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A Semidefinite Programming Relaxation for the Generalized Stable Set Problem*

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Abstract

In this paper, we generalize the theory of a convex set relaxation for the maximum weight stable set problem due to Grötschel, Lovász and Schrijver to the generalized stable set problem. We define a convex set which serves as a relaxation problem, and show that optimizing a linear function over the set can be done in polynomial time. This implies that the generalized stable set problem for perfect bidirected graphs is polynomial time solvable. Moreover, we prove that the convex set is a polytope if and only if the corresponding bidirected graph is perfect. The definition of the convex set is based on a semidefinite programming relaxation of Lovász and Schrijver for the maximum weight stable set problem, and the equivalent representation using infinitely many convex quadratic inequalities proposed by Fujie and Kojima is particularly important for our proof.

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1 Introduction

A 0-1 integer programming problem, each of whose constraints has two variables, can be transformed to the following form :

$$
\begin{array}{r|lrcll}
\text{(GSSP)} & \text{Maximize} & \boldsymbol{w}^T\boldsymbol{x} & & & \\
& \text{subject to} & x_i + x_j & \leq & 1 & ((i,j) \in E), \\
& & x_i - x_j & \leq & 0 & ((i,j) \in A), \\
& & -x_i - x_j & \leq & -1 & ((i,j) \in D), \\
& & x_i \in \{0,1\} & & & (i \in V),
\end{array}
$$

where $\boldsymbol{w} \in R^V$. Here, we call the problem *the generalized stable set problem*, because if $A = D = \emptyset$ then (GSSP) is a 0-1 integer programming formulation of the maximum weight stable set problem (SSP). In this context, (GSSP) is considered as a maximization problem on the bidirected graph $B = (V; E, A, D)$ (see Section 2.3).

(GSSP), as well as its inequality system, has been studied by Johnson and Padberg [20], Bourjolly [3], Sewell [29], Ikebe and Tamura [19], Tamura [30, 31], and others. Since the inequalities of (GSSP) represent binary relations of pairs of variables, results on (GSSP) are used in preprocessing for integer programming problems including 0-1 variables [2, 5, 16, 17, 28].

(SSP) is known to be NP-hard in general, even if $\boldsymbol{w} = \boldsymbol{e}$, the vector of all ones. On the other hand, it is also known that for some classes of graphs, including the perfect graph, (SSP) can be solved in polynomial time. Grötschel, Lovász and Schrijver [11] showed that the number $\vartheta(G, \boldsymbol{w})$ which is a weighted version of $\vartheta(G)$ $(= \vartheta(G, \boldsymbol{e}))$ introduced by Lovász [24] can be computed in polynomial time. $\vartheta(G, \boldsymbol{w})$ is an upper bound for (SSP) in general, but if G is perfect then $\vartheta(G, \boldsymbol{w})$ coincides with the optimal value of (SSP). Hence, (SSP) for perfect graphs is polynomial time solvable [11]. Grötschel, Lovász and Schrijver [13] later defined a convex set $\mathrm{TH}(G)$ containing the stable set polytope $\mathrm{STAB}(G)$ with the property that $\vartheta(G, \boldsymbol{w}) = \max\{\boldsymbol{w}^T\boldsymbol{x} \mid \boldsymbol{x} \in \mathrm{TH}(G)\}$. In [13], several representations of $\mathrm{TH}(G)$ (and $\vartheta(G, \boldsymbol{w})$) are provided and the following properties are proved :

(a) Every facet defining inequality of $\mathrm{TH}(G)$ is a positive multiple of one of the non-negativity inequalities or one of the clique inequalities.

(b) $\mathrm{TH}(G)$ is a polytope if and only if G is perfect.

The perfectness notion for (GSSP) was defined in [19, 30, 31]. In this paper, we define a convex set $\mathrm{GTH}(B)$ and show that $\mathrm{GTH}(B)$ has the almost same features as $\mathrm{TH}(G)$. The definition of $\mathrm{GTH}(B)$ is based on the representation of $\mathrm{TH}(G)$ by a semidefinite programming relaxation of (SSP) provided by Lovász and Schrijver [25]. Hence, it will be immediately implied that (GSSP) for perfect bidirected graphs can be solved in polynomial time : the fact has been already shown in [29, 30, 31] by demonstrating that (GSSP) for a perfect bidirected graph is equivalent to (SSP) for a certain perfect undirected graph. Moreover, we prove that $\mathrm{GTH}(B)$ has the properties (a)

and (b) above. Our proofs of (a) and (b) will depend on the representation of GTH(B) by convex quadratic inequalities, of which Fujie and Kojima [7] proved the equivalence to the semidefinite programming relaxation. We point out that properties (a) and (b) for GTH(B) cannot be proved as a straightforward generalization of the proofs for TH(G) in [13, 14]. The proofs for TH(G) in [13, 14] fully utilize the down-monotone property of TH(G) ($\boldsymbol{x} \in$ TH(G) and $\boldsymbol{0} \leq \boldsymbol{y} \leq \boldsymbol{x}$ implies $\boldsymbol{y} \in$ TH(G)) and the complement graph $\overline{G}$ of an undirected graph G, while GTH(B) does not satisfy the down-monotone property and the complement of a bidirected graph B cannot be defined in a straightforward way. The down-monotone property allows to introduce the antiblocker of TH(G), and Grötschel, Lovász and Schrijver [13, 14] showed that it is equivalent to TH($\overline{G}$). Our proofs in this paper are based on the recent results on the semidefinite programming, and they provide alternative proofs for TH(G) without using the antiblocker nor the complement of a graph (see the companion paper [8]).

The rest of the paper is organized as follows. In Section 2, we introduce definitions and results for the semidefinite programming relaxation, the maximum weight stable set problem and the generalized stable set problem. Section 3 devotes to state our main results in this paper. Proofs of the results are given in Section 4. Finally, concluding remarks are described in Section 5.

2 Preliminaries

2.1 A Semidefinite Programming Relaxation

Let $\mathcal{S}^n$ be the set of all $n \times n$ symmetric real matrices. We denote $\boldsymbol{A} \succeq \boldsymbol{O}$ when $\boldsymbol{A} in \mathcal{S}^n$ and $\boldsymbol{A}$ is positive semidefinite, and for $\boldsymbol{A} = (A_{ij}), \boldsymbol{B} = (B_{ij}) \in \mathcal{S}^n$, $\boldsymbol{A} \bullet \boldsymbol{B}$ stands for the "inner product" $\boldsymbol{A} \bullet \boldsymbol{B} = \sum_{i=1}^n \sum_{j=1}^n A_{ij} B_{ij}$. Then the semidefinite programming problem is formulated as follows:

$$\begin{array}{ll} \text{Maximize} & \boldsymbol{P}_0 \bullet \boldsymbol{Y} \\ \text{subject to} & \boldsymbol{P}_i \bullet \boldsymbol{Y} \sim_i d_i \quad (i = 1, \ldots, m), \\ & \boldsymbol{Y} \succeq \boldsymbol{O}, \end{array}$$

where $\boldsymbol{P}_i \in \mathcal{S}^n$ for $i = 0, 1, \ldots, m$, and $d_i \in R$ and $\sim_i \in \{\leq, =, \geq\}$ for $i = 1, \ldots, m$. Semidefinite programming problems can be solved in polynomial time within any given accuracy by ellipsoid algorithms [11, 14] or interior point algorithms [1, 21, 26, etc.].

In [25], Lovász and Schrijver proposed a procedure to make a semidefinite programming relaxation for 0-1 integer programming problems. Fujie and Kojima [7] studied the relaxation for more general nonconvex quadratic programming problems. Let

$$F = \left\{ \boldsymbol{x} \in R^n \;\middle|\; \boldsymbol{x}^T \boldsymbol{Q}_i \boldsymbol{x} + \boldsymbol{q}_i^T \boldsymbol{x} + \pi_i \leq 0 \quad (i = 1, \ldots, m) \right\}, \tag{1}$$

where $\boldsymbol{Q}_i \in \mathcal{S}^n$, $\boldsymbol{q}_i \in R^n$, $\pi_i \in R$ for $i = 1, \ldots, m$. We assume m is finite. Since none of $\boldsymbol{Q}_i \in \mathcal{S}^n$ is necessarily positive semidefinite, F is not necessarily a convex set and a problem $\max\{\boldsymbol{w}^T \boldsymbol{x} \mid \boldsymbol{x} \in F\}$

is NP-hard in general. Applying the procedure of Lovász and Schrijver [25] to the set F, we have the set

$$K_{\rm SDP}(F) = \left\{ \boldsymbol{x} \in R^n \;\middle|\; \begin{array}{l} {}^\exists\boldsymbol{X} \in \mathcal{S}^n \text{ s.t.} \\ \boldsymbol{Q}_i \bullet \boldsymbol{X} + \boldsymbol{q}_i^T\boldsymbol{x} + \pi_i \leq 0 \;\; (i = 1, \ldots, m), \\ \boldsymbol{X} - \boldsymbol{x}\boldsymbol{x}^T \succeq \boldsymbol{O} \end{array} \right\}. \tag{2}$$

Since F is represented as

$$F = \left\{ \boldsymbol{x} \in R^n \;\middle|\; \begin{array}{l} \boldsymbol{Q}_i \bullet \boldsymbol{X} + \boldsymbol{q}_i^T\boldsymbol{x} + \pi_i \leq 0 \;\; (i = 1, \ldots, m), \\ \boldsymbol{X} - \boldsymbol{x}\boldsymbol{x}^T = \boldsymbol{O} \end{array} \right\},$$

$F \subseteq K_{\rm SDP}(F)$ holds. Note that, by the Schur complement (see, for example, [18]), we have

$$\boldsymbol{X} - \boldsymbol{x}\boldsymbol{x}^T \succeq \boldsymbol{O} \iff \begin{pmatrix} 1 & \boldsymbol{x}^T \\ \boldsymbol{x} & \boldsymbol{X} \end{pmatrix} \succeq \boldsymbol{O}. \tag{3}$$

Hence

$$\begin{aligned} \max\{\boldsymbol{w}^T\boldsymbol{x} \mid \boldsymbol{x} \in K_{\rm SDP}(F)\} &= \max \left\{ \boldsymbol{w}^T\boldsymbol{x} \;\middle|\; \begin{array}{l} \boldsymbol{Q}_i \bullet \boldsymbol{X} + \boldsymbol{q}_i^T\boldsymbol{x} + \pi_i \leq 0 \;\; (i = 1, \ldots, m), \\ \boldsymbol{X} - \boldsymbol{x}\boldsymbol{x}^T \succeq \boldsymbol{O} \end{array} \right\} \\ &= \max \left\{ \boldsymbol{P}_0 \bullet \boldsymbol{Y} \;\middle|\; \begin{array}{l} \boldsymbol{E}_{00} \bullet \boldsymbol{Y} = 1, \\ \boldsymbol{P}_i \bullet \boldsymbol{Y} \leq 0 \;\; (i = 1, \ldots, m), \\ \boldsymbol{Y} \succeq \boldsymbol{O} \end{array} \right\} \end{aligned}$$

and optimizing a linear function over $K_{\rm SDP}(F)$ is a semidefinite programming problem, where

$$\boldsymbol{E}_{00} = \begin{pmatrix} 1 & \boldsymbol{0}^T \\ \boldsymbol{0} & \boldsymbol{O} \end{pmatrix}, \quad \boldsymbol{P}_0 = \begin{pmatrix} 0 & \boldsymbol{w}^T/2 \\ \boldsymbol{w}/2 & \boldsymbol{O} \end{pmatrix}, \quad \boldsymbol{P}_i = \begin{pmatrix} \pi_i & \boldsymbol{q}_i^T/2 \\ \boldsymbol{q}_i/2 & \boldsymbol{Q}_i \end{pmatrix} \;\; (i = 1, \ldots, m).$$

The following set, defined by infinitely many convex quadratic inequalities, is introduced in [7]:

$$K_{\rm CQI}(F) = \left\{ \boldsymbol{x} \in R^n \;\middle|\; \sum_{i=1}^m t_i \left(\boldsymbol{x}^T\boldsymbol{Q}_i\boldsymbol{x} + \boldsymbol{q}_i^T\boldsymbol{x} + \pi_i\right) \leq 0 \;\; \left(\begin{array}{l} t_i \geq 0 \; (i = 1, \ldots, m), \\ \displaystyle\sum_{i=1}^m t_i\boldsymbol{Q}_i \succeq \boldsymbol{O} \end{array} \right) \right\}. \tag{4}$$

Since $F \subseteq K_{\rm CQI}(F)$ holds by definition, $K_{\rm CQI}(F)$ also serves as a relaxation problem. In fact, it has been shown that $K_{\rm SDP}(F)$ and $K_{\rm CQI}(F)$ are equivalent.

Theorem 2.1 ([7, 22]) $K_{\rm SDP}(F) = K_{\rm CQI}(F)$. ■

As an immediate consequence of the theorem, we know that, if $\boldsymbol{Q}_i \succeq \boldsymbol{O}$ for $i = 1, \ldots, m$, $F = K_{\rm SDP}(F) = K_{\rm CQI}(F)$. Another consequences are shown in [7, 22]. In particular, Kojima and Tunçel [22, 23] used Theorem 2.1 to derive some features of their successive convex relaxation method for a very large class of optimization problems.

2.2 The Maximum Weight Stable Set Problem

Let $G = (V, E)$ be an undirected graph. Then the maximum weight stable set problem (SSP) is formulated as follows:

$$\text{(SSP)} \left| \begin{array}{lll} \text{Maximize} & \boldsymbol{w}^T \boldsymbol{x} & \\ \text{subject to} & x_i + x_j \leq 1 & ((i,j) \in E), \\ & x_i \in \{0,1\} & (i \in V). \end{array} \right.$$

A subset $S \subseteq V$ is called a *stable set* if $(i,j) \notin E$ for any distinct $i, j \in S$. Conversely, a subset $C \subseteq V$ is called a *clique* if $(i,j) \in E$ for any distinct $i, j \in C$. It is easy to verify that any feasible solution of (SSP) corresponds to a stable set of G, and vice versa.

(SSP) is NP-hard in general, but is polynomial time solvable for perfect graphs [11, 12, 14]. The polynomial time solvability is explained via a convex programming relaxation. To this end, let STAB(G) be the convex hull of all feasible solutions of (SSP). STAB(G) satisfies the nonnegativity inequalities

$$x_i \geq 0 \quad (i \in V)$$

and the clique inequalities

$$\sum_{i \in C} x_i \leq 1 \quad (C : \text{clique}).$$

Let us denote by QSTAB(G) the polytope with the nonnegativity inequalities and the clique inequalities. Then we have STAB(G) $\subseteq$ QSTAB(G). Moreover

Theorem 2.2 ([9, 4]) STAB(G) = QSTAB(G) *if and only if G is a perfect graph.* ■

For definition and results of perfect graphs including Theorem 2.2, see [10]. Theorem 2.2 is a polyhedral characterization of perfect graphs. Note that it is still NP-hard to optimize $\boldsymbol{w}^T \boldsymbol{x}$ over QSTAB(G) [11].

Grötschel, Lovász, and Schrijver [13] introduced the set TH(G). In [13], some equivalent representations of TH(G) are provided. One of such representations is defined by infinitely many linear inequalities:

$$\text{TH}(G) = \left\{ \boldsymbol{x} \in R^V \left| \begin{array}{l} \boldsymbol{x} \geq \boldsymbol{0}, \\ \displaystyle\sum_{i \in V} \left(\boldsymbol{u}_0^T \boldsymbol{u}_i\right)^2 x_i \leq 1 \\ \qquad \left(\begin{array}{ll} \boldsymbol{u}_i \in R^N, \ ||\boldsymbol{u}_i|| = 1 & (i \in V \cup \{0\}), \ N : \text{positive integer} \\ \boldsymbol{u}_i^T \boldsymbol{u}_j = 0 & ((i,j) \notin E) \end{array} \right) \end{array} \right. \right\}. \tag{5}$$

The name "TH" comes from the fact that Lovász' ϑ(theta)-number is characterized as $\vartheta(G, \boldsymbol{w}) = \max\{\boldsymbol{w}^T \boldsymbol{x} \mid \boldsymbol{x} \in \text{TH}(G)\}$ [13]. It can be shown that TH(G) lies between STAB(G) and QSTAB(G), that is,

Theorem 2.3 ([13, 14]) $\mathrm{STAB}(G) \subseteq \mathrm{TH}(G) \subseteq \mathrm{QSTAB}(G)$. ■

Moreover,

$$\max\{\boldsymbol{w}^T\boldsymbol{x} \mid \boldsymbol{x} \in \mathrm{TH}(G)\} \tag{6}$$

can be solved in polynomial time [11, 13, 14]. This will be also shown using Theorem 2.7 below, which states that (6) is a semidefinite programming problem.

$\mathrm{TH}(G)$ is nonpolyhedral in general, and a necessary and sufficient condition that $\mathrm{TH}(G)$ is a polytope is known. The following theorem is a key property to prove the condition.

Theorem 2.4 ([13, 14]) *Every facet defining inequality of* $\mathrm{TH}(G)$ *is a positive multiple of one of the nonnegativity inequalities or one of the clique inequalities.* ■

Theorem 2.5 ([13, 14]) $\mathrm{TH}(G)$ *is a polytope if and only if* G *is a perfect graph.* ■

An immediate corollary which characterizes perfect graphs is obtained.

Corollary 2.6 ([13, 14])

(a) $\mathrm{TH}(G) = \mathrm{STAB}(G)$ *if and only if* G *is perfect.*

(b) $\mathrm{TH}(G) = \mathrm{QSTAB}(G)$ *if and only if* G *is perfect.* ■

In [8], the authors provide alternative proofs of Theorems 2.4 and 2.5 based on a new representation of $\mathrm{TH}(G)$. Let us define the set

$$F_G = \left\{ \boldsymbol{x} \in R^V \;\middle|\; \begin{array}{ll} x_i x_j = 0 & ((i,j) \in E), \\ x_i^2 - x_i = 0 & (i \in V) \end{array} \right\}.$$

For any 0-1 solution $\boldsymbol{x}$,

$$x_i + x_j \leq 1 \iff x_i x_j = 0.$$

Moreover, we have

$$x_i \in \{0, 1\} \iff x_i^2 - x_i = 0.$$

Hence, $\mathrm{STAB}(G) = \mathrm{conv}(F_G)$ and F_G is a representation of $\mathrm{STAB}(G)$ in terms of quadratic equations. Moreover, we can define a semidefinite programming relaxation $K_{\mathrm{SDP}}(F_G)$ or $K_{\mathrm{CQI}}(F_G)$. In fact, it provides the another representation of $\mathrm{TH}(G)$.

Theorem 2.7 ([25]) $\mathrm{TH}(G) = K_{\mathrm{SDP}}(F_G)$. ■

As a result of the theorem, $\mathrm{TH}(G)$ is represented by infinitely many convex quadratic inequalities.

Corollary 2.8

$$\begin{aligned} \mathrm{TH}(G) &= K_{\mathrm{CQI}}(F_G) \\ &= \left\{ \boldsymbol{x} \in R^V \;\middle|\; \boldsymbol{x}^T\boldsymbol{M}\boldsymbol{x} - \sum_{i \in V} M_{ii} x_i \leq 0 \quad \left(\begin{array}{ll} M_{ij} = 0 & ((i,j) \notin E), \\ \boldsymbol{M} \succeq \boldsymbol{O} & \end{array} \right) \right\}. \end{aligned}$$

Proof : The assertion follows from Proposition 3.6. See also [8]. ■

2.3 The Generalized Stable Set Problem

The constraints of (GSSP) can be rewritten as

$$\left.\begin{array}{rcrll} x_i & + & x_j & \leq 1 & (i,j) \in E, \\ x_i & + & (1-x_j) & \leq 1 & (i,j) \in A, \\ (1-x_i) & + & (1-x_j) & \leq 1 & (i,j) \in D, \\ & & x_i \in \{0,1\} & & (i \in V). \end{array}\right\} \tag{7}$$

A bidirected graph $B = (V; E, A, D)$ is thus defined, where V is a set of vertices and $E \cup A \cup D$ is a set of edges. Bidirected graphs are introduced by Edmonds and Johnson [6] and the use of bidirected graphs to (GSSP) is originated with Johnson and Padberg [20].

For a bidirected graph $B = (V; E, A, D)$, each edge $e \in E \cup A \cup D$ has two ends $i, j \in V$ and two associated signs (plus or minus) at i and j. For the system (7), E consists of edges with two plus signs at their ends, A consists of edges with one plus (at i) and one minus signs (at j) at their ends, and D consists of edges with two minus signs at their ends. Conversely, for a bidirected graph $B = (V; E, A, D)$, we can associate the system (7). Hence, we will say that 0-1 solution $\boldsymbol{x} \in \{0,1\}^V$ is feasible for B if it is feasible for the corresponding system (7). We assume that, for any edge(joining i and j), exactly one of (i,j) and (j,i) is contained in $E \cup A \cup D$. Hence, each pair (i,j) appears at most once in summations in (10), and so on.

A bidirected graph is called *transitive* if the presence of two edges $e_1 = (i,j)$ and $e_2 = (j,k)$ with opposite signs at j implies that there is an edge $e_3 = (i,k)$ whose signs at i and k agree with those of e_1 and e_2, respectively. In the system (7), the transitivity means that the inequality associated with e_3 which is the sum of the inequalities associated with e_1 and e_2 is contained in (7). A bidirected graph without any selfloops nor any two edges incident to the same ends is called *simple*. Suppose a bidirected graph $B = (V; E, A, D)$ has a selfloop $e = (i,i)$. If e has one plus and one minus signs at its ends then the corresponding inequality is $x_i - x_i \leq 0$. If e has two plus signs at its ends then the corresponding inequality is $x_i + x_i \leq 1$ and x_i must be 0 for any 0-1 solution of B. Similarly, if e has two minus signs at its ends then x_i must be 1 for any 0-1 solution of B. Next suppose a transitive bidirected graph $B = (V; E, A, D)$ has two or more edges incident to i and j. Then the transitivity implies that "$x_i = x_j$ or $x_i = 1 - x_j$" or "i and/or j has selfloops". Hence, for our purposes, we have only to deal with transitive and simple bidirected graphs. Such a bidirected graph is called *closed*.

For a closed bidirected graph $B = (V; E, A, D)$, let GSTAB(B) be the convex hull of all feasible solutions of B.

Theorem 2.9 ([20]) *For any closed bidirected graph B,* GSTAB(B) *is full-dimensional.* ■

The name "GSTAB(B)" comes from the fact that it is a generalization of STAB(G).

Johnson and Padberg [20] generalized the clique inequalities for undirected graphs to bidirected graphs. A disjoint pair $C = (C^+, C^-)$ of V is called a *biclique* if an induced subgraph $B[(C^+ \cup C^-)]$ is completely connected, and

- $i,j \in C^+$ and $i \neq j \Longrightarrow (i,j) \in E$,
- $i \in C^+$ and $j \in C^- \Longrightarrow (i,j) \in A$, and
- $i,j \in C^-$ and $i \neq j \Longrightarrow (i,j) \in D$.

Proposition 2.10 ([20]) *For a biclique $C = (C^+, C^-)$, the biclique inequality*

$$\sum_{i \in C^+} x_i + \sum_{i \in C^-} (1 - x_i) \leq 1$$

is valid for STAB(B). ■

In [20], a necessary and sufficient condition that the biclique inequality defines a facet of GSTAB(B) is given. Before defining a polytope which is a generalization of QSTAB(G), we should note that bound constraints on variables (that is, $0 \leq x_i \leq 1$) are biclique inequalities. Actually, $(\{v\},\emptyset)$ is a biclique which corresponds to the inequality $x_v \leq 1$, and $(\emptyset,\{v\})$ is a biclique which corresponds to the inequality $x_v \geq 0$. Hence, as a generalization of QSTAB(G), we define a polytope

$$\text{QGSTAB}(B) = \left\{ \boldsymbol{x} \in R^V \;\middle|\; \boldsymbol{x} \text{ satisfies all biclique inequalities of } B \right\}.$$

By Proposition 2.10, we have GSTAB$(B) \subseteq$ QGSTAB(B). Based on Theorem 2.2, a bidirected graph B is defined to be *perfect* if it is closed and GSTAB$(B) =$ QGSTAB(B). The notion of this perfectness was considered, for example, in [19, 30, 31]. An interesting perfectness relation between bidirected and undirected graphs is known. For a bidirected graph $B = (V; E, A, D)$, we denote $\underline{B} = (V, E \cup A \cup D)$.

Theorem 2.11 ([15, 19, 29]) *Let B be a closed bidirected graph. Then B is perfect if and only if $\underline{B}$ is perfect.* ■

3 Results

For any 0-1 vector $\boldsymbol{x} \in \{0,1\}^V$, we have

$$\begin{array}{rcrcrcl}
x_i & + & x_j & \leq 1 & \Longleftrightarrow & x_i x_j & = 0, \\
x_i & + & (1 - x_j) & \leq 1 & \Longleftrightarrow & x_i(1 - x_j) & = 0, \\
(1 - x_i) & + & (1 - x_j) & \leq 1 & \Longleftrightarrow & (1 - x_i)(1 - x_j) & = 0.
\end{array}$$

Hence, if we define

$$F_B = \left\{ \boldsymbol{x} \in R^V \;\middle|\; \begin{array}{rl}
x_i x_j = 0 & ((i,j) \in E), \\
x_i(1 - x_j) = 0 & ((i,j) \in A), \\
(1 - x_i)(1 - x_j) = 0 & ((i,j) \in D), \\
x_i^2 - x_i = 0 & (i \in V)
\end{array} \right\}, \tag{8}$$

$\mathrm{GSTAB}(B) = \mathrm{conv}(F_B)$, i.e., F_B is a representation of $\mathrm{GSTAB}(B)$ in terms of quadratic equations. Note that, by introducing matrices $\boldsymbol{E}_{ij} = \boldsymbol{e}_i\boldsymbol{e}_j^T + \boldsymbol{e}_j\boldsymbol{e}_i^T$ $(i,j \in V, i \neq j)$ and $\boldsymbol{E}_{ii} = \boldsymbol{e}_i\boldsymbol{e}_i^T$ $(i \in V)$, (8) is written as

$$F_B = \left\{ \boldsymbol{x} \in R^V \;\middle|\; \begin{array}{ll} \boldsymbol{x}^T\boldsymbol{E}_{ij}\boldsymbol{x} = 0 & ((i,j) \in E), \\ \boldsymbol{x}^T\boldsymbol{E}_{ij}\boldsymbol{x} - 2x_i = 0 & ((i,j) \in A), \\ \boldsymbol{x}^T\boldsymbol{E}_{ij}\boldsymbol{x} - 2(x_i + x_j - 1) = 0 & ((i,j) \in D), \\ \boldsymbol{x}^T\boldsymbol{E}_{ii}\boldsymbol{x} - x_i = 0 & (i \in V), \end{array} \right\}. \tag{9}$$

As a direct generalization of Theorem 2.7, let us define

$$\begin{aligned} \mathrm{GTH}(B) &= K_{\mathrm{SDP}}(F_B) \\ &= \left\{ \boldsymbol{x} \in R^V \;\middle|\; \begin{array}{lll} {}^{\exists}\boldsymbol{X} \in \mathcal{S}^V \text{ s.t.} & X_{ij} = 0 & ((i,j) \in E), \\ & X_{ij} = x_i & ((i,j) \in A), \\ & X_{ij} = x_i + x_j - 1 & ((i,j) \in D), \\ & X_{ii} = x_i & (i \in V), \\ & \boldsymbol{X} - \boldsymbol{x}\boldsymbol{x}^T \succeq \boldsymbol{O} & \end{array} \right\}. \end{aligned}$$

Our results are Theorems 3.1, 3.3 and 3.4 below, which are generalizations of Theorem 2.3, 2.4 and 2.5, respectively. We will first show that $\mathrm{GTH}(B)$ lies between $\mathrm{GSTAB}(B)$ and $\mathrm{QGSTAB}(B)$, which is a generalization of Theorem 2.3. That is,

Theorem 3.1 $\mathrm{GSTAB}(B) \subseteq \mathrm{GTH}(B) \subseteq \mathrm{QGSTAB}(B)$.

A proof of Theorem 3.1 will be given in Section 4.2. As a corollary, we have

Corollary 3.2 *The generalized stable set problem for perfect bidirected graphs can be solved in polynomial time.*

Proof : By Theorem 3.1, we have $\mathrm{GSTAB}(B) = \mathrm{GTH}(B) = \mathrm{QGSTAB}(B)$ for a perfect bidirected graph B. Moreover, optimizing a linear function over $\mathrm{GTH}(B)$ is a semidefinite programming problem and can be solved in polynomial time. Therefore, optimizing over $\mathrm{GSTAB}(B)$ is also polynomial time solvable. ■

Corollary 3.2 has been shown in [29, 30, 31] by demonstrating a polynomial time transformation of the generalized stable set problem to a certain maximum weight stable set problem while preserving the perfectness. Our proof based on a semidefinite programming provides a more simple way to solve the generalized stable set problem for perfect bidirected graphs.

Theorem 3.3 *Every facet defining inequality of* $\mathrm{GTH}(B)$ *is a positive multiple of one of the biclique inequalities.*

Theorem 3.4 *Let B be a closed bidirected graph. Then* $\mathrm{GTH}(B)$ *is a polytope if and only if B is a perfect bidirected graph.*

Proofs of Theorems 3.3 and 3.4 will be given in Sections 4.3 and 4.4, respectively. As an immediate corollary of Theorem 3.4, we have

Corollary 3.5 *Let B be a closed bidirected graph. Then*

(a) $\mathrm{GTH}(B) = \mathrm{GSTAB}(B)$ *if and only if B is perfect.*

(b) $\mathrm{GTH}(B) = \mathrm{QGSTAB}(B)$ *if and only if B is perfect.* ■

Our proof of the theorems will depend on a representation of $\mathrm{GTH}(B)$ by convex quadratic inequalities.

Proposition 3.6 *Let B be a closed bidirected graph. Then*

$$\mathrm{GTH}(B) = \left\{ \boldsymbol{x} \in R^V \;\middle|\; \begin{array}{r} \boldsymbol{x}^T \boldsymbol{M} \boldsymbol{x} - 2\sum_{(i,j)\in A} M_{ij} x_i - 2\sum_{(i,j)\in D} M_{ij}(x_i + x_j - 1) - \sum_{i\in V} M_{ii} x_i \le 0 \\ \left(\begin{array}{ll} M_{ij} = 0 & ((i,j),(j,i) \notin E \cup A \cup D), \\ \boldsymbol{M} \succeq \boldsymbol{O} & \end{array} \right) \end{array} \right\}. \tag{10}$$

Proof : We first note that if F is described by quadratic equalities :

$$F = \left\{ \boldsymbol{x} \in R^n \;\middle|\; \boldsymbol{x}^T \boldsymbol{Q}_i \boldsymbol{x} + \boldsymbol{q}_i^T \boldsymbol{x} + \pi_i = 0 \quad (i = 1, \ldots, m), \right\}$$

then F can be rewritten as

$$F = \left\{ \boldsymbol{x} \in R^n \;\middle|\; \begin{array}{l} \boldsymbol{x}^T \boldsymbol{Q}_i \boldsymbol{x} + \boldsymbol{q}_i^T \boldsymbol{x} + \pi_i \le 0 \quad (i = 1, \ldots, m), \\ -\left(\boldsymbol{x}^T \boldsymbol{Q}_i \boldsymbol{x} + \boldsymbol{q}_i^T \boldsymbol{x} + \pi_i \right) \le 0 \quad (i = 1, \ldots, m) \end{array} \right\},$$

and it is not hard to show that

$$K_{\mathrm{CQI}}(F) = \left\{ \boldsymbol{x} \in R^n \;\middle|\; \sum_{i=1}^m t_i \left(\boldsymbol{x}^T \boldsymbol{Q}_i \boldsymbol{x} + \boldsymbol{q}_i^T \boldsymbol{x} + \pi_i \right) \le 0 \quad \left(\sum_{i=1}^m t_i \boldsymbol{Q}_i \succeq \boldsymbol{O} \right) \right\}.$$

Therefore, by Theorem 2.1 and (9),

$$\begin{aligned} \mathrm{GTH}(B) &= K_{\mathrm{CQI}}(F_B) \\ &= \left\{ \boldsymbol{x} \in R^V \;\middle|\; \begin{array}{l} \sum_{(i,j)\in E} s_{ij} \boldsymbol{x}^T \boldsymbol{E}_{ij} \boldsymbol{x} + \sum_{(i,j)\in A} t_{ij} \left(\boldsymbol{x}^T \boldsymbol{E}_{ij} \boldsymbol{x} - 2x_i \right) \\ \quad + \sum_{(i,j)\in D} u_{ij} \left(\boldsymbol{x}^T \boldsymbol{E}_{ij} \boldsymbol{x} - 2\left(x_i + x_j - 1 \right) \right) + \sum_{i\in V} h_i \left(\boldsymbol{x}^T \boldsymbol{E}_{ii} \boldsymbol{x} - x_i \right) \le 0 \\ \quad \left(\sum_{(i,j)\in E} s_{ij} \boldsymbol{E}_{ij} + \sum_{(i,j)\in A} t_{ij} \boldsymbol{E}_{ij} + \sum_{(i,j)\in D} u_{ij} \boldsymbol{E}_{ij} + \sum_{i\in V} h_i \boldsymbol{E}_{ii} \succeq \boldsymbol{O} \right) \end{array} \right\}. \end{aligned}$$

Letting $\boldsymbol{M} = \sum_{(i,j)\in E} s_{ij} \boldsymbol{E}_{ij} + \sum_{(i,j)\in A} t_{ij} \boldsymbol{E}_{ij} + \sum_{(i,j)\in D} u_{ij} \boldsymbol{E}_{ij} + \sum_{i\in V} h_i \boldsymbol{E}_{ii}$, we have the desired result. ■

Before closing the section, we notice that it is unlikely to generalize all of the results in [13, 14] concerning $\mathrm{TH}(G)$ to $\mathrm{GTH}(B)$. For example, we have not found any representation of $\mathrm{GTH}(B)$ by linear inequalities as (5). On the other hand, we can show the following representation of $\mathrm{GTH}(B)$.

Proposition 3.7

$$\mathrm{GTH}(B) = \left\{ \left((\boldsymbol{v}_0^T \boldsymbol{v}_i)^2 \,\middle|\, i \in V \right) \;\middle|\; \begin{array}{ll} \boldsymbol{v}_i \in R^N,\ ||\boldsymbol{v}_i|| = 1 & \left(\begin{array}{l} i \in V \cup \{0\}, \\ N \ : \textit{positive integer} \end{array} \right), \\ \boldsymbol{v}_i^T \boldsymbol{v}_j = 0 & ((i,j) \in E), \\ \boldsymbol{v}_i^T (\boldsymbol{v}_0 - (\boldsymbol{v}_0^T \boldsymbol{v}_j)\boldsymbol{v}_j) = 0 & ((i,j) \in A), \\ (\boldsymbol{v}_0 - (\boldsymbol{v}_0^T \boldsymbol{v}_i)\boldsymbol{v}_i)^T (\boldsymbol{v}_0 - (\boldsymbol{v}_0^T \boldsymbol{v}_j)\boldsymbol{v}_j) = 0 & ((i,j) \in D) \end{array} \right\}.$$

Proof : The proof proceeds in the same way as the one in [25, Lemma 2.17]. ■

4 Proofs

4.1 Facet Defining Convex Quadratic Inequalities

This subsection is a preparation of Section 4.3. We consider

$$K = \left\{ \boldsymbol{x} \in R^n \,\middle|\, \boldsymbol{x}^T \boldsymbol{Q} \boldsymbol{x} - \boldsymbol{q}^T \boldsymbol{x} + \pi \leq 0 \quad \left(\begin{pmatrix} \pi & \boldsymbol{q}^T/2 \\ \boldsymbol{q}/2 & \boldsymbol{Q} \end{pmatrix} \in \mathcal{P} \cap \mathcal{Q}_+ \right) \right\}, \tag{11}$$

where $\mathcal{P}$ is a closed convex cone in $\mathcal{S}^{1+n}$ and

$$\mathcal{Q}_+ = \left\{ \begin{pmatrix} \pi & \boldsymbol{q}^T/2 \\ \boldsymbol{q}/2 & \boldsymbol{Q} \end{pmatrix} \,\middle|\, \boldsymbol{Q} \in \mathcal{S}^n,\ \boldsymbol{Q} \succeq \boldsymbol{O} \right\} \subseteq \mathcal{S}^{1+n}.$$

K is a closed convex set defined by infinitely many convex quadratic inequalities. This description of K using $\mathcal{P}$ and $\mathcal{Q}_+$ is due to Kojima and Tunçel [22]. $K_{\mathrm{CQI}}(F)$ in (4) is described of the form (11) if we take

$$\mathcal{P} = \mathrm{cone}\left(\left\{ \begin{pmatrix} \pi_i & -\boldsymbol{q}_i^T/2 \\ -\boldsymbol{q}_i/2 & \boldsymbol{Q}_i \end{pmatrix} \,\middle|\, i = 1, \ldots, m \right\} \right),$$

where cone(P) denotes the cone generated by P.

Faces and facets of convex sets are treated comprehensively in [27]. However, for our purposes, we have only to consider the case that K has dimension n and facets are defined as follows. $F \subseteq K$ is a facet of K if F has dimension $n-1$ and there is an inequality $\boldsymbol{a}^T \boldsymbol{x} \leq b$ valid for K such that $F = K \cap \{\boldsymbol{x} \mid \boldsymbol{a}^T \boldsymbol{x} = b\}$. We also say that the inequality $\boldsymbol{a}^T \boldsymbol{x} \leq b$ defines a facet of K. The purpose of this subsection is to prove the following theorem :

Theorem 4.1 *Assume that K has dimension n and that F is a facet of K.*

(i) *There is* $\begin{pmatrix} \pi & \boldsymbol{q}^T/2 \\ \boldsymbol{q}/2 & \boldsymbol{Q} \end{pmatrix} \in \mathcal{P} \cap \mathcal{Q}_+$ *such that*

$$\boldsymbol{x}^T \boldsymbol{Q} \boldsymbol{x} - \boldsymbol{q}^T \boldsymbol{x} + \pi = 0 \quad ({}^\forall \boldsymbol{x} \in F).$$

(ii) $\mathrm{rank}\,\boldsymbol{Q} \leq 1$, *that is,* $\boldsymbol{Q} = \boldsymbol{O}$ *or* $\boldsymbol{Q} = \boldsymbol{p}\boldsymbol{p}^T$ *for some* $\boldsymbol{p} \in R^n$.

(iii) *If* $\boldsymbol{Q} \neq \boldsymbol{O}$ *then*

$$\boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x} - \boldsymbol{q}^T\boldsymbol{x} + \pi = (\boldsymbol{p}^T\boldsymbol{x} - \alpha)\,(\boldsymbol{p}^T\boldsymbol{x} - \beta + \alpha)$$

for some $\boldsymbol{p} \in R^n$ *and* $\alpha, \beta \in R$.

To prove (i), we need the following lemma which ensures that any boundary point in K satisfies some inequalities, taken from $\mathcal{P} \cap \mathcal{Q}_+$, with equality.

Lemma 4.2 *For any boundary point* $\widehat{\boldsymbol{x}} \in K$, *there exists* $\begin{pmatrix} \pi & \boldsymbol{q}^T/2 \\ \boldsymbol{q}/2 & \boldsymbol{Q} \end{pmatrix} \in \mathcal{P} \cap \mathcal{Q}_+$ *such that* $\widehat{\boldsymbol{x}}^T\boldsymbol{Q}\widehat{\boldsymbol{x}} - \boldsymbol{q}^T\widehat{\boldsymbol{x}} + \pi = 0$.

Proof : We may restrict ourselves to consider $\boldsymbol{P} \in \mathcal{P} \cap \mathcal{Q}_+$ with $||\boldsymbol{P}|| = 1$, where $||\cdot||$ denotes a certain matrix norm. For $j = 1, 2, \ldots$, we consider the open ball $B(\widehat{\boldsymbol{x}}, 1/j)$ with radius $1/j$ and center $\widehat{\boldsymbol{x}}$. Since $\widehat{\boldsymbol{x}}$ is on the boundary of K, there is $\boldsymbol{P}_j = \begin{pmatrix} \pi_j & \boldsymbol{q}_j^T/2 \\ \boldsymbol{q}_j/2 & \boldsymbol{Q}_j \end{pmatrix} \in \mathcal{P} \cap \mathcal{Q}_+$ which cuts off $B(\widehat{\boldsymbol{x}}, 1/j)$ (that is, $\widehat{x}^T\boldsymbol{Q}_j\widehat{\boldsymbol{x}} - \boldsymbol{q}_j^T\widehat{\boldsymbol{x}} + \pi_j \leq 0$ but there is $\widetilde{\boldsymbol{x}} \in B(\widehat{\boldsymbol{x}}, 1/j)$ with $\widetilde{x}^T\boldsymbol{Q}_j\widetilde{\boldsymbol{x}} - \boldsymbol{q}_j^T\widetilde{\boldsymbol{x}} + \pi_j > 0$). Since we assume $||\boldsymbol{P}_j|| = 1$ and $\mathcal{P} \cap \mathcal{Q}_+$ is closed, there is a subsequence $\boldsymbol{P}_{j_1}, \boldsymbol{P}_{j_2}, \ldots$ which converges to a nonzero matrix $\boldsymbol{P} = \begin{pmatrix} \pi & \boldsymbol{q}^T/2 \\ \boldsymbol{q}/2 & \boldsymbol{Q} \end{pmatrix} = \lim_{k\to\infty} \boldsymbol{P}_{j_k} \in \mathcal{P} \cap \mathcal{Q}_+$. Moreover, $\boldsymbol{P}$ must satisfy $\widehat{\boldsymbol{x}}^T\boldsymbol{Q}\widehat{\boldsymbol{x}} - \boldsymbol{q}^T\widehat{\boldsymbol{x}} + \pi = 0$. ■

We further need an easy claim to prove (i):

Claim 4.3 *Let* $\gamma(\boldsymbol{x})$ *be a convex function such that* $\gamma(\boldsymbol{x}) \leq 0$ *is valid for* K. *If* $\widehat{\boldsymbol{x}} = \sum_{j=1}^n \lambda_j \boldsymbol{x}^j$ *is a convex combination of* $\boldsymbol{x}^1, \ldots, \boldsymbol{x}^n \in K$ *and* $\gamma(\widehat{\boldsymbol{x}}) = 0$ *then* $\gamma(\boldsymbol{x}^j) = 0$ *if* $\lambda_j > 0$.

Proof : By definitions, we have

$$0 = \gamma(\widehat{\boldsymbol{x}}) \leq \sum_{j=1}^n \lambda_j \gamma(\boldsymbol{x}^j) \leq 0.$$

The first inequality follows from the convexity of γ, and the second inequality follows by definition. Hence $\sum_{j=1}^n \lambda_j \gamma(\boldsymbol{x}^j) = 0$ and the assertion holds. ■

Proof of Theorem 4.1 (i) : Let $\boldsymbol{x}^1, \ldots, \boldsymbol{x}^n$ be affinely independent points in F and $\widehat{\boldsymbol{x}} = \frac{1}{n}\sum_{j=1}^n \boldsymbol{x}^j$. Since F is convex, $\widehat{\boldsymbol{x}} \in F$. Hence, by Lemma 4.2, there is $\boldsymbol{P} = \begin{pmatrix} \pi & \boldsymbol{q}^T/2 \\ \boldsymbol{q}/2 & \boldsymbol{Q} \end{pmatrix} \in \mathcal{P} \cap \mathcal{Q}_+$ such that $\widehat{\boldsymbol{x}}^T\boldsymbol{Q}\widehat{\boldsymbol{x}} - \boldsymbol{q}^T\widehat{\boldsymbol{x}} + \pi = 0$. In the rest of this subsection, we define $g(\boldsymbol{x}) = \boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x} - \boldsymbol{q}^T\boldsymbol{x} + \pi$. Then, since $\widehat{\boldsymbol{x}}$ is represented by a convex combination of $\boldsymbol{x}^1, \ldots, \boldsymbol{x}^n$, $g(\boldsymbol{x}^j) = 0$ $(j = 1, \ldots, n)$ by Claim 4.3. Below, we will show that $g(\boldsymbol{x}) = 0$ for $\boldsymbol{x} \in F = K \cap \mathrm{aff}(\{\boldsymbol{x}^1, \ldots, \boldsymbol{x}^n\})$.

First, consider $\boldsymbol{x} \in \mathrm{conv}(\{\boldsymbol{x}^1, \ldots, \boldsymbol{x}^n\})$. Assume

$$\boldsymbol{x} = \sum_{j=1}^{n} \lambda_j \boldsymbol{x}^j, \ \sum_{j=1}^{n} \lambda_j = 1 \text{ and } \lambda_j \geq 0 \ (j = 1, \ldots, n).$$

Letting $\mu = \min\{1/(\lambda_j n) \mid \lambda_j > 0\}$, we have $\mu > 0$ and

$$\widehat{\boldsymbol{x}} = \mu \boldsymbol{x} + \sum_{j=1}^{n} \left(\frac{1}{n} - \mu \lambda_j \right) \boldsymbol{x}^j.$$

Since then $\widehat{\boldsymbol{x}}$ is represented by a convex combination of $\boldsymbol{x}$ and $\boldsymbol{x}^1, \ldots, \boldsymbol{x}^n$, we have $g(\boldsymbol{x}) = 0$ by Claim 4.3.

Next, consider $\boldsymbol{x} \in F = K \cap \mathrm{aff}(\{\boldsymbol{x}^1, \ldots, \boldsymbol{x}^n\})$. Assume

$$\boldsymbol{x} = \sum_{j=1}^{n} \lambda_j \boldsymbol{x}^j \text{ and } \sum_{j=1}^{n} \lambda_j = 1.$$

Since we have shown that $g(\boldsymbol{x}) = 0$ if $\lambda_j \geq 0$ $(j = 1, \ldots, n)$, we may assume there is j with $\lambda_j < 0$. Define

$$\boldsymbol{x}^+ = \frac{1}{\Lambda^+} \sum_{j:\lambda_j > 0} \lambda_j \boldsymbol{x}^j, \ \ \boldsymbol{x}^- = \frac{1}{\Lambda^-} \sum_{j:\lambda_j < 0} (-\lambda_j) \boldsymbol{x}^j, \ \ \Lambda^+ = \sum_{j:\lambda_j > 0} \lambda_j, \text{ and } \Lambda^- = \sum_{j:\lambda_j < 0} (-\lambda_j).$$

Then $\Lambda^- > 0$, $\Lambda^+ > 1$ and $\Lambda^+ - \Lambda^- = 1$. Since $\boldsymbol{x} = \Lambda^+ \boldsymbol{x}^+ - \Lambda^- \boldsymbol{x}^-$, we have

$$\boldsymbol{x}^+ = \frac{1}{\Lambda^+} \boldsymbol{x} + \frac{\Lambda^+ - 1}{\Lambda^+} \boldsymbol{x}^-.$$

Hence $\boldsymbol{x}^+$ is written by a convex combination of $\boldsymbol{x}$ and $\boldsymbol{x}^-$, where $\boldsymbol{x}, \boldsymbol{x}^- \in F \subseteq K$ is easily proved. Moreover, $g(\boldsymbol{x}^+) = 0$ since $\boldsymbol{x}^+ \in \mathrm{conv}(\{\boldsymbol{x}^1, \ldots, \boldsymbol{x}^n\})$. Therefore $g(\boldsymbol{x}) = 0$ by Claim 4.3. ■

Proof of Theorem 4.1 (ii) : For $j = 2, \ldots, n$, $\widehat{\boldsymbol{x}}^j = \frac{1}{2}(\boldsymbol{x}^1 + \boldsymbol{x}^j) \in F$ since F is convex. Hence

$$g(\widehat{\boldsymbol{x}}^j) = \frac{1}{2} g(\boldsymbol{x}^1) + \frac{1}{2} g(\boldsymbol{x}^j) \ (= 0).$$

More concretely, we have

$$\frac{1}{4}(\boldsymbol{x}^1 + \boldsymbol{x}^j)^T \boldsymbol{Q} (\boldsymbol{x}^1 + \boldsymbol{x}^j) - \frac{1}{2} \boldsymbol{q}^T (\boldsymbol{x}^1 + \boldsymbol{x}^j) + \pi = \frac{1}{2} \left((\boldsymbol{x}^1)^T \boldsymbol{Q} \boldsymbol{x}^1 - \boldsymbol{q}^T \boldsymbol{x}^1 + \pi \right) + \frac{1}{2} \left((\boldsymbol{x}^j)^T \boldsymbol{Q} \boldsymbol{x}^j - \boldsymbol{q}^T \boldsymbol{x}^j + \pi \right).$$

As a result, we have

$$(\boldsymbol{x}^j - \boldsymbol{x}^1)^T \boldsymbol{Q} (\boldsymbol{x}^j - \boldsymbol{x}^1) = 0 \ \text{ for } j = 2, \ldots, n. \tag{12}$$

By the positive semidefiniteness of $\boldsymbol{Q}$, $\boldsymbol{Q} = \boldsymbol{V}^T \boldsymbol{V}$ for some $\boldsymbol{V} \in R^{m \times n}$. Hence, (12) is equivalent to

$$\boldsymbol{V}(\boldsymbol{x}^j - \boldsymbol{x}^1) = 0 \ \text{ for } j = 2, \ldots, n.$$

Since $\boldsymbol{x}^1,\ldots,\boldsymbol{x}^n$ are affinely independent, $\boldsymbol{x}^2-\boldsymbol{x}^1,\ldots,\boldsymbol{x}^n-\boldsymbol{x}^1$ are linearly independent. Hence the rank of $\boldsymbol{V}$ is at most one. Since $\operatorname{rank}\boldsymbol{Q}=\operatorname{rank}\boldsymbol{V}$ holds in general [18], $\operatorname{rank}\boldsymbol{Q}\leq 1$. ■

Proof of Theorem 4.1 (iii) : By Theorem 4.1 (ii), $\boldsymbol{Q}=\boldsymbol{p}\boldsymbol{p}^T$ for some $\boldsymbol{p}\in R^n$ with $\boldsymbol{p}\neq\boldsymbol{0}$. Then by (12), we have $\boldsymbol{p}^T\left(\boldsymbol{x}^j-\boldsymbol{x}^1\right)=0$ $(j=2,\ldots,n)$. Let $\alpha=\boldsymbol{p}^T\boldsymbol{x}^1(=\boldsymbol{p}^T\boldsymbol{x}^2=\cdots=\boldsymbol{p}^T\boldsymbol{x}^n)$. Then $F=K\cap\{\boldsymbol{x}\mid\boldsymbol{p}^T\boldsymbol{x}=\alpha\}$. On the other hand,

$$\boldsymbol{q}^T\boldsymbol{x}=\boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x}+\pi=(\boldsymbol{p}^T\boldsymbol{x})^2+\pi=\alpha^2+\pi$$

for any $\boldsymbol{x}\in F$. Hence, $\boldsymbol{p}^T\boldsymbol{x}=\alpha$ and $\boldsymbol{q}^T\boldsymbol{x}=\alpha^2+\pi$ define the same hyperplane and there is $\beta\in R$ such that $\boldsymbol{q}=\beta\boldsymbol{p}$ and $\alpha^2+\pi=\beta\alpha$. Therefore,

$$\begin{aligned}\boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x}-\boldsymbol{q}^T\boldsymbol{x}+\pi &= \left(\boldsymbol{p}^T\boldsymbol{x}\right)^2-\beta\boldsymbol{p}^T\boldsymbol{x}+\beta\alpha-\alpha^2\\ &= \left(\boldsymbol{p}^T\boldsymbol{x}-\alpha\right)\left(\boldsymbol{p}^T\boldsymbol{x}-\beta+\alpha\right),\end{aligned}$$

and completes the proof. ■

4.2 Proof of Theorem 3.1

Since $F\subseteq K_{\mathrm{CQI}}(F)$ holds in general and $K_{\mathrm{CQI}}(F)$ is convex, we have $\mathrm{GSTAB}(B)\subseteq\mathrm{GTH}(B)$. To show $\mathrm{GTH}(B)\subseteq\mathrm{QGSTAB}(B)$, let $\boldsymbol{x}\in\mathrm{GTH}(B)$ and (C^+,C^-) a biclique of B. Let $\boldsymbol{p}\in R^V$ be defined by

$$p_i=\begin{cases}1 & (i\in C^+),\\ -1 & (i\in C^-),\\ 0 & (i\notin C^+\cup C^-)\end{cases}$$

and $\boldsymbol{M}=\boldsymbol{p}\boldsymbol{p}^T$. Then $M_{ij}=0$ $((i,j),(j,i)\notin E\cup A\cup D)$ by definition of the biclique, and $\boldsymbol{M}\succeq\boldsymbol{O}$. Hence, we have

$$\boldsymbol{x}^T\boldsymbol{M}\boldsymbol{x}-2\sum_{(i,j)\in A}M_{ij}x_i-2\sum_{(i,j)\in D}M_{ij}(x_i+x_j-1)-\sum_{i\in V}M_{ii}x_i\ \leq 0.$$

Moreover, if we define $\boldsymbol{x}(C)=\sum_{i\in C}x_i$ for $C\subseteq V$ then

- $\boldsymbol{p}^T\boldsymbol{x}=\boldsymbol{x}(C^+)-\boldsymbol{x}(C^-)$, $\boldsymbol{x}^T\boldsymbol{M}\boldsymbol{x}=(\boldsymbol{p}^T\boldsymbol{x})^2=(\boldsymbol{x}(C^+)-\boldsymbol{x}(C^-))^2$,
- $\displaystyle\sum_{(i,j)\in A}M_{ij}x_i=\sum_{i\in C^+,j\in C^-}p_ip_jx_i=-\sum_{i\in C^+,j\in C^-}x_i=-|C^-|\boldsymbol{x}(C^+)$,
- $\displaystyle\sum_{(i,j)\in D}M_{ij}(x_i+x_j-1)=\sum_{i,j\in C^-}p_ip_j(x_i+x_j-1)=(|C^-|-1)\boldsymbol{x}(C^-)-\frac{|C^-|\left(|C^-|-1\right)}{2}$,
- $\displaystyle\sum_{i\in V}M_{ii}x_i=\boldsymbol{x}(C^+\cup C^-)=\boldsymbol{x}(C^+)+\boldsymbol{x}(C^-)$.

Hence, we have

$$\begin{aligned}
0 \geq\ & \boldsymbol{x}^T \boldsymbol{M} \boldsymbol{x} - 2\sum_{(i,j)\in A} M_{ij}x_i - 2\sum_{(i,j)\in D} M_{ij}(x_i + x_j - 1) - \sum_{i\in V} M_{ii}x_i \\
=\ & (\boldsymbol{x}(C^+) - \boldsymbol{x}(C^-))^2 + 2|C^-|\boldsymbol{x}(C^+) - 2(|C^-| - 1)\boldsymbol{x}(C^-) + |C^-|(|C^-| - 1) - \boldsymbol{x}(C^+) - \boldsymbol{x}(C^-) \\
=\ & (\boldsymbol{x}(C^+) - \boldsymbol{x}(C^-))^2 + (2|C^-| - 1)\boldsymbol{x}(C^+) - (2|C^-| - 1)\boldsymbol{x}(C^-) + |C^-|(|C^-| - 1) \\
=\ & (\boldsymbol{x}(C^+) - \boldsymbol{x}(C^-) + |C^-|)(\boldsymbol{x}(C^+) - \boldsymbol{x}(C^-) + |C^-| - 1).
\end{aligned}$$

Therefore,

$$-|C^-| \leq \boldsymbol{x}(C^+) - \boldsymbol{x}(C^-) \leq 1 - |C^-| \tag{13}$$

holds. Since (13) is equivalent to

$$0 \leq \sum_{i\in C^+} x_i + \sum_{i\in C^-} (1 - x_i) \leq 1,$$

$\boldsymbol{x}$ satisfies the biclique inequalities and $\boldsymbol{x} \in \mathrm{QGSTAB}(B)$.

4.3 Proof of Theorem 3.3

Let $B = (V; E, A, D)$ be a closed bidirected graph and $S \subseteq V$. Then a reflection of B at S, denoted by $B{:}S$, is a bidirected graph obtained from B by reversing signs of the i side of all edges incident to i for $i \in S$. $B{:}S$ is also closed and the system (7) of $B{:}S$ is obtained from that of B by replacing all of x_i by $1 - x_i$ for $i \in S$. To show more relations, for $\boldsymbol{x} \in R^V$ and $S \subseteq V$, we define $L_S(\boldsymbol{x}) \in R^V$ as $L_S(\boldsymbol{x})_i = 1 - x_i$ if $i \in S$, $L_S(\boldsymbol{x})_i = x_i$ otherwise. Moreover, for a set $\Omega \in R^V$ and $S \subseteq V$, we define $L_S(\Omega) = \{L_S(\boldsymbol{x}) \in R^V \mid \boldsymbol{x} \in \Omega\}$.

Lemma 4.4 *For a closed bidirected graph $B = (V; E, A, D)$ and $S \subseteq V$,*

$$L_S(\mathrm{GSTAB}(B)) = \mathrm{GSTAB}(B{:}S), \tag{14}$$

$$L_S(\mathrm{QGSTAB}(B)) = \mathrm{QGSTAB}(B{:}S), \text{ and} \tag{15}$$

$$L_S(\mathrm{GTH}(B)) = \mathrm{GTH}(B{:}S). \tag{16}$$

Proof : Since 0-1 solutions of $B{:}S$ are obtained from those of B by reversing 0s and 1s in the i-th coordinate for $i \in S$, (14) follows. (15) also follows since any biclique inequality for $B{:}S$ is obtained from those of B by replacing x_i by $1 - x_i$ for $i \in S$. Hence we are left to prove (16).

Note that it is enough to show that (16) is true for a singleton $S = \{k\}$. We denote $B{:}\{k\} = (V, E', A', D')$. Let $\boldsymbol{x} \in L_{\{k\}}(\mathrm{GTH}(B))$ and $\boldsymbol{y} = L_{\{k\}}(\boldsymbol{x}) \in \mathrm{GTH}(B)$. Then $\boldsymbol{x} = \boldsymbol{D}^k\boldsymbol{y} + \boldsymbol{e}_k$, where $\boldsymbol{D}^k \in \mathcal{S}^V$ is the diagonal matrix such that $D^k_{kk} = -1$ and $D^k_{ii} = 1$ for $i \neq k$. Take any $\boldsymbol{M'} \in \mathcal{S}^V$ with $M'_{ij} = 0$ $((i,j),(j,i) \notin E' \cup A' \cup D')$ and $\boldsymbol{M'} \succeq \boldsymbol{O}$. If we define $\boldsymbol{M} = \boldsymbol{D}^k\boldsymbol{M'}\boldsymbol{D}^k$ then

$M_{ij} = 0$ ($(i,j),(j,i) \notin E \cup A \cup D$) and $\boldsymbol{M} \succeq \boldsymbol{O}$. Moreover

$$
\begin{aligned}
& \boldsymbol{x}^T \boldsymbol{M}' \boldsymbol{x} - 2 \sum_{(i,j) \in A'} M'_{ij} x_i - 2 \sum_{(i,j) \in D'} M'_{ij} (x_i + x_j - 1) - \sum_{i \in V} M'_{ii} x_i \\
= \; & \boldsymbol{y}^T \boldsymbol{M} \boldsymbol{y} + 2 \sum_{i \neq k} M'_{ik} y_i - 2 M'_{kk} y_k + M'_{kk} \\
& - 2 \sum_{\substack{(i,j) \in A' \\ i,j \neq k}} M'_{ij} y_i - 2 \sum_{(k,j) \in A'} M'_{kj} (1 - y_k) - 2 \sum_{(i,k) \in A'} M'_{ik} y_i \\
& - 2 \sum_{\substack{(i,j) \in D' \\ i,j \neq k}} M'_{ij} (y_i + y_j - 1) - 2 \sum_{(i,k) \in D'} M'_{ik} (y_i + 1 - y_k - 1) \\
& - \sum_{i \neq k} M'_{ii} y_i - M'_{kk} (1 - y_k) \\
= \; & \boldsymbol{y}^T \boldsymbol{M} \boldsymbol{y} + 2 \sum_{i \neq k} M'_{ik} y_i \\
& - 2 \sum_{\substack{(i,j) \in A' \\ i,j \neq k}} M'_{ij} y_i - 2 \sum_{(k,j) \in A'} M'_{kj} (1 - y_k) - 2 \sum_{(i,k) \in A'} M'_{ik} y_i \\
& - 2 \sum_{\substack{(i,j) \in D' \\ i,j \neq k}} M'_{ij} (y_i + y_j - 1) - 2 \sum_{(i,k) \in D'} M'_{ik} (y_i + 1 - y_k - 1) \\
& - \sum_{i \neq k} M'_{ii} y_i - M'_{kk} y_k .
\end{aligned}
$$

For (i,j) with $i,j \neq k$,

- If $(i,j) \in E'$ then $(i,j) \in E$ and the coefficient of y_i is 0.
- If $(i,j) \in A'$ then $(i,j) \in A$ and we have a term $-2M'_{ij} y_i = -2M_{ij} y_i$.
- If $(i,j) \in D'$ then $(i,j) \in D$ and we have a term $-2M'_{ij}(y_i + y_j - 1) = -2M_{ij}(y_i + y_j - 1)$.

For $i \in V \setminus \{k\}$,

- If $(i,k),(k,i) \notin E' \cup A' \cup D'$ then $M'_{ik} = 0$. Moreover, we have $(i,k) \notin E \cup A \cup D$ and $M_{ik} = -M'_{ik} = 0$.
- If $(i,k) \in E'$ then $(i,k) \in A$ and we have a term $2M'_{ik} y_k = -2M_{ik} y_k$.
- If $(i,k) \in A'$ then $(i,k) \in E$ and we have $(2M'_{ik} - 2M'_{ik}) y_k = 0$.
- If $(k,i) \in A'$ then $(k,i) \in D$ and we have a term $2M'_{ki} y_i - 2M'_{ik}(1 - y_k) = 2M'_{ki}(y_i + y_k - 1) = -2M_{ki}(y_i + y_k - 1)$.
- If $(i,k) \in D'$ then $(k,i) \in A$ and we have a term $2M'_{ik} y_i - 2M'_{ik}(y_i - y_k) = 2M'_{ik} y_k = -2M_{ik} y_k$.

For $i \in V$, $M'_{ii} = M_{ii}$. In summary,

$$\begin{aligned} & \boldsymbol{x}^T \boldsymbol{M}' \boldsymbol{x} - 2\sum_{(i,j)\in A'} M'_{ij} x_i - 2\sum_{(i,j)\in D'} M'_{ij}(x_i + x_j - 1) - \sum_{i\in V} M'_{ii} x_i \\ = \quad & \boldsymbol{y}^T \boldsymbol{M} \boldsymbol{y} - 2\sum_{(i,j)\in A} M_{ij} y_i - 2\sum_{(i,j)\in D} M_{ij}(y_i + y_j - 1) - \sum_{i\in V} M_{ii} y_i \leq 0. \end{aligned}$$

Hence we have $\boldsymbol{x} \in \mathrm{GTH}(B{:}S)$, and $L_{\{k\}}(\mathrm{GTH}(B)) \subseteq \mathrm{GTH}(B{:}S)$ is proved. The converse can be proved in the same way, and (16) holds. ■

Let $\boldsymbol{x} \in \{0,1\}^V$ be a 0-1 solution of B and $S = \{i \in V \mid x_i = 1\}$. Then, as shown in [29] or proved easily, $B{:}S = (V; E', A', D')$ satisfies $D' = \emptyset$. By the fact and Lemma 4.4, we may assume that a bidirected graph $B = (V; E, A, D)$ satisfies $D = \emptyset$ and the proof of Theorem 3.3 will be given under the assumption.

Let $F \subseteq \mathrm{GTH}(B)$ be a facet of $\mathrm{GTH}(B)$. $\mathrm{GTH}(B)$ is full-dimensional by Theorem 2.9. Hence, by Theorem 4.1, we have a representation

$$\boldsymbol{x}^T \boldsymbol{M} \boldsymbol{x} - 2\sum_{(i,j)\in A} M_{ij} x_i - \sum_{i\in V} M_{ii} x_i = \left(\boldsymbol{p}^T \boldsymbol{x} - \alpha\right)\left(\boldsymbol{p}^T \boldsymbol{x} - \beta + \alpha\right) = 0 \quad (^\forall \boldsymbol{x} \in F), \tag{17}$$

where $\boldsymbol{M} = \boldsymbol{p}\boldsymbol{p}^T \succeq \boldsymbol{O}$ and $M_{ij} = 0$ $((i,j),(j,i) \notin E \cup A)$. Note that $\boldsymbol{M} \neq \boldsymbol{O}$ since otherwise $\boldsymbol{x}^T \boldsymbol{M} \boldsymbol{x} - 2\sum_{(i,j)\in A} M_{ij} x_i - \sum_{i\in V} M_{ii} x_i = 0$ for any $\boldsymbol{x}$. From (17), we have

$$p_i^2 + 2\sum_{j:(i,j)\in A} p_i p_j = \beta p_i \ (i \in V), \tag{18}$$

$$\alpha(\beta - \alpha) = 0. \tag{19}$$

Let $C = \{i \in V \mid p_i \neq 0\} (\neq \emptyset)$. Then (18) is equivalent to

$$p_i + 2\sum_{j:(i,j)\in A} p_j = \beta \ (i \in C) \tag{20}$$

and $p_i = 0$ $(i \notin C)$.

Claim 4.5 *C induces a complete subgraph, that is, for any distinct $i, j \in C$, $(i,j) \in E \cup A$ or $(j,i) \in E \cup A$.*

Proof: $i, j \in C$ and $(i,j),(j,i) \notin E \cup A$ imply $p_i p_j = M_{ij} = 0$, contradicting $p_i \neq 0$ and $p_j \neq 0$. ■

Claim 4.6 *For $i, j \in C$, define a relation*

$$i \leq_A j \ \Leftrightarrow \ i = j \text{ or } (i,j) \in A.$$

Then $P_A = (C, \leq_A)$ forms a partially ordered set.

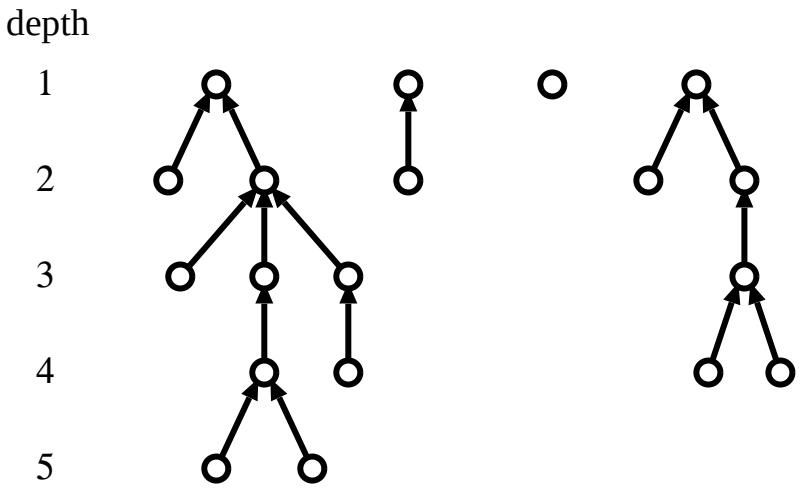

Figure 1: An Example of the Hasse Diagram of P_A.

Proof : $i \leq_A i$ holds by definition. Since B has no two edges incident to the same ends, $i \leq_A j$ and $j \leq_A i$ imply $i = j$. Finally, by the transitivity in B, $i \leq_A j$ and $j \leq_A k$ imply $i \leq_A k$. ■

We write $i <_A j$ if $i \leq_A j$ and $i \neq j$. As commonly used, if $i <_A j$ and there is no $k \in C$ such that $i <_A k <_A j$ then we say that j covers i or i is covered by j.

Claim 4.7 *For $i \in C$, i is covered by at most one element.*

Proof : Suppose i is covered by j and k. Then, by Claim 4.5, $(j,k) \in E \cup A$ or $(k,j) \in E \cup A$. However, $(j,k),(k,j) \notin E$ by the transitivity in B and $(j,k),(k,j) \notin A$ since otherwise j or k does not cover i. ■

See Figure 1 for an example of the Hasse diagram of P_A. We define a depth for each $i \in C$. Maximal elements have depth one, and inductively, elements covered by elements of depth k have depth $k+1$. We denote by $\mathrm{ODD}(C)$ (resp. $\mathrm{EVEN}(C)$) the set of elements of an odd (resp. even) depth.

Claim 4.8 *The solution of the system* (20) *is uniquely determined by*

$$p_i = \begin{cases} \beta & \text{if } i \in \mathrm{ODD}(C), \\ -\beta & \text{if } i \in \mathrm{EVEN}(C). \end{cases} \tag{21}$$

Proof : The proof will be done by induction. For i of depth one, i is a maximal element and (20) for i is equivalent to $p_i = \beta$. For $i \in C$ of depth at least two, inductively, (20) for i is equivalent to

$$\begin{aligned}
\beta &= p_i + 2 \sum_{j:(i,j)\in A} p_j \\
&= p_i + 2 \sum_{\substack{j:(i,j)\in A \\ j\in \mathrm{ODD}(C)}} p_j + 2 \sum_{\substack{j:(i,j)\in A \\ j\in \mathrm{EVEN}(C)}} p_j \\
&= p_i + 2 \sum_{\substack{j:(i,j)\in A \\ j\in \mathrm{ODD}(C)}} \beta - 2 \sum_{\substack{j:(i,j)\in A \\ j\in \mathrm{EVEN}(C)}} \beta \\
&= \begin{cases} p_i & \text{if } i \in \mathrm{ODD}(C), \\ p_i + 2\beta & \text{if } i \in \mathrm{EVEN}(C). \end{cases}
\end{aligned}$$

The last equality follows from Claim 4.7. Hence (21) is proved. ■

The inequality

$$\boldsymbol{x}^T \boldsymbol{M} \boldsymbol{x} - 2 \sum_{(i,j)\in A} M_{ij} x_i - \sum_{i\in V} M_{ii} x_i = \left(\boldsymbol{p}^T \boldsymbol{x} - \alpha\right)\left(\boldsymbol{p}^T \boldsymbol{x} - \beta + \alpha\right) \le 0 \tag{22}$$

is equivalent to

$$\beta - \alpha \le \sum_{i\in C} p_i x_i \le \alpha \tag{23}$$

or

$$\alpha \le \sum_{i\in C} p_i x_i \le \beta - \alpha. \tag{24}$$

Since (19) implies $\alpha = \beta$ or $\alpha = 0$, (23) and (24) are equivalent. Here we only consider (23). If $\alpha = \beta$ then (23) becomes

$$0 \le \sum_{i\in C} p_i x_i \le \beta, \tag{25}$$

while if $\alpha = 0$ then (23) becomes

$$\beta \le \sum_{i\in C} p_i x_i \le 0. \tag{26}$$

(25) and (26) are also equivalent because (25) implies $\beta > 0$ and (26) implies $\beta < 0$. In summary, (22) is equivalent to

$$0 \le \sum_{i\in C} p_i x_i \le \beta. \tag{27}$$

Claim 4.8 says $\beta \neq 0$. We can assume $\beta = 1$. Hence, (27) is equivalent to

$$0 \le \sum_{i\in C} p_i x_i \le 1, \tag{28}$$

where $p_i = 1$ or -1. We will end with the proof of Theorem 3.3 to show that both of the two inequalities in (28) are a summation of biclique inequalities. Since we assume $D = \emptyset$, any biclique is one of the following forms:

$$\begin{aligned}
&\text{TYPE I} && \sum_{i \in C^+} x_i \leq 1, \\
&\text{TYPE II} && \sum_{j \in C^+} x_j - x_i \leq 0, \\
&\text{TYPE II'} && -\sum_{j \in C^+} x_j + x_i \geq 0.
\end{aligned}$$

For a partially ordered set $(C, \leq_A)$ and $i \in C$, we define $C_A(i) = \{j \in C \mid j \text{ is covered by } i\}$.

Claim 4.9 $\sum_{i \in C} p_i x_i \leq 1$ *is a summation of biclique inequalities.*

Proof : Let C_1 be the set of maximal elements. Then

$$C = C_1 \cup \bigcup_{i \in \mathrm{EVEN}(C)} (\{i\} \cup C_A(i))$$

forms a partition of C.

- C_1 is a biclique of the form $(C_1, \emptyset)$ and $\sum_{i \in C_1} x_i \leq 1$ is the corresponding biclique inequality (TYPE I).
- For $i \in \mathrm{EVEN}(C)$, $(C_A(i), \{i\})$ is a biclique and $\sum_{j \in C_A(i)} x_j - x_i \leq 0$ is the corresponding biclique inequality (TYPE II).

Since

$$\sum_{i \in C} p_i x_i = \sum_{i \in C_1} x_i + \sum_{i \in \mathrm{EVEN}(C)} \left(\sum_{j \in C_A(i)} x_j - x_i \right),$$

$\sum_{i \in C} p_i x_i \leq 1$ is a summation of the biclique inequalities. Figure 2 explains the proof for the Hasse diagram of Figure 1. ■

Claim 4.10 $\sum_{i \in C} p_i x_i \geq 0$ *is a summation of biclique inequalities.*

Proof :

$$C = \bigcup_{i \in \mathrm{ODD}(C)} (\{i\} \cup C_A(i))$$

forms a partition of C, and for $i \in \mathrm{ODD}(C)$, $(C_A(i), \{i\})$ is a biclique and $-\sum_{j \in C_A(i)} x_j + x_i \geq 0$ is the corresponding biclique inequality (TYPE II'). Since

$$\sum_{i \in C} p_i x_i = \sum_{i \in \mathrm{ODD}(C)} \left(-\sum_{j \in C_A(i)} x_j + x_i \right),$$

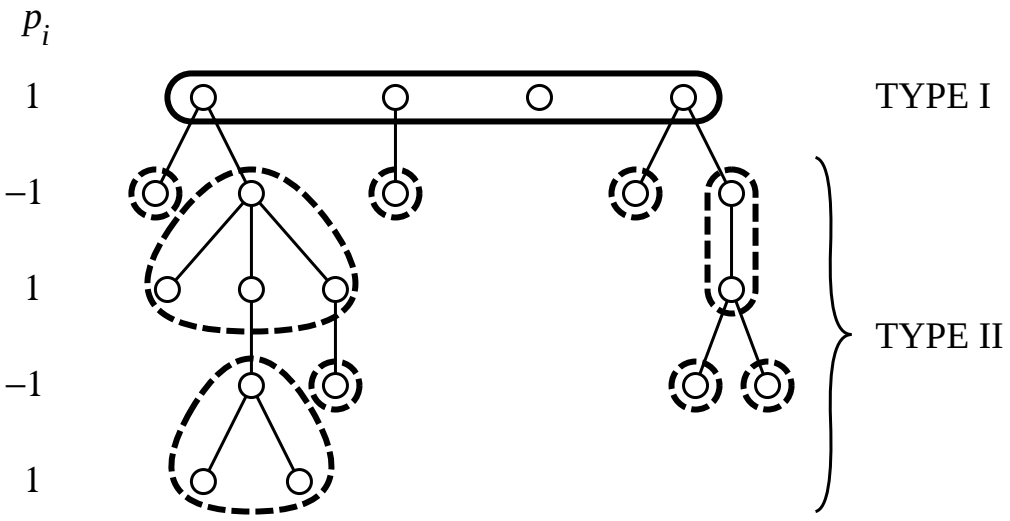

Figure 2: $\boldsymbol{p}^T\boldsymbol{x} \leq 1$ is a summation of biclique inequalities.

$\sum_{i \in C} p_i x_i \geq 0$ is a summation of the biclique inequalities. Figure 3 explains the proof for the Hasse diagram of Figure 1. ■

Suppose $\boldsymbol{p}^T\boldsymbol{x} \leq 1$ defines a facet of GTH(B) and that $\boldsymbol{p}^T\boldsymbol{x} \leq 1$ is a summation of two or more biclique inequalities. Then it never defines a facet of QGSTAB(B), which implies that it also never defines a facet of GTH(B) since GTH(B) $\subseteq$ QGSTAB(B). Hence C itself must be a biclique. Similarly, if $\boldsymbol{p}^T\boldsymbol{x} \geq 0$ defines a facet of GTH(B) then C itself must be a biclique. Therefore, every facet defining inequality of GTH(B) must be a biclique inequality, and the proof of Theorem 3.3 is completed.

4.4 Proof of Theorem 3.4

If a bidirected graph B is perfect then

$$\mathrm{GSTAB}(B) = \mathrm{GTH}(B) = \mathrm{QGSTAB}(B)$$

and GTH(B) is a polytope.

Conversely, suppose GTH(B) is a polytope. Then, by Theorem 3.3, facets of GTH(B) are biclique inequalities and GTH(B) = QGSTAB(B) holds. To complete the proof, we will show the following theorem :

Theorem 4.11 *Let $\widehat{\boldsymbol{x}}$ be a non-integral vertex of* QGSTAB(B). *Then $\widehat{\boldsymbol{x}} \notin$* GTH($B$).

Since we have had GTH(B) = QGSTAB(B), Theorem 4.11 implies that vertices of GTH(B) consist of integral vertices of QGSTAB(B). Since it is shown in [20] that conv(QGSTAB(B) $\cap \{0,1\}^V$) = GSTAB(B), GTH(B) = GSTAB(B) follows. Therefore, we have GSTAB(B) = QGSTAB(B) and B is perfect.

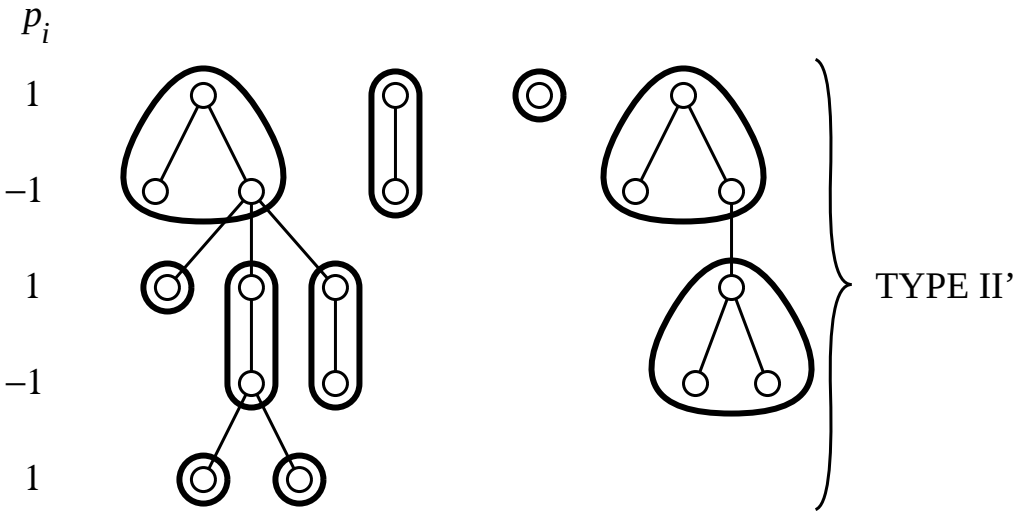

Figure 3: $\boldsymbol{p}^T\boldsymbol{x} \geq 0$ is a summation of biclique inequalities.

To prove Theorem 4.11, let $\widehat{\boldsymbol{x}}$ be a non-integral vertex of QGSTAB(B). Since $\widehat{\boldsymbol{x}}$ is a vertex of QGSTAB(G), there are $|V|$ bicliques $C_1, \ldots, C_{|V|}$ of B such that $\widehat{\boldsymbol{x}}$ is a unique solution of the system:

$$\boldsymbol{x}(C_k^+) - \boldsymbol{x}(C_k^-) = 1 - |C_k^-| \quad (k = 1, \ldots, |V|).$$

Let us denote the system by $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ and define

$$\boldsymbol{M} = \boldsymbol{A}^T\boldsymbol{A} = \sum_{k=1}^{|V|} (\chi^{C_k^+} - \chi^{C_k^-})(\chi^{C_k^+} - \chi^{C_k^-})^T,$$

where $\chi^S \in \{0,1\}^V$ is the incidence vector of $S \subseteq V$. Then $\boldsymbol{M}$ satisfies $M_{ij} = 0$ $((i,j),(j,i) \notin E \cup A \cup D)$ and $\boldsymbol{M}$ is positive definite since $\widehat{\boldsymbol{x}}$ is assumed to be a vertex of QGSTAB(B) and thus $\boldsymbol{A}$ has full-rank. Moreover,

$$\begin{aligned} f(\boldsymbol{M}, \widehat{\boldsymbol{x}}) &= \widehat{\boldsymbol{x}}^T\boldsymbol{M}\widehat{\boldsymbol{x}} - 2\sum_{(i,j)\in A} M_{ij}\widehat{x}_i - 2\sum_{(i,j)\in D} M_{ij}(\widehat{x}_i + \widehat{x}_j - 1) - \sum_{i\in V} M_{ii}\widehat{x}_i \\ &= \sum_{k=1}^{|V|} \left(\widehat{\boldsymbol{x}}(C_k^+) - \widehat{\boldsymbol{x}}(C_k^-) + |C_k^-|\right)\left(\widehat{\boldsymbol{x}}(C_k^+) - \widehat{\boldsymbol{x}}(C_k^-) + |C_k^-| - 1\right) = 0. \end{aligned} \tag{29}$$

Since $\widehat{\boldsymbol{x}}$ is supposed to be non-integral, there is $j \in V$ with $0 < \widehat{x}_j < 1$. Let us define

$$\boldsymbol{M}' = \boldsymbol{M} - \varepsilon\boldsymbol{E}_{jj}$$

for sufficiently small $\varepsilon > 0$. Since $\boldsymbol{M}$ is positive definite, $\boldsymbol{M}' \succeq \boldsymbol{O}$. Moreover, $M'_{ij} = 0$ $((i,j),(j,i) \notin E \cup A \cup D)$. Hence $f(\boldsymbol{M}', \boldsymbol{x}) \leq 0$ is valid for GTH(B). On the other hand, by (29) and $\widehat{x}_j(\widehat{x}_j - 1) < 0$,

$$f(\boldsymbol{M}', \widehat{\boldsymbol{x}}) = f(\boldsymbol{M}, \widehat{\boldsymbol{x}}) - \varepsilon\widehat{x}_j(\widehat{x}_j - 1) = -\varepsilon\widehat{x}_j(\widehat{x}_j - 1) > 0.$$

Therefore, $\widehat{\boldsymbol{x}} \notin$ GTH(B).

5 Conclusions

In this paper, we defined a convex set $\mathrm{GTH}(B)$ for a bidirected graph B, which is a generalization of $\mathrm{TH}(G)$ for an undirected graph G. We showed that $\mathrm{GTH}(B)$ has the almost same features as $\mathrm{TH}(G)$, which were shown by Grötschel, Lovász and Schrijver. However, our proofs were not straightforward generalizations. The definition of $\mathrm{GTH}(B)$ by convex quadratic inequalities played a significant role for our proofs. In fact, the representation of $\mathrm{TH}(G)$ by convex quadratic inequalities is a new result. Moreover, by using the representation, we can prove that $\mathrm{TH}(G)$ contains no non-integral vertices of $\mathrm{QSTAB}(G)$ such as Theorem 4.11, and also give alternative proofs for results of Grötschel, Lovász and Schrijver. See the companion paper [8] for details.

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