BLOCH'S CONJECTURE, DELIGNE COHOMOLOGY AND HIGHER CHOW GROUPS

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(Revised version)

Let X be a smooth projective complex surface. D. Mumford [29] showed that the kernel of the Albanese map $\mathrm{CH}_0(X)^0 \to \mathrm{Alb}(X)$ is 'huge' if $p_g(X) \neq 0$ (where $p_g(X) = \dim \Gamma(X, \Omega_X^2)$). Then S. Bloch [5] conjectured that the condition $p_g(X) = 0$ should conversely imply

(a) The Albanese map $\mathrm{CH}_0(X)^0 \to \mathrm{Alb}(X)$ is injective.

This conjecture was proved in [9] if X is not of general type, but the general case still remains open. It was suggested in [35] that condition (a) in the case $p_g(X) = 0$ would be related closely to the following conditions:

(b) $\mathrm{ind}\lim_U H^3_{\mathcal{D}}(U, \mathbb{Q}(2)) = 0$, where U runs over nonempty open subvarieties of X.

(c) The cycle map $\mathrm{CH}^2(U,1)_{\mathbb{Q}} \to H^3_{\mathcal{D}}(U, \mathbb{Q}(2))$ is surjective for any open subvarieties U of X.

Here $\mathrm{CH}^p(U,n)$ is Bloch's higher Chow group, and $H^i_{\mathcal{D}}(U, \mathbb{Q}(k))$ is $\mathbb{Q}$-Deligne cohomology. Note that condition (a) is equivalent to the injectivity of the Albanese map tensored with $\mathbb{Q}$ due to A. Roitman [32]. The equivalence of (a) and (b) has been proved by L. Barbieri-Viale and V. Srinivas [1] showing the exact sequence (see also [21], [33]):

$$H^3_{\mathcal{D}}(X, \mathbb{Z}(2)) \to \mathrm{ind}\lim_U H^3_{\mathcal{D}}(U, \mathbb{Z}(2)) \to \mathrm{CH}_0(X)^0 \to \mathrm{Alb}(X).$$

By [11], condition (a) implies the decomposability of $\mathrm{CH}^2(X,1)_{\mathbb{Q}}$ (i.e. it is generated by the image of $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{C}^*$). See also [18]. So it is conjectured that $\mathrm{CH}^2(X,1)_{\mathbb{Q}}$ is decomposable if $p_g(X) = 0$. Then it would be interesting whether the (reduced) Abel-Jacobi map

$$\mathrm{CH}^2_{\mathrm{ind}}(X,1)_{\mathbb{Q}} \to \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}, H^2(X, \mathbb{Q})(2))/\mathrm{NS}(X)_{\mathbb{Q}} \otimes_{\mathbb{Z}} \mathbb{C}^*$$

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is injective, where $\mathrm{CH}^2_{\mathrm{ind}}(X,1)_{\mathbb{Q}}$ is the quotient of $\mathrm{CH}^2(X,1)_{\mathbb{Q}}$ by the image of $\mathrm{Pic}(X)_{\mathbb{Q}}\otimes_{\mathbb{Z}}\mathbb{C}^*$. This injectivity is related to [3], 6.1, because the kernel of the map is isomorphic to

$$\mathrm{Coker}(K_2(\mathbb{C}(X))\to \mathrm{ind}\lim_U \mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Q},H^2(U,\mathbb{Q})(2)))$$

by [30], [33], where the morphism of $K_2(\mathbb{C}(X))$ is given by $d\log\wedge d\log$.

We give in this paper an explicit description of the cycle map

$$cl:\mathrm{CH}^p(X,1)\to H^{2p-1}_{\mathcal{D}}(X,\mathbb{Z}(p))$$

for a smooth complex algebraic variety X in terms of currents (3.3) by generalizing a formula of the Abel-Jacobi map for higher cycles due to A. Beilinson [3] and M. Levine [26]. See also [22], [23], [28], etc. Then we construct an example of indecomposable higher cycles such that its image by the Abel-Jacobi map is not contained in the image of $F^1H^2(X,\mathbb{C})$ by using the theory of period integrals (4.2). This is given by a fiber Z of a certain elliptic surface X over an elliptic curve such that Z is a one-point compactification of $\mathbb{G}_m$. We show that there exists a holomorphic 2-form ω on X such that $\int_\Gamma\omega\neq\int_\eta\omega$ for any topological 2-cycles with rational coefficients, where Γ is a topological 2-chain such that $\partial\Gamma$ is a generator of $H_1(Z)$. The argument uses double integrals which are the integrals of period integrals over 1-chains on the elliptic curve. See [22] for an example using the Abel-Jacobi map to real Deligne cohomology.

In this paper, we also generalize the above three conditions to the higher dimensional case, using Deligne cohomology in a generalized sense, and prove their equivalence. The notion of Deligne cohomology was first introduced by Deligne in the case X is smooth and proper. It is a natural generalization of the first two terms of the exponential sequence. See [17]. The generalization to the open or singular case was first done by A. Beilinson [2] and H. Gillet [21], where the weight filtration was not used. Later Beilinson [3] found a more natural generalization from the view point of mixed Hodge theory, which he calls absolute Hodge cohomology, and denotes by $H^i_{\mathcal{H}}(X,A(k))$, $H^i_{\mathcal{H}^p}(X,A(k))$, where A is a subring of $\mathbb{R}$. In this paper we denote them by $H^i_{\mathcal{D}}(X,A(k))'$, $H^i_{\mathcal{D}}(X,A(k))''$ respectively. See (1.1).

Let X be a connected smooth projective variety. We denote by $\mathrm{CH}^p_{\mathrm{hom}}(X)$ the subgroup of $\mathrm{CH}^p(X)$ consisting of homologically equivalent to zero cycles. For an open subvariety U of X, let $j_{U,X}:U\to X$ denote the inclusion morphism, and $j^*_{U,X}$ the pull-back of Deligne cohomology. For an integer p, we prove in this paper the equivalence of the following conditions:

(a) Griffiths' Abel-Jacobi map $\mathrm{CH}^p_{\mathrm{hom}}(X)_{\mathbb{Q}}\to J^p(X)_{\mathbb{Q}}$ is injective.

(b) $\mathrm{ind}\lim_Y H^{2p-1}_{\mathcal{D}}(X\setminus Y,\mathbb{Q}(p))'/\mathrm{Im}\,j^*_{X\setminus Y,X}=0$, where Y runs over closed subvarieties of X with pure codimension $p-1$.

(c) The cycle map $\mathrm{CH}^p(X\setminus Y,1)_{\mathbb{Q}}\to H^{2p-1}_{\mathcal{D}}(X\setminus Y,\mathbb{Q}(p))'/\mathrm{Im}\,j^*_{X\setminus Y,X}$ is surjective for any closed subvariety Y of X with pure codimension $p-1$.

Note that $H^{2p-1}_{\mathcal{D}}(X\setminus Y,\mathbb{Q}(p))'=H^{2p-1}_{\mathcal{D}}(X\setminus Y,\mathbb{Q}(p))''$ if Y has codimension $\geq p-1$, and $H^{2p-1}_{\mathcal{D}}(X\setminus Y,\mathbb{Q}(p))=H^{2p-1}_{\mathcal{D}}(X\setminus Y,\mathbb{Q}(p))'$ if furthermore $p=\dim X$. (In this

case, condition (a) is related with [31].) The pull-back $j^*_{X\setminus Y,X}$ vanishes if $H^{2p-2}(X,\mathbb{Q})$ is generated by algebraic cycle classes and Y is sufficiently large.

For the proof of the equivalence of these conditions, we study the cycle map of the higher Chow group to Deligne homology in the divisor case, and prove

0.1. Theorem. *For a variety Y of pure dimension m, the cycle map induces isomorphisms*

$$cl : \mathrm{CH}^1(Y,n) \xrightarrow{\sim} H^{\mathcal{D}}_{2m-2+n}(Y,\mathbb{Z}(m-1))'' = H^{\mathcal{D}}_{2m-2+n}(Y,\mathbb{Z}(m-1))'$$

for $n = 0, 1$, where $\mathrm{CH}^1(Y,n) = 0$ for $n > 1$. If $m = 1$ or $n = 1$, the isomorphisms hold also for $H^{\mathcal{D}}_{2m-2+n}(Y,\mathbb{Z}(m-1))$.

Indeed, using the compatibility of the cycle map with the localization sequence, this is reduced to the smooth case, where the assertion is more or less well-known. See (2.7). It is also possible to describe $\mathrm{CH}^1_{\mathrm{hom}}(Y)$ by using the normalization of Y. See (2.10). This may be useful for explicit calculation.

By the localization sequence, (0.1) for $n = 0$ immediately implies

0.2. Corollary. *For an integer p, we have a canonical exact sequence*

$$H^{2p-1}_{\mathcal{D}}(X,\mathbb{Z}(p)) \to \mathrm{ind}\lim_Y H^{2p-1}_{\mathcal{D}}(X\setminus Y,\mathbb{Z}(p))'' \to \mathrm{CH}^p_{\mathrm{hom}}(X) \to J^p(X),$$

where the inductive limit is taken over closed subvarieties Y of X with pure codimension $p-1$. Here $H^{2p-1}_{\mathcal{D}}(X\setminus Y,\mathbb{Z}(p))''$ may be replaced by $H^{2p-1}_{\mathcal{D}}(X\setminus Y,\mathbb{Z}(p))'$ in general, and by $H^{2p-1}_{\mathcal{D}}(X\setminus Y,\mathbb{Z}(p))$ if $p = \dim X$.

This expresses the kernel of Griffiths' Abel-Jacobi map in terms of Deligne cohomology (in the generalized sense) using inductive limit, and generalizes the result of [1] mentioned above. Note that $\mathrm{CH}^p(X) = \mathrm{ind}\lim_Y \mathrm{CH}^1(Y)$ where the inductive limit is taken as above.

By (0.2) the equivalence of (a) and (b) is clear. Since the cycle map is compatible with the localization sequence, (0.1) for $n = 0$ implies also the equivalence of (a) and (c). As for the equivalence of (b) and (c), it follows also from (0.1) for $n = 1$. See (2.9).

In Sect. 1, we review some elementary facts from the theories of Deligne cohomology and mixed Hodge Modules which are needed in this paper. In Sect. 2, we recall the definition of Bloch's higher Chow groups and the cycle map, and prove (0.1–2) together with the equivalence of (a), (b) and (c). In Sect. 3 we give an explicit description of the cycle map for higher cycles, and show an example of indecomposable higher cycle in Sect. 4.

In this paper a variety means a separated scheme of finite type over $\mathbb{C}$. All sheaves are considered on the associated analytic space, and $H^j(X^{\mathrm{an}},\mathbb{Q})$ is denoted by $H^j(X,\mathbb{Q})$.

1. Deligne cohomology and mixed Hodge Modules

1.1. *Deligne cohomology* (see [2, 3, 15, 17, 18, 23], etc.) Let X be a smooth variety, and $\bar{X}$ a smooth compactification of X such that $D := \bar{X}\setminus X$ is a divisor with normal crossings. Let $j : X \to \bar{X}$ denote the inclusion morphism. Let A be $\mathbb{Z}$ or $\mathbb{Q}$ for simplicity in this paper.

Then A-Deligne cohomology $H^i_{\mathcal{D}}(X,A(k))$ is defined to be the i-th hypercohomology group of

$$C^\bullet_{\tilde X\langle D\rangle}A(k) := C(\mathbf{R}j_*A_X(k)\oplus\sigma_{\ge k}\Omega^\bullet_{\tilde X}(\log D)\to\mathbf{R}j_*\Omega^\bullet_X)[-1],$$

where $A_X(k)=(2\pi i)^kA_X\subset\mathbb{C}_X$.

For a complex variety X in general, let $K=(K_A,(K_{\mathbb{Q}},W),(K_{\mathbb{C}},F,W))$ be the complex of graded-polarizable mixed A-Hodge structures corresponding by [3, 3.11] to the mixed Hodge complex calculating the cohomology of X which is defined by using a simplicial resolution of a compactification of X as in [14]. Let $\mathrm{MHS}(A)^p$ (resp. $\mathrm{MHS}(A)$) denote the abelian category of graded-polarizable (resp. not necessarily graded-polarizable) mixed A-Hodge structures. Then we define A-Deligne cohomology in the generalized sense by

$$\begin{aligned}
H^i_{\mathcal{D}}(X,A(k)) &= H^i(C(K_A(k)\oplus F^kK_{\mathbb{C}}\to K_{\mathbb{C}})[-1]),\\
H^i_{\mathcal{D}}(X,A(k))' &= \mathbf{R}\mathrm{Hom}_{\mathrm{MHS}(A)}(A,K(k)[i]),\\
H^i_{\mathcal{D}}(X,A(k))'' &= \mathbf{R}\mathrm{Hom}_{\mathrm{MHS}(A)^p}(A,K(k)[i]).
\end{aligned}$$

The last two are called absolute Hodge cohomology in [3], and denoted respectively by $H^i_{\mathcal{H}}(X,A(k))$, $H^i_{\mathcal{H}^p}(X,A(k))$. By loc. cit. we have natural morphisms

$$H^i_{\mathcal{D}}(X,A(k))''\to H^i_{\mathcal{D}}(X,A(k))'\to H^i_{\mathcal{D}}(X,A(k)).$$

Similarly, let $K'=(K'_A,(K'_{\mathbb{Q}},W),(K'_{\mathbb{C}},F,W))$ be the dual of the complex of graded-polarizable mixed A-Hodge structures corresponding by [3, 3.11] to the mixed Hodge complex calculating the cohomology with compact support of X which is defined by using a simplicial resolution of a compactification of X together with that of the divisor at infinity. Then we define A-Deligne homology in the generalized sense by

$$\begin{aligned}
H^{\mathcal{D}}_i(X,A(k)) &= H^{-i}(C(K'_A(-k)\oplus F^{-k}K'_{\mathbb{C}}\to K'_{\mathbb{C}})[-1]),\\
H^{\mathcal{D}}_i(X,A(k))' &= \mathbf{R}\mathrm{Hom}_{\mathrm{MHS}(A)}(A,K'(-k)[-i]),\\
H^{\mathcal{D}}_i(X,A(k))'' &= \mathbf{R}\mathrm{Hom}_{\mathrm{MHS}(A)^p}(A,K'(-k)[-i]).
\end{aligned}$$

We have also natural morphisms

$$H^{\mathcal{D}}_i(X,A(k))''\to H^{\mathcal{D}}_i(X,A(k))'\to H^{\mathcal{D}}_i(X,A(k)).$$

If X is smooth of pure dimension n, then $K=K'(n)[2n]$ so that

$$H^{\mathcal{D}}_i(X,A(k))=H^{2n-i}_{\mathcal{D}}(X,A(n-k)),$$

and similarly for $H^{\mathcal{D}}_i(X,A(k))'$, etc.

For a mixed A-Hodge structure $H=(H_A,(H_{\mathbb{Q}},W),(H_{\mathbb{C}},F,W))$ and an integer k, we define

$$\begin{aligned}
J(H(k)) &= H_{\mathbb{C}}/(H_A(k)_{\mathrm{free}}+F^kH_{\mathbb{C}}),\\
J'(H(k)) &= W_{2k}H_{\mathbb{C}}/((W_{2k}H_A)(k)_{\mathrm{free}}+F^kH_{\mathbb{C}}),\\
J''(H(k)) &= W_{2k-1}H_{\mathbb{C}}/(((W_{2k}H_A)(k)_{\mathrm{free}}+F^kH_{\mathbb{C}})\cap W_{2k-1}H_{\mathbb{C}}),
\end{aligned}$$

where $H_A(k)_{\rm free} = H_A(k)/H_A(k)_{\rm tor}$. We define also

$$F^kW_{2k}H_A(k) = \mathrm{Ker}(H_A(k) \to H_{\mathbb{C}}/F^kW_{2k}H_{\mathbb{C}}).$$

(Similarly for $F^kH_A(k)$.) Then we have short exact sequences

$$\begin{aligned} &0 \to J(H^{i-1}(X,A)(k)) \to H^i_{\mathcal{D}}(X,A(k)) \to F^kH^i(X,A)(k) \to 0,\\ &0 \to J(H^{i-1}(X,A)(k))' \to H^i_{\mathcal{D}}(X,A(k))' \to F^kW_{2k}H^i(X,A)(k) \to 0,\\ &0 \to J(H^{i-1}(X,A)(k))'' \to H^i_{\mathcal{D}}(X,A(k))'' \to F^kW_{2k}H^i(X,A)(k) \to 0, \end{aligned} \tag{1.1.1}$$

because $J'(H(k)) = \mathrm{Ext}^1_{\mathrm{MHS}(A)}(A,H(k))$, $J''(H(k)) = \mathrm{Ext}^1_{\mathrm{MHS}(A)^p}(A,H(k))$ by [12] and the semisimplicity of polarizable Hodge structures. We have also

$$0 \to J(H^{\rm BM}_{i+1}(X,A)(-k))'' \to H^{\mathcal{P}}_i(X,A(k))'' \to F^{-k}W_{-2k}H^{\rm BM}_i(X,A)(-k) \to 0, \tag{1.1.2}$$

etc. Here $H^{\rm BM}_i(X,A)$ denotes Borel-Moore homology.

It is known that $H^i_{\mathcal{D}}(X,A(k))$ and $H^{\mathcal{P}}_i(X,A(k))$ (together with Deligne local cohomology) satisfy the axioms of Bloch-Ogus [10]. See [2], [21], [23], etc. In particular, we have a canonical long exact sequence

$$\to H^{\mathcal{P}}_i(Y,A(k)) \to H^{\mathcal{P}}_i(X,A(k)) \to H^{\mathcal{P}}_i(U,A(k)) \to H^{\mathcal{P}}_{i-1}(Y,A(k)) \to \tag{1.1.3}$$

for a closed subvariety Y of X and $U = X \setminus Y$. (This is functorial for Y, U.) Similar assertions hold for $H^{\mathcal{P}}_i(X,A(k))'$, $H^{\mathcal{P}}_i(X,A(k))''$. (The assertion for $H^{\mathcal{P}}_i(X,\mathbb{Q}(k))''$ follows also from [37] using (1.4) below.)

Remark. Let $K = (K_{\mathbb{Z}},(K_{\mathbb{Q}},W),(K_{\mathbb{C}};F,W);K'_{\mathbb{Q}},(K'_{\mathbb{C}},W);\alpha_1,\alpha_2,\alpha_3,\alpha_4)$ be a polarizable mixed Hodge complex in the sense of [3, 3.9], where

$$\begin{aligned} \alpha_1 : K_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q} \to K'_{\mathbb{Q}}, \quad \alpha_2 : K_{\mathbb{Q}} \to K'_{\mathbb{Q}},\\ \alpha_3 : (K_{\mathbb{Q}},W) \otimes_{\mathbb{Q}} \mathbb{C} \to (K'_{\mathbb{C}},W), \quad \alpha_4 : (K_{\mathbb{C}},W) \to (K'_{\mathbb{C}},W) \end{aligned}$$

are (filtered) quasi-isomorphisms, and $(\mathrm{Gr}^W_iK_{\mathbb{Q}}, \mathrm{Gr}^W_i(K_{\mathbb{C}},F))$ with the isomorphism in the derived category $\mathrm{Gr}^W_iK_{\mathbb{Q}}\otimes_{\mathbb{Q}}\mathbb{C} = \mathrm{Gr}^W_iK_{\mathbb{C}}$ induced by α_3,α_4 is a polarizable Hodge complex of weight i in the sense of [14], i.e. $\mathrm{Gr}^W_i(K_{\mathbb{C}},F)$ is strict and $H^j(\mathrm{Gr}^W_iK_{\mathbb{Q}}, \mathrm{Gr}^W_i(K_{\mathbb{C}},F))$ is a polarizable Hodge structure of weight $i+j$.

Let $\mathrm{Dec}\,W$ be as in loc. cit. By definition, we have a canonical surjection

$$(\mathrm{Dec}\,W)_0K^j_{\mathbb{Q}} \to H^j\mathrm{Gr}^W_{-j}K_{\mathbb{Q}},$$

(and similarly for $K_{\mathbb{C}}, K'_{\mathbb{C}}$). For a $\mathbb{Q}$-Hodge structure $H = (H_{\mathbb{Q}},(H_{\mathbb{C}},F))$ of weight 0, let

$$H^{(0)} = \mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Q},H),$$

which is identified with a subgroup of $H_{\mathbb{Q}}, H_{\mathbb{C}}$. We define the subcomplex $(\mathrm{Dec}\,W)_0^{(0)}K_{\mathbb{Q}}$ of $(\mathrm{Dec}\,W)_0K_{\mathbb{Q}}$ so that $(\mathrm{Dec}\,W)_0^{(0)}K_{\mathbb{Q}}^j$ is the inverse image of $(H^j\mathrm{Gr}_{-j}^WK)^{(0)}$ by the above morphism (and similarly for $(\mathrm{Dec}\,W)_0^{(0)}K_{\mathbb{C}}$, $(\mathrm{Dec}\,W)_0^{(0)}K'_{\mathbb{C}}$).

Let $(W_0H^jK_{\mathbb{Q}})^{(0)}$ be the inverse image of $(\mathrm{Gr}_0^WH^jK_{\mathbb{Q}})^{(0)}$ by the projection of $W_0H^jK_{\mathbb{Q}}$ to $\mathrm{Gr}_0^WH^jK_{\mathbb{Q}}$. Since d_1 of the weight spectral sequence is a morphism of Hodge structures, we can show the canonical quasi-isomorphism

$$\tau_{\le j}(\mathrm{Dec}\,W)_0^{(0)}K_{\mathbb{Q}}/\tau_{<j}(\mathrm{Dec}\,W)_0^{(0)}K_{\mathbb{Q}}\to(W_0H^jK_{\mathbb{Q}})^{(0)},$$

and similarly for $K_{\mathbb{C}}, K'_{\mathbb{C}}$.

We define a complex $\Gamma(D''_{\mathcal{H}}K)$ to be the single complex associated with

$$K_{\mathbb{Z}}\oplus(\mathrm{Dec}\,W)_0^{(0)}K_{\mathbb{Q}}\oplus F^0(\mathrm{Dec}\,W)_0^{(0)}K_{\mathbb{C}}\overset{\phi}{\to}K'_{\mathbb{Q}}\oplus(\mathrm{Dec}\,W)_0^{(0)}K'_{\mathbb{C}},$$

where ϕ is induced by $(\alpha_1-\alpha_2)\oplus(\alpha_3-\alpha_4)$, and the degree of the source of ϕ is zero. Then, by an argument similar to [3], we can show the isomorphism

$$\mathrm{Hom}_{\mathcal{D}''}(\mathbb{Z},K)=H^0\Gamma(D''_{\mathcal{H}}K),\tag{1.1.4}$$

where $\mathcal{D}''$ denotes the category of polarizable mixed Hodge complexes in the sense of [3, 3.9]. So we can define $H^i_{\mathcal{D}}(X,\mathbb{Z}(k))''$ taking a mixed Hodge complex which calculates the cohomology of X as in [14]. Note that (1.1.4) implies the equivalence of categories

$$D^b\mathrm{MHS}(\mathbb{Z})^p\overset{\sim}{\to}\mathcal{D}''$$

in Lemma 3.1 of [3], and that it is easy to show the exact sequence

$$0\to\mathrm{Ext}^1_{\mathrm{MHS}(\mathbb{Z})^p}(\mathbb{Z},H^{-1}K)\to\mathrm{Hom}_{\mathcal{D}''}(\mathbb{Z},K)\to\mathrm{Hom}_{\mathrm{MHS}(\mathbb{Z})}(\mathbb{Z},H^0K)\to0,$$

using the truncation τ.

We can similarly define $\Gamma(D'_{\mathcal{H}}K)$,$\Gamma(D_{\mathcal{H}}K)$ to be the single complex associated with

$$K_{\mathbb{Z}}\oplus(\mathrm{Dec}\,W)_0K_{\mathbb{Q}}\oplus F^0(\mathrm{Dec}\,W)_0K_{\mathbb{C}}\to K'_{\mathbb{Q}}\oplus(\mathrm{Dec}\,W)_0K'_{\mathbb{C}},$$
$$K_{\mathbb{Z}}\oplus K_{\mathbb{Q}}\oplus F^0K_{\mathbb{C}}\to K'_{\mathbb{Q}}\oplus K'_{\mathbb{C}}$$

respectively. They can be defined also for a mixed Hodge complex K in the sense of [3, 3.2]. Using these, we can also define $H^i_{\mathcal{D}}(X,A(k))'$, $H^i_{\mathcal{D}}(X,A(k))$. Note that $\Gamma(D_{\mathcal{H}}K)$ is canonically isomorphic to

$$K_{\mathbb{Z}}\oplus F^0K_{\mathbb{C}}\to K'_{\mathbb{C}},$$

if there is a canonical morphism $\alpha':K_{\mathbb{Z}}\to K_{\mathbb{Q}}$ such that $\alpha_1=\alpha_2\circ\alpha'$.

1.2. Lemma. *The natural morphisms*

$$H^i_{\mathcal{D}}(X,A(k))' \to H^i_{\mathcal{D}}(X,A(k)), \quad H^i_{\mathcal{D}}(X,A(k))'' \to H^i_{\mathcal{D}}(X,A(k))'$$

are isomorphisms if the following conditions are satisfied respectively:

(1.2.1) $H^{i-1}(X,A)$ *and* $H^i(X,A)$ *have weights* $\le 2k$.

(1.2.2) $\mathrm{Gr}^W_{2k}H^{i-1}(X,\mathbb{Q})$ *is isomorphic to a direct sum of* $\mathbb{Q}(-k)$.

We have the corresponding assertion for Deligne homology where $H^{i-1}(X,A)$, $H^i(X,A)$ *and* k *are replaced respectively by* $H^{\mathrm{BM}}_{i+1}(X,A)$, $H^{\mathrm{BM}}_i(X,A)$ *and* $-k$.

Proof. This is clear by (1.1.1).

Remark. Condition (1.2.2) is satisfied for $H^{2p-2}(X\setminus Y,\mathbb{Q})$ with $i=2p-1$ and $k=p$, if X is smooth and Y is a closed subvariety of codimension $\ge p-1$. Indeed, $\mathrm{Gr}^W_{2p}H^{2p-1}_Y(X,\mathbb{Q})$ is a direct sum of $\mathbb{Q}(-p)$. A similar assertion holds also for $\mathrm{Gr}^W_{2-2m}H^{\mathrm{BM}}_{2m-1}(Y,\mathbb{Q})$ if Y is of pure dimension m.

1.3. *Mixed Hodge Modules* (see [34]). For a variety X we denote by $\mathrm{MHM}(X)$ the abelian category of mixed $\mathbb{Q}$-Hodge Modules on X, and $D^b\mathrm{MHM}(X)$ its derived category consisting of bounded complexes of mixed $\mathbb{Q}$-Hodge Modules. There is a natural functor $\mathrm{rat} : D^b\mathrm{MHM}(X) \to D^b_c(X,\mathbb{Q})$ assigning the underlying $\mathbb{Q}$-complexes where $D^b_c(X,\mathbb{Q})$ denotes the full subcategory of $D^b_c(X^{\mathrm{an}},\mathbb{Q})$ consisting of $\mathbb{Q}$-complexes whose cohomology sheaves are algebraically constructible. We denote by $H^i : D^b\mathrm{MHM}(X) \to \mathrm{MHM}(X)$ the usual cohomology functor.

For morphisms f of algebraic varieties we have canonically defined functors $f_*, f_!, f^*, f^!$ between the derived categories of mixed $\mathbb{Q}$-Hodge Modules. They are compatible with the corresponding functors of $\mathbb{Q}$-complexes via the functor rat. For a closed embedding $i : X \to Y$, the direct image i_* will be omitted sometimes in order to simplify the notation, because

(1.3.1) $$i_* : D^b\mathrm{MHM}(X) \to D^b\mathrm{MHM}(Y)$$

is fully faithful.

If $X = \mathrm{Spec}\,\mathbb{C}$ we have naturally an equivalence of categories

(1.3.2) $$\mathrm{MHM}(\mathrm{Spec}\,\mathbb{C}) = \mathrm{MHS}(\mathbb{Q})^p.$$

Here the right-hand side is as in (1.1). So $\mathrm{MHM}(\mathrm{Spec}\,\mathbb{C})$ will be identified with $\mathrm{MHS}(\mathbb{Q})^p$.

We denote by $\mathbb{Q}(j)$ the mixed Hodge structure of type $(-j,-j)$ whose underlying $\mathbb{Q}$-vector space is $(2\pi i)^j\mathbb{Q} \subset \mathbb{C}$. See [14]. For a variety X with structure morphism $a_X : X \to \mathrm{Spec}\,\mathbb{C}$, we define

(1.3.3) $$\mathbb{Q}^H_X(j) = a_X^*\mathbb{Q}(j), \quad \mathbb{D}^H_X(j) = a_X^!\mathbb{Q}(j),$$

so that $\mathbb{D}^H_X(j)$ is the dual of $\mathbb{Q}^H_X(-j)$. We will write $\mathbb{Q}^H_X$ for $\mathbb{Q}^H_X(0)$, and similarly for $\mathbb{D}^H_X$. If X is smooth of pure dimension n, we have a canonical isomorphism

(1.3.4) $$\mathbb{D}^H_X = \mathbb{Q}^H_X(n)[2n].$$

1.4. Proposition. *With the notation of* (1.1) *and* (1.3) *we have canonical isomorphisms*

$$H^i_{\mathcal{D}}(X,\mathbb{Q}(k))'' = \mathrm{Ext}^i(\mathbb{Q},(a_X)_*\mathbb{Q}^H_X(k)),$$
$$H_i^{\mathcal{D}}(X,\mathbb{Q}(k))'' = \mathrm{Ext}^{-i}(\mathbb{Q},(a_X)_*\mathbb{D}^H_X(-k)).$$

Proof. In the case $A=\mathbb{Q}$, we have canonical isomorphisms $K=(a_X)_*\mathbb{Q}^H_X$, $K'=(a_X)_*\mathbb{D}^H_X$ by [38].

1.5. *Remark.* Let X be a reduced variety of pure dimension n, and X_i be the irreducible components of X. Let $\mathrm{Rat}(X)^*=\prod \mathrm{Rat}(X_i)^*$ with $\mathrm{Rat}(X_i)$ the rational function field of X_i. Then by [19, 2.12], [23, 3.1] we have a canonical isomorphism

$$H^{\mathcal{D}}_{2n-1}(X,\mathbb{Z}(n-1)) = \{g\in \mathrm{Rat}(X)^* : \mathrm{div}\ g = 0\}, \tag{1.5.1}$$

where $\mathrm{div}\ g=\sum \mathrm{div}\ g_i$ if $g=(g_i)$ with $g_i\in\mathrm{Rat}(X_i)^*$. (Here (1.2.1–2) are satisfied.)

If X is smooth, this is due to [19, 2.12]. In this case, the left-hand side of (1.5.1) is isomorphic to $\mathrm{Ext}^1(\mathbb{Z}_X,\mathbb{Z}_X(1))$ (where Ext^1 is taken in the category of admissible variation of mixed Hodge structures), and the assertion is related with the theory of 1-motives [14], and is more or less well-known. Indeed, if X is a point, the assertion is verified by calculating the period of the mixed Hodge structure on $H^1(\mathbb{A}^1\setminus\{0\},\{1\}\cup\{x\})$ for $x\in\mathbb{C}\setminus\{0,1\}$, i.e., by using the integral of dt/t on the relative cycle connecting $\{1\}$ and $\{x\}$, where t is the coordinate of $\mathbb{A}^1$.

The general case is due to [23, 3.1]. Let $Y=\mathrm{Sing}\,X$, and $U=X\setminus Y$ with the inclusion morphisms $i:Y\to X, j:U\to X$. Then (1.1.3) gives an exact sequence

$$0\to H^{\mathcal{D}}_{2n-1}(X,\mathbb{Z}(n-1))\to H^{\mathcal{D}}_{2n-1}(U,\mathbb{Z}(n-1))\xrightarrow{\partial} H^{\mathcal{D}}_{2n-2}(Y,\mathbb{Z}(n-1)). \tag{1.5.2}$$

Let Y_i be the irreducible components of Y with dimension $n-1$. Then we have a canonical isomorphism

$$H^{\mathcal{D}}_{2n-2}(Y,\mathbb{Z}(n-1)) = \oplus_i H^{\mathcal{D}}_{2n-2}(Y_i,\mathbb{Z}(n-1)) = \oplus_i \mathbb{Z},$$

and it is enough to show that the morphism ∂ in (1.5.2) is naturally identified with

$$\mathrm{div}\ : \Gamma(U,\mathcal{O}^*_U)\to \oplus_i\mathbb{Z}.$$

2. Higher Chow groups and cycle maps

2.1. *Higher Chow groups* ([6]). Let $\Delta^n=\mathrm{Spec}(\mathbb{C}[t_0,\dots,t_n]/(\sum t_i-1))$. For a subset I of $\{0,\dots,n\}$, let $\Delta^n_I=\{t_i=0\,(i\in I)\}\subset\Delta^n$. It is naturally isomorphic to Δ^m with $m=n-|I|$ (fixing the order of the coordinates), and is called a face of Δ^n. For $0\le i\le n$, we have inclusions $\iota_i:\Delta^{n-1}\to\Delta^n$ such that its image is $\Delta^n_{\{i\}}$.

Let X be an equidimensional variety. Then $X\times\Delta^n_I$ is also called a face of $X\times\Delta^n$. Following Bloch, we define $z^p(X,n)$ to be the free abelian group with generators the

irreducible closed subvarieties of $X \times \Delta^n$ of codimension p, intersecting all the faces of $X \times \Delta^n$ properly. We have face maps

$$\partial_i : z^p(X,n) \to z^p(X,n-1),$$

induced by ι_i. Let $\partial = \sum(-1)^i \partial_i$. Then $\partial^2 = 0$, and $\mathrm{CH}^p(X,n)$ is defined to be $\mathrm{Ker}\,\partial/\mathrm{Im}\,\partial$ which is a subquotient of $z^p(X,n)$. By [6] it is isomorphic to

$$\frac{\bigcap_{0\le i\le n}\mathrm{Ker}(\partial_i : z^p(X,n) \to z^p(X,n-1))}{\partial_{n+1}(\bigcap_{0\le i\le n}\mathrm{Ker}(\partial_i : z^p(X,n+1) \to z^p(X,n)))} \tag{2.1.1}$$

Indeed, let $z^p(X,\bullet)'$ be the subcomplex of $z^p(X,\bullet)$ defined by

$$z^p(X,n)' = \bigcap\nolimits_{0\le i<n}\mathrm{Ker}(\partial_i : z^p(X,n) \to z^p(X,n-1)).$$

Then

$$z^p(X,\bullet)' \to z^p(X,\bullet) \tag{2.1.2}$$

is a quasi-isomorphism. (For this, we can consider first the subcomplex defined by $\mathrm{Ker}\,\partial_0$, using a homotopy given by the zeroth degeneracy, and then proceed inductively.)

2.2. *Functoriality.* Let $f : X \to Y$ be a proper morphism of varieties, and put $r = \dim X - \dim Y$. Then we have the pushforward functor

$$f_* : \mathrm{CH}^p(X,n) \to \mathrm{CH}^{p-r}(Y,n).$$

In fact, for a face map $\iota : \Delta^m \to \Delta^n$, Bloch showed the commutative diagram

$$\begin{array}{ccc} z^p(X,n) & \xrightarrow{\iota^*} & z^p(X,m) \\ {\scriptstyle f_*}\downarrow & & {\scriptstyle f_*}\downarrow \\ z^{p-r}(X,n) & \xrightarrow{\iota^*} & z^{p-r}(X,m) \end{array}$$

As for the pull-back, we have $f^* : \mathrm{CH}^p(Y,n) \to \mathrm{CH}^p(X,n)$ if f is flat. In the case X, Y are quasi-projective and smooth, we have $f^* : \mathrm{CH}^p(Y,n)_{\mathbb{Q}} \to \mathrm{CH}^p(X,n)_{\mathbb{Q}}$ by [27]. Here we have a quasi-isomorphic subcomplex $z^p_f(Y,\bullet)_{\mathbb{Q}}$ of $z^p(Y,\bullet)_{\mathbb{Q}}$ on which the pull-back f^* is naturally defined.

2.3. *Cycle map.* Let X be an equidimensional variety. By [2], [7], [15], etc., we have a cycle map

$$cl : \mathrm{CH}^p(X,n) \to H^{\mathcal{D}}_{2d+n}(X,\mathbb{Q}(d))'', \tag{2.3.1}$$

where $d = \dim X - p$. The target becomes $H^{2p-n}_{\mathcal{D}}(X,\mathbb{Q}(p))$ by if X is smooth. Using mixed Hodge Modules [36], the cycle map (2.3.1) is defined as follows.

Let $S^{n-1}=\bigcup\Delta^n_{\{i\}}\subset\Delta^n, U=\Delta^n\setminus S^{n-1}$ with the inclusion morphisms $i: S^{n-1}\to\Delta^n, j: U\to\Delta^n$. Then

$$(a_{\Delta^n})_*j_!\mathbb{Q}^H_U=\mathbb{Q}^H_{pt}[-n], \tag{2.3.2}$$

where $a_{\Delta^n}:\Delta^n\to pt:=\operatorname{Spec}\mathbb{C}$ is the structure morphism. Let $\zeta=\sum_k n_k[Z_k]\in\bigcap_{0\le i\le n}\operatorname{Ker}\partial_i\subset z^p(X,n)$ (see (2.1)), where Z_k are irreducible closed subvarieties of $X\times\Delta^n$. Let $d'=\dim Z_k=d+n$. Put $Z=\bigcup_k Z_k$. Then the coefficients n_k of Z_k induces a morphism

$$\mathbb{Q}^H_Z\to\oplus_k\operatorname{IC}_{Z_k}\mathbb{Q}^H[-d']\to\mathbb{D}^H_Z(-d')[-2d']\to\mathbb{D}^H_{X\times\Delta^n}(-d')[-2d'], \tag{2.3.3}$$

where $\operatorname{IC}_{Z_k}\mathbb{Q}^H$ denotes the mixed Hodge Module whose underlying perverse sheaf is the intersection complex $\operatorname{IC}_{Z_k}\mathbb{Q}$ [4]. Let $\pi: X\times\Delta^n\to X$ be the first projection, and let j denote also $\mathrm{id}\times j: X\times U\to X\times\Delta^n$ (and the same for i). Then

$$\pi_*j_!\mathbb{D}^H_{X\times U}=\mathbb{D}^H_X(n)[n]$$

by (1.3.4) and (2.3.2). So it is enough to show that (2.3.3) is uniquely lifted to

$$\mathbb{Q}^H_Z\to j_!\mathbb{D}^H_{X\times U}(-d')[-2d'],$$

i.e., the composition of (2.3.3) with

$$\mathbb{D}^H_{X\times\Delta^n}(-d')[-2d']\to i_*i^*\mathbb{D}^H_{X\times\Delta^n}(-d')[-2d']$$

is zero and $\operatorname{Hom}(\mathbb{Q}^H_Z, i_*i^*\mathbb{D}^H_{X\times\Delta^n}(-d')[-2d'-1])=0$. But they follow from the condition on proper intersection together with $\zeta\in\bigcap_{0\le i\le n}\operatorname{Ker}\partial_i$. For the well-definedness of the cycle map, it is enough to show its invariance under a deformation of cycle parametrized by $\mathbb{A}^1$ (using a blow-up of Δ^n).

Remark. The cycle map for $n=0$ is defined with integral coefficients by the composition of

$$\mathbb{Z}\to H^{\mathcal{D}}_{2d}(Z,\mathbb{Z}(d))\to H^{\mathcal{D}}_{2d}(X,\mathbb{Z}(d)) \tag{2.3.4}$$

for $\zeta=\sum_k n_k[Z_k]\in\mathrm{CH}_d(X)$, where $Z=\cup_k Z_k$. See [2], [21], etc. If X is smooth proper, this coincides with Deligne's cycle map, which is defined by the composition of

$$\mathbb{Z}\to H^{2p}_Z(X,\mathbb{Z}(p))\to K(p)[2p]$$

where K is as in (1.1). It induces Griffiths' Abel-Jacobi map

$$\mathrm{CH}^p_{\mathrm{hom}}(X)\to J^p(X):=\operatorname{Ext}^1_{\mathrm{MHS}}(\mathbb{Z},H^{2p-1}(X,\mathbb{Z})(p)). \tag{2.3.5}$$

See [17] (and also [34, (4.5.20)]), etc. Here $J^p(X)$ is Griffiths' intermediate Jacobian by [12]. (This can be defined even if X is not proper.) If $p=\dim X$, (2.3.5) is the Albanese map.

2.4. *Compatibility.* The cycle map (2.3.1) is compatible with f_* for a proper morphism f, and also with f^* for a morphism of smooth quasi-projective varieties $f: X\to Y$ in the case of rational coefficients. Indeed, this is reduced to the case of the usual Chow groups by (2.2) and the construction of the cycle map (2.3), and follows from [37].

2.5. Proposition. *Let X be a quasi-projective variety, and Y a closed subvariety. Assume X, Y are equidimensional. Let $r = \mathrm{codim}_X Y$, and $d = \dim X - p$. Then the cycle map induces a morphism of long exact sequences*

$$\begin{array}{ccccccc} \mathrm{CH}^p(X \setminus Y, n+1) & \longrightarrow & \mathrm{CH}^{p-r}(Y, n) & \longrightarrow & \mathrm{CH}^p(X, n) & \longrightarrow \\ \downarrow & & \downarrow & & \downarrow & \\ H^{\mathcal{D}}_{2d+n+1}(X \setminus Y, \mathbb{Q}(d))'' & \longrightarrow & H^{\mathcal{D}}_{2d+n}(Y, \mathbb{Q}(d))'' & \longrightarrow & H^{\mathcal{D}}_{2d+n}(X, \mathbb{Q}(d))'' & \longrightarrow \end{array}$$

where the first exact sequence is the localization sequence [8] *(choosing the sign appropriately), and the second comes from* (1.1.3).

Proof. The assertion is clear except for the commutativity of the left part of the diagram. Let $U' = \Delta^{n+1} \setminus \cup_{0 \le i \le n} \Delta^{n+1}_{\{i\}}$, and identify $\Delta^{n+1}_{\{n+1\}} \cap U'$ with $U^n := \Delta^n \setminus S^{n-1}$. Let $j : X \setminus Y \to X, j' : U' \to \Delta^{n+1}, j^n : U^n \to \Delta^n$ denote the inclusion morphisms so that we have distinguished triangles

$$\begin{aligned} &\to \mathbb{D}^H_Y \to \mathbb{D}^H_X \to j_* \mathbb{D}^H_{X \setminus Y} \to, \\ &\to j^{n+1}_! \mathbb{D}^H_{U^{n+1}} \to j'_! \mathbb{D}^H_{U'} \to j^n_! \mathbb{D}^H_{U^n}(1)[2] \to, \end{aligned}$$

where the direct images by closed embeddings are omitted to simplify the notation. See (1.3.1).

Let $\zeta \in \mathrm{CH}^p(X \setminus Y, n+1)$. By (2.1.2) and [8], it is represented by $\zeta \in z^p(X \setminus Y, n+1)'$ which is extended to $\zeta' \in z^p(X, n+1)'$ so that its restriction to $X \times \Delta^n$ is $\bar\zeta \in z^p(Y, n)'$. Then the image of ζ by the morphism of the localization sequence is $\bar\zeta$. By definition ζ' gives

$$\xi \in \mathrm{Hom}(\mathbb{Q}^H_Z, j'_! \mathbb{D}^H_{X \times U'}(-d'-1)[-2d'-2])$$

such that its restriction to $(X \setminus Y) \times \Delta^n$ vanishes. (Here $d' = \dim X - p + n$, and j' denotes also $id \times j'$.) So it induces

$$\begin{aligned} \xi' &\in \mathrm{Hom}(\mathbb{Q}^H_Z, j^n_! \mathbb{D}^H_{Y \times U^n}(-d')[-2d']), \\ \xi'' &\in \mathrm{Hom}(\mathbb{Q}^H_Z, j_* j^{n+1}_! \mathbb{D}^H_{(X \setminus Y) \times U^{n+1}}(-d'-1)[-2d'-2]), \end{aligned}$$

using the external product of the above two distinguished triangles. We see that ξ' coincides with the image of $\bar\zeta$ by the cycle map, and the second distinguished triangle induces an isomorphism $(a_{\Delta^n})_* j^n_! \mathbb{D}^H_{U^n}(1)[1] \to (a_{\Delta^{n+1}})_* j^{n+1}_! \mathbb{D}^H_{U^{n+1}}$. So the assertion is reduced to the next lemma. (Here we represent the middle terms of the distinguished triangles by the mapping cone of the morphism of the other terms so that we get short exact sequences as below.)

2.6. Lemma. *Let $\{K^{i,j,\bullet}\}$ be a square diagram of short exact sequences of complexes of an abelian category, i.e. $K^{i,j,k} = 0$ for $|i| > 1$ or $|j| > 1$, and $K^{i-1,j,k} \to K^{i,j,k} \to K^{i+1,j,k}$ is exact (and the same for the index j). Assume $H^{k-1}(K^{1,1,\bullet}) = 0$, and let $\xi \in H^k(K^{0,0,\bullet})$ such that its image in $H^k(K^{1,1,\bullet})$ vanishes. Let $\xi' \in H^k(K^{-1,1,\bullet})$, $\xi'' \in H^k(K^{1,-1,\bullet})$ such*

that the images of ξ, ξ' in $H^k(K^{0,1,\bullet})$ coincide and the images of ξ, ξ'' in $H^k(K^{1,0,\bullet})$ coincide. Then the images of ξ', ξ'' in $H^{k+1}(K^{-1,-1,\bullet})$ coincide up to sign.

Proof. Let ξ denote also an element of $K^{0,0,k}$ representing ξ. We may assume that the image of ξ in $K^{1,1,k}$ vanishes by replacing the representative if necessary. Then the image of ξ in $K^{0,1,k}$ belongs to $K^{-1,1,k}$, and it is ξ'. (Similarly for ξ''.) Let $\tilde\xi'$, $\tilde\xi''$ denote lifts of ξ', ξ'' to $K^{-1,0,k}$, $K^{0,-1,k}$ respectively. Consider $\tilde\xi = \tilde\xi' + \tilde\xi'' - \xi$ in $K^{0,0,k}$. Then its images in $K^{1,0,k}$, $K^{0,1,k}$ vanish and it belongs to $K^{-1,-1,k}$. So the assertion follows from $d\tilde\xi = d\tilde\xi' + d\tilde\xi''$.

2.7. *Proof of* (0.1). It is enough to show the assertion for $H^{\mathcal{D}}_{2m-2+n}(Y,\mathbb{Z}(m-1))''$ by (1.2). We apply (2.5) to Y and a divisor Z on Y containing $\operatorname{Sing} Y$. Let $U = Y \setminus Z$. We have $\mathrm{CH}^0(Z,n) = 0$ for $n > 0$ by (2.1.1), and $\mathrm{CH}^1(U,0) = \mathrm{CH}^1(U)$, $\mathrm{CH}^1(U,1) = \Gamma(U,\mathbb{G}_m)$ and $\mathrm{CH}^1(U,n) = 0$ for $n > 1$ by [6, 6.1]. On the other hand, $H^{\mathcal{D}}_{2m-2+n}(Y,\mathbb{Q}(m-1))'' = 0$ for $n > 1$ by (1.1.2). So the assertion follows from [19, 2.12] and [37, I, (3.4)] if Y is smooth (i.e. if $U = Y$). Note that we have the surjectivity of the cycle map $\mathrm{CH}^1(U) \to H^{\mathcal{D}}_{2m-2}(U,\mathbb{Z}(m-1))''$ in loc. cit, because $H^1(X,\mathbb{Z})$ is torsion-free.

In general, the assertion is reduced to the smooth case by using the cycle map of the localization sequence

$$0 \to \mathrm{CH}^1(Y,1) \to \mathrm{CH}^1(U,1) \to \mathrm{CH}^0(Z) \to \mathrm{CH}^1(Y) \to \mathrm{CH}^1(U) \to 0$$

to the corresponding exact sequence of Deligne homology. Here the cycle map with integral coefficients is not yet defined for $\mathrm{CH}^1(Y,1)$. But it is defined for $\mathrm{CH}^1(U,1)$ in a compatible way with the morphism to $\mathrm{CH}^0(Z)$ by using (1.5.1) and [6, 6.1].

2.8. *Remarks.* (i) In general, the isomorphism $\mathrm{CH}^1(Y) = H^{\mathcal{D}}_{2m-2}(Y,\mathbb{Z}(m-1))$ does not hold (even for a smooth Y). See [37, I, (3.5)]. (In Remark (i) of loc. cit. the assumption of the second statement should be replaced by the condition that $H^2(X,\mathbb{Q}) \cap F^1H^2(X,\mathbb{C})$ is not contained in $W_2H^2(X,\mathbb{Q})$.)

(ii) It is well known (see e.g. [28]) that any element ζ of $\mathrm{CH}^p(X,1)$ can be represented by $\sum_j (Z_j, g_j)$ where Z_j are irreducible (and reduced) subvarieties of X with pure codimension $p-1$ and g_j are rational functions on Z_j such that $\sum_j \operatorname{div} g_j = 0$.

Indeed, let $\zeta = \sum_j n_j[Z'_j] \in \mathrm{CH}^p(X,1)$ where Z'_j are closed irreducible subvarieties of $X \times \mathbb{A}^1$ with codimension $p-1$ such that the restrictions of ζ to $Z \times \{\lambda\}$ for $\lambda = 0$ and 1 are zero as cycles (without any equivalence relation). Let Z_j be the closure of $pr_1(Z'_j)$ if $\dim Z'_j = \dim pr_1(Z'_j)$, and $Z_j = \emptyset$ otherwise, where $pr_1 : X \times \mathbb{A}^1 \to X$ is the first projection. We may view ζ as a cycle on $X \times \mathbb{P}^1$ such that its restrictions to $X \times \{\lambda\}$ for $\lambda = 0$ and ∞ are zero by taking the closure in $X \times \mathbb{P}^1$ and then using an automorphism of $\mathbb{P}^1$ sending $0, 1, \infty$ to $0, \infty, 1$ respectively. Then we have a cycle Z''_j which is the graph of a rational function g_j on Z_j such that $g_j(x)$ is the n-th power of the product of the points of $Z'_j \cap \{x\} \times \mathbb{A}^1$ (using the multiplicative structure of $\mathbb{A}^1$) for a general $x \in Z_j$. We can verify that ζ is equivalent to $\sum_j [Z''_j]$ which is often denoted by $\sum_j (Z_j, g_j)$ (or $\sum_j Z_j \otimes g_j$).

(iii) If we assume $g_j = \text{const}$ in Remark (ii) above, we get a natural morphism

$$\mathrm{CH}^{p-1}(X) \otimes \mathbb{C}^* \to \mathrm{CH}^p(X,1).$$

Its image is denoted by $\mathrm{CH}^{p-1}_{\mathrm{dec}}(X,1)$, and is called the subgroup of *decomposable* higher cycles. We put

$$\mathrm{CH}^{p-1}_{\mathrm{ind}}(X,1)_{\mathbb{Q}} = \mathrm{CH}^{p-1}(X,1)_{\mathbb{Q}}/\mathrm{CH}^{p-1}_{\mathrm{dec}}(X,1)_{\mathbb{Q}}.$$

The image of $\mathrm{CH}^{p-1}_{\mathrm{dec}}(X,1)_{\mathbb{Q}}$ by the Abel-Jacobi map (see (3.4.3) below) is contained in

$$J(N^{p-1}H^{2p-2}(X,\mathbb{Q})(p))\,(= N^{p-1}H^{2p-2}(X,\mathbb{Q})\otimes\mathbb{C}^*),$$

where N is the coniveau filtration. See [22], [28], etc.

2.9. *Proof of the equivalence of* (a), (b) *and* (c). The equivalence of (b) and (c) in the introduction follows from (0.1) for $n=1$ and (2.5), which we apply to X and a closed subvariety of pure codimension $p-1$. The equivalence of (a) and (c) follows from (2.5) and (0.1) for $n=0$.

2.10. *Relation with the normalization.* We can express the subgroup $\mathrm{CH}^1_{\mathrm{hom}}(Y)$ of $\mathrm{CH}^1(Y)$ consisting of Borel-Moore homologically equivalent to zero cycles by using the normalization of Y. This may be useful for explicit calculation.

Let Y be a connected variety of pure dimension m with $Y_k\,(1\le k\le r)$ the irreducible components of Y. Let $\tilde{Y}$ be the disjoint union of the normalizations $\tilde{Y}_k$ of Y_k with $\pi:\tilde{Y}\to Y$ the natural morphism. Let $D=\{y\in Y:|\pi^{-1}(y)|>1\}$. We assume $\tilde{Y}$ is smooth, D is a smooth closed subvariety of Y with pure codimension one, and $\pi_*\mathbb{Z}_{\tilde{Y}}|_D$ is a local system. (We may assume these because $\mathrm{CH}^1(Y)$ and $H^{\mathcal{D}}_{2m-2}(Y,\mathbb{Z}(m-1))$ do not change by deleting a closed subvariety of codimension >1.)

Let $\tilde{D}=\pi^{-1}(D)$, and $\tilde{D}_i\,(i\in I), D_j\,(j\in J)$ be connected components of $\tilde{D}, D$. Put

$$I_j=\{i\in I:\tilde{D}_i\subset\pi^{-1}(D_j)\},\quad I(k)=\{i\in I:\tilde{D}_i\subset\tilde{Y}_k\}.$$

We define $\mathcal{E}_j=\mathrm{Ker}(\mathrm{Tr}:\oplus_{i\in I_j}\pi_*\mathbb{Z}_{\tilde{D}_i}\to\mathbb{Z}_{D_j})$, $E_j=H^0(D_j,\mathcal{E}_j)$, and $E=\oplus_j E_j$. Let d_i be the degree of $\tilde{D}_i$ over $\pi(\tilde{D}_i)$. Then E_j is naturally identified with

$$\{a_i\in\mathbb{Z}\,(i\in I_j):\sum_{i\in I_j}d_ia_i=0\}.$$

Let $E'=\oplus_{j\in J}H^1(D_j,\mathcal{E}_j)$. (This may have torsion which is related to the cokernel of $\mathrm{CH}^1(\tilde{Y})\to\mathrm{CH}^1(Y)$.) Then we have an exact sequence

$$\begin{aligned}0&\to H^{\mathrm{BM}}_{2m-1}(\tilde{Y},\mathbb{Z})\to H^{\mathrm{BM}}_{2m-1}(Y,\mathbb{Z})\to E(m-1)\\&\xrightarrow{\gamma} H^{\mathrm{BM}}_{2m-2}(\tilde{Y},\mathbb{Z})\to H^{\mathrm{BM}}_{2m-2}(Y,\mathbb{Z})\to E'(m-1),\end{aligned}\tag{2.10.1}$$

where γ is defined by $(a_i)\to\sum_i a_i cl([\tilde{D}_i])$. Here $cl([\tilde{D}_i])$ denotes the cycle class.

We define $E^0=\mathrm{Ker}\,\gamma\subset E$ so that we get

$$0\to H^1(\tilde{Y},\mathbb{Z})(1)\to H^{\mathrm{BM}}_{2m-1}(Y,\mathbb{Z})(1-m)\to E^0\to 0,\tag{2.10.2}$$

The associated extension class is denoted by $e \in \mathrm{Ext}^1_{\mathrm{MHS}}(E^0, H^1(\tilde{Y},\mathbb{Z})(1))$. The cycle map induces an isomorphism of exact sequences

$$\begin{array}{ccccccc} E^0 & \longrightarrow & \mathrm{CH}^1_{\mathrm{hom}}(\tilde{Y}) & \longrightarrow & \mathrm{CH}^1_{\mathrm{hom}}(Y) & \longrightarrow & 0 \\ \| & & \| & & \| & & \\ E^0 & \longrightarrow & J^1(\tilde{Y}) & \longrightarrow & J^1(Y)^{\mathrm{BM}} & \longrightarrow & 0, \end{array} \tag{2.10.3}$$

where $J^1(Y)^{\mathrm{BM}} := \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}, H^{\mathrm{BM}}_{2m-1}(Y,\mathbb{Z})(1-m))$ and $J^1(\tilde{Y})$ is as in (2.3.5) (and is a quotient of the Jacobian of a smooth compactification of $\tilde{Y}$).

Indeed, let $\mathrm{CH}^1(Y)' = \mathrm{Im}(\mathrm{CH}^1(\tilde{Y}) \to \mathrm{CH}^1(Y))$, $\mathrm{CH}^1_{\mathrm{hom}}(Y)' = \mathrm{CH}^1(Y)' \cap \mathrm{CH}^1_{\mathrm{hom}}(Y)$. Then, for the exactness of the first row, it is sufficient to show

$$\mathrm{CH}^1_{\mathrm{hom}}(Y)' = \mathrm{CH}^1_{\mathrm{hom}}(Y). \tag{2.10.4}$$

This is reduced to the case where the cycle is supported on $\mathrm{Sing}\, Y$, and follows from the localization sequence for Borel-Moore homology. The second row is induced by (2.10.2), and we can show that for $u \in \mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Z}, E^0)$,

$$e \circ u \in \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}, H^1(\tilde{Y},\mathbb{Z})(1)) \tag{2.10.5}$$

coincides with the image of $\sum_i a_i[\tilde{D}_i]$ by the Abel-Jacobi map, where $(a_i) = u(1) \in E^0$. This is verified by using a natural morphism of (2.10.2) to

$$0 \to H^1(\tilde{Y},\mathbb{Z})(1) \to H^1(\tilde{Y} \setminus \tilde{D},\mathbb{Z})(1) \to H^0(\tilde{D},\mathbb{Z}).$$

3. Abel-Jacobi map for higher cycles

3.1. *Currents.* For a complex manifold X of dimension n, let $\mathcal{C}^\bullet(X)$ denote the complex of currents on X which has the Hodge filtration F as usual. Here we normalize $\mathcal{C}^\bullet(X)$ so that $\mathcal{C}^j(X) = 0$ for $i > 0$ or $i < -2n$. It has a structure of double complex such that the Hodge filtration is given by the first degree. We have a natural morphism

$$\mathcal{E}^\bullet(X)(n)[2n] \to \mathcal{C}^\bullet(X),$$

where $\mathcal{E}^i(X)$ denotes the vector space of C^∞ i-forms on X. Let $S^\bullet(X)$ denote the complex of locally finite C^∞ chains on X where $S^{-j}(X)$ consists of locally finite j-chains so that $H^j(S^\bullet(X)) = H^{j+2n}(X,\mathbb{Z})(n)$. There is a natural morphism

$$\iota : S^\bullet(X) \to \mathcal{C}^\bullet(X). \tag{3.1.1}$$

Here the differential d of $S^\bullet(X)$ is defined in a compatible way with that of $\mathcal{C}^\bullet(X)$. So it differs from the usual boundary map ∂ by the sign $-(-1)^{\deg}$ due to the Stokes theorem. Note that the differential of a current Φ is defined by

$$(d\Phi)(\omega) + (-1)^{\deg \Phi}\Phi(d\omega) = 0 \tag{3.1.2}$$

for C^∞ forms ω with compact supports. For a smooth complex algebraic variety X, we will denote $S^\bullet(X^{\mathrm{an}})$, $\mathcal{C}^\bullet(X^{\mathrm{an}})$ by $S^\bullet(X)$, $\mathcal{C}^\bullet(X)$ to simplify the notation.

Let $\bar{X}$ be a smooth proper complex algebraic variety of dimension n, and D a divisor on $\bar{X}$ with normal crossings such that each irreducible component D_j is smooth. Put $X = \bar{X} \setminus D$. Let $\tilde{D}^{(j)}$ be the disjoint union of the intersections of j irreducible components as in [14, II]. Then we have naturally a double complex

$$\to F^k\mathcal{C}^\bullet(\tilde{D}^{(j+1)}) \to F^k\mathcal{C}^\bullet(\tilde{D}^{(j)}) \to \cdots \to F^k\mathcal{C}^\bullet(\tilde{D}^{(0)}) \to 0$$

by the dual construction of [14, III] (using the push-down of currents instead of the pull-back of forms), and the associated single complex will be denoted by $F^k\mathcal{C}^\bullet(\bar{X}\langle D\rangle)$.

We have the weight filtration W on $\mathcal{C}^\bullet(\bar{X}\langle D\rangle)$ such that

$$\mathrm{Gr}_j^W\mathcal{C}^\bullet(\bar{X}\langle D\rangle) = \mathcal{C}^\bullet(\tilde{D}^{(j)})[j].$$

We define similarly $S^\bullet(\bar{X}\langle D\rangle)$ with the filtration W such that

$$\mathrm{Gr}_j^W S^\bullet(\bar{X}\langle D\rangle) = S^\bullet(\tilde{D}^{(j)})[j].$$

Then we get the polarizable mixed Hodge complex $K(\bar{X}\langle D\rangle)$ defined by

$$(S^\bullet(X), (S^\bullet(\bar{X}\langle D\rangle)_{\mathbb{Q}}, W), (\mathcal{C}^\bullet(\bar{X}\langle D\rangle); F, W); S^\bullet(X)_{\mathbb{Q}}, (\mathcal{C}^\bullet(\bar{X}\langle D\rangle), W)),$$

which calculates the Borel-Moore homology of X. By (1.1.4) we have a canonical isomorphism

$$H^i_{\mathcal{D}}(X, \mathbb{Z}(k)) = H^{i-2n}(\Gamma(D''_{\mathcal{H}}(K(\bar{X}\langle D\rangle)(k-n)))). \tag{3.1.3}$$

See also [23], [24].

3.2. *Cycle class.* With the above notation, let $\zeta = \sum_j (Z_j, g_j) \in \mathrm{CH}^p(X, 1)$ as in Remark (ii) of (2.8). Put $d = n - p$. Let γ_j be the closure of the inverse image of

$$\{z \in \mathbb{C} \mid \mathrm{Re}\, z > 0, \mathrm{Im}\, z = 0\} \subset \mathbb{P}^1.$$

Then $\partial\gamma_j = -\mathrm{div}\, g_j$, and $\gamma := \sum_j \gamma_j$ is a (topological) cycle on $Z := \cup_j Z_j$. (Note that the orientation of γ depends on the choice of $i = \sqrt{-1}$.) We take a triangulation of γ so that it belongs to $S^{-2d-1}(X)$.

Let $\tilde{Z}_j \to Z_j$ be a resolution of singularities such that the divisor of the pull-back $\tilde{g}_j$ of g_j to $\tilde{Z}_j$ has normal crossings. Let $\pi_j : \tilde{Z}_j \to X$ denote its composition with the inclusion $i_j : Z_j \to X$. Then we have the push-down of currents $(\pi_j)_*$.

We choose a branch of $\log \tilde{g}_j$ on $\tilde{Z}_j \setminus \tilde{\gamma}_j$ where $\tilde{\gamma}_j$ is the pull-back of γ_j to $\tilde{Z}_j$. Then $\log \tilde{g}_j$ is a locally integrable function on $\tilde{Z}_j$, and defines a current on $\tilde{Z}_j$ such that

$$d(\log \tilde{g}_j) = \tilde{g}_j^{-1} d\tilde{g}_j + (2\pi i)\iota\tilde{\gamma}_j. \tag{3.2.1}$$

We can verify (3.2.1) by using the integration on the inverse image of

$$\{z \in \mathbb{C} \,|\, \min\{|\arg z|, |z|, |z|^{-1}\} > \epsilon\}$$

for $\epsilon \to 0$ as in [22]. Note that $\tilde{g}_j^{-1} d\tilde{g}_j$ is a form with locally integrable coefficients on $\tilde{Z}_j$ and $\sum_j (\pi_j)_*(\tilde{g}_j^{-1} d\tilde{g}_j)$ is a closed current, because

$$(2\pi i)^{-1} d(\tilde{g}_j^{-1} d\tilde{g}_j) = d\iota(\operatorname{div} \tilde{g}_j). \tag{3.2.2}$$

We define

$$(i_j)_*(g_j^{-1} dg_j) = (\pi_j)_*(\tilde{g}_j^{-1} d\tilde{g}_j), \quad gdg = \sum_j (i_j)_*(g_j^{-1} dg_j) \in F^{-d}\mathcal{C}^{-2d-1}(\bar{X}),$$

$$(i_j)_*(\log g_j) = (\pi_j)_*(\log \tilde{g}_j) \quad \log g = \sum_j (i_j)_*(\log g_j) \in \mathcal{C}^{-2d-2}(\bar{X}),$$

These are independent of the choice of $\tilde{Z}_j$.

Let $\bar{Z}_j$ be the closure of Z_j in $\bar{X}$. Then g_j is identified with a rational function $\bar{g}_j$ on $\bar{Z}_j$, and we can define $\bar{g}^{-1} d\bar{g} = \sum_j (\bar{i}_j)_*(\bar{g}_j^{-1} d\bar{g}_j)$, etc. similarly, where $\bar{i}_j : \bar{Z}_j \to \bar{X}$ is the inclusion morphism.

Let $\operatorname{div} \bar{g} = \sum_j \operatorname{div} \bar{g}_j$. This is supported on D, and there is a cycle $(\operatorname{div} \bar{g})^{(1)}$ on $\tilde{D}^{(1)}$ such that its image in $\bar{X}$ coincides with $\operatorname{div} \bar{g}$. Taking a triangulation, $(\operatorname{div} \bar{g})^{(1)}$ can be viewed as an element of $S^{-2d}(\tilde{D}^{(1)})$. So we get

$$(g^{-1} dg)^\wedge := (\overline{g}^{-1} d\overline{g}, -\iota(\operatorname{div} \overline{g})^{(1)}) \in F^{-d}\mathcal{C}^{-2d-1}(\bar{X}\langle D\rangle)$$

such that $d(g^{-1} dg)^\wedge = 0$. Let $\bar{\gamma}$ be the closure of γ in $\bar{X}$. Since $\partial\bar{\gamma} = -\operatorname{div} \overline{g}$, we get

$$\gamma^\wedge := (\bar{\gamma}, (\operatorname{div} \overline{g})^{(1)}) \in S^{-2d-1}(\bar{X}\langle D\rangle)$$

such that $d\iota\gamma^\wedge = 0$. Then, by (3.2.1), $d(\log g) \in \mathcal{C}^{-2d-1}(\bar{X}\langle D\rangle)$ coincides with the sum of the images of $(2\pi i)\gamma^\wedge$ and $(g^{-1} dg)^\wedge$ in $\mathcal{C}^{-2d-1}(\bar{X}\langle D\rangle)$.

3.3. Theorem. *With the above notation, $cl(\zeta) \in H_{\mathcal{D}}^{2p-1}(X, \mathbb{Z}(p))''$ corresponds by the isomorphism* (3.1.3) *to*

$$(-(2\pi i)^{-d}\gamma, -(2\pi i)^{-d}\gamma^\wedge, (2\pi i)^{-d-1}(g^{-1} dg)^\wedge; 0, (2\pi i)^{-d-1} \log g) \tag{3.3.1}$$

in $S^{-2d-1}(X)(-d) \oplus S^{-2d-1}(\bar{X}\langle D\rangle)_{\mathbb{Q}}(-d) \oplus F^{-d}\mathcal{C}^{-2d-1}(\bar{X}\langle D\rangle) \oplus S^{-2d-2}(X)_{\mathbb{Q}}(-d) \oplus \mathcal{C}^{-2d-2}(\bar{X}\langle D\rangle)$.

Proof. We first verify that (3.3.1) belongs to $H^{-2d-1}(\Gamma(D''_{\mathcal{H}}(K(\bar{X}\langle D\rangle)(-d))))$. See Remark after (1.1). Let $\bar{Z}$ be the closure of Z in $\bar{X}$, and

$$\Sigma = \cup_j \operatorname{supp} \operatorname{div} \bar{g}_j \cup \operatorname{Sing} \bar{Z}.$$

Then the canonical morphism

$$H_{\mathcal{D}}^{2p-1}(X,\mathbb{Z}(p))'' \to H_{\mathcal{D}}^{2p-1}(X\setminus\Sigma,\mathbb{Z}(p))'' \tag{3.3.2}$$

is injective by the localization sequence. So we may replace X with $X\setminus\Sigma$, and assume that Z is smooth, and hence irreducible.

Then the isomorphism (3.1.3) is compatible with the push-forward by the closed embedding $Z\to X$. Here we may assume that the closure $\bar{Z}$ of Z in $\bar{X}$ is also a good smooth compactification (i.e. $\bar{Z}\setminus Z$ is a divisor with normal crossings) by taking further blowing-ups if necessary, and that every irreducible components of $\bar{Z}\setminus Z$ is contained by only one irreducible component of $\bar{X}\setminus X$.

So the assertion is reduced to the case $X=Z$, $p=1$. Then we have isomorphisms

$$H_{\mathcal{D}}^1(X,\mathbb{Z})'' = H_{\mathcal{D}}^1(X,\mathbb{Z}) = H^1(\bar{X},C^\bullet_{\bar{X}\langle D\rangle}\mathbb{Z}(1)).$$

See (1.1-2). By definition of the cycle map (see also (1.5)), we have the commutative diagram

$$\begin{array}{ccc} \Gamma(X,\mathbb{G}_m) & \xrightarrow{\sim} & H^1(\bar{X},C^\bullet_{\bar{X}\langle D\rangle}\mathbb{Z}(1)) \\ \downarrow & & \downarrow \\ \Gamma(X^{\mathrm{an}},\mathcal{O}^*_{X^{\mathrm{an}}}) & \xrightarrow{\sim} & H^1(X,C^\bullet_X\mathbb{Z}(1)), \end{array} \tag{3.3.3}$$

where the isomorphism on the bottom row is induced by the canonical quasi-isomorphism

$$\mathcal{O}^*_{X^{\mathrm{an}}} = C(\mathbb{Z}_{X^{\mathrm{an}}}(1)\to\mathcal{O}_{X^{\mathrm{an}}}).$$

Here we may replace $H^1(X,C^\bullet_X\mathbb{Z}(1))$ with the cohomology of the single complex associated with

$$S^\bullet(X)(-d)\to\mathcal{C}^\bullet(X)/F^{-d}\mathcal{C}^\bullet(X).$$

Then it is enough to show that the image of $g\in\Gamma(X^{\mathrm{an}},\mathcal{O}^*_{X^{\mathrm{an}}})$ in this cohomology is represented by

$$(-(2\pi i)^{-d}\gamma,(2\pi i)^{-d-1}\log g).$$

But this can be verified by using a Cech resolution together with the delta functions supported on faces of a triangulation of X^{an} compatible with γ. (See [20] for the notion of integral current.)

Indeed, let $\mathcal{U}=\{U_i\}_{i\in\Lambda}$ be an open covering of X^{an} such that U_i are simply connected. We will denote by $C^\bullet_{\mathcal{U}}\mathcal{F}$ the Cech complex of a sheaf $\mathcal{F}$ associated with the covering $\mathcal{U}$, where

$$C^i_{\mathcal{U}}\mathcal{F} = \oplus_{|I|=i+1}\Gamma(U_I,\mathcal{F}) \quad \text{with } U_I=\cap_{i\in I}U_i \text{ for } I\subset\Lambda.$$

Then the element of the cohomology of the single complex associated with

$$C^\bullet_{\mathcal{U}}\mathbb{Z}_{X^{\mathrm{an}}}(1)\to C^\bullet_{\mathcal{U}}\mathcal{O}_{X^{\mathrm{an}}}$$

(where the first term has degree one) corresponding to $g \in \Gamma(X^{\rm an}, O^*_{X^{\rm an}})$ is given by

$$(\{\log(g|_{U_i}) - \log(g|_{U_j})\}_{i,j}, \{\log(g|_{U_i})\}_i) \in C^1_{\mathcal{U}}\mathbb{Z}_{X^{\rm an}}(1) \oplus C^0_{\mathcal{U}}\mathcal{O}_{X^{\rm an}}.$$

Since $\{(2\pi i)^{-d-1}((\log g)|_{U_i}) - \log(g|_{U_i}))\}_i \in C^0_{\mathcal{U}}\mathcal{C}^{-2d-2}(X)$ belongs to the image of $C^0_{\mathcal{U}}S^{-2d-2}(X)(-d)$, we get the assertion, using the triple complex

$$C^\bullet_{\mathcal{U}}S^\bullet(X)(-d) \to C^\bullet_{\mathcal{U}}(\mathcal{C}^\bullet(X)/F^{-d}\mathcal{C}^\bullet(X)).$$

This completes the proof of (3.3).

3.4. *Remark.* Assume $X = \bar{X}$. Then (3.3) implies that the image of ζ by the morphism

$$\mathrm{CH}^p(X,1) \to \mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Z}, H^{2p-1}(X,\mathbb{Z})(p)) \tag{3.4.1}$$

induced by the cycle map (2.3.1) is represented by

$$-(2\pi i)^{-d}\gamma \quad \text{and} \quad (2\pi i)^{-d-1}\sum_j (i_j)_*(g_j^{-1}dg_j).$$

Let $\mathrm{CH}^p_{\rm hom}(X,1)$ be the kernel of (3.4.1). Then cycle map (2.3.1) induces the Abel-Jacobi map for higher cycles

$$\mathrm{CH}^p_{\rm hom}(X,1) \to J(H^{2p-2}(X,\mathbb{Z})(p)) = \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}, H^{2p-2}(X,\mathbb{Z})(p)), \tag{3.4.2}$$

in the notation of (1.1). Note that $\mathrm{CH}^p(X,1)/\mathrm{CH}^p_{\rm hom}(X,1)$ is finite because the target of (3.4.1) is torsion. So (3.4.2) induces the Abel-Jacobi map for higher cycles

$$\mathrm{CH}^p(X,1)_{\mathbb{Q}} \to J(H^{2p-2}(X,\mathbb{Q})(p)). \tag{3.4.3}$$

Let $\zeta \in \mathrm{CH}^p_{\rm hom}(X,1)$. Then there exist a C^∞ chain Γ on X and a current Ξ on X such that

$$\partial\Gamma = \gamma, \quad d\Xi = \sum_j (i_j)_*(g_j^{-1}dg_j).$$

Here we may assume that Ξ is a sum of currents of type (i,j) with $i \ge -d$ by the strictness of the Hodge filtration. Then by (3.3) the image of ζ by the Abel-Jacobi map (3.4.3) is represented by

$$\Phi_\zeta = (2\pi i)^{-d-1}(\sum_j (i_j)_* \log g_j + (2\pi i)\iota\Gamma - \Xi). \tag{3.4.4}$$

(Note that $d\iota\Gamma = -\iota\gamma$ by Stokes.) Here it is enough to consider $\Phi_\zeta(\omega)$ for C^∞ forms ω with compact supports which are direct sums of forms of type (i,j) with $i \ge d+1$, because the dual of $H^{2p-2}(X,\mathbb{C})/F^pH^{2p-2}(X,\mathbb{C})$ is $F^{d+1}H^{2d+2}(X,\mathbb{C})$. Then Ξ can be neglected, and we get the Abel-Jacobi map in [3], [26]. See also [22], [23], etc.

4. Construction of indecomposable higher cycles

4.1. *Elliptic surface.* Let $f : \mathbb{A}^2 \to \mathbb{A}^1$ be a morphism defined by the polynomial $f = y^2 - x^2(x+1)$. Let $\bar{f} : Y \to S = \mathbb{P}^1$ be a desingularization of its natural compactification defined by taking the closure of the graph of f in $\mathbb{P}^2 \times \mathbb{P}^1$. Then it has three singular fibers Y_0, Y_a, Y_∞ where $a = -4/27$, and Y_0, Y_a are one-point compactifications of $\mathbb{G}_m$.

Let $\pi : \tilde{S} \to S$ be a double cover with four branch points such that one of them is $\infty \in S$ and the other three points P_i $(i = 1, 2, 3)$ are general. Let $\alpha_i \in \mathbb{C} = \mathbb{A}^1(\mathbb{C})$ corresponding to P_i. Let

$$T = \{(\alpha_1, \alpha_2, \alpha_3) \in (\mathbb{C} \setminus [a, +\infty[)^3 \mid \alpha_i \neq \alpha_j \text{ for } i \neq j\}, \tag{4.1.1}$$

where $[a, +\infty[= \{z \in \mathbb{C} \mid \mathrm{Re}\, z \geq a, \mathrm{Im}\, z = 0\}$.

Let $\tilde{f} : X \to \tilde{S}$ be a desingularization of the base change of $Y \to S$ by π. Let Z be the fiber of $\tilde{f}$ at a point over $0 \in S$. It has a rational function g defined by the canonical coordinate of $\mathbb{G}_m$, and (Z, g) is a higher cycle.

4.2. Proposition. *Assume $(\alpha_1, \alpha_2, \alpha_3)$ is very general (i.e. it is in the complement of the image of a union of countablly many closed analytic subsets of a universal covering of T). Then the image of (Z, g) in $J(H^2(X, \mathbb{Q})(p))$ by (3.4.3) does not belongs to the image of $F^1H^2(X, \mathbb{C})$. In particular, (Z, g) is indecomposable.*

Proof. Let Γ' be the topological cycle defined by the compact connected component of $\{(x, y) \in \mathbb{R}^2 \mid f(x, y) \leq 0\}$. Let Γ be the connected component of the inverse image of Γ' to X such that $\partial\Gamma$ is contained in Z. Then it is enough to show that there exists a holomorphic 2-form $\tilde{\omega}$ on X such that

$$\int_\Gamma \tilde{\omega} \neq \int_\eta \tilde{\omega} \tag{4.2.1}$$

for any $\eta \in H_2(X, \mathbb{Q})$, because the dual of $H^2(X, \mathbb{C})/F^1H^2(X, \mathbb{C})$ is $F^2H^2(X, \mathbb{C})$ and the pull-back of a holomorphic 2-form to the normalization of Z vanishes.

Let $\omega = -2dx \wedge dy/f^*dt\,(= y^{-1}dx)$ where t is the standard coordinate of $\mathbb{A}^1$. Let $S'' = S \setminus \{0, a, \infty\}$, $Y'' = \bar{f}^{-1}(S'')$. Then ω gives

$$\omega \in \bar{f}_*\omega_{Y''/S''} = F^1H^1_{\mathrm{DR}}(Y''/S'')$$

as is well-known. Furthermore, ω satisfies the Gauss hypergeometric differential equation (via the Gauss-Manin connection)

$$t(t + 4/27)\partial_t^2\omega + (2t + 4/27)\partial_t\omega + (5/36)\omega = 0. \tag{4.2.2}$$

See (4.3) below. Note that this differential equation is also satisfied by the period integrals $\int_{\gamma_t} \omega$ for multivalued horizontal families of cycles $\{\gamma_t\}$.

Since the roots of the indicial equation [13] at $0, a, \infty$ are respectively $\{0, 0\}$, $\{0, 0\}$ and $\{1/6, 5/6\}$, we see that ω can be extended to $\bar{f}_*\omega_{Y/S}$ (which is determined by $\bar{f}_*\omega_{Y''/S''}$

and the Deligne extension of $H^1_{\rm DR}(Y''/S'')$, see [25]). Similarly, $\pi^*\omega$ is extended to $\tilde f_*\omega_{X/\tilde S}$, because the roots of the indicial equation at $\pi^{-1}(\infty)$ are $1/3, 5/3$. Then it determines a nonzero element of $\tilde f_*\omega_X$ by the triviality of the invertible sheaf $\omega_{\tilde S}$. Indeed, $\tilde S$ is given by the equation

$$s^2 = (t-\alpha_1)(t-\alpha_2)(t-\alpha_3), \tag{4.2.3}$$

and $\omega_{\tilde S}$ is trivialized by $s^{-1}dt$, so that $\tilde\omega$ in (4.2.1) is given by the product of $s^{-1}dt$ with the extension of $\pi^*\omega$ to $\tilde S$.

Now we show that η in (4.2.1) may be restricted to those coming from the local system consisting of $H_1(X_s,\mathbb{Q})$. By the Poincaré duality, we may replace homology with cohomology. Consider the (perverse) Leray spectral sequence

$$E_2^{i,j} = H^i(\tilde S, {}^pH^j\mathbf{R}\tilde f_*(\mathbb{Q}_X[2])) \Rightarrow H^{i+j+2}(X,\mathbb{Q}),$$

where ${}^pH^j$ is the perverse cohomology functor [4]. This degenerates at E_2 by the decomposition theorem, and the associated Leray filtration on $H^\bullet(X,\mathbb{Q})$ is self-dual. We have furthermore the decomposition

$${}^pH^0\mathbf{R}\tilde f_*(\mathbb{Q}_X[2]) = {\rm IC}_{\tilde S}\tilde L \oplus B_\infty,$$

where $\tilde L$ is the local system defined by $H^1(X_s,\mathbb{Q})$ for $s \in \tilde S'' := \tilde S \setminus \pi^{-1}(\{0,a,\infty\})$, and B_∞ is a vector space which is identified with a sheaf supported on $\pi^{-1}(\infty)$. Note that ${\rm IC}_{\tilde S}\tilde L = \tilde K[1]$ with $\tilde K = \tilde j_*\tilde L$ where $\tilde j : \tilde S'' \to \tilde S$ is the inclusion morphism. See loc. cit.

We do not have a direct factor supported on $\pi^{-1}(0)$ or $\pi^{-1}(a)$, because Y_0, Y_a are irreducible. Since $E_2^{1,-1}, E_2^{-1,1}$ and B_∞ are of type (0,0), it is enough to consider the self pairing of $H^0(\tilde S, {\rm IC}_{\tilde S}\tilde L)$. (Note also that $\int_\eta \tilde\omega$ vanishes if η is supported on a divisor on X.)

Let U be sufficiently small contractible open neighborhood of $[a,+\infty]$ in $\mathbb{P}^1$ such that $P_i \notin \bar U$ $(i=1,2,3)$, ∂U is a smooth real curve, and $U \setminus [a,+\infty]$ is homeomorphic to the product of ∂U with an open real interval. Let $V = \mathbb{P}^1 \setminus [a,+\infty]$, $\tilde V = \pi^{-1}(V)$ and $\tilde U = \pi^{-1}(U)$. Let L be the local system defined by $H^1(Y_s,\mathbb{Q})$ for $s \in S'' := S \setminus \{0,a,\infty\}$ so that $\tilde L = \pi^*L$. Let $K = j_*L$ so that ${\rm IC}_S L = K[1]$, where $j : S'' \to S$ is the inclusion morphism. Then we have

$$h^1(\tilde S,\tilde K) = 6,\ \ h^1(\tilde V,\tilde K|_{\tilde V}) = 4,\ \ h^1(\tilde U,\tilde K|_{\tilde U}) = 4,\ \ h^1(\partial\tilde U,\tilde K|_{\partial\tilde U}) = 2, \tag{4.2.4}$$

together with the short exact sequence

$$0 \to H^1(\tilde S,\tilde K) \to H^1(\tilde U,\tilde K|_{\tilde U}) \oplus H^1(\tilde V,\tilde K|_{\tilde V}) \to H^1(\partial\tilde U,\tilde K|_{\partial\tilde U}) \to 0.$$

See (4.4) below.

Furthermore, the canonical morphism $H^1(\tilde S,\tilde K) \to H^1(\tilde S'',\tilde K|_{\tilde S''})$ is injective, and the kernel of its dual morphism is generated by the 2-cycles associated to 1-parameter families of cycles over sufficiently small circles around the points of $\pi^*(\{0,a\})$. Indeed, the cokernel

of the above morphism is given by the direct sum of $\mathrm{Coker}\,\mathrm{Var}_i$ with the notation of (4.5), and its dual is given by $\mathrm{Ker}\,\mathrm{Var}_i$. Note that the integrals of $\tilde\omega$ over such 2-cycles are zero, because the exponents of the period mapping at singular points are nonnegative by (4.2.2).

Since $H^1(\partial\tilde U,\tilde K|_{\partial\tilde U})$ comes from $H^1(\partial U,K|_{\partial U})$, we see that $\int_\eta\tilde\omega$ also vanishes for η coming from $\tilde f^{-1}(\partial\tilde U)$ (using the action of the involution of X). So it is enough to consider the integral $\int_\eta\tilde\omega$ with η belonging to some 4-dimensional subspace of $H_2(\tilde f^{-1}(\tilde S''),\mathbb{Q})$ which is a direct sum of two 2-dimensional subspaces coming from $H_2(\tilde f^{-1}(\tilde U''),\mathbb{Q})$ and $H_2(\tilde f^{-1}(\tilde V''),\mathbb{Q})$ where $\tilde U''=\tilde U\cap\tilde S''$, etc.

Let β_1,β_2,β_3, be a path on S connecting respectively $\{P_1,P_2,P_3\}$, $\{0,\infty\}$ and $\{a,\infty\}$. Let $\beta=\beta_1\cup\beta_2\cup\beta_3$, and $\tilde\beta=\pi^{-1}(\beta)$. Then by (4.2.4) and (4.5) below, we may assume that η is a 1-parameter family of C^∞ 1-cycles over $\tilde\beta$. Indeed, we can take the limit of the path around the singular points so that we get a family of cycles over a path connecting the singular points, because the exponents of the period integrals are nonnegative by (4.2.2). (For β_1 we consider the union of three paths between P_1, P_2, P_3 although it is enough to take one of them at least locally on T, because there is no canonical choice globally on T.)

By definition the integral over β_i for $i=1,2,3$ is given by

$$\int_{\beta_i}s^{-1}F_{\eta_i}(t)dt\quad\text{with } F_{\eta_i}(t)=\int_{\eta_i(t)}\omega$$

where $\eta_i(t)$ is the family of C^∞ 1-cycles (with rational coefficients) over β_i.

By (4.5), $\eta_2(t)$ and $\eta_3(t)$ belong to the image of the variation at 0 and a respectively. We may assume furthermore that β_2 is the real interval $[0,+\infty]$, and β_3 is the union of β_2 and the real interval $[a,1]$ by taking the limit. Then the restriction of the integration to the interval $[a,0]$ coincides with $\int_\Gamma\tilde\omega=\int_{\beta_0}s^{-1}F_{\eta_0}(t)dt$ multiplied by a rational number C, where $\beta_0=[a,0]$ and $\eta_0(t)=\Gamma\cap Y_t$. Indeed, $\eta_0(t)$ generates the kernel of the variation at a which coincides with the image of the variation, so that $\eta_3(t)=C\eta_0(t)$ for $t\in\,]a,0[$.

Since the images of the variations at 0 and a are different, it is enough to show

$$(1-C)\int_{\beta_0}s^{-1}F_{\eta_0}(t)dt\neq\int_{\beta_1}s^{-1}F_{\eta_1}(t)dt+\int_{\beta_2}s^{-1}F_{\eta_2}(t)dt\tag{4.2.5}$$

as multivalued functions on T in (4.1.1) for any families of cycles $\eta_i(t)$ over β_i $(i=1,2)$, where $\eta_2(t)$ is replaced with $\eta_2(t)+\eta_3(t)$ for $t\in[0,+\infty[$.

Assume the equality holds in (4.2.5). Let $\alpha_2=\alpha_1+\epsilon,\alpha_3=\alpha_1-\epsilon$, and take the limit for $\epsilon\to0$. Then first term of the right-hand side of (4.2.5) diverges by using the coordinate u such that $t-\alpha_1=\epsilon u$, but the other terms remain bounded. So we get $\eta_1(t)=0$.

If $C\neq1$, the left-hand side diverges and the right-hand side remains bounded when the α_i approach a. So we get a contradiction, and may assume $C=1$, i.e. the left-hand side is zero. In this case $\eta_2(t)\neq0$ by the definition of C (because $\eta_2(t)$ is the sum of the previous $\eta_2(t)$ and $\eta_3(t)$). Thus the assertion is reduced to

$$\int_{\beta_2}s^{-1}F_{\eta_2}(t)dt\neq0\quad\text{if }\eta_2(t)\neq\ 0,\tag{4.2.6}$$

and follows from (4.6) below (restricting the α_i to negative real numbers), because $F_{\eta_2}(t) > 0$ for $t > 0$ if $\eta_2(t)$ is a cycle with an appropriate orientation such that its image in S is the real interval $[c, +\infty]$ in the notation of (4.6).

4.3. *Proof of* (4.2.2). Let $\mathcal{O} = \mathbb{C}[x,y], \Omega^2 = \mathcal{O}dx \wedge dy$, and $G_f^{(0)} = \Omega^2/df \wedge d\mathcal{O}$ as in [16]. Then $G_f^{(0)}$ has a natural action of ∂_t^{-1} defined by $\partial_t^{-1}[\omega] = [df \wedge \eta]$ for $d\eta = \omega$. Let ω_0, ω_1 denote respectively the class of $dx \wedge dy$ and $xdx \wedge dy$ in $G_f^{(0)}$. Then we can verify

$$(t - \frac{5}{6}\partial_t^{-1})\omega_0 = \frac{2}{9}\omega_1, \quad ((t + \frac{4}{27}) - \frac{7}{6}\partial_t^{-1})\omega_1 = \frac{1}{9}\partial_t^{-1}\omega_0.$$

Applying ∂_t and ∂_t^2 respectively to these equations, and eliminating $\partial_t\omega_1$, we get (4.2.2).

4.4. *Proof of* (4.2.4). Let $S' = S \setminus \{\infty\}$, $\tilde{S}' = \tilde{S} \setminus \{\infty\}$, $U' = U \setminus \{\infty\}$, and $\tilde{U}' = \tilde{U} \setminus \{\infty\}$. Since 1 is not the eigenvalue of the monodromy at ∞ by (4.2.2), we have $H^\bullet(\tilde{S}, \tilde{K}) = H^\bullet(\tilde{S}', \tilde{K}|_{\tilde{S}'})$ (similarly for $H^\bullet(S, K)$), and $h^0(\tilde{U}', \tilde{K}|_{\tilde{U}'}) = h^0(\tilde{S}', \tilde{K}|_{\tilde{S}'}) = 0$. Then

$$h^1(\tilde{S}', \tilde{K}|_{\tilde{S}'}) = -\chi(\tilde{S}', \tilde{K}|_{\tilde{S}'}) = -2\chi(S', K|_{S'}) + 3\bullet 2 = 6.$$

because the restriction of π over S' is ramified at three points and $\operatorname{rank} L = 2$. Here the vanishing of $\chi(S', K|_{S'})$ follows from the fact that $K|_{S'}$ is isomorphic to the higher direct image of the constant sheaf on $\mathbb{A}^2$ by f (because the Leray spectral sequence degenerates in the case the base space is an affine curve). Similarly, we have

$$h^1(\tilde{U}', \tilde{K}|_{\tilde{U}'}) = -\chi(\tilde{U}', \tilde{K}|_{\tilde{U}'}) = -2\chi(U', K|_{U'}) = 4,$$

because $\chi(U', K|_{U'}) = \chi(S', K|_{S'}) - \chi(V, L|_V) = -2$ by the triviality of $L|_V$. We have also

$$\chi(\tilde{V}, \tilde{L}|_{\tilde{V}}) = 2\chi(\tilde{V}) = 2(2\chi(V) - 3) = -2,$$

by $\chi(V) = 1$, and get $h^1(\tilde{V}, \tilde{L}|_{\tilde{V}}) = 4$, because $h^0(\tilde{V}, \tilde{L}|_{\tilde{V}}) = 2$ by the triviality of $\tilde{L}|_{\tilde{V}}$. The last triviality implies also $h^1(\partial\tilde{U}, \tilde{L}|_{\partial\tilde{U}}) = 2$.

The exact sequence after (4.2.4) follows from it using the Mayer-Vietoris exact sequence.

4.5. *Remark.* Let D be an open disk, and $S = D \setminus \{P_1, \dots, P_n\}$. Take $P_0 \in S$, and choose a path γ_i between P_0 and P_i such that $\gamma_i \cap \gamma_j = \{P_0\}$. Let $f : X \to S$ be a proper smooth morphism of complex manifolds. Put $Y = f^{-1}(P_0)$. Let Var_i denote the variation $T_i - id$ around P_i which acts on $H_j(Y, \mathbb{Q})$ using γ_i. Then we have an isomorphism

$$\operatorname{Coker}(H_{j+1}(Y) \to H_{j+1}(X)) = \operatorname{Ker}(\sum_i \operatorname{Var}_i : \oplus_i H_j(Y) \to H_j(Y)), \tag{4.5.1}$$

such that an element $\{\eta_i\}$ in the right-hand side corresponds to $\sum_i \Gamma_i - \Gamma_0$ in the left-hand side, where Γ_i is a $(j+1)$-chain on X such that $\partial\Gamma_i = \operatorname{Var}_i(\eta_i)$ and $f(\Gamma_i)$ is contained in the union of γ_i and a sufficiently small circle around P_i, and Γ_0 is a $(j+1)$-chain on Y such that $\partial\Gamma_0 = \sum_i \operatorname{Var}_i(\eta_i)$. (Indeed, Γ_i is given by using a C^∞ trivialization of the restriction of f over a path on S.)

The assertion follows from the Mayer-Vietoris exact sequence by applying it, for example, to an open covering $\{U, U'\}$ of S where U is a disjoint union of simply connected open subsets U_i $(1 \le i \le n)$ such that $U_i \cap \gamma_j = \emptyset$ and $P_i \in \bar{U}_i$, and U' is a simply connected open subset such that $U_i \cap U'$ is a union of two simply connected open subsets. Here D can be replaced by an appropriate neighborhood of the union of γ_i which is a deformation retract of D. (This would be probably well-known to specialists.)

4.6. Lemma. *Let $g(x) = x^2(x+1) + t$ for $t > 0$, and $c, \alpha \pm i\beta$ be the roots of $g(x)$ which are viewed as functions of t (where $\alpha \in \mathbf{R}$ and $\beta \geq 0$). Then $\mathrm{Re}\, g(\alpha + iu) > 0$ if $-\beta < u < \beta$, and hence $\mathrm{Im} \int_{\gamma_t} \omega > 0$, where $\omega := g(x)^{-1/2} dx$ is a holomorphic form on the curve defined by $y^2 = g(x)$, and γ_t is the cycle on the curve (with an appropriate orientation) such that its image in $\mathbb{C}$ is a path connecting $\alpha - i\beta$ and $\alpha + i\beta$.*

Proof. Since $c = -(2\alpha + 1)$ and $t = -c^2(c+1) = 2\alpha(2\alpha+1)^2 > 0$, we get $\alpha > 0, c < -1$. We calculate

$$\mathrm{Re}\, g(\alpha + iu) = \alpha^2(\alpha + 1) + t - (3\alpha + 1)u^2,$$

where $\mathrm{Re}\, g(\alpha + iu) = 0$ for $u = \pm\beta$. So the first assertion follows, and the second from $d(\alpha + iu) = i du$.

4.7. *Remark.* By [30], [33], the kernel of the reduced Abel-Jacobi map

$$\mathrm{CH}^2_{\mathrm{ind}}(X, 1)_{\mathbb{Q}} \to J((H^2(X, \mathbb{Q})/N^1 H^2(X, \mathbb{Q}))(2))$$

is isomorphic to

$$\mathrm{Coker}(K_2(\mathbb{C}(X)) \to \mathrm{ind}\lim_U \mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Q}, H^2(U, \mathbb{Q})(2))) \tag{4.7.1}$$

where the morphism is given by $d\log \wedge d\log$, and the inductive limit is taken over nonempty open subvarieties U of X. Indeed, for a sufficiently large divisor Z on X, we have the short exact sequence

$$0 \to H^2(X, \mathbb{Q})/N^1 H^2(X, \mathbb{Q}) \to H^2(X \setminus Z, \mathbb{Q}) \to \mathrm{Ker}(H^3_Z(X, \mathbb{Q}) \to H^3(X, \mathbb{Q})) \to 0$$

together with $H^3_Z(X, \mathbb{Q}) = H^{\mathrm{BM}}_{2n-3}(Z, \mathbb{Q})(-n)$. So we get

$$\mathrm{CH}^1_{\mathrm{ind}}(Z, 1)_{\mathbb{Q}} = \mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Q}, H^3_Z(X, \mathbb{Q})(2))$$

by (0.1) and (1.1.1), and

$$\mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Q}, H^2(X \setminus Z, \mathbb{Q})(2)) = \mathrm{Ker}(\mathrm{CH}^1_{\mathrm{ind}}(Z, 1)_{\mathbb{Q}} \to J((H^2(X, \mathbb{Q})/N^1 H^2(X, \mathbb{Q}))(2))),$$

because $\mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Q}, H^3(X, \mathbb{Q})(2)) = 0$. Then it is enough to take the inductive limit over Z, and divide it by the image of $K(\mathbb{C}(X))$ under the tame symbol. Note that the morphism $H^2(X \setminus Z, \mathbb{Q}) \to H^3_Z(X, \mathbb{Q})$ is given by taking the residue of logarithmic forms (at least on Z_{reg}, see (2.7)), and the residue of $d\log f \wedge d\log g$ coincides with the differential of the logarithm of the image of $\{f, g\}$ by the tame symbol up to sign. It would be interesting whether (4.7.1) vanishes. See [3, 6.1].

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