EXPONENTS OF AN IRREDUCIBLE PLANE CURVE SINGULARITY

Morihiko Saito

RIMS Kyoto University, Kyoto 606-8502 Japan

Let $f : (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$ be a germ of a holomorphic function with an isolated singularity. Using the theory of mixed Hodge structure of Deligne-Steenbrink [11] on the cohomology of the Milnor fiber, we can define the *exponents* (or spectra, up to the shift by one, in the terminology of Varchenko [12]) by μ rational numbers $\{\alpha_1, \dots, \alpha_\mu\}$ such that $\exp(2\pi i(\sqrt{-1}\alpha_i)$ are the eigenvalues of the Milnor monodromy and their integral part is determined by the Hodge filtration of the mixed Hodge structure. This notion was first introduced by Steenbrink [11]. By [14] the exponents are constant under μ-constant deformation of f. In particular, they depend only on $f^{-1}(0)$. They express the vanishing order (up to the shift by one) of the period integrals of holomorphic forms on vanishing cycles by [12], [13].

Let $(V, 0) \subset (\mathbf{C}^2, 0)$ be a germ of reduced and irreducible plane curve defined by a holomorphic function f. It is known that the equisingular class of $(V, 0)$ is determined by its Puiseux pairs, and numerical invariants such as the Milnor number or the characteristic polynomial of the monodromy can be expressed in terms of the *Puiseux pairs*, cf. [2], [7]. In this note we give a formula for the exponents of f in terms of the Puiseux pairs, cf. Theorem (1.5). The proof uses the *Enriques diagram* [3], [4] of $(V, 0)$ which describes the canonical process of embedded resolution of a plane curve V by iterating point center blowing-ups. In the irreducible case, we can describe explicitly the Enriques diagram as well as the multiplicity of the pull-back of f along the irreducible components of the exceptional divisor of the resolution, using the continued fraction expansion of the Puiseux pairs. Then we can apply a formula of Steenbrink [11] to calculate the Hodge numbers of the vanishing cohomology.

As an application we can prove a conjecture of Hertling [6] that the variance (i.e. the square of the standard deviation) of the exponents is bounded by the difference between the maximal and minimal exponents divided by 12 in this case. See (5.2). This was rather unexpected, because no philosophical reason for the conjecture is known. For the proof, we need a good estimate of the average of the exponents less than 1 in the quasihomogeneous case (see (5.3)) which is sharp and leads to a simple expression like (5.2.3).

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his attention to Hertling's conjecture. This is a revised version of a paper typed in 1990 at RIMS, and its original manuscript was written in 1982.

1. Exponents

1.1. Let $f : (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$ be a germ of a holomorphic function with an isolated singularity. Let $f : X \to S$ be a good representative of f (sometimes called a Milnor fibration) defined by

$$X = \{x \in \mathbf{C}^{n+1} : |x| < \varepsilon, |f(x)| < \eta\}, \quad S = \{t \in \mathbf{C} : |t| < \eta\}$$

for $0 < \eta \ll \varepsilon \ll 1$.

As in [11], we denote by $H^n(X_\infty, \mathbf{C})$ the vanishing cohomology of f, which is (non-canonically) isomorphic to the cohomology of the Milnor fiber $H^n(X_t, \mathbf{C})$ for $t \neq 0$. In [11] Steenbrink constructed a canonical mixed Hodge structure on $H^n(X_\infty, \mathbf{C})$ using Deligne's theory of mixed Hodge structure [5]. Let F be the Hodge filtration on $H^n(X_\infty, \mathbf{C})$. Let $\mu = \dim_{\mathbf{C}} H^n(X_\infty, \mathbf{C})$, which is called the Milnor number of f. Let

$$H^n(X_\infty, \mathbf{C})_\lambda = \mathrm{Ker}(T_s - \lambda : H^n(X_\infty, \mathbf{C}) \to H^n(X_\infty, \mathbf{C})),$$

where $T = T_s T_u$ is the Jordan decomposition of the monodromy.

1.2. Definition [11]. The *exponents* of f are μ-rational numbers $\{\alpha_1, \dots, \alpha_\mu\}$ such that

$$0 < \alpha_1 \le \alpha_2 \le \cdots \le \alpha_\mu < n + 1 \tag{1.2.1}$$

and are defined by the following condition:

$$\begin{aligned} \#\{j : \mathbf{e}(-\alpha_j) = \lambda, [\alpha_j] = n - p\} &= \dim_{\mathbf{C}} \mathrm{Gr}_F^p H^n(X_\infty, \mathbf{C})_\lambda \ (\lambda \neq 1) \\ \#\{j : \mathbf{e}(-\alpha_j) = n - p + 1\} &= \dim_{\mathbf{C}} \mathrm{Gr}_F^p H^n(X_\infty, \mathbf{C})_1 \end{aligned} \tag{1.2.2}$$

where $[\alpha] = \max\{i \in \mathbf{Z} : i \le \alpha\}$ is the Gauss symbol, $\mathbf{e}(\alpha) = \exp(2\pi i \alpha)$, and F is the Hodge filtration of the mixed Hodge structure on $H^n(X_\infty, \mathbf{C})$. (See also [8].) Let

$$\chi_f(t) = \sum_{1 \le i \le \mu} t^{\alpha_i}$$

which is called the spectrum of f by a recent terminology.

Remark. By the symmetry of the exponents (cf. [11]), we have

$$\chi_f(t) = t^{n+1} \chi_f(t^{-1}). \tag{1.2.3}$$

1.3. *Example* (quasihomogeneous case). Let f be a quasihomogeneous polynomial with weight $(w_0, \dots, w_n)$, i.e., f is a linear combination of monomials $x_0^{m_0} \cdots x_n^{m_n}$ such that $\sum w_i m_i = 1$. Then

$$\Omega_f := \Omega_{X,0}^{n+1} / df \wedge \Omega_{X,0}^n \tag{1.3.1}$$

is a graded μ-dimensional vector space whose grading is induced by the weight of the coordinates and is indexed by $\mathbf{Q}$. Let $P(\Omega_f, t)$ be the Poincaré polynomial of Ω_f. By [10], we have

$$P(\Omega_f, t) = \chi_f(t). \tag{1.3.2}$$

Furthermore $P(\Omega_f, t)$ is written explicitly in terms of the weights, and we get

$$\chi_f(t) = \prod_{i=0}^{n} \frac{t^{w_i} - t}{1 - t^{w_i}} \tag{1.3.3}$$

as is well-known. This formula is generalized to the nondegenerate Newton polyhedron case by [11], [9] using the Newton filtration on Ω_f.

1.4. Definition. Let (k_i, n_i) be pairs of mutually prime positive integers such that $n_i > 1$ for $1 \le i \le g$. Let w_i be integers defined inductively by $w_1 = k_1$ and

$$w_i = w_{i-1} n_{i-1} n_i + k_i \quad \text{for } i > 1.$$

We define $\Phi_g(k_1, n_1; \dots; k_g, n_g)(t)$ by induction on g as follows:

$$\Phi_1(k, n)(t) = \left(\frac{t^{1/k} - t}{1 - t^{1/k}}\right)\left(\frac{t^{1/n} - t}{1 - t^{1/n}}\right) \qquad \text{if } g = 1, \tag{1.4.1}$$

$$\begin{aligned} \Phi_g(k_1, n_1; \dots; k_g, n_g)(t) = {} & \left(\frac{t^{1/w_g} - t}{1 - t^{1/w_g}}\right)\left(\frac{t^{1/n_g} - t}{1 - t^{1/n_g}}\right) \\ & + \left(\frac{1 - t}{1 - t^{1/n_g}}\right) \Phi_{g-1}^{<1}(t^{1/n_g}) \\ & + t^{1-1/n_g} \left(\frac{1 - t}{1 - t^{1/n_g}}\right) \Phi_{g-1}^{>1}(t^{1/n_g}) \quad \text{if } g > 1, \end{aligned} \tag{1.4.2}$$

where $\Phi_{g-1}^{<1}(t)$ and $\Phi_{g-1}^{>1}(t)$ are defined respectively by $\sum_{\alpha<1} c_\alpha t^\alpha$ and $\sum_{\alpha>1} c_\alpha t^\alpha$ with $\sum_\alpha c_\alpha t^\alpha = \Phi_{g-1}(k_1, n_1; \dots; k_{g-1}, n_{g-1})(t)$

1.5. Theorem. *Let $(V, 0) \subset (\mathbf{C}^2, 0)$ be a reduced and irreducible curve defined by a function f, and $(k_1, n_1), \dots, (k_g, n_g)$ the Puiseux pairs of V, cf. (2.1). Then*

$$\chi_f(t) = \Phi_g(k_1, n_1; \dots; k_g, n_g)(t). \tag{1.5.1}$$

Remark. The assertion means that the exponents which are smaller than 1 are given (with multiplicity) by

$$\left\{\left(\frac{1}{n_{\nu+1} \cdots n_g}\left(\frac{i}{n_\nu} + \frac{j}{w_\nu}\right) + \frac{r}{n_{\nu+1} \cdots n_g}\right)\right\} \tag{1.5.2}$$

for $0 < i < n_\nu$, $0 < j < w_\nu$, $0 \le r < n_{\nu+1} \cdots n_g$, $1 \le \nu \le g$ such that $i/n_\nu + j/w_\nu < 1$.

2. Puiseux pairs and Enriques diagram

2.1. Definition. Let $(V,0)$ be a germ of reduced and irreducible plane curve in $(\mathbf{C}^2,0)$. Let (x,y) be the coordinate system of $(\mathbf{C}^2,0)$. We have the Puiseux expansion associated with V:

$$\begin{aligned} y = & \sum_{1\le i\le[k_1/n_1]} c_{0,i}x^i \\ & + \sum_{0\le i\le[k_2/n_2]} c_{1,i}x^{(k_1+i)/n_1} \\ & + \sum_{0\le i\le[k_3/n_3]} c_{2,i}x^{k_1/n_1+(k_2+i)/n_1n_2} \\ & \qquad\vdots \\ & + \sum_{0\le i} c_{g,i}x^{k_1/n_1+k_2/n_1n_2+\cdots+(k_g+i)/n_1\cdots n_g} \end{aligned} \tag{2.1.1}$$

where $c_{j,i}\in\mathbf{C}$ such that $c_{j,0}\ne 0\,(j\ne 0)$, and $k_j,n_j\in\mathbf{Z}_+$ such that $(k_j,n_j)=1$ and $n_j>1$. In this paper, we call $(k_1,n_1),\dots,(k_g,n_g)$ the *Puiseux pairs* of $(V,0)$ with respect to the coordinates x,y. It is known that the Puiseux pairs are independent of the coordinates as long as the condition $k_1>n_1$ is satisfied (cf. for example [1], [15]). This follows also from (2.6) below. We will assume always this condition, unless the coordinates are specified explicitly. Note that the condition is always satisfied by exchanging the coordinates x,y if necessary.

We define the *modified Puiseux pairs* of $(V,0)$ with respect to the coordinates x,y by deleting $\sum_{1\le i\le[k_1/n_1]} c_{0,i}x^i$ in (2.1.1) and allowing $n_1=1$ if $c_{0,i}\ne 0$ for some i in the old expression. If $n_1>1$ (i.e., $c_{0,i}=0$ in the old expression), the modified Puiseux pairs are the same as the Puiseux pairs with respect to the coordinates. (This notion depends on the coordinates, and will be used for the inversion of Puiseux pairs, cf. (2.7).)

2.2. Let $(V,0)$ be a germ of reduced plane curve in $(\mathbf{C}^2,0)$. We have a canonical embedded resolution $\phi:(X',D)\to(\mathbf{C}^2,0)$ by iterating point center blow-ups along the points at which the total transform of V does not have normal crossings, where $D=\phi^{-1}(0)$. Let V' be the proper transform of V. By definition, $\phi^{-1}(V)=D\cup V'$ is a divisor with normal crossings, and the irreducible components $D_\alpha(\alpha\in\Lambda)$ of D are $\mathbf{P}^1$. We say that D_α is proximate to D_β if D_β is first obtained as (the proper transform of) the exceptional divisor E_β of a blow-up and then D_α is obtained as (the proper transform of) the exceptional divisor of a blow-up whose center is contained in the proper transform of E_β, cf. [3], [4], [15], [16].

2.3. Definition (cf. [3], [4]). With the above notation, the *Enriques diagram* of V is an oriented graph E consisting of white and black vertices and arrows:
(i) The white vertices correspond to the exceptional divisors $D_\alpha\,(\alpha\in\Lambda)$, and the black vertices to the irreducible components of V'.

(ii) There is an arrow from a white vertex corresponding to D_α to a white vertex corresponding to D_β if D_α is proximate to D_β. There is an arrow form a black vertex to a white vertex if the corresponding divisors intersect. There are no other arrows.

2.4. *Continued fraction expansion.* Let k, n be mutually prime positive integers such that $n > 1$. Let

$$\frac{k}{n} = a_1 + \frac{1}{a_2 + \cdots + \frac{1}{a_h}} \tag{2.4.1}$$

be the continued fraction expansion, i.e., there are positive integers $r_0, \ldots, r_h$ such that $r_0 = k, r_1 = n$ and

$$r_{i-1} = a_i r_i + r_{i+1}, \quad a_i = \left[\frac{r_{i-1}}{r_i}\right] \quad \text{for } 1 \le i \le h, \tag{2.4.2}$$

with $r_{h+1} = 0$,. Here $r_h = 1$ because $(k, n) = 1$. We have $a_1 = 0$ if $k < n$.

We define nonnegative integers $P_1, \ldots, P_h$ and $Q_1, \ldots, Q_h$ by

$$\mathbf{v}_i = a_i \mathbf{v}_{i-1} + \mathbf{v}_{i-2} \text{ for } 1 \le i \le h \tag{2.4.3}$$

with $\mathbf{v}_i = (P_i, Q_i)$, where $\mathbf{v}_{-1} = (0, 1), \mathbf{v}_0 = (1, 0)$. Then

$$P_i Q_{i-1} - P_{i-1} Q_i = (-1)^i, \tag{2.4.4}$$

and $(\mathbf{v}_{i-1}, \mathbf{v}_i)$ is a basis of $\mathbf{Z}^2$. So there exists integers $x_i, y_i (0 \le i \le h)$ such that

$$(k, n) = x_i \mathbf{v}_i + y_i \mathbf{v}_{i-1}.$$

Note that $x_0 = k, y_0 = n$ and $x_{i-1} = a_i x_i + y_i, y_{i-1} = x_i (1 \le i \le h)$ by (2.4.3). Comparing this with (2.4.2), we get $x_i = r_i$ and $(x_h, y_h) = (1, 0)$, i.e.,

$$P_h = k, Q_h = n. \tag{2.4.5}$$

This implies

$$\frac{P_i}{Q_i} = a_1 + \frac{1}{a_2 + \cdots + \frac{1}{a_i}}. \tag{2.4.6}$$

by replacing k/n with the right hand side of (2.4.6).

2.5. With the above notation, we define the oriented graph $E(k, n)$ as follows:
(i) The vertices of $E(k, n)$ consist of white vertices $\{D_{i,j} : 1 \le i \le h, 1 \le j \le a_j\}$ and a black vertex V'.
(ii) There is an arrow from $D_{i',j'}$ to $D_{i,j}$ if one of the following conditions is satisfied:
(a) $i' = i$ and $j' = j + 1$,
(b) $i' = i + 1$ and $j = a_i$,
(c) $i' = i + 2, j = a_i$ and $j' = 1$.

There is an arrow from the black vertex V' to a white vertex $D_{i,j}$ if $i = h$ and $j = a_h$. There are no other arrows.

We say that $D_{i,j}$ is an *even* (resp. *odd*) vertex if so is i, and $D_{i,j}$ is the *first* (resp. *last*) white vertex of $E(k,n)$ if $(i,j) = (1,1)$ (resp. (h,a_h)).

Let (k_i,n_i) be pairs of mutually prime positive integers such that $n_i > 1 (1 \le i \le g)$. We define an oriented graph $E(k_1,n_1;\dots;k_g,n_g)$ by induction on g. It has a unique black vertex, and there is a unique white vertex, called the *last* vertex, to which there is an arrow from the black vertex. If $g = 1$, this is $E(k_1,n_1)$ defined above. For $g > 1$, $E(k_1,n_1;\dots;k_g,n_g)$ is obtained by deleting the black vertex of $E(k_1,n_1;\dots;k_{g-1},n_{g-1})$ and the arrow between the black vertex and the last white vertex, and then identifying the last white vertex of $E(k_1,n_1;\dots;k_{g-1},n_{g-1})$ with the first vertex of $E(k_g+n_g,n_g)$.

The white vertices of $E(k_1,n_1;\dots;k_g,n_g)$ are naturally identified with the disjoint union of the white vertices of $E(k_\nu,n_\nu)(1 \le \nu \le g)$, and the vertex corresponding to $D_{i,j}$ of $E(k_\nu,n_\nu)$ will be denoted by $D_{i,j}^{(\nu)}$.

The following proposition is due to Deligne [4] and Zariski [16].

2.6. Proposition. . *Let $(V,0)$ be a germ of reduced and irreducible plane curve, and $(k_1,n_1),\dots,(k_g,n_g)$ the Puiseux pairs of $(V,0)$. Then the Enriques diagram of $(V,0)$ is $E(k_1,n_1;\dots;k_g,n_g)$. More precisely, the canonical embedded resolution $\phi : (X',D) \to (\mathbf{C}^2,0)$ of $(V,0)$ is a composition of $\psi_\nu : (X_\nu,D_\nu) \to (X_{\nu-1},D_{\nu-1})(1 \le \nu \le g)$ such that the proper transform V_ν of V by $\phi_\nu = \psi_1 \circ \cdots \circ \psi_\nu$ has modified Puiseux pairs $(k_{\nu+1},n_{\nu+1}),\dots,(k_g,n_g)$ with respect to some local coordinates x_ν,y_ν such that x_ν is the defining equation of the divisor corresponds to the last white vertex of $E(k_\nu,n_\nu)$, where $(X_g,D_g) = (X',D),(X_0,D_0) = (\mathbf{C}^2,0)$, and $D_\nu = \phi_\nu^{-1}(0)$. The resolution process of ψ_ν and ϕ_ν are expressed respectively by the oriented graphs $E(k_\nu,n_\nu)$ and $E(k_1,n_1;\dots;k_\nu,n_\nu)$.*

Remark. For $\nu < g$, the proper transform V_ν intersects transversally with the divisor corresponding to the last white vertex of $E(k_\nu,n_\nu)$ if $k_{\nu+1} > n_{\nu+1}$, and they intersect maximally otherwise. In both cases, their intersection number is equal to $n_{\nu+1}\cdots n_g$.

This proposition is a direct consequence of the following inversion formula of Puiseux pairs due to Abhyanker [1], Deligne [4] and Zariski [15]:

2.7. Lemma. *With the notation of (2.1), let $(k_1,n_1),(k_2,n_2),\dots,(k_g,n_g)$ be the modified Puiseux pairs of $(V,0)$ with respect to coordinates x,y. Then the modified Puiseux pairs of $(V,0)$ with respect to the coordinates y,x are $(n_1,k_1),(k_2,n_2),\dots,(k_g,n_g)$.*

2.8. Let $(V,0)$ be as in (2.2). Let Γ be the dual graph of the exceptional divisor D of the canonical resolution ϕ in (2.2). By definition, the dual graph consists of white and black vertices and edges connecting them such that:
(i) The vertices are the same as the Enriques diagram.
(ii) Two vertices are connected if the corresponding divisors intersect.

For mutually prime positive integers (k,n) such that $n > 1$, we define a graph $\Gamma(k,n)$ as follows:
(i) The vertices are the same as $E(k,n)$.
(ii) Two vertices $D_{i,j},D_{i',j'}$ ($i < i'$ or $i = i', j < j'$) are connected in the following cases:

(a) $i' = i, j' = j + 1$.
(b) $i' = i + 2, j = a_i, j' = 1$.
(c) $i = h - 1, i' = h, j = a_{h-1}, j' = a_h$.

The last white vertex and the black vertex are connected. The vertices are not connected in the other cases. (This means that, deleting the last white vertex, the white vertices have two connected components: one consists of even vertices and the other of odd ones, and they are connected linearly.)

We define a graph $\Gamma(k_1, n_1; \ldots; k_g, n_g)$ inductively by identifying the black vertex of $\Gamma(k_1, n_1; \ldots; k_{g-1}, n_{g-1})$ with the first odd vertex of $\Gamma(k_g, n_g)$, where the first odd vertex is the first vertex $D_{1,1}$ if $k_g > n_g$ and $D_{3,1}$ otherwise.

Let D_α, D_β be irreducible components of D in (2.2). They intersect if and only if one of them is proximate to the other and there are no vertices proximate to both. So we get

2.9. Lemma. *With the above notation, assume $(V, 0)$ irreducible with Puiseux pairs $(k_1, n_1), (k_2, n_2), \ldots, (k_g, n_g)$. Then the dual graph Γ is $\Gamma(k_1, n_1; \ldots; k_g, n_g)$.*

3. Multiplicity

3.1. With the notation of (2.2), let m_α be the multiplicity of $\phi^* f$ along $D_\alpha (\alpha \in \Lambda)$, where f is a reduced defining equation of V. Let P_α be the center of the blow-up such that D_α is the proper transform of its exceptional divisor. Let V_α be the proper transform of V at P_α. Then we have

$$m_\alpha = \sum_{\beta \leftarrow \alpha} m_\beta + \mathrm{mult}_{P_\alpha} V_\alpha, \tag{3.1.1}$$

where $\beta \leftarrow \alpha$ if D_α is proximate to D_β.

3.2. We study the ν^{th} step of the resolution $\psi_\nu : X_\nu \to X_{\nu-1}$ in the notation of (2. 6). Let $k = k_\nu, n = n_\nu$ and $n' = n_{\nu+1} \cdots n_g$. We have the continued fraction expansion (2.4.1) of k/n. Let r_i be as in (2.4.2). Then we see

(3.2.1) $r_i n'$ is the multiplicity of the proper transform of V at the center of the blow up corresponding to $D_{i,j}^{(\nu)}$.

Let $m_{i,j}^{(\nu)}$ be the multiplicity of $\phi^* f$ along $D_{i,j}^{(\nu)}$ in the notation of (2.5). Let $m_i = m_{i,a_i}^{(\nu)}$ and $\overline{m}_i = m_i / n'$ for $1 \le i \le h$. By (3.1.1)(3.2.1), we get

$$\overline{m}_i = a_i(r_i + \overline{m}_{i-1}) + \overline{m}_{i-2} \quad \text{for} \quad 1 \le i \le h, \tag{3.2.2}$$

where $\overline{m}_0 = 0$ and $\overline{m}_{-1} = m_{-1}/n'$ with m_{-1} the multiplicity of $\phi^* f$ along the divisor corresponding to the last white vertex of $E(k_{\nu-1}, n_{\nu-1})$ if $\nu > 1$, and 0 otherwise. We have $\overline{m}_i \in \mathbf{Z}$ applying (3.2.1)(3.2.2) inductively to $\nu' < \nu$.

Remark. If $k < n$, we have $a_1 = 0, \overline{m}_1 = \overline{m}_{-1}$ and $r_2 = k$. If $k = 1$, we have $h = 2, a_1 = 0, a_2 = n, r_0 = r_2 = 1$ and $r_1 = n$.

3.3. Lemma. *Let k, n be as above, and $P_1, \dots, P_h$ and $Q_1, \dots, Q_h$ as in (2.4). Then we have*

$$\overline{m}_i = \begin{cases} nP_i + \overline{m}_{-1}Q_i & \text{if } i : \text{odd} \\ (\overline{m}_{-1} + k)Q_i & \text{otherwise.} \end{cases} \tag{3.3.1}$$

Proof. We have

$$r_{i+1} = -a_i r_i + r_{i-1} \quad \text{for } 1 \le i \le h \tag{3.3.2}$$

by (2.4.2). Together with (3.2.2), this implies

$$(\overline{m}_i + r_{i+2}/2) = a_i(\overline{m}_{i-1} + r_{i+1}/2) + (\overline{m}_{i-2} + r_i/2) \quad \text{for} \quad 1 \le i \le h, \tag{3.3.3}$$

where $\overline{m}_{-1} + r_0/2 = \overline{m}_{-1} + k/2, \overline{m}_0 + r_1/2 = n/2$. So we get

$$\overline{m}_i + r_{i+2}/2 = (n/2)P_i + (\overline{m}_{-1} + k/2)Q_i \quad \text{for } 1 \le i \le h \tag{3.3.4}$$

On the other hand, we have

$$(-1)^{i+1} r_{i+1} = -nP_i + kQ_i \text{ for } 1 \le i \le h \tag{3.3.5}$$

by (3.3.2). So we get the assertion.

3.4. With the notation of (3.2), let C_0 denote the last white vertex $D^{(\nu)}_{h,a_h}$ of $E(k_\nu, n_\nu)$, and C_1, C_2, C_3 the divisors intersecting with C_0 such that
(i) C_1 is the last white vertex of $E(k_{\nu-1}, n_{\nu-1})$ if $k_\nu = 1$, and $D^{(\nu)}_{h-1,a_{h-1}}$ otherwise,
(ii) $C_2 = D^{(\nu)}_{h,a_h-1}$,
(iii) C_3 is the first odd white vertex of $E(k_{\nu+1}, n_{\nu+1})$ if $\nu < g, k_{\nu-1} \ne 1$, the last white vertex of $E(k_{\nu+1}, n_{\nu+1})$ if $\nu < g, k_{\nu-1} = 1$, and V' otherwise.

Let m'_i denote the multiplicity of $\phi^* f$ along C_i, and $\overline{m}'_i = m'_i/n'$, where $n' = n_{\nu+1} \cdots n_g, k = k_\nu, n = n_\nu$, and $\overline{m}_{-1}$ are as in (3.2). Since the intersection number of C_0 with the total transform $\sum_{\alpha \in \Lambda} m_\alpha D_\alpha + V'$ of V is zero, we have

$$\sum_i \overline{m}'_i \equiv 0 \mod \overline{m}'_0. \tag{3.4.1}$$

Since

$$\overline{m}'_2 = \overline{m}'_0 - \overline{m}'_1 - 1 \tag{3.4.2}$$

by (3.2.1), we get

$$\overline{m}'_3 \equiv 1 \mod \overline{m}'_0. \tag{3.4.3}$$

Let $P = P_{h-1}, Q = Q_{h-1}$ in the notation of (3.3). Then $kQ - nP = (-1)^h$ by (2.4). So we get

$$\overline{m}'_0 = (\overline{m}_{-1} + k)n \tag{3.4.4}$$

$$\overline{m}'_1 = \begin{cases} (\overline{m}_{-1} + k)Q - 1 & \text{if } h : \text{even} \\ (\overline{m}_{-1} + k)Q & \text{otherwise.} \end{cases} \tag{3.4.5}$$

by (3.3.1). They imply

$$\overline{m}'_2 = \begin{cases} (\overline{m}_{-1} + k)(n - Q) & \text{if } h : \text{even} \\ (\overline{m}_{-1} + k)(n - Q) - 1 & \text{otherwise.} \end{cases} \tag{3.4.6}$$

by (3.4.2). We can check that $\overline{m}'_3$ is equal to 1 if $\nu = g, 1 + \overline{m}'_0$ if $\nu < g, k_{\nu+1} = 1$, and $1 + \overline{m}'_0(1 + [n_{\nu+1}/k_{\nu+1}])$ otherwise. But it will not be used later, because (3.4.3) is sufficient.

Let w_ν be as in (1.4). Then (3.4.4) implies

$$w_\nu = \overline{m}_{-1} + k_\nu, \;\; m'_0 = w_\nu n_\nu \cdots n_g \tag{3.4.7}$$

by induction on ν. We can also check that the multiplicity on the first even vertex of $E(k_\nu, n_\nu)$ is $w_\nu n_{\nu+1} \cdots n_g$.

3.5. *Remark.* With the notation of (2.2), let d_α be the multiplicity of the determinant of $d\phi$ along D_α, and $\tilde{d}_\alpha = d_\alpha + 1, e_\alpha = \tilde{d}_\alpha / m_\alpha$. Assume V irreducible. Let $d^{(\nu)}_{i,j} = d_\alpha, \tilde{d}^{(\nu)}_{i,j} = d_\alpha$ and $e^{(\nu)}_{i,j} = e_\alpha$, if $D_\alpha = D^{(\nu)}_{i,j}$. We say that $D^{(\nu)}_{i,j}$ is odd (resp. even) if i is odd (resp. even) or $D^{(\nu)}_{i,j}$ is the last white vertex of $E(k_\nu, n_\nu)$. Then, for $D^{(\nu)}_{i,j} \neq D^{(\nu')}_{i',j'}$, we have the inequality $e^{(\nu)}_{i,j} > e^{(\nu')}_{i',j'}$ in the following cases:

(a) $D^{(\nu)}_{i,j}, D^{(\nu')}_{i',j'}$ are odd, and $(\nu, i, j) > (\nu', i', j')$ with $\nu' > 1$ or $(i, j) < (i', j')$ with $\nu = \nu' = 1$.

(b) $D^{(\nu)}_{i,j}, D^{(\nu')}_{i',j'}$ are even, $\nu = \nu'$ and $(i, j) < (i', j')$.

(c) $D^{(\nu)}_{i,j}$ is even and $D^{(\nu')}_{i',j'}$ is odd, $\nu \geq \nu' > 1$.

(d) $D^{(\nu')}_{i',j'}$ is the last white vertex of $E(k_1, n_1)$.

Here $>$ denote the lexicographic order. The case (c) follows from (a) and (b), and may be omitted. By (d), the minimum of $e^{(\nu)}_{i,j}$ is attained by the last white vertex of $E(k_1, n_1)$. We can show that it is equal to $(k_1 + n_1)/k_1 n_1 \cdots n_g$.

In fact, with the notation of (3.2), let $\tilde{d}_i = d_i + 1 \, (-1 \leq i \leq h)$ with $d_i = \tilde{d}^{(\nu)}_{i,a_i} \, (1 \leq i \leq h)$, $d_0 = 0$ and d_{-1} the multiplicity of $\det d\phi$ along the divisor corresponding to the last white vertex of $E(k_{\nu-1}, n_{\nu-1})$ if $\nu > 1$, and 0 otherwise. Then we have

$$\tilde{d}^{(\nu)}_{i,j} = j\tilde{d}_{i-1} + \tilde{d}_{i-2}, \;\; \tilde{d}_i = a_i \tilde{d}_{i-1} + \tilde{d}_{i-2} \quad \text{for } 1 \leq i \leq h,$$

and $\tilde{d}_i = P_i + \tilde{d}_{-1} Q_i$ by (2.4.3). Combining with (2.4.4)(2.4.4), we can check the assertion, showing also $e^{(\nu)}_{i,j} < 1$ inductively, which implies $n\tilde{d}_{-1}/\overline{m}_{-1} < 1$ in the above notation.

4. Proof of Theorem (1.5)

4.1. With the notation of (1.1), let

$$H_\lambda^{p,q} = \mathrm{Gr}_F^p \mathrm{Gr}_{p+q}^W H^n(X_\infty, \mathbf{C})_\lambda. \tag{4.1.1}$$

where F and W are the Hodge and weight filtrations of the mixed Hodge structure. We denote by $D_\beta (\beta \in \Lambda')$ the irreducible components of the proper transform V' of $V (cf.(2.2))$. Let $\Lambda_\alpha = \{\beta \in \Lambda \cup \Lambda' : D_\alpha \cap D_\beta \neq \emptyset\}$, and $m_\beta = 1$ for $\beta \in \Lambda'$. By [11, (3.13–14)], we have

$$\begin{aligned} H_\lambda^{0,1} &= \bigoplus_{\substack{\alpha\in\Lambda \\ 0\le c\le m_\alpha \\ \lambda=\mathbf{e}(c/m_\alpha)}} H(\alpha, c) \quad \text{with} \\ H(\alpha,c) &= H^1\Big(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}\Big(\sum_{\beta\in\Lambda_\alpha}\Big(-\frac{cm_\beta}{m_\alpha} + \Big[\frac{cm_\beta}{m_\alpha}\Big]\Big)\Big)\Big). \qquad (4.1.2) \\ &\simeq H^0\Big(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}\Big(\sum_{\beta\in\Lambda_\alpha}\Big(\frac{cm_\beta}{m_\alpha} - \Big[\frac{cm_\beta}{m_\alpha}\Big]\Big) - 2\Big)\Big). \end{aligned}$$

We have $H(\alpha, c) = 0$ if $|\Lambda_\alpha| \le 2$, and $H(\alpha, c) = \mathbf{C}$ if and only if

$$\sum_{\beta\in\Lambda_\alpha}\Big(\frac{cm_\beta}{m_\alpha} - \Big[\frac{cm_\beta}{m_\alpha}\Big]\Big) = 2. \tag{4.1.3}$$

Remark. We can show (4.1.2) using the weight spectral sequence of $H^\bullet(X_\infty, \mathbf{C})$. In fact, let $g = \phi^* f$. Then we can calculate $\mathrm{Gr}_0^W \psi_{g,\lambda}\mathbf{C}_{X'}$, where $\psi_{g,\lambda}\mathbf{C}_{X'}$ is the λ-eigenvalue part of the nearby cycle $\psi_{g,\lambda}\mathbf{C}_{X'}$ by the action of the monodromy, and W is the monodromy filtration. Its direct factor supported on D_α is an intersection complex associated with a local system of rank one whose monodromy around the intersection with D_β is λ^{-m_β}. Then we can deduce (4.1.2) using Deligne's canonical extension of the local system.

For the calculation of (4.1.2) we use the following

4.2. Lemma. *Let $\alpha_1, \alpha_2, \alpha_3$ be three real numbers such that $\alpha_1 + \alpha_2 + \alpha_3 \in \mathbf{Z}$. Then $\sum_i(\alpha_i - [\alpha_i]) = 2$ if and only if $\alpha_1 \notin \mathbf{Z}$ and $\alpha_2 + \alpha_3 - [\alpha_2 + \alpha_3] < \alpha_3 - [\alpha_3]$.*

4.3. *Proof of* (1.5). Since V is irreducible, the monodromy is semisimple and one is not an eigenvalue of the monodromy [2], [7], so that

$$H_\lambda^{0,0} = H_\lambda^{1,1} = H_1^{p,q} = 0, \tag{4.3.1}$$

cf. [11]. So it is enough to calculate $H_\lambda^{0,1}$ for $\lambda \neq 1$ by the Hodge symmetry. We have $H(\alpha, c) = 0$ unless D_α is the last white vertex of $E(k_\nu, n_\nu)$ for some $\nu, cf.(2.9)$. So we may assume that D_α is C_0 in (3.4). The multiplicities are calculated in (3.4). Let $w = w_\nu$. By

(3.4.7) applied to $\nu-1$, we get $n|\overline{m}_{-1}$, and $(w,n)=1$. With the notation of (3.2), the integer c in $[0,wnn')$ is expressed uniquely as $c=iw+jn+rwn$ for $i,j,r\in\mathbf{Z}$ such that

$$0\le i<n, 0\le iw+jn<wn, 0\le r<n'.$$

By (3.4)(4.1.3) and (4.2), $H(\alpha,c)=\mathbf{C}$ if and only if

$$\begin{aligned}
&\frac{c(n-Q)}{n}\notin\mathbf{Z},\quad \frac{cQ}{n}-\left[\frac{cQ}{n}\right]<\frac{c}{wn}-\left[\frac{c}{wn}\right] && \text{if } h:\text{even}\\
&\frac{cQ}{n}\notin\mathbf{Z},\quad \frac{c(n-Q)}{n}-\left[\frac{c(n-Q)}{n}\right]<\frac{c}{wn}-\left[\frac{c}{wn}\right] && \text{otherwise.}
\end{aligned}$$

Since $Qw\equiv(-1)^h \mod n$ by (2.4.4), this condition is equivalent to

$$i>0, j>0.$$

By definition of exponents (2.2), the contribution of C_0 to the exponents which are greater than 1 is given by

$$\left\{2-\left(\frac{1}{n_{\nu+1}\cdots n_g}\left(\frac{i}{n_\nu}+\frac{j}{w_\nu}\right)+\frac{r}{n_{\nu+1}\cdots n_g}\right)\right\}$$

for $0<i<n_\nu$, $0<j<w_\nu$, $0\le r<n_{\nu+1}\cdots n_g$ such that $i/n_\nu+j/w_\nu<1$. So we get the assertion by the symmetry of exponents, cf. (1.2.3).

5. Variance of exponents

In this section we prove the following conjecture of C. Hertling [6] in the case of irreducible plane curve singularities.

5.1. Conjecture. $V_f:=\frac{1}{\mu}\sum_i(\alpha_i-\frac{n+1}{2})^2\le(\alpha_\mu-\alpha_1)/12$ with the notation of (1.2).

Remark. V_f is the variance (i.e. the square of the standard deviation) of the exponents. Hertling (loc. cit.) showed that the equality holds in (5.1) if f is quasihomogeneous. In particular, (5.1) is true for $g=1$ in the case of irreducible plane curve singularity.

5.2. Theorem. *$V_f<(\alpha_\mu-\alpha_1)/12$ for $g>1$ in the case of irreducible plane curve singularity.*

Proof. Let $\{\alpha_i\}_{i\in\Lambda(\nu)}$ be the exponents which are less than 1 and come from the νth part of the Enriques diagram corresponding to $E(k_\nu,n_\nu)$, cf. (2.5). Here $\{\Lambda(\nu)\}_{1\le\nu\le g}$ is a partition of $\{1,\dots,\mu/2\}$. By (1.5), $\{\alpha_i\}_{i\in\Lambda(\nu)}$ are given by (1.5.2) with ν fixed.

Let $n'_\nu=n_{\nu+1}\cdots n_g$ for $0\le\nu\le g$ (where $n'_g=1$). We define

$$\begin{aligned}
\mu^{(\nu)}&=2|\Lambda(\nu)|\,(=(w_\nu-1)(n_\nu-1)n'_\nu),\\
S^{(\nu)}&=2\sum_{i\in\Lambda(\nu)}(\alpha_i-1)^2,\\
\varepsilon^{(\nu)}&=6S^{(\nu)}-\mu^{(\nu)}(1-\alpha_1).
\end{aligned}$$

Then we have to show $\sum_{\nu=1}^{g}\varepsilon^{\nu}<0$.

By (1.5), we have

$$\alpha_1=\frac{k_1+n_1}{k_1n_0'},\tag{5.2.1}$$

because $(k_1+n_1)/k_1n_1<1$. We can verify (see also [6])

$$6S^{(g)}=\mu^{(g)}\left(1-\frac{1}{w_g}-\frac{1}{n_g}\right).\tag{5.2.2}$$

Let $a=w_\nu$, $b=n_\nu$, $c=n_\nu'$, and $\Lambda(a,b)=\big\{(i,j)\in\mathbb{Z}^2:i,j>0,\frac{i}{a}+\frac{j}{b}\le 1\big\}$. Then

$$\begin{aligned}
6S^{(\nu)}&=\sum_{k=0}^{c-1}\sum_{(i,j)\in\Lambda(a,b)}\frac{12}{c^2}\left(1-\frac{i}{a}-\frac{j}{b}+k\right)^2\\
&=\sum_{(i,j)\in\Lambda(a,b)}\left(\frac{12}{c}\left(1-\frac{i}{a}-\frac{j}{b}\right)^2+12\left(1-\frac{i}{a}-\frac{j}{b}\right)\frac{c-1}{c}\right)\\
&\qquad+(a-1)(b-1)\frac{(c-1)(2c-1)}{c}\\
&\le\frac{(a-1)(b-1)}{c}\left(\left(1-\frac{1}{a}-\frac{1}{b}\right)+2(c-1)+(c-1)(2c-1)\right)\\
&\qquad-\left(1-\frac{1}{b}\right)\left(1-\frac{1}{c}\right)(a+b-1)
\end{aligned}$$

by (5.3) below together with a formula similar to (5.2.2) (where $w_g=a$, $n_g=b$). So we get

$$\varepsilon^{(\nu)}\le(a-1)(b-1)c-ab+\frac{b-1}{ac}+\frac{a-1}{b}+1+(a-1)(b-1)c\alpha_1.$$

Since $w_\nu=w_{\nu-1}n_{\nu-1}n_\nu+k_\nu$, this implies for $1<\nu<g$

$$\begin{aligned}
\varepsilon^{(\nu)}&\le w_\nu n_{\nu-1}'-w_{\nu-1}n_{\nu-2}'-w_\nu n_\nu+w_{\nu-1}n_{\nu-1}-(k_\nu-1)\left(n_\nu'-\frac{1}{n_\nu}\right)\\
&\qquad-(n_{\nu-1}'-1)+\frac{n_\nu-1}{w_\nu n_\nu'}+(w_\nu-1)(n_\nu-1)n_\nu'\alpha_1.
\end{aligned}$$

Then, after a calculation, we get a simple expression

$$\sum_{\nu=1}^{g}\varepsilon^{(\nu)}\le\sum_{\nu=1}^{g}\left(\frac{n_\nu-1}{w_\nu n_\nu'}-(k_\nu-1)\left(n_\nu'-\frac{1}{n_\nu}\right)-(n_{\nu-1}'-1)\right)+\mu\alpha_1.\tag{5.2.3}$$

By induction on j, we see

$$\sum_{\nu=1}^{j}\mu^{(\nu)}=(w_j-1)n_{j-1}'-\sum_{\nu=1}^{j}(k_\nu+n_{\nu-1}n_\nu-1)n_\nu',\tag{5.2.4}$$

where $n_0 = 0$. Combined with $w_g n_g = \sum_{\nu=1}^{g} k_\nu n'_{\nu-1} n'_\nu$, this (for $j = g$) implies

$$\mu\alpha_1 - \sum_{\nu=1}^{g}(k_\nu - 1)\left(n'_\nu - \frac{1}{n_\nu}\right) = \sum_{\nu=1}^{g}(k_\nu - 1)\left(n'_\nu - \frac{1}{n_\nu}\right)(n'_{\nu-1}\alpha_1 - 1) + \sum_{\nu=1}^{g}(n'_\nu - 1)n'_{\nu-1}\alpha_1.$$

Since $n'_{\nu-1}\alpha_1 - 1 < 0$ for $\nu > 1$ and $n'_0\alpha_1 - 1 = n_1/k_1$, it remains to show

$$(k_1 - 1)\left(n'_1 - \frac{1}{n_1}\right)\frac{n_1}{k_1} + (n'_1 - 1)\frac{n_1}{k_1} + \sum_{\nu=1}^{g}\frac{n_\nu - 1}{w_\nu n'_\nu} < n'_0 - 1.$$

But this is reduced to $\sum_{\nu=1}^{g}(n_\nu - 1)/w_\nu n'_\nu < (n_1 - 1)/k_1$, and is easily verified.

5.3. Lemma. *Let a, b be relatively prime positive integers such that $a > b$. Then*

$$\sum_{(i,j)\in\Lambda(a,b)}\left(1 - \frac{i}{a} - \frac{j}{b}\right) \le \frac{(a-1)(b-1)}{6} - \frac{(b-1)(a+b-1)}{12b}. \tag{5.3.1}$$

Proof. Let $F(a,b) = \sum_{(i,j)\in\Lambda(a,b)}\left(1 - \frac{i}{a} - \frac{j}{b}\right)$. We see

$$\begin{aligned} F(a,b) - \frac{a-b}{a}F(a-b,b) &= \sum_{(i,j)\in\Lambda(b,b)}\left(1 - \frac{i}{a} - \frac{j}{b}\right) \\ &= \frac{(b-1)(2ab - b^2 - a - b)}{6a}. \end{aligned} \tag{5.3.2}$$

Let $a = mb + k$ with $0 < k < b$. Then (5.3.2) implies

$$F(a,b) - \frac{k}{a}F(k,b) = \frac{(b-1)m}{12a}((2b^2 - b)m + 4bk - 2k - 3b). \tag{5.3.3}$$

Let $E(a,b) = (a-1)(b-1)/6 - F(a,b)$. Using (5.3.3) we can verify

$$E(a,b) - \frac{k}{a}E(b,k) = \frac{m(b-1)(a+b+k)}{12a}. \tag{5.3.4}$$

So the inequality $E(a,b) \ge (b-1)(a+b-1)/12b$ follows by induction, because

$$\frac{(k-1)(b+k-1)}{a} + \frac{m(b-1)(a+b+k)}{a} \ge \frac{(b-1)(a+b-1)}{b}. \tag{5.3.5}$$

This completes the proof of (5.2).

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