On a WKB-theoretic approach to Ablowitz-Segur's connection problem for the second Painlevé equation

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Abstract

We discuss Ablowitz-Segur's connection problem for the second Painlevé equation from the viewpoint of WKB analysis of Painlevé transcendents with a large parameter. The formula they first discovered is rederived from a suitable combination of connection formulas for the first Painlevé equation.

1 Introduction

Exact WKB analysis, initiated by Voros ([18]) and developed by Pham and his collaborators ([6], [7] etc.), is a powerful tool of analyzing global behavior of solutions of one-dimensional Schrödinger equations (cf. [14] also). In a parallel way to the case of Schrödinger equations, Aoki, Kawai and the author have recently established an analogous analysis for Painlevé transcendents (i.e., solutions of Painlevé equations) with a large parameter ([12], [3], [13], [17]). Although mathematically rigorous justification of the theory is still an open problem, this analysis has gradually turned out to be effective for studying global connection problems of Painlevé equations. In this paper, to show its validity and effectiveness, we explain an outline of the theory

and discuss Ablowitz-Segur's connection problem for the second Painlevé equation from this viewpoint.

The author would like to express his sincere gratitude to Professors T. Kawai and T. Aoki for the stimulating discussions with them. This work is supported in part by Japan Society for the Promotion of Science under Grant-in-Aid for Encouragement of Young Scientists (No. 11740087).

2 Review of WKB analysis of Painlevé transcendents

The equations treated in our WKB analysis of Painlevé transcendents are the following Painlevé equations (P_J) with a large parameter η:

$$\frac{d^2\lambda}{dt^2} = \eta^2 F_J(\lambda, t) + G_J\left(\lambda, \frac{d\lambda}{dt}, t\right) \quad (J = \mathrm{I}, \dots, \mathrm{VI}), \tag{P_J}$$

where F_J and G_J are rational functions. For example, $(P_{\mathrm{I}}) : d^2\lambda/dt^2 = \eta^2(6\lambda^2 + t)$, $(P_{\mathrm{II}}) : d^2\lambda/dt^2 = \eta^2(2\lambda^3 + t\lambda + c)$.

For these equations (P_J) we can construct the following 2-parameter family of formal solutions called instnton-type solutions ([3], [16]):

$$\lambda_J(t; \alpha, \beta) = \lambda_0(t) + \eta^{-1/2}\lambda_{1/2}(t, \eta) + \eta^{-1}\lambda_1(t, \eta) + \cdots, \tag{1}$$

where $\lambda_0(t)$ is an algebraic function determined by $F_J(\lambda_0(t), t) = 0$ and $\lambda_{j/2}(t, \eta)$ $(j \geq 1)$ has the following expansion:

$$\begin{aligned} \lambda_{1/2} &= \mu_J(t)\left\{\alpha(\theta_J(t)\eta^2)^{\alpha\beta}e^{\eta\phi_J(t)} + \beta(\theta_J(t)\eta^2)^{-\alpha\beta}e^{-\eta\phi_J(t)}\right\}, \\ \lambda_{j/2} &= \sum_{k=0}^{j} b_{j-2k}^{(j/2)}(t)(\theta_J(t)\eta^2)^{(j-2k)\alpha\beta}e^{(j-2k)\eta\phi_J(t)} \qquad (j \geq 2). \end{aligned}$$

Here

$$\phi_J(t) = \int_{t_0}^{t} \sqrt{\frac{\partial F_J}{\partial \lambda}(\lambda_0(t), t)}\, dt \qquad (t_0\text{: a fixed point}), \tag{2}$$

$\mu_J(t)$ and $\theta_J(t)$ are some functions of t (whose explicit description is given in [13]), and (α, β) denotes a pair of free parameters. (Precisely speaking, both α and β are infinite series $\sum_{n\geq 0}\eta^{-n}\alpha_n$ and $\sum_{n\geq 0}\eta^{-n}\beta_n$ since $b_{\pm 1}^{(n+1/2)}(t)$

($n \geq 1$) may contain additional free parameters (α_n, β_n). In this paper, however, we are concerned only with computations of the top degree level and accordingly consider α and β as ordinary parameters.)

Instanton-type solutions can be regarded as a substitute for WKB solutions of Schrödinger equations. For these solutions let us introduce the notion of turning points and Stokes curves in the following way:

Definition 1 (i) A turning point of (P_J) is a point r satisfying

$$F_J(\lambda_0(r), r) = \frac{\partial F_J}{\partial \lambda}(\lambda_0(r), r) = 0. \tag{3}$$

A turning point r is said to be simple if $(\partial^2 F_J/\partial\lambda^2)(\lambda_0(r), r) \neq 0$.
(ii) A Stokes curve of (P_J) is defined by the following relation:

$$\Im \int_r^t \sqrt{\frac{\partial F_J}{\partial \lambda}(\lambda_0(t), t)}dt = 0, \tag{4}$$

where r is a turning point of (P_J).

For example, in the case of $(P_{\rm I})$ $t = 0$ is a unique simple turning point and the Stokes curves are given by the relation $\Im x^{5/4} = 0$ (cf. Figure 1 below).

Stokes curves divide the complex t-plane (or rather the Riemann surface of $F_J(\lambda, t) = 0$) into several small regions. An instanton-type solution $\lambda_J(t; \alpha, \beta)$ is then expected to define an analytic solution in each region and the relation between the instanton-type solutions in two adjacent regions corresponding to the same analytic solution should be described by the so-called "connection formula". Concerning the determination of explicit form of the connection formula, we can prove the following two fundamental theorems.

Theorem 2.1 (Local reduction to $(P_{\rm I})$; [13]) *In a neighborhood of $\tilde{t}_*$, a point on a Stokes curve emanating from a simple turning point $\tilde{r}$ of (P_J), the Painlevé equation (P_J) can be transformed to $(P_{\rm I})$. To be more precise, for each instanton-type solution $\tilde{\lambda}_J(\tilde{t}; \tilde{\alpha}, \tilde{\beta})$ of (P_J) we can find an instanton-type solution $\lambda_{\rm I}(t; \alpha, \beta)$ of $(P_{\rm I})$ for which the following holds:*

$$x(\tilde{\lambda}_J(\tilde{t}; \tilde{\alpha}, \tilde{\beta}), \tilde{t}, \eta) = \lambda_{\rm I}(t(\tilde{t}, \eta); \alpha, \beta), \tag{5}$$

where $x(\tilde{x}, \tilde{t}, \eta)$ and $t(\tilde{t}, \eta)$ are formal series of the form

$$x(\tilde{x}, \tilde{t}, \eta) = \sum_{j \geq 0} \eta^{-j/2} x_{j/2}(\tilde{x}, \tilde{t}, \eta), \quad t(\tilde{t}, \eta) = \sum_{j \geq 0} \eta^{-j/2} t_{j/2}(\tilde{t}, \eta), \tag{6}$$

and the parameters obey the following relation

$$(\alpha, \beta) = (i^{\kappa}\tilde{\alpha}, i^{\kappa}\tilde{\beta}) + \mathcal{O}(\eta^{-1}) \tag{7}$$

with some integer κ. *(Here we are assuming that the end-point* t_0 *in the definition (2) of* $\tilde{\phi}_J(\tilde{t})$ *and* $\phi_{\mathrm{I}}(t)$ *is taken to be* $\tilde{r}$ *and* 0 *respectively.)*

Theorem 2.2 (Connection formula for (P_{I})**; [17])** *(As there is no essential difference among the five Stokes curves due to symmetry, we consider only a Stokes curve* $\{\arg t = -\pi/5\}$ *here.) Let* $\lambda_{\mathrm{I}}(t;\alpha,\beta)$ *(normalized so that* $t_0 = 0$ *in (2)) be an instanton-type solution of* (P_{I}) *in the region* $\{-3\pi/5 < \arg t < -\pi/5\}$ *and* $\lambda_{\mathrm{I}}(t;\alpha',\beta')$ *its analytic continuation to* $\{-\pi/5 < \arg t < \pi/5\}$ *(cf. Figure 1). Then the following relations hold between* (α,β) *and* (α',β').

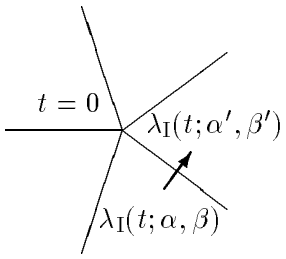

FIGURE 1

$$\begin{cases} \alpha e^{-\frac{i\pi E}{4}}\chi(E) = \alpha' e^{-\frac{i\pi E'}{4}}\chi(E'), \\ e^{\frac{i\pi E}{2}} + \beta e^{\frac{i\pi E}{2}}\chi(-E) = \beta' e^{\frac{i\pi E'}{2}}\chi(-E'), \end{cases} \tag{8}$$

where $\chi(z) = \dfrac{\sqrt{\pi}}{\Gamma(z/4+1)}2^{z/4+1}$, $E = -8\alpha\beta$, *and* $E' = -8\alpha'\beta'$.

Unfortunately, as it is not known how to give an analytic meaning to instanton-type solutions at the present stage, we have not yet succeeded in verifying the analytic version of these theorems. However, in parallel with the Schrödinger case, the connection formula at a simple turning point for (P_J) should, in principle, be derived from combination of these two theorems and repeated use of the formula for (P_J) thus obtained (i.e., obtained by substitution of (7) into (8)) according to the configuration of Stokes curves should enable us to analyze the global behavior of solutions of (P_J).

In the subsequent sections, to exemplify the validity of our approach, we discuss Ablowitz-Segur's connection problem from this viewpoint.

3 Ablowitz-Segur's connection problem

Let us consider a solution $u(z)$ of the equation

$$u'' = zu + 2u^3 \tag{9}$$

with the following asymptotic behavior for $z > 0, z \to \infty$:

$$u(z) \sim \frac{a}{2\sqrt{\pi}} z^{-1/4} \exp(-\frac{2}{3} z^{3/2}) \qquad (z \to +\infty) \tag{10}$$

where a is a constant satisfying $0 < a < 1$. It is known that, after the analytic continuation along the real axis, $u(z)$ has the following asymptotic expansion for $z \to -\infty$:

$$u(z) \sim d(-z)^{-1/4} \sin\left(\frac{2}{3}(-z)^{3/2} - \frac{3}{4} d^2 \log(-z) + \theta\right) \quad (z \to -\infty), \tag{11}$$

where d and θ are given by

$$\begin{cases} d^2 &= -\dfrac{1}{\pi}\log(1-a^2), \\ \theta &= \dfrac{\pi}{4} - \dfrac{3}{2} d^2 \log 2 - \arg\Gamma\left(1 - \dfrac{id^2}{2}\right). \end{cases} \tag{12}$$

The formula (12) was first discovered by Ablowitz and Segur ([1], [15]), then discussed by many people (e.g., [5], in particular, [11] is closely related to our approach in the sense that they discuss singular-perturbative form of equation (9)), and finally proved rigorously in [8] and [9]. See also, for example, [4] for more recent researches.

Our goal is to derive this formula (12) by using the WKB analysis explained in the preceding section.

4 WKB analysis for the second Painlevé equation

By a simple scaling transformation

$$u = \eta^{1/3}\lambda, \quad z = \eta^{2/3} t \tag{13}$$

equation (9) is transformed to our second Painlevé equation

$$\frac{d^2\lambda}{dt^2} = \eta^2(2\lambda^3 + t\lambda) \tag{P_{II}^0}$$

with the parameter c being equal to 0. Furthermore, taking as solution of (P_{II}^0) an instanton-type solution $\lambda_{\mathrm{II}}^0(t;\alpha,\beta)$ with the identically vanishing

top term $\lambda_0(t) \equiv 0$ and $\phi^0_{\rm II}(t) = \displaystyle\int_0^t \sqrt{t}dt$, we readily find that both the asymptotic solutions (10) and (11) of (9) correspond through the scaling transformation (13) to $\lambda^0_{\rm II}(t;\alpha,\beta)$ (at least at the leading order level) on condition that the parameters are related in the following manner:

$$\begin{cases} \beta = \dfrac{a}{2\sqrt{\pi}}, \quad \alpha = 0 & (\text{for } z \to \infty), \\ 4\alpha\beta = id^2, \quad \dfrac{\alpha}{\beta}(16e^{3i\pi})^{-2\alpha\beta} = -e^{2i\theta} & (\text{for } z \to -\infty). \end{cases} \tag{14}$$

Our problem is thus to solve the connection problem between $\lambda^0_{\rm II}(t;0,\beta)$ for $z \to \infty$ and $\lambda^0_{\rm II}(t;\alpha,\beta)$ for $z \to -\infty$.

Let us here draw the configuration of Stokes curves of the second Painlevé equation ($P_{\rm II}$) (cf. Figure 2). In the case of $c = 0$ there is a unique turning

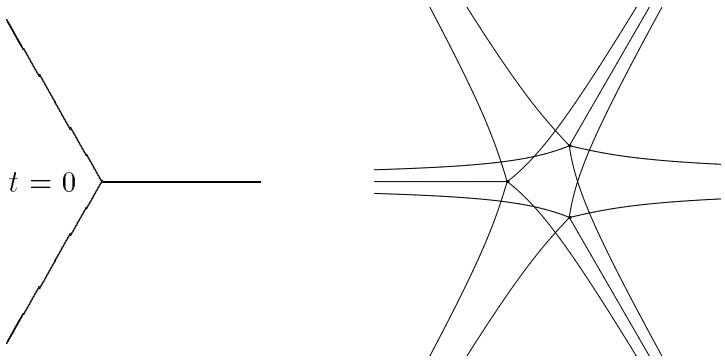

FIGURE 2 : Stokes curves of ($P_{\rm II}$) for $c = 0$ (left) and for $c \neq 0$ (right).

point at $t = 0$ and three Sokes curves (lines) emanate from it. The connection problem in question can be solved by using connection formulas on these Stokes curves. However, it is not an easy task to determine their explicit form as $t = 0$ is not a simple turning point. On the other hand, in the case of $c \neq 0$ (although the configuration itself is quite complicated) all of the three turning points $r_j = -6(c/4)^{2/3}\omega^j$ ($j = 0, 1, 2$, $\omega = e^{2i\pi/3}$) are simple and hence the connection formula (8) for ($P_{\rm I}$) is expected to be applicable to each Stokes curve. Taking this situation into account, we first consider

$(P_{\rm II})$ with a non-zero parameter c and then take a limit $c \to 0$ to study the connection problem for $(P_{\rm II}^0)$.

As is pointed out and explicitly done by using *Mathematica 3.01* in [2], the configuration of Stokes curves of $(P_{\rm II})$ should be lifted onto the Riemann surface of $F_{\rm II}(\lambda, t) = 2\lambda^3 + t\lambda + c = 0$. (See Figure 3, where a wiggly line, a solid line and a dotted line respectively designate a cut, a Stokes curve on the sheet concerned and that on another sheet). Let us consider a limit $c \to 0$ for

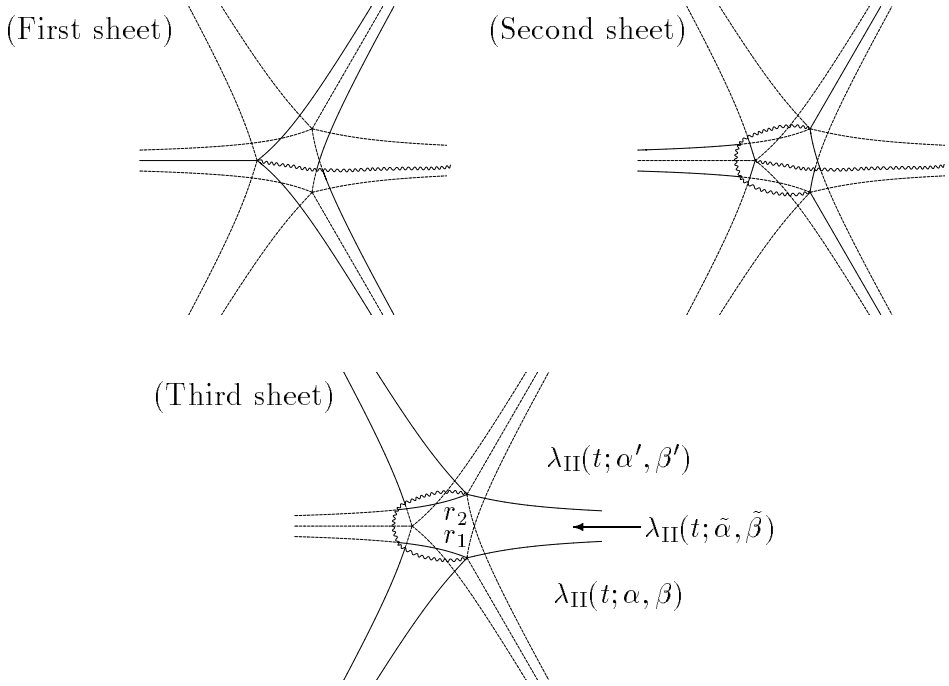

FIGURE 3 : The lift of Stokes curves of $(P_{\rm II})$ onto the Riemann surface of $2\lambda^3 + t\lambda + c = 0$.

this lift of Stokes curves. In the process of $c \to 0$ all of the turning points tend to the origin and the cut between the second and third sheets disappears. Consequently the Riemann surface at $c = 0$ becomes the disjoint union of two connected components; the double-covering part (first & second sheets) and the simple-covering part (third sheet). Note that this is consistent with the fact that $F_{\rm II}(\lambda, t)$ at $c = 0$ is factorized as $2\lambda^3 + t\lambda = \lambda(2\lambda^2 + t)$. Since we are interested in instanton-type solutions of $(P_{\rm II}^0)$ with the identically vanishing

top term $\lambda_0(t) \equiv 0$, we should discuss instanton-type solutions of $(P_{\rm II})$ on the third sheet for $c \neq 0$. The configuration of Stokes curves on the third sheet, in fact, approaches to that of $(P^0_{\rm II})$.

Having these geometric facts in mind, we now try to determine the connection formula for $(P^0_{\rm II})$ on, for example, the positive real axis, i.e., the relation between $\lambda^0_{\rm II}(t;\alpha,\beta)$ in the region $\{-2\pi/3 < \arg t < 0\}$ and $\lambda^0_{\rm II}(t;\alpha',\beta')$ in $\{0 < \arg t < 2\pi/3\}$. Let $\lambda_{\rm II}(t;\alpha,\beta)$ and $\lambda_{\rm II}(t;\alpha',\beta')$ respectively denote instanton-type solutions of $(P_{\rm II})$ in the corresponding regions on the third sheet. (Cf. Figure 3. Here we assume that the end-point t_0 in (2) is taken to be a simple turning point r_2 for both the solutions.) Note that there is an intermediate region between these two regions, an instanton-type solution in which is denoted by $\lambda_{\rm II}(t;\tilde{\alpha},\tilde{\beta})$. It then follows from the simplicity of the turning point r_2 that the relation between $\lambda_{\rm II}(t;\tilde{\alpha},\tilde{\beta})$ and $\lambda_{\rm II}(t;\alpha',\beta')$ should be described by the connection formula (8) for $(P_{\rm I})$, provided that our expectation explained at the end of Section 2 is true. That is,

$$\begin{cases} \tilde{\alpha} e^{-i\pi\tilde{E}/4}\chi(\tilde{E}) = \alpha' e^{-i\pi E'/4}\chi(E') \\ e^{i\pi\tilde{E}/2} + \tilde{\beta} e^{i\pi\tilde{E}/2}\chi(-\tilde{E}) = \beta' e^{i\pi E'/2}\chi(-E'). \end{cases} \tag{15}$$

Similarly, since the integer κ in the relation (7) of parameters is equal to 2 for the local reduction to $(P_{\rm I})$ at r_1, the relation between $\lambda_{\rm II}(t;\alpha,\beta)$ and $\lambda_{\rm II}(t;\tilde{\alpha},\tilde{\beta})$ should be described by

$$\begin{cases} \alpha e^{-i\pi E/4}\chi(E) = \tilde{\alpha} e^{-i\pi\tilde{E}/4}\chi(\tilde{E}) \\ -e^{i\pi E/2}e^{2i\pi c\eta} + \beta e^{i\pi E/2}\chi(-E) = \tilde{\beta} e^{i\pi\tilde{E}/2}\chi(-\tilde{E}). \end{cases} \tag{16}$$

(The factor $e^{2i\pi c\eta} = \exp(\eta \displaystyle\int_{r_2}^{r_1} \sqrt{6\lambda_0^2 + t}dt)$ appears due to the difference of the choice of end-points of $\phi_{\rm II}(t)$.) Eliminating $(\tilde{\alpha},\tilde{\beta})$ from equations (15) and (16) and taking a limit $c \to 0$, we thus obtain the following "connection formula for $(P^0_{\rm II})$ on the positive real axis":

$$\begin{cases} \alpha e^{-i\pi E/4}\chi(E) = \alpha' e^{-i\pi E'/4}\chi(E') \\ e^{i\pi E/2} - \beta e^{i\pi E/2}\chi(-E) = \dfrac{1 - \beta' e^{i\pi E'/2}\chi(-E')}{1 - i\alpha' e^{-i\pi E'/4}\chi(E')}. \end{cases} \tag{17}$$

(Note that $\alpha = \alpha' = 0$ and $\beta = \beta'$ satisfy (17). This reflects the well-definedness of $\lambda^0_{\rm II}(t;0,\beta)$ on the positive real axis.)

Connection formulas on the other Stokes curves can also be derived from (17) by using the rotational symmetry of angle $2\pi/3$ of (P_{II}^0). (For example, replacement $\alpha \mapsto -i\beta$, $\beta \mapsto -i\alpha$ in (17) gives the connection formula on $\{\arg t = 2\pi/3\}$.) As a consequence we obtain the following solution of the connection problem in question: Consider an instanton-type solution $\lambda_{\mathrm{II}}^0(t;0,\beta)$ of (P_{II}^0) on the positive real axis and denote by $\lambda_{\mathrm{II}}^0(t;\alpha',\beta')$ its analytic continuation to the negative real axis through the upper-half plane, then we have

$$\begin{cases} 2\sqrt{\pi}\beta &= -i\alpha' e^{-i\pi E'/2}\chi(E') \\ 2\sqrt{\pi}\beta &= \beta' e^{i\pi E'/4}\chi(-E'). \end{cases} \tag{18}$$

By a straightforward computation we find that the formula (12) certainly follows from (18) and (14) (assuming that (α,β) in the second relation of (14) is replaced by (α',β')). In other words, Ablowitz-Segur's formula can be derived from repeated use of the connection formula for (P_{I}).

We believe that this result strongly supports the validity and effectiveness of our WKB theory for Painlevé equations. To complete the theory, however, we need some analytic interpretation of instanton-type solutions. (A recent work of Joshi ([10]) may be regarded as the first step in the study of this problem.) To clarify the precise analytic meaning of the local reduction to (P_{I}) is also an important open problem in this theory.

References

[1] M. J. Ablowitz and H. Segur, "Asymptotic solution of the Korteweg-de Vries equation", *Stud. Appl. Math.* **57**(1977) 13–44.

[2] T. Aoki, "Stokes geometry of Painlevé equations with a large parameter", *RIMS Kôkyûroku* 1088(1999) 39–54.

[3] T. Aoki, T. Kawai and Y. Takei, "WKB analysis of Painlevé transcendents with a large parameter. II", in *Structure of Solutions of Differential Equations* (eds. M. Morimoto and T. Kawai), (World Scientific, 1996) 1–49.

[4] A. P. Bassom, P. A. Clarkson, C. K. Law and J. B. McLeod, "Application of uniform asymptotics to the second Painlevé transcendent", *Arch. Rat. Mech. Anal.* **143**(1998) 241–271.

[5] P. A. Clarkson and J. B. McLeod, "A connection formula for the second Painlevé transcendent", *Arch. Rat. Mech. Anal.* **103**(1988) 97–138.

[6] B. Candelpergher, J.C. Nosmas et F. Pham, *Approche de la résurgence* (Hermann, 1993).

[7] E. Delabaere and F. Pham, "Resurgent methods in semi-classical asymptotics", *Ann. Inst. H. Poincaré* **71**(1999) 1–94.

[8] P. Deift and X. Zhou, "A steepest descent method for oscillatory Riemann-Hilbert problems", *Ann. Math.* **137**(1993) 295–368.

[9] ———, "Asymptotics for the Painlevé II equation", *Comm. Pure Appl. Math.* **48**(1995) 277–337.

[10] N. Joshi, "True solutions asymptotic to formal WKB solutions of the second Painlevé equation with large parameter", in *Toward the Exact WKB Analysis of Differential Equations, Linear or Non-Linear* (eds. C. J. Howls, T. Kawai and Y. Takei), (Kyoto Univ. Press, 2000) 223–230.

[11] M. V. Karasev and A. V. Pereskokov, "On connection formulas for the second Painlevé transcendent", *Russian Acad. Sci. Izv. Math.* **42**(1994) 501–560.

[12] T. Kawai and Y. Takei, "WKB analysis of Painlevé transcendents with a large parameter. I", *Adv. in Math.* **118**(1996) 1–33.

[13] ———, "WKB analysis of Painlevé transcendents with a large parameter. III", *Adv. in Math.* **134**(1998) 178–218.

[14] ———, *Algebraic Analysis of Singular Perturbations* (Iwanami, 1998). (In Japanese. English translation will be published by Amer. Math. Soc.)

[15] H. Segur and M. J. Ablowitz, "Asymptotic solutions of nonlinear evolution equations and a Painlevé transcendent", *Physica D* **3**(1981) 165–184.

[16] Y. Takei, "Singular-perturbative reduction to Birkhoff normal form and instanton-type formal solutions of Hamiltonian systems", *Publ. RIMS, Kyoto Univ.* **34**(1998) 601–627.

[17] ———, "An explicit description of the connection formula for the first Painlevé equation", in *Toward the Exact WKB Analysis of Differential Equations, Linear or Non-Linear* (eds. C. J. Howls, T. Kawai and Y. Takei), (Kyoto Univ. Press, 2000) 271–296.

[18] A. Voros, "The return of the quartic oscillator. The complex WKB method", *Ann. Inst. H. Poincaré* **39**(1983) 211–338.