HIGHER DIRECT IMAGES OF DUALIZING SHEAVES OF LAGRANGIAN FIBRATIONS

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Abstract. We prove that the higher direct images of the dualizing sheaf of a Lagrangian fibration between smooth projective manifolds are isomorphic to the cotangent bundles of base space. As a corollary, we obtain that every Hodge number of the base space of a fibre space of an irreducible symplectic manifold is the same to that of a projective space if the base space is smooth.

1. Introduction

We begin with the definition of *Lagrangian fibrations*.

Definition **1.1.** *Let X be a Kähler manifold with a holomorphic symplectic form ω and S a normal variety. A proper surjective morphism $f : X \to S$ is said to be a Lagrangian fibration if a general fibre F of f is a Lagrangian submanifold with respect to ω, that is, the restriction of 2-form $\omega|_F$ is identically zero and* $\dim F = (1/2)\dim X$.

In this note, we investigate higher direct images of the dualizing sheaf of a Lagrangian fibration. Our result is the following.

Theorem **1.2.** *Let $f : X \to S$ be a Lagrangian fibration between smooth projective manifolds. Then*

$$R^i f_* \mathcal{O}_X \cong \Omega^i_S.$$

In particular, $R^i f_* \mathcal{O}_X$ are locally free.

Remark 1. It is known that the higher direct images of the dualizing sheaf are locally free if the discriminant locus is a normal crossing divisor [11, Theorem 2.6], [15, Theorem 2.6], and [16, Theorem 1]. However, there is a projective Lagrangian fibration whose discriminant locus is not a normal crossing divisor. The above theorem asserts that the higher direct images are locally free even if the discriminant locus is not necessarily normal crossing.

Example. Let $X := \mathrm{Hilb}^2 T$, where T is a projective $K3$ surface which has an elliptic fibration $T \to \mathbb{P}^1$. Then X admits a Lagrangian fibration over $\mathbb{P}^2$ and whose discriminant locus is not a normal crossing divisor.

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REMARK 2. The assumption projectivity is crucial in our present proof. The author suspect, however, that the most part of our results holds true for Lagrangian fibrations. Please see REMARK 4 in Section 3.

Combining Theorem 1.2 with [12, Theorem 2] and [13, Theorem 1], we obtain the following corollary.

COROLLARY **1.3.** *Let $f : X \to S$ be a fibre space of a projective irreducible symplectic manifold over a smooth projective base S. Then S is Fano and hodge numbers of S are the following:*

$$h^q(S, \Omega_S^p) = \begin{cases} 1 & (p = q) \\ 0 & (p \neq q). \end{cases}$$

Every Hodge number of S is the same as that of $\mathbb{P}^n$, where $n = \dim S$.

REMARK 3. Every Hodge number of a quadric hypersurface Q in $\mathbb{P}^{2n}$ is same to that of $\mathbb{P}^{2n-1}$ and Q is not isomorphic to $\mathbb{P}^{2n-1}$ $(n \geq 2)$. The author does not know whether there exists an example such that a fibre space of an irreducible symplectic manifold $f : X \to S$ whose base space S is not isomorphic to a projective space. Recently, Miyaoka [2] announced that S is isomorphic to a projective space if f admits a local section. Note that if f admits a local section, then S is smooth. The author would like to state one question.

QUESTION **1.4.** *Under the assumptions of Theorem 1.2, we have*

$$R^{n-1} f_* \omega_{X/S} \cong T_S,$$

which is the dual of $R^1 f_* \mathcal{O} \cong \Omega_S^1$. Note that $\omega_{X/S}$ is the relative dualizing sheaf and $n = \dim S$. Then $R^{n-1} f_* \omega_{X/S}$ is nef? In particular, $R^{n-1} f_* \omega_{X/S}$ is ample if X is an irreducible symplectic manifold?

Note that it is known that $R^i f_* \omega_{X/S}$ is weakly positive.

COROLLARY **1.5.** *If $\dim X = 4$, then T_S is nef. In particular, T_S is ample if X is an irreducible symplectic manifold.*

This paper is organized as follows. In section 2, we prepare for the proof of Theorem 1.2. Theorem 1.2, Corollary 1.3 and Corollary 1.5 are proved in section 3. Section 4 is devoted to the classification of singular fibres over codimension one points which needs for the proof of Theorem 1.2.

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2. Preliminary

We correct definitions and fundamental material which are needed in the proof of Theorem 1.2. First we consider a relative compactification of an Abelian fibration.

Proposition 2.1. *Let $f^\circ : X^\circ \to \Delta^*(t_1)\times\Delta^{n-1}(t_2,\cdots,t_n)$ be a smooth projective Abelian fibration. Assume that f° has the following properties:*

(1) *f° has a projective relative compactification $f' : Y \to \Delta^n$.*
(2) *$f'^{-1}(p)$ has a reduced component for a general point p of the discriminat locus of f'.*
(3) *The monodoromy matrix T of $R^1f^\circ_*\mathbb{C}$ around $\Delta^*(t_1)$ is unipotent.*
(4) *The rank of $T-I$ is at most one, where I is the identity matrix.*

Then there exists a projective relative compactification $f : X \to \Delta^n$ which has the following properties:

(1) *Every singular fibre of f is an Abelian variety or a cycle of several copies of a $\mathbb{P}^1$-bundle over an Abelian variety. In later case, $E_i \cap E_j$ forms a section of the ruling of E_i and E_j, where E_i and E_j are irreducible components of a singular fibre.*
(2) *The morphism $f^{-1}(D) \to D$ is a locally trivial deformation, where D is the discriminat locus of f.*
(3) *K_X is f-trivial.*
(4) *Every fibre of f is reduced.*

Proof. This proposition is essentially deduced from [17, Theorem 5.3]. However, for a complete sake and the proof of projectivity of the relative compactification is not published, we give a proof. Let $\tau(t_1,\mathbf{t})$ be the period matrix of f° where $\mathbf{t}=(t_2,\cdots,t_n)$, m is the dimension of a fibre of f°, I_k is the identity matrix of rank k and δ is the polarization matrix of f. We consider T as an element of $\mathrm{Sp}(m,\mathbb{R})$ by the conjugation

$$T' := \begin{pmatrix} I_m & 0 \\ 0 & \delta^{-1} \end{pmatrix} T \begin{pmatrix} I_m & 0 \\ 0 & \delta \end{pmatrix}.$$

Let G be the semiproduct of $\mathbb{Z}$ and $\chi := \{a+b\delta; a,b \in \mathbb{Z}^{\oplus m}\}$ with commuting relations $(a+b\delta)\cdot\gamma = \gamma\cdot(a+b\delta)T'$ for $\gamma\in\mathbb{Z}$ and $a+b\delta\in\chi$. By the assumption, f' admits a meromorphic section locally over an open set V such that $\mathrm{codim}(\Delta^n\backslash V)\geq 2$. Since $\pi_1(V)\cong\pi_1(\Delta^n)=\{1\}$, f° admits a global section. Hence the original X°

is reconstructed as follows:

$$X^\circ = (\mathbb{H} \times \Delta^{n-1} \times \mathbb{C}^m)/G$$

where $\mathbb{H}$ is an upper half space. The action of G is defined by

$$\begin{array}{rccc} \gamma: & \mathbb{H} \times \Delta^{n-1} \times \mathbb{C}^m & \to & \mathbb{H} \times \Delta^{n-1} \times \mathbb{C}^m \\ & (s, \mathbf{t}, \mathbf{z}) & \mapsto & (s+1, t, \mathbf{z}(C\tau(\exp(2\pi i s), \mathbf{t}) + D)^{-1}) \\ a+b\delta: & \mathbb{H} \times \Delta^{n-1} \times \mathbb{C}^m & \to & \mathbb{H} \times \Delta^{n-1} \times \mathbb{C}^m \\ & (s, \mathbf{t}, \mathbf{z}) & \mapsto & (s, t, \mathbf{z} + (a+b\delta) \begin{pmatrix} \tau(\exp(2\pi i s), \mathbf{t}) \\ I_m \end{pmatrix}), \end{array}$$

where

$$T' = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

LEMMA **2.2.** *If the rank of $T - I_{2m}$ is zero, then there exists a smooth Abelian fibration $f : X \to \Delta^n$ which is a relative compactification of X°. This fibration has properties of Proposition 2.1 (1), (2), (3) and (4).*

PROOF. By the assumption, the monodoromy matrix $T = I_{2m}$. Therefore the period τ can be extended holomorphically over Δ^n. We construct X as follows:

$$X = (\Delta^n \times \mathbb{C}^m)/\chi,$$

where $\chi := \{a + b\delta; a, b \in \mathbb{Z}^{\oplus 2m}\}$ and the action of χ is

$$\begin{array}{rccc} a+b\delta: & \Delta^n \times \mathbb{C}^m & \to & \Delta^n \times \mathbb{C}^m \\ & (t_1, \mathbf{t}, \mathbf{z}) & \mapsto & (t_1, \mathbf{t}, \mathbf{z} + (a+b\delta) \begin{pmatrix} \tau(t_1, \mathbf{t}) \\ I_m \end{pmatrix}). \end{array}$$

Since $f : X \to \Delta^n$ is a smooth morphism, it is obvious that X satisfies the assertions of Lemma. □

We consider the case that the rank of $T - I_{2m}$ is one. Since the discriminant locus of g is smooth, $\lim_{t_1 \to 0} \tau(t_1, \mathbf{t})$ exists in a rational boundary component F_α of Siegel upper half plane and F_α does not depend $\mathbf{t}$. ([17, Remark (4.4)]). There exists a matrix $M \in \mathrm{Sp}(m, \mathbb{Z})$ which transform F_α to $F_{g'}$ (cf. [17, (2.3)]), where $F_{g'}$ is the rational boundary component of Siegel upper half plane defined by

$$F_{g'} := \left\{ \begin{pmatrix} 0 & 0 \\ 0 & Z'' \end{pmatrix}; Z'' \in \mathrm{GL}(m - g', \mathbb{C}), I_{m-g'} - Z''\bar{Z}'' > 0 \right\}.$$

Since T' is unipotent, the transformation of T' by M can be written

$$MT'M^{-1} = \begin{pmatrix} I_m & B \\ 0 & I_m \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}, \tag{1}$$

by [17, (2.6)]. Note that $b \in \mathbb{R}^1$ because the rank of $T - I_{2m}$ is one. From this representation of the monodoromy matrix $MT'M^{-1}$, the period matrix can be written as

$$M\tau(t_1, \mathbf{t}) = \begin{pmatrix} \tau'(t_1, \mathbf{t}) & \tau'''(t_1, \mathbf{t}) \\ {}^t\tau'''(t_1, \mathbf{t}) & \tau''(t_1, \mathbf{t}) \end{pmatrix} + \frac{1}{2\pi i} \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} \log t_1,$$

where each $\tau^*(t_1, \mathbf{t})$ be a holomorphic function on Δ^n. From [17, (2.9)], $\mathrm{Im}\tau' > 0$ and $\mathrm{Im}\tau'' > 0$. We denote each element of χM^{-1} as

$$\{(\alpha_1, \alpha_2, \beta_1, \beta_2) \in \chi M^{-1}; \alpha_1, \beta_1 \in \mathbb{R}^{m-1}, \alpha_2, \beta_2 \in \mathbb{R}^1\}.$$

Let $\bar{\chi}$ be the sublattice of χM^{-1} such that

$$\bar{\chi} := \{(\alpha_1, \alpha_2, \beta_1, \beta_2) \in \chi M^{-1}; \alpha_1 = \alpha_2 = \beta_1 = 0\}.$$

and $(0, 0, 0, c)$ its generator. We define a toric variety $\mathcal{M}$ for the construction X.

DEFINITION **2.3.** *Let σ_k be a cone defined by*

$$\sigma_k := \{(u, y) \in \mathbb{R}^2; ku \leq y \leq (k+1)u\}$$

Then $\mathbb{C}[\sigma_k^\vee \cap \mathbb{Z}^2]$ is isomorphic to $\mathbb{C}$-subalgebra of $\mathbb{C}[v^{\pm 1}, t_1^{\pm 1}]$ generated by monomials $\{t_1^{-k}v, t_1^{k+1}v^{-1}\}$ and there is a natural morphism $\mathrm{Spec}\mathbb{C}[\sigma_k^\vee \cap \mathbb{Z}^2] \to \mathrm{Spec}\mathbb{C}[t_1]$. We define the toric variety $\mathcal{M}$ as follows:

$$\mathcal{M} := \bigcup_{k \in \mathbb{Z}} (\mathrm{Spec}\mathbb{C}[\sigma_k^\vee \cap \mathbb{Z}^2]) \times_{\mathbb{C}^1} \Delta^1.$$

Note that $\mathcal{M}$ has a fibration $\mathcal{M} \to \Delta^1$ and whose singular fibre is an infinite chain of $\mathbb{P}^1$.

LEMMA **2.4.** *Let $\mathcal{M}^* := \mathcal{M} \times_{\Delta^1} \Delta^*$. Then $\mathcal{M}^* \cong \Delta^* \times \mathbb{C}^*$ and the variable v is considered to be a coordinate of $\mathbb{C}^*$. We define the action of χM^{-1} on $\Delta^{n-1} \times \mathbb{C}^{m-1} \times \mathcal{M}^*$ as follows:*

$$\Gamma_{(\alpha_1, \alpha_2, \beta_1, \beta_2)} : (\mathbf{t}, t_1, \mathbf{z}, v) \mapsto (\mathbf{t}, t_1, \mathbf{z} + A, Bv),$$

where

$$\begin{aligned} A &= \alpha_1\tau'(t_1, \mathbf{t}) + \beta_1 + \alpha_2\tau'''(t_1, \mathbf{t}) \\ B &= \exp\left(\frac{2\pi i}{c}(\alpha_1 {}^t\tau'''(t_1, \mathbf{t}) + \alpha_2\tau''(t_1, \mathbf{t}) + \beta_2)\right) {t_1}^{\alpha_2 b/c}. \end{aligned}$$

Then the quotient $(\Delta^{n-1} \times \mathcal{M}^* \times \mathbb{C}^{m-1})/\chi M^{-1} \cong X^\circ$.

PROOF. We bigin with to prove that the action is well defined. It is enough to show that $\alpha_2 b/c$ is an integer for every element of χM^{-1}. For each element x of χ, $xT' \in \chi$. Hence $\chi(T' - I_{2m}) \subset \chi$ and we have $\chi M^{-1}(MT'M^{-1} - I_{2m}) \subset \chi M^{-1}$. On the contrary, $(0, 0, 0, \alpha_2 b) = (\alpha_1, \alpha_2, \beta_1, \beta_2)(MT'M^{-1} - I_{2m})$. Therefore

$(0,0,0,\alpha_2 b) \in \bar{\chi}$ for every element of χM^{-1} and $\alpha_2 b/c$ is an integer. If we consider the morphism

$$\begin{aligned} \Delta^{n-1} \times \Delta^* \times \mathbb{C}^{m-1} \times \mathbb{C} &\to \Delta^{n-1} \times \Delta^* \times \mathbb{C}^{m-1} \times \mathbb{C}^* \\ (\mathbf{t}, t_1, \mathbf{z}, z) &\mapsto (\mathbf{t}, t_1, \mathbf{z}, \exp(2\pi i z/c)), \end{aligned}$$

and rewrite the construction of X° by $MT'M^{-1}$ and $M\tau$, we obtain that $\Delta^{n-1} \times \mathcal{M}^* \times \mathbb{C}^{m-1}/\chi M^{-1} \cong X^\circ$. □

LEMMA **2.5.** *The action of χM^{-1} in Lemma 2.4 can be extended on $\Delta^{n-1} \times \mathbb{C}^{n-1} \times \mathcal{M}$. This action is properly discontinuous and fixed points free. The quotient X is a relative compactification of X° and $f : X \to \Delta^n$ has properties of Proposition 2.1 (1), (2), (3) and (4).*

PROOF. By definition, every singular fibre of $\Delta^{n-1} \times \mathcal{M} \times \mathbb{C}^{n-1} \to \Delta^n$ is the product of an infinite chain of $\mathbb{P}^1$ and $\mathbb{C}^{n-1}$. We denote each irreducible component of a singular fibre by V_k. Since the action of χM^{-1} on $\mathcal{M}$ maps $\mathrm{Spec}\mathbb{C}[\sigma_k \cap \mathbb{Z}^2]$ to $\mathrm{Spec}\mathbb{C}[\sigma_{k+(\alpha_2 b)/c} \cap \mathbb{Z}^2]$, V_k maps to $V_{k+\alpha_2 b/c}$ by this action. Hence this action is properly discontinuous and fixed point free. The quotient X is smooth and a relative compactification of X°. By construction, it is obvious that f satisfies the properties (1) and (2) of Propositon 2.1. We will prove that every fibre of f is reduced. By construction, the multiplicity of each component of a singular fibre is same. Thus it is enough to prove that every fibre has a reduced component. The section $\Delta^n \ni (\mathbf{t}, t_1) \mapsto (\mathbf{t}, t_1, 1, 0) \in \Delta^{n-1} \times \mathcal{M} \times \mathbb{C}^{n-1}$ induces a global section of f. Hence every fibre of f is reduced. Moreover, K_X is f-trivial because there is the holomorphic $n+m$-form

$$dt_1 \wedge \cdots \wedge dt_n \wedge dz_1 \wedge \cdots \wedge dz_{m-1} \wedge \frac{dv}{v}$$

which is χM^{-1}-invariant and nowhere vanishing on $\Delta^{n-1} \times \mathcal{M} \times \mathbb{C}^{m-1}$. □

The rest of assertions of Proposition 2.1 is deduced from the following Lemma.

LEMMA **2.6.** *Let $f : X \to \Delta^n$ be the relative compactification constructed in Lemma 2.2 or Lemma 2.5. Then f is projective.*

PROOF. By changing birational model, we may assume that there is a morphism $Y \to X$. Let H be a relative ample divisor on Y and H' its proper transform on X. Then H' is a big divisor on each component of a fibre of f. If $T = I_{2m}$, every fibre of f is an Abelian variety. Thus every big divisor on each component of a fibre of f is ample. Therefore H' is ample. If $T \neq I_{2m}$, we need the following claim:

CLAIM **2.7.** *Let $f : X \to \Delta^n$ be the relative compactification constructed in Lemma 2.5 and E_s an irreducible component of a singular fibre of f at s. Then there is a section e of the ruling of E_s such that $-K_{E_s} \equiv 2e$ and e is nef.*

PROOF. Let C be a smooth curve on Δ^n such that $C \cap D = \{s\}$, where D is the discriminat locus of f. We consider the restriction $f_C : X_C := X \times_{\Delta^n} C \to C$. We denote each irreducible component of the singular fibre of f_C by $E_{i,s}$. Let $E_{i+1,s}$ and $E_{i-1,s}$ are next components of $E_{i,s}$ and $F_{ij,s} := E_{i,s} \cap E_{j,s}$. Since K_{X_C} is f_C-trivial, $K_{E_{i,s}} \sim -F_{ii+1,s} - F_{ii-1,s}$. Each $E_{i,s}$ is a $\mathbb{P}^1$-bundle over an Abelian variety. Hence $-K_{E_{i,s}}$ is nef. In order to prove Claim, it is enough to show that $F_{ii+1,s}$ and $F_{ii-1,s}$ are numerically equivalent on $E_{i,s}$. Since X_s is a deformation retract of X_C, there is a corrasping morphism $c_t : X_t \to X_s$, where X_t is a general fibre of f_C. Let $E'_{i,s} := c_t^{-1}(E_{i,s})$ and $F'_{ij,s} := c_t^{-1}(F_{ij,s})$. Note that $E'_{i,s} \to E_{i,s}$ is a real blow up along $F_{ij,s}$ and $F'_{ij,s}$ is a S^1-bundle over $F_{ij,s}$. We consider the following Mayer-Vietoris sequences:

$$0 \to \mathbb{C}_{X_t} \to \oplus_{E'_{i,s}} \mathbb{C}_{E'_{i,s}} \to \oplus_{F'_{ij,s}} \mathbb{C}_{F'_{ij,s}} \to 0$$
$$0 \to \mathbb{C}_{X_s} \to \oplus_{E_{i,s}} \mathbb{C}_{E_{i,s}} \to \oplus_{F_{ij,s}} \mathbb{C}_{F_{ij,s}} \to 0$$

From the above sequence,

$$0 \to H^0(X_t) \to \oplus_{E'_{i,s}} H^0(E'_{i,s}) \to \oplus_{F'_{ij,s}} H^0(F'_{ij,s}) \stackrel{\alpha_1}{\to} H^1(X_t),$$

and

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
\to H^1(X_s) & \to & \oplus_{E_{i,s}} H^1(E_{i,s}) & \stackrel{\alpha_4}{\to} & \oplus_{F_{ij,s}} H^1(F_{ij,s}) & \to & \\
 & & \downarrow & & \downarrow & & \\
\stackrel{\alpha_1}{\to} H^1(X_t) & \stackrel{\alpha_2}{\to} & \oplus_{E'_{i,s}} H^1(E'_{i,s}) & \stackrel{\alpha_3}{\to} & \oplus_{F'_{ij,s}} H^1(F'_{ij,s}) & \to & H^2(X_t) \to \\
 & & \downarrow & & \downarrow & & \\
0 \to \oplus_{F_{ij,s}} H^0(F_{ij,s}) & \to & \oplus_{E_{i,s} \supset F_{ij,s}} H^0(F_{ij,s}) & \to & \oplus_{F_{ij,s}} H^0(F_{ij,s}) & \to 0 & \\
 & & \downarrow \beta & & \downarrow & & \\
 & & \oplus_{E_{i,s}} H^2(E_{i,s}) & \to & \oplus_{F_{ij,s}} H^2(F_{ij,s}) & &
\end{array}$$

Note that the each column is Gysin sequence. Then $\dim \mathrm{Im}(\alpha_1) = 1$ and $\dim \mathrm{Ker}(\alpha_4) = 2n-2$, because the singular fibre of f_C satisfies the properity (1) of Propositon 2.1. Since $\dim \mathrm{Im}(\alpha_1) + \dim \mathrm{Im}(\alpha_2) = 2n$, $\dim \mathrm{Ker}(\alpha_3) = 2n-1$. Hence β is not isomorphism. The morphism β sends

$$H^0(F_{ii-1,s}) \oplus H^0(F_{ii+1,s}) \ni (\lambda_1, \lambda_2) \mapsto (\lambda_1[F_{ii-1,s}] + \lambda_2[F_{ii+1,s}]) \in H^2(E_{i,s}).$$

If $F_{ii-1,s}$ and $F_{ii+1,s}$ is not numerically equivalent, β is injection. That derives a contradiction and we are done. □

We go back to the proof of Lemma. From Claim 2.7, the nef cone of each irreducible component of a singular fibre is generated by a section and divisors on an Abelian variety. Therefore every big divisor on each irreducible component of singular fibre is ample and projectivity of f can be shown similarly as in the case $T = I_{2m}$. □
We complete the proof of Proposition 2.1. □
For the using later, we prove the following Lemma.

LEMMA **2.8.** *Let $f : X \to \Delta^n$ be a projective Abelian fibration which satisfies the properties (1), (2) and (3) of Proposition 2.1. Then every bimeromorphic map $\Phi : X \dashrightarrow X$ which commutes with f can be extended to a biholomorphic morphism, that is, X is the unique relative minimal model.*

PROOF . We derive a contradiction assuming that Φ is not isomorphism. Since X is a relative minimal model and Φ commutes with f, Φ is an isomorphism over codimension one point of X. Let H be an ample divisor on X and H' the proper transform by Φ. If Φ is not isomorphism, then H' is not f-nef. Hence by [6, Theorem 3-1-1], there exists an extremal contraction $\nu : X \to \bar{X}$. Let ℓ be a rational curve which is contracted by ν and E an irreducible component of $f^{-1}(D)$ which contains ℓ. Since f satisfies the properties (1) and (2) of Proposition 2.1 and the discriminat locus D of f is simply connected, $E \to D$ factors $E \to A \to D$, where $E \to A$ is a $\mathbb{P}^1$-bundle and $A \to D$ is a smooth Abelian fibration. Hence ℓ must be a fibre of the ruling of E. Thus ν contracts every fibre of the ruling of E and the exceptional locus of ν has codimension one. That derives a contradiction. □

DEFINITION **2.9.** *Let $f : X \to \Delta^n$ be a projective Abelian fibration which satisfies the properties of Proposition 2.1 (1) - (4). We call X as a toroidal model of type I or II according to the singular fibre of f is an Abelian variety or a cycle of several copies of a $\mathbb{P}^1$-bundle over an Abelian variety.*

Next we note variations of Hodge structures. Let $f : X \to S$ be a projective morphism between smooth manifold. Assume that there exists an open set U of S such that f is smooth over U and $D := S \setminus U$ is a simple normal crossing divisor. Let $f^\circ : f^{-1}(U) \to U$ be the smooth part of f. We denote the upper canonical extension of $\mathcal{H} := R^i f^\circ_* \mathbb{C}$ by ${}^u\mathcal{H}$ and the lower canonical extension by ${}^\ell\mathcal{H}$. For the proof of Theorem 1.2, we need the following Lemma.

LEMMA **2.10.** *Let $f : X \to \Delta^n$ be a projective morphism from a smooth manifold X to a polydisk Δ^n. Assume that f has the following properties:*

(1) *The discriminant locus D of f is smooth and $E := (f^*D)_{\mathrm{red}}$ is a simple normal crossing divisor.*
(2) *The morphism $E \to D$ is a locally trivial deformation.*

Let $f^*D = \sum a_i E_i$. *Then*

$$ {}^u\mathcal{F}^1(\mathcal{H}) \cong f_*(\Omega^1_{X/\Delta^n}(\log E) \otimes \sum (a_i - 1)E_i). $$

PROOF. If $\dim S = 1$, then this assertion is [5, VIII]. Thus ${}^u\mathcal{F}^1(\mathcal{H})$ and $f_*(\Omega^1_{X/\Delta^n}(\log E) \otimes \sum (a_i - 1)D_i)$ are isomorphic in codimension one points of S. By the assumptions of f, [18, Theorem 3.5] and the argument in the proof of [11, Lemma 2.11], $f_*(\Omega^1_{X/\Delta^n}(\log E) \otimes \sum (a_i - 1)D_i)$ is locally free. Therefore we obtain the assertion of Lemma 2.10. □

3. Correspondence between higher direct images and cotangent bundles

In this section, we prove Theorem 1.2, Corollary 1.3 and Corollary 1.5. We begin with to consider the relation among $R^i f_* \mathcal{O}_X$. From the following lemma, it is enough to consider $R^1 f_* \mathcal{O}_X$ for the proof of Theorem 1.2.

LEMMA **3.1.** *Let* $f : X \to S$ *be a Lagrangian fibration between smooth projective manifolds. Assume that* $R^1 f_* \mathcal{O}_X \cong \Omega^1_S$. *Then*

$$ R^i f_* \mathcal{O}_X \cong \Omega^i_S. $$

PROOF. Since $\omega_X \cong \mathcal{O}_X$, $R^i f_* \mathcal{O}_X$ are torsion free by [10, Theorem 2.1]. Thus $R^i f_* \mathcal{O}_X$ are reflexive sheaves by [11, Corollary 3.9]. Therefore it is enough to show that there exists an open set U of S such that $R^i f_* \mathcal{O}_X|_U \cong \Omega^i_U$ and $\operatorname{codim}(S \setminus U) \geq 2$. Let D be the discriminant locus of f. We choose an open set U of S such that D is smooth in U. Since a general fibre of f is an Abelian variety (Liouville's Theorem) there exists a morphism

$$ \wedge^i R^1 f_* \mathcal{O}_X|_U \xrightarrow{\alpha_i} R^i f_* \mathcal{O}_X|_U. $$

Then α is injective because $R^1 f_* \mathcal{O}_X|_U$ is torsion free. By [11, Theorem 2.6], $R^i f_* \mathcal{O}_X|_U \cong {}^\ell\mathcal{G}r^i(\mathcal{H})$. Considering exponents of the lower extension of $\mathcal{H}$, α_n is not surjective if α_i $(0 < i < n)$ is not surjective. By [10, Corollary 7.6], $R^n f_* \mathcal{O}_X \cong \omega_S$. Combining $R^1 f_* \mathcal{O}_X \cong \Omega^1_S$, α_n is isomorphism. Hence α_i is isomorphism for $1 \leq i \leq n$. □

REMARK 4. The reflexiveness of $R^i f_* \mathcal{O}_X$ is one of crucial point why we need projectivity. This comes from the following decomposition theorem.

THEOREM **3.2** ([11, Theorem 3.1]). *Let* $f : X \to S$ *be a surjective morphism between projective variety. If* X *is smooth, then*

$$ Rf_* \omega_X \sim_{\text{q.i.s.}} \sum R^i f_* \omega_X[-i]. $$

If the above theorem valid for a projective morphism, we can replace projectivity of X by projectivity of f in the assumptions of Theorem 1.2.

Theorem 1.2 can be deduced from the following proposition.

PROPOSITION **3.3.** *Let $f : X \to S$ be a projective Lagrangian fibration with a symplectic form ω. Then there exists an open set U of S such that*

$$R^1 f_* \mathcal{O}_X|_U \cong \Omega^1_U$$

and $\operatorname{codim}(S \setminus U) \geq 2$.

PROOF. We prove Proposition 3.3 in four steps.

STEP 1. First we consider the smooth part of f. Let $U_0 := S \setminus D$, where D is the discriminant locus of f. We consider the following exact sequence:

$$\begin{array}{ccccccccc} & & & & T_{f^{-1}(U_0)} & \to & f^*T_{U_0} & \to 0 \\ & & & & \downarrow \omega & & & \\ 0 \to & f^*\Omega^1_{U_0} & \to & \Omega^1_{f^{-1}(U_0)} & \to & \Omega^1_{f^{-1}(U_0)/U_0} & \to 0 \end{array}.$$

From the above diagram, ω defines a morphism

$$f^*T_{U_0} \to \Omega^1_{f^{-1}(U_0)/U_0}$$

because every fibre of f is a Lagrangian subvariety by [14, Theorem 1]. Since ω is nondegenerate, the above morphism is isomorphism. Therefore we obtain isomorphisms

$$T_{U_0} \cong f_*\Omega^1_{f^{-1}(U_0)/U_0} \cong \mathcal{F}^1(\mathcal{H}).$$

Taking the dual of above isomorphism,

$$R^1 f_* \mathcal{O}_X|_{U_0} \cong \Omega^1_{U_0}. \tag{2}$$

STEP 2. In order to prove Proposition 3.3, we extend the isomorphism (2) over codimension one point of S. Let s be a point of D such that S and D are smooth around s. In Step 2, 3 and 4, we will show that the isomorphism (2) can be extended on a neighborhood of s. In step 2, we treat the case that $f^{-1}(s)$ satisfies the properties of Theorem 4.1 (1) and $G_1 = G_2 = \{1\}$. In this case, there exists a polydisk $s \in \Delta^n$ of S and the restriction morphism $f : X_{\Delta^n} := X \times_S \Delta^n \to \Delta^n$ is a toroidal model of type I or II (See Definition 2.9). If f is a toroidal model of type I, then f is smooth and we obtain $R^1 f_* \mathcal{O}_{X_{\Delta^n}} \cong \Omega^1_{\Delta^n}$ around s. Thus we may assume that f is a toroidal model of type II. Let

$$X^\circ := \{x \in X_{\Delta^n} | f \text{ is smooth at } x\}.$$

We consider the following diagram:

$$\begin{array}{ccccccc} & & & & T_{X^\circ} & \to & f^*T_{\Delta^n}|_{X^\circ} \to 0 \\ & & & & \downarrow \omega & & \\ 0 \to & f^*\Omega^1_{\Delta^n}|_{X^\circ} & \to & \Omega^1_{X^\circ} & \to & \Omega^1_{X^\circ/\Delta^n} & \to 0 \end{array}.$$

From the above diagram, ω defines a morphism $f^*T_{\Delta^n}|_{X^\circ} \to \Omega^1_{X^\circ/\Delta^n}$ since every fibre of f is a Lagrangian subvariety by [14, Theorem 1]. We investigate the relation between $\Omega^1_{X^\circ/\Delta^n}$ and $\Omega^1_{X_{\Delta^n}/\Delta^n}(\log E)|_{X^\circ}$. Let us consider the following diagram:

$$\begin{array}{ccccccccc} 0 \to & & f^*\Omega^1_{\Delta^n}|_{X^\circ} & \to & \Omega^1_{X^\circ} & \to & \Omega^1_{X^\circ/\Delta^n} & & \to 0 \\ & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\ 0 \to & f^*\Omega^1_{\Delta^n}(\log D)|_{X^\circ} & \to & \Omega^1_{X_{\Delta^n}}(\log E)|_{X^\circ} & \to & \Omega^1_{X_{\Delta^n}/\Delta^n}(\log E)|_{X^\circ} & & \to 0. \end{array}$$

We obtain $\Omega^1_{X^\circ/S} \cong \Omega^1_{X_{\Delta^n}/\Delta^n}(\log E)|_{X^\circ}$ if $\mathrm{Coker}\alpha \cong \mathrm{Coker}\beta$. Let x be a point of $f^{-1}(D)$. From Definition 2.9, if we choose a suitable local parameter z_i at x and t_i at $f(x)$, f is written as

$$z_0 z_1 = t_1, \quad z_i = t_i \quad (2 \leq i),$$

on a suitable neighborhood of x. The generater of $\mathrm{Coker}\alpha$ is f^*dt_1/t_1 and the generaters of $\mathrm{Coker}\beta$ are dz_0/z_0 and dz_1/z_1. The morphsim $\mathrm{Coker}\alpha \to \mathrm{Coker}\beta$ sends

$$f^*dt_1/t_1 \mapsto dz_0/z_0 + dz_1/dz_1.$$

If $x \in X^\circ$, $z_0 \neq 0$ or $z_1 \neq 0$. Hence $\mathrm{Coker}\alpha \to \mathrm{Coker}\beta$ is an isomorphsim and we obtain $\Omega^1_{X^\circ/\Delta^n} \cong \Omega^1_{X_{\Delta^n}/\Delta^n}(\log E)|_{X^\circ}$. We conclude that

$$f^*T_{\Delta^n} \cong \Omega^1_{X_{\Delta^n}/\Delta^n}(\log E),$$

because $\mathrm{codim}(X \setminus X^\circ) \geq 2$. Since $f_*\Omega^1_{X_{\Delta^n}/\Delta^n}(\log E) \cong \mathcal{F}^1(\mathcal{H})$, we obtain

$$R^1 f_* \mathcal{O}_{X_{\Delta^n}} \cong \Omega^1_{\Delta^n}.$$

Step 3. In this step, we treat that $f^{-1}(s)$ satisfies the properties of Theorem 4.1 (1) and $G_1 \neq \{1\}$ or $G_2 \neq \{1\}$. By Theorem 4.1 (1), there exists a following diagram:

$$\begin{array}{ccccccccc} X_{\Delta^n} & \overset{\eta}{\to} & \bar{X}/G_1 & \overset{\nu_1}{\leftarrow} & \bar{X} & = & \tilde{X}/G_2 & \overset{\nu_2}{\leftarrow} & \tilde{X} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \Delta^n & = & \bar{\Delta}^n/G_1 & \leftarrow & \bar{\Delta}^n & = & \tilde{\Delta}^n/G_2 & \leftarrow & \tilde{\Delta}^n. \end{array}$$

First we consider the case $G_1 = \{1\}$ and $G_2 \neq \{1\}$. Since η is isomorphism on the smooth locus of $\tilde{X}/G$, $X_{\Delta^n} \cong \tilde{X}/G_2$. For a symplectic form ω on X, $\nu_2^*\omega$ is a symplectic form on $\tilde{X}$ because ν_2 is étale. By Step 2, $\nu_2^*\omega$ defines an G_2-equivariant ismorphism $R^1\tilde{f}_*\mathcal{O}_{\tilde{X}} \cong \Omega^1_{\tilde{\Delta}^n}$. Take the G_2-invariant part, we obtain

$$R^1 f_* \mathcal{O}_{X_{\Delta^n}} \cong \Omega^1_{\Delta^n}.$$

Next we consider the case $G_1 \neq \{1\}$. Let U_1 be the smooth locus of $\bar{X}/G$. Since η is isomorphism on U_1 and $\mathrm{codim}((\bar{X}/G) \setminus U_1) \geq 2$, we define $\eta_*\omega$ as the extension of $\omega|_{U_1}$. On the contrary, ν_1 is étale in codimension one, $\nu_1^*(\eta_*\omega)$ is a symplectic form on $\bar{X}$. According to the argument in the case $G_1 = \{1\}$, $\nu_1^*(\eta_*\omega)$ defines a G_1-equivariant isomorphsim

$$R^1 f_* \mathcal{O}_{\bar{X}} \cong \Omega^1_{\bar{\Delta}^n}.$$

Take a G_1-invariant part of the above morphism, we obtain

$$R^1 f_* \mathcal{O}_{X_{\Delta^n}} \cong \Omega^1_{\Delta^n}.$$

STEP 4. In this step, we treat that $f^{-1}(s)$ satisfies the properties of Theorem 4.1 (2). If we choose a suitable neighborfood Δ^n of s, there is an Abelian fibration $\tilde{f} : \tilde{X} \to \tilde{\Delta}^n$ and the restriction $f : X_{\Delta^n} := X \times_S \Delta^n \to \Delta^n$ which satisfy the following diagram:

$$\begin{array}{ccccc} X_{\Delta^n} & = & \tilde{X}/G & \overset{\tau}{\leftarrow} & \tilde{X} \\ \downarrow & & \downarrow & & \downarrow \\ \Delta^n & = & \tilde{\Delta}^n/G & \leftarrow & \tilde{\Delta}^n, \end{array}$$

where τ is an étale morphism. Since τ is étale, $\tilde{\omega} := \tau^*\omega$ is a symplectic form where ω is a symplectic form on X_{Δ^n}. If $\tilde{\omega}$ defines an isomorphism $R^1\tilde{f}_*\mathcal{O}_{\tilde{X}} \to \Omega^1_{\tilde{\Delta}^n}$ we obtain a $R^1 f_* \mathcal{O}_{X_{\Delta^n}} \cong \Omega^1_{\Delta^n}$ by taking a G-invariant part. Thus we consider the Abelian fibration $\tilde{f} : \tilde{X} \to \tilde{\Delta}^n$. By Theorem 4.1 (2), there is a birational morphism $\nu : W \to \tilde{X}$ such that $\tilde{f} \circ \nu : W \to \tilde{\Delta}^n$ satisfies the assumptions (1) and (2) of Lemma 2.10. Let $E := ((\tilde{f} \circ \nu)^* \tilde{D})_{red}$ and $(\tilde{f} \circ \nu)^* \tilde{D} = \sum a_i E_i$. We will show that $\nu^*\tilde{\omega}$ defines the morphism

$$(\tilde{f} \circ \nu)^* T_{\tilde{\Delta}^n} \to \Omega^1_{W/\tilde{\Delta}^n}(\log E) \otimes \sum (a_i - 1) E_i. \tag{3}$$

Let w be a point of $(\tilde{f} \circ \nu)^{-1}(\tilde{D})$. On a suitable neighborhood of w, $\tilde{f} \circ \nu$ is written as

$$z_0^{a_0} \cdots z_k^{a_k} = t_1, \quad z_{k+i} = t_i \quad (i \geq 2),$$

where z_i are local parameter at w and t_i are local parameter at $\tilde{f} \circ \nu(w)$. The morphism $T_W \to (\tilde{f} \circ \nu)^* T_{\tilde{\Delta}^n}$ is written as

$$\frac{\partial}{\partial z_j} \mapsto a_j z_0^{a_0} \cdots z_j^{a_j - 1} \cdots z_k^{a_k} \frac{\partial}{\partial t_1}.$$

Let $W_0 := W \setminus \mathrm{Sing}(\tilde{f} \circ \nu)^{-1}(\tilde{D})$. If $w \in W_0$, one of z_i $(0 \leq i \leq k)$ is nonzero. Thus

$$(\tilde{f} \circ \nu)^* T_{\tilde{\Delta}^n} \otimes \sum (1 - a_i) E_i|_{W_0} \subset \mathrm{Image}(T_W|_{W_0} \to (\tilde{f} \circ \nu)^* T_{\tilde{\Delta}^n}|_{W_0}).$$

We cosider the following diagram:

$$\begin{array}{ccccccccc}
0 \to & & T_{W/\tilde{\Delta}^n} & \to & T_W & \to & (\tilde{f}\circ\nu)^*T_{\tilde{\Delta}^n} & & \\
 & & & & \downarrow & & & & \\
 & & (\tilde{f}\circ\nu)^*\Omega^1_{\tilde{\Delta}^n} & \to & \Omega^1_W & \to & \Omega^1_{W/\tilde{\Delta}^n} & \to 0 & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 \to & (\tilde{f}\circ\nu)^*\Omega^1_{\tilde{\Delta}^n}(\log \tilde{D}) & \to & \Omega^1_W(\log E) & \to & \Omega^1_{W/\tilde{\Delta}^n}(\log E) & \to 0. & &
\end{array}$$

By [14, Theorem 1], the restriction of $\nu^*\tilde{\omega}$ to every irreducible component of every fibre is identically zero. Thus $\nu^*\tilde{\omega}(\alpha, *) = 0$ in $\Omega^1_{W/\tilde{\Delta}^n}$ for all elements $\alpha \in T_{W/\tilde{\Delta}^n}$. Hence $\nu^*\tilde{\omega}$ defines the morphism

$$(\tilde{f}\circ\nu)^*T_{\tilde{\Delta}^n} \otimes \sum(1-a_i)E_i|_{W_0} \to \Omega^1_{W/\tilde{\Delta}^n}(\log E)|_{W_0}.$$

Since $\mathrm{codim}(W \setminus W_0) \geq 2$, we obtain the morphsim (3). Let $\tilde{X}^\circ := \{\tilde{x} \in \tilde{X}; \tilde{f} \text{ is smooth at } \tilde{x}\}$ and V the open set of $\tilde{X}$ such that $\nu^{-1}(V) \to V$ is an isomorphism. Note that $a_i = 1$ if $E_i \cap \nu^{-1}(X^\circ) \neq \emptyset$. By similar argument in Step 2, we obtain

$$\Omega^1_{W/\tilde{\Delta}^n}(\log E) \otimes (\sum(a_i-1)E_i)|_{\nu^{-1}(\tilde{X}^\circ)} \cong \Omega^1_{\nu^{-1}(\tilde{X}^\circ)/\tilde{\Delta}^n}.$$

Since $\nu^*\tilde{\omega}$ is nondegenerate on $\nu^{-1}(V)$, the morphsim (3) is an isomorphism on $\nu^{-1}(\tilde{X}^\circ \cap V)$. Thus we obtain an isomorphism

$$\tilde{f}^*T_{\tilde{\Delta}^n}|_{\tilde{X}^\circ\cap V} \to \nu_*(\Omega^1_{W/\tilde{\Delta}^n}(\log E) \otimes \sum(a_i-1)E_i)|_{\tilde{X}^\circ\cap V}. \tag{4}$$

By Theorem 4.1 (2), every fibre of $\tilde{f}$ is reduced. Hence $X^\circ = X \setminus \mathrm{Sing}(\tilde{f}^{-1}(\tilde{D}))$ and $\mathrm{codim}(\tilde{X}\setminus\tilde{X}^\circ) \geq 2$. Combining with $\mathrm{codim}(\tilde{X}\setminus V) \geq 2$, $\mathrm{codim}(\tilde{X}\setminus(\tilde{X}^\circ\cap V)) \geq 2$. Since $\tilde{f}^*T_{\tilde{\Delta}^n}$ is locally free, we obtain the morphism (4) is an isomorphism on $\tilde{X}$ and

$$T_{\tilde{\Delta}^n} \cong \tilde{f}_*(\Omega^1_{W/\tilde{\Delta}^n}(\log E) \otimes \sum(a_i-1)E_i).$$

By Lemma 2.10, $\tilde{f}_*(\Omega^1_{W/\tilde{\Delta}^n}(\log E) \otimes \sum(a_i-1)E_i) \cong {}^u\mathcal{F}^1(\mathcal{H})$. We conclude that $T_{\tilde{\Delta}^n} \cong {}^u\mathcal{F}^1(\mathcal{H})$ and $R^1\tilde{f}_*\mathcal{O}_{\tilde{X}} \cong \Omega^1_{\tilde{\Delta}^n}$. We complete the proof of Proposition 3.3. □

Proof of Theorem 1.2. Under the assumption of Theorem 1.2, $R^1f_*\mathcal{O}_X$ is a reflexive sheaf by [11, Corollary 3.9]. Hence we obtain $R^1f_*\mathcal{O}_X \cong \Omega^1_S$ by Proposition 3.3. Then by Lemma 3.1, we obtain the assertion of Theorem 1.2. □

Proof of Corollary 1.3. By [12, Theorem 2] and [13, Theorem 1], $f : X \to S$ is a Lagrangian fibration and S is a Fano manifold. From [11, Corollary 3.2],

$$h^k(X, \omega_X) = \sum_{p+q=k} h^q(S, R^pf_*\omega_X).$$

Since $\omega_X \cong \mathcal{O}_X$, the right hand side of the above equation is the sum of Hodge numbers of S by Theorem 1.2. By Hodge conjugate $h^k(X, \mathcal{O}_X) = h^0(X, \Omega_X^k)$. Moreover

$$h^0(X, \Omega_X^k) = \begin{cases} 0 & k \equiv 1 \pmod 2 \\ 1 & k \equiv 0 \pmod 2 \end{cases}$$

since X is an irreducible symplectic manifold. From the assumption that S is projective, $h^{p,p}(S) \geq 1$. Combining the above results, we obtain the assertion of Corollary 1.3. □

Proof of Corollary 1.5. We classify S by using the classification of algebraic surfaces. We begin with to prove that S does not contain curves whose selfintersection number are negative. Assume the contrary. Then there is a birational morphism $\pi : S \to \bar{S}$ which contracts curves to points. The composition morphism $\pi \circ f$ is a Lagrangian fibration and not equidimensional. This contradicts to [14, Theorem 1]. Next we investigate the irregularity and Kodaira dimension of S. From [8, Theorem 1.1], $\kappa(S) \leq 0$. By similar argument in the proof of Corollary 1.3, $2q(S) = q(X)$. Since $\kappa(X) = 0$, $q(X) \leq 4$ and $q(S) \leq 2$. If $q(S) = 2$, X is an Abelian variety. From the following Claim, S is an Abelian surface.

Claim **3.4.** *Let X be an Abelian variety and $f : X \to S$ be a fibre space. Then S is an Abelian variety.*

Proof. Let F be a general fibre of f. Since K_F is trivial, F is an Abelian variety by [7, Theorem 12]. We consider the following exact sequence:

$$0 \to F \to X \to X/F \to 0$$

Then the quotient morphism is nothing but $X \to S$. Thus $X/F \cong S$ and S is an Abelian variety. □

We go back to the proof of Corollary 1.5. Since S does not contain curves whose selfintersection number are negative and $\kappa(S) \leq 0$, S is a $\mathbb{P}^1$-bundle over an elliptic curve or a hyperelliptic surface if $q(S) = 1$. Moreover, S is $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$ if $q(S) = 0$. From these classification, we obtain assertions of Colollary 1.5. □

4. Classification of singular fibres over codimension one points

In this section, we classify singular fibres over codimension one points of a projective Lagrangian fibration.

Theorem **4.1.** *Let $f : X \to S$ be a projective Lagrangian fibration and D the discriminat locus of f. Assume that $\dim X = 2n$, S and D is smooth at s. Then there exists a polydisk $s \in \Delta^n$ of S and the restriction morphism $f_{\Delta^n} : X_{\Delta^n} \to \Delta^n$ has one of the following two properties:*

(1) *There is a toroidal model $\tilde{X}$ of type I or II (See Definition 2.9), an Abelian fibration $\bar{X} \to \bar{\Delta}^n$ and an action of a cyclic group G_2 (resp. G_1) on $\tilde{X}$ (resp. $\bar{X}$) which satisfy the following diagram.*

$$\begin{array}{ccccccc} X_{\Delta^n} & \stackrel{\eta}{\to} & \bar{X}/G_1 & \stackrel{\nu_1}{\leftarrow} & \bar{X} & = & \tilde{X}/G_2 & \stackrel{\nu_2}{\leftarrow} & \tilde{X} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \Delta^n & = & \bar{\Delta}^n/G_1 & \leftarrow & \bar{\Delta}^n & = & \tilde{\Delta}^n/G_2 & \leftarrow & \tilde{\Delta}^n, \end{array}$$

where η is isomorphism on the smooth locus of $\bar{X}/G_1$ and $\bar{X}/G_1$ is normal. Moreover ν_1 is étale in codimension one and ν_2 is étale.

(2) *There is an Abelian fibration $\tilde{f} : \tilde{X} \to \tilde{\Delta}^n$ and an action of a cyclic group G on $\tilde{X}$ which satisfy the following properties:*

(a) *X and $\tilde{X}$ satisfy the following diagram:*

$$\begin{array}{ccccc} X_{\Delta^n} & = & \tilde{X}/G & \stackrel{\tau}{\leftarrow} & \tilde{X} \\ \downarrow & & \downarrow & & \downarrow \\ \Delta^n & = & \tilde{\Delta}^n/G & \leftarrow & \tilde{\Delta}^n, \end{array}$$

where τ is an étale morphism.

(b) *For the discriminant locus $\tilde{D}$ of $\tilde{f}$, $\tilde{f}^*\tilde{D}$ is a reduced divisor.*

(c) *There is a birational morphsim $\nu : W \to \tilde{X}$ such that $\tilde{f} \circ \nu : W \to \tilde{\Delta}^n$ satisfies the assumptions (1) and (2) of Lemma 2.10.*

REMARK 5. In the case that $\dim X = 2$, f is a minimal elliptic fibration and f satisfies the properties of Theorem 4.1(2) if the singular fibre of f is a Kodaira singular fibre of type II, III or IV. If the singular fibre of f is the other types, f satisfies the properties of Theorem 4.1(1).

Before the proof of Theorem 4.1, we investigate the monodormy around s.

LEMMA **4.2.** *Let $f : X \to S$ be a projective Lagrangian fibration. Assume that the discriminant locus of f and S are smooth at s. Then there is a polydisk $s \in \Delta^n$ of S and the restriction $f_{\Delta_n} : X_{\Delta^n} := X \times_S \Delta^n \to \Delta^n$ has the following properties:*

(1) *There is a toroidal model $f^+ : X^+ \to \tilde{\Delta}^n$ of type I or II and the action of a cyclic group G on X^+ which commutes with f^+ and satisfies the following diagram:*

$$\begin{array}{ccc} X_{\Delta^n} & \stackrel{\alpha}{\dashrightarrow} & X^+/G \\ \downarrow & & \downarrow \\ \Delta^n & = & \tilde{\Delta}^n/G, \end{array}$$

where α is a bimeromorphic map.

(2) *Let $\tilde{D}$ be the ramification locus of $\tilde{\Delta}^n \to \Delta^n$, Then*

$$\dim F^1 \mathrm{Gr}_1^W H^1(X_{\tilde{s}}^+, \mathbb{C})^G \geq n - 1,$$

where $X_{\tilde{s}}^+$ be a fibre of f^+ over a point $\tilde{s}$ of $\tilde{D}$.

PROOF. (1) By the assumption that the discriminant locus is smooth at s, we may assume that S and D is smooth. Let ω be a symplectic form on X and $\nu : Y \to X$ a birational morphism such that $E = (f \circ \nu)^* D$ is a simple normal crossing divisor. We denote each component of E by E_i.

CLAIM **4.3.** *Let $E := \sum E_i$ be a simple normal crossing variety. Then*

$$\mathrm{Gr}_k^W H^k(E, \mathbb{C}) = \{(\alpha_i) \in \oplus H^k(E_i, \mathbb{C}); \alpha_i|_{E_i \cap E_j} = \alpha_j|_{E_i \cap E_j}\}.$$

PROOF. Let

$$E^{[k]} := \coprod_{i_0 < \cdots < i_k} (E_{i_0} \cap \cdots \cap E_{i_k})^{\sim},$$

where $\sim$ means the normalization. For an index set $I = \{i_0, \cdots i_k\}$, we define an inclusion δ_j^I

$$\delta_j^I : E_{i_0} \cap \cdots \cap E_{i_k} \to E_{i_0} \cap \cdots \cap E_{i_{j-1}} \cap E_{i_{j+1}} \cap \cdots E_{i_k} \quad (0 \leq j \leq k).$$

We consider the following spectral sequence [4, Chapter 4]:

$$E_1^{p,q} = H^q(E^{[p]}, \mathbb{C}) \Longrightarrow E^{p+q} = H^{p+q}(E, \mathbb{C}),$$

where $D : E_1^{p,q} \to E_1^{p+1,q}$ is defined by the

$$\bigoplus_{|I|=p} \sum_{j=0}^{p} (-1)^j (\delta_j^I)^*.$$

Since this spectral sequence degenerates at E_2 level ([4, Chapter 4.8]), we deduce

$$\mathrm{Gr}_k^W(H^k(E, \mathbb{C})) = \mathrm{Ker}(\oplus_i H^k(E_i, \mathbb{C}) \xrightarrow{D} \oplus_{i<j} H^k((E_i \cap E_j)^{\sim}, \mathbb{C})).$$

Thus we obtain the assertion of Claim 4.3 from the definition of D. □

CLAIM **4.4.** *Let U be an open set of D such that $f \circ \nu|_{(f \circ \nu)^{-1}(U)}$ is a locally trivial deformation and every fibre of $f \circ \nu|_{(f \circ \nu)^{-1}(U)}$ is a simple normal crossing variety. Then every point s' of U,*

$$\dim F^1 \mathrm{Gr}_1^W H^1(Y_{s'}, \mathbb{C}) \geq n - 1,$$

where $Y_{s'}$ is the fibre of $f \circ \nu$ at s'.

PROOF OF CLAIM. First we show that there is a morphism

$$(f \circ \nu)^* T_U \to \Omega^1_{E_i/U} \tag{5}$$

for the component $E_i \cap (f \circ \nu)^{-1}(U) \neq \emptyset$. By the choice of U, $E_i \to U$ is a smooth morphism. Thus there is a following exact sequence:

$$0 \to T_{E_i/U} \to T_{E_i} \to (f \circ \nu)^* T_U \to 0.$$

From the following diagram,

$$\begin{array}{ccccccccc} 0 & \to & T_{E_i} & \to & T_Y & \to & N_{E_i/Y} & \to & 0 \\ & & & & \downarrow & \nu^*\omega & & & \\ 0 & \to & I_{E_i}/I_{E_i}^2 & \to & \Omega^1_Y & \to & \Omega^1_{E_i} & \to & 0, \end{array}$$

we define $T_{E_i} \to \Omega^1_{E_i}$ by $\nu^*\omega$. By [14, Theorem 1], the restriction $\nu^*\omega$ to each component of a fibre of $f \circ \nu$ is identically zero. Hence $\nu^*\omega(\alpha, *) = 0$ in $\Omega^1_{E_i/U}$ for every element $\alpha \in T_{E_i/U}$ and there is the morphism (5). We take a point $s' \in U$ and denote local parameters of U at s' by t_j, $(2 \leq j \leq n)$. Let $E_{i,s'}$ be the fibre of $E_i \to U$ over s'. From the morphism (5), $\alpha_{ij} := \nu^*\omega(\partial/\partial t_j, *)|_{E_{i,s'}}$ defines an element of $H^0(E_{i,s'}, \Omega^1_{E_{i,s'}})$. We will show that α_{ij}, $(2 \leq j \leq n)$ are linearly independent in $H^0(E_{i,s'}, \Omega^1_{E_{i,s'}})$ for non ν-exceptional E_i. Assume the contrary. There exists a linear combination $\gamma := \sum \lambda_j \partial/\partial t_j$ such that $\gamma \neq 0$ and $\nu^*\omega(\gamma, *)|_{E_{i,s'}} = 0$ in $H^0(E_{i,s'}, \Omega^1_{E_{i,s'}})$. From the definition of U, $E_{i,s'}$ is not contain ν-exceptional locus if E_i is not ν-exceptional. Thus there is an open set V of $E_{i,s}$ such that $\nu^*\omega$ is nondegenerate over V. For every point $y \in V$,

$$T_{E_{i,s'},y} = \{\alpha \in T_{Y,y}; \nu^*\omega(\alpha, \beta) = 0, \quad \forall \beta \in T_{E_{i,s'},y}\},$$

because $\nu(E_{i,s'})$ is a Lagrangian subvariety. However, $\gamma \notin T_{E_{i,s'},y}$ and $\nu^*\omega(\gamma, *)|_{E_{i,s'}} = 0$ in $H^0(E_{i,s'}, \Omega^1_{E_{i,s'}})$. That derives a contradiction. If $E_{i,s'} \cap E_{i',s'} \neq \emptyset$, $\alpha_{i,j} = \alpha_{i',j}$ on $E_{i,s'} \cap E_{i',s'}$ by definiton. Thus the assertion of Claim 4.4 follows by Claim 4.3. □

We go back to the proof of Lemma 4.2 (1). We choose a neighborhood $\Delta^n(t_1, \cdots, t_n)$ of s such that D is defined by $t_1 = 0$ and $\Delta^n \cap U \neq \emptyset$. Let $(f \circ \nu)^*D = \sum e_i E_i$ and $e := \text{L.C.M.}\{e_i; E_i \cap (f \circ \nu)^{-1}(U) \neq \emptyset\}$. We consider a e-fold cyclic cover

$$d : \tilde{\Delta}^n \ni (u, t_2, \cdots, t_n) \mapsto (t_1 = u^e, t_2, \cdots, t_n) \in \Delta^n$$

and the normalization $\hat{Y}$ of $Y \times_{\Delta^n} \tilde{\Delta}^n$. Since $\hat{Y}$ is projective and $g^{-1}(u)$ has a reduced component for a general point u of $d^{-1}(D)$, the Abelian fibration $g : \hat{Y} \to \Delta^n$ satisfies the assumptions of Lemma 2.8 except the monodromy conditions. Let $\tilde{T}$ be the monodromy matrix of $R^1 g^\circ_* \mathbb{C}$, where g° is smooth part of g. We will show that $\tilde{T}$ is unipotent and the rank of $\tilde{T} - I$ is at most one. Let s' be a point of $\Delta^n \cap U$. We choose an unit disk Δ^1 in Δ^n such that $\Delta^1 \cap D = \{s'\}$ and put $\tilde{\Delta}^1 = d^{-1}(\Delta^1)$. We concentrate the restriction morphism

$$g_{\tilde{\Delta}^1} : \hat{Y}_{\tilde{\Delta}^1} := \hat{Y} \times_{\tilde{\Delta}^n} \tilde{\Delta}^1 \to \tilde{\Delta}^1,$$

because the monodromy matrix of $R^1(g_{\tilde{\Delta}^1})^\circ_* \mathbb{C}$ is $\tilde{T}$, where $g^\circ_{\tilde{\Delta}^1}$ is smooth part of $g_{\tilde{\Delta}^1}$. By [9, Theorem 11*], if we take a suitable resolution $\tilde{Y}_{\tilde{\Delta}^1}$ of $\hat{Y}_{\tilde{\Delta}^1}$, every fibre of $\tilde{g}_{\tilde{\Delta}^1} : \tilde{Y}_{\tilde{\Delta}^1} \to \tilde{\Delta}^1$ is reduced and simple normal crossing.

CLAIM **4.5.** *Let $Z_{\tilde{\Delta}^1}$ be a resolution of $\hat{Y}_{\tilde{\Delta}^1}$. We denote the fibre of $f\circ\nu$ at s' by $Y_{s'}$ and the fibre of $Z_{\tilde{\Delta}^1}\to\tilde{\Delta}^1$ at $\tilde{s'}:=d^{-1}(s')$ by $Z_{\tilde{\Delta}^1,\tilde{s'}}$ Then the induced morphism $H^1(Y_{s'},\mathbb{C})\to H^1(Z_{\tilde{\Delta}^1,\tilde{s'}},\mathbb{C})$ is injection.*

PROOF OF CLAIM. Let $Y_{\Delta^1}:=Y\times_{\Delta^n}\Delta^1$. Since Y_{Δ^1} and $Z_{\tilde{\Delta}^1}$ are deformation retract to $Y_{s'}$ and $Z_{\tilde{\Delta}^1,\tilde{s'}}$ respectively, it is enough to prove that $H^1(Y_{\Delta^1},\mathbb{C})\to H^1(Z_{\tilde{\Delta}^1},\mathbb{C})$ is injective. From the isomorphism $\hat{Y}_{\tilde{\Delta}^1}/G\cong Y_{\Delta^1}$, $H^1(Y_{\Delta^1},\mathbb{C})\cong H^1(\hat{Y}_{\tilde{\Delta}^1},\mathbb{C})^G$ and $H^1(Y_{\Delta^1},\mathbb{C})\to H^1(\hat{Y}_{\tilde{\Delta}^1},\mathbb{C})$ is injection. Moreover $\hat{Y}_{\tilde{\Delta}^1}$ is a homology manifold because $\hat{Y}_{\tilde{\Delta}^1}$ has only quotient singularities. ([18, Proposition 1.4]). Hence $H^1(\hat{Y}_{\tilde{\Delta}^1},\mathbb{C})\to H^1(Z_{\tilde{\Delta}^1},\mathbb{C})$ is injection by [3, Thèoréme 8.2.4]. □

We go back to the proof of Lemma 4.2. From Claim 4.5 and Claim 4.4,

$$\dim F^1\mathrm{Gr}_1^W H^1(\tilde{Y}_{\tilde{s'}},\mathbb{C})\geq n-1.$$

Since $\tilde{g}$ is a semistable degeneration, $\tilde{T}$ is unipotent. We consider the Clemens-Schmid exact sequence.

$$0\to\mathrm{Gr}_1^W(H^1(\tilde{Y}_{\tilde{\Delta}^1,\tilde{s'}},\mathbb{C}))\to\mathrm{Gr}_1^W(\tilde{Y}_{\tilde{\Delta}^1,\tilde{u}},\mathbb{C}))\overset{\tilde{T}-I}{\to}\mathrm{Gr}_1^W(\tilde{Y}_{\tilde{\Delta}^1,\tilde{u}},\mathbb{C})),$$

where $\tilde{Y}_{\tilde{\Delta}^1,\tilde{u}}$ is a general fibre of $\tilde{g}_{\tilde{\Delta}^1}$. From the above exact sequence,

$$\dim\mathrm{Gr}_1^W H^1(\tilde{Y}_{\tilde{\Delta}^1,\tilde{u}},\mathbb{C})\geq 2n-2,$$

Moreover, the limit Hodge structure of $H^1(\tilde{Y}_{\tilde{\Delta}^1,\tilde{u}},\mathbb{C})$ is as follows:

$$0\subset W_0=\mathrm{Im}(\tilde{T}-I)\subset W_1=\mathrm{Ker}(\tilde{T}-I)\subset W_2=H^1(\tilde{Y}_{\Delta_1,\tilde{u}}).$$

Since $\dim W_0=\dim W_2/W_1$, the rank of $\tilde{T}-I$ is at most one. Therefore $\hat{Y}$ satisfies the assumptions of Proposition 2.1 and there exists a toroidal model $f^+:X^+\to\tilde{\Delta}^n$ of type I or II such that X^+ is bimeromorphic to $\hat{Y}$. The Galois group G of the covering d acts on $\hat{Y}$ and therefore act on $\tilde{X}$ bimeromorphically. This action commutes with f^+. By Lemma 2.8, the action of G is biholomorphic and X^+/G is bimeromorphic to X_{Δ^n} over Δ^n.

(2) Let T be the monodoromy matrix of $R^1(f\circ\nu)^\circ_*\mathbb{C}$ around D and $\tilde{D}:=d^{-1}(D)$. For the fibre of $X^+_{\tilde{s}}$ of f^+ over $\tilde{s}\in\tilde{D}$, the action G on $H^1(X^+_{\tilde{s}},\mathbb{C})$ is determined by the monodoromy matrix T. Thus the function

$$\mu(\tilde{s}):\tilde{s}\mapsto\dim F^1\mathrm{Gr}_1^W H^1(X^+_{\tilde{s}},\mathbb{C})^G$$

is constant on $\tilde{D}$. For the proof of the assertion of Lemma 4.2 (2), it is enough to show $\mu(\tilde{s'})\geq n-1$ for $\tilde{s'}=d^{-1}(s')$. We consider the restriction morphism $X^+_{\tilde{\Delta}^1}:=X^+\times_{\tilde{\Delta}^n}\tilde{\Delta}^1\to\tilde{\Delta}^1$. Then there is a G-equivariant bimeromorphic map $\hat{Y}_{\tilde{\Delta}^1}\dashrightarrow X^+_{\tilde{\Delta}^1}$. We take a G-equivariant resolution of indeterminacy $Z_{\tilde{\Delta}^1}$ of $\hat{Y}_{\tilde{\Delta}^1}\dashrightarrow X^+_{\tilde{\Delta}^1}$. By Claim 4.5, $H^1(Y_{\Delta^1,s'},\mathbb{C})\to H^1(Z_{\tilde{\Delta}^1,\tilde{s'}},\mathbb{C})$ is injection. By Claim 4.4.

$\dim F^1\mathrm{Gr}_1^W H^1(Z_{\tilde{\Delta}^1,\tilde{s}'},\mathbb{C})^G \geq n-1$. Because $Z_{\tilde{\Delta}^1} \to X^+_{\tilde{\Delta}^1}$ is G-equivariant bimeromorphic morphism, there is a G-equivariant isomorphism $H^1(Z_{\tilde{\Delta}^1},\mathbb{C}) \cong H^1(X^+_{\tilde{\Delta}^1},\mathbb{C})$. Since $Z_{\tilde{\Delta}^1}$ and $X^+_{\tilde{\Delta}^1}$ are deformation retract to $Z_{\tilde{\Delta}^1,\tilde{s}'}$ and $X^+_{\tilde{\Delta}^1,\tilde{s}'}$ respectively,

$$F^1\mathrm{Gr}_1^W H^1(Z_{\tilde{\Delta}^1,\tilde{s}'},\mathbb{C})^G \cong F^1\mathrm{Gr}_1^W H^1(X^+_{\tilde{s}'},\mathbb{C})^G.$$

Therefore we obtain $\mu(\tilde{s}') \geq n-1$. □

Proof of Theorem 4.1. Let $f : X \to S$ be a projective Lagrangian fibration. By Lemma 4.2, there exists a toroidal model $f^+ : X^+ \to \tilde{\Delta}^n$ and an action of a cyclic group G. The assertions of Theorem 4.1 follows the following Proposition.

Proposition 4.6. (1) *If f^+ is a toroidal model of type I, then $f^{-1}(s)$ satisfies the properties of Theorem 4.1 (1) or (2).*

(2) *If f^+ is a toroidal model of type II, then $f^{-1}(s)$ is satisfies the properties of Theorem 4.1 (1).*

Proof of Proposition 4.6. (1) We begin with to prove that the representation $G \to H^0(X^+_{\tilde{s}},\mathbb{C})$ is not trivial, where $X^+_{\tilde{s}}$ is the fibre of f^+ at $\tilde{s}$. Assume the contrary. Since every fibre of f^+ is an Abelian variety, the action of G on X^+ is fixed point free. Thus X^+/G is smooth and $K_{X^+/G}$ is numerically trivial. Since X^+/G has no rational curve, X^+/G is the unique relative minimal model. On the contrary, X_{Δ^n} is a relative minimal model. Therefore $X_{\Delta^n} \cong X^+/G$. However, $K_{X^+/G} \not\sim 0$ because K_{X^+} is not G-invariant. This derives a contradiction, because $K_{X_{\Delta^n}} \sim 0$. We need the following Lemma to prove Proposition 4.6 (1).

Lemma 4.7. *Let $X^+_{\tilde{D}} := (f^+)^{-1}(\tilde{D})$, where $\tilde{D}$ is the ramification locus of $\tilde{\Delta}^n \to \Delta^n$. Then $X^+_{\tilde{D}} \to \tilde{D}$ decompose a G-equivariant smooth elliptic fibration $X^+_{\tilde{D}} \to A'$ and a G-equivariant smooth Abelian fibration $A' \to \tilde{D}$. Moreover they satisfy the following diagram:*

$$\begin{array}{ccccc} X^+_{\tilde{D}} & \to & A' & \to & \tilde{D} \\ \downarrow & & \downarrow & & \downarrow \\ X^+_{\tilde{D}}/G & \to & A & \to & D, \end{array}$$

where D is the branch locus of $\tilde{\Delta}^n \to \Delta^n$, $A \to D$ are smooth $n-1$ dimensional Abelian fibration and $A'/G \cong A$.

Proof. Let F be a fibre of $X^+_{\tilde{D}} \to \tilde{D}$. First we show that F/G is smooth and $q(F/G) = n-1$. Let T be the representation matrix of $\rho : G \to \mathrm{Aut}H^0(F,\Omega^1_F)$. Since G is a finite cyclic group and Lemma 4.2 (2), $T = \mathrm{diag}(\zeta,1,\cdots,1)$ under suitable coordinate of $H^0(F,\Omega^1_F)$. Because F is an Abelian variety, the action of G is written $(z_1,z_2,\cdots,z_n) \mapsto (\zeta z_1,z_2,\cdots,z_n)$ around a fixed point p of F, where $(z_1,\cdots,z_n)$ is a local coordinate of p and ζ is a n-th root of unity. Note that $\zeta \neq 1$

because ρ is not trivial representation. The branch locus of the quotient map $F \to F/G$ are smooth divisors of F and F/G is smooth. Since $\dim H^0(F, \Omega^1_F)^G = n-1$, $q(F/G) = n-1$.

Next we consider relative Jacobian of $X^+_{\tilde{D}}/G \to D$. Since $q(F/G) = n-1$, there is a $n-1$-dimensional smooth Abelian fibration $A \to D$ which factor $X^+_{\tilde{D}}/G \to D$. We prove that $\alpha : X^+_{\tilde{D}}/G \to A$ is surjective and every fibre is connected. We consider the following diagram

$$\begin{array}{ccccc} X^+_{\tilde{D}} & \to & B_1 & & \\ \downarrow & & \downarrow & & \\ X^+_{\tilde{D}}/G & \to & B_2 & \to & A, \end{array}$$

where $X^+_{\tilde{D}}/G \to B_2$ is the stein factorization of α and $X^+_{\tilde{D}} \to B_1$ is the stein factorization of $X^+_{\tilde{D}} \to B_2$. Let A_p (resp. $B_{1,p}$, $B_{2,p}$) be the fibre of $A \to D$ (resp. $B_1 \to D$, $B_2 \to D$) at p. Note that the fibre X^+_p of $X^+_{\tilde{D}} \to \tilde{D}$ at p is an Abelian variety and $X^+_p \to B_{1,p}$ is surjective and connected fibre. By Claim 3.4, $B_{1,p}$ is an Abelian variety. Since $B_{1,p} \to B_{2,p}$ is a finite morphism, $\kappa(B_{2,p}) \leq \kappa(B_{1,p}) = 0$ by [7, Corollary 9]. From [7, Theorem 13], there is an étale morhpism $\tilde{B}_{2,p} \to B_{2,p}$ and $\tilde{B}_{2,p}$ is isomorphic to the product of an Abelian variety and a variety of general type. Thus $0 \leq \kappa(\tilde{B}_{2,p}) = \kappa(B_{2,p})$. Therefore $\kappa(\tilde{B}_{2,p}) = \kappa(B_{2,p}) = 0$, $\tilde{B}_{2,p}$ is an Abelian variety and $K_{B_{2,p}}$ is numerically trivial. Since $B_{2,p} \to A_p$ is finite and A_p is an Abelian variety, $B_{2,p}$ is an Abelian variety. By the universal properties of relative Jacobian, $B_2 \cong A$. The stein factorization $X^+_{\tilde{D}} \to A'$ of $X^+_{\tilde{D}} \to A$ is the desired morphism. □

We go back to the proof of Proposition 4.6 (1). By Lemma 4.7, there is a G-equivariant smooth Abelian fibration $\pi' : A' \to \tilde{D}$. Since $A'/G \cong A$, the action of G on A' is a translation on each fibre of π'. Let g be a generater of G and m the smallest integer such that the action of g^m on each fibre of π' is trivial.

First we consider the case that $H = \{1\}$. In this case, the action of G is fix point free. Hence X^+/G is smooth. Moreover X^+/G is the unique relative minimal model over Δ^n since it has no rational curves. On the contrary, X_{Δ^n} is a relative minimal model over Δ^n, $X_{\Delta^n} \cong X^+/G$. Thus X satisfies the properties of Theorem 4.1 (1) if we put $\tilde{X} = X^+$, $G_1 = \{1\}$ and $G_2 = G$.

Next we consider the case that $H \neq \{1\}$. Since the action of H on each fibre of π' is trivial and $X^+_{\tilde{D}} \to A'$ is a smooth elliptic fibration, every singular locus of X^+/H forms a multisection of $X^+_{\tilde{D}}/H \to A'$ and $X^+_{\tilde{D}}/H \to A'$ is a $\mathbb{P}^1$-bundle. For every singular point p of X^+/H, $(p, X^+/H) \cong$ (surface cyclic quotient singularity) $\times$ $(\mathbb{C}^{2n-2}, 0)$. The list of surface cyclic quotient singularities which occur above is found in [1, Table 5 (pp 157)]. By definition, the action of G/H on each fibre of π' is non trivial translation. Thus the action of G/H on X^+/H is fix point free and the

quotient morphism $X^+/H \to X^+/G$ is an étale morphism. Every singular locus also forms a multisection of $X^+_{\tilde{D}}/G \to A$ and $X^+_{\tilde{D}}/G \to A$ is a $\mathbb{P}^1$-bundle. For every singular point p of X^+/G, $(p, X^+/G) \cong$ (surface cyclic quotient singularity) $\times$ $(\mathbb{C}^{2n-2}, 0)$. We will construct a suitable resolution X^+/G and a relative minimal model over Δ^n according to singularities of X^+/G.

If every singular point p of X^+/G, $(p, X^+/G) \cong$ (Du Val singularity)$\times(\mathbb{C}^{2n-2}, 0)$, we take the minimal resolusion $\eta : X' \to X^+/G$ by using the minimal resolution of Du Val singualrities. Then X' is a relative minimal model over Δ^n. The morphism $\eta^*(X^+_{\tilde{D}}/G) \to A$ is a locally trivial deformation whose fibre is a Kodaira singular fibre of type I_0^*, II^*, III^* or IV^*. Combining that $A \to D$ is a smooth Abelian fibration, we conclude that every irreducible component of $\eta^*(X^+_{\tilde{D}}/G)$ is a $\mathbb{P}^1$-bundle over a smooth Abelian fibration. By similar argument in Lemma 2.8, X' is the unique minimal model over Δ^n. Hence $X_{\Delta^n} \cong X'$. It is verified that X satisfies the properties of Theorem 4.1 (1) by $\tilde{X} = \bar{X} = X^+$, $G_1 = G$ and $G_2 = \{1\}$.

If some singular point p of X^+/G, $(p, X^+/G) \ncong$ (Du Val singularity)$\times(\mathbb{C}^{2n-2}, 0)$, we consider the minimal resolution $w : W \to X^+/H$, which is obtained from the minimal resolution of surface quotient singularities. Then $w^*(X^+_{\tilde{D}}/H) \to A'$ is a locally trivial deformation whose fibre is a tree of $\mathbb{P}^1$. (cf. [1, pp 158]) Since $A' \to \tilde{D}/H$ is a smooth Abelian fibration, $w^*(X^+_{\tilde{D}}/H)$ is a simple normal crossing divisor and the morphism $w^*(X^+_{\tilde{D}}/H) \to \tilde{D}/H$ is a locally trivial deformation. Therefore W satisfies the assumption (1) and (2) of Lemma 2.10. Every irreducible component of $w^*(X^+_{\tilde{D}}/H)$ is a $\mathbb{P}^1$-bundle over a smooth Abelian fibration. By similar argument in Lemma 2.8, W is the unique minimal resolution of X^+/H. Thus the action of G/H on X^+/G can be lifted on W and $W \to X^+/H$ is G/H-equivariant morphism. Hence $w^*(X^+_{\tilde{D}}/H) \to A'$ is G/H-equivariant. By contracting $\mathbb{P}^1$-bundles along its ruling, (cf. [1, pp 158]) we obtain a G/H-equivariant relative minimal model $\tilde{X}$ of W over $\tilde{\Delta}^n/H$. Let $\tilde{f} : \tilde{X} \to \Delta^n/H$ and $\tilde{X}_{\tilde{D}/H} := \tilde{f}^{-1}(\tilde{D}/H)$. By construction, $\tilde{X}$ is smooth and every singular fibre of $\tilde{X}$ is reduced. The morphism $\tilde{X}_{\tilde{D}/H} \to A'$ is a G/H-equivariant locally trivial deformation whose fibre is a Kodaira singular fibre of type II, III or IV. Thus the normalization of every irreducible component of $\tilde{X}_{\tilde{D}/H}$ is a $\mathbb{P}^1$-bundle over a smooth Abelian fibration. By similar argument in Lemma 2.8, $\tilde{X}$ is the unique minimal model. Since the action of G/H on A' is non trivial translation, the action of G/H on $\tilde{X}$ is fixed point free. Thus $\tilde{X} \to \tilde{X}/(G/H)$ is an étale morphism and $\tilde{X}/(G/H)$ is a relative minimal model. Moreover, $\tilde{X}/(G/H)$ is the unique relative minimal model because $\tilde{X}$ is so. Hence $\tilde{X}/(G/H) \cong X_{\Delta^n}$. Therefore $\tilde{X}$, X and W satisfies the properties of Theorem 4.1 (2). We finish the proof of Proposition 4.6 (1).

(2) First we prove the following Lemma.

LEMMA **4.8.** *Let $\tilde{D}$ be the discriminat locus of $\tilde{f}$ and $\tilde{f}^*\tilde{D} = \sum E_i$. We define the subgroup H of G as follows:*

$$H := \{g \in G; g(E_i) = E_i \text{ for all } i.\}$$

Then there is the unique minimal relative minimal model $\bar{f} : \bar{X} \to \tilde{\Delta}^n/H$ of $X^+/H \to \tilde{\Delta}^n/H$ which have the properties of Proposition 2.1 (1), (2) and (3). Moreover

(1) *There is an action of G/H on $\bar{X}$. If an element g of G/H preserves every irreducible component of every singular fibre, then $g = 1$.*
(2) *For every point of s of the discriminat locus of $\bar{f}$,*

$$\dim F^1\mathrm{Gr}^1H^1(\bar{X}_s, \mathbb{C}) \geq n - 1,$$

where $\bar{X}_s$ is the fibre of $\bar{f}$ at s.
(3) *There exists an étale morphism $\nu : \hat{X} \to \bar{X}$ and an cyclic group G' such that $\hat{X}$ is a toroidal model of type II and $\hat{X}/G' \cong \bar{X}$.*

PROOF. We begin with investigation of the action of H on each E_i. Let $X^+_{\tilde{s}}$ be the fibre of f^+ over $\tilde{s} \in \tilde{D}$. We denote each irreducible component of $X^+_{\tilde{s}}$ by $E_{i,\tilde{s}}$ and $F_{ij,\tilde{s}} := E_{i,\tilde{s}} \cap E_{j,\tilde{s}}$. From the following Claim, the action of H on $H^0(E_{i,\tilde{s}}, \Omega^1_{E_{i,\tilde{s}}})$ and $H^0(F_{ij,\tilde{s}}, \Omega^1_{F_{ij,\tilde{s}}})$ are trivial.

CLAIM **4.9.** *Let $E = \sum E_{i,\tilde{s}}$ be a cycle of several copies of a $\mathbb{P}^1$-bundle over an Abelian variety such that $F_{ij,\tilde{s}} := E_{i,\tilde{s}} \cap E_{j,\tilde{s}}$ forms a section of the ruling of $E_{i,\tilde{s}}$ and $E_{j,\tilde{s}}$. Assume that there is an action of a cyclic group G on E such that $\dim F^1\mathrm{Gr}^W_1H^1(E, \mathbb{C})^G \geq n - 1$. Then*

(1) *If $F_{ij,\tilde{s}}$ is G-stable, then every element of $H^0(F_{ij,\tilde{s}}, \Omega^1_{F_{ij,\tilde{s}}})$ is G-invariant.*
(2) *If $E_{i,\tilde{s}}$ is G-stable, then every element of $H^0(E_{i,\tilde{s}}, \Omega^1_{E_{i,\tilde{s}}})$ is G-invariant.*

PROOF. Assume that some $F_{ij,\tilde{s}}$ is G-stable and there exists a non G-invariant element of $H^0(F_{ij,\tilde{s}}, \Omega^1_{F_{ij,\tilde{s}}})$. Since every $E_{i,\tilde{s}}$ is a $\mathbb{P}^1$-bundle over an Abelian variety and every $F_{ij,\tilde{s}}$ forms a section of the ruling, there is a G-equivariant isomorphism $F^1\mathrm{Gr}^W_1H^1(E, \mathbb{C}) \cong H^0(F_{ij,\tilde{s}}, \Omega^1_{F_{ij,\tilde{s}}})$ by Claim 4.3. Thus there exists a non G-invariant element in $F^1\mathrm{Gr}^W_1H^1(E, \mathbb{C})$. On the contrary, $\dim F^1\mathrm{Gr}^W_1H^1(E, \mathbb{C}) = n - 1$. Hence $F^1\mathrm{Gr}^W_1H^1(E, \mathbb{C})^G = F^1\mathrm{Gr}^W_1H^1(E, \mathbb{C})$. That is a contradiction. If some $E_{i,\tilde{s}}$ is G-stable, we obtain the assertion of Claim by similar argument. □

We go back to the proof of Lemma 4.8. Since $F_{ij,\tilde{s}}$ is an Abelian variety, the action of H is translation. If the action of H on $F_{ij,\tilde{s}}$ is not trivial, the action of H is fixed point free. We put $\bar{X} := X^+/H$. It is easy to check that X^+/H satisfies the assertions of Lemma.

We consider the case that the action of H on $F_{ij,\tilde{s}}$ is trivial. In this case, every point of $F_{ij} := E_i \cap E_j$ is fixed by H. We choose a point $p \in F_{ij}$ and a local

coordinate z_i, $(1 \leq i \leq 2n)$ at p such that E_i (resp. E_j) is defined by $z_1 = 0$. (resp. $z_2 = 0$.) Since E_i and E_j are stable under the action of H, the action of H can be written as

$$(z_1, z_2, z_3, \cdots, z_{2n}) \mapsto (az_1, bz_2, z_3, \cdots, z_{2n}),$$

where $a, b \in \mathbb{C}^*$. Thus for every singular point q of X^+/H,

$$(q, X^+/H) \cong (\text{surface cyclic quotient singularity}) \times (\mathbb{C}^{2n-2}, 0).$$

The morphism $((f^+)^*\tilde{D})/H \to \tilde{D}/H$ is a locally trivial deformation. Every irreducible component of every fibre is a $\mathbb{P}^1$-bundle over an Abelian variety. The singular locus of X^+/H is contained in $\cup_{i<j}(F_{ij}/H)$. We take the minimal resolution $\nu : Z_0 \to X^+/H$ by using the minimal resolution of a surface cyclic quotient singularity. Then the morphism $\nu^*(((f^+)^*\tilde{D})/H) \to \tilde{D}/H$ is a locally trivial deformation. Since exceptional locus of the minimal resolution of a surface cyclic quotient singularity is a chain of $\mathbb{P}^1$, every fibre of $\nu^*(((f^+)^*\tilde{D})/H) \to \tilde{D}/H$ is a cycle of several copies of a $\mathbb{P}^1$-bundle over an Abelian variety. Hence Z_0 satisfies the properties (1) and (2) of Proposition 2.1. Every irreducible component of $\nu^*(((f^+)^*\tilde{D})/H)$ is a $\mathbb{P}^1$-bundle over a smooth Abelian fibration because $\pi(\tilde{D}/H) = \{1\}$. By similar argument in Lemma 2.8, Z_0 is the unique minimal resolution of X^+/H. Therefore the action of G/H can be lifted on Z_0. Let $z_0 : Z_0 \to Z_1$ be a G/H-equivariant extremal contraction. Let ℓ be a curve which is contracted by z_0. We denote the irreducible component of $\nu^*(((f^+)^*\tilde{D}/H))$ which contains ℓ by E_0. Since E_0 is a $\mathbb{P}^1$-bundle over a smooth Abelian fibration, ℓ is a fibre of the ruling of E_0. Thus z_0 is the contraction of $\mathbb{P}^1$-bundles along its ruling. Hence Z_1 is smooth and it satisfies the properties (1) and (2) of Proposition 2.1. Iterating this process, we obtain a G/H-equivariant minimal model $\bar{X}$. By construction, $\bar{X}$ has the properties (1), (2) and (3) of Proposition 2.1. By Lemma 2.8, $\bar{X}$ is the unique relative minimal model. Let $\bar{g}$ be an element of G/H. By the definition of H, $\bar{g} = 1$ if every irreducible component of singular fibres of $Z_0 \to \tilde{\Delta}^n/H$ are stable under the action of $\bar{g}$. Since at least one $\bar{g}$-orbit is not contracted in the contracting process $Z_0 \to \bar{X}$, $\bar{X}$ satisfies the assertion (1) of Lemma.

For the proof of (2) and (3) of Lemma, we consider the restriction of $\bar{X}$. Let C be a smooth curve of $\tilde{\Delta}^n/H$ which is stable under the action of G/H and which intersects transversally with the discriminat locus of $\bar{f}$ at s. We consider the restrictions $(X^+/H)_C := X^+/H \times_{\tilde{\Delta}^n/H} C$, $(Z_0)_C := Z_0 \times_{\tilde{\Delta}^n/H} C$ and $\bar{X}_C := \bar{X} \times_{\tilde{\Delta}^n/H} C$. Since X^+ satisfies the property (2) of Proposition 2.1, $(X^+/H)_C$ has only quotient singularities. Hence there is an injection $H^1((X^+/H)_C, \mathbb{C}) \to H^1((Z_0)_C, \mathbb{C})$. Thus $H^1((X^+/H)_s, \mathbb{C}) \to H^1((Z_0)_s, \mathbb{C})$ is an injection because $(X^+/H)_C$ and $(Z_0)_C$

are deformation retract to $(X^+/H)_s$ and $(Z_0)_s$ respectively. There is the G/H-equivariant bimeromorphic morphism $(Z_0)_C \to (X^+/H)_C$. Moreover

$$F^1\mathrm{Gr}_1^W H^1((X^+/H)_s, \mathbb{C})^{G/H} \cong F^1\mathrm{Gr}_1^W H^1(X_s^+, \mathbb{C})^G.$$

We obtain

$$\dim F^1\mathrm{Gr}_1^W H^1((Z_0)_s, \mathbb{C})^{G/H} \geq n-1,$$

from Lemma 4.2 (2). Since $(Z_0)_C \to \bar{X}_C$ is G/H-equivariant bimeromorphic morphism, there is a G/H-equivariant isomorphism $H^1(\bar{X}_C, \mathbb{C}) \to H^1((Z_0)_C, \mathbb{C})$. Hence $F^1\mathrm{Gr}_1^W H^1((Z_0)_s, \mathbb{C})^{G/H} \cong F^1\mathrm{Gr}_1^W H^1(\bar{X}_s, \mathbb{C})^{G/H}$ because $\bar{X}_C$ is deformation retract to $\bar{X}_s$. We obtain the assertion (2) of Lemma.

For the proof of the assertion (3) of Lemma, we need the following Claim.

CLAIM **4.10.** *Let $\bar{f} : \bar{X} \to \tilde{\Delta}^n/H$ and $\bar{f}^*(\tilde{D}/H) = \sum e_i \bar{E}_i$. Then $e_i = m$ for all i.*

PROOF. Assume the contrary. Let $\bar{E}_0$ be the component such that e_0 attains minimum value of e_i and $\bar{E}_i$, $(i = 1, 2)$ next components. We may assume that $e_1 > e_0$. We consider the restriction $\bar{X}_C$ and denote the restriction $\bar{E}_i$ to $\bar{X}_C$ by $\bar{E}_{C,i}$. Since $\bar{f}$ satisfies the properties (1), (2) and (3) of Proposition 2.1, each $\bar{E}_{C,i}$ is a $\mathbb{P}^1$-bundle over an Abelian variety and $K_{\bar{E}_{C,0}} \sim (K_{\bar{X}_C} + \bar{E}_{C,0})|_{\bar{E}_{C,0}} \sim \bar{E}_{C,0}|_{\bar{E}_{C,0}}$. Let ℓ be a fibre of the ruling of $\bar{E}_{C,0}$. Since each intersection $\bar{E}_{C,i} \cap \bar{E}_{C,j}$ forms a section of the ruling of $\bar{E}_{C,i}$.

$$(\sum e_i \bar{E}_{C,i})\ell = e_1 + e_2 + e_0 \bar{E}_{C,0}.\ell = 0.$$

Because ℓ is a fibre of the ruling of $\bar{E}_{C,0}$, $K_{\bar{E}_{C,0}}.\ell = -2$. From the assumption $e_1 > e_0$ and $e_2 \geq e_0$, this derives a contradiction. □

We choose a coordinate $(t_1, \cdots, t_n)$ of $\tilde{\Delta}^n/H$ such that $\tilde{D}/H$ is defined by $t_1 = 0$. Let $d : \tilde{\Delta}^n(s_1, t_2, \cdots, t_n) \to (\tilde{\Delta}^n/H)(t_1, \cdots, t_n)$ be the m-fold cyclic covering defined by $s_1^m = t_1$, G' the galois group of d and $\tilde{X} := \bar{X} \times_{\tilde{\Delta}^n/H} \tilde{\Delta}^n$. Then $\tilde{X} \to \bar{X}$ is étale, $\tilde{X}$ is smooth and every fibre is reduced. (cf. [18, Lemma 2.2]) We complete the proof of Lemma. □

We go back to the proof of Proposition 4.6 (2). Let us consider the action of G/H on $\bar{X}$, which is constructed in Lemma 4.8. Since the dual graph Γ of every singular fibre of $\bar{X}$ is the Dynkin diagram of type $\tilde{A}_n$, the action of G/H on Γ is rotation or reflection. If the action of G/H is rotation, the action of G/H on $\bar{X}$ is fixed point free by the assertion (1) of Lemma 4.8. The quotient $\bar{X}/(G/H)$ is the unique minimal model because $\bar{X}$ is so by Lemma 4.8. Hence $\bar{X}/(G/H) \cong X_{\Delta^n}$. Therefore X, $\bar{X}$ and $\tilde{X}$ satisfy the properties of Theorem 4.1 (1).

We consider the case that the action of G/H on Γ is reflection. If the action of G/H is fixed point free, we prove that $\bar{X}/(G/H) \cong X$ and X satisfies the properties (1) of Theorem 4.1 by similar argument in the case that the action of

G/H on Γ is rotation. Thus we assume that G/H has fixed points. We use same notations as in Lemma 4.8 and Claim 4.10. Since $\bar{X}$ satisfies the properties (1) and (2) of Proposition 2.1, the morphism $\bar{f}^*(\check{D}/H) \to \check{D}/H$ is a locally trivial deformation and every fibre of $\bar{f}^*(\check{D}/H) \to \check{D}/H$ is a cycle of several copies of a $\mathbb{P}^1$-bundle over an Abelian variety. Since $\pi_1(\check{D}/H) = \{1\}$, each $\bar{E}_i$ is a $\mathbb{P}^1$-bundle over a smooth Abelian fibration. From the assumption that the action G/H on Γ is reflection, there are two G/H-stable component in $\bar{f}^*(\check{D}/H)$. If $\bar{E}_i$ is G/H-stable, every one form on $\bar{E}_{i,s}$ is G/H-invariant by Lemma 4.8 (2) and Claim 4.9, where $\bar{E}_{i,s}$ is a fibre of $\bar{E}_i \to \check{D}/H$. Since $\bar{E}_{i,s}$ is a $\mathbb{P}^1$-bundle over an Abelian variety, every fibre of the ruling of $\bar{E}_i$ is G/H-stable and the fixed locus on $\bar{E}_i$ forms multisection of the ruling of $\bar{E}_i$. If $\bar{F}_{ij} := \bar{E}_i \cap \bar{E}_j$ is G/H-stable, every one form of $\bar{F}_{ij,s} := \bar{E}_{i,s} \cap \bar{E}_{j,s}$ is G/H-invariant by Lemma 4.8 (2) and Claim 4.9. We obtain that the action of G/H on $\bar{F}_{ij}$ is trivial because $\bar{F}_{ij,s}$ is an Abelian variety and we assume that there exists a fixed point. Since $\bar{F}_{ij}$ forms a section of the ruling of $\bar{E}_i$ and $\bar{E}_j$, every connected component of the fixed locus of $\bar{X}$ forms a multisection of the ruling of some $\bar{E}_i$. For every singular point p of $\bar{X}/(G/H)$, $(p, \bar{X}/(G/H)) \cong (A_1\text{-singularity}) \times (\mathbb{C}^{2n-2}, 0)$. We obtain a relative minimal model $\bar{f}^+ : \bar{X}^+ \to \Delta^n$ of $\bar{X}/(G/H)$ over $\check{\Delta}^n/G = \Delta^n$ by blowing up along singular locus. Let D be the discriminat locus of $\bar{f}^+$. Since every irreducible component of $(\bar{f}^+)^*(D)$ is a $\mathbb{P}^1$-bundle over a smooth Abelian fibration, we obtain that $\bar{X}^+$ is the unique relative minimal model by similar argument in Lemma 2.8. Hence $X_{\Delta^n} \cong \bar{X}^+$. From Lemma 4.8, there is an étale covering $\check{X} \to \bar{X}$ such that $\check{X}$ is a toroidal model of type II. Thus X satisfies the properties of Theorem 4.1 (1). □

We complete the proof of Theorem 4.1. □

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