COMBINATORICS OF THE $\widehat{\mathfrak{sl}}_2$ SPACES OF COINVARIANTS II

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ABSTRACT. The spaces of coinvariants are quotient spaces of integrable $\widehat{\mathfrak{sl}}_2$ modules by subspaces generated by actions of certain subalgebras labeled by a set of points on a complex line. When all the points are distinct, the spaces of coinvariants essentially coincide with the spaces of conformal blocks in the WZW conformal field theory and their dimensions are given by the Verlinde rule.

We describe monomial bases for the $\widehat{\mathfrak{sl}}_2$ spaces of coinvariants, In particular, we prove that the spaces of coinvariants have the same dimensions when all the points coincide. We establish recursive relations satisfied by the monomial bases and the corresponding characters of the spaces of coinvariants. For the proof we use filtrations of the $\widehat{\mathfrak{sl}}_2$ modules, and further filtrations on the adjoint graded spaces for the first filtrations.

This paper is the continuation of [FKLMM].

1. INTRODUCTION

Let e, h, f be the standard basis of the Lie algebra $\mathfrak{sl}_2$, and $e(x) = \sum_{i\in\mathbb{Z}} e_i x^i$, $h(x) = \sum_{i\in\mathbb{Z}} h_i x^i$, $f(x) = \sum_{i\in\mathbb{Z}} f_i x^i$ the currents of $\widehat{\mathfrak{sl}}_2 = \mathfrak{s}l_2 \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$. By $\mathfrak{a}^{(M,N)}$ we denote the Lie subalgebra of $\widehat{\mathfrak{sl}}_2$ generated by e_i $(i \geq M)$ and f_i $(i \geq N)$, where M, N are non-negative integers. Let $L_{k,l}$ be the integrable irreducible representation of $\widehat{\mathfrak{sl}}_2$ of level k and highest weight $l \in \{0, \dots, k\}$. We suppose that the highest weight vector of $L_{k,l}$ is annihilated by e_i $(i \leq 0)$, h_j and f_j $(j < 0)$.

In this paper, we study the space of coinvariants

$$L_{k,l}^{(M,N)} = L_{k,l}/\mathfrak{a}^{(M,N)} L_{k,l}.$$

We prove that $L_{k,l}^{(M,N)}$ is a finite-dimensional space and its dimension is given by the Verlinde formula. Namely, we have the recurrence formula

$$\dim L_{k,l}^{(M,N)} = \sum_{l',l''} \dim L_{k,l'}^{(M,N-1)}, \tag{1.1}$$

where the summation is over the pairs of non-negative integers (l', l'') such that l has the same parity as $l' + l''$ and

$$|l' - l''| \leq l \leq \min(l' + l'', 2k - l' - l'').$$

We prove (1.1) in the following way.

Recall the Verlinde algebra for the $\widehat{sl}_2$-theory of level k. It is a commutative algebra with the basis $\{\pi_0, \dots, \pi_k\}$. The defining relations are

$$\pi_{l''} \cdot \pi_{l'} = \sum_{\substack{l \equiv l' + l'' \bmod 2 \\ |l' - l''| \leq l \leq \min(l' + l'', 2k - l' - l'')}} \pi_l.$$

First, we prove that $\dim L_{k,l}^{(M,N)}$ is greater than or equal to the number $d_{k,l}^{(M+N)}$ defined by the formula

$$(\pi_0 + \cdots + \pi_k)^N = \sum_{0 \le l \le k} d_{k,l}^{(N)} \pi_l. \tag{1.2}$$

More precisely, we have the following result. Consider a pair of polynomials $P(t), Q(t)$ with degrees $\deg P = M$ and $\deg Q = N$. Let $\mathfrak{a}(P,Q)$ be the subalgebra of $\widehat{\mathfrak{sl}}_2$ generated by the subspaces $\mathbb{C}e \otimes P(t)\mathbb{C}[t]$ and $\mathbb{C}f \otimes Q(t)\mathbb{C}[t]$. It is clear that $\mathfrak{a}(t^M, t^N)$ is exactly $\mathfrak{a}^{(M,N)}$. Consider the family of spaces of coinvariants:

$$L_{k,l}(P,Q) = L_{k,l}/\mathfrak{a}(P,Q)L_{k,l}.$$

It is well-known (for the proof see, e.g., the appendix of [FKLMM]) that (1.1) is true if we replace $\dim L_{k,l}^{(M,N)}$ by $\dim L_{k,l}(P,Q)$ for "generic" polynomials P, Q. The word "generic" means that all $M+N$ zeros of the polynomials P, Q are distinct. It is easy to see that for arbitrary P, Q the dimension of $L_{k,l}(P,Q)$ is greater than or equal to the dimension at a "generic" point. Formula (1.1) actually means that the family of vector spaces $L_{k,l}(P,Q)$ forms a vector bundle over the space of parameters (P,Q). In other words, the dimension does not "jump" at special points.

Let us try to understand why the formula (1.1) is valid. The reason may be that there exists some natural filtration in $L_{k,l}^{(M,N)}$ such that the corresponding graded space is isomorphic to the direct sum $\oplus_{l',l''} L_{k,l'}^{(M,N-1)}$. We do not know such a filtration, nevertheless what we show in this paper is rather similar.

We replace $\widehat{\mathfrak{sl}}_2$ by the Lie algebra $\widehat{\mathcal{H}} = \mathcal{H} \otimes \mathbb{C}[t, t^{-1}]$, where $\mathcal{H}$ is the three-dimensional Heisenberg algebra with the basis $\{\bar{e}, \bar{h}, \bar{f}\}$ and the relations $[\bar{e}, \bar{f}] = \bar{h}$ and $[\bar{h}, \bar{e}] = [\bar{h}, \bar{f}] = 0$. As explained in [FKLMM], $\widehat{\mathcal{H}}$ is the adjoint graded Lie algebra associated to $\widehat{\mathfrak{sl}}_2$ with respect to the filtration. Let $F_0 = \mathbb{C}c \oplus \mathbb{C}d$, $F_1 = F_0 \oplus \left(\oplus_{i \in \mathbb{Z}} \mathbb{C}e_i\right) \oplus \left(\oplus_{j \in \mathbb{Z}} \mathbb{C}f_j\right)$ and $F_2 = F_3 = \cdots = \widehat{\mathfrak{sl}}_2$. The graded Lie algebra $F_0 \oplus F_1/F_0 \oplus F_2/F_1$ is isomorphic to the sum $\widehat{\mathcal{H}} \oplus \mathbb{C}c \oplus \mathbb{C}d$.

The filtration $\{F_i\}$ in $\widehat{\mathfrak{sl}}_2$ naturally induces a filtration in any cyclic representation of $\widehat{\mathfrak{sl}}_2$. Let M be a cyclic representation and v a cyclic vector in M. We define the filtration $\{F_i(M)\}_{i \ge 0}$ by the following inductive procedure: $F_0(M) = \mathbb{C}v$, $F_{i+1}(M) = F_1 \cdot F_i(M)$ $(F_1 \subset \widehat{\mathfrak{sl}}_2)$. It is evident that the adjoint graded space $\mathrm{Gr}(M) = \oplus_i F_{i+1}(M)/F_i(M)$ is a cyclic representation of $\widehat{\mathcal{H}}$.

We describe explicitly the $\widehat{\mathcal{H}}$-module $\mathrm{Gr}(L_{k,l})$.

In the algebra $\widehat{\mathcal{H}}$ we have the basis $\bar{e}_i, \bar{h}_i, \bar{f}_i$ $(i \in \mathbb{Z})$ and the corresponding currents $\bar{e}(x), \bar{h}(x), \bar{f}(x)$. Recall that in $L_{k,l}$ the following relations hold: $e(x)^{k+1} = f(x)^{k+1} = 0$. From this we deduce that in $\mathrm{Gr}(L_{k,l})$ $\bar{e}(x)^{k+1} = \bar{f}(x)^{k+1} = 0$. Define now the $\widehat{\mathcal{H}}$-module $\bar{L}_{k,l}$ as follows. Let $\bar{M}$ be the quotient of the module induced from the one-dimensional trivial representation $\mathbb{C}\bar{v}$ of the subalgebra spanned by $\bar{e}_i$ $(i \le 0)$, $\bar{h}_j$ and $\bar{f}_j$ $(j < 0)$. Then, $\bar{L}_{k,l}$ is the quotient of $\bar{M}$ by the submodule generated by the vectors $\bar{e}(x)^{k+1} w$, $\bar{f}(x)^{k+1} w$ $(w \in \bar{M})$ and $\bar{f}_0^{l+1} \bar{v}$. We prove the isomorphism of $\widehat{\mathcal{H}}$-modules $\mathrm{Gr}(L_{k,l}) \simeq \bar{L}_{k,l}$.

For simplicity let us assume that $M > 0$ in this introduction. In the main text, the case $M = 0$ is also handled with a proper modification. For the $\widehat{\mathcal{H}}$-module $\bar{L}_{k,l}$ we define the spaces of coinvariants $\bar{L}_{k,l}^{(M,N)}$ and show that $\dim \bar{L}_{k,l}^{(M,N)} = \dim L_{m,n}^{(M,N)}$. Actually, we prove that (1.1) is true for $\dim \bar{L}_{k,l}^{(M,N)}$. For this purpose we define a bigger set of representations of $\widehat{\mathcal{H}}$ depending on three

parameters l_1, l_2, l_3. For this set of $\widehat{\mathcal{H}}$-modules, we define coinvariants and prove an analogue of (1.1) using suitable filtrations. Such recursive description of the spaces $\bar{L}_{k,l}^{(M,N)}$ gives us a collection of monomial bases in $\bar{L}_{k,l}^{(M,N)}$ (and therefore in $L_{k,l}^{(M,N)}$).

The plan of this paper is as follows. In Section 2, we give some preliminary definitions and propositions. In Section 3, the bijection between the combinatorial and Verlinde paths is given. In Section 4 several mappings between Heisenberg modules are given. In Section 5, the recursion relation for Heisenberg modules is proved. In Section 6, some miscellaneous results are given. In Appendix, we give several lemmas for zero vectors in Heisenberg modules.

2. Preliminaries

2.1. **Spaces of coinvariants.** Let $\mathfrak{sl}_2$ be the Lie algebra with basis $\{e, f, h\}$ such that

$$[e, f] = h, \qquad [e, h] = -2e, \qquad [f, h] = 2f.$$

Let $(\ ,\ )$ be the Killing form normalized by $(h, h) = 2$. The algebra $\widehat{\mathfrak{sl}}_2 = \mathfrak{sl}_2 \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$ is spanned by $\{g_i = g \otimes t^i, c, d \ ; \ g = e, f, h \in \mathfrak{sl}_2, i \in \mathbb{Z}\}$, where c is a central element and d is the scaling operator, $[d, g_i] = ig_i$. The commutators are given by (notice our normalization of the central element)

$$\begin{aligned}[X \otimes f(t), Y \otimes g(t)] &= [X, Y] \otimes f(t)g(t) + c \cdot (X, Y)\mathrm{res}_{t=\infty} \frac{df}{dt}(t)g(t) \\ &= [X, Y] \otimes f(t)g(t) - c \cdot (X, Y)\mathrm{res}_{t=0} \frac{df}{dt}(t)g(t).\end{aligned}$$

Consider the Verma module $M_{k,l} = U(\widehat{\mathfrak{sl}}_2)v_k[l]$, with $k \in \mathbb{Z}_{>0}$ and $l \in \{0, \ldots, k\}$, generated by the highest weight vector $v = v_k[l]$ satisfying

$$\begin{aligned}&cv = kv, \quad h_0 v = lv, \quad dv = 0, \\ &e_i v = 0 \ (i \leq 0), \quad h_i v = f_i v = 0 \ (i < 0).\end{aligned}$$

The integrable module $L_{k,l}$ is the quotient of $M_{k,l}$ by the submodule generated by the vectors

$$e(z)^{k+1} w \quad (\text{or equivalently } f(z)^{k+1} w) \quad \text{where } w \in M_{k,l}.$$

Here we used the generating series notations $e(z) = \sum_{i \in \mathbb{Z}} e_i z^i$, etc.. The module $L_{k,l}$ is an irreducible $\widehat{\mathfrak{sl}}_2$ module.

Remark 2.1.1. Usually, the module $L_{k,l}$ is defined as the quotient of $M_{k,l}$ by the relations $f_0^{l+1} v = e_1^{k-l+1} v = 0$. These relations follow from the relation $f(z)^{k+1} = 0$. Indeed, the condition $f(z)^{k+1} v = 0$ and $f_i v = 0$ $(i < 0)$ implies $f_0^{k+1} v = 0$. Then, applying $\mathrm{ad} e_0$ to $f_0^{k+1} v = 0$ and using the identity $[e_0, f_0^{j+1}]v = (j+1)(l-j) f_0^j v$ and $h_0 v = lv$, we obtain $f_0^{l+1} v = 0$. Similarly, we have $e_1^{k-l+1} v = 0$.

The module $L_{k,l}$ is graded by h_0 and d. We denote its character by $\chi_{k,l}$,

$$\chi_{k,l} = \mathrm{ch}_{q,z} L_{k,l} = \mathrm{trace}_{L_{k,l}} \left(q^d z^{h_0} \right).$$

Let $M, N \in \mathbb{Z}_{\geq 0}$, $\mathbf{z} = (z_1, \ldots, z_M) \in \mathbb{C}^M$ and $\mathbf{z}' = (z'_1, \ldots, z'_N) \in \mathbb{C}^N$, where z_i, z'_i are not necessarily distinct. Consider the following Lie subalgebra of $\widehat{\mathfrak{sl}}_2$:

$$\mathfrak{sl}_2^{(M,N)}(\mathbf{z}; \mathbf{z}') = \Big[\mathbb{C}e \otimes \mathbb{C}[t] P(t)\Big] \oplus \Big[\mathbb{C}h \otimes \mathbb{C}[t] P(t) Q(t)\Big] \oplus \Big[\mathbb{C}f \otimes \mathbb{C}[t] Q(t)\Big],$$

where $P(t) = \prod_{i=1}^{M}(t - z_i)$ and $Q(t) = \prod_{i=1}^{N}(t - z'_i)$.

The corresponding space of coinvariants is defined by

$$L_{k,l}^{(M,N)}(\mathbf{z};\mathbf{z}') = L_{k,l}/\mathfrak{sl}_2^{(M,N)}(\mathbf{z};\mathbf{z}')L_{k,l}.$$

The space $L_{k,l}^{(M,N)}(\mathbf{z};\mathbf{z}')$ is finite-dimensional, and for generic $\mathbf{z},\mathbf{z}'$ the dimension is given in terms of the Verlinde algebra. Let us recall its definition.

The level k Verlinde algebra associated to $\mathfrak{sl}_2$ is a complex $k+1$-dimensional commutative associative algebra with basis $\{\pi_l;\ 0 \le l \le k\}$ and multiplication

$$\pi_l \cdot \pi_{l'} = \sum_{\substack{i=|l-l'| \\ i+l-l' \text{: even}}}^{\min(2k-l-l',l+l')} \pi_i. \tag{2.1}$$

For $N \in \mathbb{Z}_{>0}$, define the positive integer $d_{k,l}^{(N)}$ by

$$(\pi_0 + \pi_1 + \cdots + \pi_k)^N = \sum_l d_{k,l}^{(N)} \pi_l. \tag{2.2}$$

Theorem 2.1.2. ([FKLMM]) *If the points $z_1,\dots,z_M,z'_1,\dots,z'_N$ are distinct, then*

$$\dim L_{k,l}^{(M,N)}(\mathbf{z};\mathbf{z}') = d_{k,l}^{(M+N)}.$$

Theorem 2.1.2 is proved in Appendix of [FKLMM].

We denote

$$L_{k,l}^{(M,N)} = L_{k,l}^{(M,N)}((0,\dots,0),(0,\dots,0)).$$

The space $L_{k,l}^{(M,N)}$ inherits from $L_{k,l}$ a grading by h_0 and d. We define the character

$$\chi_{k,l}^{(M,N)} = \mathrm{ch}_{q,z} L_{k,l}^{(M,N)} = \mathrm{trace}_{L_{k,l}^{(M,N)}}\left(q^d z^{h_0}\right). \tag{2.3}$$

We denote by $\left(L_{k,l}^{(M,N)}\right)_{s,e}$ the weight subspace of $L_{k,l}^{(M,N)}$ where $h_0 = s$ and $d = e$.

2.2. **A deformation argument.** We will need the following deformation lemma.

Lemma 2.2.1. ([FKLMM]) *Let V be a vector space with filtration*

$$0 = F_{-1} \subset F_0 \subset F_1 \subset \cdots \subset V, \qquad V = \bigcup_j F_j.$$

Let $T_i : V \to V$, $(i \in I)$, be a set of linear maps with degree $d_i \ge 0$, i.e. $T_i(F_j) \subset F_{j+d_i}$.

Let $\mathrm{Gr}^F(V) = \oplus_j \mathrm{Gr}_j^F(V) = \oplus_j F_j/F_{j-1}$ and let $\overline{T}_i$ be the induced graded maps. Then, we have the inequality for the dimensions of the spaces of coinvariants:

$$\dim\left(V/\sum_i T_i V\right) \le \dim\left(\mathrm{Gr}^F(V)/\sum_i \overline{T}_i\,\mathrm{Gr}^F(V)\right). \tag{2.4}$$

Proof. See [FKLMM]. □

Remark 2.2.2. Note that if in a particular case, (2.4) can be shown to be an equality, and the image of $\sqcup_j \{v_\alpha^{(j)} \in F_j; 1 \le \alpha \le \dim \mathrm{Gr}_j^F(V)/\sum_i \bar{T}_i \mathrm{Gr}_{j-d_i}^F(V)\}$ forms a basis in $\mathrm{Gr}^F(V)/\sum_i \overline{T}_i \mathrm{Gr}^F(V)$, then the image of the same set of vectors forms a basis in $V/\sum_i T_i V$.

Proposition 2.2.3. *We have an inequality*

$$\dim L_{k,l}^{(M,N)}(\mathbf{z};\mathbf{z}') \le \dim L_{k,l}^{(M,N)}.$$

Proof. Proposition 2.2.3 follows from Lemma 2.2.1 applied to the vector space $L_{k,l}$ with the filtration induced by the d-grading, and the set of operators

$$\{e \otimes P(t)t^i, f \otimes Q(t)t^i,\ i \ge 0\}.$$

□

One of the main results which we prove in this paper is that in fact

$$\dim L_{k,l}^{(M,N)}(\mathbf{z};\mathbf{z}') = \dim L_{k,l}^{(M,N)} \tag{2.5}$$

(see Theorem 5.4.4).

3. Combinatorial and Verlinde paths

We establish the bijection between two sets of "paths", the combinatorial paths and the Verlinde paths. The former will be used to describe the monomial basis of $L_{k,l}^{(0,N)}(\mathbf{z})$. The latter arises in the well-known result in the $\widehat{\mathfrak{sl}}_2$-invariant conformal field theory. The bijection leads to a recursive formula for the combinatorial paths. This will be a key result in our proof of Theorem 5.4.4 given in Section 5.

3.1. **Combinatorial paths.** Let $\mathcal{C}_{k,l}^{(N)}$ be the set of all $(\mathbf{a};\mathbf{b}) = (a_{N-1},\dots,a_0;b_{N-1},\dots,b_1\} \in \{0,\dots,k\}^{2N-1}$ such that

$$a_0 \le l, \tag{3.1}$$

$$a_i + b_{i+1} + a_{i+1} \le k, \qquad b_i + a_i + b_{i+1} \le k \qquad (i \ge 0), \tag{3.2}$$

$$\sum_{s=i}^{j} b_s \le k + \sum_{s=i+1}^{j-2} a_s \qquad (-1 \le i < j \le \infty), \tag{3.3}$$

where we set $b_\infty = k, b_{-1} = l, b_0 = a_{-1} = a_\infty = 0$ and $a_i = b_i = 0$ for $N \le i < \infty$.

Note, in particular, that

$$b_{N-1} = 0$$

because of (3.3) with $i = N-1, j = \infty$.

In the above definition we assumed that N is an positive (i.e., $N > 0$) integer. For $N = 1$ we have $a_0 = l$ and $\sharp(\mathcal{C}_{k,l}^{(1)}) = 1$. There is a canonical injection $\mathcal{C}_{k,l}^{(N)} \to \mathcal{C}_{k,l}^{(N+1)}$. We define

$$\mathcal{C}_{k,l}^{(0)} = \begin{cases} \mathcal{C}_{k,0}^{(1)} & \text{if } l = 0; \\ \emptyset & \text{otherwise.} \end{cases} \tag{3.4}$$

We call elements $(\mathbf{a};\mathbf{b}) \in \mathcal{C}_{k,l}^{(N)}$ combinatorial paths of length N, level k and weight l.

It is convenient to write combinatorial paths $(\mathbf{a};\mathbf{b}) \in \mathcal{C}^{(N)}_{k,l}$ as follows:

$$\begin{pmatrix} b_\infty = k & \cdots & 0 & & b_{N-1} & \cdots & & b_2 & & b_1 & & 0 & & b_{-1} = l \\ \dots & & 0 & & a_{N-1} & & \cdots & a_2 & & a_1 & & a_0 & & 0 & \end{pmatrix}$$

One of the main purposes of this section is to prove that the cardinality of the set $\mathcal{C}^{(N)}_{k,l}$ is given by the Verlinde number $d^{(N)}_{k,l}$ defined by (1.2):

$$\sharp(\mathcal{C}^{(N)}_{k,l}) = d^{(N)}_{k,l}.$$

As shown in the appendix of [FKLMM], we know that the Verlinde number $d^{(M+N)}_{k,l}$ gives the dimension of the space $L^{(M,N)}_{k,l}(\mathbf{z};\mathbf{z}')$ when $(\mathbf{z};\mathbf{z}')$ are distinct. Therefore, we can think of $\mathcal{C}^{(N)}_{k,l}$ as a set which parametrizes a basis of coinvariants. In fact, in Section 5 we will prove that the images of vectors

$$\{f_{N-1}^{a_{N-1}} h_{N-1}^{b_{N-1}} \dots f_1^{a_1} h_1^{b_1} f_0^{a_0} v_k[l]\,;\, (\mathbf{a};\mathbf{b}) \in \mathcal{C}^{(N)}_{k,l}\} \tag{3.5}$$

form a basis in $L^{(0,N)}_{k,l}(\mathbf{z})$ for all $\mathbf{z} \in \mathbb{C}^N$ (Corollary 5.4.10).

One can check that for $k = 1$ the sets $\mathcal{C}^{(N)}_{1,0}$ and $\mathcal{C}^{(N)}_{1,1}$ coincide with sets $\mathcal{C}^{0,N}$ and $\mathcal{C}^{1,N}$ in [FKLMM]. For $k = 1$, Theorem 5.4.4 and Corollary 5.4.10 are proved in [FKLMM] (see Corollary 5.2.2 and Theorem 5.3.1 therein).

3.2. **Verlinde paths.** In this section we recall the definition and some elementary properties of Verlinde paths.

A triple $(\alpha,\beta,\gamma) \in \{0,\dots,k\}^3$ is called an admissible Verlinde triple (of level k) if it can be written as

$$(\alpha,\beta,\gamma) \mapsto \begin{pmatrix} & \beta & \\ \alpha & & \gamma \end{pmatrix} = \begin{pmatrix} & y+z & \\ x+y & & x+z \end{pmatrix}, \qquad x+y+z \le k, \qquad x,y,z \in \mathbb{Z}_{\ge 0}. \tag{3.6}$$

Then the numbers x, y, z are determined uniquely,

$$x(\alpha,\beta,\gamma) = \frac{\alpha+\gamma-\beta}{2}, \qquad y(\alpha,\beta,\gamma) = \frac{\alpha+\beta-\gamma}{2}, \qquad z(\alpha,\beta,\gamma) = \frac{\beta+\gamma-\alpha}{2}. \tag{3.7}$$

If (α,β,γ) is an admissible triple then any permuted triple, e.g., (γ,α,β), is also admissible. Note, however, that formula (3.7) is not symmetric in α,β,γ.

Now, formula (2.1) for multiplication in the Verlinde algebra can be rewritten as

$$\pi_l \cdot \pi_{l''} = \sum_{l' :\, (l,l'',l') \text{ is admissible}} \pi_{l'}. \tag{3.8}$$

A sequence $(\boldsymbol{\alpha};\boldsymbol{\beta}) = (\alpha_1,\dots,\alpha_N;\beta_1,\dots\beta_{N-1}) \in \{0,\dots,k\}^{2N-1}$ is called a Verlinde path of length N, weight l and level k if $\alpha_1 = l$ and $(\alpha_i,\beta_i,\alpha_{i+1})$, $i = 1,\dots,N-1$ are admissible Verlinde triples of level k. We set $\beta_N = \alpha_N$, $\alpha_i = \beta_i = 0$ for $i > N$, so that $(\alpha_i,\beta_i,\alpha_{i+1})$ form Verlinde triples for all $i \in \mathbb{Z}_{>0}$.

We denote $\mathcal{P}^N_{k,l}$ the set of all Verlinde paths of length N, level k and weight l.

We define the following two maps associated with Verlinde paths. The first is the "multiplication" map given by

$$\begin{aligned} m_{\mathcal{P}} : \mathcal{P}^N_{k_1,l_1} \times \mathcal{P}^N_{k_2,l_2} &\to \mathcal{P}^N_{k_1+k_2,l_1+l_2}, \\ ((\boldsymbol{\alpha}^{(1)};\boldsymbol{\beta}^{(1)}),(\boldsymbol{\alpha}^{(2)};\boldsymbol{\beta}^{(2)})) &\mapsto (\boldsymbol{\alpha}^{(1)}+\boldsymbol{\alpha}^{(2)};\boldsymbol{\beta}^{(1)}+\boldsymbol{\beta}^{(2)}). \end{aligned} \tag{3.9}$$

The map $m_{\mathcal{P}}$ is well-defined. This is obvious from the definition of the admissibility (3.6).

For an admissible triple (l,l'',l'), we have a well-defined injective left concatenation map

$$c_{\mathcal{P}}(l,l'',l') : \mathcal{P}^N_{k,l'} \to \mathcal{P}^{N+1}_{k,l}, \qquad (\boldsymbol{\alpha};\boldsymbol{\beta}) \mapsto (l,\boldsymbol{\alpha};l'',\boldsymbol{\beta}). \tag{3.10}$$

Note that multiplication and concatenation commute:

$$m_{\mathcal{P}}(c_{\mathcal{P}}(l_1,l_1'',l_1'),c_{\mathcal{P}}(l_2,l_2'',l_2')) = c_{\mathcal{P}}(l_1+l_2,l_1''+l_2'',l_1'+l_2')m_{\mathcal{P}}. \tag{3.11}$$

We have a recursive formula for Verlinde paths.

$$\mathcal{P}^{N+1}_{k,l} = \bigsqcup_{l'',l' :\, (l,l'',l') \text{ is admissible}} c_{\mathcal{P}}(l,l'',l')\mathcal{P}^N_{k,l'}. \tag{3.12}$$

Lemma 3.2.1. *The multiplication map $m_{\mathcal{P}}$ is surjective.*

Proof. The statement is clear for $N = 2$, i.e., for Verlinde triples. The general case follows from (3.11) and (3.12). □

Lemma 3.2.2. *The cardinality of set $\mathcal{P}^N_{k,l}$ is given by the Verlinde number $d^{(N)}_{k,l}$ defined by (2.2).*

Proof. By (3.8) and (3.12), the numbers computing the cardinality of $\mathcal{P}^N_{k,l}$ satisfy the same recursion relation as the Verlinde numbers $d^{(N)}_{k,l}$. □

3.3. **Recursion of combinatorial paths.** In this section we establish properties of the set of combinatorial paths $\mathcal{C}^{(N)}_{k,l}$ similar to the properties of the set of Verlinde paths $\mathcal{P}^N_{k,l}$ described in the previous section.

We have an obvious "multiplication" map (compare to (3.9)):

$$\begin{aligned} m_{\mathcal{C}} : \mathcal{C}^{(N)}_{k_1,l_1} \times \mathcal{C}^N_{k_2,l_2} &\to \mathcal{C}^{(N)}_{k_1+k_2,l_1+l_2}, \\ ((\mathbf{a}^{(1)};\mathbf{b}^{(1)}),(\mathbf{a}^{(2)};\mathbf{b}^{(2)})) &\mapsto (\mathbf{a}^{(1)}+\mathbf{a}^{(2)};\mathbf{b}^{(1)}+\mathbf{b}^{(2)}). \end{aligned} \tag{3.13}$$

We will prove in Theorem 3.3.2 that it is surjective.

Next, we would like to obtain maps $c_{\mathcal{C}}(l,l'',l')$ similar to $c_{\mathcal{P}}(l,l'',l')$ in (3.10). We require three properties for these maps. First, we would like to have a property similar to the recursion relation (3.12):

$$\mathcal{C}^{(N+1)}_{k,l} = \bigsqcup_{l'',l' :\, (l,l'',l') \text{ is admissible}} c_{\mathcal{C}}(l,l'',l')\mathcal{C}^{(N)}_{k,l'}. \tag{3.14}$$

Second, we would like the maps $c_{\mathcal{C}}(l,l'',l')$ to be extensions of the form

$$c_{\mathcal{C}}(l',l'',l) : \mathcal{C}^{(N)}_{k,l'} \to \mathcal{C}^{(N+1)}_{k,l}, \qquad (\mathbf{a};\mathbf{b}) \mapsto (\mathbf{a},a;\mathbf{b},b), \tag{3.15}$$

where a, b depend only on the Verlinde triple (l,l'',l') and a_0. This condition is natural because it is similar to (3.10).

The properties (3.14), (3.15) uniquely determine the $c_{\mathcal{C}}(l,l'',l')$ for level 1. However, we need an additional property to fix the maps for the general level.

We require the surjectivity of the multiplication map. However, it does not uniquely fix the map. We choose *a priori* the map (3.15) so that Proposition 3.3.2 holds. This choice will be rederived *a posteriori* when we study the filtration of Heisenberg modules in Section 5.

Now we describe such maps.

For each admissible triple of level k, (l,l'',l'), we define maps $c_{\mathcal{C}}(l,l'',l')$ by formula (3.15) where

$$a = a(l,l'',l';a_0) = y(l,l'',l'), \tag{3.16}$$

$$b = b(l,l'',l';a_0) = z(l,l'',l') - \Big(a_0 - x(l,l'',l')\Big)^+. \tag{3.17}$$

Here t^+ means the positive part of t:

$$t^+ = \max(t,0), \tag{3.18}$$

and x,y,z are given by (3.7).

Proposition 3.3.1. *The maps $c_{\mathcal{C}}(l,l'',l')$ given by the formula (3.15) and (3.16) are well-defined. The combinatorial paths enjoy the recursion (3.14).*

Proof. For $(\mathbf{a};\mathbf{b}) \in \mathcal{C}^{(N)}_{k,l'}$ we have to check that $(\mathbf{a},a;\mathbf{b},b) \in \mathcal{C}^{(N+1)}_{k,l}$, where a,b are given by (3.16). First, $a = y(l,l'',l') \le l$, because $a = l - x(l,l'',l')$. Among the conditions (3.2) for $(\mathbf{a},a;\mathbf{b},b)$ we have new cases only for the triples (a_0,b,a) and (b_1,a_0,b). They follow from

$$a + b + a_0 = y + z + a_0 - (a_0 - x)^+ \le y + z + x \le k,$$

$$a_0 + b_1 + b = a_0 + b_1 + z - (a_0 - x)^+ \le b_1 + z + x = b_1 + l' \le k.$$

(The last inequality follows from (3.3) with $i = -1, j = 1$.) Among the conditions (3.3) for $(\mathbf{a},a;\mathbf{b},b)$ we have new cases only for $i = 1,0,-1$. The case $i = 0$ is weaker than $i = 1$. The case $i = 1$ (resp., $i = -1$) follows from $b \le l'$ (resp., $b + l \le l' + a$) and the condition (3.3) for $(\mathbf{a};\mathbf{b})$ with $i = -1$.

Next, we have to check that the images of the maps $c_{\mathcal{C}}(l,l'',l')$ are disjoint. Suppose they are not, and $a(l,l'',l';a_0) = a(l,l''_1,l'_1;a_0)$ and $b(l,l'',l',a_0) = b(l,l''_1,l'_1,a_0)$ for some l,a_0,l'',l''_1,l',l'_1. Then, the first equality implies $l'' - l' = l''_1 - l'_1$. Therefore, we have $x(l,l'',l') = x(l,l''_1,l'_1)$. It follows that the positive part terms, i.e., $(a_0 - x)^+$, in the second equality are equal and after its cancellation $l'' + l' = l''_1 + l'_1$. Therefore, we have $l'' = l''_1$ and $l' = l'_1$.

Finally, let $(\mathbf{a},a;\mathbf{b},b) \in \mathcal{C}^{(N+1)}_{k,l}$. Note that

$$(\mathbf{a},a,\mathbf{b},b) = c_{\mathcal{C}}(l,l'',l')(\mathbf{a};\mathbf{b}),$$

where

$$l' = x + z = l - a + b + (a_0 - (l-a))^+,$$
$$l'' = y + z = a + b + (a_0 - (l-a))^+.$$

Moreover, the triple (l,l'',l') is admissible, because

$$x + y + z = l + b + (a_0 - (l-a))^+ \le a_0 + a + b \le k,$$

by condition (3.2) for $(\mathbf{a},a;\mathbf{b},b)$. Therefore, it suffices to show that $(\mathbf{a};\mathbf{b}) \in \mathcal{C}^{(N)}_{k,l'}$.

This is also straightforward. The combinatorial path $(\mathbf{a};\mathbf{b})$ looks like

$$\begin{pmatrix} b_\infty = k & \cdots & 0 & & b_{N-1} & \cdots & & b_2 & & b_1 & & 0 & & l' \\ \dots & & 0 & & a_{N-1} & & \cdots & a_2 & & a_1 & & a_0 & & 0 \end{pmatrix},$$

and the combinatorial path $(\mathbf{a},a;\mathbf{b},b)$ looks like

$$\begin{pmatrix} b_\infty = k & \cdots & 0 & & b_{N-1} & \cdots & & b_2 & & b_1 & & b & & 0 & & l \\ \dots & & 0 & & a_{N-1} & & \cdots & a_2 & & a_1 & & a_0 & & a & & 0 \end{pmatrix}. \tag{3.19}$$

The relation $a_0 \le l'$ follows from $b \ge 0$. Conditions (3.2) are clearly preserved. Condition (3.3) for $(\mathbf{a};\mathbf{b})$ is also clear in all places except when $i = 0, -1$; moreover the case $i = 0$ obviously follows from the case $i = -1$. If $l \le a + a_0$ then we have $l' = b + a_0$ and condition (3.3) for $(\mathbf{a};\mathbf{b})$ with $i = -1$ follows from condition (3.3) for $(\mathbf{a},a;\mathbf{b},b)$ with $i = 1$. If $l \ge a + a_0$ then we have $l' + a = b + l$ and condition (3.3) for $(\mathbf{a};\mathbf{b})$ with $i = -1$ follows from condition (3.3) for $(\mathbf{a},a;\mathbf{b},b)$ with $i = -1$. □

Let $\mathcal{C}^{(N)}_{k,l}[i] \subset \mathcal{C}^{(N)}_{k,l}$ be the set of all combinatorial paths such that $a_0 = i$. We have

$$\mathcal{C}^{(N)}_{k,l} = \bigsqcup_{i=0}^{l} \mathcal{C}^{(N)}_{k,l}[i].$$

The recursion (3.14) splits into

$$\mathcal{C}^{(N+1)}_{k,l}[i] = \bigsqcup_{l',l'',i'} c_{\mathcal{C}}(l,l'',l')\mathcal{C}^{(N)}_{k,l'}[i'], \tag{3.20}$$

where the sum is over l', l'', i' such that (l,l'',l') is Verlinde triple, $i = (l + l'' - l')/2$ and $i' \le l'$.

Next we turn to the multiplication map. The analog of the commutative property (3.11) does not hold. However we have the following

Proposition 3.3.2. *The multiplication map $m_{\mathcal{C}}$ given by (3.13) is surjective. Moreover, let $(\mathbf{a};\mathbf{b}) \in \mathcal{C}^{(N)}_{k,l'}$ and let (l,l'',l') be an admissible triple of level k. Let $(l,l'',l') = (l_1,l_1'',l_1') + (l_2,l_2'',l_2')$, where $(l_1,l_1'',l_1'),(l_2,l_2'',l_2')$ are admissible triples of levels k_1,k_2, $k_1 + k_2 = k$. Then there exist $(\mathbf{a}^{(1)};\mathbf{b}^{(1)}) \in \mathcal{C}^{(N)}_{k_1,l_1'}$, $(\mathbf{a}^{(2)};\mathbf{b}^{(2)}) \in \mathcal{C}^{(N)}_{k_2,l_2'}$ such that*

$$m_{\mathcal{C}}((\mathbf{a}^{(1)};\mathbf{b}^{(1)}),(\mathbf{a}^{(2)};\mathbf{b}^{(2)})) = (\mathbf{a};\mathbf{b}),$$

$$m_{\mathcal{C}}(c_{\mathcal{C}}(l_1,l_1'',l_1')(\mathbf{a}^{(1)};\mathbf{b}^{(1)}), c_{\mathcal{C}}(l_2,l_2'',l_2')(\mathbf{a}^{(2)};\mathbf{b}^{(2)})) = c_{\mathcal{C}}(l,l'',l')(\mathbf{a};\mathbf{b}).$$

Proof. First, we prove that for any combinatorial path $(\mathbf{a};\mathbf{b}) \in \mathcal{C}^{N}_{k,l'}$ and for any level k Verlinde triple (l,l'',l'), we have a decomposition of $(\mathbf{a};\mathbf{b})$,

$$(\mathbf{a};\mathbf{b}) = \sum_{j=1}^{k} (\mathbf{a}_j;\mathbf{b}_j),$$

where $(\mathbf{a}_j;\mathbf{b}_j) \in \mathcal{C}^{(N)}_{1,l_j'}$, $j = 1,\dots,k$ and $l_1' + \cdots + l_k' = l'$; and a decomposition of the triple (l,l'',l'),

$$(l,l'',l') = \sum_{j=1}^{k} (l_j,l_j'',l_j'),$$

where (l_j, l_j'', l_j'), $j = 1, \dots, k$, are level 1 triples, such that we have

$$\sum_{j=1}^{k} c_{\mathcal{C}}(l_j, l_j'', l_j')(\mathbf{a}_j; \mathbf{b}_j) = c_{\mathcal{C}}(l, l'', l')(\mathbf{a}; \mathbf{b}). \tag{3.21}$$

We prove this statement by induction on N using the induction hypothesis that any level k combinatorial path of length N can be represented as a sum of k combinatorial paths of level 1. It is obvious that the decomposition (3.21) for N implies the above hypothesis for $N+1$.

The decomposition of $(\mathbf{a}; \mathbf{b})$ is not unique. We can choose any decomposition $(\mathbf{a}_j; \mathbf{b}_j) \in \mathcal{C}^{(N)}_{1,l_j'}$. Among the pairs $(l_j', (a_j)_0)$, we have a_0 pairs $(1,1)$, $l' - a_0$ pairs $(1,0)$ and $k - l'$ pairs $(0,0)$. We call these boundary pairs. They are independent of the choice of the decomposition.

The decomposition of (l, l'', l') is unique. We have x triples $(1,0,1)$, y triples $(1,1,0)$, z triples $(0,1,1)$ and $k - x - y - z$ triples $(0,0,0)$, where x, y, z are given by (3.7). We want to label these triple as (l_j, l_j'', l_j') so that the equality (3.21) holds.

Recall (3.16). The key property in the proof is that for level 1 we have $a(1,1,0;0) = b(0,1,1;0) = 1$ and $a = b = 0$ in all other cases.

Now we match up the level 1 triples with the boundary pairs. The triples $(0,0,0)$ and $(1,1,0)$ must match up with the pairs $(0,0)$ because of the consistency of l_j'. Their numbers are the same, and we match up arbitrarily among them.

If $a_0 \geq x$ we match up all $(1,0,1)$ with $(1,1)$, and we match up $(0,1,1)$ with $(1,0)$ or $(1,1)$. If $a_0 \leq x$ we match up all $(0,1,1)$ with $(1,0)$, and we match up $(1,0,1)$ with $(1,0)$ or $(1,1)$.

It is easy to see (3.21) by using the above remark on a, b for level 1.

Now, we prove the statement of the proposition using the result obtained above.

One can group $\mathcal{L} = \{(l_j, l_j'', l_j')\}_{j=1,\dots,k}$ as $\mathcal{L} = \mathcal{L}^{(1)} \sqcup \mathcal{L}^{(2)}$ where $\mathcal{L}^{(i)} = \{(l_j^{(i)}, l_j^{(i)''}, l_j^{(i)'})\}_{j=1,\dots,k_i}$ so that $(l_i, l_i'', l_i') = \sum_{j=1,\dots,k_i} (l_j^{(i)}, l_j^{(i)''}, l_j^{(i)'})$. We have the corresponding grouping $\{(\mathbf{a}_j^{(i)}, \mathbf{b}_j^{(i)})\}_{j=1,\dots,k_i}$. Then, setting $(\mathbf{a}^{(i)}, \mathbf{b}^{(i)}) = \sum_{j=1,\dots,k_i} (\mathbf{a}_j^{(i)}, \mathbf{b}_j^{(i)})$, we have

$$\sum_{j=1}^{k_i} c_{\mathcal{C}}(l_j^{(i)}, l_j^{(i)''}, l_j^{(i)'})(\mathbf{a}_j^{(i)}; \mathbf{b}_j^{(i)}) = c_{\mathcal{C}}(l_i, l_i'', l_i')(\mathbf{a}^{(i)}; \mathbf{b}^{(i)}).$$

The statement follows from this. □

Remark 3.3.3. As we noted, the surjectivity of the multiplication map does not determine the map $c_{\mathcal{C}}(l, l'', l')$ uniquely. On the other hand it is *not* possible to define a map satisfying the commutativity similar to (3.11). This requirement is too strong. Proposition 3.3.2 gives the approriate requirement in order to fix the map $c_{\mathcal{C}}(l, l'', l')$.

3.4. **Bijection between the combinatorial paths and the Verlinde paths.** In this section we use the results of the previous section to construct bijections ι between the sets of Verlinde paths and the sets of combinatorial paths.

Define a map $\iota^N_{k,l} : \mathcal{P}^N_{k,l} \to \mathcal{C}^{(N)}_{k,l}$ as follows. We map $(\boldsymbol{\alpha}; \boldsymbol{\beta})$ to $(\mathbf{a}; \mathbf{b})$ with

$$\begin{aligned} a_{i-1} &= y(\alpha_i, \beta_i, \alpha_{i+1}) && (i = 1, \dots, N), \\ b_i &= z(\alpha_i, \beta_i, \alpha_{i+1}) - (y(\alpha_{i+1}, \beta_{i+1}, \alpha_{i+2}) - x(\alpha_i, \beta_i, \alpha_{i+1}))^+ && (i = 1, \dots, N-1), \end{aligned} \tag{3.22}$$

where x, y, z are given by (3.7) and $(\ )^+$ by (3.18).

Theorem 3.4.1. *The map $\iota^N_{k,l} : \mathcal{P}^N_{k,l} \to \mathcal{C}^{(N)}_{k,l}$ is well-defined and bijective.*

Proof. By the construction $\iota^N_{k,l}$ intertwines the maps $c_{\mathcal{P}}(l,l'',l')$ and $c_{\mathcal{C}}(l,l'',l')$. Now the statement follows from (3.12) and (3.14) by induction on N. □

Corollary 3.4.2. *The cardinality of $\mathcal{C}^{(N)}_{k,l}$ is equal to the Verlinde number $d^{(N)}_{k,l}$.*

Proof. This follows from Theorem 3.4.1 and Lemma 3.2.2. □

We define a triple grading on the Verlinde paths by using the functions $e, s_1, s_2 : \mathcal{P}^N_{k,l} \to \mathbb{Z}_{\geq 0}$ given by

$$\begin{aligned} e(\boldsymbol{\alpha};\boldsymbol{\beta}) &= \sum_{i=1}^{N-1} i(a_i + b_i), \\ s_1(\boldsymbol{\alpha};\boldsymbol{\beta}) &= \sum_{i=1}^{N-1} b_i, \\ s_2(\boldsymbol{\alpha};\boldsymbol{\beta}) &= a_0 + \sum_{i=1}^{N-1} (a_i + b_i), \end{aligned} \tag{3.23}$$

where a_i, b_i are given by (3.22).

Note that the grading is local in the sense that the grading functions are given by sums over terms a_i, b_i which depend only on two neighboring Verlinde triples $(\alpha_i, \beta_i, \alpha_{i+1})$ and $(\alpha_{i+1}, \beta_{i+1}, \alpha_{i+2})$. It is also intriguing that the form of e is similar to the one-dimensional configuration sum appearing in the corner transfer matrix method in solvable lattice models (see, e.g., [B], [DJKMO]).

Define the corresponding character

$$\chi^N_{k,l}(q, z_1, z_2) = \sum_{(\boldsymbol{\alpha};\boldsymbol{\beta}) \in \mathcal{P}^N_{k,l}} q^{e(\boldsymbol{\alpha};\boldsymbol{\beta})} z_1^{s_1(\boldsymbol{\alpha};\boldsymbol{\beta})} z_2^{s_2(\boldsymbol{\alpha};\boldsymbol{\beta})}.$$

In Section 5 we show that

$$\chi^{(0,N)}_{k,l}(q, z) = z^l \chi^N_{k,l}(q, z^2, z^{-2}),$$

where $\chi^{(0,N)}_{k,l}(q,z)$ is defined by (2.3). The meaning of the grading by z_1, z_2 will be clarified in Section 6.4.

3.5. **Recursion relations.** In this section we describe recursion relations for the characters. We consider two types of the recursion relations which correspond to the insertion of a Verlinde triple either at the beginning of the Verlinde path (left concatenation) or at the end (right concatenation).

First, let us consider the right concatenation. Assume that $N \geq 2$, and consider $(\boldsymbol{\alpha},\boldsymbol{\beta}) \in \mathcal{P}^N_{k,l}$ and an admissible triple (α,β,γ) such that $\alpha = \alpha_N$. Then the differences of the gradings of the original path $(\boldsymbol{\alpha};\boldsymbol{\beta})$ and the concatenated path $(\boldsymbol{\alpha},\gamma;\boldsymbol{\beta},\beta)$ depend only on the new triple (α,β,γ) and the difference $\beta_{N-1} - \alpha_{N-1}$. Note that $\alpha + \beta_{N-1} - \alpha_{N-1} = 2z(\alpha_{N-1}, \beta_{N-1}, \alpha)$ as in (3.7).

Therefore, we split $\chi^N_{k,l}(q, z_1, z_2)$ into terms which depend on the values of α_N and $z(\alpha_{N-1}, \beta_{N-1}, \alpha)$. Namely, we define the partial character

$$\chi^N_{k,l}[*; i, j](q, z_1, z_2) = \sum_{\substack{(\boldsymbol{\alpha};\boldsymbol{\beta}) \in \mathcal{P}^{N+}_{k,l}, \\ \alpha_N = i,\ \alpha_N + \beta_{N-1} - \alpha_{N-1} = 2j}} q^{e(\boldsymbol{\alpha};\boldsymbol{\beta})} z_1^{s_1(\boldsymbol{\alpha};\boldsymbol{\beta})} z_2^{s_2(\boldsymbol{\alpha};\boldsymbol{\beta})} \quad (N \geq 2).$$

The partial character $\chi^N_{k,l}[*;i,j]$ is nontrivial only for $0 \le j \le i \le k$.

Proposition 3.5.1. *The partial characters $\chi^N_{k,l}[*;i,j](q,z_1,z_2)$ satisfy the following recursion relation*

$$\chi^{N+1}_{k,l}[*;i,j](q,z_1,z_2) = \sum_{i',j'} R^{(N)}_k((i,j),(i',j'))\chi^N_{k,l}[*;i',j'](q,z_1,z_2),$$

where

$$R^{(N)}_k((i,j),(i',j')) = q^{Ni-(N-1)(i-j-j')^+} z_1^{j'-(j+j'-i)^+} z_2^{i-(i-j-j')^+},$$

if $(i',2j-i+i',i)$ is admissible, and $R^{(N)}_k((i,j),(i',j')) = 0$ otherwise.

Proof. Follows from (3.23). □

Example 3.5.2. For $k=1$, the matrix $R^{(N)}_1$ is given by

$$\begin{array}{c} \\ (0,0) \\ (1,0) \\ (1,1) \end{array} \begin{array}{c} \begin{array}{ccc} (0,0) & (1,0) & (1,1) \end{array} \\ \begin{pmatrix} 1 & 1 & 1 \\ 0 & q & q^N z_1 z_2 \\ q^N z_2 & 0 & 0 \end{pmatrix} \end{array}.$$

Similarly, for the left concatenation (see (3.10)) we define the partial character

$$\chi^N_{k,l}[i;*](q,z_1,z_2) = \sum_{\substack{(\boldsymbol{\alpha};\boldsymbol{\beta})\in\mathcal{P}^N_{k,l},\\ \alpha_1+\beta_1-\alpha_2=2i}} q^{e(\boldsymbol{\alpha};\boldsymbol{\beta})} z_1^{s_1(\boldsymbol{\alpha};\boldsymbol{\beta})} z_2^{s_2(\boldsymbol{\alpha};\boldsymbol{\beta})} \quad (N \ge 1). \tag{3.24}$$

The partial character $\chi^N_{k,l}[i;*]$ is nontrivial only for $i \in \{0,\dots,l\}$.

Proposition 3.5.3. *The partial characters $\chi^N_{k,l}[i;*](q,z_1,z_2)$ satisfy the following recursion relation*

$$\chi^{N+1}_{k,l}[i;*](q,z_1,z_2) = \sum_{l',i'} L_k((l,i),(l',i'))\chi^N_{k,l'}[i';*](q,z_1,qz_2), \tag{3.25}$$

where

$$L_k((l,i),(l',j)) = (qz_1z_2)^{l'+i-l-(i'+i-l)^+} q^{i'} z_2^i,$$

if $(l,2i-l+l',l')$ is admissible, and $L_k((l,i),(l',i')) = 0$ otherwise.

Proof. Follows from (3.23). □

Example 3.5.4. For $k=1$, the matrix L_1 is given by

$$\begin{array}{c} \\ (0,0) \\ (1,0) \\ (1,1) \end{array} \begin{array}{c} \begin{array}{ccc} (0,0) & (1,0) & (1,1) \end{array} \\ \begin{pmatrix} 1 & qz_1z_2 & q \\ 0 & 1 & q \\ z_2 & 0 & 0 \end{pmatrix} \end{array}$$

Note that the matrices $R^{(N)}_k$ and L_k are nondegenerate square matrices whose size is $k(k+1)/2$. The matrix L_k does not depend on N, but the corresponding recursion relation mixes different values of l and includes shifts with respect to z_2. The matrix $R^{(N)}_k$ does not mix l but depends on N. Note also that at $q = z_1z_2 = 1$ their ranks degenerate to $k+1$. In this special case our recursion relation is essentially the recursion relation for Verlinde numbers $d^N_{k,l}$ which is given by

multiplication by an element in $k+1$ dimensional Verlinde algebra. This is called the Verlinde rule. In this sense, our recursion relation can be thought of as a graded version of of the Verlinde rule.

Now we describe the relation between the two recursions above. The key observation is the following lemma.

Lemma 3.5.5. *Let* $(\boldsymbol{\alpha};\boldsymbol{\beta}) = (\alpha_1,\dots,\alpha_N;\beta_1,\dots,\beta_{N-1})$ *be a Verlinde path of length* N*. Then*

$$(\bar{\boldsymbol{\alpha}};\bar{\boldsymbol{\beta}}) = (\alpha_N,\alpha_{N-1},\dots,\alpha_1;\beta_{N-1},\dots,\beta_1)$$

is also a Verlinde path of length N *and*

$$\begin{aligned} e(\bar{\boldsymbol{\alpha}};\bar{\boldsymbol{\beta}}) + e(\boldsymbol{\alpha};\boldsymbol{\beta}) &= (N-1)\,(s_2(\boldsymbol{\alpha};\boldsymbol{\beta}) + l - i), \\ s_1(\bar{\boldsymbol{\alpha}};\bar{\boldsymbol{\beta}}) + l' - i' &= s_1(\boldsymbol{\alpha};\boldsymbol{\beta}) + l - i, \\ s_2(\bar{\boldsymbol{\alpha}};\bar{\boldsymbol{\beta}}) + l' - i' &= s_2(\boldsymbol{\alpha};\boldsymbol{\beta}) + l - i, \end{aligned}$$

where

$$l' = \alpha_N, \qquad l = \alpha_1, \qquad i' = \frac{1}{2}(\alpha_N + \beta_{N-1} - \alpha_{N-1}), \qquad i = \frac{1}{2}(\alpha_1 + \beta_1 - \alpha_2).$$

Proof. The assertion follows from formula (3.23). Indeed,

$$a_i + b_i = \frac{1}{2}(\beta_i + \beta_{i+1} + 2\alpha_{i+1} - \alpha_i - \alpha_{i+2} - (\beta_i + \beta_{i+1} - \alpha_i - \alpha_{i+2})^+) = \bar{a}_{N-1-i} + \bar{b}_{N-1-i},$$

$i = 1,\dots,N-2$. Also, we always have $b_0 = b_{N-1} = 0$. Therefore, for energy e and spin s_2 we only have to compensate the contribution from a_{N-1} and a_0.

To prove the equality for s_1 we note that

$$s_2(\boldsymbol{\alpha},\boldsymbol{\beta}) - s_1(\boldsymbol{\alpha},\boldsymbol{\beta}) = \frac{1}{2}(\alpha_1 + \alpha_N + \sum_{j=1}^{N-1}\beta_j) = s_2(\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{\beta}}) - s_1(\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{\beta}}).$$

□

Note that for the Verlinde path $(\boldsymbol{\alpha},\boldsymbol{\beta})$ in Lemma 3.5.5, we have $l = \bar{a}_{N-1}$, $l' = a_{N-1}$, $i = a_0$ and $i' = \bar{a}_0$.

We decompose the partial characters further and define

$$\chi_k^N[i,l;i',l'](q,z_1,z_2) = \sum_{\substack{(\boldsymbol{\alpha};\boldsymbol{\beta})\in\mathcal{P}_{k,l}^N,\ \alpha_1+\beta_1-\alpha_2=2i \\ \alpha_N=l',\ \alpha_N+\beta_{N-1}-\alpha_{N-1}=2i'}} q^{e(\boldsymbol{\alpha};\boldsymbol{\beta})} z_1^{s_1(\boldsymbol{\alpha};\boldsymbol{\beta})} z_2^{s_2(\boldsymbol{\alpha};\boldsymbol{\beta})} \quad (N \geq 2).$$

Then, we have

$$\sum_{i=0}^{l} \chi_k^N[i,l;i',l'] = \chi_{k,l}^N[*;i',l'],$$

$$\sum_{\substack{i',l'=0 \\ i'\leq l'}}^{k} \chi_k^N[i,l;i',l'] = \chi_{k,l}^N[i;*],$$

$$\sum_{\substack{i',l',i=0 \\ i'\leq l',\ i\leq l}}^{k} \chi_k^N[i,l;i',l'] = \chi_{k,l}^N.$$

Then, left and right concatenations act on vectors of the form

$$\{\chi_k^N[i,l;i',l']\}_{i,l,i',l'\in\{1,\dots,k\},\ 0\le i\le l,\ 0\le i'\le l'}$$

as block diagonal matrices with $k(k+1)/2$ blocks, each being equal to L_k and $R_k^{(N)}$ (plus a shift $z_2 \mapsto qz_2$) respectively. By Lemma 3.5.5, these two actions are conjugated by a transformation $\chi_k^N[i,l;i',l'] \to \chi_k^N[i',l';i,l]$,

$$\chi_k^N[i,l;i',l'](q,z_1,z_2) = q^{(N-1)(l'-i')}(z_1z_2)^{l'-i'-l+i}\,\chi_k^N[i',l';i,l](q^{-1},z_1,z_2q^{N-1}).$$

In the subsequent paper we will present explicit fermionic formulas for the characters in terms of q-binomial coefficients.

4. Heisenberg modules

In this section we discuss the reduction of the problem of computing coinvariants for $\widehat{\mathfrak{sl}}_2$ to the problem of computing coinvariants for affine Heisenberg algebra $\widehat{\mathcal{H}}$. Then we begin a study of the structure of "integrable" Heisenberg modules.

4.1. **Abelianization.** In this section we discuss the idea of passage from $\widehat{\mathfrak{sl}}_2$ module to $\widehat{\mathcal{H}}$ module.

Let us recall that the Verma module $M_{k,l}$ is a quotient of the free module generated by v over the universal enveloping algebra of $\widehat{\mathfrak{sl}}_2$ by the highest weight conditions for v. The highest weight conditions in the Verma module $M_{k,l}$ have the form $e_iv = 0$ $(i \le 0)$, $h_iv = 0$ $(i \le -1)$, $h_0v = lv$ and $f_iv = 0$ $(i \le -1)$. The integrable module $L_{k,l}$ is the quotient of the Verma module $M_{k,l}$ by the submodule generated by the elements of the form $e(z)^{k+1}w, f(z)^{k+1}w$ for $w \in M_{k,l}$. The space of (M,N)-coinvariants $L_{k,l}^{(M,N)}$ is the further quotient of $L_{k,l}$ by $\operatorname{Im} e_i$ $(i \ge M)$ and $\operatorname{Im} f_i$ $(i \ge N)$.

We call a vector in $L_{k,l}^{(M,N)}$ of the form $\mathbf{m}v$ where $\mathbf{m}$ is a monomial in $U(\widehat{\mathfrak{sl}}_2)$ as a monomial vector or simply as a monomial. Our goal is to obtain a monomial basis of $L_{k,l}^{(M,N)}$, i.e., a set of monomial vectors in $L_{k,l}^{(M,N)}$ which forms a basis. In principle, one can rewrite given monomials to linear combinations of other monomials by using the above relations in the hope that we can find a monomial basis as a result. However, this is not a unique procedure, and in general, we do not know any canonical way of doing such reduction, which eventually leads to a monomial basis.

In [FKLMM] an approach using a filtration of $L_{k,l}$ and its adjoint graded space $\operatorname{Gr}(L_{k,l})$ was used in the special case $k = 1$. In this paper we further develop this method.

Let us explain the basic idea in this approach. If we consider a filtration such that each generators e_i, f_i, h_i $(i \in \mathbb{Z})$ of $\widehat{\mathfrak{sl}}_2$ has a fixed degree with respect to the filtration, we can induce its graded action on $\operatorname{Gr}(L_{k,l})$. We consider the space of coinvariants in $\operatorname{Gr}(L_{k,l})$ with respect to the induced actions of e_i $(i \ge M)$ and f_i $(i \ge N)$. For an appropriate choice of filtration, the relations in terms of the induced actions simplify, and we may find a monomial basis of the space of coinvariants $\operatorname{Gr}(L_{k,l})^{(M,N)}$ by exploiting the induced actions. If, in addition, the equality of dimensions

$$\dim L_{k,l}^{(M,N)} = \dim \operatorname{Gr}(L_{k,l})^{(M,N)}$$

holds, the same set of monomials gives a basis of $L_{k,l}^{(M,N)}$.

Now, let us discuss a concrete example of filtration.

Let $\mathfrak{g}$ be a Lie algebra with a Lie subalgebra $\mathfrak{a} \subset \mathfrak{g}$ and V a cyclic $\mathfrak{g}$-module with a cyclic vector v. Consider the filtration $A(V) = (A_i(V))_{i\geq 0}$ defined inductively by

$$A_0(V) = \mathbb{C}v, \quad A_{i+1}(V) = A_i(V) + \mathfrak{g} \cdot A_i(V).$$

Then the induced actions of elements of $\mathfrak{g}$ on $\mathrm{Gr}^A(V)$ with respect to this filtration commute. We call the passage from the module V to $\mathrm{Gr}^A(V)$ the abelianization procedure. Now by Lemma 2.2.1 we have the inequality of dimensions of coinvariants

$$\dim V/\mathfrak{a}V \leq \dim \mathrm{Gr}^A(V)/\overline{\mathfrak{a}}\,\mathrm{Gr}^A(V), \tag{4.1}$$

where $\overline{\mathfrak{a}}$ is the induced action of $\mathfrak{a}$.

In the case we consider in this paper, $\mathfrak{g} = \widehat{\mathfrak{sl}}_2$, $\mathfrak{a} = (\oplus_{i\geq M} e_i) \oplus (\oplus_{i\geq M+N} h_i) \oplus (\oplus_{i\geq N} f_i)$ and $V = L_{k,l}$, the equality in (4.1) does not hold. In other words, the abelianization loses the information about the character of the original coinvariants. The dimension of the coinvariants jumps in the abelian limit.

We will successfully exploit a finer filtration given by the actions of $\{e_i\}$ and $\{f_i\}$ (but not $\{h_i\}$). The associated graded spaces are Heisenberg modules, which are more tractable than $\widehat{\mathfrak{sl}}_2$-modules. Moreover, it turns out that the dimension of the coinvariants does not jump in this limit if $M, N \geq 1$. (In the cases $M = 0, N > 0$ and $M > 0, N = 0$ the dimension does not jump either if one chooses an appropriate cyclic vector in $L_{k,l}$.)

4.2. **Affine Heisenberg algebra.** In this section, we derive an action of the affine Heisenberg algebra $\widehat{\mathcal{H}}$ on the associated graded space of the integrable $\widehat{\mathfrak{sl}}_2$-modules, filtered by the action of the currents $e(z)$ and $f(z)$.

The affine Heisenberg algebra $\widehat{\mathcal{H}}$ is a Lie algebra generated by the elements e_i, h_i, f_i $(i \in \mathbb{Z})$ satisfying the relations

$$[e_i, f_i] = h_{i+j}, \qquad [h_i, e_j] = [h_i, f_j] = 0.$$

Here and below we use the same symbols e_i, f_i and h_i for the generators of $\widehat{\mathcal{H}}$ as those of $\widehat{\mathfrak{sl}}_2$.

An $\widehat{\mathcal{H}}$-module V is called a highest weight module if it is generated by a highest weight vector v satisfying

$$e_i v = 0, \qquad h_i v = 0, \qquad f_i v = 0 \qquad (i \leq -1).$$

A highest weight $\widehat{\mathcal{H}}$- module V is called level-k integrable if the action satisfies

$$e(z)^{k+1} w = f(z)^{k+1} w = 0, \tag{4.2}$$

for any vector $w \in V$. For simplicity, we call a level k integrable highest weight module an $\widehat{\mathcal{H}}_k$-module.

Let V be an $\widehat{\mathcal{H}}_k$-module. Since $(\mathrm{ad}\, e_0) f(z) = h(z)$ and $(\mathrm{ad}\, f_0) e(z) = -h(z)$, we have the relations

$$e(z)^{k+1-a} h(z)^a w = f(z)^{k+1-a} h(z)^a w = 0 \quad (0 \leq a \leq k+1) \tag{4.3}$$

for all $w \in V$.

The $\widehat{\mathcal{H}}_k$ module V is graded by

$$\deg v = (0,0,0), \quad \deg e_i = (1,0,i), \quad \deg f_i = (0,1,i), \quad \deg h_i = (1,1,i),$$

where v is the highest weight vector in V. Let $V_{m,n,e}$ be the subspace of degree (m,n,e) and $V_{m,n} = \oplus_e V_{m,n,e}$.

Below, we will introduce various $\widehat{\mathcal{H}}$-linear mappings $\mu : V \to W$ among $\widehat{\mathcal{H}}_k$-modules. All these maps are graded maps. If the highest weight vector v of V is mapped to a vector in a subspace of some degree (m,n,e) then the mapping μ has degree (m,n,e).

Let $L_{k,l}$ be the integrable $\widehat{\mathfrak{sl}}_2$ module of level k and highest weight l. Define a filtration $F(L_{k,l}) = (F_i(L_{k,l}))_{i\geq 0}$ of $L_{k,l}$ inductively by

$$F_0(L_{k,l}) = \mathbb{C}v_k[l], \quad F_{i+1}(L_{k,l}) = F_i(L_{k,l}) + \mathfrak{c} \cdot F_i(L_{k,l}),$$

where

$$\mathfrak{c} \;=\; \Big(\bigoplus_{i\in\mathbb{Z}} \mathbb{C}e_i\Big) \bigoplus \Big(\bigoplus_{i\in\mathbb{Z}} \mathbb{C}f_i\Big) \subset \widehat{\mathfrak{sl}}_2.$$

We define the induced action of e_i, f_i and h_i on the associated graded space to have degree 1 for e_i and f_i, and 2 for h_i.

Proposition 4.2.1. *The induced actions of e_i, f_i, h_i $(i \in \mathbb{Z})$, define an $\widehat{\mathcal{H}}_k$-module structure on the vector space* $\mathrm{Gr}^F(L_{k,l})$.

Proof. Let us denote the induced actions by $\bar{e}_i$, $\bar{f}_i$, $\bar{h}_i$. We will show $[\bar{h}_i, \bar{e}_j] = 0$. (The proof of $[\bar{h}_i, \bar{f}_j] = 0$ is similar, and the rest of the relations are straightforward.) We have $[h_i, e_j] = 2e_{i+j}$. The mapping $[\bar{h}_i, \bar{e}_j]$ sends $\mathrm{Gr}^F_n(L_{k,l})$ to $\mathrm{Gr}^F_{n+3}(L_{k,l})$. For $v \in F_n(L_{k,l})$, let $\bar{v}^{(n)}$ be the corresponding element in $\mathrm{Gr}^F_n(L_{k,l})$. Then we have

$$[\bar{h}_i, \bar{e}_j]\bar{v}^{(n)} = \overline{[h_i, e_j]v}^{(n+3)} = 2\overline{e_{i+j}v}^{(n+3)} = 0.$$

□

We define an involution ι of $\widehat{\mathfrak{sl}}_2$ by

$$\iota(e_i) = f_i, \qquad \iota(f_i) = e_i, \qquad \iota(h_i) = -h_i. \tag{4.4}$$

and an involution of $L_{k,l}$ which we also denote ι by

$$\begin{aligned} \iota(v_k[l]) &= u_k[l], \\ \iota(x \cdot y) &= \iota(x) \cdot \iota(y), \end{aligned}$$

for $x \in \widehat{\mathfrak{sl}}_2$ and $y \in L_{k,l}$. Here we set

$$u_k[l] = \frac{f_0^l}{l!} v_k[l]. \tag{4.5}$$

The involution ι induces a new filtration E of $L_{k,l}$ from F. Namely, it is given by a procedure similar to F above,

$$E_0(L_{k,l}) = \mathbb{C}u_k[l], \quad E_{i+1}(L_{k,l}) = E_i(L_{k,l}) + \mathfrak{c} \cdot E_i(L_{k,l}).$$

The adjoint graded space $\mathrm{Gr}^E(L_{k,l})$ is also an $\widehat{\mathcal{H}}_k$-module.

4.3. **Heisenberg modules.** In this section we define three types of $\widehat{\mathcal{H}}_k$-modules which differ only by the annihilation conditions for the highest weight vector.

Let $l_1,l_2,l_3 \geq -1$. Let $V_k[l_1,l_2,l_3]$ be the $\widehat{\mathcal{H}}_k$-module generated by the highest weight vector $v_V = v_V[l_1,l_2,l_3]$, satisfying the relations

$$e_i v_V = 0 \quad (i \leq 0), \qquad f_i v_V = 0 \quad (i \leq -1), \tag{4.6}$$

$$e_1^{l_1+1} v_V = f_0^{l_2+1} v_V = h_0^{l_3+1} v_V = 0. \tag{4.7}$$

Let $U_k[l_1,l_2,l_3]$ be the $\widehat{\mathcal{H}}_k$-module generated by the highest weight vector $v_U = v_U[l_1,l_2,l_3]$, satisfying the relations

$$e_i v_U = 0 \qquad (i \leq -1), \qquad f_i v_U = 0 \qquad (i \leq 0), \tag{4.8}$$

$$e_0^{l_1+1} v_U = f_1^{l_2+1} v_U = h_0^{l_3+1} v_U = 0. \tag{4.9}$$

Let $W_k[l_1,l_2,l_3]$ be the $\widehat{\mathcal{H}}_k$-module generated by the highest weight vector $v_W = v_W[l_1,l_2,l_3]$, satisfying the relations

$$e_i v_W = f_i v_W = 0 \qquad (i \leq 0), \tag{4.10}$$

$$e_1^{l_1+1} v_W = f_1^{l_2+1} v_W = h_1^{l_3+1} v_W = 0. \tag{4.11}$$

The modules $V_k[l_1,l_2,l_3]$, $U_k[l_1,l_2,l_3]$ and $W_k[l_1,l_2,l_3]$ are related through simple automorphisms of $\widehat{\mathcal{H}}$ as follows.

Define the shift automorphism T of $\widehat{\mathcal{H}}$ by

$$T(e_i) = e_i, \qquad T(f_i) = f_{i+1}, \qquad T(h_i) = h_{i+1}. \tag{4.12}$$

We have an isomorphism of linear spaces

$$T : V_k[l_1,l_2,l_3] \xrightarrow{\sim} W_k[l_1,l_2,l_3], \tag{4.13}$$

determined by the properties

$$\begin{aligned} T(v_V) &= v_W, \\ T(x \cdot y) &= T(x) \cdot T(y) \quad (x \in \widehat{\mathcal{H}},\ y \in V_k[l_1,l_2,l_3]). \end{aligned}$$

Define involution ι of $\widehat{\mathcal{H}}$ (cf. (4.4)) by

$$\iota(e_i) = f_i, \qquad \iota(f_i) = e_i, \qquad \iota(h_i) = -h_i. \tag{4.14}$$

We have an isomorphism of linear spaces

$$\iota : V_k[l_1,l_2,l_3] \xrightarrow{\sim} U_k[l_2,l_1,l_3], \tag{4.15}$$

determined by the properties

$$\begin{aligned} \iota(v_V) &= v_U, \\ \iota(x \cdot y) &= \iota(x) \cdot \iota(y) \quad (x \in \widehat{\mathcal{H}},\ y \in V_k[l_1,l_2,l_3]). \end{aligned}$$

Note that if $l_i = -1$ for some i, then we have

$$V_k[l_1,l_2,l_3] = U_k[l_1,l_2,l_3] = W_k[l_1,l_2,l_3] = 0.$$

The formulas (4.2), (4.3) imply

$$e_1^{k+1} v = f_0^{k+1} v = h_0^{k+1} v = 0,$$

for any highest weight vector in a $\widehat{\mathcal{H}}_k$-module. Therefore, we have

$$V_k[l_1,l_2,l_3] = V_k[\min(l_1,k),\min(l_2,k),\min(l_3,k)].$$

From Lemma 7.1.1 we obtain another relation

$$V_k[l_1,l_2,l_3] = V_k[l_1,l_2;\min(l_1,l_2,l_3)].$$

Similarly we have

$$\begin{aligned} U_k[l_1,l_2,l_3] &= U_k[\min(l_1,k),\min(l_2,k),\min(l_3,k)],\\ U_k[l_1,l_2,l_3] &= U_k[l_1,l_2;\min(l_1,l_2,l_3)], \end{aligned}$$

and

$$\begin{aligned} W_k[l_1,l_2,l_3] &= W_k[\min(l_1,k),\min(l_2,k),\min(l_3,k)],\\ W_k[l_1,l_2,l_3] &= W_k[l_1,l_2;\min(l_1,l_2,l_3)]. \end{aligned}$$

Introduce the following simplified notation:

$$\begin{aligned} \overline{V}_k[l_1,l_2] &= V_k[l_1,l_2;\min(l_1,l_2)],\\ \overline{U}_k[l_1,l_2] &= U_k[l_1,l_2;\min(l_1,l_2)],\\ \overline{W}_k[l_1,l_2] &= W_k[l_1,l_2;\min(l_1,l_2)]. \end{aligned}$$

These are the modules where h condition on the highest weight vector in (4.7),(4.9),(4.11) follows from e and f conditions. So, these are the "simplest" and the largest modules.

Also introduce the following notations.

$$\begin{aligned} U_k[l_1,l_2] &= U_k[l_1,l_2,0],\\ V_k[l_1,l_2] &= V_k[l_1,l_2,0],\\ W_k[l_1,l_2] &= W_k[l_1,l_2,l_1+l_2-k]. \end{aligned}$$

These are the modules which are directly related to the $\widehat{\mathfrak{sl}}_2$ modules studied in this paper (see Section 5.3).

4.4. **Filtration and recurrence relations.** In this section we will determine the structure of the $\widehat{\mathcal{H}}_k$-module $W_k[l_1,l_2,l_3]$ by recurrence relations. We construct a filtration of $W_k[l_1,l_2,l_3]$ such that each graded component of the adjoint graded space is isomorphic to $T(W_k[l_1',l_2'])$ for some l_1',l_2' such that $0 \le l_1',l_2' \le k$ and $l_1'+l_2' \ge k$.

There is a significant difference between the $\widehat{\mathfrak{sl}}_2$-modules $L_{k,l}$ and the $\widehat{\mathcal{H}}_k$-modules $U_k[l_1,l_2,l_3]$, $V_k[l_1,l_2,l_3]$ and $W_k[l_1,l_2,l_3]$. The modules $L_{k,l}$ are irreducible, while the Heisenberg modules contain infinitely many submodules. We are interested in the self-similarity of subquotients of $\widehat{\mathcal{H}}_k$-modules, which will allow us to construct a basis of $\widehat{\mathcal{H}}_k$-modules by a recursion procedure.

First we consider a relatively simple case, when a submodule of a $\widehat{\mathcal{H}}_k$ is isomorphic to $W_k[l_1',l_2']$ for some l_1',l_2'.

Let v be the highest weight vector in a highest weight $\widehat{\mathcal{H}}_k$-module. We denote the submodule generated by a set of vectors $\mathbf{m}_i v$ $(i \in I)$ by $\langle \mathbf{m}_i \ (i \in I)\rangle$.

We have the following short exact sequences:

$$\begin{aligned} &0 \to \langle f_0^{l_2}\rangle \to V_k[l_1,l_2] \to V_k[l_1,l_2-1] \to 0 \quad (l_1+l_2 \le k),\\ &0 \to \langle e_0^{l_1}\rangle \to U_k[l_1,l_2] \to U_k[l_1-1,l_2] \to 0 \quad (l_1+l_2 \le k). \end{aligned}$$

Proposition 4.4.1. *The following are well-defined $\widehat{\mathcal{H}}$-linear mappings.*

$$W_k[l_1+l_2, k-l_2] \to V_k[l_1,l_2], \quad \mathbf{m}v_W \mapsto \mathbf{m}f_0^{l_2}v_V \quad (l_1+l_2 \le k), \tag{4.16}$$

$$W_k[k-l_1, l_1+l_2] \to U_k[l_1,l_2], \quad \mathbf{m}v_W \mapsto \mathbf{m}e_0^{l_1}v_U \quad (l_1+l_2 \le k), \tag{4.17}$$

Proof. We give the proof for the first map. The second one is similar.

We need to show the validity of the relations (4.10) and (4.11). The relation $e_0 f_0^{l_2} v = 0$ follows from $h_0 v = 0$. The relation $e_1^{l_1+l_2+1} f_0^{l_2} v = 0$ follows from a variant of Lemma 7.1.2. Finally, the relation $f_1^{k-l_2+1} f_0^{l_2} v = 0$ follows from the integrability condition (4.2), and $h_1^{l_1+1} f_0^{l_2} v = 0$ from a variant of Lemma 7.1.4. □

Corollary 4.4.2. *The sequences*

$$W_k[l_1+l_2, k-l_2] \to V_k[l_1,l_2] \to V_k[l_1,l_2-1] \to 0 \quad (l_1+l_2 \le k), \tag{4.18}$$

$$W_k[k-l_1, l_1+l_2] \to U_k[l_1,l_2] \to U_k[l_1-1,l_2] \to 0 \quad (l_1+l_2 \le k) \tag{4.19}$$

are exact.

Proof. Corollary 4.4.2 follows from Proposition 4.4.1. □

Later, we will prove that (4.16) and (4.17) are in fact injective (see Corollary 5.4.9).

Now we consider a more complicated situation. The module $W_k[l_1,l_2,l_3]$ has a filtration, such that each of the corresponding subquotients is isomorphic to an $\widehat{\mathcal{H}}_k$-module of the form $T(W_k[l_1',l_2'])$ for some l_1', l_2'.

Fix l_1, l_2, l_3 such that $0 \le l_1, l_2 \le k$ and $0 \le l_3 \le \min(l_1,l_2)$, and denote the highest weight vector of $W_k[l_1,l_2,l_3]$ by v.

Lemma 4.4.3. *The vector $h_1^a f_1^c v \in W_k[l_1,l_2,l_3]$ is zero unless*

$$0 \le a \le l_3, \quad 0 \le c \le l_2 - a. \tag{4.20}$$

Proof. The relation $h_1^a f_1^c v = 0$ for $a > l_3$ follows from $h_1^{l_3+1} v = 0$. The relation $h_1^a f_1^c v = 0$ for $a + c > l_2$ follows from $f_1^{l_2+1} v = e_0 v = 0$ by a variant of Lemma 7.1.1. □

Now, we define a filtration of $W_k[l_1,l_2,l_3]$. Consider submodules of the form $\langle h_1^a f_1^c \rangle$. Since these submodules are not linearly ordered with respect to inclusion, we define a two-step filtration. First, define the filtration by $\langle h_1^a \rangle$, then refine this filtration by using $h_1^a f_1^c$.

Set

$$W_{a,c} = \langle h_1^a f_1^c, h_1^{a+1} \rangle \subset W_k[l_1,l_2,l_3].$$

Note that

$$\begin{aligned} W_{a,c} &\supset W_{a,c+1}, \\ W_{a,0} &= \langle h_1^a \rangle, \\ W_{a,c} &= W_{a+1,0} \text{ if } c > l_2 - a. \end{aligned}$$

Thus we have a filtration of $W[l_1,l_2,l_3]$:

$$W[l_1,l_2,l_3] = W_{0,0} \supset W_{0,1} \supset \cdots \supset W_{1,0} \supset W_{1,1} \supset \cdots \supset W_{l_3,0} \supset \cdots \supset W_{l_3,l_2-l_3+1} = 0. \tag{4.21}$$

We call the filtration of $W_k[l_1,l_2,l_3]$ given by (4.21) the canonical filtration of the first kind, and denote it by $C(W_k[l_1,l_2,l_3])$.

Proposition 4.4.4. *For each (a,c) satisfying (4.20) there exists an $\widehat{\mathcal{H}}$-linear surjection*

$$\begin{aligned} T(W_k[l_1',l_2']) &\to \mathrm{Gr}^C_{a,c}(W_k[l_1,l_2,l_3]), \\ T(\mathbf{m}v_W[l_1',l_2',l_1'+l_2'-k]) &\mapsto T(\mathbf{m})v_{a,c}, \\ v_{a,c} &= h_1^a f_1^c v_W[l_1,l_2,l_3]. \end{aligned} \tag{4.22}$$

Here

$$l_1' = \min(k-a, l_1+c-a), \quad l_2' = k-c. \tag{4.23}$$

Proof. It is enough to show that the vector $v_{a,c}$ satisfies

$$\begin{aligned} &e_i v_{a,c} = 0 \quad (i \le 0), \qquad f_i v_{a,c} = 0 \quad (i \le 1), \\ &e_1^{l_1'+1} v_{a,c} = 0, \quad f_2^{l_2'+1} v_{a,c} = 0, \quad h_2^{l_1'+l_2'-k+1} v_{a,c} = 0. \end{aligned}$$

First, note that the quotient module $\mathrm{Gr}^C_{a,c}(W_k[l_1,l_2,l_3])$ is generated by $h_1^a f_1^c v$ where $v = v_W[l_1,l_2,l_3]$. Note also that that $h_1^a f_1^{c+1} v = 0$ and $h_1^{a+1} f_1^{c-1} v = 0$.

If $i \le 0$, then $e_i v = 0$. Therefore, we have

$$e_i v_{a,c} = e_i h_1^a f_1^c v = c h_1^a h_{i+1} f_1^{c-1} v = 0$$

because

$$\begin{aligned} h_{i+1} v &= 0 \qquad (i<0), \\ h_1^{a+1} f_1^{c-1} v &= 0 \qquad (i=0). \end{aligned}$$

If $i \le 1$, we have

$$f_i v_{a,c} = f_i h_1^a f_1^c v = 0$$

because

$$\begin{aligned} f_i v &= 0 \qquad i \le 0, \\ h_1^a f_1^{c+1} v &= 0 \qquad (i=1). \end{aligned}$$

We have

$$f_2^{l_2'+1} v_{a,c} = h_1^a f_2^{k-c+1} f_1^c v = 0$$

because of integrability condition (4.2).

Consider the case $0 \le c \le k - l_1$. We have $l_1' = l_1 + c - a$. From Lemma 7.1.3 we have

$$e_1^{l_1'+1} v_{a,c} = e_1^{l_1+c-a+1} h_1^a f_1^c v = 0.$$

From Lemma 7.1.4 we have

$$h_2^{l_1'+l_2'-k+1} v_{a,c} = h_2^{l_1-a+1} h_1^a f_1^c v = 0.$$

Finally, consider the case $k - l_1 \le c \le l_2 - a$. We have $l_1' = k - a$, and by using (4.3), we obtain

$$\begin{aligned} e_1^{l_1'+1} v_{a,c} &= e_1^{k-a+1} h_1^a f_1^c v = 0, \\ h_2^{l_1'+l_2'-k+1} v_{a,c} &= h_2^{k-a-c+1} h_1^a f_1^c v = 0. \end{aligned}$$

□

In Section 6.1, we will prove that the map (4.22) is an isomorphism.

5. Recursion for the coinvariants

In this section we prove the main theorem (Theorem 5.4.4), which states that the dimension of the space of the $\widehat{\mathfrak{sl}}_2$ coinvariants $L_{k,l}^{(M,N)}$ is given by the Verlinde rule. The key idea is to derive a recursion relation for the spaces of the Heisenberg coinvariants with respect to the parameters M, N.

5.1. **The spaces of (M,N)-coinvariants.** In Section 2 we have introduced the space of (M,N)-coinvariants $L_{k,l}^{(M,N)}$. In this section we define similar spaces for $\widehat{\mathcal{H}}_k$-modules and prove that the properties of $\widehat{\mathcal{H}}_k$ modules described in Section 4.4 are valid for the coinvariants as well.

For $M, N \geq 0$, we set

$$\mathfrak{a}^{(M,N)} = \left(\bigoplus_{i\geq M} \mathbb{C}e_i\right) \oplus \left(\bigoplus_{i\geq M+N} \mathbb{C}h_i\right) \oplus \left(\bigoplus_{i\geq N} \mathbb{C}f_i\right) \subset \widehat{\mathcal{H}}.$$

For an $\widehat{\mathcal{H}}_k$-module V, the space of (M,N)-coinvariants is defined by

$$V^{(M,N)} = V/\mathfrak{a}^{(M,N)}V.$$

Remark 5.1.1. For simplicity of notation, we denote the image of the highest weight vector v of an $\widehat{\mathcal{H}}_k$-module V in the space of (M,N)-coinvariants by v.

Define

$$W_k^{(M,N)}[l_1,l_2,l_3] = \bigoplus_{m,n} W_k^{(M,N)}[l_1,l_2,l_3]_{m,n},$$

$$W_k^{(M,N)}[l_1,l_2,l_3]_{m,n} = \begin{cases} 0 \text{ if } M=0 \text{ and } n-2m < k-l_1; \\ 0 \text{ if } N=0 \text{ and } m-2n < k-l_2; \\ \oplus_e W_k[l_1,l_2,l_3]_{m,n,e}^{(M,N)} \text{ otherwise.} \end{cases} \tag{5.1}$$

Namely, we cut off some weight spaces of $W_k[l_1,l_2,l_3]^{(M,N)}$ when either M or N is zero. Note, in particular, that

$$W_k^{(0,0)}[l_1,l_2,l_3] = \begin{cases} \mathbb{C}\cdot v \text{ if } l_1=l_2=k; \\ 0 \text{ otherwise.} \end{cases} \tag{5.2}$$

Note also the difference in notation between $W_k^{(M,N)}[l_1,l_2,l_3]$ and $W_k[l_1,l_2,l_3]^{(M,N)}$. They are equal if $M,N>0$.

Remark 5.1.2. The reason for the above cutoff is to obtain the spaces satisfying the recursion relations (6.2). One can easily check that the cutoff condition is equivalent to the restriction $\sum_i a_i - \sum_i b_i \geq l$ for the monomial basis (5.34) following from (3.3).

Now we establish the coinvariant version of Corollary 4.4.2. First we deal with the cutoffs.

Lemma 5.1.3. *Consider the mapping*

$$W_k[k-l_1,l_1+l_2]_{m,n,e}^{(0,N)} \to U_k[l_1,l_2]_{m+l_1,n,e}^{(0,N)}, \quad \mathbf{m}v_W \mapsto \mathbf{m}e_0^{l_1}v_U$$

induced from (4.17). If $n-2m<l_1$, the image of this mapping is zero.

Proof. Take an element of $W_k[k-l_1, l_1+l_2]^{(0,N)}_{m,n,e}$ of the form

$$f_{i_1}\cdots f_{i_K}h_{j_1}\cdots h_{j_{K'}}v,$$

where v is the highest weight vector of $W_k[k-l_1, l_1+l_2]$. These vectors form a spanning set.

Suppose that $n-2m<l_1$, i.e., $K-K'<l_1$. Let us prove that in $U_k[l_1,l_2]^{(0,N)}$ we have

$$f_{i_1}\cdots f_{i_K}h_{j_1}\cdots h_{j_{K'}}e_0^{l_1}u=0,$$

where the vector u is the highest weight vector of $U_k[l_1,l_2]$. Note that we have

$$e_0^{l_1+1}u=0.$$

This is the only property of u we need for the proof. We use induction on K'. If $K'=0$ then $K<l_1$. Using $\operatorname{ad}e_0(f_i)=h_i$ and $\operatorname{ad}e_0(h_i)=0$, we have

$$(\operatorname{ad}e_0)^{l_1}(f_{i_1}\cdots f_{i_K})u=0.$$

Therefore, we have

$$f_{i_1}\cdots f_{i_K}e_0^{l_1}u=(-\operatorname{ad}e_0)^{l_1}(f_{i_1}\cdots f_{i_K})u=0.$$

If $K'>0$, we have

$$f_{i_1}\cdots f_{i_K}h_{j_1}\cdots h_{j_{K'}}e_0^{l_1}u \;=\; f_{i_1}\cdots f_{i_K}h_{j_1}\cdots h_{j_{K'-1}}e_0f_{j_{K'}}e_0^{l_1}u. \tag{5.3}$$

Note that $K-(K'-1)<l_1+1$. Changing K' to $K'-1$, l_1 to l_1+1 and u to $f_{j_{K'}}u$, we apply the induction hypothesis. Thus we obtain

$$f_{i_1}\cdots f_{i_K}h_{j_1}\cdots h_{j_{K'-1}}e_0^{l_1+1}f_{j_{K'}}u \;=\; 0. \tag{5.4}$$

From (5.3) and (5.4) we have

$$\begin{aligned} f_{i_1}\cdots f_{i_K}h_{j_1}\cdots h_{j_{K'}}e_0^{l_1}u &= f_{i_1}\cdots f_{i_K}h_{j_1}\cdots h_{j_{K'-1}}e_0[f_{j_{K'}},e_0^{l_1}]u\\ &= -l_1f_{i_1}\cdots f_{i_K}h_{j_1}\cdots h_{j_{K'}}e_0^{l_1}u\\ &= 0. \end{aligned}$$

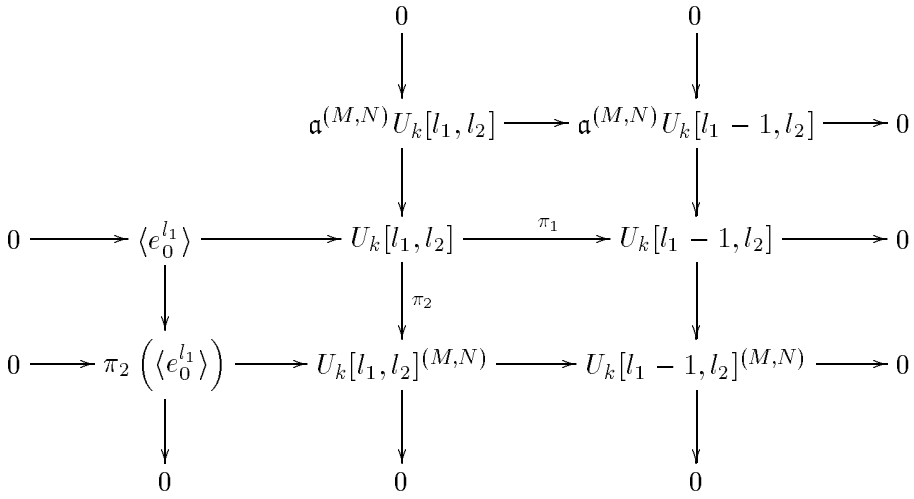
□

Lemma 5.1.4. *Suppose that $l_1+l_2\le k$. The sequences*

$$\begin{aligned} W_k^{(M,N)}[k-l_1,l_1+l_2]\to U_k[l_1,l_2]^{(M,N)}\to U_k[l_1-1,l_2]^{(M,N)}\to 0 \quad (M\ge 0, N>0),\\ W_k^{(M,N)}[l_1+l_2,k-l_2]\to V_k[l_1,l_2]^{(M,N)}\to V_k[l_1,l_2-1]^{(M,N)}\to 0 \quad (M>0, N\ge 0) \end{aligned}$$

are exact.

Proof. We consider the first sequence. Consider the following commutative diagram.



The columns and the first and second rows are exact. One can check that the third row is exact by the standard diagram chasing.

Lemma 5.1.4 follows from the exactness of the third row and the surjectivity of (4.17). It is clear from Lemma 5.1.3 that the modification of the spaces (5.1) does not break the exactness. □

5.2. **Recursion of coinvariants.** In this section we establish a version of Proposition 4.4.4 for the spaces of coinvariants which leads to a recursion for the space of coinvariants.

For $M, N \geq 0$, consider the canonical surjection

$$\pi : W_k[l_1,l_2,l_3] \to W_k^{(M,N)}[l_1,l_2,l_3]. \tag{5.5}$$

We induce a filtration of $W_k^{(M,N)}[l_1,l_2,l_3]$ from (4.21):

$$W_{a,c}^{(M,N)} \quad = \quad \pi(W_{a,c}). \tag{5.6}$$

We call (5.6) the canonical filtration of $W_k^{(M,N)}[l_1,l_2,l_3]$ and denote it by $C_{a,c}(W_k^{(M,N)}[l_1,l_2,l_3])$.

Proposition 5.2.1. *For the same (a,c) and (l_1',l_2') as in Proposition 4.4.4, we have a surjection*

$$T(W_k^{(M,N-1)}[l_1',l_2']) \to \mathrm{Gr}_{a,c}^C(W_k^{(M,N)}[l_1,l_2,l_3]). \tag{5.7}$$

Proof. First, we consider the spaces of coinvariants with respect to $\mathfrak{a}^{(M,N)}$ without the cutoff (5.1). We use the surjection

$$\bar{\pi} : W_k[l_1,l_2,l_3] \to W_k[l_1,l_2,l_3]/\mathfrak{a}^{(M,N)}W_k[l_1,l_2,l_3].$$

Noting that $T(\mathfrak{a}^{(M,N-1)}) = \mathfrak{a}^{(M,N)}$, we can induce a surjection from (4.22).

$$\begin{aligned} T(W_k[l_1',l_2']/\mathfrak{a}^{(M,N-1)}W_k[l_1',l_2']) \quad &\to \quad (W_{a,c}/W_{a,c+1})/\mathfrak{a}^{(M,N)}(W_{a,c}/W_{a,c+1}) \\ &\simeq \quad W_{a,c}/(W_{a,c+1}+\mathfrak{a}^{(M,N)}W_{a,c}). \end{aligned}$$

We continue to another surjection

$$\begin{aligned} &\to \quad W_{a,c}/\left(W_{a,c+1} + \mathfrak{a}^{(M,N)} W_k[l_1,l_2,l_3] \cap W_{a,c}\right) \\ &\simeq \quad (W_{a,c} + \mathfrak{a}^{(M,N)} W_k[l_1,l_2,l_3])/(W_{a,c+1} + \mathfrak{a}^{(M,N)} W_k[l_1,l_2,l_3]) \\ &\simeq \quad \bar{\pi}(W_{a,c})/\bar{\pi}(W_{a,c+1}). \end{aligned}$$

Therefore, we obtain the surjection

$$T(W_k[l_1',l_2']/\mathfrak{a}^{(M,N-1)} W_k[l_1',l_2']) \quad \to \quad \bar{\pi}(W_{a,c})/\bar{\pi}(W_{a,c+1}).$$

Finally, we show that the cutoff (5.1) does not break the surjectivity. There are two cases we must check: the case $(M,0) \to (M,1)$ and $(0,N-1) \to (0,N)$. In the first case, the weight spaces that are cut off in (5.1) are mapped to zero. The proof is similar to the proof of Lemma 5.1.3.

In the second case the proof is more elaborate. If $l_1 + c \le k$, the weight spaces that are cut off in the $(0,N-1)$-coinvariants, are mapped to the weight spaces that are also cut off. However, if $l_1 + c > k$ (and therefore $l_1' = k - a$), this is no longer true. In this case we show that the weight spaces that are cut off are mapped to zero.

Let $K, K' \in \mathbb{Z}_{\ge 0}$, and $\mu : \{1,\dots,K+K'\} \to \mathbb{Z}$. For a subset J of $\{1,\dots,K+K'\}$, we denote

$$f(\mu;J) = \prod_{j\in J} f_{\mu(j)}, \quad h(\mu;J) = \prod_{j\in J} h_{\mu(j)}.$$

We set $w(\mu) = f(\mu;\{1,\dots,K\})h(\mu;\{K+1,\dots,K+K'\})$.

We claim that if $K - K' < a$, then

$$w(\mu) h_1^a f_1^c v = 0 \text{ in } W_{a,c}^{(0,N)}/W_{a,c+1}^{(0,N)}. \tag{5.8}$$

First consider the case when $K < K'$. Since $e_0 h_1^a f_1^c v = 0$ in $W_{a,c}/W_{a,c+1}$, it is enough to show the equality

$$w(\mu) u \equiv 0 \bmod \operatorname{Im} e_0 \tag{5.9}$$

in an arbitrary $\widehat{\mathcal{H}}$-module assuming that $K' = K+1$ and $e_0 u = 0$.

Let P_K be the set of all subsets of $\{1,\dots,2K+1\}$ with the cardinality $K+1$, and denote by J^c the complement of $J \in P_K$. We have

$$[f(\mu;J), e_0] h(\mu;J^c) u \equiv 0 \bmod \operatorname{Im} e_0.$$

Define the matrix M indexed by P_K:

$$M_{J,I} = \begin{cases} 1 \text{ if } I^c \subset J; \\ 0 \text{ otherwise.} \end{cases}$$

Note that $I^c \subset J$ is equivalent to $J = I^c \sqcup \{j\}$ for some $j \in J$ because $\sharp(I^c) = \sharp(J) - 1$. Therefore, M is the incidence matrix of the term $f(\mu;I^c)h(\mu;I)u$ in $[f(\mu;J),e_0]h(\mu;J^c)u$. It is easy to show that the matrix M is non-degenerate (e.g., by calculating its inverse). Therefore, (5.9) follows.

If $K \ge K'$, we start with showing the equality

$$(\operatorname{ad} e_0)^{K'} \big(f(\mu;\{1,\dots,K+K'\})\big) h_1^a f_1^c v \equiv 0 \bmod \operatorname{Im} e_0. \tag{5.10}$$

If $K' > 0$, (5.10) follows from $e_0 h_1^a f_1^c v = 0$ in $W_{a,c}/W_{a,c+1}$. If $K' = 0$, we have $K < a$. Since

$$(c+1) X h_1^a f_1^c v = [X, e_0] h_1^{a-1} f_1^{c+1} v,$$

(5.10) follows by induction on K.

Now we show (5.8) by using (5.10). For $\sigma \in \mathfrak{S}_{K+K'}$ we define $\mu^\sigma : \{1,\dots,K+K'\} \to \mathbb{Z}$ by $\mu^\sigma(n) = \mu(\sigma(n))$. The left hand side of (5.10) is a positive linear combination of $w(\mu^\sigma)h_1^a f_1^c v$ ($\sigma \in \mathfrak{S}_{K+K'}$). We will show that

$$w(\mu^\sigma)h_1^a f_1^c v = w(\mu)h_1^a f_1^c v \tag{5.11}$$

for any $\sigma \in \mathfrak{S}_{K+K'}$. Then, our assertion (5.8) follows from (5.10).

The symmetry (5.11) is true for the transpositions $1 \leftrightarrow 2 \leftrightarrow \cdots \leftrightarrow K$ and $K+1 \leftrightarrow K+2 \leftrightarrow \cdots \leftrightarrow K+K'$. Therefore, it is enough to show the symmetry with respect to the transposition $1 \leftrightarrow K+K'$.

We use the notations

$$\begin{aligned} (n)_m &= (n+1)\cdots(n+m), \\ F^{(n)} &= (-\mathrm{ad}\, e_0)^n\big(f(\mu;\{1,\dots,K\})\big). \end{aligned}$$

For $p,q \in \mathbb{Z}$ satisfying $0 \le p \le K$, $0 \le q \le K'-1$ and $q \le p \le q+K-K'+1$, we set

$$[p,q;Q] = F^{(p)} \prod_{i\in Q} f_{\mu(i)} \prod_{i\in Q^c} h_{\mu(i)} h_1^{a-p+q} f_1^{c+p-q} v.$$

Here Q is a subset $Q \subset \{K+1,\dots,K+K'\}$ such that $\sharp(Q) = q$, and Q^c is its complement in $\{K+1,\dots,K+K'\}$. We have $[0,0;\emptyset] = w(\mu)h_1^a f_1^c v$. Note also that $p-q \le K-K'+1 \le a$.

We will prove that $[p,q;Q]$ is in fact independent of Q and

$$[p,q;Q] = (p-q)_q (c)_{p-q} [0,0;\emptyset]. \tag{5.12}$$

In particular, we have

$$[K,K'-1;\{K+1,\dots,K+K'-1\}] = (K-K'+1)_{K'-1}(c)_{K-K'+1}[0,0;\emptyset],$$

and therefore $w(\mu)h_1^a f_1^c v$ is symmetric with respect to the transposition $1 \leftrightarrow K+K'$.

The proof is by induction on the pair of integers (q,p) in the lexicographic ordering. For the induction steps we use the identities

$$Xh_1^a f_1^c v = \frac{1}{(c)_r}(-\mathrm{ad}\, e_0)^r(X) h_1^{a-r} f_1^{c+r} v \qquad (0 \le r \le a), \tag{5.13}$$

$$\begin{aligned} F^{(p)} f_{j_1}\dots f_{j_q} h_{j_{q+1}}\dots h_{j_{K'}} h_1^{a-r} f_1^{c+r} v &= F^{(p+1)} f_{j_1}\dots f_{j_{q+1}} h_{j_{q+2}}\dots h_{j_{K'}} h_1^{a-r} f_1^{c+r} v \\ &\quad - \sum_{1\le s\le q} F^{(p)}\Big(\prod_{\substack{1\le t\le q \\ t\ne s}} f_{j_t}\Big) f_{j_{q+1}} h_{j_s} h_{j_{q+2}}\dots h_{j_{K'}} h_1^{a-r} f_1^{c+r} v \\ &\quad + \begin{cases} 0 & \text{if } r=0; \\ -(c+r)F^{(p)} f_{j_1}\dots f_{j_{q+1}} h_{j_{q+2}}\dots h_{j_{K'}} h_1^{a-r+1} f_1^{c+r-1} v & \text{if } 1 \le r \le a. \end{cases} \end{aligned} \tag{5.14}$$

Using (5.13) with $r=p$ we obtain (5.12) with $q=0$. Because of the induction hypothesis, we can rewrite (5.14) with $r = p-q$ as

$$[p,q] = [p+1,q+1;Q] - q[p,q] + \begin{cases} 0 & \text{if } p=q; \\ (c+p-q)[p,q+1] & \text{otherwise.} \end{cases}$$

Here $[p,q]$ denotes the right hand side of (5.12) and Q is an arbitrary subset of $\{K+1,\dots,K+K'\}$ such that $\sharp(Q)=q+1$. From this follows that $[p+1,q+1;Q]$ is independent of Q and it is given by (5.12). □

Corollary 5.2.2. *Suppose that for each (a,c) satisfying $0\le a\le l_3, 0\le c\le l_2-a$ and (l_1',l_2') given by (4.23) we have a set of monomials $\{\mathbf{m}\}$ such that the set of vectors $\{\mathbf{m}v_W[l_1',l_2',l_1'+l_2'-k]\}\subset W[l_1',l_2']$ forms a basis. Then, the mapping*

$$\bigoplus_{\substack{0\le a\le l_3\\ 0\le c\le l_2-a}} W_k^{(M,N-1)}[l_1',l_2'] \to W_k^{(M,N)}[l_1,l_2,l_3] \tag{5.15}$$

which sends $\mathbf{m}v_W[l_1',l_2',l_1'+l_2'-k]$ to $T(\mathbf{m})h_1^a f_1^c v_W[l_1,l_2,l_3]$ is a surjection.

5.3. **Connection between Heisenberg and $\widehat{\mathfrak{sl}}_2$ modules.** In the course of the paper we will prove that the Heisenberg modules $\mathrm{Gr}^F(L_{k,l})$ and $\mathrm{Gr}^E(L_{k,l})$ are isomorphic to $V_k[k-l,l]$ and $U_k[l,k-l]$, respectively. Now we are ready to establish surjectivity.

Lemma 5.3.1. *There exist surjective maps of $\widehat{\mathcal{H}}$-modules,*

$$V_k[k-l,l] \to \mathrm{Gr}^F(L_{k,l}), \tag{5.16}$$

$$U_k[l,k-l] \to \mathrm{Gr}^E(L_{k,l}), \tag{5.17}$$

such that the highest weight vectors $v_V\in V_k[k-l,l]$ and $v_U\in U_k[l,k-l]$ are mapped to the image v_F of $v_k[l]$ in $\mathrm{Gr}^F(L_{k,l})$ and to the image v_E of $u_k[l]$ in $\mathrm{Gr}^E(L_{k,l})$ respectively.

Proof. We consider the first map. We need to show that the $\widehat{\mathcal{H}}_k$-action on $\mathrm{Gr}^F(L_{k,l})$ satisfies $e_1^{k-l+1}v_F=0$, $f_0^{l+1}v_F=0$ and $h_0v_F=0$. The first two relations follow from the corresponding relations for $v_k[l]\in L_{k,l}$. The last one is proved in a similar manner as the proof of $[\bar h_i,\bar e_j]=0$ in Proposition 4.2.1, by using the relation $h_0v_k[l]=lv_k[l]$ in $L_{k,l}$. □

Corollary 5.3.2. *For $M,N\ge 0$, there are surjective maps of coinvariants,*

$$V_k[k-l,l]^{(M,N)} \to \mathrm{Gr}^F(L_{k,l})^{(M,N)}, \tag{5.18}$$

$$U_k[l,k-l]^{(M,N)} \to \mathrm{Gr}^E(L_{k,l})^{(M,N)}. \tag{5.19}$$

Proof. Corollary 5.3.2 follows from Lemma 5.3.1. □

We will show that (5.16), (5.17) (and therefore (5.18), (5.19)) are isomorphism in Corollary 5.4.7.

In order to construct a recursion we need to "decompose" the spaces of the coinvariants into smaller pieces.

Lemma 5.3.3. *There exist surjective linear maps*

$$\bigoplus_{i=0}^{l} W_k[k-l+i,k-i] \to V_k[k-l,l],$$

$$\bigoplus_{i=0}^{l} W_k[k-l+i,k-i] \to U_k[l,k-l].$$

Moreover, for $M, N \geq 0$, these maps can be factorized to the surjective maps of coinvariants,

$$\bigoplus_{i=0}^{l} W_k^{(M,N)}[k-l+i, k-i] \quad \to \quad V_k[k-l,l]^{(M,N)} \qquad (M>0), \tag{5.20}$$

$$\bigoplus_{i=0}^{l} W_k^{(M,N)}[k-l+i, k-i] \quad \to \quad U_k[l,k-l]^{(M,N)} \qquad (N>0). \tag{5.21}$$

Proof. Lemma 5.3.3 follows from Corollary 4.4.2 and Lemma 5.1.4. □

We will see below (Corollary 5.4.8) that all these maps are, in fact, isomorphisms.

We note that if $0 < l \leq k$ and $M = 0$ the dimension of $V[k-l,l]^{(0,N)}$ is greater than the dimension of $L_{k,l}^{(0,N)}$. This can be seen from the following lemma.

Lemma 5.3.4. *For $s > -l$,*

$$\left(L_{k,l}^{(0,N)}\right)_{s,e} = 0.$$

Proof. The $\widehat{\mathfrak{sl}}_2$-module $L_{k,l}$ is generated by the vector $u_k[l] = f_0^l v_k[l]$. The set of vectors of the form

$$f_{i_1} \cdots f_{i_K} h_{j_1} \cdots h_{j_{K'}} u_k[l] \in \left(L_{k,l}^{(0,N)}\right)_{-2K-l,e}$$

is a spanning set of $L_{k,l}^{(M,N)}$. In fact, the operators e_i can be eliminated either by using the highest weight condition $e_i u_k[l] = 0$ $(i \leq -1)$ or by taking the quotient with respect to the subspace $\operatorname{Im} e_i$ $(i \geq 0)$. The statement of the lemma follows. □

The weight space $V_k[k-l,l]_{m,n,e}^{(0,N)}$ is mapped to $\operatorname{Gr}^F(L_{k,l})_{l+2m-2n,e}^{(M,N)}$ by (5.16). If $l+2m-2n > -l$, by Lemma 5.3.4 the space $\operatorname{Gr}^F(L_{k,l})_{l+2m-2n,e}^{(0,N)}$ is zero. It happens, in particular, for $l > 0$ and $m = n = 0$. On the other hand, the space $V_k[k-l,l]_{0,0}^{(0,N)}$ is spanned by the the highest weight vector and is one dimensional.

Note that in the case $N > 0$, even if we have $M = 0$, we can use the $U_k[l,k-l]^{(M,N)}$ instead. If $M = N = 0$, the space of $\widehat{\mathfrak{sl}}_2$-coinvariants is easily calculated. The result is

$$L_{k,l}^{(0,0)} \quad = \quad \begin{cases} \mathbb{C} \cdot v_k[0] \text{ if } l = 0; \\ 0 \text{ otherwise.} \end{cases}$$

5.4. **Dimension counting.** In this section we show that the recursion for the spaces $W^{(M,N)}[l_1, l_2]$ coincides with the recursion for the combinatorial paths $\mathcal{C}_{k,l}^{(M+N)}$. This gives us the upper bound of $\dim L_{k,l}^{(M,N)}$ by $d_{k,l}^{(M+N)}$. Since we have already shown that the same number gives the lower bound, we complete the proof of the equality $\dim L_{k,l}^{(M,N)} = d_{k,l}^{(M+N)}$.

We start with

Lemma 5.4.1.

$$\dim V_k[k-l,l]^{(M,N)} \geq \dim L_{k,l}^{(M,N)} \geq d_{k,l}^{(M+N)},$$
$$\dim U_k[l,k-l]^{(M,N)} \geq \dim L_{k,l}^{(M,N)} \geq d_{k,l}^{(M+N)}.$$

Proof. We show the first line of inequalities, the second one is proved similarly.

Using Lemma 2.2.1 we have

$$\dim \mathrm{Gr}^F(L_{k,l})^{(M,N)} \geq \dim L_{k,l}^{(M,N)}.$$

From Lemma 5.3.1 we have

$$\dim V_k[k-l,l]^{(M,N)} \geq \dim \mathrm{Gr}^F(L_{k,l})^{(M,N)}.$$

This lemma follows from these inequalities along with Theorem 2.1.2 and Proposition 2.2.3. □

Corollary 5.4.2.

$$\sum_{i=0}^{l} \dim(W^{(M,N)}[k-l+i,k-i]) \geq d_{k,l}^{(M+N)}.$$

Proof. This corollary follows from Lemmas 5.4.1 and 5.3.3. □

Now we show that the recursive relation described by Proposition 5.2.1 for the spaces of coinvariants $W^{(M,N)}[l_1,l_2]$ coincides with the recursion of the combinatorial sets $\mathcal{C}_{k,l}^{(N)}$ (see (3.14) and Proposition 3.3.1).

Lemma 5.4.3. *The set of $\mathcal{C}_{k,l'}^{(N)}[i']$ which appear in the summation of the right hand side of (3.20) coincides with the set of $W_k^{(M,N-1)}[l_1',l_2']$ which appear in the summation of the left hand side of (5.15) in the case $l_1 = k-l+i, l_2 = k-i, l_3 = k-l$ by the identification of $\mathcal{C}_{k,l'}^{(N)}[i']$ with $W_k^{(M,N-1)}[k-l'+i', k-i']$.*

Proof. Since $l'' = 2i-l+l'$, $l'+l'' \geq l$ and $l+l' \geq l''$ are equivalent to $i \geq 0$ and $i \leq l$, respectively. Therefore, the admissibility of the triple (l,l'',l') reduces to

$$l+l'+l'' \leq 2k, \qquad l''+l'-l \geq 0.$$

Namely, the sum in (3.20) is taken over l' and i' such that

$$l-i \leq l' \leq k-i, \qquad 0 \leq i' \leq l'.$$

This is equal to the sum over

$$0 \leq i' \leq k-i, \qquad \max(l-i,i') \leq l' \leq k-i. \tag{5.22}$$

Now consider the surjection

$$\bigoplus_{\substack{0\leq a\leq k-l \\ 0\leq c\leq k-i-a}} W_k^{(M,N-1)}[l_1',l_2'] \to W_k^{(M,N)}[k-l+i,k-i] \tag{5.23}$$

The sum in the left hand side is equivalent to the sum over

$$0 \leq c \leq k-i, \qquad 0 \leq a \leq \min(k-l, k-i-c). \tag{5.24}$$

Set

$$i' = c, \tag{5.25}$$

$$l' = \begin{cases} a+l-i & \text{if } 0 \leq c \leq l-i; \\ a+c & \text{if } l-i \leq c \leq k-i. \end{cases} \tag{5.26}$$

Then, we have

$$l'_1 = k - l' + i', \quad l'_2 = k - i'. \tag{5.27}$$

The sum (5.22) is equal to (5.24) by this identification. □

Now we are ready to prove

Theorem 5.4.4. *For $M, N \geq 0$,*

$$\dim L_{k,l}^{(M,N)} = d_{k,l}^{(M+N)}. \tag{5.28}$$

Proof. We will prove the above equality by induction on (M, N) in three kinds steps. The first steps are $(0, N) \to (0, N+1)$, the second $(M, 0) \to (M, 1)$, and the third $(M, N) \to (M, N+1)$.

In the first and the third steps, we assume that

$$\dim W_k^{(M,N-1)}[k-l+i, k-i] = \sharp(\mathcal{C}_{k,l}^{(M+N-1)}[i]),$$

and prove that

$$\dim W_k^{(M,N)}[k-l+i, k-i] = \sharp(\mathcal{C}_{k,l}^{(M+N)}[i]). \tag{5.29}$$

In the second steps, we assume that

$$\dim W_k^{(M,0)}[k-i, k-l+i] = \sharp(\mathcal{C}_{k,l}^{(M)}[i]), \tag{5.30}$$

and prove that

$$\dim W_k^{(M,1)}[k-l+i, k-i] = \sharp(\mathcal{C}_{k,l}^{(M+1)}[i]).$$

In each step, we also prove the equality of the form (5.28). The base of the induction is $(M, N) = (0, 0)$, where (5.29) is valid because of (3.4) and (5.2). Note also that the assumption (5.30) follows from (5.29) with (M, N) replaced with $(0, M)$, which is proved in the first steps.

Now, we show the first and the third induction steps at the same time.

Using Theorem 2.1.2, Proposition 2.2.3, Corollary 5.3.2 and Lemma 5.3.3, we have the chain of inequalities:

$$d_{k,l}^{(M+N)} = \dim L_{k,l}^{(M,N)}(\mathbf{z}, \mathbf{z}') \leq \dim L_{k,l}^{(M,N)} \leq \sum_{i=0}^{l} \dim W_k^{(M,N)}[k-l+i, k-i].$$

Using (5.23) we obtain

$$\begin{aligned} \dim W_k^{(M,N)}[k-l+i, k-i] &\leq \\ \sum_{\substack{l-i \leq l' \leq k-i \\ 0 \leq i' \leq l'}} \dim W_k^{(M,N-1)}[k-l'+i', k-i'] &= \sum_{\substack{l-i \leq l' \leq k-i \\ 0 \leq i' \leq l'}} \sharp(\mathcal{C}_{k,l'}^{(M+N-1)}[i']) \\ &= \sharp(\mathcal{C}_{k,l}^{(M+N)}[i]). \end{aligned} \tag{5.31}$$

Summing up these inequalities for $i = 0, \ldots, l$ and finally using Corollary 3.4.2 we obtain

$$\sum_{i=0}^{l} \dim W_k^{(M,N)}[k-l+i, k-i] \leq \sum_{i=0}^{l} \sharp(\mathcal{C}_{k,l}^{(M+N)}[i]) = \sharp(\mathcal{C}_{k,l}^{(M+N)}) = d_{k,l}^{(M+N)}.$$

In particular, we obtain (5.29) and (5.28).

In the second steps we proceed with $N = 1$. We only append one formula to (5.31):

$$\sum_{\substack{l-i\le l'\le k-i\\ 0\le i'\le l'}} \dim W_k^{(M,0)}[k-l'+i', k-i'] = \sum_{\substack{l-i\le l'\le k-i\\ 0\le i'\le l'}} \dim W_k^{(M,0)}[k-i', k-l'+i'] = \sum_{\substack{l-i\le l'\le k-i\\ 0\le i'\le l'}} \sharp(\mathcal{C}_{k,l'}^{(M)}[i'])$$

The rest of proof is similar. □

Corollary 5.4.5. *The maps (4.22) and (5.7) are isomorphisms if* $l_3 = l_1 + l_2 - k$.

Proof. The corollary follows from the proof of Theorem 5.4.4. □

In Section 6, we will prove that (5.15) is an isomorphism in all other cases, too.

Corollary 5.4.6. *Suppose* $N > 0$. *We have the equality*

$$\dim L_{k,l}^{(M,N)} = \dim \mathrm{Gr}^E(L_{k,l})^{(M,N)}. \tag{5.32}$$

The mapping

$$\bigoplus_{i=0}^{l} W_k^{(M,N)}[k-l+i, k-i] \to \mathrm{Gr}^E(L_{k,l})^{(M,N)} \tag{5.33}$$

obtained by the composition of (5.21) and (5.19), is an isomorphism.

Proof. This is proved in the proof of Theorem 5.4.4. □

Corollary 5.4.7. *The maps (5.16), (5.17) are isomorphisms of* $\widehat{\mathcal{H}}$ *modules.*

Proof. The corollary follows from the proof of Theorem 5.4.4. □

Corollary 5.4.8. *For* l_1, l_2 *such that* $0 \le l_1, l_2 \le k \le l_1 + l_2$, *we have the short exact sequences*

$$0 \to W_k^{(M,N)}[l_1+l_2, k-l_2] \to V_k[l_1,l_2]^{(M,N)} \to V_k[l_1, l_2-1]^{(M,N)} \to 0 \qquad (M>0, N\ge 0),$$
$$0 \to W_k^{(M,N)}[k-l_1, l_1+l_2] \to U_k[l_1,l_2]^{(M,N)} \to U_k[l_1-1, l_2]^{(M,N)} \to 0 \qquad (N\ge 0, M>0).$$

Proof. By Lemma 5.1.4, all we have to show is the injectivity of the mappings

$$W_k^{(M,N)}[l_1+l_2, k-l_2] \to V_k[l_1,l_2]^{(M,N)},$$
$$W_k^{(M,N)}[k-l_1, l_1+l_2] \to U_k[l_1,l_2]^{(M,N)}.$$

The injectivity follows from the proof of Theorem 5.4.4. □

Corollary 5.4.9. *For* l_1, l_2 *such that* $0 \le l_1, l_2 \le k \le l_1 + l_2$, *we have the short exact sequences*

$$0 \to W_k[l_1+l_2, k-l_2] \to V_k[l_1,l_2] \to V_k[l_1, l_2-1] \to 0,$$
$$0 \to W_k[k-l_1, l_1+l_2] \to U_k[l_1,l_2] \to U_k[l_1-1, l_2] \to 0.$$

Proof. This follows from Corollary 5.4.8 by letting $M, N \to \infty$. □

Corollary 5.4.10. *The set of monomial vectors*

$$\{f_{N-1}^{a_{N-1}} h_{N-1}^{b_{N-1}} \dots f_1^{a_1} h_1^{b_1} f_0^{a_0} v_k[l] \in L_{k,l}^{(0,N)}(\mathbf{z}) \ ; \ (\mathbf{a};\mathbf{b}) \in \mathcal{C}_{k,l}^{(N)}\} \tag{5.34}$$

forms a basis of $L_{k,l}^{(0,N)}(\mathbf{z})$ *for all* $\mathbf{z} \in \mathbb{C}^N$.

Proof. The recursion (5.15) gives us a recursive way of constructing monomial basis of the space of coinvariants $W_k^{(M,N)}[l_1,l_2]$. As shown in Lemma 5.4.3, for each (i,l) satisfying $0 \le i \le l$, the mapping (5.25), (5.26) gives a bijection between the set of (a,c) satisfying $0 \le a \le k-l, 0 \le c \le k-i-a$ and the set of (i',l') satisfying $0 \le i' \le l'$ and such that $(l,l''=2i-l+l',l')$ is admissible. Recall also that under this identification, (l_1',l_2') given by (5.27) satisfies (4.23) with $l_1 = k-l+i$.

Suppose that we have a set of monomials $\{\mathbf{m}\}$ for each (i',l') such that for each l' the union $\sqcup_{0\le i'\le l'}\{\mathbf{m}f_0^{i'}v_k[l']\}$ forms a basis of $\mathrm{Gr}^E(L_{k,l'})^{(M,N-1)}$. The inverse map of the bijection (5.33) (with N,l,i replaced by $N-1,l',i'$) maps the basis $\{\mathbf{m}f_0^{i'}v_k[l']\}$ to the basis $\{\mathbf{m}v_W[l_1',l_2',l_1'+l_2'-k]\} \subset W_k^{(M,N-1)}[l_1',l_2']$. By Corollary 5.2.2 we have the basis $\{T(\mathbf{m})h_1^a f_1^c v_w[k-l+i,k-i,k-l]\}$ of $W_k^{(M,N)}[k-l+i,k-i]$. By (5.33) we then have the basis $\{T(\mathbf{m}f_0^{i'})h_1^a f_1^c v_k[l]\}$ of $\mathrm{Gr}^E(L_{k,l})^{(M,N)}$.

In this way we obtain the basis of $\mathrm{Gr}^E(L_{k,l})^{(M,N)}$ from the basis of $\mathrm{Gr}^E(L_{k,l'})^{(M,N-1)}$. Noting that $i=(l+l''-l')/2, a = i+l'-l-(i'-l+i)^+$ and comparing these formulas with (3.16), (3.17) we see that this recursion gives exactly the basis (5.34). □

Example 5.4.11. As an example, we list the $k=1$ monomials for $M+N \le 3$:

(M,N)	$W_1^{(M,N)}[1,1]$	$W_1^{(M,N)}[1,0]$	$W_1^{(M,N)}[0,1]$
$(0,0)$	1	$\emptyset$	$\emptyset$
$(0,1)$	1	1	$\emptyset$
$(1,0)$	1	$\emptyset$	1
$(0,2)$	$1, f_1$	1	f_1
$(1,1)$	$1, h_1$	1	1
$(2,0)$	$1, e_1$	e_1	1
$(0,3)$	$1, f_1, f_2, f_2h_1$	$1, f_2$	f_1, f_2
$(1,2)$	$1, f_1, h_1, h_2$	$1, h_2$	$1, f_1$
$(2,1)$	$1, e_1, h_1, h_2$	$1, e_1$	$1, h_2$
$(3,0)$	$1, e_1, e_2, e_2h_1$	e_1, e_2	$1, e_2$

Remark 5.4.12. For small values of (M,N) we have the following results. (Here, we assume that $l_3 \le \min(l_1,l_2)$.)

$$
\begin{aligned}
W^{(0,0)}[l_1,l_2,l_3] &= \begin{cases} \mathbb{C}\cdot 1 & \text{if } l_1=l_2=k; \\ 0 & \text{otherwise,} \end{cases} \\
W^{(0,1)}[l_1,l_2,l_3] &= \begin{cases} \mathbb{C}\cdot 1 & \text{if } l_1=k; \\ 0 & \text{otherwise,} \end{cases} \\
W^{(1,0)}[l_1,l_2,l_3] &= \begin{cases} \mathbb{C}\cdot 1 & \text{if } l_2=k; \\ 0 & \text{otherwise,} \end{cases} \\
W^{(1,1)}[l_1,l_2,l_3] &= \oplus_{0\le a\le l_3}\mathbb{C}h_1^a.
\end{aligned}
$$

Note also that if we are interested in the case $M,N>0$, we can avoid the argument on cut-off and apply the recursion relation (5.15) with the base $(M,N)=(1,1)$.

6. Miscellaneous results

In this section we give several results which follow from the recursion of coinvariants (5.15). Some of them will play an important role in our next paper where we will compute the characters of the coinvariants.

6.1. **Generalized recursion.** In this section we prove the decomposition of the space of coinvariants $W_k^{(M,N)}[l_1,l_2,l_3]$ for generic l_3. For this purpose we need to introduce another filtration of $W_k[l_1,l_2,l_3]$. We will show that the adjoint graded spaces of the two filtrations are canonically isomorphic.

The filtration we used in Section 5, i.e., the canonical filtration of the first kind, is such that we take the first filtration with respect to h_1^a and then the second one with respect to f_1^c. We will define the filtration in the opposite order. Namely, we consider the subspaces

$$C'_{a,c}(W_k[l_1,l_2,l_3]) = \langle h_1^a f_1^c, f_1^{a+c+1}\rangle \subset W_k[l_1,l_2,l_3]$$

for (a,c) satisfying (4.20). This region is equivalently written as

$$0 \le c \le l_2, \quad 0 \le a \le \min(l_3, l_2 - c).$$

We abbreviate $C'_{a,c}(W_k[l_1,l_2,l_3])$ to $W'_{a,c}$. We define l_1', l_2' as before, i.e., by (4.23).

Note the inclusion $\langle h_1^a f_1^{b-a}\rangle \supset \langle h_1^{a+1} f_1^{b-a-1}\rangle$ is valid. We have the filtration:

$$\begin{aligned} W_k[l_1,l_2,l_3] = W'_{0,0} &\supset \dots \\ &\supset W'_{0,c} \supset W'_{1,c-1} \supset W'_{\min(l_3,c),c-\min(l_3,c)} \supset W'_{0,c+1} \\ &\qquad \cdots \supset W'_{l_3,l_2-l_3} \supset W'_{0,l_2+1} = 0. \end{aligned}$$

The arguments in Sections 4 and 5 can be repeated for the canonical filtration of the second kind. We describe the results without giving proofs. The proofs can be repeated without modifications by the following reason.

Set $u_0 = h_1^a f_1^c v_W \in W_k[l_1,l_2,l_3]$, $u_1 = h_1^a f_1^c v_W \in \langle h_1^a f_1^c, h_1^{a+1}\rangle / \langle h_1^a f_1^{c+1}, h_1^{a+1}\rangle$ and $u_2 = h_1^a f_1^c v_W \in \langle h_1^a f_1^c, f_1^{a+c+1}\rangle / \langle h_1^{a+1} f_1^{c-1}, f_1^{a+c+1}\rangle$. In the proofs, we use additional properties of u_1 or u_2, which are not satisfied by u_0. Namely, we use

$$e_0 u_i = 0, \quad f_1 u_i = 0 \quad (i = 1,2).$$

These are equivalent to $h_1^a f_1^{c+1} v_W = 0$ and $h_1^{a+1} f_1^{c-1} v_W = 0$, respectively. They are valid in both of the quotient spaces.

Proposition 6.1.1. *For each (a,c) satisfying (4.20) there exists an $\widehat{\mathcal{H}}$-linear surjection*

$$\begin{aligned} T(W_k[l_1',l_2']) &\to \mathrm{Gr}^{C'}_{a,c}(W_k[l_1,l_2,l_3]), \\ T(m v_W[l_1',l_2',l_1'+l_2'-k]) &\mapsto T(m) v_{a,c}, \\ v_{a,c} &= h_1^a f_1^c v_W[l_1,l_2,l_3]. \end{aligned} \tag{6.1}$$

Here l_1', l_2' are given by (4.23).

Proposition 6.1.2. *For the same (a,c) and (l_1',l_2') as in Proposition 6.1.1, we have a surjection*

$$T(W_k^{(M,N-1)}[l_1',l_2']) \to \mathrm{Gr}^{C'}_{a,c}(W_k^{(M,N)}[l_1,l_2,l_3]). \tag{6.2}$$

Proposition 6.1.3. *If $l_3 = l_1 + l_2 - k$, the maps (6.1) and (6.2) are isomorphisms.*

Now, we remove the above restriction on l_3.

Proposition 6.1.4. *The maps* (6.1) *and* (6.2) *are isomorphisms.*

Proof. Consider the exact sequence obtained by the same argument as in the proof of Lemma 5.1.4.

$$0 \to \pi(\langle h_1^{l_3+1} \rangle) \to W_k^{(M,N)}[l_1,k] \to W_k^{(M,N)}[l_1,k,l_3] \to 0.$$

Here $\pi : W_k[l_1,k] \to W_k^{(M,N)}[l_1,k]$ is the canonical surjection and $\langle h_1^{l_3+1} \rangle$ is the submodule of $W_k[l_1,k]$ generated by $h_1^{l_3+1} v_W[l_1,k,l_1]$.

The subspace $\langle h_1^{l_3+1} \rangle$ appears in the canonical filtration of the first kind as $W_{l_3+1,0}$. Therefore, by Corollary 5.4.5. we know the exact decomposition of the adjoint graded spaces for $\pi(\langle h_1^{l_3+1} \rangle)$ and $W_k^{(M,N)}[l_1,k]$. In particular, we have

$$\dim \pi(\langle h_1^{l_3+1} \rangle) = \sum_{\substack{l_3+1 \le a \le l_1 \\ 0 \le c \le k-a}} \dim W_k^{(M,N-1)}[l_1', l_2']$$

and

$$\dim W_k^{(M,N)}[l_1,k] = \sum_{\substack{0 \le a \le l_1 \\ 0 \le c \le k-a}} \dim W_k^{(M,N-1)}[l_1', l_2'].$$

Therefore, we have

$$\dim W_k^{(M,N)}[l_1,k,l_3] = \sum_{\substack{0 \le a \le l_3 \\ 0 \le c \le k-a}} \dim W_k^{(M,N-1)}[l_1', l_2'].$$

From this the statement of the proposition for $l_2 = k$ follows.

Next, consider the exact sequence

$$0 \to \pi(\langle f_1^{l_2+1} \rangle) \to W_k^{(M,N)}[l_1,k,l_3] \to W_k^{(M,N)}[l_1,l_2,l_3] \to 0.$$

The subspace $\langle f_1^{l_2+1} \rangle$ appears in the canonical filtration of the second kind as W'_{0,l_2+1}. Applying the same argument as before, we obtain the statement of the proposition for general l_2. □

Let us summarize the results obtained in Propositions 4.4.4, 5.2.1 and 6.1.4.

Theorem 6.1.5. *Let l_1, l_2, l_3 be such that $0 \le l_1, l_2 \le k$, $0 \le l_3 \le \min(l_1, l_2)$. There exists an isomorphism of vector spaces*

$$\rho : \bigoplus_{a,c} W_k[l_1', l_2'] \to W_k[l_1,l_2,l_3]$$

where

$$l_1' = l_1 + c - a - (l_1 + c - k)^+, \quad l_2' = k - c,$$

and the sum in the left hand side is taken over $0 \le a \le l_3$, $\quad 0 \le c \le l_2 - a$. This map is explicitly described in Proposition 4.4.4. Moreover, the induced map

$$\rho_k^{(M,N)} : \bigoplus_{a,c} W_k^{(M,N-1)}[l_1', l_2'] \to W_k^{(M,N)}[l_1,l_2,l_3]$$

is also an isomorphism of vector spaces.

It is convenient to write the above theorem informally:

$$W_k^{(M,N)}[l_1,l_2,l_3] \;=\; \bigoplus_{\substack{0\le a\le l_3\\ 0\le c\le l_2-a}} T(W_k^{(M,N-1)}[l_1',l_2'])h_1^a f_1^c.$$

Note that because the roles of e_i and f_i in W modules are similar, $\iota(W_k[l_1,l_2]) = W_k[l_2,l_1]$, we also have a recursion relations on M. Namely,

$$W_k^{(M,N)}[l_2,l_1,l_3] \;=\; \bigoplus_{\substack{0\le a\le l_3\\ 0\le c\le l_2-a}} T'(W_k^{(M-1,N)}[l_2',l_1'])h_1^a e_1^c,$$

where T' is such the automorphism of $\widehat{\mathcal{H}}$ that $\iota T = T'\iota$, namely

$$T'(e_i) = e_{i+1}, \qquad T'(f_i) = f_i, \qquad T(h_i) = h_{i+1}. \tag{6.3}$$

Remark 6.1.6. In fact we have constructed many bases of the (M_0,N_0) coinvariants. Namely, we start from $(0,0)$ case and choose a path on the (M,N) plane from $(0,0)$ to (M_0,N_0), increasing either M or N by one. Then we apply the recursions in the order according to this path and get a basis. Clearly, the bases corresponding to different paths are different.

6.2. **Short exact sequences.** In this section we establish several short exact sequences.

We have the following exact sequences:

$$0 \to \langle h_0\rangle \to \overline{V}_k[l_1,l_2] \to V_k[l_1,l_2] \to 0 \quad (l_1+l_2\le k), \tag{6.4}$$

$$0 \to \langle h_0\rangle \to \overline{U}_k[l_1,l_2] \to U_k[l_1,l_2] \to 0 \quad (l_1+l_2\le k), \tag{6.5}$$

$$0 \to \langle h_1^{l_1+l_2-k+1}\rangle \to \overline{W}_k[l_1,l_2] \to W_k[l_1,l_2] \to 0, \quad (l_1+l_2\ge k). \tag{6.6}$$

Proposition 6.2.1. *The following are well-defined and surjective $\widehat{\mathcal{H}}$-linear mappings.*

$$\overline{V}_k[l_1-1,l_2-1] \;\to\; \langle h_0\rangle \subset \overline{V}_k[l_1,l_2] \quad (l_1+l_2\le k), \tag{6.7}$$

$$\overline{U}_k[l_1-1,l_2-1] \;\to\; \langle h_0\rangle \subset \overline{U}_k[l_1,l_2] \quad (l_1+l_2\le k), \tag{6.8}$$

$$\overline{W}_k[k-l_2-1,k-l_1-1] \;\to\; \langle h_1^{l_1+l_2-k+1}\rangle \subset \overline{W}_k[l_1,l_2] \quad (l_1+l_2\ge k). \tag{6.9}$$

Each of these mappings is defined to be such that the highest weight vector of the left hand side is mapped to the indicated generator of the submodule in the right hand side.

Proof. We have to show the validity of the relations (4.6) and (4.7), or (4.10) and (4.11). We will give the proofs for the non-trivial ones. The relations $e_1^{l_1}h_0v = f_0^{l_2}h_0v = 0$ for (6.7) follow from Lemma 7.1.1 with $c=0$. The relations $e_1^{k-l_2}h_1^{l_1+l_2-k+1}v = f_1^{k-l_1}h_1^{l_1+l_2-k+1}v = 0$ for (6.9) follow from Lemma 7.1.3 with $c=0$. □

Proposition 6.2.2. *For l_1,l_2 such that $0\le l_1\le l_2\le k$ and $l_1+l_2\ge k$, we have the exact sequence*

$$0 \to W_k^{(M,N)}[k-l_2-1,k-l_1-1,k-l_2-1] \to W_k^{(M,N)}[l_1,l_2,l_1] \to W_k^{(M,N)}[l_1,l_2] \to 0.$$

Proof. The proof goes similarly as in Proposition 6.1.4 except for the difference in the formulas of l_1',l_2' between the first term and the second and third terms.

The difference of the ranges of (a,c) for $W_k^{(M,N)}[l_1,l_2,l_1]$ and $W_k^{(M,N)}[l_1,l_2,l_1+l_2-k]$ is given by

$$l_1+l_2-k+1\le a\le l_1, 0\le c\le l_2-a.$$

In this range, we have

$$l_1 + c - k < 0.$$

Therefore, we have

$$l_1' = l_1 + c - a, l_2' = k - c.$$

On the other hand, the range of $(\bar{a}, \bar{c})$ for $W_k^{(M,N)}[k - l_2 - 1, k - l_1 - 1, k - l_2 - 1]$ is

$$0 \leq \bar{a} \leq k - l_2 - 1, 0 \leq \bar{c} \leq k - l_1 - 1 - \bar{a}.$$

Noting that $l_1 + l_2 \geq k$ we have

$$k - l_2 - 1 + \bar{c} - k < 0.$$

Therefore, we have

$$l_1' = k - l_2 - 1 + \bar{c} - \bar{a}, l_2' = k - \bar{c}.$$

By the correspondence

$$a = \bar{a} + l_1 + l_2 - k + 1, c = \bar{c},$$

these two ranges of (l_1', l_2') coincide. □

Lemma 6.2.3. *For l_1, l_2 such that $0 \leq l_1, l_2 \leq k$ and $l_1 + l_2 \leq k$ we have an exact sequence*

$$0 \to W_k^{(M,N)}[l_1 - 1, l_2 - 1, l_3 - 1] \to W_k^{(M,N)}[l_1, l_2, l_3] \to W_k^{(M,N)}[l_1, l_2, 0] \to 0, \tag{6.10}$$

where $l_3 = \min(l_1, l_2)$.

Proof. The proof goes similarly as in the previous proposition.

Noting that $l_1 + l_2 \leq k$, we have

$$l_1' = l_1 + c - a, l_2' = k - c, \tag{6.11}$$

for $W_k^{(M,N)}[l_1, l_2, l_3]$ and $W_k^{(M,N)}[l_1, l_2, 0]$. The range of (a, c) for $W_k^{(M,N)}[l_1, l_2, l_3]$ is

$$0 \leq a \leq l_3, 0 \leq c \leq l_2 - a.$$

The subset corresponding to $a = 0$ is exactly the range for $W_k^{(M,N)}[l_1, l_2, 0]$.

Similarly, we have

$$l_1' = l_1 - 1 + \bar{c} - \bar{a}, l_2' = k - \bar{c}, \tag{6.12}$$

for $W_k^{(M,N)}[l_1 - 1, l_2 - 1, l_3 - 1]$ where

$$0 \leq \bar{a} \leq l_3 - 1, 0 \leq c \leq l_2 - 1 - \bar{a}.$$

By the correspondence

$$\bar{a} = a - 1, \bar{c} = c,$$

the range of (6.12) overlaps the range of (6.11) exactly when

$$1 \leq a \leq l_3, 0 \leq c \leq l_2 - a.$$

□

Corollary 6.2.4. *The triples*

$$0 \to \overline{V}_k[l_1-1,l_2-1] \to \overline{V}_k[l_1,l_2] \to V_k[l_1,l_2] \to 0 \quad (l_1+l_2 \le k), \tag{6.13}$$

$$0 \to \overline{U}_k[l_1-1,l_2-1] \to \overline{U}_k[l_1,l_2] \to U_k[l_1,l_2] \to 0 \quad (l_1+l_2 \le k), \tag{6.14}$$

$$0 \to \overline{W}_k[k-l_2-1,k-l_1-1] \to \overline{W}_k[l_1,l_2] \to W_k[l_1,l_2] \to 0, \quad (l_1+l_2 \ge k) \tag{6.15}$$

are exact.

Proof. The exactness of triple (6.15) follows from Propositions 6.2.1 and 6.2.2. The other two cases are obtained from (6.10) by applying the automorphisms T^{-1} and ιT^{-1}. □

6.3. **Injectivity of coproduct.** Recall that the coproduct Δ in the universal enveloping algebra of Lie algebra $\widehat{\mathfrak{sl}}_2$ is defined by the condition $\Delta(g) = g \otimes 1 + 1 \otimes g$ for $g \in \widehat{\mathfrak{sl}}_2$.

There exists an $\widehat{\mathfrak{sl}}_2$-linear map

$$\begin{aligned} \Delta : L_{k^{(1)}+k^{(2)},l^{(1)}+l^{(2)}} &\to L_{k^{(1)},l^{(1)}} \otimes L_{k^{(2)},l^{(2)}}, \\ \Delta(v_{k^{(1)}+k^{(2)}}[l^{(1)}+l^{(2)}]) &= v_{k^{(1)}}[l^{(1)}] \otimes v_{k^{(2)}}[l^{(2)}]. \end{aligned}$$

This map is obviously injective because of the irreducibility of $L_{k^{(1)}+k^{(2)},l^{(1)}+l^{(2)}}$. Similar mappings also exist for the $\widehat{\mathcal{H}}_k$-modules. However, the injectivity is not clear (and non-trivial) because they are not irreducible modules. In this section, we prove the injectivity of such maps for $W_k[l_1,l_2,l_3]$. In fact we will show the injectivity of the coproduct for the spaces $W_k^{(M,N)}[l_1,l_2,l_3]$. As a corollary, we obtain the injectivity of the map

$$L^{(M,N)}_{k^{(1)}+k^{(2)},l^{(1)}+l^{(2)}} \to L^{(M,N)}_{k^{(1)},l^{(1)}} \otimes L^{(M,N)}_{k^{(2)},l^{(2)}}.$$

Remark 6.3.1. Recall the vector $u_k[l] = \frac{f_0^l}{l!} v_k[l]$ given by (4.5). The above map Δ is equivalently characterized by the condition

$$\Delta(u_{k^{(1)}+k^{(2)}}[l^{(1)}+l^{(2)}]) = u_{k^{(1)}}[l^{(1)}] \otimes u_{k^{(2)}}[l^{(2)}].$$

Define the coproduct Δ in the universal enveloping algebra of the affine Heisenberg algebra $\widehat{\mathcal{H}}$ by the condition $\Delta(g) = g \otimes 1 + 1 \otimes g$ for $g \in \widehat{\mathcal{H}}$.

Let $k^{(j)}, l_i^{(j)}$ $(i = 1,2,3,\ j = 1,2)$ be such that $k^{(1)} + k^{(2)} = k$ and $l_i^{(1)} + l_i^{(2)} = l_i$ $(1 \le i \le 3)$. Let

$$\Delta \ : \ W_k[l_1,l_2,l_3] \to W_{k^{(1)}}[l_1^{(1)},l_2^{(1)},l_3^{(1)}] \otimes W_{k^{(2)}}[l_1^{(2)},l_2^{(2)},l_3^{(2)}]$$

be the $\widehat{\mathcal{H}}$-linear map such that $\Delta(v) = v^{(1)} \otimes v^{(2)}$, where v, $v^{(1)}$ and $v^{(2)}$ are highest weight vectors of $W_k[l_1,l_2,l_3]$, $W_{k^{(1)}}[l_1^{(1)},l_2^{(1)},l_3^{(1)}]$ and $W_{k^{(2)}}[l_1^{(2)},l_2^{(2)},l_3^{(2)}]$ respectively.

The coproduct Δ descends to the map $\Delta^{(M,N)}$ on the coinvariants

$$\Delta^{(M,N)} : W_k^{(M,N)}[l_1,l_2,l_3] \to W^{(M,N)}_{k^{(1)}}[l_1^{(1)},l_2^{(1)},l_3^{(1)}] \otimes W^{(M,N)}_{k^{(2)}}[l_1^{(2)},l_2^{(2)},l_3^{(2)}].$$

Note that the cutoff (5.1) does not break the well-definedness of the map.

For simplicity, we suppress the dependence of these mappings on $k^{(i)}$, $l_i^{(j)}$ in our notation,.

We start from

Lemma 6.3.2. *Suppose that we have filtrations of vectors spaces:*

$$W^{(i)} = W_0^{(i)} \supset W_1^{(i)} \supset \cdots \supset W_m^{(i)} \supset W_{m+1}^{(i)} = 0 \quad (i = 1,2).$$

We have the filtration of $W = W^{(1)} \otimes W^{(2)}$ *consisting of the subspaces* $W_n = \sum_{a+b=n} W^{(1)}_a \otimes W^{(2)}_b$. *Then, the mapping*

$$W_n \to \oplus_{a+b=n} \mathrm{Gr}_a(W^{(1)}) \otimes \mathrm{Gr}_b(W^{(2)}) \tag{6.16}$$

induced from the canonical surjections

$$\pi_{a,b} : W^{(1)}_a \otimes W^{(2)}_b \to \mathrm{Gr}_a(W^{(1)}) \otimes \mathrm{Gr}_b(W^{(2)})$$

is well-defined and induces the isomorphism

$$\mathrm{Gr}_n(W) \to \oplus_{a+b=n} \mathrm{Gr}_a(W^{(1)}) \otimes \mathrm{Gr}_b(W^{(2)}).$$

Proof. Suppose that an element $w \in W_n$ can be expressed in two different ways: $w = \sum_i x_i = \sum_i y_i$ where $x_i, y_i \in W^{(1)}_i \otimes W^{(2)}_{n-i}$. We must show that $\pi_{i,n-i}(x_i - y_i) = 0$ for each i.

Consider a pair of vectors spaces and their subspaces: $V \supset V_1, U \supset U_1$. Then, we have $(V \otimes U_1) \cap (V_1 \otimes U) = V_1 \otimes U_1$.

Using this one can prove that

$$x_i - y_i \in W^{(1)}_i \otimes W^{(2)}_{n-i+1} + W^{(1)}_{i+1} \otimes W^{(2)}_{n-i}.$$

The welldefinedness of (6.16) follows from this, and the rest of the lemma is easy. □

Proposition 6.3.3. *The mapping* $\Delta^{(M,N)}$ *is injective.*

Proof. It is enough to prove the case $k^{(1)} = k - 1$ and $k^{(2)} = 1$. There are four cases:

(i) $\Delta^{(M,N)} : W^{(M,N)}_k[l_1, l_2, l_3] \to W^{(M,N)}_{k-1}[l_1 - 1, l_2, l_3] \otimes W^{(M,N)}_1[1, 0, 0]$,

(ii) $\Delta^{(M,N)} : W^{(M,N)}_k[l_1, l_2, l_3] \to W^{(M,N)}_{k-1}[l_1, l_2 - 1, l_3] \otimes W^{(M,N)}_1[0, 1, 0]$,

(iii) $\Delta^{(M,N)} : W^{(M,N)}_k[l_1, l_2, l_3] \to W^{(M,N)}_{k-1}[l_1 - 1, l_2 - 1, l_3] \otimes W^{(M,N)}_1[1, 1, 0]$,

(iv) $\Delta^{(M,N)} : W^{(M,N)}_k[l_1, l_2, l_3] \to W^{(M,N)}_{k-1}[l_1 - 1, l_2 - 1, l_3 - 1] \otimes W^{(M,N)}_1[1, 1, 1]$.

We use induction on (M, N). If $(M, N) = (0, 0)$, the assertion is clear from (5.2).

In the induction, we use the recurrence relations for the space $W^{(M,N)}_1[l_1, l_2, l_3]$ given by Theorem 6.1.5. We have the following bijections:

$$\begin{aligned}
W^{(M,N-1)}_1[1,1] &\to \mathrm{Gr}_{0,0}(W^{(M,N)}_1[1,0,0]),\\
W^{(M,N-1)}_1[0,1] &\to \mathrm{Gr}_{0,0}(W^{(M,N)}_1[0,1,0]),\\
W^{(M,N-1)}_1[1,0] &\to \mathrm{Gr}_{0,1}(W^{(M,N)}_1[0,1,0]),\\
W^{(M,N-1)}_1[1,1] &\to \mathrm{Gr}_{0,0}(W^{(M,N)}_1[1,1,0]),\\
W^{(M,N-1)}_1[1,0] &\to \mathrm{Gr}_{0,1}(W^{(M,N)}_1[1,1,0]),\\
W^{(M,N-1)}_1[1,1] &\to \mathrm{Gr}_{0,0}(W^{(M,N)}_1[1,1,1]),\\
W^{(M,N-1)}_1[1,0] &\to \mathrm{Gr}_{0,1}(W^{(M,N)}_1[1,1,1]),\\
W^{(M,N-1)}_1[0,1] &\to \mathrm{Gr}_{1,0}(W^{(M,N)}_1[1,1,1]).
\end{aligned}$$

Let us prove the injectivity of (i). We use the following abbreviated notations. $W = W^{(M,N)}_k[l_1, l_2, l_3]$, $W^{(1)} = W^{(M,N)}_{k-1}[l_1 - 1, l_2, l_3]$, $W^{(2)} = W^{(M,N)}_1[1, 0, 0]$. We denote the highest weight vectors in

$W, W^{(1)}, W^{(2)}$, respectively, by $v, v^{(1)}, v^{(2)}$. Note that $h_1 v^{(2)} = f_1 v^{(2)} = 0$, and hence

$$\Delta(h_1^a f_1^c)(v^{(1)} \otimes v^{(2)}) = (h_1^a f_1^c v^{(1)}) \otimes v^{(2)}. \tag{6.17}$$

Consider the following mapping diagram.

$$\begin{array}{ccc}
C_{a,c}(W) & \longrightarrow & C_{a,c}(W^{(1)}) \otimes C_{0,0}(W^{(2)}) \\
\downarrow & & \downarrow \\
\mathrm{Gr}_{a,c}(W) & \longrightarrow & \mathrm{Gr}_{a,c}(W^{(1)}) \otimes \mathrm{Gr}_{0,0}(W^{(2)}) \\
\uparrow & & \uparrow \\
W^{(M,N-1)}[l_1', l_2'] & \longrightarrow & W^{(M,N-1)}[l_1' - 1, l_2' - 1] \otimes W_1^{(M,N-1)}[1,1]
\end{array}$$

The first and the third horizontal arrows are the coproduct maps. The downward vertical arrows are canonical surjections, and the upward vertical arrows are the canonical bijections of Proposition 6.1.4. The second horizontal arrow is induced from either of the other horizontal arrows. Noting that the shift automorphism T and the coproduct Δ are commutative, one can check that the induced map is unique, and thereby we have a commutative diagram.

By the induction hypothesis, the third horizontal arrow is injective. Therefore, the second horizontal arrow is also injective. From this follows that the kernel of the first horizontal arrow is included in a smaller subset in the filtration. Repeating this argument, i.e., using induction on (a, c), one can show the injectivity of the coproduct map $\Delta^{(M,N)}$.

In other three cases, the relation (6.17) is modified. Let us consider the case (ii). Using similar abbreviated notations, we have

$$\Delta(h_1^a f_1^c)(v^{(1)} \otimes v^{(2)}) = (h_1^a f_1^c v^{(1)}) \otimes v^{(2)} + c(h_1^a f_1^{c-1} v^{(1)}) \otimes f_1 v^{(2)}. \tag{6.18}$$

This induces a mapping of the form

$$C_{a,c}(W) \to C_{a,c}(W^{(1)}) \otimes C_{0,0}(W^{(2)}) + C_{a,c-1}(W^{(1)}) \otimes C_{0,1}(W^{(2)}). \tag{6.19}$$

Here we use $W = W_k^{(M,N)}[l_1, l_2, l_3], W^{(1)} = W_{k-1}^{(M,N)}[l_1, l_2 - 1, l_3], W^{(2)} = W_1^{(M,N)}[0,1,0]$.

If $c \neq 0$, we consider the following mapping diagram.

$$\begin{array}{ccc}
C_{a,c}(W) & \longrightarrow & C_{a,c}(W^{(1)}) \otimes C_{0,0}(W^{(2)}) + C_{a,c-1}(W^{(1)}) \otimes C_{0,1}(W^{(2)}) \\
\downarrow & & \downarrow \\
\mathrm{Gr}_{a,c}(W) & \longrightarrow & \mathrm{Gr}_{a,c-1}(W^{(1)}) \otimes \mathrm{Gr}_{0,1}(W^{(2)}) \\
\uparrow & & \uparrow \\
W^{(M,N-1)}[l_1', l_2'] & \longrightarrow & W^{(M,N-1)}[l_1' - 1, l_2'] \otimes W_1^{(M,N-1)}[1,0]
\end{array}$$

The first horizontal arrow is $\frac{1}{c}\Delta$. The downward vertical arrow in the right is assured by Lemma 6.3.2. The rest of proof is similar.

If $c = 0$ the second term in the right hand side of (6.18) is absent. In this case, we use

$$\begin{array}{ccc}
C_{a,c}(W) & \longrightarrow & C_{a,c}(W^{(1)}) \otimes C_{0,0}(W^{(2)}) \\
\downarrow & & \downarrow \\
\mathrm{Gr}_{a,c}(W) & \longrightarrow & \mathrm{Gr}_{a,c}(W^{(1)}) \otimes \mathrm{Gr}_{0,0}(W^{(2)}) \\
\uparrow & & \uparrow \\
W^{(M,N-1)}[l_1', l_2'] & \longrightarrow & W^{(M,N-1)}[l_1', l_2' - 1] \otimes W_1^{(M,N-1)}[0, 1]
\end{array}$$

The rest of the proof is similar.

In the cases (iii) and (iv), the proof will go with $\mathrm{Gr}_{a,c}(W^{(1)}) \otimes \mathrm{Gr}_{0,0}(W^{(2)})$ using the bijection of the form

$$W^{(M,N-1)}[l_1' - 1, l_2' - 1] \to \mathrm{Gr}_{a,c}(W^{(1)}).$$

□

Now we are ready to show the injectivity of coproduct for the $\widehat{\mathfrak{sl}}_2$ spaces of coinvariants.

Theorem 6.3.4. *The mapping*

$$\Delta^{(M,N)} \quad : \quad L^{(M,N)}_{k^{(1)}+k^{(2)},l^{(1)}+l^{(2)}} \to L^{(M,N)}_{k^{(1)},l^{(1)}} \otimes L^{(M,N)}_{k^{(2)},l^{(2)}}$$

is injective.

Proof. We assume that $N > 0$. Consider the following commutative diagram.

$$\begin{array}{ccc}
L^{(M,N)}_{k,l} & \longrightarrow & L^{(M,N)}_{k^{(1)},l^{(1)}} \otimes L^{(M,N)}_{k^{(2)},l^{(2)}} \\
{\scriptstyle \iota_{k,l}}\uparrow & & \uparrow{\scriptstyle \iota_{k^{(1)},l^{(1)}} \otimes \iota_{k^{(2)},l^{(2)}}} \\
E_i(L_{k,l}) & \longrightarrow & \sum_{i_1+i_2=i} E_{i_1}(L_{k^{(1)},l^{(1)}}) \otimes E_{i_2}(L_{k^{(2)},l^{(2)}}) \\
\downarrow{\scriptstyle \pi_{k,l}} & & \downarrow{\scriptstyle \pi_{k^{(1)},l^{(1)}} \otimes \pi_{k^{(2)},l^{(2)}}} \\
\mathrm{Gr}^E_i(L_{k,l})^{(M,N)} & \longrightarrow & \oplus_{i_1+i_2=i} \mathrm{Gr}^E_{i_1}(L_{k^{(1)},l^{(1)}})^{(M,N)} \otimes \mathrm{Gr}^E_{i_2}(L_{k^{(2)},l^{(2)}})^{(M,N)}
\end{array}$$

The horizontal arrows are coproducts, the vertical up-arrows are compositions of canonical injections and surjections, and the vertical down-arrows are canonical surjections. The only non-trivial one is

$$\sum_{i_1+i_2=i} E_{i_1}(L_{k^{(1)},l^{(1)}}) \otimes E_{i_2}(L_{k^{(2)},l^{(2)}}) \to \oplus_{i_1+i_2=i} \mathrm{Gr}^E_{i_1}(L_{k^{(1)},l^{(1)}})^{(M,N)} \otimes \mathrm{Gr}^E_{i_2}(L_{k^{(2)},l^{(2)}})^{(M,N)}$$

which follows from Lemma 6.3.2.

We are to prove that the first horizontal arrow is injective. We know that the third horizontal arrow is injective (Proposition 6.3.3 up to the automorphisms T' given by (6.3)). Note also that because of (5.32) we can apply Remark 2.2.2.

Suppose $w \in L^{(M,N)}_{k,l}$ belongs to the kernel of $\Delta^{(M,N)}$. If $w \neq 0$, we take the smallest values i such that w belongs to the image of $E_i(L_{k,l})$. Take a preimage $w' \in E_i(L_{k,l})$ of w. We have

$i > 0$ because $\Delta^{(M,N)}(u_k[l]) \neq 0$. We show a contradiction by finding $w'' \in E_{i-1}(L_{k,l})$ satisfying $w = \iota_{k,l}(w'')$.

For simplicity set $\iota = \iota_{k^{(1)},l^{(1)}} \otimes \iota_{k^{(2)},l^{(2)}}$ and $\pi = \pi_{k^{(1)},l^{(1)}} \otimes \pi_{k^{(2)},l^{(2)}}$. Since $\Delta^{(M,N)}(w) = 0$, $\Delta(w')$ belongs to $\operatorname{Ker}\iota$. We will show that

$$\operatorname{Ker}\iota \subset \operatorname{Ker}\pi. \tag{6.20}$$

Prepare a set of vectors $B_j^{(a)} = \{v_{j,\alpha}^{(a)} \in E_j(L_{k^{(a)},l^{(a)}}); 1 \leq \alpha \leq \dim \operatorname{Gr}_j^E(L_{k^{(a)},l^{(a)}})^{(M,N)}\}$ for each (a,j) such that $\pi_{k^{(a)},l^{(a)}}(B_j^{(a)})$ forms a basis of $\operatorname{Gr}_j^E(L_{k^{(a)},l^{(a)}})^{(M,N)}$. Because of Remark 2.2.2 $\iota_{k^{(a)},l^{(a)}}(B_j^{(a)})$ forms a basis of $L^{(M,N)}_{k^{(a)},l^{(a)}}$.

Suppose that $v = \sum_{i_1+i_2=i} v_{i_1,i_2}$ where $v_{i_1,i_2} \in E_{i_1}(L_{k^{(1)},l^{(1)}}) \otimes E_{i_2}(L_{k^{(2)},l^{(2)}})$. We write

$$\pi(v_{i_1,i_2}) = \sum_{\alpha,\beta} c_{i_1,i_2}^{\alpha,\beta} \pi_{k^{(1)},l^{(1)}}(v_{i_1,\alpha}^{(1)}) \otimes \pi_{k^{(2)},l^{(2)}}(v_{i_2,\beta}^{(2)}).$$

We have

$$v_{i_1,i_2} - \sum_{\alpha,\beta} c_{i_1,i_2}^{\alpha,\beta} v_{i_1,\alpha}^{(1)} \otimes v_{i_2,\beta}^{(2)} \in \operatorname{Ker}\pi \cap E_{i_1}(L_{k^{(1)},l^{(1)}}) \otimes E_{i_2}(L_{k^{(2)},l^{(2)}}).$$

Note that

$$\begin{aligned}&\operatorname{Gr}_{i_a}^E(L_{k^{(a)},l^{(a)}})^{(M,N)}\\ &\quad\simeq E_{i_a}(L_{k^{(a},l^{(a)}})/\Big(\sum_{j\geq M} e_j E_{i_a-1}(L_{k^{(a)},l^{(a)}}) + \sum_{j\geq N} f_j E_{i_a-1}(L_{k^{(a)},l^{(a)}}) + E_{i_a-1}(L_{k^{(a)},l^{(a)}})\Big).\end{aligned}$$

From this we conclude that $v - \sum_{\substack{i_1+i_2=i\\ \alpha,\beta}} c_{i_1,i_2}^{\alpha,\beta} v_{i_1,\alpha}^{(1)} \otimes v_{i_2,\beta}^{(2)}$ belongs to

$$\begin{aligned}&\sum_{i_1+i_2=i-1} E_{i_1}(L_{k^{(1)},l^{(1)}}) \otimes E_{i_2}(L_{k^{(2)},l^{(2)}})\\ &+ \sum_{i_1+i_2=i} \Bigg(\sum_{j\geq M} \Big(e_j E_{i_1-1}(L_{k^{(1)},l^{(1)}})\Big) \otimes E_{i_2}(L_{k^{(2)},l^{(2)}}) + \sum_{j\geq N} \Big(f_j E_{i_1-1}(L_{k^{(1)},l^{(1)}})\Big) \otimes E_{i_2}(L_{k^{(2)},l^{(2)}})\\ &+ \sum_{j\geq M} E_{i_1}(L_{k^{(1)},l^{(1)}}) \otimes \Big(e_j E_{i_2-1}(L_{k^{(2)},l^{(2)}})\Big) + \sum_{j\geq N} E_{i_1}(L_{k^{(1)},l^{(1)}}) \otimes \Big(f_j E_{i_2-1}(L_{k^{(2)},l^{(2)}})\Big)\Bigg).\end{aligned}$$

Repeating the argument for $E_{i-1}(L_{k,l})$ and so on, we obtain

$$v = \sum_{\substack{i_1+i_2\leq i\\ \alpha,\beta}} c_{i_1,i_2}^{\alpha,\beta} v_{i_1,\alpha}^{(1)} \otimes v_{i_2,\beta}^{(2)} \bmod \operatorname{Ker}\iota \cap \operatorname{Ker}\pi.$$

If $v \in \operatorname{Ker}\iota$, thus we have $c_{i_1,i_2}^{\alpha,\beta} = 0$ for all α, β, i_1, i_2, and thereby $v \in \operatorname{Ker}\pi$.

We return to $w' \in E_i(L_{k,l})$ which satisfies $\Delta(w') \in \operatorname{Ker}\pi$. Using (6.20) and the injectivity of the third horizontal arrow, we conclude that $\pi_{k,l}(w') = 0$.

This implies

$$w' \in \sum_{j\geq M} e_j E_{i-1}(L_{k,l}) + \sum_{j\geq N} f_j E_{i-1}(L_{k,l}) + E_{i-1}(L_{k,l}).$$

Therefore, we can replace w' by $w'' \in E_{i-1}(L_{k,l})$ keeping the property $\iota(w'') = w$. □

6.4. **Character identities.** In this section we collect the relations between characters of coinvariants of different modules. All these equations are easy corollaries of the results proved in the paper.

Recall that for a $\widehat{\mathcal{H}}_k$ module V we have a triple grading such that

$$\deg v = (0,0,0), \quad \deg e_i = (1,0,i), \quad \deg f_i = (0,1,i), \quad \deg h_i = (1,1,i),$$

where v is the highest weight vector in V. Let as before $V_{m,n,e}$ be the subspace of degree (m,n,e) and $V_{m,n} = \oplus_e V_{m,n,e}$.

Define the character of V by

$$\chi(V)(q,z_1,z_2) = \sum_{m,n,e \in \mathbb{Z}_{\geq 0}} z_1^m z_2^n q^e \dim V_{m,n,e}.$$

For $M,N \geq 0$, the space of coinvariants $V^{(M,N)}$ is a quotient of V by a graded ideal. Therefore, we have a well-defined induced character of the coinvariant space $\chi(V^{(M,N)})(q,z_1,z_2)$.

In the limit $M,N \to \infty$ we obviously obtain the character of the whole module,

$$\chi(V^{(\infty,\infty)})(q,z_1,z_2) = \chi(V)(q,z_1,z_2).$$

Below we assume that $M,N \geq 0$, moreover, $N > 0$ for the relations which include $V_k^{(M,N)}[l_1,l_2,l_3]$ and $M > 0$ for the relations which include $U_k^{(M,N)}[l_1,l_2,l_3]$

First, we give the relation between the characters of the Heisenberg and $\widehat{\mathfrak{sl}}_2$ coinvariants.

We have (see Corollaries 5.3.2, 5.4.7)

$$z^l \chi(V_k[k-l,l]^{(M,N)})(q,z^2,z^{-2}) = \chi_{k,l}^{(M,N)} \qquad (M>0, N \geq 0), \tag{6.21}$$

$$z^{-l} \chi(U_k[l,k-l]^{(M,N)})(q,z^2,z^{-2}) = \chi_{k,l}^{(M,N)} \qquad (M \geq 0, N>0). \tag{6.22}$$

In particular,

$$\chi(V_k[k-l,l]^{(M,N)}(1,1,1) = d_{k,l}^{(M+N)}, \qquad \chi(U_k[l,k-l]^{(M,N)}(1,1,1) = d_{k,l}^{(M+N)}.$$

By Lemma 5.3.3 and Corollary 5.4.8, the spaces W refine these identities,

$$\sum_{i=0}^{l} z_2^i \chi(W_k^{(M,N)}[k-l+i,k-i])(q,z_1,z_2) = \chi(V_k[k-l,l]^{(M,N)})(q,z_1,z_2). \tag{6.23}$$

Also, we have

$$z_2^i \chi(W_k^{(0,N)}[k-l+i,k-i])(q,z_1,z_2) = \chi_{k,l}^N[i,*](q,z_1,z_2)$$

(see (3.24) for the definition of the right hand side).

From (4.13), (4.15), we have

$$\begin{aligned} \chi(W_k[l_1,l_2,l_3]^{(M,N)})(q,z_1,z_2) &= \chi(V_k[l_1,l_2,l_3]^{(M,N-1)})(q,z_1,qz_2), \\ \chi(U_k[l_1,l_2,l_3]^{(M,N)})(q,z_1,z_2) &= \chi(V_k[l_2,l_1,l_3]^{(N,M)})(q,z_2,z_1). \end{aligned}$$

The short exact sequences (4.18), (4.19), (6.13), (6.14), (6.15) and (6.10) give us the following relations.

$$\begin{aligned}
\chi(V_k[l_1,l_2]^{(M,N)}) &= \chi(\overline{V}_k[l_1,l_2]^{(M,N)}) - z_1z_2\chi(\overline{V}_k[l_1-1,l_2-1]^{(M,N)}),\\
\chi(U_k[l_1,l_2]^{(M,N)}) &= \chi(\overline{U}_k[l_1,l_2]^{(M,N)}) - z_1z_2\chi(\overline{U}_k[l_1-1,l_2-1]^{(M,N)}),\\
\chi(W_k[l_1,l_2]^{(M,N)}) &= \chi(\overline{W}_k[l_1,l_2]^{(M,N)}) - (qz_1z_2)^{l_1+l_2-k+1}\chi(\overline{W}_k[k-l_2-1,k-l_1-1]^{(M,N)})
\end{aligned}$$

and

$$\begin{aligned}
\chi(V_k[l_1,l_2]^{(M,N)}) &= \chi(V_k[l_1,l_2-1]^{(M,N)}) + z_2^{l_2}\chi(W_k[l_1+l_2,k-l_2]^{(M,N)}),\\
\chi(U_k[l_1,l_2]^{(M,N)}) &= \chi(U_k[l_1-1,l_2-1]^{(M,N)}) + z_1^{l_1}\chi(W_k[k-l_1,l_1+l_2]^{(M,N)}),\\
\chi(W_k[l_1,l_2,l_3]^{(M,N)}) &= \chi(W_k[l_1,l_2,0]^{(M,N)}) + qz_1z_2\chi(W_k[l_1-1,l_2-1,l_3-1]^{(M,N)}).
\end{aligned}$$

The main recursion (see Theorem 6.1.5) gives us the following relation

$$\chi(W_k[l_1,l_2,l_3]^{(M,N)})(q,z_1,z_2) = \sum_{\substack{0\le a\le l_3\\ 0\le c\le l_2-a}} q^{a+c}z_1^a z_2^{a+c}\chi(W_k[l_1',l_2']^{(M,N-1)})(q,z_1,qz_2).$$

In the case $M=0$, $l_3 = l_1+l_2-k$, this relation coincides with (3.25).

7. Appendix

In this appendix we give several technical lemmas about some vectors being zero in the $\widehat{\mathcal{H}}_k$-modules.

7.1. **Zero vector lemmas.**

Let V be a $\widehat{\mathcal{H}}_k$-module and $u \in V$ be a vector.

Lemma 7.1.1. *If* $e_1^{a+1}u = f_{-1}u = 0$, *then*

$$e_1^{a-c}h_0^{c+1}u = 0, \qquad 0 \le c \le a.$$

If $f_0^{b+1}u = e_0u = 0$, *then*

$$f_0^{b-c}h_0^{c+1}u = 0, \qquad 0 \le c \le b.$$

Proof. Since $(\mathrm{ad}f_{-1})^c(e_1^{a+1})u = 0$, by using $[e_1,f_{-1}] = h_0$ and $[h_0,f_{-1}] = 0$, we obtain the first equation of the lemma. The proof of second one is similar. □

Lemma 7.1.2. *If* $e_1^a u = 0$, *then*

$$e_1^{a+b}f_1^b u = 0.$$

Proof. First, we show

$$e_1^{a+b-1}f_1^b e_1 u = 0 \tag{7.1}$$

by induction on b. The case $b=0$ is the assumption of the Lemma. Suppose (7.1) is true for $b-1$. Then

$$\begin{aligned}
e_1^{a+b-1}f_1^b e_1 u &= e_1[e_1^{a+b-2},f_1]f_1^{b-1}e_1u\\
&= (a+b-2)h_2e_1^{a+b-2}f_1^{b-1}e_1u\\
&= 0,
\end{aligned}$$

and (7.1) holds for b. Next, we have

$$\begin{aligned} e_1^{a+b} f_1^b u &= e_1^{a+b-1}[e_1, f_1^b]u \\ &= b h_2 e_1^{a+b-1} f_1^{b-1} u. \end{aligned}$$

Therefore, the lemma follows by induction on b. □

Lemma 7.1.3. *If $e_1^a u = f_0 u = 0$, then*

$$e_1^{a-b+c} h_1^b f_1^c u = 0.$$

Proof. We use induction on b. The case $b = 0$ follows from Lemma 7.1.2. We have

$$\begin{aligned} e_1^{a-b+c} h_1^b f_1^c u &= e_1^{a-b+c} h_1^{b-1}[e_1, f_0] f_1^c u \\ &= -e_1^{a-b+c} h_1^{b-1} f_0 e_1 f_1^c u. \end{aligned}$$

Using the induction hypothesis, we have

$$\begin{aligned} e_1^{a-b+c} h_1^b f_1^c u &= [f_0, e_1^{a-b+c}] h_1^{b-1} e_1 f_1^c u \\ &= -(a-b+c) e_1^{a-b+c} h_1^b f_1^c u. \end{aligned}$$

The assertion follows from this. □

Lemma 7.1.4. *If $e_1^a u = h_1^b f_1^{c+1} u = f_0 u = 0$, then*

$$h_2^{a-b} h_1^b f_1^c u = 0.$$

Proof. First, we prove

$$h_1^b f_1^{c+1+n} e_1^n u = 0$$

by induction on n. In fact,

$$\begin{aligned} h_1^b f_1^{c+1+n} e_1^n u &= [h_1^b f_1^{c+1+n}, e_1] e_1^{n-1} u \\ &= -(c+1+n) h_2 h_1^b f_1^{c+n} e_1^{n-1} u \\ &= 0. \end{aligned}$$

Next, we proceed as follows:

$$\begin{aligned} h_2 h_1^b f_1^{c+n} e_1^n u &= h_1^b f_1^{c+n}[e_1, f_1] e_1^n u \\ &= [h_1^b f_1^{c+n}, e_1] f_1 e_1^n u - h_1^b f_1^{c+n+1} e_1^{n+1} u \\ &= -(c+n) h_2 h_1^b f_1^{c+n} e_1^n u - h_1^b f_1^{c+n+1} e_1^{n+1} u. \end{aligned}$$

By repeating this argument, the assertion reduces to showing that

$$h_1^b f_1^{c+a-b} e_1^{a-b} u = 0.$$

This follows from Lemma 7.1.3 with $c = 0$. □

References

[B] R.J. Baxter, *Exactly solvable models in Statistical mechanics*, Academic Press, London, 1982.

[DJKMO] E. Date, M. Jimbo, A. Kuniba, T. Miwa, and M. Okado, *One dimensional configuration sums in vertex models and affine Lie algebra characters*, Lett. Math. Phys., 17 (1989), 69-77.

[FL] B. Feigin and S. Loktev, *On generalized Kostka polynomials and quantum Verlinde rule*, math.QA/9812093.

[FM] B. Feigin and T. Miwa,*Extended vertex operator algebras and monomial bases*, math.QA/9901067.

[FS] B. Feigin and A. Stoyanovsky, *Quasi-particles models for the representations of Lie algebras and geometry of flag manifold*, hep-th/9308079, RIMS 942; *Functional models for the representations of current algebras and the semi-infinite Schubert cells*, Funct. Anal. Appl. **28** (1994), 55–72.

[Fi] M. Finkelberg, *An equivalence of fusion categories*, Geom. Funct. Anal. **6**(1996)249-267.

[FKLMM] B. Feigin, R. Kedem, S. Loktev, T. Miwa, E. Mukhin, *Combinatorics of the $\widehat{\mathfrak{sl}}_2$ spaces of coinvariants*, math-ph/9908003, RIMS 1243.

[KKMM] R. Kedem, T. Klassen, B. McCoy, E. Melzer, *Fermionic sum representations for conformal field theory characters*. Phys. Lett. **B 307** (1993), 68–76.

[S] G. Segal, Geometric aspect of quantum field theory, in Proceedings of the ICM at Kyoto, 1990.

[St] A. Stoyanovsky, *Lie algebra deformation and character formulas*, Func. Anal. Its Appl.,**32**(1998)66-68.

[TUY] A. Tsuchiya, K. Ueno and Y. Yamada,*Conformal field theory on the universal family of stable curves with gauge symmetry*, Adv. Stud. Pure Math., **19**(1989)459-466.

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