\vee = \langle v_1^*, \cdots v_l^*, \pm v_{l+1}^*, \cdots, \pm v_n^* \rangle.$$

Setting $z_j = z^{v_j^*}$, we have

$$X = \mathrm{Spec} K[z_1, \cdots, z_l, z_{l+1}^{\pm 1}, \cdots, z_n^{\pm 1}].$$

The K^*-action corresponds to a one-parameter subgroup

$$a = (\alpha_1, \cdots, \alpha_n) \in N,$$

i.e., the action of $t \in K^*$ is given by

$$t(z_1, \cdots, z_n) = (t^{\alpha_1} z_1, \cdots, t^{\alpha_n} z_n).$$

Let $H_{\alpha,X}$ be the ideal generated by all the monomials of K^*-character α. Then we see obviously that $H_{\alpha,X}$ is K^*-equivariant and that

$$(H_{\alpha,X})_q \subset J_{\alpha,q} \text{ for } q \in X - F_X^*.$$

Thus in order to prove $H_{\alpha,X}$ is the α-torific ideal over X we only have to show the opposite inclusion

$$J_{\alpha,q} \subset (H_{\alpha,X})_q.$$

Remark first that $J_{\alpha,q}$ is generated by all the monomials in $z_1, \cdots, z_n$ of G_q-character α. (Note that $z_1 - z_1(q), \cdots, z_n(q) - z_n(q)$ are generators of the maximal ideal m_q and are all semi-invariant functions with respect to the action of G_q. Indeed, if $z_i(q) \neq 0$, then z_i has the trivial G_q-character. Then using the argument in the proof of Lemma 3-1-7, we see that $J_{\alpha,q}$ is generated by all the monomials in $z_1 - z_1(q), \cdots, z_n(q) - z_n(q)$ of G_q-character α and hence generated by all the monomials in $z_1, \cdots, z_n$ of G_q-character α).

Let $f \in \mathcal{O}_{X,q}$ be a monomial in $z_1, \cdots, z_n$ of G_q-character α. We claim that there exists a monomial g in $z_1, \cdots, z_n$ invertible at q, i.e., $g \in \mathcal{O}_{X,q}^*$, such that $f \cdot g$ is a monomial of K^*-character α.

In fact, let $b_q \in \mathbb{Z}_{\geq 0}$ be the non-negative generator of the subgroup of $\mathbb{Z}$ generated by the following set

$$A_q := \{\alpha_j ; \alpha_j \neq 0 \text{ with } z_j(q) \neq 0\}.$$

The stabilizer $G_q \subset K^*$ consists of the b_q-th roots of unity. Let α_f be the K^*-character of the monomial f. Then $\alpha - \alpha_f$ is a multiple of b_q, say $b_f b_q$. By the

description of the set A_q, we have a monomial g in $z_1, \cdots, z_n$, invertible at q, of K^*-character $b_f b_q$. Then $f \cdot g$ is a monomial of K^*-character α by construction.

Since $f \cdot g$ is a monomial regular at q, it is regular in a K^*-invariant open neighborhood U_q of q. Since X is affine and hence quasi-elementary, we conclude that U_q must contain $\pi^{-1}(U)$, where $\pi : X \to X//K^*$ is the quotient morphism and where U is some open neighborhood of $\pi(q)$. Take a global function h on $X//K$ (which is identified with a global K^*-invariant function on X) such that $h \equiv 0$ on $X//K^* - U$ and $h(\pi(q)) = 1$ (See Mumford-Fogarty-Kirwan [1] Corollary 1.2.). Then $f \cdot g \cdot h^l$ (for $l >> 0$) is a global regular function of K^*-character α, hence a linear combination of monomials in $z_1, \cdots, z_n$ of K^*-character α.

Since f is an arbitrary monomial of G_q-character α and since g and h are invertible at q, we conclude that $J_{\alpha,q}$ generated by all the monomials of G_q-character α is also generated by all the monomials of K^*-character α. Therefore, we have

$$J_{\alpha,q} \subset (H_{\alpha,X})_q.$$

This completes the proof for Proposition 3-1-8 and hence the proof for Theorem 3-1-2 via Steps 1, 2 and 3.

§3-2. The torifying property of the torific ideal

In this section, we discuss the torifying property of the torific ideal (which is actually the product of several α-torific ideals defined in the previous section, where the characters α are taken from a certain finite set), i.e., how we can induce the torification of the K^*-action by blowing up the torific ideal, as announced at the beginning of Chapter 3.

First we explain the **torification principle** for affine nonsingular toric varieties X, showing the torifying property for them. The explanation of the torification principle should also provide the motivation for the definition of the torific ideal, including that of the α-torific ideal of the previous section. Secondly, utilizing on Luna's locally toric charts and based upon the torifying property for toric varieties, we discuss the torifying property of the torific ideal on a quasi-elementary cobordism in general.

Observation 3-2-1 (Torification Principle for Toric Varieties).

Let $X = X(N, \sigma)$ be an affine nonsingular toric variety with K^* acting as a one-parameter subgroup. As in Proposition 3-1-8, we take a $\mathbb{Z}$-basis of N so that

$$\begin{aligned} \sigma &= \langle v_1, \cdots, v_l \rangle \\ \sigma^\vee &= \langle v_1^*, \cdots, v_l^*, \pm v_{l+1}^*, \cdots, \pm v_n^* \rangle \\ X &= \mathrm{Spec} K[z_1, \cdots, z_l, z_{l+1}^{\pm}, \cdots, z_n^{\pm}] \text{ with } z_j = z^{v_j^*} \end{aligned}$$

and that the K^*-action corresponds to

$$a = (\alpha_1, \cdots, \alpha_n) \in N.$$

We have the coordinate divisors

$$D_{v_1} = \{z_1 = 0\}, \cdots, D_{v_l} = \{z_l = 0\}$$

corresponding to the 1-dimensional faces (generated by the v_j) of σ. As X is a toric variety containing the torus T_X, obviously

$$(T_X = X - (D_{v_1} \cup \cdots \cup D_{v_l}), X)$$

has the toric and hence toroidal structure with respect to which the K^*-action is toric and hence toroidal.

Let I be an ideal generated by some monomials and let

$$\mu_X : \tilde{X} \to X$$

be the normalization of the blowup of X along I. Since I is toric having monomial generators, the torus-action lifts to the blowup of X along I and to the normalization $\tilde{X}$, which is thus again a toric variety $\tilde{X} = X(N, \tilde{\sigma})$ where $\tilde{\sigma}$ is a fan obtained by subdividing σ. (We define a toric variety to be a NORMAL variety which contains a torus, as a dense open subset, whose action on itself extends to the entire variety.) The K^*-action also lifts to $\tilde{X}$ being a one-parameter subgroup.

Let $\tilde{D}$ to be the divisor defined by the principal ideal $\mu_X^{-1}(I) \cdot \mathcal{O}_{\tilde{X}}$. Then as $\tilde{X}$ is a toric variety containing the torus $T_X = T_{\tilde{X}}$,

$$(T_{\tilde{X}} = \tilde{X} - \{(D_{v_1} \cup \cdots \cup D_{v_l}) \cup \tilde{D}\}, \tilde{X})$$

has the toric and hence toroidal structure with respect to which the K^*-action is toric and hence toroidal. (We use the same notation D_{v_j} for the strict transform on $\tilde{X}$ of the divisor D_{v_j} on X, indicating both are the closures of the orbit corresponding to the 1-dimensional ray generated by v_j in $\tilde{X}$ and X, respectively.)

We look for a situation on $\tilde{X}$ (and the conditions on I which yields that situation on $\tilde{X}$) where

$$(U_{\tilde{X}} = \tilde{X} - \tilde{D}, \tilde{X})$$

has a toroidal structure with respect to which the K^*-action is toroidal. That is to say, after removing those coordinate divisors among $D_{v_1}, \cdots, D_{v_l}$ which are NOT contained in $\tilde{D}$ from the toric boundary divisor $(D_{v_1} \cup \cdots \cup D_{v_l}) \cup \tilde{D}$ we still want to preserve a toroidal stucture on $(U_{\tilde{X}}, \tilde{X})$ with respect to which the K^*-action remains toroidal.

Recall that the original coordinate divisors $D_{v_1}, \cdots, D_{v_l}$ correspond to a locally toric chart of the quasi-elementary cobordism B and are not chosen canonically and that we have to remove those which are NOT contained in the canonically constructed boundary divisor, which corresponds to $\tilde{D}$ which arises from the torific ideal defined canonicaly only in terms of the K^*-action.

Lemma 3-2-2. *We consider a fixed coordinate divisor D_{v_j} where $j = 1, \cdots, l$. Suppose that we have the following situation $(\heartsuit_j)$:*

$$(\heartsuit_j)\begin{cases} \text{For any affine (open) toric subvariety } Z \subset \tilde{X}, \\ \text{we have } (T_Z, Z) = (T_{\mathbb{A}^1}, \mathbb{A}^1) \times (T_{Z'_j}, Z'_j) \text{as toric varieties,} \\ \text{provided that } D_{v_j} \not\subset \tilde{D} \text{ and } Z \cap D_{v_j} \neq \emptyset, \\ \text{where } \mathbb{A}^1 \text{ has the standard toric structure with torus } T_{\mathbb{A}^1} = \mathbb{A}^1 - \{0\} \\ \text{and } Z'_j \text{ is an affine toric variety with the torus } T_{Z'_i} \\ \text{and where the } K^*\text{-action on } \mathbb{A}^1 = \mathrm{Spec}K[x_j] \text{ is trivial} \\ \text{and } K^* \text{ acts on } Z'_j \text{ as a one-parameter subgroup in } T_{Z'_j} \\ \text{and we have } Z \cap D_{v_j} = \{x_j = 0\}. \end{cases}$$

Then

$$(U_{\tilde{X},j} = \tilde{X} - \{(D_{v_1} \cup \cdots \cup \overset{\vee}{D}_{v_j} \cup \cdots \cup D_{v_l}) \cup \tilde{D}\}, \tilde{X})$$

has a toroidal structure with respect to which the K^*-action is toroidal.

Proof.

If $D_{v_j} \subset \tilde{D}$, then for any affine (open) toric subvariety $Z \subset \tilde{X}$ we have

$$(T_Z, Z) = Z \cap (U_{\tilde{X},j}, \tilde{X}) = Z \cap (U_{\tilde{X}}, \tilde{X})$$

has the toric and hence toroidal structure with respect to which the K^*-action is toric and hence toroidal. Since these Z's cover the entire $\tilde{X}$, the claim follows.

So we may assume $D_{v_j} \not\subset \tilde{D}$. Let $Z \subset \tilde{X}$ be an affine (open) toric subvariety.

If $Z \cap D_{v_j} = \emptyset$, then

$$(T_Z, Z) = Z \cap (U_{\tilde{X},j}, \tilde{X}) = Z \cap (U_{\tilde{X}}, \tilde{X})$$

has the toric and hence toroidal structure with respect to which the K^*-action is toric and hence toroidal.

If $Z \cap D_{v_j} \neq \emptyset$, then by the situation $(\heartsuit_j)$ we have

$$\begin{aligned} Z \cap (U_{\tilde{X},j}, \tilde{X}) &= (\mathbb{A}^1, \mathbb{A}^1) \times (U_{Z'_j}, Z'_j) \\ &= (U_{Z_{\{0\}}}, Z_{\{0\}}) \cup (U_{Z_{\{1\}}}, Z_{\{1\}}) \\ &\quad \text{where} \\ &\quad (U_{Z_{\{0\}}}, Z_{\{0\}}) := (\mathbb{A}^1 - \{0\}, \mathbb{A}^1 - \{0\}) \times (U_{Z'_j}, Z'_j) \text{ and} \\ &\quad (U_{Z_{\{1\}}}, Z_{\{1\}}) := (\mathbb{A}^1 - \{1\}, \mathbb{A}^1 - \{1\}) \times (U_{Z'_j}, Z'_j). \end{aligned}$$

Since the K^*-action on $(\mathbb{A}^1, \mathbb{A}^1) = (\mathbb{A}^1 - \{0\}, \mathbb{A}^1 - \{0\}) \cup (\mathbb{A}^1 - \{1\}, \mathbb{A}^1 - \{1\})$ is trivial, we have

$$(U_{Z_{\{0\}}}, Z_{\{0\}}), (U_{Z_{\{1\}}}, Z_{\{1\}}) \overset{K^*\text{-equivariant}}{\cong} (\mathbb{A}^1 - \{0\}, \mathbb{A}^1 - \{0\}) \times (U_{Z'_j}, Z'_j).$$

Since Z's with $Z \cap D_{v_j} = \emptyset$ and $Z_{\{0\}}, Z_{\{1\}}$'s with $Z \cap D_{v_j} \neq \emptyset$ cover the entire $\tilde{X}$, the claim follows.

Remark 3-2-3. *Let $S = \{j; D_{v_j} \in \tilde{D}\}$ be the set of indices of the primitive vectors generating the 1-dimensional cones in $\tilde{\sigma}$ which correspond to the divisors contained in $\tilde{D}$. Then $(\heartsuit_j)$ is equivalent to the following $(\heartsuit_{j,fan})$ in terms of the geometry of the fan $\tilde{\sigma}$:*

$(\heartsuit_{j,fan})$ *For any cone $\sigma_Z \in \tilde{\sigma}$,*
we have $\sigma_Z = \langle v_j \rangle \oplus \tau_{Z'_j}$ as cones,
provided that $j \notin S$ and $v_j \in \sigma_Z$,
where the cone $\tau_{Z'_j} = \langle w'_k s \rangle$ is generated by
the extremal rays w_k's of σ_Z
and is contained in a hyperplane H_j which also contains $a \in N$
and where we have the exact sequence
$$0 \to H_j \cap N \to N \to \mathbb{Z} \to 0$$
with the image of $v_j \in N$ to be $\pm 1 \in \mathbb{Z}$.

Now it is immediate to see that the blowup of the α_j-torific ideal $I_{\alpha_j,X}$ yields the situation $(\heartsuit_j)$. (See the proof of Theorem 3-2-6.) This justifies and actually motivated the definition of the α-torific ideal in the previous section.

In order to see that we can remove all the coordinate divisors which are not contained in $\tilde{D}$ simultaneously, not just one $D_{v_j} \not\subset \tilde{D}$ at a time, and hence

$$(U_{\tilde{X}}, \tilde{X})$$

has a toroidal structure, we need the lemma below.

Lemma 3-2-4. *Suppose we have the following situation $(\heartsuit)$:*

$(\heartsuit)$ *For any affine (open) toric subvariety $Z \subset \tilde{X}$,*
we have $(T_Z, Z) = \prod_j (T_{\mathbb{A}^1_j}, \mathbb{A}^1_j) \times (T_{Z'}, Z')$as toric varieties,
with the indices j varying among those with $D_{v_j} \not\subset \tilde{D}$ and $Z \cap D_{v_j} \neq \emptyset$,
where $\mathbb{A}^1_j$ has the standard toric structure with torus $T_{\mathbb{A}^1_j} = \mathbb{A}^1_j - \{0\}$
and Z' is an affine toric variety with the torus $T_{Z'}$
and where the K^*-action on $\mathbb{A}^1_j = \mathrm{Spec} K[x_j]$ is trivial
and K^* acts on Z' as a one-parameter subgroup in $T_{Z'}$
and we have $Z \cap D_{v_j} = \{x_j = 0\}$.

Then
$$(U_{\tilde{X}} = \tilde{X} - \tilde{D}, \tilde{X})$$
has a toroidal structure with respect to which the K^*-action is toroidal.

The proof is identical to that of Lemma 3-2-2.

Lemma 3-2-5. *The collection of the situations* $(\heartsuit_j)$ *for all* $j = 1, \cdots, l$ *implies the situation* $(\heartsuit)$.

Proof.

We prove that the collection of the situations $(\heartsuit_{j,fan})$ implies the following situation $(\heartsuit_{fan})$, which is equivalent to $(\heartsuit)$ and described in terms of the geometry of the fan $\tilde{\sigma}$:

$(\heartsuit_{fan})$ For any cone $\sigma_Z \in \tilde{\sigma}$,
we have $\sigma_Z = \Sigma_j \langle v_j \rangle \oplus \tau_{Z'}$ as cones,
with the indices varying among those $j \notin S$ with $v_j \in \sigma_Z$
where the cone $\tau_{Z'} = \langle w'_k s \rangle$ is generated by
the extremal rays w_k's (other than those v_j's with $j \notin S$) of σ_Z
and is contained in a linear space L (of codimension $\#\{j; j \notin S, v_j \in \sigma_Z\}$)
which also conains $a \in N$
and where we have the exact sequence

$$0 \to L \cap N \to N \to \mathbb{Z}^{\oplus \#\{j; j \notin S, v_j \in \sigma_Z\}} \to 0$$

with the image of $\{v_j\,; j \notin S, v_j \in \sigma_Z\} \in N$
forming a $\mathbb{Z}$-basis of $\mathbb{Z}^{\oplus \#\{j; j \notin S, v_j \in \sigma_Z\}}$.

Now having the exact sequences for the indices $j \notin S$ with $v_j \in \sigma_Z$

$$0 \to H_j \cap N \to N \to \mathbb{Z} \to 0$$

where $v_j \in N$ maps to $\pm 1 \in \mathbb{Z}$, it is elementary to see that we have the exact sequence with $L = \cap_{\{j; j \notin S, v_j \in \sigma_Z\}} H_j$

$$0 \to L \cap N \to N \to \mathbb{Z}^{\oplus \#\{j; j \notin S, v_j \in \sigma_Z\}} \to 0$$

such that $\{v_j\,; j \notin S, v_j \in \sigma_Z\}$ maps to a $\mathbb{Z}$-basis of $\mathbb{Z}^{\oplus \#\{j; j \notin S, v_j \in \sigma_Z\}}$.

This completes the proof for Lemma 3-2-5.

Therefore, in order to achieve the situation $(\heartsuit)$, we only have to verify the collection of the situations $(\heartsuit_j)$ for $j = 1, \cdots, l$, which should follow from taking the blowup μ_X to be the "collection" of the blowups of the $I_{\alpha_j, X}$ for $j = 1, \cdots, l$. This motivates us to define the torific ideal I to be the product of these:

$$I := \prod_{j=1}^{l} I_{\alpha_j, X}.$$

(Actually in Definition 3-2-7 we may multiply some more α-torific ideals with $\alpha \neq 1, \cdots, l$. See also the definition of the torific ideal for the quasi-elementary cobordism B.)

Now we present the verification of the torifying property of the torific ideal for toric varieties according to the torification principle.

Lemma 3-2-6. *There exists a finite set $\mathfrak{C}$ of integers so that at each point $q \in B$ it contains all the characters of G_q acting on the tangent space $T_{B,q}$ (modulo b_q).*

Proof.

In the case of $B = X$ being a affine nonsingular toric variety, the requirement is equivalent to $\{\alpha_1, \cdots, \alpha_l\} \subset \mathfrak{C}$ in the notation of Observation 3-2-1. The claim for the case of a general quasi-elementary cobordism B follows from this and the fact that B can be covered by a finite number of Luna's locally toric charts.

Definition 3-2-7. *Let $\mathfrak{C}$ be a finite set of integers as above. We define the torific ideal $I_{\mathfrak{C}}$ with respect to $\mathfrak{C}$ to be*

$$I_{\mathfrak{C}} = \prod_{\alpha \in \mathfrak{C}} I_\alpha.$$

Note that the torific ideal $I_{\mathfrak{C}}$ does depend on the choice of $\mathfrak{C}$ but we usually suppress this, call it the torific ideal and denote it by I without the subscript.

Theorem 3-2-8. *Let X be an affine nonsingular toric variety with K^* acting as a one-parameter subgroup. Suppose that the K^*-action is effective and that $X_+ \cap X_- \neq \emptyset$. Let*

$$\mu_X : X^{tor} \to X$$

be the normalization of the blowup of the torific ideal I_X for X and D^{tor} the divisor defined by the principal ideal $\mu_X^{-1}(I_X) \cdot \mathcal{O}_{X^{tor}}$. Then we have the following:

(i) $(U_{X^{tor}} := X^{tor} - D^{tor}, X^{tor})$ has a toroidal structure with respect to which the induced K^*-action is toroidal. More precisely, we have the situation ($\heartsuit$) for X^{tor}.

(ii) X^{tor} is a quasi-elementary cobordism.

(iii) Denote by $\{\sigma_{Z_{max}}\}$ the set of the maximal cones of σ^{tor}, where σ^{tor} is the fan corresponding to the toric variety $X^{tor} = X(N, \sigma^{tor})$. Then each of the affine toric varieties $\{X(N, \sigma_{Z_{max}})\}$ satisfies the condition ($\star$):

($\star$) If a K^*-orbit $O(q)$ lies in $X(N, \sigma_{Z_{max}})$, then its closure $\overline{O(q)}$ also lies in $X(N, \sigma_{Z_{max}})$.

As $\{X(N, \sigma_{Z_{max}}\}$ cover $X^{tor} = X(N, \sigma^{tor})$, the quotients $\{X(N, \sigma_{Z_{max}})/K^* = X(\pi(N), \pi(\sigma_{Z_{max}}))\}$ cover X^{tor}/K^*. Moreover, we have the commutative diagram of Cartesian products

$$\begin{array}{ccccc}
X(\pi(N), \pi(\partial_- \sigma_{Z_{max}})) & & & & X(\pi(N), \pi(\partial_+ \sigma_{Z_{max}})) \\
& \searrow & & \swarrow & \\
\downarrow & & X(\pi(N), \pi(\sigma_{Z_{max}})) & & \downarrow \\
(X^{tor})_-/K^* & & & & (X^{tor})_+/K^* \\
& \searrow & \downarrow & \swarrow & \\
& & X^{tor}/K^*. & &
\end{array}$$

Proof.

First note that the assumption of the K^*-action being effective and $X_+ \cap X_- \neq \emptyset$ guarantees that the α-torific ideal $I_{\alpha,X}$ is NOT zero for any $\alpha \in \mathbb{Z}$. (Actually this is the only place where we use the assumption. The following argument is valid as long as all the α-torific ideals appearing in the product to define the torific ideal are not zero.)

We use the same notation as in Observation 3-2-1. By Lemma 3-2-4 and Lemma 3-2-5 we only have to show the situations $(\heartsuit_j)$ for $j = 1, \cdots, l$.

First note that μ_X is characterized by the following properties:

(o) μ_X is proper birational,

(i) X^{tor} is normal,

(ii) $\mu_X^{-1}(I_{\alpha,X}) \cdot \mathcal{O}_{X^{tor}}$ is principal for each $\alpha \in S$, and

(iii) for any $\mu' : X' \to X$ satisfying the properties (o), (i) and (ii), there exists a unique $\nu' : X' \to X^{tor}$ with $\mu' = \mu_X \circ \nu'$.

Fix $j = 1, \cdots, l$.

First take

$$\mu_j : X_j \to X$$

to be the normalization of the blowup of X along $I_{\alpha_j,X}$. Secondly, let

$$\tau_j : X_j^{tor} \to X_j$$

to be the normalization of the blowup of X_j along the ideal

$$\prod_{\alpha \in S, \alpha \neq \alpha_j} \mu_j^{-1}(I_{\alpha,X}) \cdot \mathcal{O}_{X_j}.$$

Then by the characterization above, we have $X_j^{tor} = X^{tor}$ and

$$\mu_X = \mu_j \circ \tau_j.$$

We choose monomial generators of K^*-character α_j for $I_{\alpha_j,X}$ to be

$$v_j^*, m_1, \cdots, m_s$$

where $m_1, \cdots, m_s$ do not contain v_j^*, i.e.,

$$m_1, \cdots, m_s \in v_j^{\perp} \cap \sigma^{\vee} = \langle v_1^*, \cdots, \overset{\vee}{v_j^*}, \cdots, v_l^*, \pm v_{l+1}^*, \cdots, v_n^* \rangle.$$

Case: v_j^* generates $I_{\alpha_j,X}$ (and hence we do not have any of $m_1, \cdots, m_s$).

In this case, we have

$$D_{v_j} \subset D^{tor}.$$

Case: v_j^* does not generate $I_{\alpha_j,X}$.

In this case, X_j is covered by affine toric varieties which are the normalization of the affine charts of the following types for the blowup of X along $I_{\alpha_j,X}$.

∘ the chart obtained by inverting v_j^*: Over this chart, there is NO intersection with D_{v_j}.

∘ the chart(s) obtained by inverting, say m_1: The affine coordinate ring for the chart (before normalization) is

$$K[v_j^* - m_1, m_2 - m_1, \cdots, m_s - m_1, v_1^*, \cdots, \overset{\vee}{v_j^*}, \cdots, v_l^*, \pm v_{l+1}^*, \cdots, \pm v_n^*]$$

where

$$m_2 - m_1, \cdots, m_s - m_1, v_1^*, \cdots, \overset{\vee}{v_j^*}, \cdots, v_l^*, \pm v_{l+1}^*, \cdots, \pm v_n^* \in v_j^\perp \cap M.$$

Therefore, the normalization has the form

$$K[v_j^* - m_1] \otimes K[\{{y'}_{h'}\}]$$

where the monomials $\{{y'}_{h'}\}$ are taken from $v_j^\perp \cap M$.

Lemma 3-2-9. *Over this chart,*

$$\mu_j^{-1}(I_{\alpha,X}) \cdot \mathcal{O}_{X_j} \quad \forall \alpha \in S, \alpha \neq \alpha_j$$

is generated by monomials in $v_j^\perp \cap M$.

Proof.

Let $m \in I_{\alpha,X}$ be a monomial in $v_1^*, \cdots, v_l^*, \pm v_{l+1}^*, \cdots, \pm v_n^*$ of K^*-character α. Write

$$\begin{aligned} m &= cv_j^* + m' \\ &= c(v_j^* - m_1) + cm_1 + m' \end{aligned}$$

where $c \in \mathbb{Z}_{\geq 0}$ and m' is a monomial in $v_j^\perp \cap M$. Since $v_j^* - m_1$ has the trivial K^*-character, $cm_1 + m'$ has the K^*-character α and it is a monomial in $v_j^\perp \cap M$. As these monomials $\{m\}$ generate $\mu_j^{-1}(I_{\alpha,X})$, so do $\{cm_1 + m'\}$ and we have the claim.

By the above lemma, over this chart X^{tor}, which is obtained as the normalization of the blowup of X_j along $\prod_{\alpha \in S, \alpha \neq \alpha_j} \mu_j^{-1}(I_{\alpha,X}) \cdot \mathcal{O}_{X_j}$, is covered by the affine toric charts of the form

$$\mathrm{Spec} K[v_j^* - m_1] \otimes K[\{y_h\}]$$

where the monomials $\{y_h\}$ are taken from $v_j^\perp \cap M$ and where in the exact sequence

$$0 \to v_j^\perp \cap M \to M \to \mathbb{Z} \to 0$$

the image of $v_j^* - m_1 \in M$ is $\pm 1 \in \mathbb{Z}$. These affine charts satisfy the condition in the situation $(\heartsuit_j)$.

Varying the charts (i.e., inverting the monomial generators $m_1, \cdots, m_s$), we see that the fan σ^{tor}, where $X^{tor} = X(N, \sigma^{tor})$, is covered by the cones satisfying the condition in the situation $(\heartsuit_{j,fan})$. Therefore, we conclude that any cone in σ^{tor} satisfies the condition in the situation $(\heartsuit_j)$, that is to say, we have the situation $(\heartsuit_j)$.

This concludes the proof of the assertion (i) of Theorem 3-2-8.

Now we verify the assertion (ii) that X^{tor} is a quasi-elementary cobordism.

Suppose, on the contrary, that there is a point $q \in X^{tor}$ such that q is not a fixed point $q \notin F_{X^{tor}}$ and that both limits $\lim_{t\to\infty} t(q)$ and $\lim_{t\to 0} t(q)$ exists in X^{tor}.

Then it is necessarily the case that $\mu_X(q) \in X$ is not a fixed point.

In fact, consider the natural finite morphism

$$f : X^{tor} \to X \times \mathbb{P}^N$$

where the $N+1$ homogeneous coordinates $(f_0, \cdots, f_N)$ of $\mathbb{P}^N$ are given by the monomial generators of the torific ideal I_X of the same K^*-character $\prod_{\alpha\in S} \alpha$. If $\mu_X(q)$ is a fixed point, then $f(q) = \mu_X(q) \times (f_0(q), \cdots, f_N(q))$ is also a fixed point in $X \times \mathbb{P}^N$. As f is a finite morphism, this would imply that q is a fixed point, a contradiction.

Since μ_X is K^*-equivariant and continuous, both limits $\lim_{t\to\infty} t(\mu_X(q)) = \mu_X(\lim_{t\to\infty} t(q))$ and $\lim_{t\to 0} t(\mu_X(q)) = \mu_X(\lim_{t\to 0} t(q))$ exist in X, implying that X is not quasi-elementary. This is a contradiction, as X is affine and hence quasi-elementary.

Finally we verify the assertion (iii).

Observe that $q \in X^{tor}$, not being a fixed point, has the limit $\lim_{t\to\infty} t(q)$ in X^{tor} (resp. $\lim_{t\to 0} t(q)$) if and only if for the cone σ_Z whose corresponding orbit $O(\sigma_Z)$ contains q we have a cone $(\sigma_Z)^\infty$ (resp. $(\sigma_Z)^0$) such that there exists a point $x \in \sigma_Z$ with $x + \epsilon a \in (\sigma_Z)^\infty$ (resp. $x - \epsilon a \in (\sigma_Z)^0$) for all sufficiently small positive number ϵ. From this observation and from the fact that X^{tor} is quasi-elementary, it follows that $X(N, \sigma_{Z_{max}})$ satisfies the condition $(\star)$ for each of the maximal cones $\sigma_{Z_{max}}$ in σ^{tor}. Thus we have the inclusion

$$X(N, \sigma_{Z_{max}})/K^* \hookrightarrow X^{tor}/K^*$$

and the commutative diagram follows from the main example.

This completes the proof of Theorem 3-2-8.

Remark 3-2-10.

(i) By the argument for the assertion (iii) we see that X^{tor} being quasi-elemetary is equivalent to the condition that NO two maximal cones $\sigma_{Z_{1,max}}$ and $\sigma_{Z_{2,max}}$ with one being on top of the other: there exists a point $x \in \sigma_{Z_{1,max}} \cap \sigma_{Z_{2,max}}$ such that $x + \epsilon a \in \sigma_{Z_{1,max}}$ and $x - \epsilon a \in \sigma_{Z_{2,max}}$ for all sufficiently small positive number ϵ.

This in turn is equivalent to saying that the subdividion of σ to obtain σ^{tor} "comes from downstairs": Let $\pi : N_{\mathbb{R}} \to N_{\mathbb{R}}/\mathbb{R}\cdot a$ be the projection. Then there is a fan $\pi(\sigma)^{tor}$ obtained by subdividing $\pi(\sigma)$ so that the fan σ^{tor} is covered by the cones $\{\pi^{-1}(\tau) \cap \sigma; \tau \in \pi(\sigma)^{tor}\}$.

The last condition follows easily from the observation below, giving an alternative interpretation for the assertion (iii):

If $f_0, \cdots, f_N$ are ther monomial generators of the torific ideal I_X, then the blowup of X along I_X is covered by (the normalizations of) the affine charts of the form (say, inverting f_0)

$$\mathrm{Spec} K[\sigma^\vee \cap M][\frac{f_1}{f_0}, \cdots, \frac{f_N}{f_0}]$$

where $\frac{f_1}{f_0}, \cdots, \frac{f_N}{f_0}$ all have the trivial K^*-character, i.e., they are all K^*-invariants.

(ii) For a special choice of S so that it contains $\pm\beta$ where β is a nonzero character β which is divisible by all α_j for $j = 1, \cdots, l$ we can show that

$$\pi(\partial_- \sigma^{tor}) = \pi(\partial_+ \sigma^{tor})$$

and hence the two toroidal embeddings, obtained by excluding those coordinate divisors $D_{\pi(v_j)}$ with $j \notin S$ from the boundaries, also coincide

$$(U_{X^{tor}}, X^{tor})_-/K^* = (U_{X^{tor}}, X^{tor})_+/K^*.$$

As we do not use this fact in our proof, we only refer the reader to Abramovich-Karu-Matsuki-Włodarczyk [1] for a proof.

Example 3-2-11.

Here we give an easy example to demonstrate the torification principle for toric varieties.

Take $X = X(N, \sigma) = \mathbb{A}^3$ where σ is the regular cone generated by the standard $\mathbb{Z}$-basis

$$\sigma =< v_1, v_2, v_3 > \subset N_{\mathbb{R}}$$

and where $t \in K^*$ acts on X as

$$t(z_1, z_2, z_3) = (t^2 z_1, t z_2, t^{-1} z_3),$$

i.e., it corresponds to the one-parameter subgroup

$$a = (2, 1, -1) \in N.$$

We have the following monomial generators of the α-torific ideals for $\alpha = 2, 1, -1$

$$\begin{aligned} I_{2,X} &= \{z_1, z_1^2\} \\ I_{1,X} &= \{z_2, z_1 z_3\} \\ I_{-1,X} &= \{z\}. \end{aligned}$$

Set the torific ideal I_X (with respect to the set $\mathfrak{C} = \{2, 1, -1\}$) to be

$$I_X = I_{2,X} I_{1,X} I_{-1,X}.$$

Then the normalization $(\mathbb{A}^3)^{tor} = X(N, \sigma^{tor})$ of the blowup of X along the torific ideal I_X is described by the fan σ^{tor} consisting of the following three maximal cones

$$\sigma_1 =< v_1, v_3, 2v_1 + v_2 >$$
$$\sigma_2 =< v_2, v_1 + v_2, v_2 + v_3 >$$
$$\sigma_3 =< v_3, 2v_1 + v_2, v_1 + v_2, v_2 + v_3 > .$$

We have

$$D^{tor} = D_{v_3} \cup D_{v_1+v_2} \cup D_{v_2+v_3} \cup D_{2v_1+v_2}.$$

Thus the toroidal structure $(U_{X^{tor}}, X^{tor})$ is obtained from

$$(T_{X^{tor}} = X^{tor} - \{(D_{v_1} \cup D_{v_2}) \cup (D_{v_3} \cup D_{v_1+v_2} \cup D_{v_2+v_3} \cup D_{2v_1+v_2}), X^{tor})$$

by removing D_{v_1} and D_{v_2} from the boundary divisor. In fact, we observe the following.

The cone σ_1 has the product structure

$$\sigma_1 = \langle v_1 \rangle \oplus \langle v_3, 2v_1 + v_2 \rangle,$$

from which we have

$$X(N, \sigma_1) = \mathrm{Spec} K[v_1^* - 2v_2^*] \otimes K[v_2^*, v_3^*] \cong \mathbb{A}_1 \times Z'.$$

with $D_{v_1} \cap X(N, \sigma_1)$ defined by $\{v_1^* - 2v_2^* = 0\}$.

Thus we see that even after removing D_{v_1} from the boundary divisor

$$(U_{X^{tor}} \cap X(N, \sigma_1), X(N, \sigma_1)) = (\mathbb{A}_1 - \{0\}, \mathbb{A}_1) \times (T_{Z'}, Z') \cup (\mathbb{A}_1 - \{1\}, \mathbb{A}_1) \times (T_{Z'}, Z')$$

has the toroidal structure with respect to which the K^*-action is toroidal.

The cone σ_2 has the product structure

$$\sigma_1 = \langle v_2 \rangle \oplus \langle v_1 + v_2, v_2 + v_3 \rangle,$$

from which we have

$$X(N, \sigma_1) = \mathrm{Spec} K[v_2^* - (v_1^* + v_3^*)] \otimes K[v_1^*, v_3^*] \cong \mathbb{A}_1 \times Z''.$$

with $D_{v_2} \cap X(N, \sigma_1)$ defined by $\{v_2^* - (v_1^* + v_3^*) = 0\}$.

Thus we see that even after removing D_{v_2} from the boundary divisor

$$(U_{X^{tor}} \cap X(N, \sigma_2), X(N, \sigma_2)) = (\mathbb{A}_1 - \{0\}, \mathbb{A}_1) \times (T_{Z''}, Z'') \cup (\mathbb{A}_1 - \{1\}, \mathbb{A}_1) \times (T_{Z''}, Z'')$$

has the toroidal structure with respect to which the K^*-action is toroidal.

The cone σ_3 does not contain v_1 or v_2 and hence

$$(U_{X^{tor}} \cap X(N, \sigma_3), X(N, \sigma_3)) = (T_{X(N,\sigma_3)}, X(N, \sigma_3))$$

has the original toric and hence toroidal structure with respect to which the K^*-action is toric and hence toroidal.

Once the torification principle for toric varieties is established, it is straightforward to verify the torifying property of the torific ideal for a quasi-elementary cobordism via Luna's locally toric charts.

Theorem 3-2-12. *Let B be a nonsingular quasi-elementary cobordism with K^* acting effectively. Let*

$$\mu : B^{tor} \to B$$

be the normalization of the blowup of B along the torific ideal I and D^{tor} the divisor defined by the principal ideal $\mu^{-1}(I) \cdot \mathcal{O}_B$. Then

$$(U_{B^{tor}} = B^{tor} - D^{tor}, B^{tor})$$

has a toroidal structure with respect to which the induced K^*-action is toroidal.

Moreover, B^{tor} is a quasi-elementary cobordism, which induces a V-toroidal birational map φ as below:

$$\begin{array}{ccc} (U_{B^{tor}}, B^{tor})_-/K^* & \overset{\varphi}{\dashrightarrow} & (U_{B^{tor}}, B^{tor})_+/K^* \\ & \searrow \quad \swarrow & \\ & (U_{B^{tor}}, B^{tor})/K^* & \end{array}$$

Proof.

We take Luna's locally toric chart for $p \in B - F^*$

$$X \overset{\eta}{\leftarrow} V \overset{i}{\rightarrow} Y = U_p \subset B$$

where η and i are strongly étale with i being surjective and where $Y = U_p$ satisfies the condition $(\star)$. (Note that the collection $\{U_p ; p \in B - F^*\}$ provides an open covering of B.)

We use the same notation as in the proof of Claim 3-1-6.

Since for any point $q \in Y - F_Y^* = Y \cap (B - F^*)$ (which implies $q_V \in V - F_V^*$ and $q_X \in X - F_X^*$), we have

$$\widehat{(\eta^* I_{\alpha,X})}_{q_V} = \widehat{(I_{\alpha,V})}_{q_V} = \widehat{(i^* I_\alpha)}_{q_V}$$

with all sides being characterized as the ideal in $\widehat{\mathcal{O}_{V,q_V}}$ generated by all the semi-invariant functions of $G_{q_V}(= G_q = G_{q_X})$-character α (See Lemma 3-1-7.), which implies

$$(\eta^* I_{\alpha,X})_{q_V} = (I_{\alpha,V})_{q_V} = (i^* I_\alpha)_{q_V}.$$

Since $\eta^* I_{\alpha,X}, I_{\alpha,V}$ and $i^* I_\alpha$ are K^*-equivariant, this implies that the above equality holds not only for $q \in Y - F_Y$ but also for all $q \in Y$, i.e.,

$$\eta^* I_{\alpha,X} = I_{\alpha,V} = i^* I_\alpha.$$

Thus we conclude that the torific ideal on X pulls back to the torific ideal on V, coinciding with the torific ideal on B pulled back to V, i.e.,

$$\eta^* I_X = I_V = i^* I.$$

This gives rise to the commutative diagram of Cartesian products

$$\begin{array}{ccccccc}
(U_{X^{tor}}, X^{tor}) & \xleftarrow{\eta^{tor}} & (U_{V^{tor}}, V^{tor}) & \xrightarrow{i^{tor}} & (U_{Y^{tor}}, Y^{tor}) & \subset & (U_{B^{tor}}, B^{tor}) \\
\mu_X \downarrow & & \mu_V \downarrow & & \mu_Y \downarrow & & \\
X & \xleftarrow{\eta} & V & \xrightarrow{i} & Y & \subset & B.
\end{array}$$

For any point $r \in Y^{tor}$, we can find by the torification principle for the toric variety X an affine open set $Z_{\prod\{\delta_j\}}$ such that

$$r_{X^{tor}} \in (U_{Z_{\prod\{\delta_j\}}}, Z_{\prod\{\delta_j\}}) := Z_{\prod\{\delta_j\}} \cap (U_{X^{tor}}, X^{tor}) \cong \prod (\mathbb{A}^1 - \{\delta_j\}, \mathbb{A}^1 - \{\delta_j\}) \times (T_{Z'}, Z')$$

where Z' is an affine toric variety with the torus $T_{Z'}$ and the δ_i are either 0 or 1.

In order to obtain Luna's toroidal chart for $q \in Y^{tor} \subset B^{tor}$

$$(U_{X_q^{tor}}, X_q^{tor}) \leftarrow (U_{V_q^{tor}}, V_q^{tor}) \rightarrow (U_{Y_q^{tor}}, Y_q^{tor}) \subset (U_{Y^{tor}}, Y^{tor}) \subset (U_{B^{tor}}, B^{tor}),$$

we set

$$\begin{aligned}
(U_{X_q^{tor}}, X_q^{tor}) &:= (U_{Z_{\prod\{\delta_j\}}}, Z_{\prod\{\delta_j\}}) \\
(U_{V_q^{tor}}, V_q^{tor}) &:= (U_{Z_{\prod\{\delta_j\}}}, Z_{\prod\{\delta_j\}}) \times_{X//K^*} V//K^* \\
(U_{Y_q^{tor}}, Y_q^{tor}) &:= (U_{V_q^{tor}}, V_q^{tor})
\end{aligned}$$

when $p = \mu_Y(q) \in Y$ is a fixed point and hence i and i^{tor} are isomorphisms by construction of Luna's locally toric charts, and

$$\begin{aligned}
(U_{X_q}, X_q) &:= (U_{Z_{\prod\{\delta_j\}}}, Z_{\prod\{\delta_j\}}) \\
(U_{V_q}, V_q) &:= (U_{Z_{\prod\{\delta_j\}}}, Z_{\prod\{\delta_j\}}) \times_{X//K^*} V//K^* \cap (i^{tor})^{-1}(Y_r) \\
(U_{Y_q}, Y_q) &:= Y_q \cap (U_Y, Y)
\end{aligned}$$

where Y_q is a small K^*-invariant affine open neighborhood of q so that $Y_q \subset i^{tor}(V_q)$ when $\mu_Y(q) = p \in Y$ is not a fixed point and hence all the orbits in Y, V, X and $Y^{tor}, V^{tor}, X^{tor}$ are closed by construction of Luna's locally toric charts.

Therefore, $(U_{B^{tor}}, B^{tor})$ has a toroidal structure with respect to which the induced K^*-action is toroidal.

It remains to prove that B^{tor} is a quasi-elementary cobordism.

Suppose that B^{tor} is not quasi-elementary, i.e., there is a point $q \in B^{tor}$ such that q is not a fixed point and that both limits $\lim_{t\to\infty} t(q)$ and $\lim_{t\to 0} t(q)$ exist in B^{tor}. Take $r \in B - F^*$ so that $Y = U_r$ (in Luna's locally toric chart for r) cotains $\mu(q)$. Note that since Y satisfies the condition $(\star)$, $Y^{tor} = U_r^{tor} \subset B^{tor}$ also satisfies the condition $(\star)$. In particular, both limits sit inside of Y^{tor}. On the other hand, since both η^{tor} and i^{tor} are strongly étale as so are η and i, this implies that $q_{X^{tor}} \in X^{tor}$ is not a fixed point and that both limits $\lim_{t\to\infty} t(q_{X^{tor}})$

and $\lim_{t\to 0} t(q_{X^{tor}})$ exist in X^{tor}. This contradicts the fact by Theorem 3-2-8 that X^{tor} is quasi-elementary.

The commutative diagram indicating φ is V-toroidal is an immediate consequence of the toroidal structure $(U_{B^{tor}}, B^{tor})$ (with respect to which the induced K^*-action is toroidal) and the fact B^{tor} being quasi-elementary.

This completes the proof of Theorem 3-2-12.

CHAPTER 4.
RECOVERY OF NONSINGULARITY

Let us recall what we have established so far.

In Chapter 2, we constructed a birational cobordism B for a given birational map $\phi : X_1 \dashrightarrow X_2$ (after replacing it with some projective birational morphism by eliminating points of indeterminacy by sequences of blowups with smooth centers). The birational cobordism gives rise to the decomposition of ϕ into a sequence of (V-)locally toric birational maps

$$\phi : X_1 = W_1 \overset{\varphi_1}{\dashrightarrow} W_2 \overset{\varphi_2}{\dashrightarrow} \cdots W_i \overset{\varphi_i}{\dashrightarrow} W_{i+1} \cdots \overset{\varphi_{s-2}}{\dashrightarrow} W_{s-2} \overset{\varphi_{s-1}}{\dashrightarrow} W_s = X_2$$

where each φ_i corresponds to the commutative diagram

$$\begin{array}{ccc} W_i & \overset{\varphi_i}{\dashrightarrow} & W_{i+1} \\ \| & & \| \\ (B_{a_i})_-/K^* & & (B_{a_i})_+/K^* \\ & \searrow \quad \swarrow & \\ & (B_{a_i})/K^* & \end{array}$$

induced by the quasi-elementary cobordism B_{a_i}, associated to the value a_i of the strictly increasing function from the set of the connected components of the fixed point set of B to their characters in $\mathbb{Z}$.

The main object of Chapter 3 was to transform locally toric birational maps into toroidal birational transformations. By blowing up the torific ideal, we obtain a toroidal structure $(U_{B^{tor}_{a_i}}, B^{tor}_{a_i})$ with respect to which the K^*-action is toroidal and hence which induces the (V-)toroidal birational map φ_i^{tor} as in the commutative diagram below

$$\begin{array}{ccc} (U_{W^{tor}_{i-}}, W^{tor}_{i-}) & \overset{\varphi_i^{tor}}{\dashrightarrow} & (U_{W^{tor}_{i+}}, W^{tor}_{i+}) \\ \| & & \| \\ (U_{B^{tor}_{a_i}}, B^{tor}_{a_i})_-/K^* & & (U_{B^{tor}_{a_i}}, B^{tor}_{a_i})_+/K^* \\ & \searrow \quad \swarrow & \\ & (U_{B^{tor}_{a_i}}, B^{tor}_{a_i})/K^* & \end{array}$$

Since the blowup morphism $\mu : B_{a_i}^{tor} \to B_{a_i}$ gives rise to a commutative diagram

$$\begin{array}{ccccc}
(U_{B_{a_i}^{tor}}, B_{a_i}^{tor})_-/K^* & & & & (U_{B_{a_i}^{tor}}, B_{a_i}^{tor})_+/K^* \\
& \searrow & & \swarrow & \\
\downarrow & & (U_{B_{a_i}^{tor}}, B_{a_i}^{tor})/K^* & & \downarrow \\
(B_{a_i})_-/K^* & & & & (B_{a_i})_+/K^* \\
& \searrow & \downarrow & \swarrow & \\
& & (B_{a_i})/K^* & &
\end{array}$$

we actually factor φ_i into a sequence of birational maps

$$\varphi_i : W_i \overset{f_{i-}}{\leftarrow} (U_{W_{i-}^{tor}}, W_{i-}^{tor}) \overset{\varphi_i^{tor}}{\dashrightarrow} (U_{W_{i+}^{tor}}, W_{i+}^{tor}) \overset{f_{i+}}{\to} W_{i+1}.$$

If $W_i, W_{i-}^{tor}, W_{i+}^{tor}$ and W_{i+1} were all nonsingular and both f_{i-} and f_{i+} were sequences of blowups with smooth centers, then we would only have to apply the weak factorization theorem of Morelli [1,2] or Włodarczyk [1] (See Abramovich-Matsuki-Rashid [1] Matsuki [2].) to the toroidal birational map φ_i^{tor} between nonsingular toroidal embeddings to finish the proof of the weak factorization theorem. But in general these varieties may all be singular, while f_{i-} and f_{i+} may not be sequences of blowups with smooth centers.

The purpose of Chapter 4 is to bring this SINGULAR situation back to a NONSINGULAR one by an application of the canonical resolution of singularities.

More precisely, we construct the following commutative diagram

$$\begin{array}{ccccccc}
W_i^{res} & \xleftarrow{p_{i-}} & W_{i-}^{can} & \overset{\varphi_i^{can}}{\dashrightarrow} & W_{i+}^{can} & \xrightarrow{p_{i+}} & W_{i+1}^{res} \\
{\scriptstyle r_{i-}}\downarrow & & \downarrow{\scriptstyle h_{i-}} & & \downarrow{\scriptstyle h_{i+}} & & \downarrow{\scriptstyle r_{i+}} \\
W_i & \xleftarrow{f_{i-}} & W_{i-}^{tor} & \overset{\varphi_i^{tor}}{\dashrightarrow} & W_{i+}^{tor} & \xrightarrow{f_{i+}} & W_{i+1}
\end{array}$$

where

(i) $W_i^{res}, W_{i-}^{can}, W_{i+}^{can}$ and W_{i+1}^{res} are all nonsingular,

(ii) the morphisms p_{i-} and p_{i+} are sequences of blowups with smooth centers, and

(iii) by setting

$$U_{W_{i-}^{can}} = h_{i-}^{-1}(U_{W_{i-}^{tor}})$$
$$U_{W_{i+}^{can}} = h_{i+}^{-1}(U_{W_{i+}^{tor}})$$

the birational map

$$\varphi_i^{can} : (U_{W_{i-}^{can}}, W_{i-}^{can}) \dashrightarrow (U_{W_{i+}^{can}}, W_{i+}^{can})$$

is V-toroidal via the commutative diagram

$$\begin{array}{ccc} (U_{W_{i-}^{can}}, W_{i-}^{can}) & \overset{\varphi_i^{can}}{\dashrightarrow} & (U_{W_{i+}^{can}}, W_{i+}^{can}) \\ {\scriptstyle h_{i-}}\downarrow & & \downarrow{\scriptstyle h_{i+}} \\ (U_{W_{i-}^{tor}}, W_{i-}^{tor}) & \overset{\varphi_i^{tor}}{\dashrightarrow} & (U_{W_{i+}^{can}}, W_{i+}^{tor}) \\ & \searrow \qquad \swarrow & \\ & (U_{B_{a_i}^{tor}}, B_{a_i}^{tor})/K^*. & \end{array}$$

After the construction, we only have to apply the weak factorization theorem of toroidal birational maps to φ_i^{can} to finish the proof of the weak factorization theorem.

§4-1. Application of the canonical resolution of singularities

Remark 4-1-1.

(i) By the **canonical resolution of singularities**, we mean an algorithm which, given a variety X, provides a uniquely determined sequence of blowups with smooth centers $r : X^{res} \to X$ from a nonsingular variety, satisfying the conditions:

($\spadesuit^{res}-0$) The centers of blowups are taken only over the singular locus $Sing(X)$.

($\spadesuit^{res}-1$) If $Y \to X$ is a smooth morphism of varieties, then the canonical resolution of Y is obtained by pulling back that of X, i.e., we have a Cartesian product

$$\begin{array}{ccc} Y & \longleftarrow & Y \times_X X^{res} = Y^{res} \\ \downarrow & & \downarrow \\ X & \longleftarrow & X^{res}. \end{array}$$

In particular, the canonical resolution can be pulled back by an open immersion or an étale morphism.

The condition ($\spadesuit^{res}-1$) also implies that any automorphism of X (not necessarily over K) lifts to that of the canonical resolution X^{res}.

Moreover, the condition ($\spadesuit^{res}-1$) implies that a family of automorphisms θ_g of a variety X parametrized by a smooth variety $g \in G$ lifts to the family of automorphisms of the canonical resolution X^{res} (for example an action of an algebraic group G on a variety X). In fact, applying the condition ($\spadesuit^{res}-1$) to the map parametrizing the automorphisms of X

$$\theta : G \times X \to X \text{ given by } \theta(g,x) = \theta_g(x)$$

and applying it also to the projection onto the second factor $pr_2 : G \times X \to X$ we obtain the map parametrizing the automorphisms of X^{res}

$$\theta^{res} : G \times X^{res} = (G \times X) \times_{pr_2 X} X^{res} = (G \times X) \times_{\theta X} X^{res} = Y^{res} \to X^{res}.$$

(Actually since all the nonsingular points are étale equivalent, the condition ($\spadesuit^{res}-0$) is also a consequnce of the condition ($\spadesuit^{res}-1$). However, we state the

condition ($\spadesuit^{res}-0$) separately in order to bring it explicitly to the attention of the reader.)

(ii) By the **canonical principalization of ideals**, we mean an algorithm which, given an ideal sheaf I on a nonsingular variety X, provides a uniquely determined sequence of blowups with smooth (and admissible) centers $p : X^{can} \to X$ from a nonsingular variety with the property $p^{-1}(I) \cdot \mathcal{O}_{X^{can}}$ being principal, satisfying the conditions:

($\spadesuit^{can}-0$) The centers of blowups are taken over the support of $\mathcal{O}_X/I$.

($\spadesuit^{can}-1$) If $f : Y \to X$ is a smooth morphism of nonsingular varieties (not necessarily over the base field K), then the canonical principalization of the ideal sheaf f^*I is obtained by pulling back that of I, i.e., we have a Cartesian product

$$\begin{array}{ccc} Y & \longleftarrow & Y \times_X X^{can} = Y^{can} \\ \downarrow & & \downarrow \\ X & \longleftarrow & X^{can}. \end{array}$$

In particular, the canonical principalization (of the corresponding ideals) can be pulled back by an open immersion or an étale morphism.

Also the condition ($\spadesuit^{can}-1$) implies that any automorphism of X stabilizing the ideal I (not necessarily over K) lifts to that of the canonical principalization X^{can}.

Moreover, the condition ($\spadesuit^{can}-1$) implies that a family of automorphisms θ_g of a variety X parametrized by a smooth variety $g \in G$ lifts to the family of automorphisms of the canonical pricipalization X^{can} (for example an action of an algebraic group G on a variety X), if the ideal I is stable under the auomorphisms, i.e., $\theta_g^* I = I \ \ \forall g \in G$. In fact, applying the condition ($\spadesuit^{can}-1$) to the map parametrizing the automorphisms of X

$$\theta : G \times X \to X \text{ given by } \theta(g,x) = \theta_g(x)$$

and applying it to the projection onto the second factor $pr_2 : G \times X \to X$ and also noting that the stability implies $\theta^* I = pr_2^* I$, we obtain the map parametrizing the automorphisms of X^{can} as before

$$\theta^{can} : G \times X^{can} = (G \times X) \times_{pr_2 X} X^{can} = (G \times X) \times_{\theta X} X^{can} = Y^{can} \to X^{can}.$$

(Again the condition ($\spadesuit^{can}-0$) is also a consequnce of the condition ($\spadesuit^{can}-1$).)

(iii) All the existing algorithms for the canonical resolution of singularities and/or canonical principalization of ideals (cf. Bierstone-Milman [1] Villamayor [1] Encinas-Villamayor [1] and Hironaka [2] among others) satisfy the requirements in (i) and (ii) as above. As these are the only properties we utilize in our proof, one may choose any one of the algorithms and we denote the resulting varieties X^{res} and X^{can} without specifying the choice.

We go back to the construction of the commutative diagram for the purpose of recovery of nonsingularity.

We take W_i^{res} and W_{i+1}^{res} to be the canonical resolution of singularities of W_i and W_{i+1}, respectively.

Lemma 4-1-2. *The morphism $f_{i-} : W_{i-}^{tor} \to W_i$ is a projective morphism (in the sense of Grothendieck [1] but not in the sense of Hartshorne [1]) with an effective Cartier divisor E_{i-}^{tor} such that*

(i) the support of E_{i-}^{tor} lies in $\{D^{tor} \cap (B_{a_i}^{tor})_-\}/K^*$, where D^{tor} is the divisor defined by the principal ideal $\mu^{-1}(I) \cdot \mathcal{O}_{B_{a_i}^{tor}}$ with $\mu : B_{a_i}^{tor} \to B_{a_i}$ being the normalization of the blowup of the torific ideal I,

(ii) $\mathcal{O}_{W_{i-}^{tor}}(-E_{i-}^{tor})$ is f_{i-}-ample, and

(iii) the morphism f_{i-} is the blowup of the ideal

$$I_{i-} = (f_{i-})_* \mathcal{O}_{W_{i-}^{tor}}(-E_{i-}^{tor}).$$

Similarly, the morphism $f_{i+} : W_{i+}^{tor} \to W_{i+1}$ is a projective morphism with an effective Cartier divisor satisfying the conditions (i), (ii) and (iii) with the negative sign $-$ replaced by the positive sign $+$.

Proof.

Since $\mu : B_{a_i}^{tor} \to B_{a_i}$ is the normalization of the blowup of B_{a_i} along the torific ideal I, we can take an effective Cartier divisor E_i^{tor} such that the principal ideal $\mu^{-1}(I) \cdot \mathcal{O}_{B_{a_i}^{tor}} = \mathcal{O}_{B_{a_i}^{tor}}(-E_i^{tor})$ is μ-ample and hence $\mu|_{(B_{a_i}^{tor})_-}$-ample when restricted to $\mu|_{(B_{a_i}^{tor})_-} : (B_{a_i}^{tor})_- \to (B_{a_i})_-$.

Now if we look at Luna's locally toric chart satisfying the condition $(\star)$

$$X = X(N, \sigma) = X \xleftarrow{\eta} V \xrightarrow{i} Y = U \subset B_{a_i},$$

the morphisms

$$(B_{a_i}^{tor})_- \to (B_{a_i})_-$$
$$(B_{a_i}^{tor})_-/K^* \to (B_{a_i})_-/K^*$$

correspond to the toric morphisms

$$X(N, \partial_- \sigma^{tor}) \to X(N, \partial_- \sigma)$$
$$X(\pi(N), \pi(\partial_- \sigma^{tor})) \to X(\pi(N), \pi(\partial_- \sigma)),$$

where $\pi : \mathbb{N}_{\mathbb{R}} \to \mathbb{N}_{\mathbb{R}}/a \cdot \mathbb{R}$ is the projection (cf. the notation in Proposition 3-1-8 or Observation 3-2-1).

Remark also that since the torific ideal I corresponds to the toric torific ideal I_X, i.e., $\eta^* I_X = i^* I$, the Cartier divisor E_i^{tor} corresponds to the toric Cartier divisor $(E_i^{tor})_{X^{tor}}$ with $\eta^*(E_i^{tor})_{X^{tor}} = i^* E_i^{tor}$.

Say, the toric Cartier divisor $(E_i^{tor})_{X^{tor}}$ is associated to a piecewise linear function $\psi_{(E_i^{tor})_{X^{tor}}}$ on the fan σ^{tor} (cf. Fulton [1] or Oda [1]). Since $\pi : \partial_- \sigma^{tor} \to \pi(\partial_- \sigma^{tor})$ is a linear isomorphism of fans (without taking the lattices N and $\pi(N)$ into consideration), there exists a piecewise linear function $\psi_{(E_{i-}^{tor})_{(X^{tor})_-/K^*}}$ on the fan $\pi(\partial_- \sigma^{tor})$ such that

$$\psi_{(E_i^{tor})_{X^{tor}}}|_{\partial_- \sigma^{tor}} = \pi^*(\psi_{(E_{i-}^{tor})_{(X^{tor})_-/K^*}}).$$

Take $(E_{i-}^{tor})_{(X^{tor})_-/K^*}$ to be the effective $\mathbb{Q}$-Cartier divisor associated to the piecewise linear function $\psi_{(E_{i-}^{tor})_{(X^{tor})_-/K^*}}$. (Recall that $(E_{i-}^{tor})_{(X^{tor})_-/K^*}$ is only guaranteed to be $\mathbb{Q}$-Cartier, since in general the map of lattices $\pi : N|_{\partial_-\sigma^{tor}} \to \pi(N)|_{\pi(\partial_-\sigma^{tor})}$ is not an isomorphism.)

Observe the following implications:

$[\mathcal{O}_{X^{tor}}(-(E_i^{tor})_{X^{tor}})$ is μ_X-ample]

$\Longrightarrow$

$[\mathcal{O}_{X^{tor}}(-(E_i^{tor})_{X^{tor}})|_{(X^{tor})_-}$ is $\mu_X|_{(X^{tor})_-}$-ample]

$\Longleftrightarrow$

$[-\psi_{(E_i^{tor})_{X^{tor}}|_{\partial_-\sigma^{tor}}}$ is strictly convex when restricted to each cone in $\partial_-\sigma$ with respect to $\partial_-\sigma^{tor}]$

$\Longleftrightarrow$

$[-\psi_{(E_{i-}^{tor})_{(X^{tor})_-/K^*}}$ is strictly convex when restricted to each cone in $\pi(\partial_-\sigma)$ with respect to $\pi(\partial_-\sigma^{tor})]$

$\Longleftrightarrow$

$[\mathcal{O}_{(X^{tor})_-/K^*}(-(E_{i-}^{tor})_{(X^{tor})_-/K^*})$ is $(f_{i-})_X$-ample where $(f_{i-})_X$ is the morphism $(f_{i-})_X : (X^{tor})_-/K^* \to X_-/K^*.]$

Therefore, if we take E_{i-}^{tor} to be the effective $\mathbb{Q}$-Cartier divisor on W_{i-}^{tor} corresponding to $(E_{i-}^{tor})_{X^{tor}}$, then the conditions (i) and (ii) are satisfied.

By replacing it with a sufficiently divisible and high multiple, we see that there exists an effective Cartier divisor E_{i-}^{tor} satisfying all the conditions (i), (ii) and (iii).

This completes the proof of Lemma 4-1-2.

We take W_{i-}^{can} (resp. W_{i+}^{can}) to be canonical principalization of the ideal $(r_{i-})^{-1}(I_{i-}) \cdot \mathcal{O}_{W_i^{res}}$ (resp. $(r_{i+})^{-1}(I_{i+}) \cdot \mathcal{O}_{W_{i+1}^{res}}$).

Proposition 4-1-3. *Let*

$$h_{i-} : W_{i-}^{can} \to W_{i-}^{tor}$$
$$h_{i+} : W_{i+}^{can} \to W_{i+}^{tor}$$

be the induced morphisms and set

$$U_{W_{i-}^{can}} = h_{i-}^{-1}(U_{W_{i-}^{tor}})$$
$$U_{W_{i+}^{can}} = h_{i+}^{-1}(U_{W_{i+}^{tor}}).$$

Then

(i) both $(U_{W_{i-}^{can}}, W_{i-}^{can})$ *and* $(U_{W_{i+}^{can}}, W_{i+}^{can})$ *have the toroidal structures,*

(ii) the induced birational map $\varphi_i^{can} : (U_{W_{i-}^{can}}, W_{i-}^{can}) \dashrightarrow (U_{W_{i+}^{can}}, W_{i+}^{can})$ *is V-toroidal via the commutative diagram*

$$\begin{array}{ccc}
(U_{W_{i-}^{can}}, W_{i-}^{can}) & & (U_{W_{i+}^{can}}, W_{i+}^{can}) \\
{\scriptstyle h_{i-}}\downarrow & & \downarrow{\scriptstyle h_{i+}} \\
(U_{W_{i-}^{tor}}, W_{i-}^{tor}) & & (U_{W_{i+}^{tor}}, W_{i+}^{tor}) \\
\| & & \| \\
(U_{B_{a_i}^{tor}}, B_{a_i}^{tor})_-/K^* & & (U_{B_{a_i}^{tor}}, B_{a_i}^{tor})_+/K^* \\
& \searrow \quad \swarrow & \\
& (U_{B_{a_i}^{tor}}, B_{a_i}^{tor})/K^*. &
\end{array}$$

Proof.

We take Luna's locally toric chart for $p \in B_{a_i} - F_{a_i}^*$ as before

$$X \xleftarrow{\eta} V \xrightarrow{i} Y = U_p \subset B_{a_i}$$

where η and i are strongly étale with i being surjective and where $Y = U_p$ satisfies the condition $(*)$.

This gives rise to the commutative diagram of Cartesian products

$$\begin{array}{ccccccc}
X^{tor} & \xleftarrow{\eta^{tor}} & V^{tor} & \xrightarrow{i^{tor}} & Y^{tor} & \subset & B_{a_i}^{tor} \\
{\scriptstyle \mu_X}\downarrow & & {\scriptstyle \mu_V}\downarrow & & {\scriptstyle \mu_Y}\downarrow & & \\
X & \xleftarrow{\eta} & V & \xrightarrow{i} & Y & \subset & B_{a_i}.
\end{array}$$

Since the canonical resolution is pulled-back by étale morphisms and since so does the canonical principalization of ideals, noting

$$\{\eta^{tor}\}^*\{\mu_X^{-1} I_X \cdot \mathcal{O}_{X^{tor}}\} = \{i^{tor}\}^*\{\mu_Y^{-1} I_Y \cdot \mathcal{O}_{Y^{tor}}\},$$

which implies

$$\begin{aligned}
\{\eta_-/K^*\}^*(I_{i-})_{X_-/K^*} &= \{i_-/K^*\}^*(I_{i-})_{Y_-/K^*} \\
\{\eta_+/K^*\}^*(I_{i+})_{X_+/K^*} &= \{i_+/K^*\}^*(I_{i+})_{Y_+/K^*},
\end{aligned}$$

we have the commutative diagram of Cartesian products:

Therefore, we only have to show the assertion assuming that the quasi-elementary cobordism $B_{a_i} = X$ is a nonsingular affine toric variety on which K^* acts effectively as a one-parameter subgroup.

Note first that the morphisms

$$f_{i-} : W_{i-}^{tor} = (X^{tor})_-/K^* \to W_i = X_-/K^*$$
$$f_{i+} : W_{i+}^{tor} = (X^{tor})_+/K^* \to W_i = X_+/K^*$$

are equivariant (toric) ones between toric varieties by construction, that so are the morphisms

$$r_{i-} : W_i^{res} = (X_-/K^*)^{res} \to W_i = X_-/K^*$$
$$r_{i+} : W_i^{res} = (X_+/K^*)^{res} \to W_{i+1} = X_+/K^*$$

since the canonical resolution of singularities preserves the action of the embedded torus (See the condition ($\spadesuit^{res} - 1$).), and that so are the morphisms

$$p_{i-} : W_{i-}^{can} = (X_-/K^*)^{can} \to W_i^{res} = (X_-/K^*)^{res}$$
$$p_{i+} : W_{i+}^{can} = (X_+/K^*)^{can} \to W_i^{res} = (X_+/K^*)^{res}$$

since the ideals I_{i-} and I_{i+} (and their pull-backs by r_{i-} and r_{i+}) are both toric by construction (cf. Lemma 4-1-2) and since the canonical principalization also preserves the action of the embedded torus (See the condition ($\spadesuit^{can} - 1$).). Therefore, we conclude that the morphisms

$$h_{i-} : W_{i-}^{can} = (X_-/K^*)^{can} \to W_{i-}^{tor} = (X_-/K^*)^{tor}$$
$$h_{i+} : W_{i+}^{can} = (X_+/K^*)^{can} \to W_{i+}^{tor} = (X_+/K^*)^{tor}.$$

are also equivariant (toric).

We denote the corresponding fans and associated troic varieties by

$$\begin{aligned}
W_i &= X(\pi(N), \pi(\partial_-\sigma)) \\
W_i^{res} &= X(\pi(N), \Sigma_i^{res}) \\
W_{i-}^{tor} &= X(\pi(N), \Sigma_-^{tor}) \\
W_{i-}^{can} &= X(\pi(N), \Sigma_-^{can}) \\
W_{i+1} &= X(\pi(N), \pi(\partial_+\sigma)) \\
W_{i+1}^{res} &= X(\pi(N), \Sigma_{i+1}^{res}) \\
W_{i+}^{tor} &= X(\pi(N), \Sigma_+^{tor}) \\
W_{i+}^{can} &= X(\pi(N), \Sigma_+^{can})
\end{aligned}$$

Let $\{\sigma_{Z_{max}}\}$ be the set of the maximal cones of σ^{tor}, where σ^{tor} is the fan corresponding to the toric variety $X^{tor} = X(N, \sigma^{tor})$. By Theorem 3-2-8 (i) we have the situation ($\heartsuit_{fan}$). Therefore, for any $\sigma_{Z_{max}}$ we have the decomposition

$$\sigma_{Z_{max}} = \Sigma_{\{j; j \notin S, v_j \in \sigma_{Z_{max}}\}} \langle v_j \rangle \oplus \tau_{Z'_{max}} \text{ as cones,}$$

where the cone $\tau_{Z'_{max}}$ is generated by the extremal rays $w'_k s$ (other than those v_j's with $j \notin S$) of $\sigma_{Z_{max}}$ and is contained in a linear space L (of codimension $\#\{j; j \notin S, v_j \in \sigma_{Z_{max}}\}$) which also contains $a \in N$ and where we have the exact sequence

$$0 \to L \cap N \to N \to \mathbb{Z}^{\oplus \#\{j; j \notin S, v_i \in \sigma_{Z,max}\}} \to 0$$

with the image of $\{v_j; j \notin S, v_i \in \sigma_{Z_{max}}\}$ forming a $\mathbb{Z}$-basis of $\mathbb{Z}^{\#\{j; j \notin S, v_j \in \sigma_{Z_{max}}\}}$.

Therefore, under the projection map $\pi : N_{\mathbb{R}} \to N/a \cdot N_{\mathbb{R}}$ we have

$$\pi(N) = \Sigma_{\{j; j \notin S, v_j \in \sigma_{Z,max}\}} \mathbb{Z} \cdot v_j \oplus \pi(N \cap L)$$

and

$$\begin{aligned}
\pi(\partial_- \sigma_{Z_{max}}) &= \Sigma_{\{j; j \notin S, v_j \in \sigma_{Z_{max}}\}} \langle v_j \rangle \oplus \pi(\partial_- \tau_{Z'_{max}}) \\
\pi(\sigma_{Z_{max}}) &= \Sigma_{\{j; j \notin S, v_j \in \sigma_{Z_{max}}\}} \langle v_j \rangle \oplus \pi(\tau_{Z'_{max}}) \\
\pi(\partial_+ \sigma_{Z_{max}}) &= \Sigma_{\{j; j \notin S, v_j \in \sigma_{Z_{max}}\}} \langle v_j \rangle \oplus \pi(\partial_+ \tau_{Z'_{max}}).
\end{aligned}$$

By Theorem 3-2-8 (iii) we have the commutative diagram of Cartesian products

$$\begin{array}{ccccc}
X(\pi(N), \pi(\partial_- \sigma_{Z_{max}})) & & & & X(\pi(N), \pi(\partial_+ \sigma_{Z_{max}})) \\
& \searrow & & \swarrow & \\
\downarrow & & X(\pi(N), \pi(\sigma_{Z_{max}})) & & \downarrow \\
(X^{tor})_-/K^* & & & & (X^{tor})_+/K^* \\
& \searrow & \downarrow & \swarrow & \\
& & X^{tor}/K^*. & &
\end{array}$$

Since the affine toric varieties $\{X(\pi(N), \pi(\sigma_{Z_{max}}))\}$ cover X^{tor}/K^*, we conclude that $\{X(\pi(N), \pi(\partial_- \sigma_{Z_{max}}))\}$ cover $(X^{tor})_-/K^* = X(\pi(N), \Sigma^{tor}_-))$. That is to say, the cones $\{\pi(\partial_- \sigma_{Z_{max}})\}$ cover the fan Σ^{tor}_-. Thus the fan Σ^{can}_-, which is a refinement of Σ^{tor}_-, is the union of the refinements $\pi(\partial_- \sigma_{Z_{max}})^{can}$ of $\pi(\partial_- \sigma_{Z_{max}})$. Also we have the identical statement replacing the $-$ sign with the $+$ sign.

We prove that the following situations $(\heartsuit_{\pi(\partial_- fan)})$ and $(\heartsuit_{\pi(\partial_+ fan)})$ hold:

$$(\heartsuit_{\pi(\partial_- fan)}) \begin{cases}
\pi(\partial_- \sigma_{Z_{max}})^{can} \text{has the product structure} \\
\pi(\partial_- \sigma_{Z_{max}})^{can} = \Sigma_{\{j; j \notin S, v_j \in \sigma_{Z_{max}}\}} \langle v_j \rangle \oplus \pi(\partial_- \tau_{Z'_{max}})^{can} \\
\text{inherited from the product structure of } \pi(\partial_- \sigma_{Z_{max}}) \text{ and } \pi(\sigma_{Z_{max}}) \\
\pi(\partial_- \sigma_{Z_{max}}) = \Sigma_{\{j; j \notin S, v_j \in \sigma_{Z_{max}}\}} \langle v_j \rangle \oplus \pi(\partial_- \tau_{Z'_{max}}) \\
\pi(\sigma_{Z_{max}}) = \Sigma_{\{j; j \notin S, v_j \in \sigma_{Z_{max}}\}} \langle v_j \rangle \oplus \pi(\tau_{Z'_{max}}) \\
\text{where } \pi(\partial_- \tau_{Z'_{max}})^{can} \text{ is a refinement of } \pi(\partial_- \tau_{Z'_{max}}).
\end{cases}$$

The situation $(\heartsuit_{\pi(\partial_+ fan)})$ is idetical to $(\heartsuit_{\pi(\partial_- fan)})$, replacing the $-$ sign with the $+$ sign.

In fact, since the toroidal structure

$$X(\pi(N),\pi(\sigma_{Z_{max}}))\cap(U_{X^{tor}},X^{tor})=(U_{X(\pi(N),\pi(\sigma_{Z_{max}}))},X(\pi(N),\pi(\sigma_{Z_{max}}))$$

is covered by the open sets of the form

$$\prod_{\{j;j\in S,v_j\in\sigma_{Z_{max}}\}}(\mathbb{A}^1-\{\delta_j\},\mathbb{A}^1-\{\delta_j\})\times(T_{X(\pi(N\cap L),\pi(\partial_-\tau_{Z'_{max}}))},X(\pi(N\cap L),\pi(\partial_-\tau_{Z'_{max}}))$$
$$\cong\prod_{\{j;j\in S,v_j\in\sigma_{Z_{max}}\}}(\mathbb{A}^1-\{0\},\mathbb{A}^1-\{0\})\times(T_{X(\pi(N\cap L),\pi(\partial_-\tau_{Z'_{max}}))},X(\pi(N\cap L),\pi(\partial_-\tau_{Z'_{max}})),$$

the situations $(\heartsuit_{\pi(\partial_- fan)})$ and $(\heartsuit_{\pi(\partial_+ fan)})$ would imply that the morphisms h_{i-} and h_{i+} over these open sets are toric morphisms of the form

$$h_{i-}:\prod(\mathbb{A}^1-\{0\},\mathbb{A}^1-\{0\})\times(T_{X(\pi(N\cap L),\pi(\partial_-\tau_{Z'_{max}}))},X(\pi(N\cap L),\pi(\partial_-\tau_{Z'_{max}})^{can})$$
$$\to$$
$$\prod(\mathbb{A}^1-\{0\},\mathbb{A}^1-\{0\})\times(T_{X(\pi(N\cap L),\pi(\partial_-\tau_{Z'_{max}}))},X(\pi(N\cap L),\pi(\partial_-\tau_{Z'_{max}}))$$
$$h_{i+}:\prod(\mathbb{A}^1-\{0\},\mathbb{A}^1-\{0\})\times(T_{X(\pi(N\cap L),\pi(\partial_+\tau_{Z'_{max}}))},X(\pi(N\cap L),\pi(\partial_+\tau_{Z'_{max}})^{can})$$
$$\to$$
$$\prod(\mathbb{A}^1-\{0\},\mathbb{A}^1-\{0\})\times(T_{X(\pi(N\cap L),\pi(\partial_+\tau_{Z'_{max}}))},X(\pi(N\cap L),\pi(\partial_+\tau_{Z'_{max}}))$$

where the morphisms on the first factors are identities and where the morphisms on the second factors are induced by the refinements of the cones and that hence h_{i-} and h_{i+} are toroidal. Since we have the diagram of toric morphisms

$$\begin{array}{ccc}
X(\pi(N\cap L),\pi(\partial_-\tau_{Z'_{max}})^{can}) & \dashrightarrow & X(\pi(N\cap L),\pi(\partial_+\tau_{Z'_{max}})^{can}) \\
\downarrow & & \downarrow \\
X(\pi(N\cap L),\pi(\partial_-\tau_{Z'_{max}})) & & X(\pi(N\cap L),\pi(\partial_+\tau_{Z'_{max}})) \\
\searrow & & \swarrow \\
& X(\pi(N\cap L),\pi(\tau_{Z'_{max}})), &
\end{array}$$

we also conclude that φ_i^{can} is V-toroidal.

Now we only discuss the verification of the situation $(\heartsuit_{\pi(\partial_- fan)})$, as that of $(\heartsuit_{\pi(\partial_+ fan)})$ is identical replacing the $-$ sign with the $+$ sign.

Note that, induced by the product structure

$$\sigma_{Z_{max}}=\langle v_j\rangle\oplus\tau_{Z'_{max,j}},$$

where the cone $\tau_{Z'_{max,j}}$ is contained in a hyperplane H_j which also contains $a\in N$ and where we have the exact sequence

$$0\to H_j\cap N\to N\to\mathbb{Z}\to 0$$

with the image of $v_j \in N$ to be $\pm 1 \in \mathbb{Z}$, we have the product structure

$$\pi(\partial_- \sigma_{Z_{max}}) = \langle v_j \rangle \oplus \pi(\partial_- \tau_{Z'_{max,j}}).$$

Since

$$\pi(L) = \cap_{j \notin S, v_j \in \sigma_{Z_{max}}} \pi(H_j),$$

it is straightforward to observe that in order to prove the situation $(\heartsuit_{\pi(\partial_- fan)})$ it suffices to show the collection of the situations $(\heartsuit_{j,\pi(\partial_- fan)})$ for all $j = 1, \cdots, l$:

$$(\heartsuit_{j,\pi(\partial_- fan)}) \begin{cases} \text{Provided } j \notin S \text{ and } v_j \in \sigma_{Z_{max}}, \\ \pi(\partial_- \sigma_{Z_{max}})^{can} \text{has the product structure} \\ \pi(\partial_- \sigma_{Z_{max}})^{can} = \langle v_j \rangle \oplus \pi(\partial_- \tau_{Z'_{max,j}})^{can} \\ \text{inherited from the product structure of } \pi(\partial_- \sigma_{Z_{max}}) \text{ and } \pi(\sigma_{Z_{max}}) \\ \pi(\partial_- \sigma_{Z_{max}}) = \langle v_j \rangle \oplus \pi(\partial_- \tau_{Z'_{max,j}}) \\ \pi(\sigma_{Z_{max}}) = \langle v_j \rangle \oplus \pi(\tau_{Z'_{max,j}}) \\ \text{where } \pi(\partial_- \tau_{Z'_{max,j}})^{can} \text{ is a refinement of } \pi(\partial_- \tau_{Z'_{max,j}}). \end{cases}$$

Now we prove the situation $(\heartsuit_{j,\pi(\partial_- fan)})$ $(j = 1, \cdots, l)$ in the form of the following proposition.

Proposition 4-1-4. *Let $\pi(\partial_- \sigma_{Z_{max}})^{can}$ be the subfan of Σ_-^{can} lying over the cone $\pi(\partial_- \sigma_{Z_{max}})$ of Σ_-^{tor}. Let v_j $(j = 1, \cdots, l)$ be such primitive vector with $v_j \notin D^{tor}$ and $v_j \in \sigma_{Z_{max}}$. Then any new extremal ray $w \in \pi(\partial_- \sigma_{Z_{max}})^{can}$ to subdivide $\pi(\partial_- \sigma_{Z_{max}})$ sits inside of the cone $\pi(\partial_- \tau_{Z'_{max,j}})$, where $\pi(\partial_- \sigma_{Z_{max}})$ has the product structure*

$$\pi(\partial_- \sigma_{Z_{max}}) = \langle v_j \rangle \oplus \pi(\partial_- \tau_{Z'_{max,j}})$$

induced from the product structure of $\sigma_{Z_{max}}$

$$\sigma_{Z_{max}} = \langle v_j \rangle \oplus \tau_{Z'_{max,j}}.$$

Therefore, we have the situation $(\heartsuit_{j,\pi(\partial_- fan)})$.

proof.

We use the same notation as in Proposition 3-1-8 or Observation 3-2-1.
We take a $\mathbb{Z}$-basis $\{v_1, \cdots, v_n\}$ of the lattice N so that

$$\sigma = \langle v_1, \cdots, v_l \rangle$$
$$\sigma^\vee = \langle v_1^*, \cdots v_l^*, \pm v_{l+1}^*, \cdots, \pm v_n^* \rangle.$$

Setting $z_i = z^{v_i^*}$, we have

$$X = \mathrm{Spec} K[z_1, \cdots, z_l, z_{l+1}^{\pm 1}, \cdots, z_n^{\pm 1}].$$

The K^*-action corresponds to a one-parameter subgroup

$$a = (\alpha_1, \cdots, \alpha_n) \in N,$$

i.e., the action of $t \in K^*$ is given by

$$t(z_1, \cdots, z_n) = (t^{\alpha_1} z_1, \cdots, t^{\alpha_n} z_n).$$

Then by the proof of Theorem 3-2-8 the affine toric variety associated to a maximal cone $\sigma_{Z,max} \in \sigma^{tor}$ has the form

$$\mathrm{Spec}K[v_j^* - m_1] \otimes K[\{y_h\}]$$

where the monomials $\{y_h\}$ are taken from $v_j^{\perp} \cap M$ and where in the exact sequence

$$0 \longrightarrow v_j^{\perp} \cap M \longrightarrow M \longrightarrow \mathbb{Z} \longrightarrow 0$$

the image of $v_j^* - m_1 \in M$ is $\pm 1 \in \mathbb{Z}$.

As the dual statement in terms of the cones, we obtain the product structure

$$\sigma_{Z,max} = \langle v_j \rangle \oplus \tau_{Z'_{max,j}}$$

where the cone $\tau_{Z'_{max,j}}$ is defined by the equation $(v_j^* - m_1, \cdot) = 0$. Therefore, in the induced product structure

$$\Sigma_-^{tor} \supset \partial_- \sigma_{Z,max} = \langle v_j \rangle \oplus \partial_- \tau_{Z'_{max,j}}$$

the cone $\partial_- \tau_{Z'_{max,j}}$ is also defined by the equation $(v_j^* - m_1, \cdot) = 0$. (Note that since $(v_j^* - m_1, a) = 0$, we have $v_j^* - m_1 \in M \cap a^{\perp} = Hom_{\mathbb{Z}}(\pi(N), \mathbb{Z})$.)

Therefore, in order to show that a new extremal ray $w \in \pi(\partial_- \sigma_{Z_{max}})^{can}$ to subdivide $\pi(\partial_- \sigma_{Z_{max}})$ sits inside of $\partial_- \tau_{Z'_{max,j}}$ it suffices to show

$$(v_j^* - m_1, w) = \mathrm{ord}_{E_w}(z_j \cdot z^{-m_1}) \leq 0,$$

where $\mathrm{ord}_{E_w}(r)$ denotes the order of the rational function r on the divisor (necessarily exceptional for h_{i-}) corresponding to the ray w.

Now consider the family of automorphisms θ_c parametrized by $c \in K$ of the ring $K[z_1, \cdots, z_l, z_{l+1}^{\pm 1}, \cdots, z_n^{\pm 1}]$ (or equivalently the action of $K = \mathbb{A}^1$ on $\mathrm{Spec}K[z_1, \cdots, z_l, z_{l+1}^{\pm 1}, \cdots, z_n^{\pm 1}]$) defined by

$$\theta_c^*(z_j) = z_j + c \cdot z^{m_1}$$
$$\theta_c^*(z_k) = z_k \text{ for } k \neq j.$$

Observe that these automorphisms preserve the torific ideals because z_j and m_1 has the same character. It follows that θ_c induces a family of automorphisms of W_{i-}^{tor} mapping the divisor E_{i-}^{tor} to itself. Therefore, the automorphisms also preserve I_{i-}. (See Lemma 4-1-2.)

Therefore, by the conditions $(\spadesuit^{res} - 1)$ and $(\spadesuit^{can} - 1)$ of the canonical resolution and canonical principaliztion, we conclude that θ_c induces an automorphism of W_{i-}^{can} which preserves the exceptional divisor E_w. (Note that the set of exceptional

divisors for h_{i-} are discrete, while the family is parametrized by a continuous family $\mathbb{A}^1$. See Remark 4-1-1.)

Then, denoting the induced automorphism by the same θ_c by abuse of notation, we have

$$\begin{aligned}\mathrm{ord}_{E_w}(z_1 \cdot z^{-m_1}) &= \mathrm{ord}_{E_w}(\theta_c^*(z_1 \cdot z^{-m_1}) \\ &= \mathrm{ord}_{E_w}((z_1 + c \cdot z^{m_1}) \cdot z^{-m_1}) \\ &= \mathrm{ord}_{E_w}(z_1 \cdot z^{-m_1} + c).\end{aligned}$$

Since the above equality holds for any $c \in K$, we conclude

$$\mathrm{ord}_{E_w}(z_1 \cdot z^{-m_1}) \leq 0 \text{ (and hence } = 0)$$

as required.

This completes the proof of Proposition 4-1-4 and hence that of Proposition 4-1-3.

§4-2. Conclusion of the proof for the Weak Factorization Theorem

Since $X_1 = W_1$ and $W_s = X_2$ are nonsingular, we have

$$X_1 = W_1 \overset{p_{1-}}{\tilde{\leftarrow}} W_1^{res} \text{ and } W_s^{res} \overset{p_{s-1,+}}{\tilde{\rightarrow}} W_s = X_2.$$

For each $i = 1, \cdots, s-1$ we have constructed the commutative diagram

$$\begin{array}{ccccccc} W_i^{res} & \xleftarrow{p_{i-}} & W_{i-}^{can} & \overset{\varphi_i^{can}}{\dashrightarrow} & W_{i+}^{can} & \xrightarrow{p_{i+}} & W_{i+1}^{res} \\ {\scriptstyle r_{i-}}\downarrow & & {\scriptstyle h_{i-}}\downarrow & & \downarrow{\scriptstyle h_{i+}} & & \downarrow{\scriptstyle r_{i+}} \\ W_i & \xleftarrow{f_{i-}} & W_{i-}^{tor} & \overset{\varphi_i^{tor}}{\dashrightarrow} & W_{i+}^{tor} & \xrightarrow{f_{i+}} & W_{i+1} \end{array}$$

where

(i) $W_i^{res}, W_{i-}^{can}, W_{i+}^{can}$ and W_{i+1}^{res} are all nonsingular,

(ii) the morphisms p_{i-} and p_{i+} are sequences of blowups with smooth centers, and

(iii) W_{i-}^{can} and W_{i+}^{can} have toroidal structures so that

$$\varphi_i^{can} : (U_{W_{i-}^{can}}, W_{i-}^{can}) \dashrightarrow (U_{W_{i+}^{can}}, W_{i+}^{can})$$

is V-toroidal.

Theorem 4-2-1 (Weak Factorization Theorem for Toroidal Birational Maps). *Let*

$$\varphi : (U_X, X) \dashrightarrow (U_Y, Y)$$

be a proper toroidal biraional map between nonsingular toroidal embeddings. Then φ can be factored into a sequence of blowups and blowdowns with centers being smooth irreducible closed strata

$$\begin{aligned}(U_X, X) = (U_{V_1}, V_1) &\dashrightarrow (U_{V_2}, V_2) \dashrightarrow \cdots \\ &\dashrightarrow (U_{V_i}, V_i) \dashrightarrow (U_{V_{i+1}}, V_{i+1}) \dashrightarrow \\ &\cdots \dashrightarrow (U_{V_{l-1}}, V_{l-1}) \dashrightarrow (U_{V_l}, V_l) = (U_Y, Y)\end{aligned}$$

where all the toroidal embeddings are toroidal over (U_Y, Y) *and where there is an index* i_0 *such that for all* $i \leq i_0$ *the map* $V_i \dashrightarrow X$ *is a projective morphism and for all* $i \geq i_0$ *the map* $V_i \dashrightarrow Y$ *is a projective morphism. In particular, if both* X *and* Y *are projective then all the* V_i *are projective.*

Proof.

If $\varphi : (U_X, X) \to (U_Y, Y)$ is a proper toroidal birational morphism between nonsingular toroidal embeddings without self-intersections (which is necessarily allowable in the sense of Kempf-Knudsen-Mumford-SaintDonat [1] (cf. Abramovich-Karu [1] or Chapter 2)), then the theorem is exactly the refined statement in Matsuki [2] of what is proved in Abramovich-Matsuki-Rashid [1] Włodarczyk [1].

In general, by definition of a proper birational map φ being toroidal there exists a toroidal embedding (U_V, V) which dominates (U_X, X) and (U_Y, Y) by proper toroidal birational morphisms

$$(U_X, X) \overset{\varphi_X}{\leftarrow} (U_V, V) \overset{\varphi_Y}{\rightarrow} (U_Y, Y).$$

(Note that in case φ is V-toroidal we can take (U_V, V) to consist of the normalization V of the graph $\Gamma_\varphi \subset X \times Y$ and the open set $U_V = p_X^{-1}(U_X) = p_Y^{-1}(U_Y)$ where $\varphi_X = p_X : V \to X$ and $\varphi_Y = p_Y : V \to Y$ are the projections.)

By Lemma 1-2-6 there exists a sequence of toroidal blowups of (U_X, X) with smooth centers so that the result is a toroidal embedding without self-intersection. Since all the blowups are toroidal and since φ_X is also toroidal, we can take the corresponding sequence of toroidal blowups of (U_V, V) so that the result dominates the blowup of (U_X, X). We can apply the same procedure to φ_Y. By eliminating the self-intersection of and resolving the singularities of (U_V, V) by a sequence of toroidal blowups, we finally conclude that we may assume that (U_X, X),(U_V, V) and (U_Y, Y) are all nonsingular toroidal embeddings without self-intersections. (Note that in case φ is V-toroidal, then φ_X and φ_Y are automatically allowable by constructruction, without referring to the result of Abramovich-Karu [1] (cf. Lemma 1-2-7).)

By the same argument as in the proof of Lemma 1-4-1, using elimination of points of indeterminacy for the inverses of toroidal birational morphisms, we may also assume that both φ_X and φ_Y are projective morphisms.

Therefore, we may assume $\varphi : (U_X, X) \to (U_Y, Y)$ is a projective toroidal birational morphism (between nonsingular toroidal embeddings without self-intersections), which corresponds to a map of conical complexes $\Delta_\varphi : \Delta_X \to \Delta_Y$ and which is the blowup of a toroidal ideal $J \subset \mathcal{O}_Y$.

We apply the construction of Theorem 2-2-2 to obtain a birational cobordism B embedded in a compactified birational cobordism $\overline{B}$ projective over Y. By construction, $B \subset \overline{B}$ have the natural toroidal structures $(U_B, B) \subset (U_{\overline{B}}, \overline{B})$, which are induced by that of $(U_Y, Y) \times (U_{\mathbb{P}^1}, \mathbb{P}^1)$ where $U_{\mathbb{P}^1} = \mathbb{P}^1 - (\{0\} \cup \{\infty\}) = K^*$ (By taking some further toroidal blowups if necessary, we may assume all of these toroidal embeddings are without self-intersections.) and which correspond to conical complexes $\Sigma \subset \overline{\Sigma} \subset (N_{\Delta_Y})^+_{\mathbb{Q}} = (N_{\Delta_X})^+_{\mathbb{Q}}$ (See Abramovich-Matsuki-Rashid [1] Matsuki [2] for the notation.). Since $\overline{B}$ is projective over Y and since B is a birational cobordism between X and Y, Σ is a simplicial and collapsible cobordism

between Δ_X and Δ_Y in the sense of Abramovich-Matsuki-Rashid [1] Matsuki [2] Morelli [1].

Now we apply the process of π-desingularization to Σ (cf. Theorem 5.1 in Abramovich-Matsuki-Rashid [1] and its toroidal version) to obtain a simplicial, collapsible and π-nonsingular cobordism $\tilde{\Sigma}$ between Δ_X and Δ_Y. Remark that since the process of π-desingularization does not affect $\partial_-\Sigma$ and $\partial_+\Sigma$, which are already π-nonsingular, we can think of $\tilde{\Sigma}$ as a conical subcomplex of $\widetilde{\overline{\Sigma}} = (\overline{\Sigma} - \Sigma) \cup \tilde{\Sigma}$, which is projective over Δ_{W_2}. Let $C \subset \overline{C}$ be the toroidal embeddings corresponding to the conical complexes $\tilde{\Sigma} \subset \widetilde{\overline{\Sigma}}$.

Since $\overline{C}$ is a compactified birational cobordism projective over W_2, we can apply the Geometric Invariant Theory argument of §2-3 and §2-4 to obtain an increasing sequence of integers $a_1, \cdots, a_{s-1}$, associated quasi-elementary cobordisms C_{a_i} and linearizations ρ_{a_i} of the action of $K^* = U_{\mathbb{P}^1} \subset \mathbb{P}^1$ such that we have the descriptions of the semi-stable loci (cf. Proposition 2-3-1)

$$(\overline{C}, \rho_{a_i})^{ss} = C_{a_i}, (\overline{C}, \rho_{a_i+1})^{ss} = (C_{a_i})_+, (\overline{C}, \rho_{a_i-1})^{ss} = (C_{a_i})_-$$

and that the triangle of birational maps

$$\begin{array}{ccc} W_i = (C_{a_i})_-/K^* & \stackrel{\varphi_i}{\longrightarrow} & (C_{a_i})_+/K^* = W_{i+1} \\ & \searrow \quad \swarrow & \\ & C_{a_i}//K^* & \end{array}$$

is induced by a change of linearizations of K^*. This in turn provides a factorization of φ into toroidal birational maps among toroidal embeddings (U_{W_i}, W_i) PROJECTIVE over Y

$$\begin{aligned} (U_X, X) = (U_{W_0}, W_0) &\dashrightarrow (U_{W_1}, W_1) \dashrightarrow \cdots \\ &\dashrightarrow (U_{W_i}, W_i) \stackrel{\varphi_i}{\dashrightarrow} (U_{W_{i+1}}, W_{i+1}) \dashrightarrow \\ &\cdots \dashrightarrow (W_{s-1} \dashrightarrow (U_{W_s}, W_s) = (U_Y, Y) \end{aligned}$$

such that the (U_{W_i}, W_i) correspond to the refinements Δ_{W_i} (of Δ_Y), which are the projections of the lower faces of some conical subcomplexes in $\tilde{\Sigma}$, that hence all the W_i are nonsingular since $\tilde{\Sigma}$ is π-nonsingular and that φ_i corresponds to the simultaneous bistellar operations (cf. Abramovich-Matsuki-Rashid [1] Morelli [1]) associated to the circuits $\sigma_{i1}, \cdots, \sigma_{ib_i}$. Remark that since (the closed star of) these circuits correspond to the fixed points of the action of K^* at the same level a_i, there are no two among them on top of each other, i.e., there are no points $p_j \in \mathrm{RelInt}(\overline{\mathrm{Star}(\sigma_{ij})})$ and $p_k \in \mathrm{RelInt}(\overline{\mathrm{Star}(\sigma_{ik})})$ with $j \neq k$ and $\pi(p_j) = \pi(p_k)$. This implies that these bistellar operations are independent of each other. In other words, there are DISJOINT smooth toroidal centers $c^i_{i,1}, \cdots, c^i_{i,b_i} \subset W_i$ whose simultaneous blowups and blowdowns into DISJOINT smooth toroidal centers

$c^{i+1}_{i,1}, \cdots, c^{i+1}_{i,b_i} \subset W_{i+1}$ correspond to the birational map φ_i. Set

$$\begin{cases} V_{i-1,0} &= W_i \\ V_{i,j} &= \text{the blowup of } W_i \text{ along } c^i_{i,1}, \cdots, c^i_{i,j} \subset W_i \text{ for } j = 1, \cdots, b_i - 1 \\ V_{i,b_i} &= \text{the blowup of } W_i \text{ along } c^i_{i,1}, \cdots, c^i_{i,b_i} \subset W_i \\ &= \text{the blowup of } W_{i+1} \text{ along } c^{i+1}_{i,1}, \cdots, c^{i+1}_{i,b_i} \subset W_{i+1} \\ V_{i,b_i+j} &= \text{the blowup of } W_{i+1} \text{ along } c^{i+1}_{i,1}, \cdots, c^{i+1}_{i,b_i-j} \subset W_{i+1} \text{ for } j = 1, \cdots, b_i - 1 \\ V_{i,b_i+b_i} &= W_{i+1}. \end{cases}$$

Now all the $V_{i,k}$ for $k = 0, \cdots, b_i + b_i$ are projective either over W_i or W_{i+1}, which in turn are projective over Y. Moreover, the $V_{i,k}$ for $k = 1, \cdots, b_i + b_i$ are either blowups or blowdowns of the $V_{i,k-1}$ with smooth toroidal and irreducible centers.

Therefore, we conclude that the W_i together with the refinement given by the $V_{i,k}$ as above provides the desired weak factorization of φ.

We apply Theorem 4-2-1 to the proper and V-toroidal birational map φ_i^{can} to finish the proof of the weak factorization theorem.

(**Preservation of the open set where ϕ is an isomorphism**)

Suppose $\phi : X_1 \dashrightarrow X_2$ induces an isomorphism on an open set $\phi : X_1 \supset U \xrightarrow{\sim} U \subset X_2$. Then

Step 1: In the process of elimination of points of indeterminacy in order to be reduced to the case where ϕ is a projective birational morphism, all the centers of blowups are taken outside of U.

Step 2: In the process of constructing the cobordism $B_\phi(X_1, X_2)$ out of the product $X_2 \times \mathbb{P}^1$, all the centers of blowups are taken outside of $U \times \mathbb{P}^1$.

Step 3: In the process of torification, since the torific ideals are trivial over $U \times \mathbb{P}^1 \cap B_{a_i}$ for all i, all the modifications are made outside of $U \times \mathbb{P}^1 \cap B_{a_i}$.

Therefore, through Steps 1,2 and 3, the open set U remains untouched inside of all the W_i and $W^{tor}_{i\pm}$.

Step 4: In the process of obtaining the W^{res}_i or W^{res}_{i+1} from $W^{tor}_{i\pm}$ all the centers of blowups are taken over the singular locus and hence outside of U.

In the process of obtaining the $W^{can}_{i\pm}$ from W^{res}_i or W^{res}_{i+1} all the centers are taken over the quotient (by the K^*-action) of the locus where the torific ideals are not trivial and hence outside of U.

Step 5: Finally in the process of factoring the toroidal birational maps φ_i^{can} : $(U_{W^{can}_{i-}}, W^{can}_{i-}) \dashrightarrow (U_{W^{can}_{i+}}, W^{can}_{i+})$ all the centers of blowups are taken within the boundary divisors and hence outside of U.

Thus we conclude that all through the construction of our factorization the open set U is preserved without being modified.

(**Preservation of the projectivity**)

Step 1: In the process of elimination of points of indeterminacy in order to be reduced to the case where ϕ is a projective morphism via Lemma 1-4-2, by replacing $\phi : X_1 \to X_2$ with $\phi' : X_1' \to X_2'$, X_1' as well as all the intermediate varieties in the sequence of blowups starting from X_1 are projective over X_1, while X_2' as well as all the intermediate varieties in the sequence of blowups starting from X_2 are projective over X_2.

Step 2: In the proces of constructing the birational cobordism $B_{\phi'}(X_1', X_2')$, all the W_i are projective over X_2' via the Geometric Invariant Theory interpretation in Chapter 2.

Steps 3, 4: All the $W_{i\pm}^{tor}, W_i^{res}$ and $W_{i\pm}^{can}$ are also projective over X_2', as they are obtained from W_i by sequences of blowups.

Step 5: Now by the use of the refined version of the weak factorization theorem applied to the toroidal birational maps $\varphi_i^{can} : (U_{W_{i-}^{can}}, W_{i-}^{can}) \dashrightarrow (U_{W_{i-}^{can}}, W_{i+}^{can})$ all the varieties between W_{i-}^{can} and W_{i+}^{can} are also projective over X_2' in the factorization.

Thus we conclude that in the factorization we construct there is an index i_o such that all the intermediate varieties V_i for $i \leq i_o$ are projective over X_1 and that V_i for $i_o \leq i$ are projective over X_2. In particular, if both X_1 and X_2 are projective, they are all projective.

This completes the proof of the Weak Factorization Theorem and Chapter 4.

CHAPTER 5. GENERALIZATIONS

In this chapter we provide several generalizations of the weak factorization theorem, specifying the modifications we have to make to the arguments in Steps 1 through 5 of the strategy of the proof (cf. §0-1. Outline of the strategy) according to the different purposes.

In §5-1, we establish the weak factorization theorem for bimeromorphic maps between compact complex manifolds. Except for a couple of places where we have to avoid the use of theorems only applicable in the algebraic category and replace them with some arguments valid in the analytic category, the proof goes with little change and in fact is even simpler. For example, instead of discussing with Luna's locally toric charts where we have to require the property of being strongly étale in the algebraic category, we can take $\mathbb{C}^*$-equivariant analytic isomorphisms as local charts where the induced analytic isomorphisms for the quotients are automatic.

In §5-2, we consider the case where a birational map $\phi : X_1 \dashrightarrow X_2$ is equivariant with respect to the action of a group G on X_1 and X_2 and establish the equivariant weak factorization theorem. (Remark that the action does NOT have to be over the base field K.) We note here that in order to preserve the equivariance we may have to blow up (or blow down) several smooth irreducible centers simultaneously in the equivariant factorization. Since most of the constructions in our proof are canonical, the equivariance comes almost free under the group action in the main steps of the strategy and the proof is reduced to the equivariant version of the factorization of toroidal birational maps, for which we give a proof utilizing an idea of Abramovich-Wang [1] and hence generalizing the result of Morelli [1][2] Abramovich-Matsuki-Rashid [1] (See also Abramovich-Karu-Matsuki-Włodarczyk [2].).

In §5-3, we establish the weak factorization theorem over a base field K which may NOT be algebraically closed. By taking the algebraic closure $\overline{K}$ of K and considering the action of the Galois group $G = Gal(\overline{K}/K)$ after base change, it is almost a direct corollary of the arguments for the equivariant case, except for the technical difficulty of dealing with the possibility that after base change $X_1 \times_{\mathrm{Spec}K} \mathrm{Spec}\overline{K}$ and $X_2 \times_{\mathrm{Spec}K} \mathrm{Spec}\overline{K}$ may not stay irreducible but split into several smooth irreducible components.

In §5-4, we establish the weak factorization theorem in the logarithmic category, factoring a proper birational map between nonsingular toroidal embeddings into blowups and blowdowns with smooth and ADMISSIBLE centers. This has an application to the study of the behavior of the Hodge structures under birational transformations, e.g., it provides an easy proof for a theorem of Batyrev [1] claiming the invariance of the Betti numbers among birational nonsingular minimal models.

§5-1. Factorization of bimeromorphic maps

Theorem 5-1-1 (Weak Factorization Theorem of Bimeromorphic Maps). *Let $\phi : X_1 \dashrightarrow X_2$ be a bimeromorphic map between compact complex manifolds. Let $X_1 \supset U \subset X_2$ be a common open subset over which ϕ is an isomorphism. Then ϕ can be factored into a sequence of blowups and blowdowns with smooth irreducible centers disjoint from U. That is to say, there exists a sequence of bimeromorphic maps between compact complex manifolds*

$$X_1 = V_1 \overset{\psi_1}{\dashrightarrow} V_2 \overset{\psi_2}{\dashrightarrow} \cdots \overset{\psi_{i-1}}{\dashrightarrow} V_i \overset{\psi_i}{\to} V_{i+1} \overset{\psi_{i+1}}{\to} \cdots \overset{\psi_{l-2}}{\dashrightarrow} V_{l-1} \overset{\psi_{l-1}}{\to} V_l = X_2$$

where

(i) $\phi = \psi_{l-1} \circ \psi_{l-2} \circ \cdots \circ \psi_2 \circ \psi_1$,

(ii) ψ_i are isomorphisms over U, and

(iii) either $\psi_i : V_i \dashrightarrow V_{i+1}$ or $\psi_i^{-1} : V_{i+1} \dashrightarrow V_i$ is a morphism obtained by blowing up a smooth irreducible center disjoint from U.

Furthermore, if both X_1 and X_2 are projective, then we can choose a factorization so that all the intermediate complex manifolds V_i are projective. More precisely, in the factorization we construct, there exists an index i_o such that the bimeromorphic maps $V_i \dashrightarrow X_1$ are projective morphisms for $i \leq i_o$ and that the bimeromorphic maps $V_i \dashrightarrow X_2$ are projective morphisms for $i \geq i_o$.

Proof.

We only specify the modifications we have to make at each step of the strategy for the proof described in Chapter 0. Introduction. (Note that if X_1 and X_2 are projective, then both are algebraic and hence so is ϕ. Thus the "Furthermore" part is already proved but is stated explicitly again only for the sake of completeness.)

Step 1. Elimination of points of indeterminacy

Thanks to Hironaka [3], Lemma 1-4-1 holds without any change in the category of compact complex manifolds. Note that by construction there exists a line bundle of the form $\mathcal{A} = \mathcal{O}_{X_1'}(\Sigma(-a_i)E_i)$, where the E_i are g_1-exceptional divisors with $a_i > 0$, such that $\mathcal{A}$ is relatively ample for $\phi' : X_1' \to X_2'$, i.e., we have a projective embedding over X_2'

$$\phi' : X_1' \hookrightarrow \mathbb{P}(\phi_*'(\mathcal{A}^{\otimes l})) \to X_2' \text{ for some sufficiently large } l \in \mathbb{N}.$$

Note that $J = \phi_*'(\mathcal{A}^{\otimes l}) = \phi_*'\mathcal{O}_{X_1'}(l \cdot (\Sigma(-a_i)E_i)) \subset \phi_*'\mathcal{O}_{X_1'} = \mathcal{O}_{X_2'}$ is an ideal sheaf of $\mathcal{O}_{X_2'}$, which coincides with $\mathcal{O}_{X_2'}$ outside of U.

Therefore, by replacing $\phi : X_1 \dashrightarrow X_2$ with $\phi' : X_1' \to X_2'$, we may assume as in the algebraic case that $\phi : X_1 \to X_2$ is a projective morphism which is the blowup of X_2 along an ideal sheaf $J \subset \mathcal{O}_{X_2}$ with the support of $\mathcal{O}_{X_2}/J$ being disjoint from U.

Step 2. Construction of a birational cobordism

Theorem 2-2-2 holds for a projective bimeromorphic morphism $\phi : X_1 \to X_2$ described as in Step 1. Though the proof goes almost without any change, we show

the existence of such a coherent sheaf $\mathcal{E}$ as described in the assertion (ii) with the splitting $\mathcal{E} = \oplus_{b\in\mathbb{Z}}\mathcal{E}_b$ into the eigenspaces inductively as follows (since we would like to avoid the complete reducibility argument for a coherent sheaf $\mathcal{E}$ under a K^*-action, which is directly applicable only to an algebraic coherent sheaf under an algebraic K^*-action):

Recall that we obtain $\overline{B} = W_m$ by a sequence of successive blowups with $\mathbb{C}^*$ -invariant centers:

$$\overline{B} = W_m \to \cdots \to W_{i+1} \stackrel{\mu_i}{\to} W_i \to \cdots \to W_1 = W \to W_0 = X_2 \times \mathbb{P}^1.$$

We denote the induced projection onto X_2 by

$$\tau_i : W_i \to X_2.$$

We start with the product $W_0 = X_2 \times \mathbb{P}^1$. Here we obviously have a relatively (very) ample line bundle $\mathcal{L}_0 = p_2^*\mathcal{O}_{\mathbb{P}^1}$ and a coherent sheaf $\mathcal{E}_0 = (\tau_0)_*\mathcal{L}_0$ with the splitting

$$\begin{aligned}\mathcal{E} &= (\tau_0)_*\mathcal{L}_0 = (p_1)_*\mathcal{L}_0\\ &= \oplus_{b\in\mathbb{Z}}(\mathcal{E}_0)_b = \mathcal{O}_{X_2}\cdot T_0 \oplus \mathcal{O}_{X_2}\cdot T_1\end{aligned}$$

where T_0 and T_1 are the homogeneous coordinates of $\mathbb{P}^1$, on which $t \in K^*$ acts as $t^*(T_0) = T_0$ and $t^*(T_1) = t\cdot T_1$ by definition.

Suppose W_i is embedded equivariantly into $\mathbb{P}(\mathcal{E}_i)$ where $\mathcal{E}_i = (\tau_i)_*\mathcal{L}_i$ is a coherent sheaf with the splitting $\mathcal{E}_i = \oplus_{b\in\mathbb{N}}(\mathcal{E}_i)_b$ into the eigenspaces for some relatively (very) ample line bundle $\mathcal{L}_i$ with a $\mathbb{C}^*$-action.

Since the center of the blowup $W_{i+1} \stackrel{\mu_i}{\to} W_i$ is $\mathbb{C}^*$-invariant (See the property $(\spadesuit^{res}-1)$ of the canonical resolution.), we conclude the following: The defining ideal sheaf I_i of the center has the property that there exists a finite open covering $\{U\}$ of X_2 and a sufficiently large integer $k_i \in \mathbb{N}$ such that

$$(\tau_i)^*(\tau_i)_*\{I_i \otimes \mathcal{L}_i^{\otimes k_i}\} \to I_i \otimes \mathcal{L}_i^{\otimes k_i}$$

is surjective and that over each open set U of the covering the sheaf $(\tau_i)^*(\tau_i)_*\{I_i \otimes \mathcal{L}_i^{\otimes k_i}\}$ is generated by a finite number of sum of products, each of which has the same character as the others, of elements in the eigenspaces $\Gamma(U, (\mathcal{E}_i)_b)$ with coefficients in $\Gamma(U, \mathcal{O}_{X_2})$. It follows immediately that if we set

$$\begin{aligned}\mathcal{L}_{i+1} &= \mu_i^{-1}I_i\cdot\mathcal{O}_{W_{i+1}} \otimes \mu_i^*\mathcal{L}_i^{\otimes k_i}\\ \mathcal{E}_{i+1} &= (\tau_{i+1})_*\mathcal{L}_{i+1}\end{aligned}$$

then $\mathcal{L}_{i+1}$ has a natural K^*-action with the splitting $\mathcal{E}_{i+1} = \oplus_{b\in\mathbb{N}}(\mathcal{E}_{i+1})_b$ and the equivariant embedding $W_{i+1} \hookrightarrow \mathbb{P}(\mathcal{E}_{i+1})$.

The rest of the proof for Theorem 2-2-2 is identical to the one in the algebraic setting.

The GIT interpretation of the cobordism B, obtained from the compactified birational (bimeromorphic) cobordism $\overline{B}$ projective over X_2 constructed as above,

holds without any change in the analytic setting, leading to the definition of the quasi-elementary pieces B_{a_i}.

In order to reduce the analysis in general to the one on toric varieties, we took Luna's locally toric (toroidal) charts in the algebraic setting. In order to show the existence of these charts, we used some theorems of Sumihiro (Equivariant completion theorem & Covering by K^*-invariant affine spaces), Luna's Fundamental Lemma and/or Luna's Étale Slice Theorem, all of which are directly applicable only in the algebraic setting (cf. Proposition 1-3-4 and Lemma 2-4-4). We repalce Luna's locally toric charts by $\mathbb{C}^*$-equivariant analytic isomorphisms via the following lemma.

Lemma 5-1-2. *Let B_{a_i} be a quasi-elementary cobordism associated to the birational (bimeromorphic) cobordism B obtained from the compactified birational (bimeromorphic) cobordism $\overline{B}$ projective over X_2 constructed as above. Then for every point $p \in B_{a_i} - F^*_{a_i}$, where $F_{a_i} = B^{\mathbb{C}^*}_{a_i}$ is the fixed point set and $F^*_{a_i}$ is the set as defined in Notation 2-1-3, there exists a $\mathbb{C}^*$-equivariant analytic isomorphism*

$$B_{a_i} \supset U_p \overset{\eta_p}{\tilde{\to}} V_p \subset X_p$$

where X_p is a nonsingular affine toric variety with $\mathbb{C}^*$-acting as a one-parameter subgroup and where $U_p \subset B_{a_i}$ (resp. $V_p \subset X_p$) is a $\mathbb{C}^*$-invariant open subset (with respect to the usual topology) satisfying the following condition $(\star)$:

$(\star)$ If a $\mathbb{C}^*$-orbit $O(q)$ lies in U_p (resp. V_p), then its closure $\overline{O(q)}$ in B_{a_i} (resp. in X_p) also lies in U_p (resp. in V_p).

(As a consequence, we obtain

$$U_p//\mathbb{C}^* \supset B_{a_i}//\mathbb{C}^* \overset{\eta_p//\mathbb{C}^*}{\tilde{\to}} V_p//\mathbb{C}^* \subset X_p//K^*$$

providing not only $\mathbb{C}^*$-equivariant analytic local isomorphisms of B_{a_i} to (nonsingular) toric varieties bot also analytic local isomorphisms of the quotient $B_{a_i}//\mathbb{C}^*$ to (not necessarily nonsingular) toric varieties.)

Proof.

It follows from Theorem 2-2-2 that, locally over X_2, $\overline{B}$ can be embedded equivariantly into $X_2 \times \mathbb{P}^N$, i.e., there exists a open neighborhood U_{X_2} of $\tau_l(p)$ such that

$$\tau_l^{-1}(U_{X_2}) \hookrightarrow U_{X_2} \times \mathbb{P}^N,$$

where $\mathbb{C}^*$ acts only on the second factor $\mathbb{P}^N$ with $t \in \mathbb{C}^*$ acting on the homogeneous coordinates $T_0, \cdots, T_N$ of $\mathbb{P}^N$ by characters

$$t^*(T_j) = t^{b_j} T_j\,.$$

Case: $p \in B_{a_i} - F^*_{a_i}$ is a fixed point, i.e., $p \in F_{a_i}$.

In this case, $T_j(p) = 0$ if $b_j \neq a_i$. Take j_o with $T_{j_o}(p) \neq 0$ and $b_{j_0} = a_i$. Since $p \in B_{a_i} \cap \tau_l^{-1}(U_{X_2}) \subset U_{X_2} \times \mathbb{P}^N$ is a nonsingular point, we may choose a regular

coordinate system at p consisting of a regular coordinate system at $\tau_l(p)$ of U_{X_2} and $\{T_j/T_{j_0}; j \in J\}$ for some subset $J \subset \{0, 1, \cdots, N\}$. Consider the $\mathbb{C}^*$-equivariant morphism

$$\eta : B_{a_i} \cap \tau_l^{-1}(U_{X_2}) \cap \{T_{j_o} \neq 0\} (\hookrightarrow U_{X_2} \times \mathbb{P}^N \cap \{T_{j_o} \neq 0\}) \to \mathbb{A}^{\dim X_2 + \#J} = X_p$$

where $\mathbb{A}^{\dim X_2+\#J}$ is the nonsingular affine toric variety X_p with affine coordinates corresponding to the regular coordinate system at p choesn as above and the action of $t \in \mathbb{C}^*$ is given by

$$t^*(T_j/T_{j_0}) = t^{b_j - b_{j_o}} T_j/T_{j_0},$$

leaving others $\mathbb{C}^*$-invariant. By construction, there exist open neighborhoods

$$\begin{aligned} p &\in U_p''' \subset B_{a_i} \cap \tau_l^{-1}(U_{X_2}) \cap \{T_{j_o} \neq 0\} \\ \eta(p) &\in V_p''' \subset X_p \end{aligned}$$

such that η induces an analytic isomorphism between them

$$\eta|_{U_p'''} : U_p''' \xrightarrow{\sim} V_p'''.$$

Let $\mathbb{C}_1^* = \{z \in \mathbb{C}^*; |z| = 1\}$ be the unit circle in $\mathbb{C}^*$. Since p is a fixed point for the action of $\mathbb{C}^*$, we may choose a small open neighborhood (with respect to the usual topology) $p \in U_p'' \subset U_p'''$ such that $t(q) \in U_p'''$ for all $q \in U_p''$ and $t \in \mathbb{C}_1^*$. This implies that

$$Stab(q) = Stab(\eta(q)) \text{ for all } q \in U_p''.$$

Let

$$\begin{aligned} U_p' &= \cup_{t \in \mathbb{C}^*} t(U_p'') \\ V_p' &= \cup_{t \in \mathbb{C}^*} t(\eta(U_p'')) \end{aligned}$$

be the two $\mathbb{C}^*$-invariant neighborhood of p and $\eta(p)$, respectively. From the construction it follows that

$$\eta|_{U_p'} : U_p' \xrightarrow{\sim} V_p'$$

is a $\mathbb{C}^*$-equivariant analytic isomorphism.

Let $J' = \{j \in J; b_j - b_{j_o} \neq 0\}$ be the subset of J. Then F_{X_p} has the description

$$F_{X_p} = \{x \in X_p; T_j/T_{j_o}(x) = 0 \text{ for } j \in J'\}.$$

Let $\sigma : X_p = \mathbb{A}^{\dim X_2+\#J} \to \mathbb{A}^{\dim X_2+\#J'}$ be the obvious projection given by $\prod_{j \in J} T_j/T_{j_o}(x) \mapsto \prod_{j \in J'} T_j/T_{j_o}(x)$, leaving the rest of the coodinates coming from U_{X_2} unchanged. Take a small open neighborhood $W \subset \mathbb{A}^{\dim X_2+\#J'}$ of $\sigma \circ \eta(p)$ so that $\sigma^{-1}(W) \cap F_{X_p} \subset V_p'$. It follows that

$$\begin{aligned} U_p &= U_p' \cap \eta^{-1}(V_p' \cap \sigma^{-1}(W)) \subset B_{a_i} \\ V_p &= V_p' \cap \sigma^{-1}(W) \subset X_p \end{aligned}$$

are $\mathbb{C}^*$-invariant open subsets satisfying the condition $(\star)$ with a $\mathbb{C}^*$-equivariant analytic isomorphism

$$\eta_p : U_p \to V_p.$$

Case: $p \in B_{a_i} - F^*_{B_{a_i}}$ is NOT a fixed point, i.e., $p \notin F_{a_i}$.

Take a limit point $q = \lim_{t\to 0} t(p)$ or $\lim_{t\to\infty} t(p) \in \overline{B}$ so that $q \in F_{\overline{B}}$ is a fixed point of $\overline{B}$. By the same argument as in the previous case, we can find a $\mathbb{C}^*$-equivariant analytic isomorphism

$$\eta_q : U'_q \to V'_q$$

where $U'_q \subset \overline{B}$ and $V'_q \subset X_q$ are $\mathbb{C}^*$-invariant open neighborhoods of $q \in \overline{B}$ and $\eta_q(q) \in X_q$, respectively, and where X_q is a nonsingular affine toric variety with $\mathbb{C}^*$ acting as a one-parameter subgroup.

Let $p_2 : U_{X_2} \times \mathbb{P}^N \to \mathbb{P}^N$ be the second projection. Since $p \in B_{a_i} - F^*_{B_{a_i}}$ and $p \notin F_{B_{a_i}}$, we conclude $p_2(p) \in (\mathbb{P}^N)_{a_i} - F^*_{(\mathbb{P}^N)_{a_i}}$ and $p_2(p) \notin F_{(\mathbb{P}^N)_{a_i}}$, noting that

$$\begin{aligned}
\tau_l^{-1}(U_{X_2}) \cap B_{a_i} &= \tau_l^{-1}(U_{X_2}) \cap \overline{B} \cap p_2^{-1}((\mathbb{P}^N)_{a_i}) \text{ and} \\
\tau_l^{-1}(U_{X_2}) \cap F_{B_{a_i}} &= \tau_l^{-1}(U_{X_2}) \cap B_{a_i} \cap p_2^{-1}(F_{(\mathbb{P}^N)_{a_i}}).
\end{aligned}$$

In the proof of Proposition 1-3-5, we already saw that there exists an affine $\mathbb{C}^*$-invariant open neighborhood $p_2(p) \in U_{(\mathbb{P}^N)_{a_i}} \subset (\mathbb{P}^N)_{a_i}$ such that $U_{(\mathbb{P}^N)_{a_i}}$ satisfies the condition $(\star)$ and that $U_{(\mathbb{P}^N)_{a_i}}$ has no fixed points.

Note also that since $\eta_q(p)$ is not a fixed point in X_q, there exists a nonsingular affine toric open subvariety $X_p \subset X_q$ such that $\eta_q(p) \in X_p$ and that X_p has no fixed points.

It follows that

$$\begin{aligned}
U_p &= U'_q \cap p_2^{-1}(U_{(\mathbb{P}^N)_{a_i}}) \cap \eta_q^{-1}(X_p) \subset B_{a_i} \\
V_p &= \eta_q(U_p) \subset X_p
\end{aligned}$$

are $\mathbb{C}^*$-invariant open subsets satisfying the condition $(\star)$ with a $\mathbb{C}^*$-equivariant analytic isomorphism $\eta_p = \eta_q|_{U_p} : U_p \xrightarrow{\sim} V_p$.

This completes the proof of Lemma 5-1-2.

It is worthwhile noting that, in the notation of the proof of Lemma 5-1-2, $\tau_l^{-1}(U_{X_2}) \cap B_{a_i}//\mathbb{C}^* \hookrightarrow U_{X_2} \times (\mathbb{P}^N)_{a_i}//\mathbb{C}^*$ is embedded as a closed analytic subvariety. In fact, let $\mathcal{I}$ be the ideal sheaf defining $\tau_l^{-1}(U_{X_2})$ inside of $U_{X_2} \times \mathbb{P}^N$. Let $\mathcal{O}_{\mathbb{P}^N}(1)$ be the very ample sheaf on $\mathbb{P}^N$. For sufficiently large $l \in \mathbb{N}$, we have an exact sequence

$$\begin{aligned}
0 &\to p_{1*}\{\mathcal{I} \otimes p_2^*\mathcal{O}_{\mathbb{P}^N}(l)\} \\
&\to (p_1)_*\{p_2^*\mathcal{O}_{\mathbb{P}^N}(l)\} = \mathcal{O}_{U_{X_2}} \otimes H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(l)) \\
&\to p_{1*}\{\mathcal{O}_{\tau_l^{-1}(U_{X_2})} \otimes p_2^*\mathcal{O}_{\mathbb{P}^N}(l)\} \to 0
\end{aligned}$$

where

$$p_{1*}\{\mathcal{I}\otimes p_2^*\mathcal{O}_{\mathbb{P}^N}(l)\}\otimes\mathbb{C}(y)\to H^0(p_2^{-1}(y),\mathcal{I}\otimes p_2^*\mathcal{O}_{\mathbb{P}^N}(l)|_{p_2^{-1}(y)})$$
$$(p_1)_*\{p_2^*\mathcal{O}_{\mathbb{P}^N}(l)\}\otimes\mathbb{C}(y)\to H^0(p_2^{-1}(y),p_2^*\mathcal{O}_{\mathbb{P}^N}(l)|_{p_2^{-1}(y)})\cong H^0(\mathbb{P}^N,\mathcal{O}_{\mathbb{P}^N}(l))$$

are isomorphisms for any $y\in U_{X_2}$. It is easy to see inductively as before that the construction can be carried out so that $p_{1*}\{\mathcal{I}\otimes p_2^*\mathcal{O}_{\mathbb{P}^N}(l)\}$ splits into the eigenspaces with respect to the action of $\mathbb{C}^*$. We may also assume that $H^0(\mathbb{P}^N,\mathcal{O}_{\mathbb{P}^N}(l))^{\mathbb{C}^*}$ generates the homogeneous invariant ring $\oplus_{m\geq 0}H^0(\mathbb{P}^N,\mathcal{O}_{\mathbb{P}^N}(ml))^{\mathbb{C}^*}$. (Remark that $\mathbb{C}^*$ is acting via the twist ρ_{a_i}.) Now by Fogarty-Mumford-Kirwan [1] Chapter 1 §2, in each fiber $p_2^{-1}(y)\cong\mathbb{P}^N$, the invariant homogeneous polynomials $H^0(p_2^{-1}(y),\mathcal{I}\otimes p_2^*\mathcal{O}_{\mathbb{P}^N}(l)|_{p_2^{-1}(y)})^{\mathbb{C}^*}$ defines the image of B_{a_i} under the quotient map $U_{X_2}\times(\mathbb{P}^N)_{a_i}\to U_{X_2}\times(\mathbb{P}^N)_{a_i}//\mathbb{C}^*$. That is to say, the invariant part $p_{1*}\{\mathcal{I}\otimes p_2^*\mathcal{O}_{\mathbb{P}^N}(l)\}^{\mathbb{C}^*}$ defines the image of B_{a_i} under the quotient map globally inside of $U_{X_2}\times(\mathbb{P}^N)_{a_i}//\mathbb{C}^*\hookrightarrow U_{X_2}\times\mathbb{P}(H^0(\mathbb{P}^N,\mathcal{O}_{\mathbb{P}^N}(l))^{\mathbb{C}^*}$, verifying the assertion.

It is also worthwhile noting that the quotient $\pi: B_{a_i}\to B_{a_i}//\mathbb{C}^*$ is characterized by the following three properties:

(i) set-theoretically $B_{a_i}//\mathbb{C}^*$ is the set of equivalence classes of $\mathbb{C}^*$-orbits in B_{a_i} where two orbits $O(p)$ and $O(q)$ are equivalent if and only if $\overline{O(p)}\cap\overline{O(q)}\neq\emptyset$ (Note that B_{a_i} is quasi-elementary.),

(ii) $U\subset B_{a_i}//\mathbb{C}^*$ is open if and only if $\pi^{-1}(U)\subset B_{a_i}$ is open (with respect to the usual topology), and

(iii) $\Gamma(U,\mathcal{O}_{B_{a_i}//\mathbb{C}^*})=\Gamma(\pi^{-1}(U),\mathcal{O}_{B_{a_i}})^{\mathbb{C}^*}$.

For each $p\in B_{a_i}-F^*_{B_{a_i}}$ we choose $\eta_p: U_p\to V_p$ as constructed in Lemma 5-1-2. Then these U_p form an open covering of B_{a_i} such that $U_p=\pi^{-1}(\pi(U_p))$ by the condition $(\star)$ and that $\pi(U_p)=U_p//\mathbb{C}^*\xrightarrow{\sim}V_p//\mathbb{C}^*$ is an open subset of an affine (not necessarily nonsingular) toric variety $X_p//\mathbb{C}^*$.

Step 3. Torification

Step 4. Recovery of nonsingularity

The arguments for these two steps work verbatim if we replace Luna's locally toric charts with the $\mathbb{C}^*$-equivariant anlytic isomorphisms $\eta_p: U_p\to V_p\subset X_p$ as described in Lemma 5-1-2.

This completes the proof for the weak factorization theorem for bimeromorphic maps.

§5-2. Equivariant factorization under group action

Theorem 5-2-1 (Equivariant Weak Factorization Theorem). *Let X_1 and X_2 be complete nonsingular varieties over an algebraically closed field K of characteristic zero. Let G be a group acting on X_1 and X_2. (Note that the action does NOT have to be over the base field K.) Let $\phi: X_1\dashrightarrow X_2$ be a birational map which is equivariant under the action of G. Let $X_1\supset U\subset X_2$ be a common open set over which ϕ is an isomorphism. Then ϕ can be factored into a sequence of equivariant blowups and blowdowns with smooth G-invariant centers disjoint from*

U. (The center may be reducible, i.e., a collection of several disjoint smooth irreducible components, which is G-invariant as a whole.) That is to say, there exists a sequence of G-equivarint birational maps between complete nonsingular varieties with G-actions

$$X_1 = V_1 \overset{\psi_1}{\dashrightarrow} V_2 \overset{\psi_2}{\dashrightarrow} \cdots \overset{\psi_{i-1}}{\dashrightarrow} V_i \overset{\psi_i}{\to} V_{i+1} \overset{\psi_{i+1}}{\to} \cdots \overset{\psi_{l-2}}{\dashrightarrow} V_{l-1} \overset{\psi_{l-1}}{\to} V_l = X_2$$

where

(i) $\phi = \psi_{l-1} \circ \psi_{l-2} \circ \cdots \circ \psi_2 \circ \psi_1$,

(ii) ψ_i are isomorphisms over U, and

(iii) either $\psi_i : V_i \dashrightarrow V_{i+1}$ or $\psi_i^{-1} : V_{i+1} \dashrightarrow V_i$ is a morphism obtained by blowing up smooth G-invariant center disjoint from U.

Furthermore, if both X_1 and X_2 are projective, then we can choose a factorization so that all the intermediate varieties V_i are projective. More precisely, in the factorization we construct, there exists an index i_o such that the birational maps $V_i \dashrightarrow X_1$ are projective birational morphisms for $i \leq i_o$ and that the birational maps $V_i \dashrightarrow X_2$ are projective birational morphisms for $i_o \leq i$.

Proof.

Again we only specify the modifications we have to make at each step of the strategy for the proof described in Chapter 0. Introduction.

Step 1. Elimination of points of indeterminacy

Though not explicitly stated in Hironaka [2,3] (cf. Bierstone-Milman [1] Encinas-Villamayor [1]), its method for elimination of points of indeterminacy works equivariantly under a group action. Thus Lemma 1-4-2 holds in the equivariant case as well and hence we may assume that $\phi : X_1 \to X_2$ is a projective birational morphism which is the blowup of X_2 along a G-invariant ideal sheaf $J \subset \mathcal{O}_{X_2}$ (and hence ϕ is G-equivariant) with the support of $\mathcal{O}_{X_2}/J$ being disjoint from U.

Step 2. Construction of a birational cobordism

Since the canonical resolution of singularities is G-equivariant by the condition $(\spadesuit^{res} - 1)$, the construction in Theorem 2-2-2 is also G-equivariant. We remark that the morphism $\tau : \overline{B} \to X_2$ is projective and G-equivariant as well as K^*-equivariant. We also remark that a relatively ample line bundle $\mathcal{L}$ on $\overline{B}$ over X_2 with a K^*-linearization (We set $\mathcal{E} = \tau_*(\mathcal{L}^{\otimes l})$ in the proof of Theorem 2-2-2.) can be chosen so that it comes with a G-linearization which commutes with the K^*-linearization. In fact, $\mathcal{L}$ can be chosen to be of the form $\mathcal{O}_{\overline{B}}(n \cdot p_2^*(\infty) - E)$ where $p_2^*(\infty)$ is the pull-back of the hyperplane divisor $\infty \in \mathbb{P}^1$ and where E is an exceptional divisor for the birational morphism $\overline{B} \to W_0 = X_2 \times \mathbb{P}^1$ and is G-invariant. We identify the local sections with the elements in the function field

$$\Gamma(U, \mathcal{L}) = \{s_f = f \in K(\overline{B}); div(f) + n \cdot p_2^*(\infty) - E|_U \geq 0\}$$

and choose the linearizations of G and K^* to be the one induced from the actions on the function field $K(W_0 = X_2 \times \mathbb{P}^1) = K(\overline{B})$, i.e.,

$$g^* s_f := g^* f \in \Gamma(g^{-1}(U), \mathcal{L}) \text{ for } g \in G$$
$$t^* s_f := t^* f \in \Gamma(t^{-1}(U), \mathcal{L}) \text{ for } t \in K^*.$$

As the actions of G and K^* on $X_2 \times \mathbb{P}^1$ obviously commute with each other, so do the linearizations defined as above. It also follows then that the linearization of G on $\mathcal{L}$ commutes with the twisted linearizations discussed in §2-3. This implies that the quasi-elememtary pieces B_{a_i} are G-invariant. The rest of the argument goes without any change.

Step 3. Torification

Since the definition of the torific ideals is canonical in terms of the action of K^*, which commutes with that of G, the torific ideals are G-invariant. Thus the torification is G-equivariant. The argument in this part goes without any change.

Step 4. Recovery of Nonsingularity

First we look at Lemma 4-1-2. Note that $f_{i-} : W_i^{tor} \to W_i$ is the blowup of the G-invariant ideal sheaf $I_{i-} = (f_{i-})_* \mathcal{O}_{W_i^{tor}}(-E_i^{tor})$, where E_i^{tor} is a G-invariant effective Cartier divisor, since the torific ideal is G-invariant by Step 3. (The same claim holds replacing the negative sign $-$ with the positive sign $+$ for the subscripts.) Secondly we look at Proposition 4-1-3. Since the canonical resolution of singularities and the canonical principalization of (G-invariant) ideals are G-equivariant, we see that $h_{i-} : W_{i-}^{can} \to W_i^{tor}$ and $h_{i+} : W_{i+}^{can} \to W_i^{tor}$ are also G-equivariant. Therefore, we also conclude that $\varphi_i^{can} : (U_{W_{i-}^{can}}, W_{i-}^{can}) \dashrightarrow (U_{W_{i+}^{can}}, W_{i+}^{can})$ is G-equivariant V-toroidal birational map.

Thus the proof of the equivariant weak factorization theorem is reduced to the following:

Theorem 5-2-2 (Equivariant Weak Factorization Theorem for Toroidal Birational Maps). *Let (U_{W_1}, W_1) and (U_{W_2}, W_2) be nonsingular toroidal embeddings. Let G be a group acting on them as automorphisms of toroidal embeddings. Let*

$$\varphi : (U_{W_1}, W_1) \dashrightarrow (U_{W_2}, W_2)$$

be a proper toroidal birational map which is G-equivariant. Then ϕ can be factored into a sequence of toroidal blowups and blowdowns with G-invariant smooth centers (which may be reducible) disjoint from U. That is to say, there exists a sequence of G-equivariant birational maps between nonsingular toroidal embeddings

$$\begin{aligned}(U_{W_1}, W_1) = (U_{V_1}, V_1) &\overset{\psi_1}{\dashrightarrow} (U_{V_2}, V_2) \overset{\psi_2}{\dashrightarrow} \cdots \\ &\overset{\psi_{i-1}}{\to} (U_{V_i}, V_i) \overset{\psi_i}{\to} (U_{V_{i+1}}, V_{i+1}) \overset{\psi_{i+1}}{\to} \\ &\cdots \overset{\psi_{l-2}}{\dashrightarrow} (U_{V_{l-1}}, V_{l-1}) \overset{\psi_{l-1}}{\to} (U_{V_l}, V_l) = (U_{W_2}, W_2)\end{aligned}$$

where

(i) $\phi = \psi_{l-1} \circ \psi_{l-2} \circ \cdots \circ \psi_2 \circ \psi_1$,

(ii) ψ_i are isomorphisms over U, and

(iii) either $\psi_i : (U_{V_i}, V_i) \dashrightarrow (U_{V_{i+1}}, V_{i+1})$ or $\psi_i^{-1} : (U_{V_{i+1}}, V_{i+1}) \dashrightarrow (U_{V_i}, V_i)$ is a morphism obtained by blowing up a G-invarinat smooth center (which may be reducible) disjoint from U.

Furthermore, if both W_1 and W_2 are projective, then we can choose a factorization so that all the intermediate complex manifolds V_i are projective. More precisely, in the factorization we construct, there exists an index i_o such that the bimeromorphic maps $V_i \dashrightarrow W_1$ are projective morphisms for $i \leq i_o$ and that the bimeromorphic maps $V_i \dashrightarrow W_2$ are projective morphisms for $i \geq i_o$.

Proof.

First note that the process of Lemma 1-2-6, making a general nonsingular toroidal embedding into the one without self-intersection, is G-equivariant. Thus we may assume both (U_{W_1}, W_1) and (U_{W_2}, W_2) are without self-intersection.

Since φ is toroidal, by Definition 1-2-3 there exists a toroidal embedding (U_Z, Z) which dominates (U_{W_1}, W_1) and (U_{W_2}, W_2) by proper toroidal birational morphisms

$$(U_{W_1}, W_1) \stackrel{f_1}{\leftarrow} (U_Z, Z) \stackrel{f_2}{\rightarrow} (U_{W_2}, W_2).$$

First by the process of Lemma 1-2-6, we may assume that (U_Z, Z) is a nonsingular toroidal embedding without self-intersection. By Kempf-Knudsen-Mumford-SaintDonat [1] Abramovich-Karu [1], the morphisms f_1 and f_2 correspond to the refinements of the conical complexes

$$\Delta_{f_1} : \Delta_Z \to \Delta_{W_1}$$
$$\Delta_{f_2} : \Delta_Z \to \Delta_{W_2}.$$

Remark that we have a natural homomorphism

$$h : G \to Aut(\Delta_{W_1}),$$

sending $g \in G$, considered as an automorphism $g : (U_{W_1}, W_1) \to (U_{W_1}, W_1)$, to an automorphism of the conical complex $h(g) = \Delta_g : \Delta_{W_1} \to \Delta_{W_1}$. Since $Aut(\Delta_{W_1})$ is a finite group, so is $\overline{G} := G/Ker(h)$. Let $\Delta_{\bar{f}_1} : \Delta_{\tilde{Z}} \to \Delta_{W_1}$ be the smallest common refinement of $\{\Delta_{\overline{g}} \circ \Delta_{f_1} : \Delta_Z \to \Delta_{W_1}; \overline{g} \in \overline{G}\}$. Though G may not act on the original (U_Z, Z), by replacing (U_Z, Z) with $(U_{\tilde{Z}}, \tilde{Z})$ associated to the refinement $\Delta_{\bar{f}_1} : \Delta_{\tilde{Z}} \to \Delta_{W_1}$ (and then taking the canonical resolution of singularities, if necessary), we may assume from the beginning that G acts on the toroidal embedding (U_Z, Z) and $f_1 : (U_Z, Z) \to (U_{W_1}, W_1)$ is G-equivariant and hence that $f_2 : (U_Z, Z) \to (U_{W_2}, W_2)$ is G-equivariant as well.

Thus we only have to provide factorization for a G-equivariant toroidal birational morphism $f : (U_Z, Z) \to (U_W, W)$ between nonsingular toroidal embeddings without self-intersection, associated to the refinement $\Delta_f : \Delta_Z \to \Delta_W$ (cf. the argument in the proof of Theorem 4-2-1). Let $\overline{G} \subset Aut(\Delta_W)$ be the subgroup of the automorphism group of the conical complex Δ_W induced by the action of G. Note that $\overline{G}$ acts on Δ_Z as well as on Δ_W and that Δ_f is $\overline{G}$-equivariant.

Now we use an idea of Abramovich-Wang [1]. Take the barycentric star subdivision $\Delta_b : \Delta_{\hat{W}} \to \Delta_W$ (cf. Definition 2.1 in Abramovich-Matsuki-Rashid [1]), which is $\overline{G}$-equivariant. It is straightforward to see that the corresponding morphism $b : (U_{\hat{W}}, \hat{W}) \to (U_W, W)$ between nonsingular toroidal embeddings without

self-intersection is the blowup of a G-invariant toroidal ideal I on (U_W, W) (cf. the argument in the proof of Theorem 2-2-2). Let $\lambda : \hat{Z} \to Z$ be the canonical principalization of $f^{-1}I \cdot \mathcal{O}_Z$, which is a G-invariant and toroidal ideal since f is G-equivariant and toroidal. Thus $\lambda : (U_{\hat{Z}} = \lambda^{-1}(U_Z), \hat{Z}) \to (U_Z, Z)$ is a G-equivariant and toroidal morphism between nonsingular toroidal embeddings without self-intersection by the condition ($\spadesuit^{can} - 1$) of the canonical principalization, and hence so is the induced morphism $\hat{f} : (U_{\hat{Z}}, \hat{Z}) \to (U_{\hat{W}}, \hat{W})$. Let $\Delta_{\hat{f}} : \Delta_{\hat{Z}} \to \Delta_{\hat{W}}$ be the associated map of the conical complexes, which is $\overline{G}$-equivariant. Observe that, thanks to the process of taking the barycentric star subdivision, the conical complex $\Delta_{\hat{W}}$ has the following property ($\natural$):

($\natural$) If a cone $\sigma \in \Delta_{\hat{W}}$ is mapped to itself by some $\overline{g} \in \overline{G}$, then $\overline{g}$ acts as an identity on the cone σ.

Observe also that the conical complex $\Delta_{\hat{Z}}$ satisfies the same property ($\natural$).

It follows that the quotient $\Delta_{\hat{W}}/\overline{G}$ (resp. $\Delta_{\hat{Z}}/\overline{G}$) is again a nonsingular conical complex and the projection $\Delta_{\hat{W}} \to \Delta_{\hat{W}}/\overline{G}$ (resp. $\Delta_{\hat{Z}} \to \Delta_{\hat{Z}}/\overline{G}$) maps the cones in $\Delta_{\hat{W}}$ (resp. $\Delta_{\hat{Z}}$) isomorphically to the cones in $\Delta_{\hat{W}}/\overline{G}$ (resp. $\Delta_{\hat{Z}}/\overline{G}$) and we have a map of nonsingular conical complexes $\Delta_{\hat{f}}/\overline{G} : \Delta_{\hat{Z}}/\overline{G} \to \Delta_{\hat{W}}/\overline{G}$.

Now apply the combinatorial algorithm of Morelli [1][2] Abramovich-Matsuki-Rashid [1] (See also Matsuki [2].) to obtain a weak factorization between the conical complexes $\Delta_{\hat{W}}/\overline{G}$ and $\Delta_{\hat{Z}}/\overline{G}$ (cf. Theorem 4-2-1). We can pull back this weak factorization to obtain a $\overline{G}$-equivariant weak factorization between $\Delta_{\hat{W}}$ and $\Delta_{\hat{Z}}$. Geometrically interpreted, we have the assertion of the theorem.

This completes the proof of Theorem 5-2-2 and hence the proof of the equivariant weak factorization theorem.

Example 5-2-3.

The following example was communicated to the author by Prof. Hauser in the form of a question.

Let $W = \mathbb{A}^3$ with coordinates (z_1, z_2, z_3). First we blowup the union of the z_1-axis $\{z_2 = z_3 = 0\}$ and the z_2-axis $\{z_1 = z_3 = 0\}$ and then blowup the unique singular point of type $\{xy - zw = 0\}$ in the resulting variety to obtain a projective birational morphism $\phi : Z \to W$. There is an obvious $G = \mathbb{Z}/(2) = \langle i \rangle$-symmetry (action) of $W = \mathbb{A}^3$ defined by

$$i^* z_1 = z_2, i^* z_2 = z_1, i^* z_3 = z_3,$$

which switches the z_1-axis and z_2-axis, and which lifts to the action of Z. The morphism ϕ is G-equivariant.

Now in a sequence providing the weak factorization of ϕ, at some stage we have to blowup the (strict transform of) z_1-axis (looking backwards starting from W). If the sequence is G-equivariant, we have to blowup the (strict transform of) z_2-axis, i.e., the G-conjugate of the (strict transform of) z_1-axis at the same time, having at least two irreducible components for the center of blowup.

If we regard $W = \mathbb{A}^3 = X(N, \Delta_W)$ as a toric variety defined by the regular cone $\Delta_W = \langle v_1, v_2, v_3 \rangle$, then the morphism $\phi : Z = X(N, \Delta_Z) \to W = X(N, \Delta_W)$ also

has a toric description where the subdivision Δ_Z has the two sequences of maps to Δ_W, each of which corresponds to a sequence of blowdowns with smooth centers starting from Z and end with W.

Picture 5-2-3.

However, these two sequences of factorization are not G-equivariant. Thus, as in the proof of the equivariant weak factorization theorem, we take the barycentric subdivision $\hat{\Delta}_W$ of Δ_W and then pull it back to a subdivision of $\tilde{\sigma}$. We take a sequence of smooth star subdivisions of Δ_Z in a G-equivariant way so that the resulting nonsingular fan $\hat{\Delta}_Z$ maps G-equivariantly to Δ_W. We can now take the quotient of these subdivisions by G. We find a sequence of factorization between these quotients, e.g. by Morelli's algorithm. Finally we pull back this sequence to the one between the original subdivisions $\hat{\Delta}_Z$ and $\hat{\Delta}_W$ to obtain a G-equivariant sequence of weak factorization. The process is demonstrated in the following picture.

Picture 5-2-4.

§5-3. Factorization over a non-algebraically closed field

Theorem 5-3-1 (Weak Factorization Theorem over a NON-algebraically closed field). *Let $\phi : X_1 \dashrightarrow X_2$ be a birational map between varieties smooth and proper over a field K, which may not be algebraically closed, of characteristic zero. Let $X_1 \supset U \subset X_2$ be a common open subset over which ϕ is an isomorphism. Then ϕ can be factored into a sequence of blowups and blowdowns with irreducible centers smooth over K and disjoint from U. That is to say, there exists a sequence of birational maps between varieties smooth and proper over K*

$$X_1 = V_1 \overset{\psi_1}{\dashrightarrow} V_2 \overset{\psi_2}{\dashrightarrow} \cdots \overset{\psi_{i-1}}{\dashrightarrow} V_i \overset{\psi_i}{\to} V_{i+1} \overset{\psi_{i+1}}{\to} \cdots \overset{\psi_{l-2}}{\dashrightarrow} V_{l-1} \overset{\psi_{l-1}}{\to} V_l = X_2$$

where

(i) $\phi = \psi_{l-1} \circ \psi_{l-2} \circ \cdots \circ \psi_2 \circ \psi_1$,

(ii) ψ_i are isomorphisms over U, and

(iii) either $\psi_i : V_i \dashrightarrow V_{i+1}$ or $\psi_i^{-1} : V_{i+1} \dashrightarrow V_i$ is a morphism obtained by blowing up an irreducible center smooth over K and disjoint from U.

Furthermore, if both X_1 and X_2 are projective, then we can choose a factorization so that all the intermediate varieties V_i are projective. More precisely, in the factorization we construct, there exists an index i_o such that the birational maps $V_i \dashrightarrow X_1$ are projective birational morphisms for $i \leq i_o$ and that the birational maps $V_i \dashrightarrow X_2$ are projective birational morphisms for $i_o \leq i$.

Proof.

We specify the modifications we have to make at each step of the proof of the strategy for the proof described in Chapter 0. Introduction.

Step 1. Elimination of points of indeterminacy

Hironaka's method for elimination of points of indeterminacy works over ANY field of characteristic zero (cf. Remark 1-4-2). Hence again via Lemma 1-4-1, we may assume that $\phi : X_1 \to X_2$ is a projective birational morphism which is the blowup of of X_2 along an ideal sheaf $J \subset \mathcal{O}_{X_2}$ defined over K with the support of $\mathcal{O}_{X_2}/J$ being disjoint from U.

Step 2. Construction of a birational cobordism

Since the canonical resolution of singularities also works over any field of characteristic zero, the construction of a birational cobordism goes without any change.

Step 3. Torification

Step 4. Recovery of Nonsingularity

Here we carry the argument by taking the base change from the original field K to its algebraic closure $\overline{K}$.

As X_1 and X_2 share a common open subset U, after base change

$$\begin{aligned}\overline{X_1} &:= X_1 \times_{\mathrm{Spec}\ K} \mathrm{Spec}\ \overline{K}\\ (\text{resp. } \overline{X_2} &:= X_2 \times_{\mathrm{Spec}\ K} \mathrm{Spec}\ \overline{K},\\ \overline{\{B\}} &:= B \times_{\mathrm{Spec}\ K} \mathrm{Spec}\ \overline{K})\\ \overline{U} &:= U \times_{\mathrm{Spec}\ K} \mathrm{Spec}\ \overline{K})\end{aligned}$$

it splits into a finite number of disjoint components smooth over $\overline{K}$

$$(\overline{X_1})_{g_j} \quad (\text{resp. } (\overline{X_2})_{g_j}, \quad (\overline{\{B\}})_{g_j}, (\overline{U})_{g_j})$$

such that they are conjugate to each other under the action of the Galois group $G = Gal(\overline{K}/K)$ where we set G_e to be the decomposition group of some component $(\overline{X_1})_e$, i.e.,

$$G_e := \{g \in G; g((\overline{X_1})_e) = (\overline{X_1})_e\}$$

and where $C = \{g_o = e, \cdots, g_j, \cdots\}$ is a complete representative modulo G_e, and that $(\overline{\{B\}})_{g_j}$ is a birational cobordism for the birational map $(\overline{\varphi})_{g_j} : (\overline{X_1})_{g_j} \dashrightarrow (\overline{X_2})_{g_j}$ respecting the open subset $(\overline{U})_{g_j}$. (Note that we use the notation $\overline{\{B\}}$ in order to distinguish it from the compactified birational cobordism $\overline{B}$.)

Now we apply the argument for the equivariant weak factorization in §5-2 to the birational map $(\overline{\varphi})_e : (\overline{X_1})_e \dashrightarrow (\overline{X_2})_e$ over an algebraically closed field $\overline{K}$, induced from the birational cobordism $(\overline{\{B\}})_e$, to obtain a G_e-equivariant factorization

$$\begin{aligned}(\overline{X_1})_e = (\overline{V_1})_e &\overset{(\overline{\psi_1})_e}{\dashrightarrow} (\overline{V_2})_e \overset{(\overline{\psi_2})_e}{\dashrightarrow}\\ &\cdots \overset{(\overline{\psi_{i-1}})_e}{\to} (\overline{V_i})_e \overset{(\overline{\psi_i})_e}{\to} (\overline{V_{i+1}})_e \overset{(\overline{\psi_{i+1}})_e}{\to} \cdots\\ &\overset{(\overline{\psi_{l-2}})_e}{\dashrightarrow} (\overline{V_{l-1}})_e \overset{(\overline{\psi_{l-1}})_e}{\to} (\overline{V_l})_e = (\overline{X_2})_e\end{aligned}$$

where

(i) $(\overline{\phi})_e = (\overline{\psi_{l-1}})_e \circ (\overline{\psi_{l-2}})_e \circ \cdots \circ (\overline{\psi_2})_e \circ (\overline{\psi_1})_e$,

(ii) $(\overline{\psi_i})_e$ are isomorphisms over $(\overline{U})_e$, and

(iii) either $(\overline{\psi_i})_e : (\overline{V_i})_e \dashrightarrow (\overline{V_{i+1}})_e$ or $(\overline{\psi_i})_e^{-1} : (\overline{V_{i+1}})_e \dashrightarrow (\overline{V_i})_e$ is a morphism obtained by blowing up smooth G_e-invariant center $(\overline{C_i})_e$ disjoint from $(\overline{U})_e$.

Taking the conjugate induced by the action of the element $g_j \in G$ in the Galois group, we obtain the corresponding factorization for $(\overline{\psi_i})_{g_j} : (\overline{X_1})_{g_j} \dashrightarrow (\overline{X_2})_{g_j}$

$$\begin{aligned}(\overline{X_1})_{g_j} = (\overline{V_1})_{g_j} &\overset{(\overline{\psi_1})_{g_j}}{\dashrightarrow} (\overline{V_2})_{g_j} \overset{(\overline{\psi_2})_{g_j}}{\dashrightarrow}\\ &\cdots \overset{(\overline{\psi_{i-1}})_{g_j}}{\to} (\overline{V_i})_{g_j} \overset{(\overline{\psi_i})_{g_j}}{\to} (\overline{V_{i+1}})_{g_j} \overset{(\overline{\psi_{i+1}})_{g_j}}{\to} \cdots\\ &\overset{(\overline{\psi_{l-2}})_{g_j}}{\dashrightarrow} (\overline{V_{l-1}})_{g_j} \overset{(\overline{\psi_{l-1}})_{g_j}}{\to} (\overline{V_l})_{g_j} = (\overline{X_2})_{g_j}.\end{aligned}$$

Collectively, we obtain the factorization over $\overline{K}$ for

$$\overline{\varphi} : \overline{X_1} \dashrightarrow \overline{X_2}$$

into blowups with centers $\coprod_{g_j \in C} (\overline{C_i})_{g_j}$, which are G-invariant and smooth over $\overline{K}$.

Therefore, we obtain the factorization over K for $\phi : X_1 \dashrightarrow X_2$

$$X_1 = V_1 \overset{\psi_1}{\dashrightarrow} V_2 \overset{\psi_2}{\dashrightarrow} \cdots \overset{\psi_{i-1}}{\dashrightarrow} V_i \overset{\psi_i}{\to} V_{i+1} \overset{\psi_{i+1}}{\to} \cdots \overset{\psi_{l-2}}{\dashrightarrow} V_{l-1} \overset{\psi_{l-1}}{\to} V_l = X_2$$

where

(i) $\phi = \psi_{l-1} \circ \psi_{l-2} \circ \cdots \circ \psi_2 \circ \psi_1$,

(ii) ψ_i are isomorphisms over U, and

(iii') either $\psi_i : V_i \dashrightarrow V_{i+1}$ or $\psi_i^{-1} : V_{i+1} \dashrightarrow V_i$ is a morphism obtained by blowing up a center C_i smooth over K and disjoint from U.

In order to guarantee that the centers of blowups to be irreducible and satisfy the condition (iii), instead of blowing up C_i which may be reducible, we blow up each irreducible component one by one and replace the original sequence with the corresponding refinement.

The "furthermore" part concerning the projectivity is also obvious from the construction.

This completes the proof of the weak factorization theorem over a (possibly) NON-algebraically closed field.

§5-4. Factorization in the logarithmic category

Theorem 5-4-1 (Weak Factorization Theorem in the Logarithmic Category). *Let (U_{X_1}, X_1) and (U_{X_2}, X_2) be complete nonsingular toroidal embeddings over an algebraically closed field K of characteristic zero. Let $\phi : (U_{X_1}, X_1) \dashrightarrow (U_{X_2}, X_2)$ be a birational map which is an isomorphism over $U_{X_1} = U_{X_2}$. Then ϕ can be factored into a sequence of blowups and blowdowns with smooth ADMISSIBLE and irreducible centers disjoint from $U_{X_1} = U_{X_2}$. That is to say, there exists a sequence of birational maps between complete nonsingular toroidal embeddings*

$$\begin{aligned}(U_{X_1}, X_1) = (U_{V_1}, V_1) &\overset{\psi_1}{\dashrightarrow} (U_{V_2}, V_2) \overset{\psi_2}{\dashrightarrow} \\ &\cdots \overset{\psi_{i-1}}{\to} (U_{V_i}, V_i) \overset{\psi_i}{\to} (U_{V_{i+1}}, V_{i+1}) \overset{\psi_{i+1}}{\to} \cdots \\ &\overset{\psi_{l-2}}{\dashrightarrow} (U_{V_{l-1}}, V_{l-1}) \overset{\psi_{l-1}}{\to} (U_{V_l}, V_l) = (U_{X_2}, X_2)\end{aligned}$$

where

(i) $\phi = \psi_{l-1} \circ \psi_{l-2} \circ \cdots \circ \psi_2 \circ \psi_1$,

(ii) ψ_i are isomorphisms over U_{V_i}, and

(iii) either $\psi_i : (V_i, U_{V_i}) \dashrightarrow (U_{V_{i+1}}, V_{i+1})$ or $\psi_i^{-1} : (U_{V_{i+1}}, V_{i+1}) \dashrightarrow (U_{V_i}, V_i)$ is a morphism obtained by blowing up a smooth irreducible center C_i disjoint from $U_{V_i} = U_{V_{i+1}}$ and transversal to the boundary $D_{V_i} = V_i - U_{V_i}$ or $D_{V_{i+1}} = V_{i+1} - U_{V_{i+1}}$, i.e., at each point $p \in V_i$ or $p \in V_{i+1}$ there exists a regular coordinate system $\{x_1, \cdots, x_n\}$ in a neighborhood $p \in U_p$ such that $D_{V_i} \cap U_p$ (or $D_{V_{i+1}} \cap U_p$) $= \{\prod_{j \in J} x_j = 0\}$ and $C_i \cap U_p = \{\prod_{j \in J \cup J'} x_j = 0\}$ for some subsets $J, J' \subset \{1, \cdots, n\}$.

Furthermore, if both (U_{X_1}, X_1) *and* (U_{X_2}, X_2) *are projective, then we can choose a factorization so that all the intermediate toroidal embeddings* (U_{V_i}, V_i) *are projective. More precisely, in the factorization we construct, there exists an index* i_o *such that the birational maps* $(U_{V_i}, V_i) \dashrightarrow (U_{X_1}, X_1)$ *are projective birational morphisms for* $i \leq i_o$ *and that the birational maps* $(U_{V_i}, V_i) \dashrightarrow (U_{X_2}, X_2)$ *are projective birational morphisms for* $i_o \leq i$.

Proof.

We specify the modifications we have to make at each step of the proof of the strategy for the proof described in Chapter 0. Introduction.

Step 1. Elimination of points of indeterminacy

Lemma 1-4-1 holds in the above-mentioned logarithmic category where all the centers of the blowups necessary for elimination of points of indeterminacy can be taken to be admissible as well as smooth. Thus we may assume that $\phi : (U_{X_1}, X_1) \to (U_{X_2}, X_2)$ is a projective morphism which is the blowup of X_2 along an ideal sheaf $J \subset \mathcal{O}_{X_2}$ with the support of $\mathcal{O}_{X_2}/J$ being disjoint from $U_{X_2} = U_{X_1}$.

Step 2. Construction of a birational cobordism

In an identical manner to the proof of Theorem 2-2-2, we can construct a compactified birational cobordism $\tau : (U_{\overline{B}}, \overline{B}) \to (U_{X_2}, X_2)$ for ϕ projective over X_2 such that $\tau^{-1}(D_{X_2}) = D_{\overline{B}}$ is a divisor with only normal crossings where $D_{X_2} = X_2 - U_{X_2}$ and $D_{\overline{B}} = \overline{B} - U_{\overline{B}}$ and that

$$(U_B, B)_+/K^* = (U_{X_2}, X_2)$$
$$(U_B, B)_-/K^* = (U_{X_1}, X_1)$$

as toroidal embeddings.

When we construct a locally toric chart for a point $p \in B$ (Though (U_B, B) is a nonsingular toroidal embedding, the action of K^* may not be toroidal. Thus we still call the chart a locally toric chart), we use the method (A) in Proposition 1-3-4 so that $U_p \cap D_B = V_p \cap D_B$ is included in $\eta_p^{-1}(D_{X_p})$, where $D_B = B - U_b$ and $D_{X_p} = X_p - U_{X_p}$. This can be done, in the notation of the proof (A) of Proposition 1-3-4, by choosing a regular system of parameters

$$f_1, \cdots, f_n \in m_q$$

around the limit point $q \in \overline{O(p)}$, consisting of eigenfunctions, so that each irreducible component D of $V_q \cap D_{\overline{B}}$ is defined by some coordinate functions among $f_1, \cdots, f_n$ (shrinking V_q if necessary). In fact, since the whole boundary $D_{\overline{B}}$ is invariant under the action of K^*, so is D and hence its defining ideal $I_D \subset A(V_q)$ is also K^*-invariant. Since K^* is reductive, the ideal I_D splits into the eigenspaces according to the characters under the action of K^*. As D is nonsingular at q, there exists an eigenfunction $f_D \in I_D$ and a coordinate function , say f_1, such that

$$f_D \bmod m_q = f_1 \bmod m_q.$$

By replacing f_1 with f_D in the regular system of parameters, we achieve the goal.

Step 3. Torification
Step 4. Recovery of Nonsingularity

In the torification of the quasi-elementary cobordism $B = B_{a_i}$, after blowing up the torific ideal to obtain $\mu : B^{tor} \to B$, we only have to show that

$$(B^{tor} - \{D^{tor} \cup \mu^{-1}(D_B)\}, B^{tor})$$

has a toroidal structure with respect to which the induced action of K^* is toroidal. But this follows immediately, since already

$$(B^{tor} - D^{tor}, B^{tor})$$

has a toroidal structure with respect to which the induced action of K^* is toroidal and since the additional boundary divisors coming from D_B correspond to the toric coordinate divisors in the locally toric charts chosen as above. (In the language of Torification Principle for Toric Varieties, we do NOT have to "remove" the divisors coming from D_B from the boundary.) The argument for recovery of nonsingularity also goes without any change, considering the toroidal structures with the divisors coming from D_B added to the boundaries.

This completes the proof of the weak factorization theorem in the logarithmic category.

Corollary 5-4-2. *The Hodge numbers are birational invariants for nonsingular minimal models, i.e., if X_1 and X_2 are projective nonsingular varieties over $\mathbb{C}$ whose canonical divisors are nef and if they are birational to each other, then we have*

$$h^{p,q}(X_1) = h^{p,q}(X_2) \quad \forall p, q.$$

Proof.

This application of the weak factorization theorem to a theorem of Batyrev was communicated to us by J. Denef, F. Loeser and W. Veys. It is easy to see that X_1 and X_2, being birational and minimal, are isomorphic outside of some codimension 2 loci. By some sequences of blowups with admissible (in the usual sense) centers over the specified loci, we can reach the situation of Theorem 5-4-1, where again the birational map is factored as a sequence of blowups and blowdowns with smooth and admissible (in the sense of Theorem 5-4-1) centers over the same specified loci. Then it is straightforward to see the required invariance of the Hodge numbers via the invariance of Batyrev's stringy E-function through such blowups and blowdowns with admissible centers. We refer the details of the argument to the papers in the related field (cf. Batyrev [1] Denef-Loeser [1] Kollár [2] Wang [1]).

There is also a slightly different approach to the above theorem by considering the Grothendieck group of the Hodge structures and looking for some elementary but explicit invariant form within the group under blowups and blowdowns with admissible centers (cf. Arapura-Matsuki [1]).

Remark 5-4-3.

(i) We can also combine some of the generalizations above and prove, e.g.,

∘ an equivariant version of the weak factorization theorem in the logarithmic category or in the usual category over a (possibly) NON-algebraically closed field of characteric zero

∘ the weak factorization theorem of bimeromorphic maps in the logarithmic category with or without a group action in consideration.

The modifications we have to make are simple combinations of those discussed in §5-1 through §5-4 and we leave the details to the reader.

(ii) One can try to present the generalizations in §5-1 through §5-4 in a more unified manner, including the case of algebraic spaces (cf. Section 5 in Abramovich-Karu-Matsuki-Włodarczyk [1]). We chose, however, our rather repetitive manner in order to emphasize minor but subtle differences.

CHAPTER 6. PROBLEMS

In this chapter, we discuss some problems related to our proof of the weak factorization theorem for birational maps.

§6-1. Effectiveness of the construction

As our proof factorizes a given birational map explicitly via the birational cobordism, it is not merely existential but also constructive. On the other hand, if one asks the following effectiveness question, our construction falls short of giving an affirmative answer at several places.

Question 6-1-1. *Suppose that nonsingular projective varieties $X_1 \subset \mathbb{P}^{n_1}, X_2 \subset \mathbb{P}^{n_2}$ as well as a birational map between them $\phi : X_1 \dashrightarrow X_2$ are given by a specific set of equations in terms of homogeneous coordinates of the projective spaces. Can we give a factorization of ϕ into blowups and blowdowns with smooth centers in an effective way from the given set of equations ?*

We list the places where our construction falls short of being effective (or rather to say, the places where the AUTHOR does not know how to carry out the process in an effective way).

Step 1: Reduction to the case where ϕ is a projective birational morphism

Hironaka's method of elimination of points of indeterminacy in Lemma 1-4-1

∘ the construction of the ideal I in Remark 1-4-2 on X_1 whose blowup factors through X_2 (after taking the graph of ϕ and assuming ϕ^{-1} is a morphism)

∘ canonical principalization of the ideal I (Canonical resolution of singularities as well as canonical principalization of ideals are constructive as presented in Bierstone-Milman [1] Villamayor [1] Encinas-Villamayor [1], while the author does not know if their algorithms can be made effective.)

Step 2: Factorization into locally toric birational maps

Construction of a birational cobordism in Theorem 2-2-2

∘ the construction of the ideal $J \subset \mathcal{O}_{X_2}$ whose blowup gives the projective morphism $\phi : X_1 \to X_2$, more specifically, finding the generators of J in terms of the homogeneous coordinates for X_2

∘ canonical resolution of singularities

∘ the construction of a relatively ample line bundle $\mathcal{L}$ on $\overline{B}$, equipped with a K^*-action so that for sufficiently large $l \in \mathbb{N}$ the direct image sheaf $\mathcal{E} = \tau_*(\mathcal{L}^{\otimes l})$ has the decomposition into the eigen spaces $\mathcal{E} = \oplus_{b \in \mathbb{Z}} \mathcal{E}_b$ and that the compactified birational cobordism $\overline{B}$ has an equivariant embedding $\overline{B} \hookrightarrow \mathbb{P}(\mathcal{E})$

Actually the effectiveness for the above could be an immediate consequence of the effectiveness for canonical resolution. In fact, in the process of canonical

resolution using any one of the algorithms given by Bierstone-Milman [1] Villamayor [1] Encinas-Villamayor [1]

$$\overline{B} = W_m \to \cdots \to W_{i+1} \stackrel{\mu_i}{\to} W_i \to \cdots \to W_1 = X_2 \times \mathbb{P}^1$$

if we can find inductively an equivariant embedding

$$W_i \subset X_2 \times \mathbb{P}^{k_i} \subset \mathbb{P}^{n_2} \times \mathbb{P}^{k_i}$$

where K^* acts only on the second factor $\mathbb{P}^{k_i}$ with the semi-invariant homogeneous coordinates, which induces a linearization on $\mathcal{L}_i = p_2^* \mathcal{O}_{\mathbb{P}^{k_i}}(1)$, and if we can find effectively some generators consisting of bihomogeneous polynomials (in terms of homogeneous coordinates for $\mathbb{P}^{n_2}$ and $\mathbb{P}^{k_i}$) of some fixed bidegree for the ideal of the center of blowup for μ_i, which is K^*-invariant, then it is easy to realize the inductive situation for W_{i+1}. Then we can set $\mathcal{L} = \mathcal{L}_m$ and $l = 1$.

∘ how to find Luna's locally toric charts as described in Lemma 2-4-4 satisfying the condition $(\star)$

Step 3: Factorization into toroidal birational maps

Torification

Once Luna's toric charts satisfying the condition $(\star)$ are chosen, all the analysis is reduced to the toric case, where the process of torification is effective as well as constructive.

Step 4: Recovery of nonsingularity

Again via Luna's locally toric (toroidal) charts all the analysis here is reduced to the toric case, where the process, including the canonical resolution of singularities and canonical principalization of (toric) ideals, is effective as well as constructive.

Step 5. Factorization of toroidal birational maps among nonsingular toroidal embeddings

Morelli's combinatorial algorithm

There are two places where the algorithm for the construction of Morelli's polyhedral cobordisms (without the projectivity assertion) refers to some existential theorems in Morelli [1][2] Abramovich-Matsuki-Rashid [1].

∘ Toric Moishezon Theorem

Though we refer to Hironaka's elimination of points of indeterminacy in the two papers above, the algorithm in Deconcini-Procesi [1] of elimination of points of indeterminacy for toric birational maps is effective as well as constructive.

∘ Sumihiro's equivariant completion theorem

In the toric case, "filling up" cones between the lower face and upper face of the "vacuum" for the fan representing the birational cobordism can be made effective as well as constructive as in Matsuki-Rashid [1].

However, we can replace this construction with our construction in Chapter 2 (with the projectivity assertion), which is effective as well as constructive in the toric or toroidal situation. See Matsuki [2].

§6-2. Strong Factorization Conjecture

As we mentioned at the beginning of these notes, Kalle Karu pointed out that the algorithm presented in Abramovich-Matsuki-Rashid [1] Morelli [1][2] for the strong factorization fails to work. Thus the strong factorization conjecture remains open even for toric birational maps in dimension ≥ 3, though we have some candidate for an algorithm in dimension 3. (Without a rigorous proof, this candidate is more of a working hypothesis at the moment. See Matsuki [2].)

We would like to emphasize, however, that even assuming an algorithm for the strong factorization for toric or toroidal birational maps, our current proof falls short of providing an argument for the strong factorization of general birational maps, due to the incompatibility of the toroidal structures introduced on the quasi-elementary cobordisms after torification in Chapter 3.

An affirmative answer to the toroidalization conjecture of the next section, via the method of elimination of points of indeterminacy and (canonical) resolution of singularities, would eliminate this incompatibility trouble and completely reduce the problem to the toric or toroidal case.

§6-3. Toroidalization Conjecture

Conjecture 6-3-1 (Toroidalization Algorithm Conjecture). *Let $\phi : (U_X, X) \to (U_Y, Y)$ be a proper (not necessarily birational) morphism between nonsingular toroidal embeddings such that $\phi^{-1}(Y - U_Y) = X - U_X$ and that $\phi|_{U_X} : U_X \to U_Y$ is smooth. Then there should exist an algorithm which specifies the way to take sequences of blowups $\sigma_X : (U_{X'}, X') \to (U_X, X)$ and $\sigma_Y : (U_{Y'}, Y') \to (U_Y, Y)$ with smooth (and possibly admissible (!?)) centers in the boundaries so that the induced morphism between nonsingular toroidal embeddings $\phi' : (U_{X'}, X') \to (U_{Y'}, Y')$ is toroidal.*

As we briefly said in the introduction, a morphism between toroidal embeddings being TOROIDAL (cf. Abramovich-Karu [1] Karu [1]) is equivalent to its being log-smooth (cf. Kato [1] Iitaka [1]). Thus the above conjecture asserts that in the logarithmic category of toroidal embeddings we should have an algorithm which resolves the singularities of a morphism $\phi : (U_X, X) \to (U_Y, Y)$, much like the usual ones which resolve the singularities of varieties. The algorithms for resolution of singularities by Bierstone-Milman [1] Villamoyor [1] Encinas-Villamayor [1] and others take advantages of their internal inductional structures together with some ingenious introduction of invariants measuring how singular your variety is. Our hope here is that, going away from the original factorization problem which only applies to the birational maps, via the flexibility of the toroidalization conjecture dealing with morphisms between toroidal embeddings of any dimensions we should start seeing the internal inductional structure with some invariants measuring how singular your morphism is.

Though the toroidalization conjecture can be checked in dimension 2 as in Abkulut-King [1] Abramovich-Karu [1] Abramovich-Karu-Matsuki-Włodarczyk [1] Karu [1] and Cutkosky-Piltant [1], none of the existing methods fully reveals this (conjectural) inductional structure and hence yields a proof even in dimension 3. Probably the approach of Cutkosky-Piltant [1] is the closest to our hope in spirit

and the recent solution by Cutkosky [4] to the case where $\dim X = 3$ and $\dim Y = 2$ should provide us more information toward the general solution.

Instead of starting from a morphism between nonsingular toroidal emebeddings, we could restate the toroidalization conjecture for a proper morphism $\phi : X \to Y$ between any nonsingular varieties claiming after appropriate sequences of blowups $\sigma_X : X' \to X$ and $\sigma_Y : Y' \to Y$ the induced morphism $\phi' : (U_{X'}, X') \to (U_{Y'}, Y')$ is toroidal for some appropriate choice of open subsets $U_{X'}$ and $U_{Y'}$. In this restated form, the toroidalization conjecture in the case $\dim Y = 1$ is equivalent to resolution of singularities of hypersurface singularities, revealing another close connection between the problem of toroidalization and that of resolution of singularities. (Note that any proper morphism between NONSINGULAR toroidal embeddings stated in Conjecture 6-2-1 is automatically toroidal in the case when $\dim Y = 1$ and hence trivial.)

We end this section by giving a short proof of the toroidalization conjecture in dimension 2, which is slightly different from the above mentioned proofs as yet provides the full statement, including the minimality assertion of Cutkosky-Piltant [1].

Theorem 6-3-2 (Toroidalization Conjecture in dimension 2). *The toroidalization conjecture holds in dimension 2. Moreover, among all the toroidalized morphisms $\phi' : (U_{X'}, X') \to (U_{Y'}, Y')$ obtained from $\phi : (U_X, X) \to (U_Y, Y)$, there is the minimum one $\phi' : (U_{X'_{min}}, X'_{min}) \to (U_{Y'_{min}}, Y'_{min})$ dominated by all such, i.e., we have a commutative diagram*

$$\begin{array}{ccc} (U_{X'}, X') & \stackrel{\phi}{\to} & (U_{Y'}, Y') \\ \downarrow & & \downarrow \\ (U_{X'_{min}}, X'_{min}) & \stackrel{\phi_{min}}{\to} & (U_{Y'_{min}}, Y'_{min}). \end{array}$$

Proof.

If $\dim Y \leq 1$, then f is already toroidal and there is nothing to prove. So the only issue is to prove the statement when $\dim X = \dim Y$ and hence ϕ is a generically finite map.

Let $\hat{\phi} : \hat{X} = Spec\phi_*\mathcal{O}_X \to Y$ be the Stein factorization, $\tilde{X} \to \hat{X}$ the minimal resolution and let $\psi : X \to \tilde{X}$ and $\tilde{\phi} : \tilde{X} \to Y$ be the induced morphisms. Now by Abhyankar's lemma, the morphism $\hat{\phi} : (U_{\hat{X}}, \hat{X}) \to (U_Y, Y)$ is toroidal where $U_{\hat{X}} = \hat{\phi}^{-1}(U_Y)$ and hence so is $\tilde{\phi} : (U_{\tilde{X}}, \tilde{X}) \to (U_Y, Y)$ where $U_{\tilde{X}} = \tilde{\phi}^{-1}(U_Y)$. Now for the exceptional divisors for $\psi = \psi_0$ we classify them into two classes: if an exceptional divisor is obtained after a sequence of toroidal point blowups, i.e, the centers of blowups are the singular points of the pull-backs of $D_{\tilde{X}} = \tilde{X} - U_{\tilde{X}}$ (a normal crossing divisor), then we say the divisor is toroidal, otherwise non-toroidal. Let p be the center on Y of a non-toroidal exceptional divisor for ψ. We take the point blowup $\tau_1 : Y_1 \to Y_0 = Y$ and let $\sigma_1 : X_1 \to X_0 = X$ be the minimal sequence of point blowups for the elimination of points of indeterminacy for $X_0 \dashrightarrow Y_1$. Let $\phi_1 : (U_{X_1}, X_1) \to (U_{Y_1}, Y_1)$ be the induced morphism where $U_{Y_1} = \tau_1^{-1}(U_{Y_0})$ and $U_{X_1} = \phi_1^{-1}(U_{Y_1})$. Remark that by taking the Stein factorization of ϕ_1 and the

minimal resolution the morphism ψ_1 can be defined similarly. Observe that if p is a toroidal center, i.e., a singular point of the normal crossing divisor D_{Y_0}, then the only non-toroidal exceptional divisors for ϕ_1 are the strict transforms of the non-toroidal exceptional divisors for $\phi_0 = \phi$ and in particular the number of the non-toroidal exceptional divisors does not increase. Observe also that if p is not a toroidal center, then the non-toroidal exceptional divisors for ϕ_1 are contained in the strict transforms of the non-toroidal exceptional divisors for ϕ_0 and that at least one of the strict stransforms of the non-toroidal exceptional divisors for ϕ_0 is not exceptional (but toroidal) for ϕ_1. As the number of the non-toroidal exceptional divisors at the stage 0 is finite and as the centers of the non-toroidal exceptional divisors cannot keep being toroidal, this process has to come to an end after finitely many steps, where we have the toroidal morphism $\phi'(U_{X'}, X') \to (U_{Y'}, Y')$. Since all the steps in the construction are necessary steps to make the given morphism toroidal, it is also minimal.

Seeing a solution to the (weak) factorization problem of birational maps is hardly an end to this exciting field surrounding resolution of singularities, toroidalization, strong factorization, semistable reduction, log category ... etc. and we feel that we just had a sneak preview of what is still to come.

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