On the exact steepest descent method — a new method for the description of Stokes curves

To the memory of the late Professor Tosio KATO

by

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1 Introduction

No algorithmic way of describing Stokes regions, i.e., domains where the Borel sum of a WKB solution is well-defined, is known for higher order ordinary differential equations with a large parameter. (See, e.g., [AKT2].) The main difficulty lies in the necessity of "new Stokes curves", which emanate from possible crossing points of Stokes curves, as Berk, Nevins and Roberts ([BNR]) first observed in discussing a connection formula for WKB solutions. Berk et al. further presented a charming idea of applying the steepest descent method to the inverse Fourier transform of a WKB solution for the Fourier-transformed operator. (See also [WSW] for some related subject.) They have confirmed that this approach is effective near saddle points of the integrand ([BNR, p. 992]), but they have observed several troubles if a steepest descent path hits a turning point of the Fourier-transformed operator ([BNR, p. 1001]).

The principal aim of this article is to examine the idea of [BNR] from the viewpoint of the exact WKB analysis, i.e., with the Borel resummation of WKB solutions (cf. [V], [S], [DDP] and references cited there; see also [BH] for a related subject "hyperasymptotics"), and to propose a new notion "an exact steepest descent path", which naturally fits in with the Borel resummation method. We then present some examples which illustrate how the troubles [BNR] observed are naturally resolved by the use of exact steepest descent paths. We note that the notion of an exact steepest descent path is introduced here to exactly take into account the effect of exponentially small terms pertaining to Stokes phenomena of WKB solutions; this explains why we have succeeded in resolving the troubles that [BNR] encountered far away from saddle points.

2 Notational preliminaries

The operators we study in this paper are supposed to have the following form:

$$P = \sum_{\substack{0 \le j \le m \\ 0 \le k \le n}} a_{jk}\, x^k\, \eta^{m-j} \left(\frac{d}{dx}\right)^j, \tag{1}$$

where a_{jk} is a constant and η is a large parameter. Let $\hat{\psi}(\xi,\eta)$ denote the Laplace transform of a function $\psi(x,\eta)$, i.e.,

$$\psi(x,\eta) = \int e^{\eta x\xi}\,\hat{\psi}(\xi,\eta)d\xi, \tag{2}$$

and let $\hat{P}$ denote the Laplace transform of P, i.e.,

$$\hat{P} = \sum a_{jk}\,\eta^{m-k}\left(-\frac{d}{d\xi}\right)^k \xi^j. \tag{3}$$

[For the sake of the simplicity of notations we use the Laplace transform instead of the Fourier transform, that is, we use $e^{\eta x\xi}$ instead of $e^{i\eta x\xi}$. We also assume $m \geq n$, as our purpose is to analyze the structure of WKB solutions of a higher order operator P in terms of the exact WKB analysis for $\hat{P}$.] In what follows we let $p(x,\xi)$ denote

$$\sum_{\substack{0\leq j\leq m\\ 0\leq k\leq n}} a_{jk}\,x^k\,\xi^j. \tag{4}$$

Solving the equation $p(x,\xi)=0$ in the form $x = x_k(\xi)$ $(k=1,\dots,n)$, we obtain n WKB solutions $\hat{\psi}_k(\xi,\eta)$ of the equation $\hat{P}\hat{\psi}=0$ so that they have the following form

$$\hat{\psi}_k = \exp(\eta\int^{\xi}(-x_k(\xi))d\xi + \cdots)/\sqrt{\eta}. \tag{5}$$

Here the normalization factor $\eta^{-1/2}$ is introduced for the sake of convenience in handling its Borel transform, and $\cdots$ designates terms with non-positive powers of η.

Thus the object we are interested in is of the form

$$\varphi = \int \exp(\eta(x\xi - \int^{\xi} x_k(\xi)d\xi) + \cdots)/\sqrt{\eta}\,d\xi. \tag{6}$$

The specification of the path of integration in (6) will be given in §3. As $\hat{\psi}_k$ and φ are formal series in η^{-1}, their meaning should be given, say, by the Borel resummation. Hence we introduce operators P_B and $\hat{P}_B$ respectively by

$$P_B = \sum a_{jk}\,x^k \frac{\partial^j}{\partial x^j}\frac{\partial^{m-j}}{\partial y^{m-j}} \tag{7}$$

and

$$\hat{P}_B = \sum a_{jk} \frac{\partial^{m-k}}{\partial \tilde{y}^{m-k}} \left(-\frac{\partial}{\partial \xi}\right)^k \xi^j. \tag{8}$$

It then follows from the definition of the Borel transformation that

$$\hat{P}_B \, \hat{\psi}_{k,B} = 0 \tag{9}$$

for the Borel transform $\hat{\psi}_{k,B}$ of $\hat{\psi}_k$. Note that the Borel transform $\hat{\psi}_{k,B}$ is computed after applying the Taylor expansion of $\exp(\cdots)$; it then has the form,

$$\left(\exp(-\eta \int^{\xi} x_k(\xi) d\xi)\right) (c_0 + c_1 \eta^{-1} + c_2 \eta^{-2} + \cdots)/\sqrt{\eta} \tag{10}$$

and hence its Borel transform is, by definition,

$$\sum_{l=0}^{\infty} \frac{c_l}{\Gamma(l+1/2)} (\tilde{y} - \int^{\xi} x_k(\xi) d\xi)^{l-1/2}. \tag{11}$$

It also follows from the definition that the Borel resummed WKB solution $\hat{\psi}_k$ is given by

$$\hat{\psi}_k = \int \exp(-\eta \tilde{y}) \hat{\psi}_{k,B}(\xi, \tilde{y}) d\tilde{y} \tag{12}$$

where the integration is performed along the path

$$\tilde{y} = \int_{\hat{a}}^{\xi} x_k(\xi) d\xi + v, \quad v \geq 0. \tag{13}$$

Here $\hat{a}$ is an arbitrarily fixed point to normalize $\hat{\psi}_k$. (Cf., e.g., [V].)

So far we have discussed WKB solutions $\hat{\psi}_k$ of the equation $\hat{P}\hat{\psi} = 0$. It is clear that, if we solve the equation $p(x, \xi) = 0$ in the form $\xi = \xi_j(x)$ $(j = 1, \ldots, m)$ we obtain m WKB solutions $\psi_j(x, \eta)$ $(j = 1, \ldots, m)$ of $P\psi = 0$ in the form

$$\psi_j = \exp(\eta \int^{x} \xi_j(x) dx + \cdots)/\sqrt{\eta}, \tag{14}$$

or possibly with another normalization factor η^{-1} instead of $\eta^{-1/2}$, as we will see in §3. Its Borel transform $\psi_{j,B}$ is obtained in the same manner as above.

3 Local correspondence between $\hat{\psi}_{k,B}$ and $\psi_{j,B}$ near a saddle point

Let us first fix the path of integration in (6); as $\hat{\psi}_k(\xi,\eta)$ has been given its meaning through the Borel resummation, we have only to fix the path of integration in ξ-space. We stipulate that the ξ-integration should be performed along a steepest descent path for $\mathrm{Re}\, f_k$ that passes through a saddle point of f_k with f_k being $x\xi - \int_{\hat{a}}^{\xi} x_k(\xi)d\xi$. A saddle point ξ_* of f_k satisfies $x = x_k(\xi_*)$, and hence $\xi_* = \xi_j(x)$ holds for some j. Barring the accidental coincidence of ξ_* with a turning point of $\hat{P}$, we find

$$\frac{dx_k(\xi)}{d\xi}\Big|_{\xi=\xi_j(x)} \neq 0 \tag{15}$$

and hence

$$\frac{\partial^2 f_k}{\partial \xi^2}\Big|_{\xi=\xi_j(x)} \neq 0. \tag{16}$$

Thus the ξ-integration is performed along the path described by

$$f_k(x,\xi) = f_k(x,\xi_j(x)) - u^2, \quad u\text{: real}, \tag{17}$$

i.e.,

$$(x\xi - \int_{\hat{a}}^{\xi} x_k(\xi)d\xi) - (x\xi_j(x) - \int_{\hat{a}}^{\xi_j(x)} x_k(\xi)d\xi) = -u^2, \quad u\text{: real}. \tag{18}$$

Fixing the path of integration in (6) in this way, and substituting the Borel sum of $\hat{\psi}_k$ into the integrand, we rewrite the integral (6), by introducing $y = \tilde{y} - x\xi$, in the following way:

$$\varphi = \iint \exp(-\eta y)\hat{\psi}_{k,B}(\xi, y + x\xi)dy\, d\xi. \tag{19}$$

Here the domain of integration is described by the relation (18) and

$$y = -x\xi_j(x) + \int_{\hat{a}}^{\xi_j(x)} x_k(\xi)d\xi + v + u^2, \tag{20}$$

which follows from (13) and (18). To see the geometric meaning of (20), let $s(x)$ denote

$$-x\xi_j(x) + \int_{\hat{a}}^{\xi_j(x)} x_k(\xi)d\xi. \tag{21}$$

Then we find

$$\begin{aligned} \frac{ds(x)}{dx} &= -\xi_j(x) - x\frac{d\xi_j}{dx} + x_k(\xi_j(x))\frac{d\xi_j}{dx} \\ &= -\xi_j(x). \end{aligned} \tag{22}$$

Thus (20) is rewritten as

$$y = -\int_a^x \xi_j(x)dx + v + u^2, \tag{23}$$

where a is a fixed point. If we set $w = v + u^2$, the positivity assumption on v entails $u^2 \leq w$. Hence ξ obeying the relation (18) can range only over a compact set for a fixed pair (x, w) $(w \geq 0)$. Therefore the integral

$$\chi(x, y) = \int \hat{\psi}_{k,B}(x, y + x\xi)d\xi \tag{24}$$

is well-defined if y satisfies

$$y = -\int_a^x \xi_j(x)dx + w, \quad w \geq 0 \tag{25}$$

and if the ξ-integration is done along the path described by (18). Thus the integral (6) can be written as

$$\int \exp(-\eta y)\chi(x, y)dy \tag{26}$$

where y ranges over the half-line $\{y \in \mathbb{C}; y = -\int_a^x \xi_j(x)dx + w, w \geq 0\}$. We shall further confirm that $\chi(x, y)$ is the Borel transform of a WKB solution of the equation $P\psi = 0$. In view of the particular form of the integral (26), we will then be convinced that the integral (6) should play an important role in describing the domain on which the Borel sum of a WKB solution of $P\psi = 0$ is well-defined.

In order to verify that χ is the Borel transform of a WKB solution of $P\psi = 0$, let us first show $P_B\chi = 0$ by the integration by parts. In fact, we have

$$
\begin{aligned}
(27) \qquad P_B\chi &= \sum a_{jl}\, x^l \left(\frac{\partial}{\partial x}\right)^j \left(\frac{\partial}{\partial y}\right)^{m-j} \int \hat{\psi}_{k,B}(\xi, y + x\xi) d\xi \\
&= \sum a_{jl} \int \xi^j \left(\frac{\partial}{\partial y}\right)^{j+m-j} x^l\, \hat{\psi}_{k,B}(\xi, y + x\xi) d\xi \\
&= \sum a_{jl} \int \xi^j \left(\frac{\partial}{\partial y}\right)^{m-l} \left(x\frac{\partial}{\partial y}\right)^l \hat{\psi}_{k,B}(\xi, y + x\xi) d\xi \\
&= \sum a_{jl} \left(\frac{\partial}{\partial y}\right)^{m-l} \int \left(\frac{\partial}{\partial \zeta}\right)^l (\xi^j \hat{\psi}_{k,B}(\xi, y + x\zeta)) \Big|_{\zeta=\xi} d\xi.
\end{aligned}
$$

By repeatedly using the following identity

$$
\begin{aligned}
(28) \qquad & \frac{\partial}{\partial \xi}(\xi^j \varphi(\xi, y + x\xi)) \\
&= \frac{\partial}{\partial \xi}(\xi^j \varphi(\xi, y + x\zeta)) \Big|_{\zeta=\xi} + \frac{\partial}{\partial \zeta}(\xi^j \varphi(\xi, y + x\zeta)) \Big|_{\zeta=\xi}
\end{aligned}
$$

together with the integration by parts applied to the integral interpreted as an contour integral, we find

$$
\begin{aligned}
(29) \qquad P_B\chi &= \sum a_{jl} (\frac{\partial}{\partial y})^{m-l} \int (-\frac{\partial}{\partial \xi})^l (\xi^j \hat{\psi}_{k,B}(\xi, y + x\zeta))|_{\zeta=\xi} d\xi \\
&= 0,
\end{aligned}
$$

as $\hat{P}_B \hat{\psi}_{k,B}(\xi, y) = \sum a_{jl} (\frac{\partial}{\partial y})^{m-l} (-\frac{\partial}{\partial \xi})^l \xi^j \hat{\psi}_{k,B}(\xi, y) = 0$. Furthermore a mathematical theory ([SKK, p.422, Proposition 4.2.4]) tells us that $\chi(x, y)$ has a singularity of the form $\alpha(x, y) \log(y + \int_a^x \xi_j(x) dx) + \beta(x, y)$ with α and β holomorphic near $y = -\int_a^x \xi_j(x) dx$, as $\hat{\psi}_{k,B}(\xi, \tilde{y})$ has the singularity of the form $\tilde{\alpha}(\xi, \tilde{y})(\tilde{y} - \int_{\hat{a}}^{\xi} x_k(\xi) d\xi)^{-1/2} + \tilde{\beta}(\xi, \tilde{y})$ with $\tilde{\alpha}$ and $\tilde{\beta}$ holomorphic near $\tilde{y} = \int_{\hat{a}}^{\xi} x_k(\xi) d\xi$. This result combined with the relation $P_B\chi = 0$ entails that χ is the Borel transform of a WKB solution of $P\psi = 0$ that has the form $\exp(\eta \int_a^x \xi_j(x) dx + \cdots)/\eta$.

Summing up, the integral (6) relates $\hat{\psi}_{k,B}$ to $\psi_{j,B}$ unless extra singularities become relevant to the domain of integration specified by (13) and (18). As

we will see in the next section, other singularities of $\hat{\psi}_{k,B}(\xi, y)$ really sneak into the domain in question if the steepest descent path (18) crosses a Stokes curve for $\hat{P}$. In this sense, the discussion in this section concerns the local correspondence of $\hat{\psi}_{k,B}$ and $\psi_{j,B}$ near the saddle point $\xi = \xi_j(x)$. It is worth noting that this correspondence is what mathematicians call "the quantized Legendre transformation" with a generating function $\delta(y - \tilde{y} + x\xi)d\xi d\tilde{y}$ (cf., e.g., [K[3], p.226, Example 4.2.5]).

4 Correspondence of WKB solutions in the large — introduction of the notion of exact steepest descent paths

When the steepest descent path for $\mathrm{Re}\, f_k$ given in §3 that passes through the saddle point $\xi = \xi_j(x)$ crosses a Stokes curve for $\hat{P}$, say at $\xi = \xi_0$, the singularity of $\hat{\psi}_{k,B}(\xi, \tilde{y})$ located at $\tilde{y} = \int_{\hat{a}}^{\xi} x_{k'}(\xi)d\xi$ $(k' \neq k)$ may become relevant to the integral (19) by the definition of a Stokes curve in the exact WKB analysis (cf., e.g., [V]). If $\hat{\psi}_k$ is dominant over $\hat{\psi}_{k'}$ along the Stokes curve, the integral (19) will acquire an additional exponentially small term as a consequence of the "connection formula". To give the concrete description of the added term is the purpose of this section. Our description is based on a precise computation done for the case P is a harmonic oscillator ([T1]).

As the exact WKB analysis is complete only for the second-order operators, we now assume that $n = 2$, i.e., $\hat{P}$ is of the second order. (Practically speaking, we can deal with more general operators. See, e.g., Example 2 in §5.) When $\hat{P}$ is of the second order, we can, and really do, choose $\hat{a}$, the end point of the integral, to be the turning point from which the Stokes curve in question emanates so that $\hat{\psi}_k$ has the form

$$(-\eta x_k(\xi) + \cdots)^{-1/2} \exp\Big(\int_{\hat{a}}^{\xi} (-\eta x_k(\xi) + \cdots)d\xi\Big), \tag{30}$$

where $\cdots$ designates terms with odd (and negative) powers of η. We also note that the Stokes curve in question is given by

$$\mathrm{Im} \int_{\hat{a}}^{\xi} (x_k(\xi) - x_{k'}(\xi))d\xi = 0 \tag{31}$$

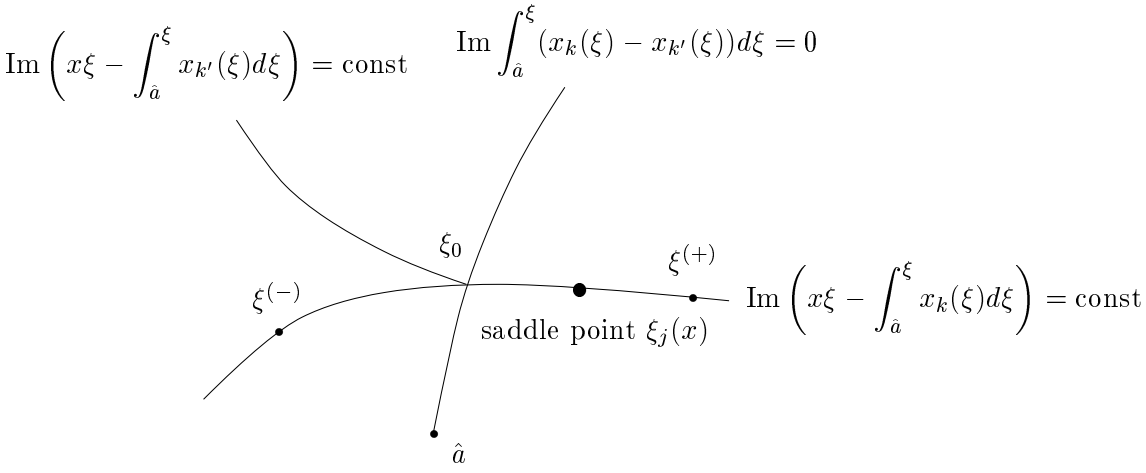

Figure 1 : Crossing of a steepest descent path and a Stokes curve.

(cf. Figure 1). Normalizing $\hat{\psi}_{k'}$ in the same manner as above, the dominant (Borel resummed) term $\hat{\psi}_k$ acquires an exponentially small term

$$\pm i\hat{\psi}_{k'} \tag{32}$$

when it crosses the Stokes curve (31). (See [AKT1].) The sign depends on the configuration of the steepest descent path and the Stokes curve, and we assume without loss of the generality that the sign is $+$. This exponentially small term appears as a consequence of the fact that at $\xi = \xi_0$ the singularity of $\hat{\psi}_{k,B}(\xi, \tilde{y})$ located at $\tilde{y} = \int_{\hat{a}}^{\xi} x_{k'}(\xi)d\xi$ hits the path of integration (13) that defines the Borel sum of $\hat{\psi}_k(\xi, \eta)$. Otherwise stated, if we define u_0 and v_0 respectively by

$$u_0^2 = -x\xi_0 + \int_{\hat{a}}^{\xi_0} x_k(\xi)d\xi + x\xi_j(x) - \int_{\hat{a}}^{\xi_j(x)} x_k(\xi)d\xi \tag{33}$$

and

$$v_0 = \int_{\hat{a}}^{\xi_0} (x_{k'}(\xi) - x_k(\xi))d\xi, \tag{34}$$

then the integrand of (19) finds a new singularity on its domain of integration when $(u, v) = (u_0, v_0)$. Note that $v_0 \geq 0$ holds because of the fact that ξ_0 is on the Stokes curve in question together with the fact that $\hat{\psi}_k$ is dominant over $\hat{\psi}_{k'}$ there. To fix the configuration of several points to be discussed below, we assume $u_0 < 0$. To analyze the integral (24) with the explicit use of ξ-variable (rather than the u-variable) we introduce the symbol $\xi^{(\pm)}$ to denote the end-points of the integral (24). For the sake of definiteness we suppose that $\xi^{(-)}$ corresponds to u that is negative and smaller than u_0. Note also that ξ_0 corresponds to u_0. If we let ξ_* denote the singular point of $\hat{\psi}_{k,B}(\xi, \tilde{y})$ that is described by $\tilde{y} = \int_{\hat{a}}^{\xi} x_{k'}(\xi) d\xi$, our task is then to describe concretely the integral I obtained by the integration from ξ_0 to ξ_* in Figure 2 (b) below.

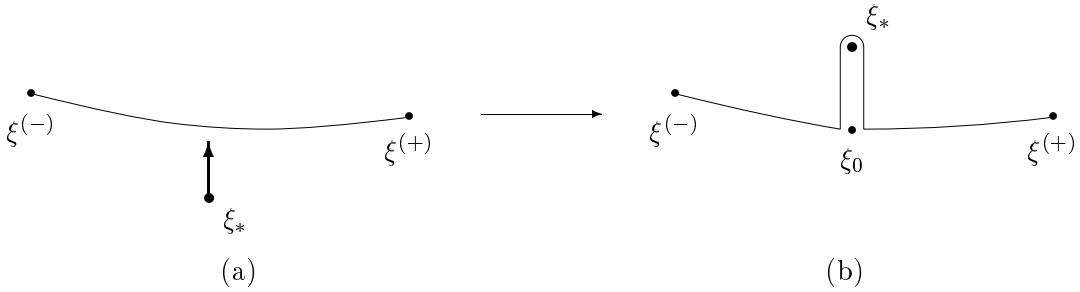

Figure 2 : Singular point ξ_* of $\hat{\psi}_{k,B}(\xi, \tilde{y})$ hitting the path of integration.

In view of the connection formula (32), this integral should be related to $i\hat{\psi}_{k',B}$. To find the precise form of the relation let us consider the following integral (35) with the domain of integration prescribed by (36) and (37) below; in what follows we show that the integral I is coincident with the integral (35) with this particular choice of the domain of integration. This integral was first found in the course of concrete computations for harmonic oscillators ([T1]):

$$(35) \qquad \begin{aligned} & i \iint \exp(\eta(x\xi - \tilde{y}))\hat{\psi}_{k',B}(\xi, \tilde{y}) d\xi d\tilde{y} \\ & \quad = i \int \exp(-\eta y)(\int \hat{\psi}_{k',B}(\xi, y + x\xi) d\xi) dy, \end{aligned}$$

where the domain of integration is given by

$$\tilde{y} = \int_{\hat{a}}^{\xi} x_{k'}(\xi)d\xi + \tilde{v},\ \tilde{v} \geq 0 \tag{36}$$

and

$$x\xi - \int_{\hat{a}}^{\xi} x_{k'}(\xi)d\xi - x\xi_0 + \int_{\hat{a}}^{\xi_0} x_{k'}(\xi)d\xi = -\tilde{u},\ \tilde{u} \geq 0. \tag{37}$$

The geometric meaning of (37) is worth mentioning: ξ obeying (37) is on the steepest descent path for $\mathrm{Re}\, f_{k'}$ with $f_{k'} \underset{\mathrm{def}}{=} x\xi - \int_{\hat{a}}^{\xi} x_{k'}(\xi)d\xi$ that passes through the point $\xi = \xi_0$. Note that ξ_0 is not a saddle point of $f_{k'}$. Let us now try to find where ξ ranges for fixed x and y; it is clear that

$$\begin{aligned} y &= \tilde{y} - x\xi \\ &= -x\xi_0 + \int_{\hat{a}}^{\xi_0} x_{k'}(\xi)d\xi + \tilde{u} + \tilde{v} \end{aligned} \tag{38}$$

follows from (36) and (37). Since we know by (22) that

$$-x\xi_j(x) + \int_{\hat{a}}^{\xi_j(x)} x_k(\xi)d\xi = -\int_a^x \xi_j(x)dx \tag{39}$$

holds for some a, we can rewrite (38) as follows:

(40)

$$y = -\int_a^x \xi_j(x)dx + x\xi_j(x) - \int_{\hat{a}}^{\xi_j(x)} x_k(\xi)d\xi - x\xi_0 + \int_{\hat{a}}^{\xi_0} x_{k'}(\xi)d\xi + \tilde{u} + \tilde{v}.$$

On the other hand, (33) and (34) entail

$$-x\xi_0 + \int_{\hat{a}}^{\xi_0} x_{k'}(\xi)d\xi + x\xi_j(x) - \int_{\hat{a}}^{\xi_j(x)} x_k(\xi)d\xi = u_0^2 + v_0. \tag{41}$$

Hence we find

$$y = -\int_a^x \xi_j(x)dx + w,\ w = w_0 + \tilde{u} + \tilde{v}, \tag{42}$$

where w_0 denotes $u_0^2 + v_0$. Since $\tilde{v}$ is supposed to be non-negative, the ξ-integration in (35) is done over the region

$$0 \leq \tilde{u} \leq w - w_0. \tag{43}$$

Let us now consider the geometric meanings of the points corresponding to $\tilde{u} = 0$ or $\tilde{u} = w - w_0$. Clearly $\xi = \xi_0$ is the point corresponding to $\tilde{u} = 0$. Let ξ'_* denote the point that corresponds to $\tilde{u} = w - w_0$. Then it follows from the definition of $\tilde{u}$ that

$$x\xi'_x - \int_{\hat{a}}^{\xi'_*} x_{k'}(\xi)d\xi = x\xi_0 - \int_{\hat{a}}^{\xi_0} x_{k'}(\xi)d\xi - (w - w_0). \tag{44}$$

Using the definition of w_0, we find this is equal to

$$x\xi_j(x) - \int_{\hat{a}}^{\xi_j(x)} x_k(\xi)d\xi - w, \tag{45}$$

which has another expression

$$\int_a^x \xi_j(x)dx - w \tag{46}$$

by (39). Thus we find

$$\begin{aligned} x\xi'_* - \int_{\hat{a}}^{\xi'_*} x_{k'}(\xi)d\xi &= \int_a^x \xi_j(x)dx - w \\ &= -y \end{aligned} \tag{47}$$

by (42), that is, ξ'_* is a point satisfying the relation (47) on the steepest descent path for $\mathrm{Re}\, f_{k'}$ that emanates from $\xi = \xi_0$. Summing up, we can rewrite the integral (35) as follows:

$$\int_L \exp(-\eta y) \left(\int_{\xi_0}^{\xi'_*} i\widehat{\psi}_{k',B}(\xi, y + x\xi)\, d\xi \right) dy \tag{48}$$

where $L = \{y \in \mathbb{C}; y = -\int_a^x \xi_j(x)dx + w,\ w \geq w_0\}$.

Before relating this integral with the connection formula for $\widehat{P}$, we note the following property of ξ'_*; the relation (47) implies that, at $\xi = \xi'_*$, $\tilde{y} = y + x\xi$ attains the value $\int_{\hat{a}}^{\xi'_*} x_{k'}(\xi)d\xi$. This is a singular point of $\widehat{\psi}_{k',B}(\xi, \tilde{y})$, which

coincides with a singular point of $\widehat{\psi}_{k,B}(\xi,\tilde{y})$ that is described by $\int_{\hat{a}}^{\xi_*} x_{k'}(\xi)d\xi$, i.e., the singularity considered in Figure 2. Thus ξ'_* is actually equal to ξ_*, an end-point of the integral I we encountered in Figure 2. Otherwise stated, the point $\xi = \xi_*$ also lies on the steepest descent path for $\mathrm{Re}\, f_{k'}$ that emanates from ξ_0. Hence the integral I in question is expressed as

$$\int_\gamma \widehat{\psi}_{k,B}(\xi, y + x\xi)d\xi, \tag{49}$$

where γ denotes a path of integration that starts from ξ_0 and comes back to ξ_0 by encircling the portion of the steepest descent path for $\mathrm{Re}\, f_{k'}$ whose boundaries are ξ_0 and ξ_*. Now let us recall the precise meaning of the connection formula (32) in the exact WKB analysis (cf. [V]): if we let $\Delta\widehat{\psi}_{k,B}$ denote the discontinuity of $\widehat{\psi}_{k,B}(\xi,\tilde{y})$ along the cut $\{\tilde{y} \in \mathbb{C}; \tilde{y} = \int_{\hat{a}}^{\xi} x_{k'}(\xi)d\xi + \tilde{v}, \tilde{v} \geq 0\}$, then

$$\Delta\widehat{\psi}_{k,B} = i\widehat{\psi}_{k',B}. \tag{50}$$

Since $(\Delta\widehat{\psi}_{k,B})(\xi, y + x\xi)$ is coincident with the integrand of the integral (49) because of the choice of the path γ, the integral I is equal to the integral (48) by (50). Therefore we conclude that the concrete form of the integral I is given by the integral (35) whose domain of integration is specified by (36) and (37).

The discussion given in this section tells us that, if we consider the integral (24) globally with respect to y, we are forced to consider not only the steepest descent path C_k for $\mathrm{Re}\, f_k$ passing through the saddle point of f_k but also another steepest descent path $C_{k'}$ for $\mathrm{Re}\, f_{k'}$ that is bifurcated from C_k at the crossing point of C_k and a Stokes curve for $\widehat{P}$. The necessity of integration over a portion of C'_k and the connection formula for WKB solutions of $\widehat{P}\widehat{\psi} = 0$ are two sides of the same coin, and hence the integral along C_k and that along $C_{k'}$ are, so to speak, equal partners. Therefore, if $C_{k'}$ hits some saddle point, say $\xi = \xi_l(x)$, then some anomalies should be observed in the Borel sum of WKB solutions of the starting equation $P\psi = 0$, that is, the point x should be on a Stokes curve for P, ordinary or new. Thus we are led to "Exact Steepest Descent Path Ansatz", or "ESDP Ansatz" for short, which is given below. To state ESDP Ansatz, we prepare the following definition of an exact steepest descent path. There we use a terminology "a Stokes curve of type $(k > k')$" to mean that $\widehat{\psi}_k$ is dominant over $\widehat{\psi}_{k'}$ along the Stokes curve.

Definition of an exact steepest descent path: Let f_l denote $x\xi - \int_{\hat{a}}^{\xi} x_l(\xi)d\xi$. An exact steepest descent path C is, by definition, the union of portions of steepest descent paths obtained by the following procedure:

Start with a steepest descent path C_k of $\mathrm{Re}\, f_k$ for some k that passes through a saddle point. If C_k crosses a Stokes curve (for $\widehat{P}$) of type $(k > k')$, consider the steepest descent path $C_{k'}$ for $\mathrm{Re}\, f_{k'}$ which starts from the crossing point. If $C_{k'}$ crosses another Stokes curve of type $(k' > k'')$, consider another steepest descent path $C_{k''}$ for $\mathrm{Re}\, f_{k''}$ in the same manner, and so on.

Exact Steepest Descent Path Ansatz: Stokes phenomena for Borel resummed WKB solutions of $P\psi = 0$ are observed at x if and only if an exact steepest descent path connects two saddle points $\xi_k(x)$ and $\xi_l(x)$.

It is evident that this Ansatz is a natural generalization of the results obtained for a Laplace-type operator P (i.e., an operator whose Laplace transform is of the first order; cf. [AKT4], [T2] and references cited there).

Remark 1. Introduction of exact steepest descent paths automatically resolves the spurious changes of topological configurations of steepest descent paths that occur when they hit a turning point of $\hat{P}$. For example, as is shown in Figure 3, when a steepest descent path C_k hits a simple turning point $\hat{a}$, no topological change of configurations occurs for the exact steepest descent path $C_k \cup C_{k'}$ although the shape of C_k itself changes abruptly. (See also [AKT5, §4].)

5 Computer-assisted study of the Exact Steepest Descent Path Ansatz

The purpose of this section is to present a computer-assisted study of ESDP Ansatz in some illuminating examples. See [AKT5] for some more detailed explanation of the figures given below. We note [AKT5] contains discussions on some other less peculiar examples; the examples given below are most illuminating ones among them. All computations are done by using *Mathematica* (ver. 3.0.1 or 4.0.2.1).

Example 1. Consider the following equation:

$$\left(\frac{d^3}{dx^3} + \eta^2(i - 4x^2)\frac{d}{dx} + \eta^3\left(\frac{i}{2} - x^2\right)\right)\psi = 0. \tag{51}$$

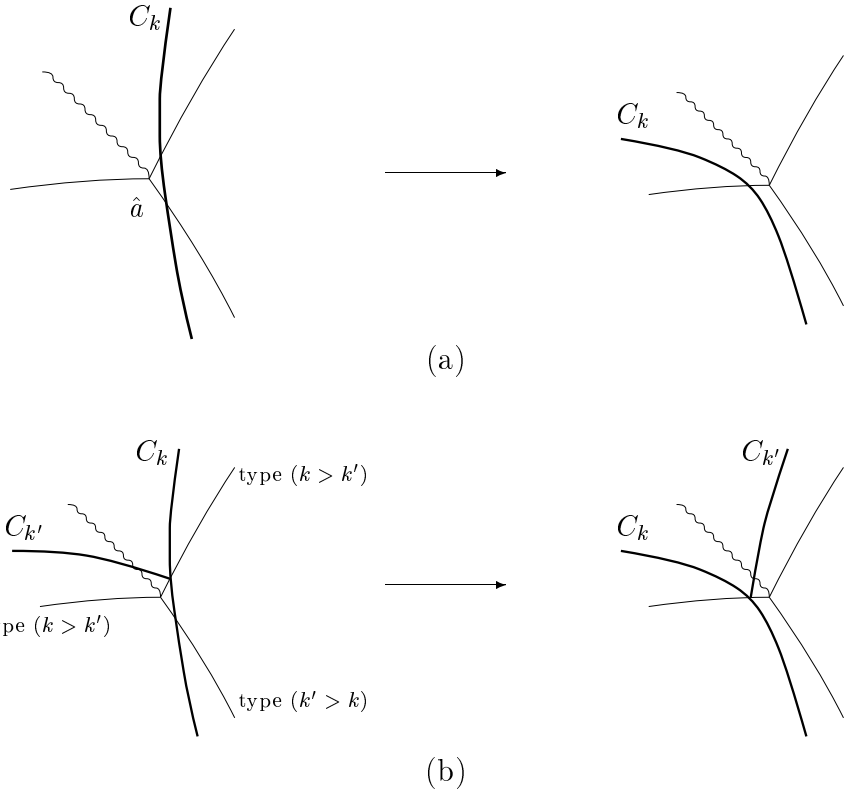

Figure 3 : Schematic illustration of a change of configurations of ordinary steepest descent paths (a) and of exact ones (b) when they hit a simple turning point $\hat{a}$; a lightfaced line and a wiggly line respectively designate a Stokes curve and a cut (defining the Riemann surface of $x_k(\xi)$).

If we draw ordinary Stokes curves (i.e., if we ignore temporarily new turning points and new Stokes curves), we obtain Figure 4 below.

(i) Let us consider the problem near the point A where two Stokes curves cross; let x_k $(k = 0, 1, \ldots, 11)$ denote the point

$$c + 0.05e^{\pi ik/6} \tag{52}$$

where c stands for the coordinate of A, and consider the ordinary steepest descent paths for the integral (19) (or rather (6)) with $x = x_k$. With the aid of a computer we find Fig. 5.k that illustrates the configuration for $x = x_k$. We seek for a topological change of configurations which occurs when a steepest

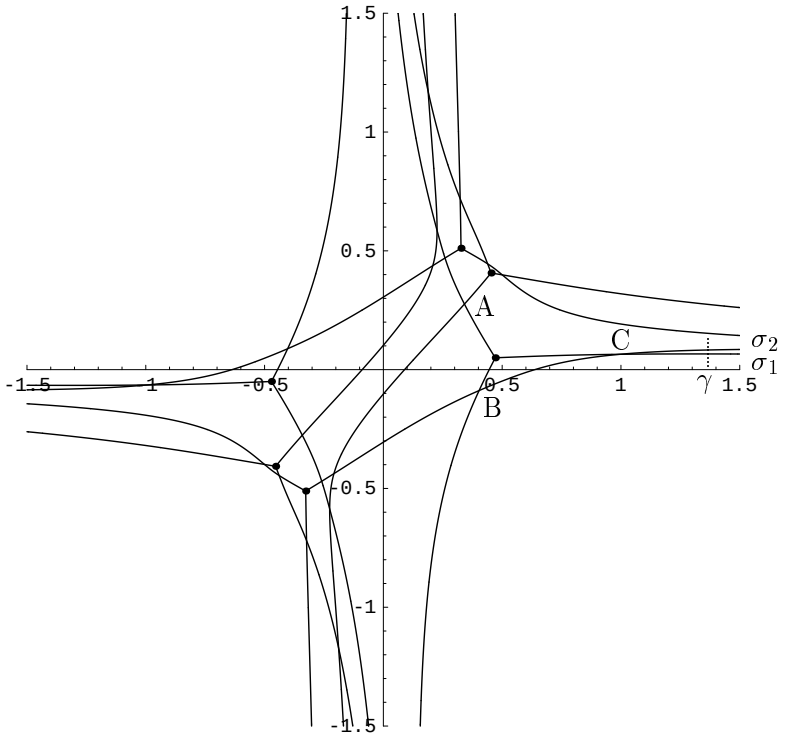

Figure 4 : Stokes curves of Equation (51).

descent path hits another saddle point. (The change that occurs when a steepest descent path hits a turning point is spurious and should be ignored; cf. Remark 1 in §4.) We then observe five changes of the configuration of this kind: between $x = x_1$ and $x = x_2$, around $x = x_4$, between $x = x_7$ and $x = x_8$, around $x = x_{10}$, and between $x = x_6$ and $x = x_7$. Each of the first four corresponds to one of four portions of Stokes curves meeting at the point A, while the fifth change that occurs between $x = x_6$ and $x = x_7$ can be attributed to a new Stokes curve passing through the ordered crossing point A (in the sense of [BNR]). Thus the (ordinary) steepest descent method applied to the integral (6) correctly detects Stokes curves in this case.

(ii) In spite of the success shown in (i), the method fails near B, another crossing point of Stokes curves in Figure 4. Choosing c to be the coordinate of the point B, we find the Figure 6.k ($k = 0, 1, \ldots, 11$) that shows the

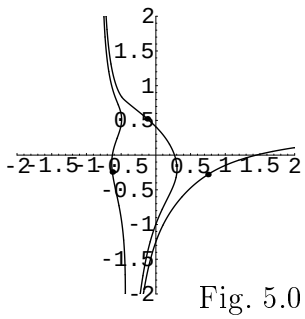

Fig. 5.0

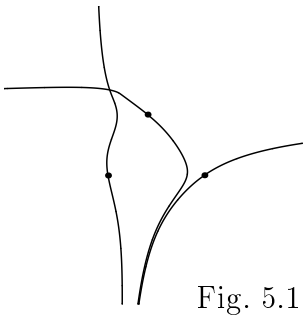

Fig. 5.1

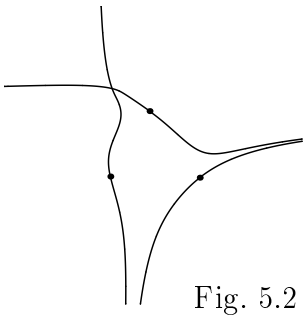

Fig. 5.2

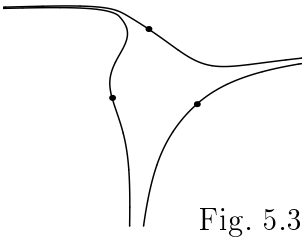

Fig. 5.3

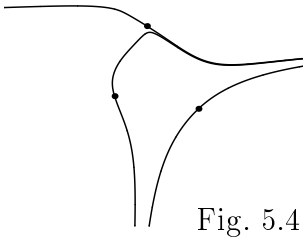

Fig. 5.4

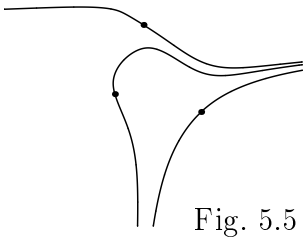

Fig. 5.5

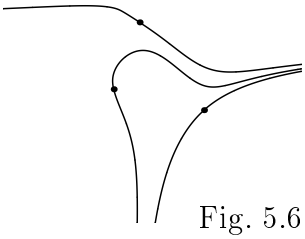

Fig. 5.6

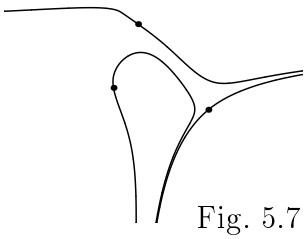

Fig. 5.7

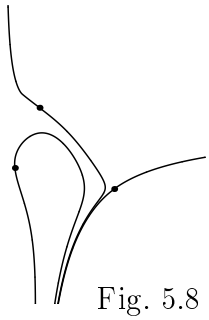

Fig. 5.8

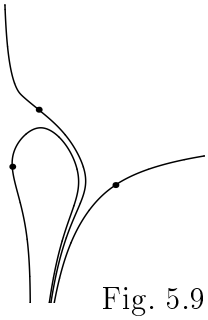

Fig. 5.9

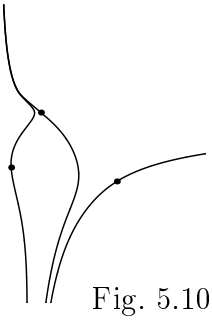

Fig. 5.10

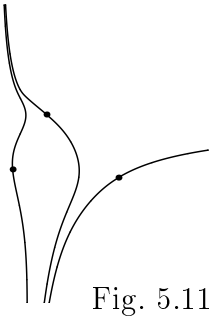

Fig. 5.11

Figure 5 : Ordinary steepest descent paths around the crossing point A.

configuration of the ordinary steepest descent paths for the integral (6). In Figure 6 one can observe only three relevant topological changes of the configuration: around $x = x_2$, around $x = x_7$, and between $x = x_8$ and $x = x_9$. However, the reasoning of [BNR, §2] (cf. also [AKT2, pp.83–84]) indicates that three Stokes segments meet at a point only if one of them or all of them are actually irrelevant to Stokes phenomena. Otherwise stated, the ordinary steepest descent method applied to the integral (6) cannot correctly detect the Stokes phenomena in this case.

Contrary to this failure, if we use exact steepest descent paths, then we can correctly detect the points where the Stokes phenomena occur, as the subsequent Figure 7 shows. Fig. 7 visualizes the movement between $x = x_0$ and $x = x_1$ of configuration of the exact steepest descent paths of the integral (19); in order not to make the figures too complicated, some irrelevant paths are omitted. The heavy line designates the *exact* steepest descent path, which plays a crucially important role.

(iii) Another evidence strongly in support of ESDP Ansatz can be found near the point C, where two Stokes curves cross. Let x_k $(k = 0, 1, \cdots, 8)$ denote the point $1.37 + \sqrt{-1}(0.059 + 0.00525k)$; these points lie on the path γ that is designated by a dotted line in Fig. 4. Then we find Fig. 8.k $(k = 0, 1, \ldots, 8)$ that describes the configuration of ordinary steepest descent paths for $x = x_k$. In Fig. 8 we observe a topological change of the configuration only near $x = x_1$, which evidently corresponds to the situation where γ crosses an ordinary Stokes curve σ_1 in Fig. 4; we cannot observe any changes that might be attributed to the crossing of γ and another ordinary Stokes curve σ_2 or a possibly relevant new Stokes curve σ_0 emanating from the crossing point C.

Let us now add an exact steepest descent path to Fig. 8.3 and Fig. 8.4; the resulting figures are respectively Fig. 9.3 and Fig. 9.4. In contrast with Fig. 8.3 and Fig. 8.4, they clearly show a topological difference of configurations. This topological change that occurs between $x = x_3$ and $x = x_4$ corresponds to the crossing of γ and a new Stokes curve σ_0 that emanates from C.

Similarly we find Fig. 9.5 and Fig. 9.6 by adding an exact steepest descent path respectively to Fig. 8.5 and Fig. 8.6. They again show a topological change of configurations, which occurs between $x = x_5$ and $x = x_6$. (As Fig. 9.5 and Fig. 9.6 seem to be too complicated to decipher, we also present Fig. 9.5 bis and Fig. 9.6 bis which contain only the added exact steepest descent path, so that the reader's attention may be concentrated upon it.)

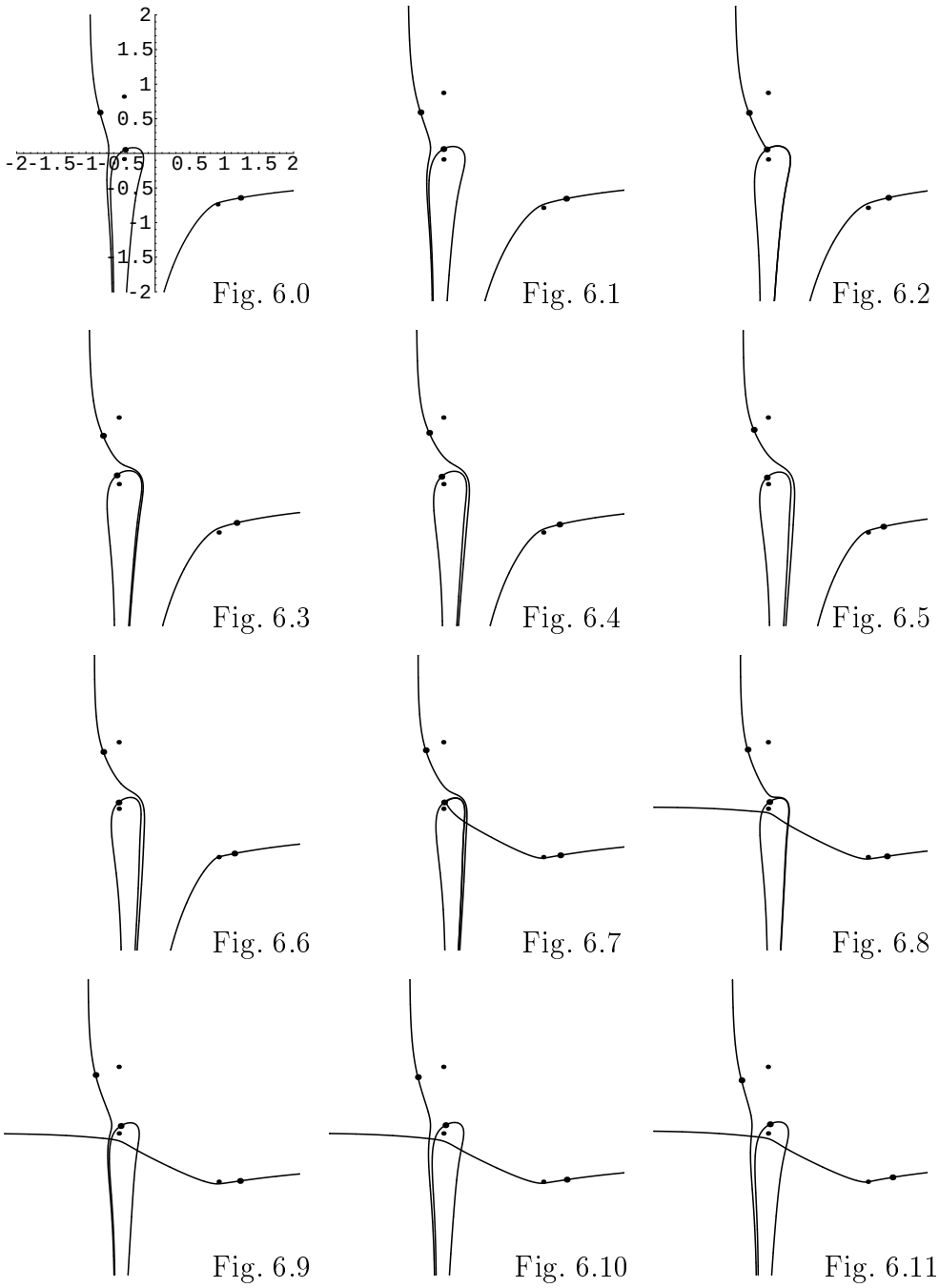

Figure 6 : Ordinary steepest descent paths around the crossing point B.

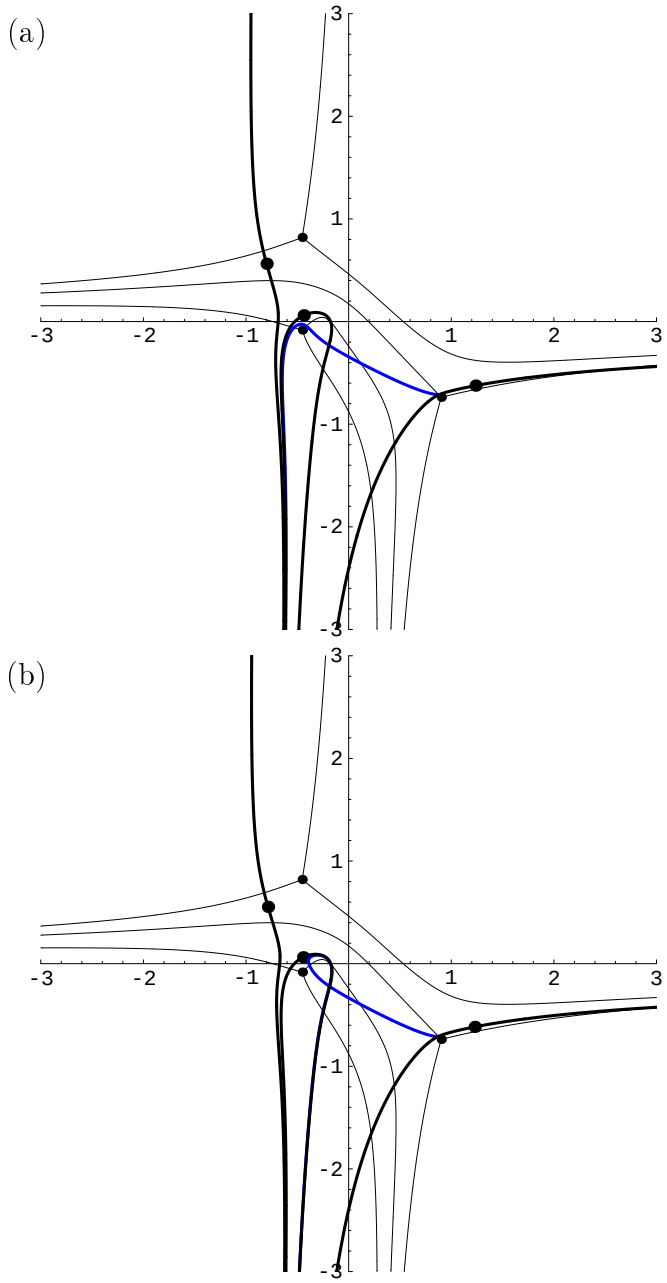

Figure 7 : Exact steepest descent paths for $x = 0.45936 - 0.067718\ i$ (a) and for $x = 0.45597 - 0.0600894\ i$ (b).

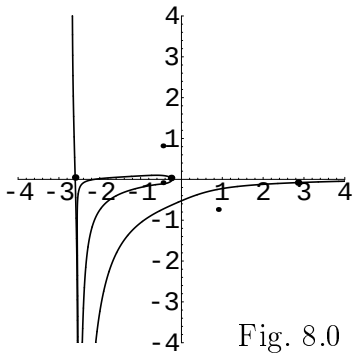

Fig. 8.0

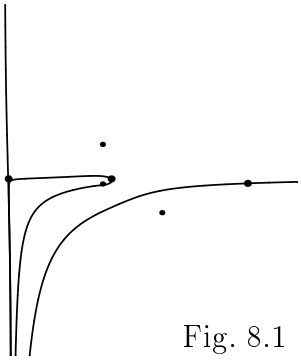

Fig. 8.1

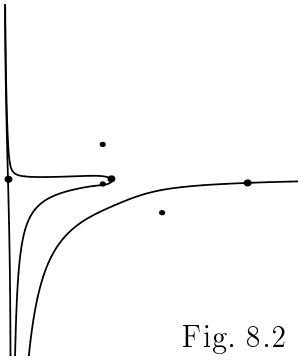

Fig. 8.2

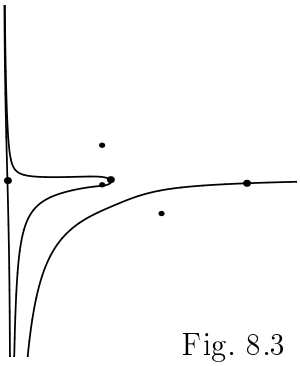

Fig. 8.3

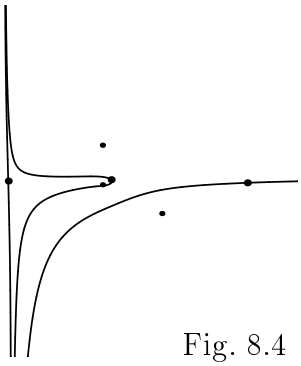

Fig. 8.4

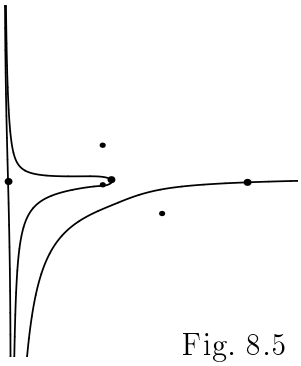

Fig. 8.5

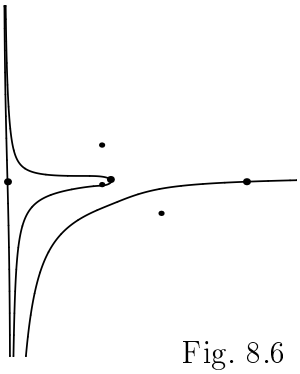

Fig. 8.6

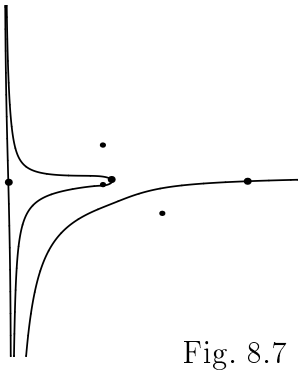

Fig. 8.7

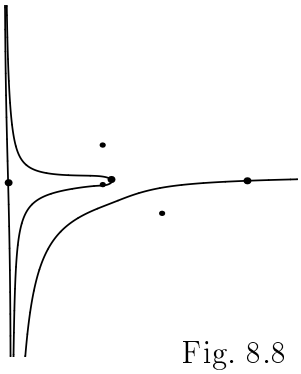

Fig. 8.8

Figure 8 : Ordinary steepest descent paths for the point x_k on the path γ.

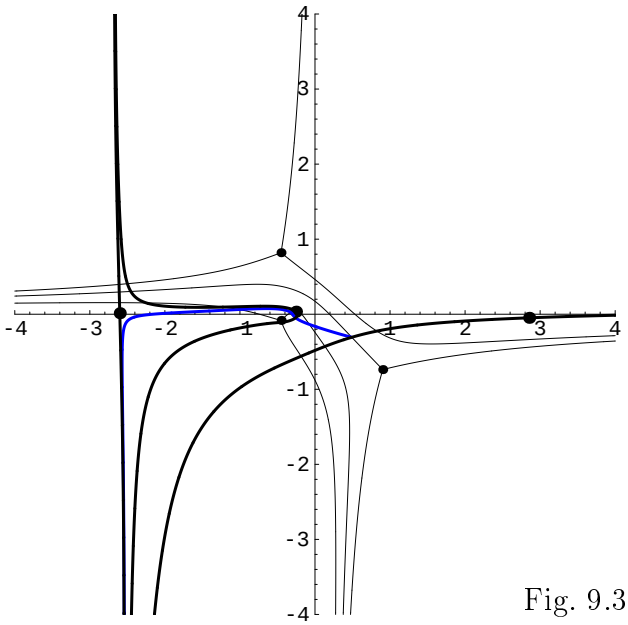

Fig. 9.3

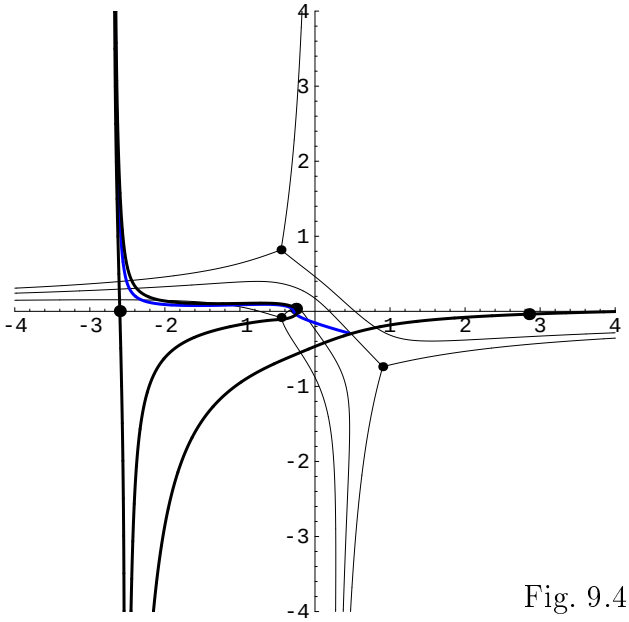

Fig. 9.4

Figure 9 (i) : Exact steepest descent paths for $x = 1.37 + 0.07475\, i$ (Fig. 9.3) and for $x = 1.37 + 0.08\, i$ (Fig. 9.4).

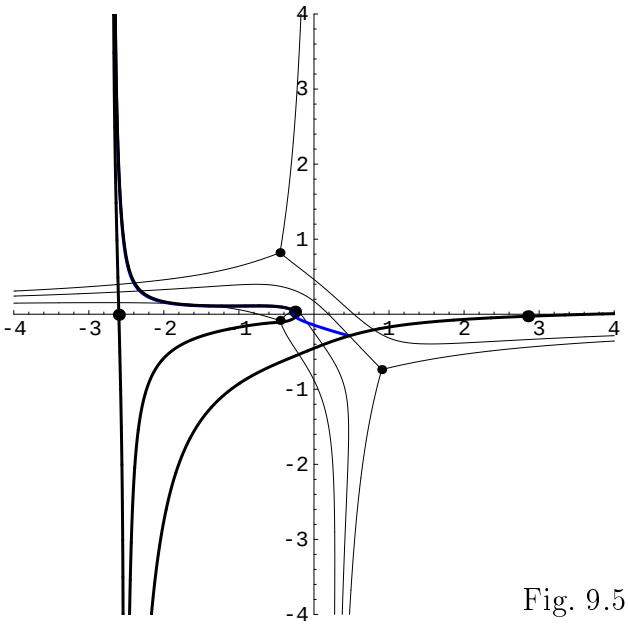

Fig. 9.5

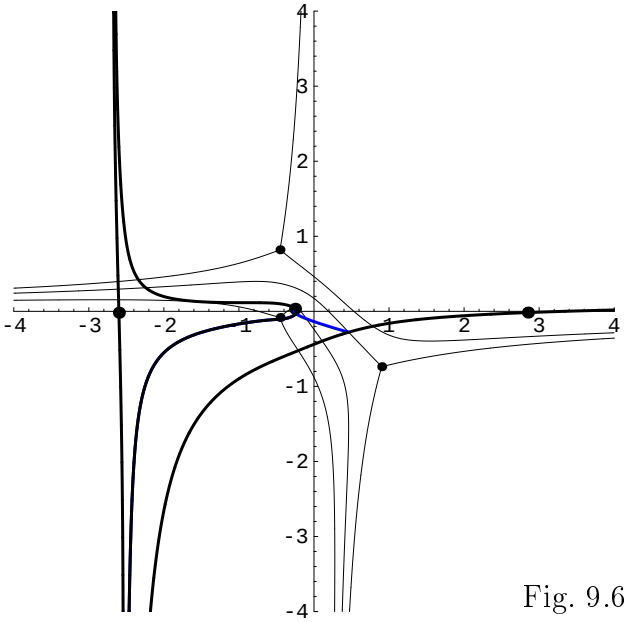

Fig. 9.6

Figure 9 (ii) : Exact steepest descent paths for $x = 1.37 + 0.08525\ i$ (Fig. 9.5) and for $x = 1.37 + 0.0905\ i$ (Fig. 9.6).

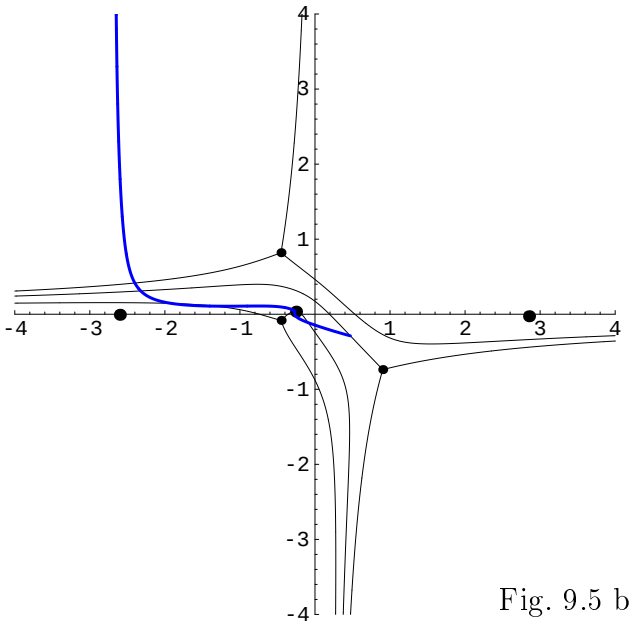

Fig. 9.5 bis

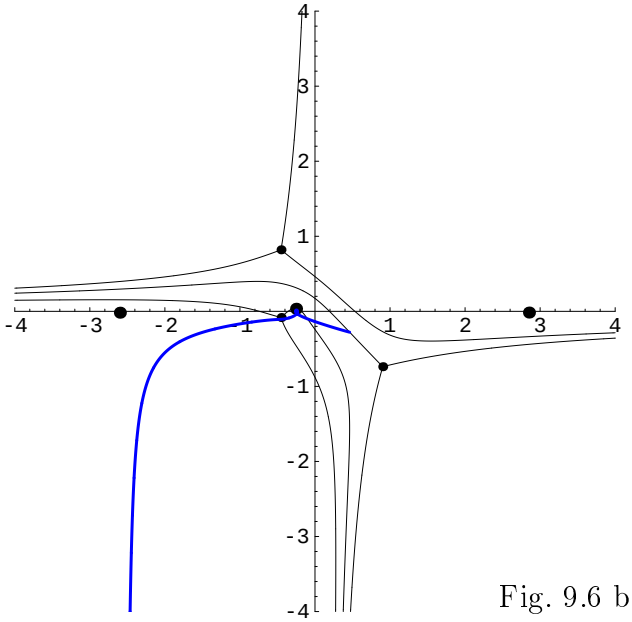

Fig. 9.6 bis

Figure 9 (ii bis) : Added exact steepest descent paths for
$x = 1.37 + 0.08525\ i$ (Fig. 9.5 bis) and for $x = 1.37 + 0.0905\ i$ (Fig. 9.6 bis).

This change that occurs between $x = x_5$ and $x = x_6$ corresponds to the crossing of γ and the ordinary Stokes curve σ_2.

Thus the addition of an exact steepest descent path neatly explains delicate issues concerning the Stokes phenomena near the point C.

Example 2. In order to show that ESDP Ansats may be valid even if $\widehat{P}$ is of higher order, let us study the following Carroll-Hioe equation:

$$
\begin{aligned}
(53)\quad & \frac{d^3\psi}{dx^3} + 2i(r_1+r_2+r_3)\eta x\frac{d^2\psi}{dx^2} \\
& + \left\{-4(r_1r_2+r_2r_3+r_3r_1)\eta^2x^2 + 2ic_1\eta + \frac{1}{4}[(\Omega_{12})^2+(\Omega_{23})^2]\eta^2\right\}\frac{d\psi}{dx} \\
& + \left\{-8ir_1r_2r_3\eta^3x^3 - 4c_2\eta^2x + \frac{1}{2}[(\Omega_{12})^2r_3+(\Omega_{23})^2r_1]\eta^3x\right\}\psi = 0,
\end{aligned}
$$

where $r_1, r_2, r_3, \Omega_{12}, \Omega_{23}, c_1$ and c_2 are some parameters. In what follows all figures are drawn for the following values of these parameters. (There is no particular meaning in this choice.)

$$
\begin{aligned}
(54)\qquad & r_1 = -2+i, \quad r_2 = \frac{1}{2}+2i, \quad r_3 = 1-2i, \\
& \Omega_{12} = -3+4i, \quad \Omega_{23} = 1-3i, \quad \text{and} \\
& c_1 \text{ and } c_2 \text{ are arbitrarily fixed.}
\end{aligned}
$$

Although $\hat{P}$ is of the third order, Carroll-Hioe ingeniously gave an integral representation of solutions of (53). Hence we can describe the complete Stokes geometry for (53) by analyzing the integral again with the aid of a computer ([AKT3]). The result is shown in Fig. 10 below. (A dotted line indicates that no Stokes phenomena occur across that portion of the curve.)

Choosing c in (52) to be the coordinate of the point A in Fig. 10, we illustrate the configuration of ordinary steepest descent paths for the integral that represents a solution of (53) and the integral (6) respectively in Fig. 11 and in Fig. 12. In Fig. 12 we cannot observe any topological change of the configuration caused by the interaction of steepest descent paths and saddle points between $x = x_3$ and $x = x_4$, which is observed in Fig. 11. If we use exact steepest descent paths in addition to ordinary ones, the topological change can be observed in the following Fig. 13 that describes the behavior between $x = x_3$ and $x = x_4$.

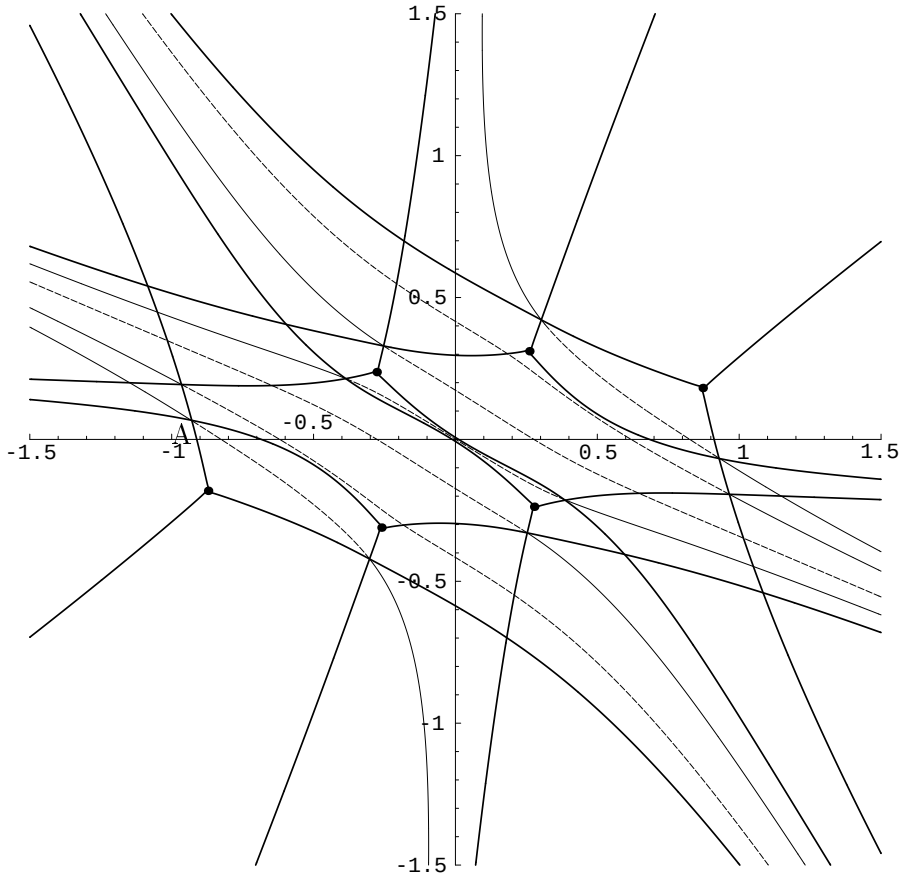

Figure 10 : Stokes geometry for Equation (53).

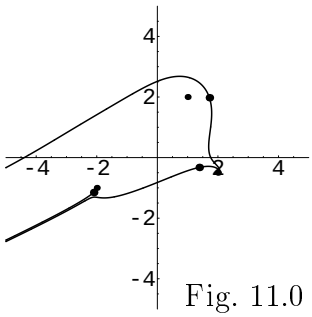

Fig. 11.0

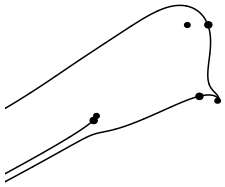

Fig. 11.1

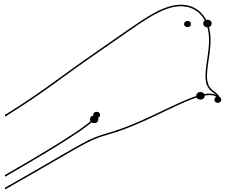

Fig. 11.2

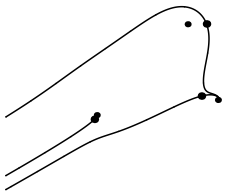

Fig. 11.3

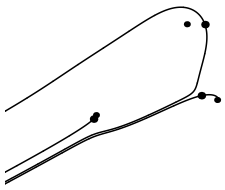

Fig. 11.4

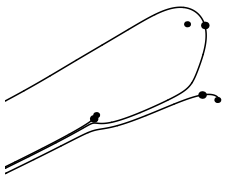

Fig. 11.5

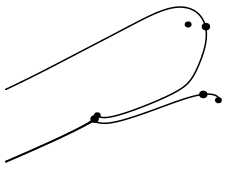

Fig. 11.6

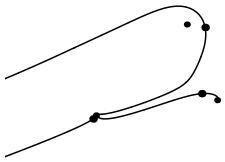

Fig. 11.7

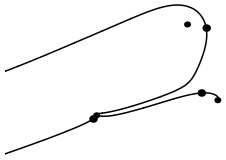

Fig. 11.8

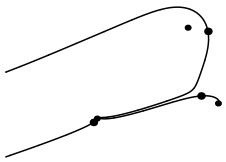

Fig. 11.9

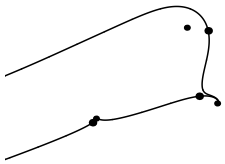

Fig. 11.10

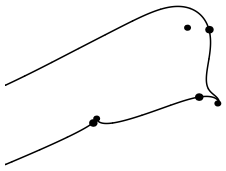

Fig. 11.11

Figure 11 : Ordinary steepest descent paths for the integral representing a solution around the crossing point A.

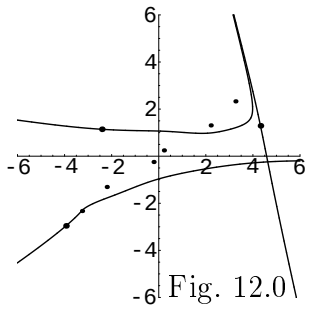

Fig. 12.0

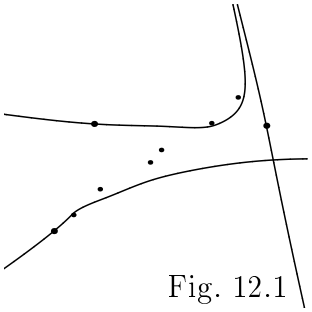
Fig. 12.1

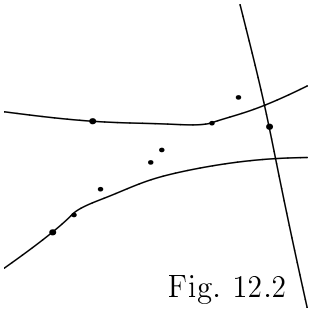
Fig. 12.2

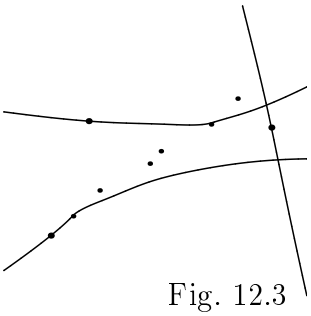
Fig. 12.3

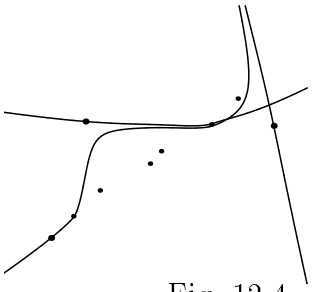
Fig. 12.4

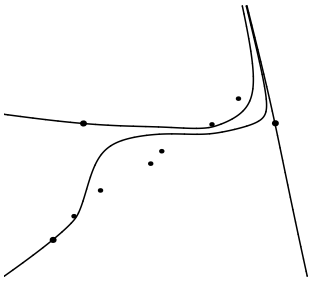
Fig. 12.5

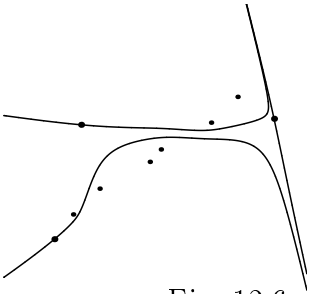
Fig. 12.6

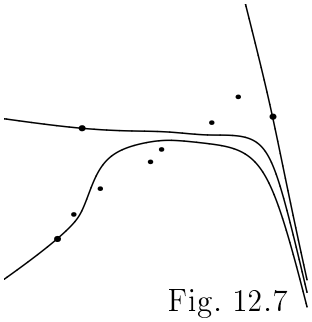
Fig. 12.7

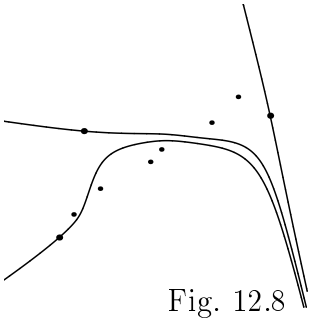
Fig. 12.8

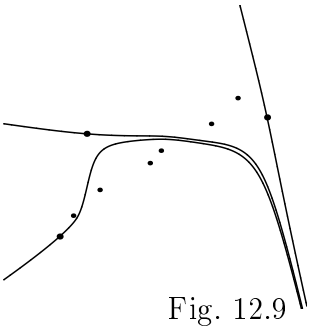
Fig. 12.9

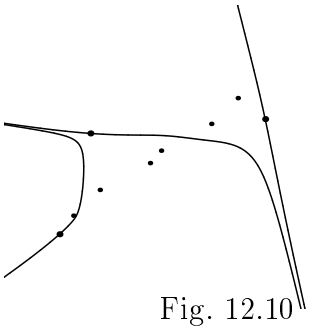
Fig. 12.10

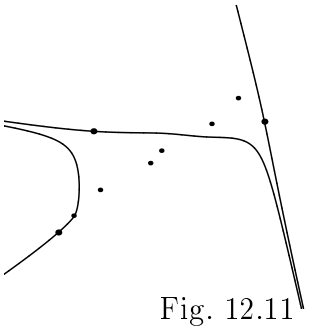
Fig. 12.11

Figure 12 : Ordinary steepest descent paths for the integral (6) around the crossing point A.

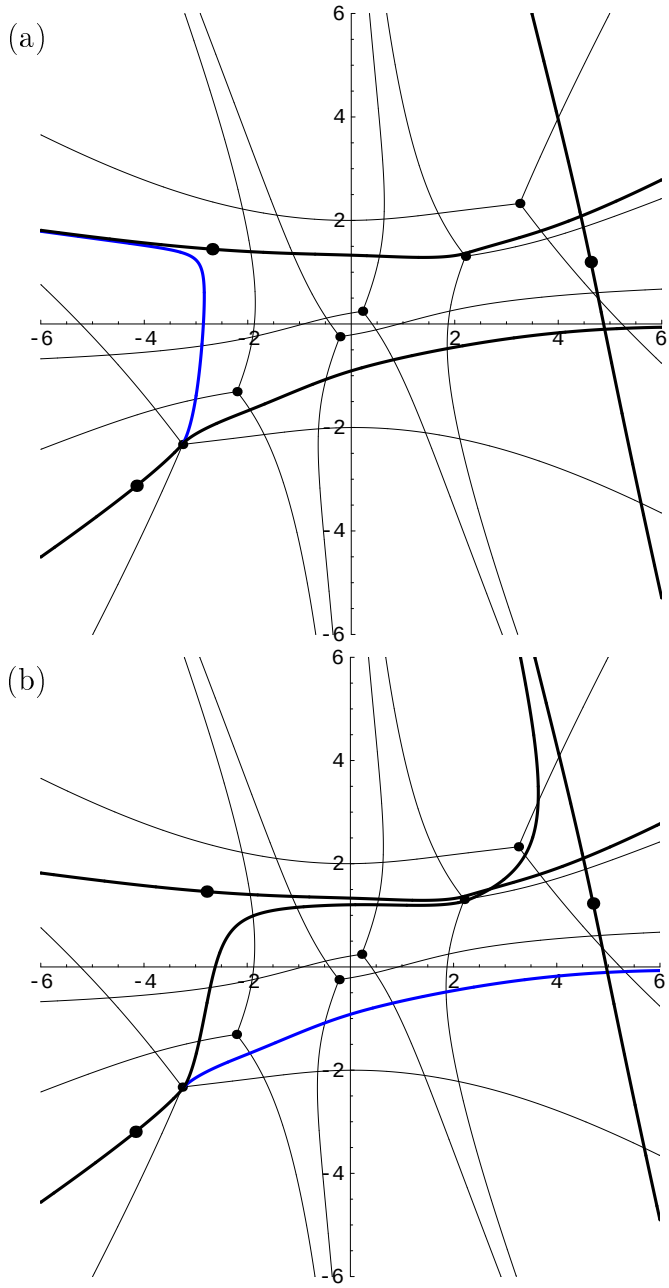

Figure 13 : Exact steepest descent paths for $x = -0.978776 + 0.247106\ i$ (a) and for $x = -0.998081 + 0.247106\ i$ (b).

6 Concluding remarks

As several examples in the preceding section have shown, ESDP Ansatz is a new and powerful proposition in WKB analysis. In particular, we note the disappearance of a pathological change of configurations of steepest descent paths which is observed when an ordinary steepest descent path hits a turning point for $\hat{P}$ (see Remark 1 in §4), if we use exact steepest descent paths; the effect of a turning point is automatically built in the definition of an exact steepest descent path through its interrelationship with a Stokes curve that emanates from the turning point. This fact nicely fits in with the fact that a turning point is not a singular point of the equation $\hat{P}\hat{\psi} = 0$; the singularity at a turning point that appears in the WKB expansion is a spurious one to disappear by the Borel resummation.

Another important implication of ESDP Ansatz is that Stokes phenomena for the operator P occur only on a Stokes curve of the form (62) below. The reasoning is as follows: Suppose that a Stokes phenomenon is observed at x. ESDP Ansatz then asserts that

$$\mathrm{Im}(x\xi_j(x) - \int_{\hat{a}}^{\xi_j(x)} x_k(\xi)d\xi) = \mathrm{Im}(x\xi_0 - \int_{\hat{a}}^{\xi_0} x_k(\xi)d\xi), \tag{55}$$

$$\mathrm{Im}(x\xi_{j'}(x) - \int_{\hat{a}}^{\xi_{j'}(x)} x_{k'}(\xi)d\xi) = \mathrm{Im}(x\xi_0 - \int_{\hat{a}}^{\xi_0} x_{k'}(\xi)d\xi), \tag{56}$$

and

$$\mathrm{Im}\int_{\hat{a}}^{\xi_0} (x_k(\xi) - x_{k'}(\xi))d\xi = 0 \tag{57}$$

hold for some j, j', k and k', a turning point $\hat{a}$ and some ξ_0, where $x_k(\xi)$ and $x_{k'}(\xi)$ solve the equation $p(x,\xi) = 0$ given by (4) and $\xi_j(x)$ and $\xi_{j'}(x)$ also solve the same equation in ξ. Let us suppose that $\mathcal{R} = \{(x,\xi) \in \mathbb{C}^2 : p(x,\xi) = 0\}$ is non-singular and that $x_k(\xi)$ etc. are situated with a turning point a of P as in Figure 14. Here we note that relations (55) and (56) are slightly weaker than the actual situation that ESDP Ansatz requires to occur, because a level surface of $\mathrm{Im}\, f_k$ or $\mathrm{Im}\, f_{k'}$ is not necessarily connected. As we see below, these weaker conditions still suffice to entail the required relation (62).

Integration of the 1-form $d(x\xi) = xd\xi + \xi dx$ on the portion C_1 in Fig. 14

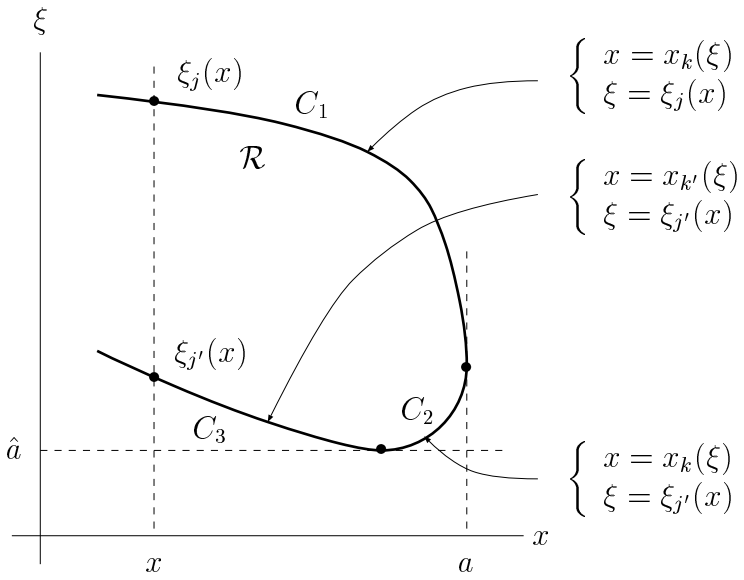

Figure 14 : Schematic illustration of $\mathcal{R} = \{(x, \xi) : p(x, \xi) = 0\}$.

gives us

$$x\xi_j(x) - a\xi_j(a) = \int_{\xi_j(a)}^{\xi_j(x)} x_k(\xi)d\xi + \int_a^x \xi_j(x)dx. \tag{58}$$

Similar calculations on the portions C_2 and C_3 respectively lead to

$$a\xi_{j'}(a) - x_k(\hat{a})\hat{a} = \int_{\hat{a}}^{\xi_{j'}(x)} x_k(\xi)d\xi + \int_{x_k(\hat{a})}^{a} \xi_{j'}(x)dx \tag{59}$$

and

$$x_{k'}(\hat{a})\hat{a} - x\xi_{j'}(x) = \int_{\xi_{j'}(x)}^{\hat{a}} x_{k'}(\xi)d\xi + \int_x^{x_{k'}(\hat{a})} \xi_{j'}(x)dx. \tag{60}$$

Combining these relations, we obtain

$$\begin{aligned} x\xi_j(x) - x\xi_{j'}(x) = &\int_{\hat{a}}^{\xi_j(x)} x_k(\xi)d\xi \\ &+ \int_{\xi_{j'}(x)}^{\hat{a}} x_{k'}(\xi)d\xi + \int_a^x \xi_j(x)dx + \int_x^a \xi_{j'}(x)dx. \end{aligned} \tag{61}$$

Considering the imaginary part of the relation (61) together with (55), (56) and (57), we conclude

$$\mathrm{Im} \int_a^x (\xi_j(x) - \xi_{j'}(x))dx = 0. \tag{62}$$

This relation enhances our belief in ESDP Ansatz.

References

[AKT1] T. Aoki, T. Kawai and Y. Takei: The Bender-Wu analysis and the Voros theory, "Special Functions", Springer-Verlag, 1991, pp. 1–29.

[AKT2] ———: New turning points in the exact WKB analysis for higher-order ordinary differential equations, "Analyse algébrique des perturbations singulières. I", Hermann, 1994, pp. 69–84.

[AKT3] ———: On the exact WKB analysis for the third order ordinary differential equations with a large parameter, Asian J. Math., **2**(1998), 625–640.

[AKT4] ———: On a complete description of the Stokes geometry for higher order ordinary differential equations with a large parameter via integral representations, "Toward the Exact WKB Analysis of Differential Equations, Linear or Non-Linear", Kyoto Univ. Press, 2000, pp. 11–14.

[AKT5] ———: Toward the exact steepest descent method, to appear in RIMS Kôkyûroku "Exact Steepest Descent Method". (In Japanese.)

[BH] M. V. Berry and C. J. Howls: Hyperasymptotics for integrals with saddles, Proc. Roy. Soc. Lond., **A 434**(1991), 657–675.

[BNR] H. L. Berk, W. M. Nevins and K. V. Roberts: New Stokes lines in WKB theory, J. Math. Phys., **23**(1982), 988–1002.

[DDP] E. Delabaere, H. Dillinger et F. Pham: Résurgence de Voros et périodes des courbes hyperelliptiques, Ann. Inst. Fourier (Grenoble), **43**(1993), 163–199.

[K^3] M. Kashiwara, T. Kawai and T. Kimura: Foundations of Algebraic Analysis, Princeton Univ. Press, 1986.

[S] H. J. Silverstone: JWKB connection-formula problem revisited via Borel summation, Phys. Rev. Lett., **55**(1985), 2523–2526.

[SKK] M. Sato, T. Kawai and M. Kashiwara: Microfunctions and pseudo-differential equations, Lect. Notes in Math., **287**, Springer, 1973, pp. 265–529.

[T1] Y. Takei: Exact WKB analysis of the harmonic oscillator and its Fourier transform, to appear in RIMS Kôkyûroku "Microlocal Analysis for Schrödinger Equations and Related Topics".

[T2] ———: Integral representation for ordinary differential equations of Laplace type and exact WKB analysis, to appear in RIMS Kôkyûroku "Exact Steepest Descent Method".

[V] A. Voros: The return of the quartic oscillator. The complex WKB method, Ann. Inst. Henri Poincaré, **39**(1983), 211–338.

[WSW] T. Watanabe, H. Sanuki and M. Watanabe: New treatment of eigenmode analysis for an inhomogeneous Vlasov plasma, J. Phys. Soc. Japan, **47**(1979), 286–293.