MONODROMY FILTRATION AND POSITIVITY

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Let $f : X \to S$ be a projective morphism of complex manifolds with relative dimension n such that S is an open disk and f is semistable (i.e. $X_0 := f^{-1}(0)$ is a divisor with normal crossings). Then J. Steenbrink [19] constructed a limit mixed Hodge structure by using a resolution of the nearby cycle sheaf. This limit mixed Hodge structure coincides with the one obtained by W. Schmid [17] using the SL_2-orbit theorem, because the weight filtration coincides with the (shifted) monodromy filtration. See also [5], [8], [15], etc.

A similar construction was then given by M. Rapoport and T. Zink [14] in the case X is projective and semistable over a henselian discrete valuation ring R of mixed characteristics. (We may assume X semistable by [2].) Then Deligne's conjecture on the monodromy filtration [3, I] is expressed as

0.1. Conjecture. The obtained weight filtration coincides with the (shifted) monodromy filtration.

This conjecture is proved so far in the case $n \le 2$ by [14] and in some other cases (see e.g. [16]). We cannot apply the same argument as in the complex analytic case, because there is not a good notion of positivity for l-adic sheaves. In the equal characteristic case, however, the conjecture was proved by Deligne [4] (assuming f is the base change of a projective morphism to a curve). In this paper we try to examine how much the arguments in the complex analytic case can be extended to the mixed characteristic case.

Let $Y^{(i)}$ denote the disjoint union of the intersections of i irreducible components of X_0 (which are assumed smooth). Let k be the residue field of R which is assumed to be a finite field. Let $\bar{k}$ be an algebraic closure of k. Let l be a prime number different from the characteristic of k. Then the E_1-term of the weight spectral sequence of Rapoport and Zink is given by direct sums of the l-adic cohomology groups of $Y^{(i)}_{\bar{k}}$ which are Tate-twisted appropriately, and the differential d_1 consists of the Cech restriction morphisms and the co-Cech Gysin morphisms which are denoted respectively by ρ and γ in this paper. The primitive cohomology of $Y^{(i)}_{\bar{k}}$ has a canonical pairing induced by the Poincaré duality and the hard Lefschetz theorem [4] (choosing and fixing an ample divisor class of f). Our first observation is

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0.2. Proposition. *The conjecture (0.1) is true if the restrictions of the canonical pairing to the intersections of the primitive part with* $\mathrm{Im}\,\rho$ *or* $\mathrm{Im}\,\gamma$ *are nondegenerate.*

So the problem is reduced to the study of the canonical pairing on the primitive part. A similar assertion (involving only the restriction morphism) was conjectured in the introduction of [14]. As is remarked there, the assumption of (0.2) is similar to an assertion which is equivalent to the hard Lefschetz theorem [4]. Note that the converse of (0.2) is also true if $n = 3$.

In the complex analytic case, the hypothesis of (0.2) is trivially satisfied due to the positivity of polarizations of Hodge structures. We can argue similarly if we have a kind of "positivity" in the characteristic $p > 0$. The positivity for zero-dimensional varieties is clear (because the pairing is defined over the subfield $\mathbb{Q}$ of $\mathbb{Q}_l$). In the one-dimensional case, this notion is provided by the theory of Riemann forms for abelian varieties [13] as follows.

Let A be an abelian variety over a field k, and $A^\vee$ the dual of A. Then we have a canonical pairing of Tate modules

$$T_l A_{\bar{k}} \otimes_{\mathbb{Z}_l} T_l A^\vee_{\bar{k}} \to \mathbb{Z}_l(1).$$

For a divisor D on A, we have a morphism $\varphi_D : A \to A^\vee$ such that $\varphi_D(a) \in A^\vee(\bar{k})$ for $a \in A(\bar{k})$ is given by $T_a^* D_{\bar{k}} - D_{\bar{k}}$, where T_a is the translation by a.

We say that a pairing of a $\mathbb{Q}_l$-module V with a continuous action of $G := \mathrm{Gal}(\bar{k}/k)$ is abelian-positive if there exists an abelian variety with an ample divisor D such that V is isomorphic to $T_l A_{\bar{k}} \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ up to a Tate twist as a $\mathbb{Q}[G]$-module and the pairing corresponds to the one on $T_l A_{\bar{k}}$ defined by the canonical pairing and φ_D. Since φ_D is an isogeny for D ample, it is easy to see the nondegeneracy of the restriction of an abelian-positive pairing to any subgroup corresponding to the Tate module of an abelian subvariety.

The following result of Deligne [6] was mentioned and used in an essential way by Rapoport-Zink for their proof of (0.1) in the case $n = 2$.

0.3. Theorem (Deligne)**.** *Let C be a smooth projective curve over k, and J its Jacobian. Then the Poincaré duality of $H^1(C_{\bar{k}}, \mathbb{Z}_l)$ is naturally identified with the pairing of $T_l J_{\bar{k}}$ induced by the theta divisor. In particular, it is abelian-positive.*

We can apply this to the case $n \geq 3$, if the $H^j(Y^{(i)}_{\bar{k}}, \mathbb{Q}_l)$ have smaller level in the following sense. Let Y be a smooth projective variety over a finite field k (which may be non connected). The primitive cohomology $H^j(Y_{\bar{k}}, \mathbb{Q}_l)^{\mathrm{prim}}$ has a canonical pairing as above. We say that $H^j(Y_{\bar{k}}, \mathbb{Q}_l)^{\mathrm{prim}}$ has cohomological level $\leq r$ if there is a smooth projective variety Y' such that $\dim Y' \leq r$ and $H^j(Y_{\bar{k}}, \mathbb{Q}_l)^{\mathrm{prim}}$ is isomorphic as a $\mathbb{Q}_l[G]$-module to a direct factor of $H^{j'}(Y'_{\bar{k}}, \mathbb{Q}_l)^{\mathrm{prim}}(m)$ (where $j = j' - 2m$). In this case, we say further that the canonical pairing on the primitive part comes from level $\leq r$ if it corresponds to the canonical pairing on $H^{j'}(Y'_{\bar{k}}, \mathbb{Q}_l)^{\mathrm{prim}}(m)$ up to a constant multiple, and that a $\mathbb{Q}_l[G]$-submodule has strictly smaller level in $H^j(Y_{\bar{k}}, \mathbb{Q}_l)^{\mathrm{prim}}$ if it corresponds to a submodule of $H^{j'}(Y'_{\bar{k}}, \mathbb{Q}_l)^{\mathrm{prim}}(m)$ contained in $N^1 H^{j'}(Y'_{\bar{k}}, \mathbb{Q}_l)^{\mathrm{prim}}(m)$, where N denotes the coniveau filtration.

Now assume

(i) The $H^j(Y^{(i)}_{\bar{k}}, \mathbb{Q}_l)^{\mathrm{prim}}$ have cohomological level ≤ 2 for any i, j.
(ii) The intersections of the primitive parts with $\mathrm{Im}\,\rho$ or $\mathrm{Im}\,\gamma$ have strictly smaller level if j is even.
(iii) The canonical pairing on the primitive parts comes from level ≤ 2 if j is even, and corresponds to an abelian-positive pairing up to a constant multiple otherwise.

Then (0.3) implies

0.4. Corollary. *If the above three conditions are satisfied, then (0.1) is true.*

Since the conditions are trivially satisfied in the case $n = 2$, we get a proof of the result of Rapoport and Zink [14] without using the theory of Néron model [7]. If $n = 3$, it is also possible to consider weaker conditions on the nondegeneracy of the restrictions of the canonical pairing to

$$H^3(Y^{(1)}_{\bar{k}}, \mathbb{Q}_l)^{\mathrm{prim}} \cap \mathrm{Im}\,\gamma, \quad H^2(Y^{(2)}_{\bar{k}}, \mathbb{Q}_l)^{\mathrm{prim}} \cap \mathrm{Im}\,\rho.$$

Actually, (0.1) for $n = 3$ is equivalent to these conditions. They are trivially satisfied if $H^2(Y^{(1)}_{\bar{k}}, \mathbb{Q}_l)^{\mathrm{prim}} = H^1(Y^{(2)}_{\bar{k}}, \mathbb{Q}_l) = 0$. However, we do not know any method to prove these conditions for $n = 3$ in general, unless the above three conditions are satisfied. Note that the first condition on the nondegeneracy for the intersection with $\mathrm{Im}\,\gamma$ is equivalent to

$$\mathrm{Ker}\, l\gamma \cap \mathrm{Ker}\,\rho \subset \mathrm{Ker}\,\gamma \quad \text{in } H^1(Y^{(2)}_{\bar{k}}, \mathbb{Q}_l).$$

It is expected that the above condition (iii) for $H^3(Y^{(1)}_{\bar{k}}, \mathbb{Q}_l)^{\mathrm{prim}}$ is satisfied if $Y^{(1)}_{\bar{k}}$ is rational.

For higher dimensional varieties, we do not know the notion of positivity except for the standard conjectures [10]. Using the theory in loc. cit., we can show

0.5. Theorem. *The conjecture (0.1) is true if the standard conjectures hold for $Y^{(i)}$ and $Y^{(i)} \times_k Y^{(i)}$.*

It would be interesting whether we can prove (0.1) without assuming the standard conjectures (by using arguments similar to [9]).

In Sect. 1 and Sect. 2, we review the theory of graded or bigraded modules of Lefschetz type and prove (0.2). In Sect. 3, we show an outline of the proof of (0.3) for the reader's convenience, and prove (0.4) and (0.5) in Sect. 4.

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1. Bigraded Modules of Lefschetz Type

1.1. Let Q be a field, and $M^{\bullet,\bullet}$ be a bigraded $Q[L_1, L_2]$-module of Lefschetz type, i.e. it is a finite dimensional bigraded vector space over Q having commuting actions of L_1, L_2 with bidegrees $(2, 0)$ and (0.2) respectively such that

$$L_1^i : M^{-i,j} \xrightarrow{\sim} M^{i,j}\,(i > 0), \quad L_2^j : M^{i,-j} \xrightarrow{\sim} M^{i,j}\,(j > 0).$$

Then we have the Lefschetz decomposition for the first index

$$M^{i,j} = \oplus_k L_1^k M_0^{i-2k,j}, \tag{1.1.1}$$

where $M_0^{-i,j} = \mathrm{Ker}\, L_1^{i+1} \subset M^{-i,j}$ for $i \geq 0$, and 0 otherwise. We define

$$C_{i,k}^j = L_1^k M_0^{-i-k,j},$$

so that $C_{i,k}^j = 0$ for $i < 0$ or $k < 0$, and $C_{i,k}^j = L_1^k C_{i+k,0}^j$ with $C_{i,0}^j = M_0^{-i,j}$.

Let d be a differential of bidegree $(1,1)$ on $M^{\bullet,\bullet}$ which commutes with L_1, L_2 and satisfies $d^2 = 0$. Then, using (1.1) together with the commutativity of L_1 and d, we get a decomposition $d = d' + d''$ such that

$$d' : C_{i,k}^\bullet \to C_{i-1,k}^\bullet[1], \quad d'' : C_{i,k}^\bullet \to C_{i,k+1}^\bullet[1]$$

are differentials which anti-commute with each other. Here [1] denotes the shift of index by 1. We have morphisms of graded $Q[L_2]$-modules

$$\gamma_i : C_{i,0}^\bullet \to C_{i-1,0}^\bullet[1]\,(i > 0), \;\; \rho_i : C_{i,0}^\bullet \to C_{i+1,0}^\bullet[1]\,(i \geq 0)$$

such that d', d'' are identified with γ_{i+k}, ρ_{i+k} respectively.

Consider the following condition for ρ_i and γ_{i+1} $(i \geq 0)$:

(C) There are direct sum decompositions

$$\mathrm{Im}\, \rho_i = (\mathrm{Im}\, \rho_i)_0 \oplus (\mathrm{Im}\, \rho_i)_1, \quad \mathrm{Im}\, \gamma_{i+1} = (\mathrm{Im}\, \gamma_{i+1})_0 \oplus (\mathrm{Im}\, \gamma_{i+1})_1,$$

such that $\rho_i(\mathrm{Im}\, \gamma_{i+1})_1 = \gamma_{i+1}(\mathrm{Im}\, \rho_i)_1 = 0$, and the restriction of ρ_i, γ_{i+1} to $(\mathrm{Im}\, \gamma_{i+1})_0$ and $(\mathrm{Im}\, \rho_i)_0$ respectively are injective.

Note that the condition implies the isomorphisms for $i \geq 0$

$$\rho_i : (\mathrm{Im}\, \gamma_{i+1})_0 \xrightarrow{\sim} \mathrm{Im}\, \rho_i\gamma_{i+1}, \quad \gamma_{i+1} : (\mathrm{Im}\, \rho_i)_0 \xrightarrow{\sim} \mathrm{Im}\, \gamma_{i+1}\rho_i,$$

and $\mathrm{Im}\, \gamma_{i+1}\rho_i = \mathrm{Im}\, \rho_{i-1}\gamma_i$ for $i > 0$.

We define graded $Q[L_2]$-submodules $Z_{i,k}^\bullet, B_{i,k}^\bullet$ of $C_{i,k}^\bullet$ by

$$Z_{i,k}^\bullet = \begin{cases} L_1^k(\mathrm{Ker}\, \rho_{i+k}) & \text{if } i = 0, \\ L_1^k(\mathrm{Ker}\, \rho_{i+k} \cap \mathrm{Ker}\, \gamma_{i+k}) & \text{if } i > 0, \end{cases}$$

$$B_{i,k}^\bullet = \begin{cases} L_1^k(\mathrm{Im}\, \rho_{i+k-1}) & \text{if } i = 0, k > 0, \\ L_1^k(\mathrm{Im}\, \gamma_{i+k+1}\rho_{i+k}) & \text{if } i > 0 \text{ or } k = 0, \end{cases}$$

so that $Z_{i,k}^\bullet \subset \mathrm{Ker}\, d, B_{i,k}^\bullet \subset \mathrm{Im}\, d$.

1.2. *Remark.* In the application, $C_{i,0}^j = H^{n-i+j}(Y_{\bar{k}}^{(i+1)}, \mathbb{Q}_l(-i))$ in the notation of the introduction, and ρ_i, γ_{i+1} are respectively the Cech restriction and co-Cech Gysin morphisms, which are dual of each other. Furthermore, L_1 is the logarithm of the monodromy, and L_2 is given by the ample divisor class of f.

1.3. Proposition. *If condition (C) is satisfied, we have isomorphisms*

$$\oplus_{i,k} Z^{\bullet}_{i,k}/B^{\bullet}_{i,k} \xrightarrow{\sim} \operatorname{Ker} d/\operatorname{Im} d, \tag{1.3.1}$$

$$L_1^{i-k} : Z^{\bullet}_{i,k}/B^{\bullet}_{i,k} \xrightarrow{\sim} Z^{\bullet}_{k,i}/B^{\bullet}_{k,i}\ (i > k). \tag{1.3.2}$$

Proof. We first show

$$\operatorname{Ker} d = \sum_{i,k} Z^{\bullet}_{i,k} + \operatorname{Im} d. \tag{1.3.3}$$

Let $m = \sum_k L_1^k m_{i+2k} \in \sum_k C^{\bullet}_{i+k,k}$ with $m_{i+2k} \in C^{\bullet}_{i+2k,0}$ for $k \geq \max\{0, -i\}$. Then $dm = 0$ means

$$\rho_{i+2k}(m_{i+2k}) = \gamma_{i+2k+2}(m_{i+2k+2}) \quad \text{for } k \geq \max\{0, -i\},$$

together with $\gamma_i(m_i) = 0$ if $i > 0$. By condition (C) there are $n_{i+2k+1} \in C^{\bullet}_{i+2k+1,0}$ for $k \geq \max\{0, -i\}$ such that

$$\rho_{i+2k}(m_{i+2k}) = \rho_{i+2k}\gamma_{i+2k+1}(n_{i+2k+1}).$$

Then we can replace m with $m - d(\sum_k L_1^k n_{i+2k+1})$ so that

$$\rho_{i+2k}(m_{i+2k}) = \gamma_{i+2k+2}(m_{i+2k+2}) = 0$$

for $k \geq \max\{0, -i\}$. This shows (1.3.3).

Now we show

$$\sum_{i,k} B^{\bullet}_{i,k} = \sum_{i,k} Z^{\bullet}_{i,k} \cap \operatorname{Im} d. \tag{1.3.4}$$

Let $n = \sum_k L_1^k n_{i+2k+1} \in \sum_k C^{\bullet}_{i+k+1,k}$ with $n_{i+2k+1} \in C^{\bullet}_{i+2k+1,0}$ for $k \geq \max\{0, -i-1\}$. Assume $dn \in \sum_{i,k} Z^{\bullet}_{i,k}$. Then

$$\gamma_{i+2k+2}\rho_{i+2k+1}(n_{i+2k+1}) = 0 \quad \text{for } k \geq \max\{0, -i\},$$

and $\rho_{i+2k+1}(n_{i+2k+1}) \in \operatorname{Im} \rho_{i+2k+1}\gamma_{i+2k+2}$ by condition (C). So we get (1.3.4) and (1.3.1).

The isomorphism (1.3.2)is equivalent to

$$(\operatorname{Ker} \gamma_i \cap \operatorname{Ker} \rho_i)/\operatorname{Im} \gamma_{i+1}\rho_i \xrightarrow{\sim} \operatorname{Ker} \rho_i/\operatorname{Im} \rho_{i-1} \quad \text{for } i > 0. \tag{1.3.5}$$

This is equivalent further to

$$\operatorname{Ker} \gamma_i \cap \operatorname{Ker} \rho_i \cap \operatorname{Im} \rho_{i-1} = \operatorname{Im} \gamma_{i+1}\rho_i,$$
$$(\operatorname{Ker} \gamma_i \cap \operatorname{Ker} \rho_i) + \operatorname{Im} \rho_{i-1} = \operatorname{Ker} \rho_i.$$

The first equality is clear by condition (C). For the second, let $m \in \operatorname{Ker}\rho_i$. Then $\rho_{i-1}\gamma_i(m) = \gamma_{i+1}\rho_i(m) = 0$, and $\gamma_i(m) = \gamma_i\rho_{i+1}(n) \in \operatorname{Im}\gamma_i\rho_{i+1}$ by (C). So replacing m with $m - \rho_{i+1}(n)$ we get the second equality.

1.4. *Proof of* (0.2). By [14], the E_1-term of the weight spectral sequence has a structure of bigraded $\mathbb{Q}_l$-modules of Lefschetz type as noted in (1.2). Then we can apply (2.4) below, because $\rho_j\gamma_{j+1}\rho_j = -\gamma_{j+2}\rho_{j+1}\rho_j = 0$. So (0.2) follows from (1.3).

1.5. *Remarks.* (i) The restriction of the canonical pairing to

$$H^1(Y_{\bar{k}}^{(i)}, \mathbb{Q}_l)^{\mathrm{prim}} \cap \operatorname{Im}\gamma_i, \quad H^1(Y_{\bar{k}}^{(i)}, \mathbb{Q}_l)^{\mathrm{prim}} \cap \operatorname{Im}\rho_{i-1}$$

is always nondegenerate by (3.4) below.

We have the nondegeneracy of the pairing also for $H^2(Y_{\bar{k}}^{(i)}, \mathbb{Q}_l)^{\mathrm{prim}} \cap \operatorname{Im}\gamma_i$. Indeed, taking a closed subvariety Z of X_0 with dimension $i+1$ which is defined by an intersection of general hyperplane sections, we can define $Z_{\bar{k}}^{(i)}, Z_{\bar{k}}^{(i+1)}$ similarly so that $\dim Z_{\bar{k}}^{(i)} = 2, \dim Z_{\bar{k}}^{(i+1)} = 1$, and we have a cartesian diagram

$$\begin{array}{ccc} Z_{\bar{k}}^{(i+1)} & \longrightarrow & Y_{\bar{k}}^{(i+1)} \\ \downarrow & & \downarrow \\ Z_{\bar{k}}^{(i)} & \longrightarrow & Y_{\bar{k}}^{(i)} \end{array}$$

Since the restriction morphism $H^2(Y_{\bar{k}}^{(i)}, \mathbb{Q}_l)^{\mathrm{prim}} \to H^2(Z_{\bar{k}}^{(i)}, \mathbb{Q}_l)^{\mathrm{prim}}$ is injective, the assertion is reduced to the case $\dim Y_{\bar{k}}^{(i)} = 2$ by the same argument as in (3.5). So we get the assertion, because the standard conjectures $I(Y, L)$ and $D(Y)$ are proved in this case. See (4.2).

(ii) In the case $n = 3$, the above Remark (i) implies that for the proof of (0.1) it is enough to show the nondegeneracy of the restriction of the canonical pairing to:
(a) $H^3(Y_{\bar{k}}^{(1)}, \mathbb{Q}_l)^{\mathrm{prim}} \cap \operatorname{Im}\gamma_1$,
(b) $H^2(Y_{\bar{k}}^{(2)}, \mathbb{Q}_l)^{\mathrm{prim}} \cap \operatorname{Im}\rho_0$.
In this case, we can see that they are also necessary as follows. Let

$$H^{i,j} = \operatorname{Ker}(d : M^{i,j} \to M^{i+1,j+1})/\operatorname{Im}(d : M^{i-1,j-1} \to M^{i,j}) \tag{1.5.1}$$

with the notation of (1.1). Let ρ_i^j, γ_i^j denote the restriction of ρ_i, γ_i to

$$C_i^j := C_{i,0}^j \,(= H^{n-i+j}(Y_{\bar{k}}^{(i+1)}, \mathbb{Q}_l(-i))).$$

Let $C_i^{j,\mathrm{prim}}$ denote the primitive part. Then, if the pairing is degenerate on (a), we have

$$\dim H^{-1,-1} < \dim H^{-1,1} = \dim H^{1,-1},$$

because the degeneracy implies that

$$\dim(\mathrm{Im}\,\gamma_1^{-1} \cap C_0^{0,\mathrm{prim}}) > \mathrm{rank}\,\rho_0^0\gamma_1^{-1},$$

using the direct sum decomposition

$$\mathrm{Im}\,\gamma_1^{-1} = (\mathrm{Im}\,\gamma_1^{-1} \cap C_0^{0,\mathrm{prim}}) \oplus \mathrm{Im}\,\gamma_1^{-1}\rho_0^{-2}.$$

This decomposition follows from

$$\dim \mathrm{Im}\,\gamma_1^{-1}/(\mathrm{Im}\,\gamma_1^{-1} \cap C_0^{0,\mathrm{prim}}) = \mathrm{rank}\,\gamma_1^1 = \mathrm{rank}\,\rho_0^{-2}$$

together with the injectivity of the restriction of $l\gamma_1^{-1} = \gamma_1^1 l$ to $\mathrm{Im}\,\rho_0^{-2}$. The last injectivity is equivalent to

$$C_1^{-1} = \mathrm{Im}\,\rho_0^{-2} \oplus \mathrm{Ker}\,l\gamma_1^{-1},$$

and follows from (3.4). Note that (3.4) implies also

$$\mathrm{rank}\,\gamma_2^0 = \mathrm{rank}\,\rho_1^{-1} = \mathrm{rank}\,\gamma_2^0\rho_1^{-1} = \mathrm{rank}\,\rho_0^0\gamma_1^{-1}.$$

So we see also that the nondegeneracy of the pairing on (a) is equivalent to

$$\mathrm{Ker}\,l\gamma_1^{-1} \cap \mathrm{Ker}\,\rho_1^{-1} = \mathrm{Ker}\,\gamma_1^{-1},$$

where the inclusion $\supset$ is clear by the above equality of the rank.

If the condition does not hold for (b), then $L_1 : H^{-1,0} \to H^{1,0}$ is not bijective, although they have the same dimension.

(iii) Let X be a smooth projective variety of pure dimension 3 over a filed k. Then $N^1H^3(X_{\bar{k}}, \mathbb{Q}_l)$ has cohomological level ≤ 1, because a surface admits a resolution of singularities. (Here we may have to replace k with a finite extension when we take a general hyperplane section.) But it is not clear whether the canonical pairing on it comes from level ≤ 1.

Indeed, let Y be a divisor with normal crossings on X such that the irreducible components Y_j are smooth. Let $\tilde{Y}$ be the disjoint union of Y_j, and $a : \tilde{Y} \to X$ the canonical morphism. (We may assume that $N^1H^3(X_{\bar{k}}, \mathbb{Q}_l) = a_*H^1(\tilde{Y}, \mathbb{Q}_l)$ if the singularities of divisors can be resolved by iterating blowing-ups with smooth centers.) Let S_X denote the canonical pairing on $H^3(X_{\bar{k}}, \mathbb{Q}_l)$. Then the pairing $S_X \circ (a_* \otimes a_*)$ on $H^1(\tilde{Y}, \mathbb{Q}_l)$ is identified with $S_{\tilde{Y}} \circ (id \otimes a^*a_*)$, and the (i,j)-component

$$(a^*a_*)_{i,j} : H^1(Y_i, \mathbb{Q}_l) \to H^3(Y_j, \mathbb{Q}_l)(-1)$$

of a^*a_* is given by the composition of the restriction and Gysin morphisms for $Y_i \cap Y_j \to Y_i$ and $Y_i \cap Y_j \to Y_j$ respectively if $i \neq j$, and by the cup product with the Chern class of the normal bundle of Y_i otherwise. To relate with the cohomology of a curve, however, we have to take further a smooth hyperplane section of Y_j.

(iv) Let X be as in Remark (iii) above. Assume there are birational morphisms $\pi : X' \to X$ and $\pi' : X' \to \mathbb{P}^3$ such that π' is a composition of blowing-ups with smooth centers of pure dimension 0 or 1. Then $H^3(X_{\bar{k}}, \mathbb{Q}_l)$ has cohomological level ≤ 1, and the canonical pairing on it comes from level ≤ 1. This follows from Remark (v) below, because $H^3(X_{\bar{k}}, \mathbb{Q}_l)$ does not change by blowing-ups with point centers.

(v) Let X be a smooth projective variety of pure dimension n over a field k, and $\pi : X' \to X$ be the blow-up along a smooth center Z with pure codimension 2. Then we have a canonical isomorphism

$$H^j(X'_{\bar{k}}, \mathbb{Q}_l) = H^j(X_{\bar{k}}, \mathbb{Q}_l) \oplus H^{j-2}(Z_{\bar{k}}, \mathbb{Q}_l)(-1)$$

such that the canonical pairing $S_{X'} : H^j(X'_{\bar{k}}, \mathbb{Q}_l) \otimes H^{2n-j}(X'_{\bar{k}}, \mathbb{Q}_l) \to \mathbb{Q}_l(-n)$ is identified with the sum of S_X and $-S_Z$. Indeed, we have a canonical isomorphism

$$\pi_* \mathbb{Q}_{l, X'_{\bar{k}}}[n] = \mathbb{Q}_{l, X_{\bar{k}}}[n] \oplus \mathbb{Q}_{l, Z_{\bar{k}}}(-1)[n-2]$$

by the decomposition theorem [1], and the Poincaré duality is compatible with this decomposition. We can verify the sign using, for example, an argument similar to Sect. 5 of [15].

2. Graded Modules of Lefschetz Type

2.1. Let $M^\bullet$ be a graded vector space over Q having an action of L with degree 2. Then $M^\bullet$ is called n-symmetric if $L^j : M^{n-j} \xrightarrow{\sim} M^{n+j}$ for $j > 0$. We say that $M^\bullet$ is a graded $Q[L]$-module of Lefschetz type, if it is a finite dimensional graded vector space over Q having an action of L with degree 2 such that $M^\bullet$ is 0-symmetric. We have the Lefschetz decomposition $M^j = \oplus_k L^k M_0^{j-2k}$ with $M_0^{-j} = \operatorname{Ker} L^{j+1} \subset M^{-j}$.

Let $f : M^\bullet \to N^\bullet[1]$ be a morphism of graded $Q[L]$-modules of Lefschetz type of degree one (i.e. $f(M^j) \subset N^{j+1}$ and $fL = Lf$). Then we have

$$f(M_0^{-j}) \subset N_0^{-j+1} \oplus L N_0^{-j-1}. \tag{2.1.1}$$

So there exists $f_+^{-j} : M_0^{-j} \to N_0^{-j+1}, f_-^{-j} : M_0^{-j} \to N_0^{-j-1}$ such that $f | M_0^{-j} = f_+^{-j} + L f_-^{-j}$. We have

$$\operatorname{Im} f \cap N_0^{-j+1} = f_+^{-j}((f_-^{-j})^{-1}(\operatorname{Im} f \cap N_0^{-j-1})). \tag{2.1.2}$$

Indeed, let $m = m' + Lm''$ with $m' \in M_0^{-j}, m'' \in M^{-j-2}$. Then (2.1.2) is clear.

We define

$$(\operatorname{Im} f)_{(0)} = \oplus_{j,k} L^k (\operatorname{Im} f \cap N_0^{-j}), \quad (\operatorname{Im} f)_{(1)} = \operatorname{Im} f / (\operatorname{Im} f)_{(0)}. \tag{2.1.3}$$

Then $(\operatorname{Im} f)_{(0)}$ is 0-symmetric, and its primitive part is given by $\operatorname{Im} f \cap N_0^{-j}$.

2.2. Lemma. $(\mathrm{Im}\, f)_{(1)}$ *is* 1*-symmetric.*

Proof. We proceed by induction on $\#\{j : M_0^{-j} \neq 0\}$. The assertion is clear if this number is one.

Let $\bar{N} = N/(\mathrm{Im}\, f)_{(0)}$, and $\bar{f} : M \to \bar{N}[1]$ be the composition of f with the canonical projection. Then we can verify

$$(\mathrm{Im}\, \bar{f})_{(0)} = 0, \tag{2.2.1}$$

by using (2.1.2). So it is enough to show that $\mathrm{Im}\, \bar{f}$ is 1-symmetric.

Let $j' = \min\{j : M_0^{-j} \neq 0\}$, and put

$$M' = \sum_{j>j'} (\sum_k L^k M_0^{-j}), \quad M'' = \sum_k L^k M_0^{-j'},$$

so that $M = M' \oplus M''$. Let $\bar{f}' : M' \to \bar{N}[1], \bar{f}'' : M'' \to \bar{N}[1]$ be the restrictions of $\bar{f}$ to M' and M'' respectively. Then $(\mathrm{Im}\, \bar{f}')_{(0)} = (\mathrm{Im}\, \bar{f}'')_{(0)} = 0$ by (2.2.1), and $\mathrm{Im}\, \bar{f}'$ and $\mathrm{Im}\, \bar{f}''$ are 1-symmetric by inductive hypothesis.

We can verify furthermore

$$\mathrm{Im}\, \bar{f}' \cap \sum_k L^k \bar{N}_0^{-j'-1} = \sum_k L^k (\mathrm{Im}\, \bar{f} \cap \bar{N}_0^{-j'-1}). \tag{2.2.2}$$

by using the Lefschetz decomposition of an element of M'. (Here we may assume $M^j = 0$ if $j - j'$ is odd.) Since the right-hand side of (2.2.2) is zero by (2.2.1), we get $\mathrm{Im}\, \bar{f}' \cap \mathrm{Im}\, \bar{f}'' = 0$. So the assertion follows.

2.3. Lemma. *Let $M^\bullet, N^\bullet$ be graded $Q[L]$-module of Lefschetz type, and $f : M^\bullet \to N^\bullet[1], g : N^\bullet \to M^\bullet[1]$ a morphism of degree one. Assume we have a nondegenerate pairing of Q-modules*

$$\Phi_M : M^j \otimes_Q M^{-j} \to Q, \quad \Phi_N : N^j \otimes_Q N^{-j} \to Q,$$

such that $\Phi_N \circ (f \otimes id)$ coincides with $\Phi_M \circ (id \otimes g)$ up to a nonzero constant multiple, and $\Phi_M \circ (id \otimes L) = \Phi_M \circ (L \otimes id)$ (and the same for Φ_M). Then

$$\dim(\mathrm{Im}\, f)^j_{(0)} = \dim(\mathrm{Im}\, g)^{j+1}_{(1)}, \quad \dim(\mathrm{Im}\, g)^j_{(0)} = \dim(\mathrm{Im}\, f)^{j+1}_{(1)},$$

Proof. *Let a_j, d_j, c_j, b_j denote respectively the above dimensions so that $a_j = a_{-j}$, etc. Then*

$$a_{j+1} + b_j = \dim(\mathrm{Im}\, f)^{j+1} = \dim(\mathrm{Im}\, g)^{-j} = c_{-j} + d_{-j-1} = c_j + d_{j+1},$$

by definition, and we get $a_{j+1} - d_{j+1} = c_j - b_j$ for any $j \in \mathbb{Z}$. This implies $a_{j+1} - d_{j+1} = c_j - b_j = 0$ by the symmetry of a_j, etc.

2.4. Proposition. *With the notation and assumption of (2.3), assume $fgf = 0$ (or equivalently, $gfg = 0$), and the restrictions of the pairings $\Phi_M \circ (id \otimes L^j), \Phi_N \circ (id \otimes L^j)$ to $\operatorname{Im} g \cap M_0^{-j}$ and $\operatorname{Im} g \cap N_0^{-j}$ respectively are nondegenerate. Then the compositions*

$$(\operatorname{Im} g)_{(0)} \xrightarrow{f} \operatorname{Im} f \to (\operatorname{Im} f)_{(1)}, \quad (\operatorname{Im} f)_{(0)} \xrightarrow{g} \operatorname{Im} g \to (\operatorname{Im} g)_{(1)} \tag{2.4.1}$$

are isomorphisms, and we have canonical decompositions

$$\operatorname{Im} f = (\operatorname{Im} f)_{(0)} \oplus (\operatorname{Im} f)_{(1)}, \quad \operatorname{Im} g = (\operatorname{Im} g)_{(0)} \oplus (\operatorname{Im} g)_{(1)}, \tag{2.4.2}$$

such that the restriction of g to $(\operatorname{Im} f)_{(0)}$ is injective, and that to $(\operatorname{Im} f)_{(1)}$ is zero, and similarly for the restriction of f.

Proof. It is enough to show the injectivity of

$$f : (\operatorname{Im} g)_{(0)} \to N^\bullet, \quad g : (\operatorname{Im} f)_{(0)} \to M^\bullet, \tag{2.4.3}$$

because this implies $f(\operatorname{Im} g)_{(0)} \cap (\operatorname{Im} f)_{(0)} = g(\operatorname{Im} f)_{(0)} \cap (\operatorname{Im} g)_{(0)} = 0$, and the assertion follows from (2.3). We show the assertion for f. Using the filtration of $M^\bullet, N^\bullet$ defined by

$$\sum_{j \le p} (\sum_k L^k M_0^{-j}), \quad \sum_{j \le p+1} (\sum_k L^k N_0^{-j}),$$

it is enough to show the injectivity of

$$f \circ L^j : \operatorname{Im} g \cap M_0^{-j} \to L^{j+1} N_0^{-j-1}.$$

Assume $fL^j g(n) = 0$ for $n \in N^{-j-1}$ such that $g(n) \in M_0^{-j}$. By assumption, there exists $n' \in N^{-j-1}$ such that $g(n') \in M_0^{-j}$ and

$$\Phi_M(g(n), L^j g(n')) \neq 0.$$

But it is equal to $\Phi_N(fL^j g(n), n') = 0$ up to a nonzero constant multiple. This is a contradiction, and the assertion follows.

3. Abelian-Positivity

3.1. *Canonical pairing.* Let A be an abelian variety over a field k. We denote by $A^\vee$ its dual variety, and by $T_l A_{\bar k}$ the Tate module of $A_{\bar k} := A \otimes_k \bar k$ where l is a prime number different form $\operatorname{char} k$. Using the Kummer sequence, we have a canonical isomorphism

$$H^1(A_{\bar k}, \mu_n) = A^\vee(\bar k)_n \,(:= \operatorname{Ker}(n : A^\vee(\bar k) \to A^\vee(\bar k))), \tag{3.1.1}$$

where n is an integer prime to char k. Then, passing to the limit, we get

$$H^1(A_{\bar k}, \mathbb{Z}_l(1)) = T_l A^\vee_{\bar k}. \tag{3.1.2}$$

Since the left-hand side of (3.1.1) is identified with

$$\mathrm{Hom}(A(\bar{k})_n, \mu_n),$$

we get the canonical pairing in [13]

$$A(\bar{k})_n \otimes A^\vee(\bar{k})_n \to \mu_n, \quad T_l A_{\bar{k}} \otimes_{\mathbb{Z}_l} T_l A^\vee_{\bar{k}} \to \mathbb{Z}_l(1). \tag{3.1.3}$$

To get a pairing of $T_l A_{\bar{k}}$, we take a divisor D on A which induces a morphism

$$\varphi_D : A \to A^\vee, \tag{3.1.4}$$

such that $\varphi_D(a) \in A^\vee(\bar{k})$ for $a \in A(\bar{k})$ is given by $T_a^* D_{\bar{k}} - D_{\bar{k}}$, where T_a is the translation by a. See loc. cit. Note that φ_D depends only on the algebraic equivalence class of D, and φ_D is an isogeny if and only if D is ample.

If the pairing is induced by an ample divisor on A, its restriction to $T_l B_{\bar{k}}$ for any abelian subvariety B of A is nondegenerate, because we have the commutative diagram

$$\begin{array}{ccc} B & \longrightarrow & A \\ \downarrow{\scriptstyle \varphi_{D|_B}} & & \downarrow{\scriptstyle \varphi_D} \\ B^\vee & \longleftarrow & A^\vee \end{array} \tag{3.1.5}$$

3.2. *Compatibility of the cycle class.* Let A be an abelian variety over k, X a smooth projective variety over k, and D a divisor on $A \times_k X$ such that its restriction to $\{0\} \times X$ is rationally equivalent to zero. Let P be the Picard variety of X. Then D induces a morphism of abelian varieties

$$\Psi_D : A \to P,$$

such $\Psi_D(a) \in P(\bar{k})$ for $a \in A(\bar{k})$ is defined by the restriction of $D_{\bar{k}}$ to $\{a\} \times X_{\bar{k}}$. See also [18].

Let $cl(D)^{1,1} \in H^1(A_{\bar{k}}, R^1(pr_1)_*\mu_n)$ denote the (1,1)-component of the cycle class of D, where n is an integer prime to $\mathrm{char}\, k$, and pr_1 is the first projection. Assume $\mathrm{NS}(X)$ is torsion-free. Then $R^1(pr_1)_*\mu_n$ is a constant sheaf on $A_{\bar{k}}$ with fiber $H^1(X_{\bar{k}}, \mu_n) = P(\bar{k})_n$, and we get

$$cl(D)^{1,1} \in H^1(A_{\bar{k}}, R^1(pr_1)_*\mu_n) = \mathrm{Hom}(A(\bar{k})_n, P(\bar{k})_n).$$

3.3. Theorem (Deligne)**.** *The induced morphism $\Psi_D : A(\bar{k})_n \to P(\bar{k})_n$ coincides with $-cl(D)^{1,1}$.*

(The proof is essentially the same as in [6].)

3.4. Theorem (Deligne)**.** *Let C be a smooth projective curve over a field k having a k-rational point, and J its Jacobian. Then we have a canonical isomorphism $H^1(C_{\bar{k}}, \mathbb{Z}_l(1)) = T_l J(\bar{k})$ such that the Poincaré duality of $H^1(C_{\bar{k}}, \mathbb{Z}_l(1))$ is identified with the pairing of $T_l J(\bar{k})$ given by the canonical pairing (3.1.3) together with the theta divisor on J.*

Proof. Choosing a k-rational point of C, we have a morphism $f : C \to J$. It is well-known that this induces isomorphisms

$$f^* : H^1(J_{\bar{k}}, \mathbb{Z}_l) \xrightarrow{\sim} H^1(C_{\bar{k}}, \mathbb{Z}_l), \quad f_* : H^1(C_{\bar{k}}, \mathbb{Z}_l) \xrightarrow{\sim} H^{2g-1}(J_{\bar{k}}, \mathbb{Z}_l(g-1)).$$

These are independent of the choice of the k-rational point of C, because a translation on J acts trivially on the cohomology of J. Since f_* and f^* are dual of each other, the pairing on $H^1(C_{\bar{k}},\mathbb{Z}_l)$ corresponds to that on $H^1(J_{\bar{k}},\mathbb{Z}_l)$ given by the Poincaré duality and $f_*\circ f^*$. But the canonical pairing (3.1.3) is also identified with this Poincaré duality.

Let $\Gamma = m^*\Theta - pr_1^*\Theta - pr_2^*\Theta$, where Θ is the theta divisor, and $m : J\times_k J \to J$ is the multiplication. Let $\Gamma' \in \mathrm{CH}^g(J\times_k J)$ be the diagonal of $f(C)\subset J$ so that $f_*\circ f^*$ is identified with Γ'. Then, by (3.3), it is enough to show that the action of the composition of correspondences $\Gamma'\circ\Gamma$ on $H^{2g-1}(J_{\bar{k}},\mathbb{Z}_l)$, or equivalently, on the the Albanese variety of J, is the multiplication by -1. (Note that this implies also the bijectivity of f^* and f_*.)

Let $C^{(j)}$ denote the j-th symmetric power of C. Since $f : C^{(g)} \to J$ is birational [21], there is a nonempty Zariski-open subset U of J such that for $a\in U(\bar{k})$, there exists uniquely $\{c_1,\dots,c_g\}\in C^{(g)}(\bar{k})$ satisfying

$$-a-\sum_{i\neq j} f(c_i) = f(c_j).$$

Since $\Theta = f(C^{(g-1)})$, this implies that $T_a^*\Theta^-\cap f(C) = \{f(c_j)\}$, where $\Theta^- = (-1)^*\Theta$. But $\varphi_\Theta = \varphi_{\Theta^-}$, because the action of the multiplication by -1 on $\mathrm{NS}(J)$ is the identity. So the assertion follows.

3.5. *Remark.* Let Y be a smooth projective variety with an ample divisor class L. Then the canonical pairing on $H^1(Y_{\bar{k}},\mathbb{Q}_l)$, defined by the Poincaré duality and L, is abelian-positive. Indeed, we may assume L is very ample, and take C a smooth closed subvariety of dimension 1 by taking the intersection of general hyperplane section. Then the composition of the restriction and Gysin morphisms

$$H^1(Y_{\bar{k}},\mathbb{Q}_l)\to H^1(C_{\bar{k}},\mathbb{Q}_l)\to H^{2n-1}(Y_{\bar{k}},\mathbb{Q}_l(n-1))$$

coincides with L^{n-1}, where $n=\dim X$. So the pairing on $H^1(Y_{\bar{k}},\mathbb{Q}_l)$ is identified with the restriction of the natural pairing on $H^1(C_{\bar{k}},\mathbb{Q}_l)$ to $H^1(Y_{\bar{k}},\mathbb{Q}_l)$.

4. Standard Conjectures

4.1. Let Y be a smooth projective variety over a field k. We fix an ample divisor class L of X. Then L acts on étale cohomology. Let $n=\dim Y$. By the hard Lefschetz theorem [4], we have

$$L^j : H^{n-j}(Y_{\bar{k}},\mathbb{Q}_l)\xrightarrow{\sim} H^{n+j}(Y_{\bar{k}},\mathbb{Q}_l(j))\ \ (j>0),$$

which implies the Lefschetz decomposition

$$H^j(Y_{\bar{k}},\mathbb{Q}_l) = \oplus_k L^k H^{j-2k}(Y_{\bar{k}},\mathbb{Q}_l(-k))^{\mathrm{prim}}. \tag{4.1.1}$$

This induces a morphism

$$\Lambda : H^j(Y_{\bar{k}},\mathbb{Q}_l)\to H^{j-2}(Y_{\bar{k}},\mathbb{Q}_l(-1)),$$

such that for $m \in H^j(Y_{\bar{k}}, \mathbb{Q}_l)^{\mathrm{prim}}$, we have $\Lambda(L^k m) = L^{k-1}m$ if $k > 0$, and 0 otherwise. The standard conjecture $B(X)$ asserts that Λ is algebraically defined as an action of a correspondence.

We denote by $A^j(Y)$ the coimage of the cycle map

$$cl : \mathrm{CH}^j(Y)_{\mathbb{Q}} \to H^{2j}(Y_{\bar{k}}, \mathbb{Q}_l(j)),$$

so that we have the injective morphism $A^j(Y) \to H^{2j}(Y_{\bar{k}}, \mathbb{Q}_l(j))$. Then the standard conjecture $A(X, L)$ asserts the isomorphism

$$L^{n-2i} : A^i(Y) \xrightarrow{\sim} A^{n-i}(Y) \ \ (i < n/2).$$

This follows from $B(X)$, and implies that the Lefschetz decomposition is compatible with the subspace $A^j(Y)$ so that

$$A^j(Y) = \oplus_k L^k A^{j-k}(Y)^{\mathrm{prim}}. \tag{4.1.2}$$

The standard conjecture $I(X, L)$ asserts that the pairing

$$(-1)^j \langle L^{n-2j} a, b \rangle \quad \text{for } a, b \in A^j(Y)^{\mathrm{prim}}$$

is positive definite for $j \leq n/2$. Note that under the assumption $I(X, L), A(X, L)$ is equivalent to $D(X)$ which asserts the coincidence of the homological and numerical equivalences for the cycles on X. It is known that $D(X)$ is true for divisors (due to Matsusaka), and $I(X, L)$ is true for surfaces (due to Hodge, Segre, Bronowski, Grothendieck). See [10].

By the Lefschetz decomposition, we have an isomorphism

$$^* : H^{n+j}(Y_{\bar{k}}, \mathbb{Q}_l) \xrightarrow{\sim} H^{n-j}(Y_{\bar{k}}, \mathbb{Q}_l(-j))$$

such that for $m \in H^i(Y_{\bar{k}}, \mathbb{Q}_l(-k))^{\mathrm{prim}}$, we have

$$^*(L^k m) = (-1)^{i(i+1)/2} L^{n-i-k} m.$$

Combined with the Poincaré duality, this induces a pairing on $H^\bullet(Y_{\bar{k}}, \mathbb{Q}_l)$ defined by $\langle m, {}^*n \rangle$ for $m, n \in H^j(Y_{\bar{k}}, \mathbb{Q}_l)$.

For a nonzero correspondence $\Gamma \in A^n(Y \times_k Y) \subset \mathrm{End}(H^\bullet(Y_{\bar{k}}, \mathbb{Q}_l))$, we define Γ' to be the transpose with respect to this pairing. Then Γ' is algebraic and

$$\mathrm{Tr}(\Gamma' \circ \Gamma) > 0, \tag{4.1.3}$$

if $B(Y)$ and $I(Y \times_k Y, L \otimes 1 + 1 \otimes L)$ are satisfied. See loc. cit. (This is an analogue of the positivity of the Losati involution [13].)

4.2. *Proof of* (0.4). We have to show that the three conditions of (0.4) implies the assumption of (0.2). The assertion for the case of level ≤ 1 is clear, because ρ and γ comes from a morphism of abelian varieties [20].

So we may assume j even and $H^j(Y_{\bar{k}}^{(i)},\mathbb{Q}_l)^{\rm prim}$ has level 2. Then we may replace $H^j(Y_{\bar{k}}^{(i)},\mathbb{Q}_l)^{\rm prim}$ with $H^2(S_{\bar{k}},\mathbb{Q}_l)^{\rm prim}$ for a smooth projective variety S of dimension 2, and the assertion follows, because $I(X,L)$ and $D(X)$ are true in this case.

4.3. *Proof of* (0.5). We assume the standard conjecture $B(Y), I(Y\times_k Y, L\otimes 1+1\otimes L)$ for $Y=Y^{(i)}$. Then $A^n(Y\times_k Y)$ is a semisimple algebra [10], and we have the projectors to the primitive parts. So the intersection of the primitive part with the image of the Cech restriction or co-Cech Gysin morphism is defined in the sense of Grothendieck motive [11], [12], and corresponds to a projector π of $A^n(Y\times_k Y)$. Let

$$\iota : A^n(Y\times_k Y)\to {\rm End}(H^\bullet(Y_{\bar{k}},\mathbb{Q}_l))$$

denote the canonical injection in (4.1).

We show in general that for a projector π of $A^n(Y\times_k Y)$, the restriction of the pairing to ${\rm Im}\,\iota(\pi)$ is nondegenerate if ${\rm Im}\,\iota(\pi)$ is contained in $H^j(Y_{\bar{k}},\mathbb{Q}_l)^{\rm prim}$. We have to show

$$(4.3.1)\qquad\qquad {\rm Im}\,\iota(\pi)\cap {\rm Ker}\,\iota(\pi')=0.$$

This is clear by (4.1.3) if π corresponds to a simple motive, because this intersection is defined as a motive (and the forgetful functor associating the underlying vector space commutes with Im, Ker and the intersection). In general, we consider a simple submotive of ${\rm Im}\,\pi$. Let π_0 be a projector defining it. Then (4.3.1) holds for π_0, and we get a orthogonal decomposition

$$H^j(Y_{\bar{k}},\mathbb{Q}_l)^{\rm prim}={\rm Im}\,\iota(\pi_0)\oplus({\rm Ker}\,\iota(\pi_0')\cap H^j(Y_{\bar{k}},\mathbb{Q}_l)^{\rm prim}),$$

which is defined motivically, and is compatible with ${\rm Im}\,\iota(\pi)$, i.e.

$${\rm Im}\,\iota(\pi)={\rm Im}\,\iota(\pi_0)\oplus({\rm Ker}\,\iota(\pi_0')\cap{\rm Im}\,\iota(\pi)).$$

Therefore, replacing $H^j(Y_{\bar{k}},\mathbb{Q}_l)^{\rm prim}$ with ${\rm Ker}\,\iota(\pi_0')\cap H^j(Y_{\bar{k}},\mathbb{Q}_l)^{\rm prim}$, we can proceed by induction. This completes the proof of (0.5).

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