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Discrete Convexity and Equilibria in Economies with Indivisible Goods and Money

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Abstract We consider a variant of the standard Arrow–Debreu model which contains a number of indivisible goods and one perfectly divisible good (numeraire or money). The objective of this paper is to clarify a general mathematical mechanism which guarantees the existence of an equilibrium in such a model. It is shown that the crucial condition for the existence of an equilibrium is that the sets of supply and demand of indivisible goods should belong to a class of "discrete convexity." The class of generalized polymatroids provides one of the most interesting classes of discrete convexity.

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1 Introduction

We consider here an economy with production in which there are a number of indivisible goods and one perfectly divisible good (numeraire or money). This means that the commodity space of the model takes the form of $\mathbb{Z}^K \times \mathbb{R}$, where K is a finite set of types of indivisible goods.

Several models of exchange economies with indivisible goods and money have been considered in the literature. In 1970, Henry ([13]) proved the existence of a competitive equilibrium in an exchange economy with one indivisible good and money. He also gave an example of an economy with two indivisible goods in which no competitive equilibrium exists. In 1982, Kaneko ([14]) showed the existence of an equilibrium in an extension of the Shapley–Shubik model. In 1984, Quinzii ([20]), Gale ([11]) and Svensson ([24]) generalized his result for house markets. There has recently been a renewed interest in exchange economies with indivisibilities and money. Van der Laan *et al.* ([15]) consider a multi-product generalization of Gale's setup. Bevia *et al.* ([2]) consider an economy with several types of indivisibles and specific transferable utilities. Danilov *et al.* ([5]) and Bikhchandani and Mamer ([3]) provide a necessary and sufficient condition for the existence of equilibria in economies with transferable utilities.

These studies, dealing with specific structures, do not provide general existence conditions comparable to those for economies with divisible goods only. As is well known, the conditions for the case of divisible goods boil down to continuity and convexity. It may be hoped, analogously, that the existence of equilibria in economies with indivisibles should be connected with some appropriate notion of convexity in the lattices of integer points.

The aim of the present paper is to provide a general existence theorem for an equilibrium in the Arrow–Debreu model with indivisibles and money. It turns out that the rationale for the existence of an equilibrium is that the sets of supply and demand of indivisible goods should be discrete-convex sets (in the sense to be made precise later). In particular, our approach provides a unified view on the afore-mentioned existence results.

Two different kinds can be differentiated among the conditions in the existence issue with indivisibles. The first consists of "continuity" assumptions. They should imply the upper semi-continuity of demands and supplies with respect to price and consequently guarantee a proper behavior at the boundary of the price space. This *topological* part is common to both divisible and indivisible cases. The second kind of conditions should control the shape of demands and supplies for a given price. In economies with divisible goods, the demand and supply should form convex subsets

in an Euclidean space. In economies with indivisibles, on the other hand, the demand and supply sets should belong to some *class of discrete convex sets*.

Let us explain the mathematical essence in a simpler case of an exchange economy with indivisibles and money, whereas the full model involving production is treated later. Let $D_h(p) \subset \mathbb{Z}^K$ denote the "discrete part" of the h-th agent's demand for a given price $p \in \mathbb{R}^K_+$, where $h \in H$ with H denoting the set of agents. Replace the "discrete" demand sets $D_h(p)$ by their convex hulls $\mathrm{co}(D_h(p))$ and consider the convexified economy (with divisible goods) in which demands are represented by $\mathrm{co}(D_h(p))$, $h \in H$. Assume that the sets $\mathrm{co}(D_h(p))$, $h \in H$, are "well-behaved" with respect to price p. Then the standard fixed-point argument leads to the existence of an equilibrium price p^* for the convexified economy. This means that the total initial endowment vector $\sum_{h\in H} W_h$ (an integral vector) belongs to the (Minkowski) sum of the convexified demands $\sum_{h\in H} \mathrm{co}(D_h(p^*)) = \mathrm{co}(\sum_{h\in H} D_h(p^*))$. In general, this does not imply that $\sum_{h\in H} W_h$ belongs to the sum of the initial "discrete" demands $\sum_{h\in H} D_h(p^*)$. Our requirement of discrete convexity on the demand set ensures that this implication holds indeed. Then it follows that the price p^* is the equilibrium price in the discrete model.

Hence, we can assert the existence of an equilibrium in the discrete model, if we can ensure the equality:

$$\mathrm{co}(\sum_{h\in H} (D_h(p^*)) \cap \mathbb{Z}^K = \sum_{h\in H} D_h(p^*). \qquad (*)$$

As a special case with $|H| = 1$, this implies an identity $D_h(p) = \mathrm{co}(D_h(p)) \cap \mathbb{Z}^K$ for each individual demand, where a subset of $\mathbb{Z}^K$ having this property is called *pseudoconvex*. It turns out, however, that the pseudoconvexity of individual demands alone is not enough for the equality $(*)$. Indeed, a simple counterexample (see Example 1 in Section 3) demonstrates that the sum of pseudoconvex sets can fail to be pseudoconvex.[2] Thus we need to identify subclasses of pseudoconvex sets which are closed with respect to sum.

A class $\mathcal{D}$ of pseudoconvex subsets is said to be a *class of discrete convexity* if it is closed with respect to sum. Accordingly, if the sets $D_h(p^*)$, $h \in H$, of individual demands belong to such a class $\mathcal{D}$, then the equality $(*)$ holds, and, consequently, the discrete model has an equilibrium.

Modulo the topological part of assumptions, our existence theorem (Theorem 1) states that an equilibrium in a discrete economy with money exists if the discrete parts of the demands and supplies belong to some class of discrete convexity. Thus,

[2]Emmerson [7] assumed (mistakenly) that the equality $(*)$ always holds.

the discrete convexity of supplies and demands is, in our opinion, the essential reason for the equilibrium existence in economies with indivisibilities. (Similar arguments can be found in [5] for a general case and in [3] for a case of transferable utilities.)

Having found both the rationale and the proper condition, we are to find non-trivial interesting classes of discrete convexity. Fortunately, there are many classes of discrete convexity. The most interesting one for the problem considered here is the class of integral generalized polymatroids (see Section 5). We also give a few examples of utility functions generating polymatroidal demands. In particular, in all the afore-mentioned models [13, 11, 20, 24, 5, 2, 15], in which the existence of an equilibrium is proved, the demand sets belong to the class of discrete convexity associated with integral generalized polymatroids.

Remark. The authors became aware, after submitting this paper, of an article by Gul and Stacchetti ([12]), which proves a fairly general existence theorem for economies with indivisibles presented in single units and money. Demands and supplies in this model also turn out to be sets belonging to the class of discrete convexity associated with integral generalized polymatroids. We will address this question in another paper.

2 Model and Definitions

We consider a production economy with a finite number of consumers and producers, a finite number of indivisible goods, and one perfectly divisible good. Indivisible commodities form only a part of the whole economy, and the remaining part is represented by means of a perfectly divisible aggregate good, called money. We denote by K the set of indivisible goods. The commodity space of the model takes the form of $\mathbb{Z}_+^K \times \mathbb{R}$.

The finite set of producers is denoted by L. A producer $l \in L$ is described by a cost function $c_l : \mathbb{Z}_+^K \to \mathbb{R} \cup \{+\infty\}$, where $c_l(0) = 0$ and c_l is a monotone function. This means that producers use only this aggregate divisible good as productive input. The producer l produces a vector $Z \in \mathbb{Z}_+^K$ of discrete goods at the cost of using $c_l(Z)$ of money. If $c_l(Z) = +\infty$, then producer l cannot produce Z.

The finite set of consumers is denoted by H. A consumer $h \in H$ has a preference $\preceq_h$ on $\mathbb{Z}_+^K \times \mathbb{R}_+$, which is a complete, closed, and transitive binary relation. We assume that preferences are increasing in indivisibles and strictly increasing in money. Each consumer $h \in H$ is endowed with a vector of initial endowment

$(W_h, w_h) \in \mathbb{Z}_+^K \times \mathbb{R}_+$, where $W_h \in \mathbb{Z}_+^K$ and $w_h \in \mathbb{R}_+$, and receive a share $\theta_{lh} \geq 0$ from producer $l \in L$, where $\sum_{h \in H} \theta_{lh} = 1$ for any $l \in L$.

Thus, a discrete economy $\mathcal{E}$ is defined as a tuple

$$\mathcal{E} = \{(\preceq_h, (W_h, w_h), (\theta_{lh}), l \in L),\ h \in H; (c_l),\ l \in L\}.$$

The markets and the production are assumed to be perfectly competitive. We normalize the prices in such a way that the price of money is equal to 1. We define the prices of indivisible goods as a positive linear functional on $\mathbb{Z}_+^K$ represented by a nonnegative vector $p = (p_k)_{k \in K} \in \mathbb{R}_+^K$ with $p_k \in \mathbb{R}_+$, that is, $p(X) = \sum_{k \in K} p_k X^k$ for $X = (X^k)_{k \in K}$ with $X^k \in \mathbb{Z}_+$.

Agents are price-takers: Given a price p, each producer $l \in L$ maximizes the profit by solving the following program:

$$\max_{Y \in \mathbb{Z}_+^K} (p(Y) - c_l(Y)). \tag{1}$$

We put $\pi_l(p) = \max(p(Y) - c_l(Y))$. Each consumer $h \in H$ seeks a best element (X_h, m_h) with respect to the preference $\preceq_h$ in the budget set

$$B_h(p) = \{(X, m) \in \mathbb{Z}_+^K \times \mathbb{R}_+ \mid p(X) + m \leq \beta_h(p)\},$$

where the income $\beta_h(p)$ is defined by

$$\beta_h(p) = p(W_h) + w_h + \sum_{l \in L} \theta_{lh} \pi_l(p).$$

Definition 1 *A tuple* $((X_h, m_h)_{h \in H}; (Y_l)_{l \in L};\ p)$ *is a* competitive equilibrium *in the economy* $\mathcal{E}$ *if*
a) for each $l \in L$, Y_l *is a solution to (1),*
b) for each $h \in H$, (X_h, m_h) *is a best element in the budget set* $B_h(p)$ *with respect to the preference* $\preceq_h$,
c) all markets clear:

$$\sum_{h \in H} X_h = \sum_{h \in H} W_h + \sum_{l \in L} Y_l, \tag{2}$$

$$\sum_{h \in H} m_h + \sum_{l \in L} c_l(Y_l) = \sum_{h \in H} w_h. \tag{3}$$

As the preferences are strictly increasing with respect to money, the clearing of the indivisible goods markets implies that of the money market.

3 Discrete convexity and the existence issue

Here we present a general existence theorem for production economies with indivisibles and money. In order to emphasize the discrete aspect concerning a "proper" notion of convexity in the lattices of integral vectors, we state the theorem in a conditional form, omitting topological issues.

Given a price p, denote by

$$S_l(p) = \mathrm{Argmax}_{Y \in \mathbb{Z}_+^K}(p(Y) - c_l(Y))$$

the supply of producer $l \in L$. Each consumer h chooses a best element $(X_h, m_h) \in B_h(p)$ with respect to the preference $\preceq_h$. Monotonicity of the preference implies $p(X_h) + m_h = \beta_h(p)$. Denote by $D_h(p)$ the set of all $X_h \in \mathbb{Z}_+^K$ such that $(X_h, \beta_h(p) - p(X_h))$ is a best element in the budget set $B_h(p)$ with respect to the preference $\preceq_h$. An equilibrium exists if there is a price p^* such that the inclusion

$$\sum_{h \in H} W_h \in \sum_{h \in H} D_h(p^*) - \sum_{l \in L} S_l(p^*) \tag{4}$$

holds true, where the sum is defined as the Minkowski sum, namely, $A \pm B = \{a \pm b \,|\, a \in A,\, b \in B\}$ for two sets A and B.

We now consider the "convexification" of the discrete economy $\mathcal{E}$. That is, we consider a new economy $\mathrm{co}(\mathcal{E})$ with divisible goods, in which individual demands and supplies are the convex hulls of demands and supplies of the initial economy $\mathcal{E}$. It should be clear that the commodity space of $\mathrm{co}(\mathcal{E})$ is $\mathbb{R}^K \times \mathbb{R}$. In the convexified economy, the demand of consumer h for price p is the set $\{(x, \beta_h(p) - p(x)),\, x \in \mathrm{co}(D_h(p))\}$, where $\mathrm{co}(\cdot)$ denotes the convex hull of a set. The supplies in $\mathrm{co}(\mathcal{E})$ are given by $\mathrm{co}(S_l(p))$, $l \in L$. The profits of the producers in $\mathrm{co}(\mathcal{E})$ are equal to their profits in $\mathcal{E}$.

Let us assume[3)] that we have an equilibrium in the economy $\mathrm{co}(\mathcal{E})$, i.e., there exists a price p^* such that

$$\sum_{h \in H} W_h \in \sum_{h \in H} \mathrm{co}(D_h(p^*)) - \sum_{l \in L} \mathrm{co}(S_l(p^*)). \tag{5}$$

When (5) implies (4), p^* is an equilibrium price also for the discrete economy $\mathcal{E}$. But this implication does not hold for arbitrary discrete sets. This is why an equilibrium might fail to exist in the discrete economy in spite of the existence of an equilibrium

[3)] There are many results on the existence of equilibria in economies with divisible goods, see, for example [1, 16]; we shall also establish an existence result in the next section.

in the convexified economy. We can see this in the following simple example.
Example 1. Consider an exchange economy with two agents and two types of goods, $H = \{1, 2\}$ and $K = \{1, 2\}$. The utilities of the agents are

$$u_1((X^1, X^2), m) = 2\min(X^1 + X^2,\, 1) + m, \quad u_2((X^1, X^2), m) = 2\min(X^1, X^2) + m.$$

Since these utilities are quasi-linear in money, we do not specify the individual initial endowments but define only the aggregate endowment $\Omega = (1, 1)$ of indivisible goods. In the convexified economy, there exists a unique equilibrium price $p^* = (1, 1)$. At this price p^*, the demands of the agents are given, respectively, by $\mathrm{co}(\{(1,0),(0,1)\})$ and $\mathrm{co}(\{(0,0),(1,1)\})$, and accordingly, $x_1 = (1/2, 1/2)$, $x_2 = (1/2, 1/2)$ are the equilibrium allocations.

But this price p^* is not an equilibrium price of the initial discrete economy. In fact, at the price $p^* = (1, 1)$ the demands of the agents are $D_1(p^*) = \{(1,0),(0,1)\}$ and $D_2(p^*) = \{(0,0),(1,1)\}$, respectively. The initial endowment $(1, 1)$ belongs to the sum of convexified demands $\mathrm{co}(\{(1,0),(0,1)\}) + \mathrm{co}(\{(0,0),(1,1)\})$, but does not belong to the sum of their discrete counterparts, $\{(1,0),(0,1)\} + \{(0,0),(1,1)\} = \{(0,1),(1,0),(1,2),(2,1)\}$. For those sets, (5) does not imply (4). □

We now introduce the main notion.

Definition 2 *A set $\mathcal{D}$ of subsets of $\mathbb{Z}^K$ is said to be a* class of discrete convexity *if it satisfies the following axioms DC1 and DC2.*

DC1. For any $A \in \mathcal{D}$, we have $A = (\mathrm{co}(A)) \cap \mathbb{Z}^K$.

DC2. For any A and $B \in \mathcal{D}$, we have $A \pm B \in \mathcal{D}$.

Given a class $\mathcal{D}$ of discrete convexity, suppose that $D_h(p^*)$, $h \in H$, and $S_l(p^*)$, $l \in L$, all belong to $\mathcal{D}$. Then, the implication (5) $\Rightarrow$ (4) holds. In fact, because of DC2, the set $A = \sum_{h\in H} D_h(p^*) - \sum_{l\in L} S_l(p^*)$ belongs to the class $\mathcal{D}$. Since $\sum_{h\in H} W_h \in \mathbb{Z}^K_+$ and $\sum_{h\in H} W_h \in \mathrm{co}(A) = \sum_{h\in H} \mathrm{co}(D_h(p^*)) - \sum_{l\in L} \mathrm{co}(S_l(p^*))$, we conclude that DC1 implies $\sum_{h\in H} W_h \in A$.

Thus, we obtain the following conditional theorem.

Theorem 1 *Let $\mathcal{E}$ be a discrete economy. Assume*
1) there exists an equilibrium price p^* for the convexified economy $\mathrm{co}(\mathcal{E})$;
2) there exists a class $\mathcal{D}$ of discrete convex sets such that, at the price p^*, the demands $D_h(p^*)$, $h \in H$, and supplies $S_l(p^*)$, $l \in L$, all belong to $\mathcal{D}$.
Then p^* is an equilibrium price for $\mathcal{E}$.

Discrete convexity of demands and supplies is the main condition, which is to be added to the topological conditions, to ensure the existence of an equilibrium in a discrete economy with money. We do not claim that this condition is a necessary condition in a formal sense; it could occur that an equilibrium exists without discrete convexity. However, we observe that, just as the convexity is a "necessary" condition for economies with divisible goods, the discrete convexity is a "necessary" condition for economies with indivisibles and money. We can also say that our existence result does not rely upon any specifics of initial endowments nor upon the number of agents.

In Section 4, we provide some topological conditions, ensuring that the first condition in Theorem 1 is satisfied, and in Section 5, we consider an interesting class of discrete convexity associated with integral g-polymatroids.

Let us demonstrate the use of Theorem 1 in the following example. In this example we highlight the discrete-mathematical reason underlying the existence of an equilibrium in Gale's model ([11]).

Example 2. Gale considers an exchange economy in which each agent owns one indivisible object (one may think of the objects as houses). This allows us to identify the set K of objects with the set H of agents. In other words, the initial endowment W_h of agent $h \in H$ is equal to the h-th unit vector, i.e., $W_h = \mathbf{1}_h$, where, for $S \subseteq K$, $\mathbf{1}_S$ denotes the vector in $\mathbb{R}^K$ with coordinates $(\mathbf{1}_S)_k = 1$ if $k \in S$ and 0 otherwise; and $\mathbf{1}_h$ is an abbreviation for $\mathbf{1}_{\{h\}}$.

The agents' preferences are such that no agent desires more than one object. Gale does not introduce the agents' preferences explicitly in his model, but instead specifies agents' demand at each price.

Given a price $p \in \mathbb{R}^H_+$, the demand $D_h(p)$ of an agent h is a subset of $\{0\} \cup \{\mathbf{1}_h \mid h \in H\}$ (this holds because agents desire no more than one object). From the Gale's assumptions it follows that the demand correspondences $p \mapsto D_h(p)$ are upper semicontinuous and have "regular" behavior with respect to the boundary values of prices. In other words, the assumptions are "topological" indeed. They guarantee the existence of an equilibrium in the convexification of his exchange economy, i.e., there exist $p^* \in \mathbb{R}^H_+$ and $x_h \in \mathrm{co}(D_h(p^*))$, $h \in H$, such that

$$\sum_{h \in H} x_h = \mathbf{1}_H. \tag{6}$$

The equality (6) breaks down to the following list of equalities

$$\sum_{h \in H} x_{hh'} = 1, \quad h' \in H, \tag{7}$$

where $x_{hh'}$ denotes the h'-th coordinate (component) of x_h. Moreover, every x_h, $h \in H$, being a convex combination of vectors of the set $\{0\} \cup \{\mathbf{1}_h \mid h \in H\}$, satisfies the inequality

$$\sum_{h' \in H} x_{hh'} \leq 1, \quad h \in H. \tag{8}$$

By (7), all inequalities in (8) are tight. Hence, the matrix $(x_{hh'})$ is doubly stochastic. By Birkhoff's theorem, there exists a permutation $\pi : H \to H$ with $x_{h\pi(h)} > 0$, $h \in H$. This means that $\mathbf{1}_{\pi(h)} \in D_h(p^*)$. Therefore, the bundle $(p^*, (X_h = \mathbf{1}_{\pi(h)})_{h \in H})$ is an equilibrium of the discrete economy.

In the above, there is a clear distinction between topological and discrete arguments. The use of a fixed-point theorem is the topological argument, whereas the use of the Birkhoff theorem is the discrete one. Of course, the calling for the Birkhoff theorem is due to a particular feature of the agents' demands in the model: agents need no more than one item, i.e., demands are subsets of $\{0\} \cup K$.

In our set-up, the topological part remains unchanged. But the discrete part, i.e., the transition from a "divisible" equilibrium allocation to an indivisible allocation, differs. Instead of resorting to the Birkhoff theorem, we make use of the fact that the sets $D_h(p)$ consist of integer points of some faces of the simplex $\Delta = \mathrm{co}(0, \mathbf{1}_1, \ldots, \mathbf{1}_{|H|})$. The faces of Δ are the simplest integral g-polymatroids, and, as we will see later in Section 5, the class of integral g-polymatroids is a class of discrete convexity. Therefore, the demands in Gale's model are discrete-convex sets, and this is the "discrete" reason underlying the existence of an equilibrium in Gale's model. □

4 Existence of equilibria in the convexified economy

Here we describe a convexification procedure for a discrete economy and provide some conditions which guarantee the existence of an equilibrium in the associated convexified economy.

4.1 Convexification

Suppose that $\mathcal{E}$ denotes a discrete economy, as in Section 2. We shall convexify $\mathcal{E}$ through the convexification of the cost functions and preferences.

Let $f : \mathbb{Z}^K \to \mathbb{R} \cup \{+\infty\}$ be a function. Denote by $\mathrm{epi}(f) := \{(x,t) \in \mathbb{Z}^K \times \mathbb{R} \mid t \geq f(x)\}$ the epigraph of f. The *convexification* of f is a function $\mathrm{co}(f) : \mathbb{R}^K \to \mathbb{R} \cup \{+\infty\}$ of which the epigraph is the closed convex closure of $\mathrm{epi}(f)$, that is, $\mathrm{epi}(\mathrm{co}(f)) = \overline{\mathrm{co}(\mathrm{epi}(f))}$, where the epigraph of a function $g : \mathbb{R}^K \to \mathbb{R} \cup \{+\infty\}$ is defined naturally as $\mathrm{epi}(g) := \{(x,t) \in \mathbb{R}^K \times \mathbb{R} \mid t \geq g(x)\}$. Alternatively, we can define $\mathrm{co}(f)$ as the supremum of affine functions $h : \mathbb{R}^K \to \mathbb{R} \cup \{+\infty\}$ such that $h(X) \leq f(X)$ for every $X \in \mathbb{Z}^K$. We implicitly assume here that such affine functions exist. Clearly, $\mathrm{co}(f)$ is a closed convex function. (Recall that a function is said to be closed if it has a closed epigraph).

Definition 3 *A function $f : \mathbb{Z}^K \to \mathbb{R} \cup \{+\infty\}$ is said to be* pseudoconvex *if*
a) $\mathrm{co}(\mathrm{epi}(f))$ *is a closed subset of* $\mathbb{R}^K \times \mathbb{R}$, *and*
b) for every $X \in \mathbb{Z}^K$, *there holds* $f(X) = \mathrm{co}(f)(X)$.
A function $f : \mathbb{Z}^K \to \mathbb{R} \cup \{-\infty\}$ is said to be pseudoconcave *if $-f$ is pseudoconvex.*

It follows from *a)* that, for any $x \in \mathbb{R}^K$, there exist points $X_0, X_1, \ldots, X_n \in \mathbb{Z}^K$, where $n = |K|$, and weights $\alpha_0, \alpha_1, \ldots, \alpha_n$ ($\alpha_k \geq 0$, $\sum_{k=0}^n \alpha_k = 1$) such that $\mathrm{co}(f)(x) = \sum_{k=0}^n \alpha_k f(X_k)$.

The convexification of the cost functions is straightforward from the definition. To define a convexification of preferences, we need some assumptions.

Assumption 1. For any consumer $h \in H$ and for any $Y \in \mathbb{Z}^K_+$ there holds

$$(W_h, w_h) \succ_h (Y, 0). \qquad \square$$

This assumption may be compared to "abundance of money", and, in particular, it states that each consumer has a positive endowment of money (this assumption is rather strong but it seems necessary, though it could be weakened).

Remark. Since equilibrium allocations are individually rational allocations, the shape of the preference of the consumer h on the set $\{(X,m) \in \mathbb{Z}^K_+ \times \mathbb{R}_+ \,|\, (X,m) \prec_h (W_h, w_h)\}$ is irrelevant.

Assumption 2. For any consumer $h \in H$ and for any (X,m) with $m \geq 0$, there exists a certain amount of money $m' \in \mathbb{R}_+$ such that

$$(\mathbf{0}, m') \succeq_h (X, m). \qquad \square$$

This assumption means that money can substitute any quantity of indivisible goods, and, in particular, it implies that any bundle (X,m) has a money equivalent. In fact, if $t_h(X,m)$ denotes the infimum of such m', then, it is easy to see that,

$(\mathbf{0}, t_h(X,m)) \sim_h (X,m)$. Denote by m_h^0 the money equivalent of the initial endowment (W_h, w_h). According to Assumption 1, for any $m \geq m_h^0$ and $X \in \mathbb{Z}_+^K$, there exists a unique number $q_h^m(X) \geq 0$ such that $(\mathbf{0}, m) \sim_h (X, q_h^m(X))$. The graph of $q_h^m(X)$ is the "indifference curve" of $\preceq_h$ passing through the point $(\mathbf{0}, m)$. Thus, we can represent a preference $\preceq$ (satisfying Assumption 2) together with the money equivalent of a vector of initial endowments by the family of functions q^m on $\mathbb{Z}_+^K$ for $m \geq m_0$.

Lemma 1 *The family of functions $q^m(X)$ has the following properties:*
1) $q^m(\mathbf{0}) = m$,
2) $q^m(X)$ is decreasing in X,
3) $q^m(X)$ is (strictly) increasing in m,
4) $q^m(X) \to +\infty$ as $m \to +\infty$,
5) $q^m(X)$ is continuous in m.
Conversely, a family of functions $q^m(X)$ with properties 1)–5) defines a preference $\preceq$ which is continuous, increasing in X, strictly increasing in money, and satisfies Assumption 2. □

Proposition 1 *Suppose that functions $q^m : \mathbb{Z}_+^K \to \mathbb{R}$ $(m \geq m_0)$ have the properties 1)–5) and are pseudoconvex. Then the functions $\mathrm{co}(q^m) : \mathbb{R}^K \to \mathbb{R}$ also satisfy properties 1)–5).*

Proof. Properties 1)–4) for $\mathrm{co}(q^m)$ are almost obvious. Let us prove the property 5). Let $m \geq m_0$ and let m' be sufficiently close to m. First, consider the case of $m' \geq m$. For $x \in \mathbb{R}_+^K$ let X_k and α_k be such that $\mathrm{co}(q^m)(x) = \sum_{k=0}^n \alpha_k q^m(X_k)$. Since $\mathrm{co}(q^{m'})(X_k) \leq \mathrm{co}(q^m)(X_k) + \epsilon$ for $k \in \{0, 1, \ldots, n\}$, we have

$$\mathrm{co}(q^{m'})(x) \leq \sum_k \alpha_k \mathrm{co}(q^{m'})(X_k) \leq \sum_k \alpha_k \mathrm{co}(q^m)(X_k) + \epsilon.$$

Next, consider the case of $m' \leq m$ with m' tending to m. For $x \in \mathbb{R}_+^K$ let X'_k and α'_k be the corresponding points and weights for m' such that $x = \sum_k \alpha'_k X'_k$ and

$$\mathrm{co}(q^{m'})(x) = \sum_k \alpha'_k \mathrm{co}(q^{m'})(X'_k).$$

We can suppose that each sequence α'_k (with k fixed) converges to some (limit) weight α_k. Let $\alpha_0, \ldots, \alpha_s$ be positive and $\alpha_{s+1}, \ldots, \alpha_n$ be equal to 0. Then we can suppose that sequences $X'_0, \ldots, X'_s$ also converge (and stabilize) to some (finite)

points $X_0, \ldots, X_s$. In fact, if, say, X'_0 is unbounded, then $\sum_k \alpha'_k X'_k$ moves away from the point x, but this cannot happen. Now,

$$\begin{aligned}
&\mathrm{co}(q^m)(x) - \mathrm{co}(q^{m'})(x) \\
&\leq \sum_k \alpha'_k q^m(X'_k) - \sum_k \alpha'_k q^{m'}(X'_k) \\
&= \sum_k \alpha'_k (q^m(X'_k) - q^{m'}(X'_k)) \\
&= \sum_{k=0}^{s} \alpha'_k (q^m(X'_k) - q^{m'}(X'_k)) + \sum_{k=s+1}^{n} \alpha'_k (q^m(X'_k) - q^{m'}(X'_k)).
\end{aligned}$$

When m' is sufficiently close to m, the first term is small because it is equal to $\sum_{k=0}^{s} \alpha'_k (q^m(X_k) - q^{m'}(X_k))$. The second term is also small, since $\alpha'_{s+1}, \ldots, \alpha'_n$ are small and $q^m(X'_k) - q^{m'}(X'_k)$ are bounded (because $q^m(X'_k) \leq q^{m'}(\mathbf{0})$ and $q^{m'}(X'_k) \geq 0$). Q.E.D.

ASSUMPTION 3. The functions q_h^m for $h \in H$, $m \geq m_0^h$ and the cost functions c_l for $l \in L$ are pseudoconvex functions. □

It follows from Lemma 1 and Proposition 1 that, for each $h \in H$, the family of functions $\mathrm{co}(q_h^m)$ determines a preference on $\mathbb{R}_+^K \times \mathbb{R}_+$, which we call the convexification of $\preceq_h$ and denote as $\mathrm{co}(\preceq_h)$.

We have thus constructed an economy $\mathrm{co}(\mathcal{E})$, of which the commodity space is $\mathbb{R}^K \times \mathbb{R}$, the initial endowments are (W_h, w_h), $h \in H$, the preferences are $\mathrm{co}(\preceq_h)$, $h \in H$, and the cost functions are $\mathrm{co}(c_l)$, $l \in L$.

In addition, we assert that, for any price $p \in \mathbb{R}_+^K$, it holds that

$$\check{S}_l(p) = \mathrm{co}(S_l(p)), \quad l \in L, \tag{9}$$

$$\check{D}_h(p) = \mathrm{co}(D_h(p)), \quad h \in H, \tag{10}$$

where

$$\check{S}_l(p) = \mathrm{Argmax}_{y \in \mathbb{R}_+^K} p(y) - \mathrm{co}(c_l)(y), \quad l \in L,$$

and $\check{D}_h(p)$ for $h \in H$ is the set of all $x \in \mathbb{R}_+^K$ such that $(x, \beta_h(p) - p(x))$ is a best element in the budget set

$$\overline{B}_h(p) = \{(x', m) \in \mathbb{R}_+^K \times \mathbb{R}_+ \,|\, p(x') + m \leq \beta_h(p)\}$$

with respect to the preference $\mathrm{co}(\preceq_h)$.

We prove the equality (9), where the subscript l is omitted. The inclusion $\subset$ is trivial. Assume $x \in \check{S}(p)$, that is,

$$p(x) - \mathrm{co}(c)(x) \geq p(y) - \mathrm{co}(c)(y)$$

for any $y \in \mathbb{R}^K_+$. Let $x = \sum_k \alpha_k X_k$ be the convex representation such that $\mathrm{co}(c)(x) = \sum_k \alpha_k c(X_k)$ holds. Then, for each k,

$$p(x) - \mathrm{co}(c)(x) \geq p(X_k) - \mathrm{co}(c)(X_k) = p(X_k) - c(X_k). \tag{11}$$

By multiplying (11) with α_k and adding over k, we obtain

$$p(x) - \mathrm{co}(c)(x) \geq \sum_k \alpha_k (p(X_k) - c(X_k)) = p(x) - c(x).$$

We see from this that $p(x) - \mathrm{co}(c)(x) = p(X_k) - c(X_k)$ if $\alpha_k \neq 0$. Therefore

$$p(X_k) - c(X_k) \geq p(Y) - c(Y)$$

for every k and $Y \in \mathbb{Z}^K_+$, that is, $X_k \in S(p)$ and $x \in \mathrm{co}(S(p))$. Thus the equality (9) has been proven, and the second equality (10) can be shown similarly.

Hence, the economy $\mathrm{co}(\mathcal{E})$ is the convexification of $\mathcal{E}$ in the sense of Section 3. We need the last assumption in order to ensure the existence of an equilibrium in the convexified economy $\mathrm{co}(\mathcal{E})$.

ASSUMPTION 4. The total endowment of discrete goods is strictly positive. i.e., $\sum_{h \in H} W_h > 0$. The cost functions have the property that $c_l(Y) \to +\infty$ as $||Y|| \to +\infty$ for each $l \in L$. □

The second part of Assumption 4 is the "no-free-lunch" condition. It holds, for example, if $c_l(Y) = +\infty$ for $Y \neq \mathbf{0}$.

Proposition 2 *Let $\mathcal{E}$ be a discrete economy and let Assumptions 1–4 hold. Then an equilibrium exists in the convexified economy* $\mathrm{co}(\mathcal{E})$.

See Appendix for a proof.

To combine this proposition with the conditional Theorem 1, we strengthen Assumption 3 to Assumption 3′ below. Namely, we assume that the supply and demand sets belong to some class of discrete convexity $\mathcal{D}$ for any price.

Definition 4 *Let $\mathcal{D}$ be a class of discrete convexity in $\mathbb{Z}^K$. A pseudoconvex function $f : \mathbb{Z}^K \to \mathbb{R} \cup \{+\infty\}$ is said to be* $\mathcal{D}$-convex *if*

$$\mathrm{Argmax}_{X \in \mathbb{Z}^K}(p(X) - f(X)) \in \mathcal{D}$$

for any linear functional $p \in (\mathbb{R}^K)^*$.

ASSUMPTION 3′. The functions q_h^m for $m \in \mathbb{R}_+$, $h \in H$ and the cost functions c_l for $l \in L$ are $\mathcal{D}$-convex functions for some class of discrete convexity $\mathcal{D}$. □

Combinination of Proposition 2 and Theorem 1 yields the following existence theorem.

Theorem 2 *Let $\mathcal{E}$ be a discrete economy and let Assumptions 1, 2, 3′ and 4 hold. Then a competitive equilibrium exists in $\mathcal{E}$.*

4.2 The case of transferable utilities

The special case of an economy with transferable utility is more tractable, in that an equilibrium can be obtained as a solution to an auxiliary optimization problem. Moreover, we use this optimization problem in our proof of Proposition 2.

Recall that a preference $\preceq$ on $\mathbb{R}_+^K \times \mathbb{R}$ is said to be *transferable* (quasi-linear in money), if it can be represented by a utility function of the form $U(X, m) = u(X)+m$ with some function $u : \mathbb{R}_+^K \to \mathbb{R}$. Let $\widehat{\mathcal{E}}$ be an economy with divisible goods and transferable utilities. Consumers' utility functions are $u_h : \mathbb{R}_+^K \to \mathbb{R}$, $h \in H$, and producers' cost functions are $c_l : \mathbb{R}_+^K \to \mathbb{R} \cup \{+\infty\}$, $l \in L$. It is assumed that the functions u_h $(h \in H)$ are closed concave, and that functions c_l $(l \in L)$ are closed convex.

Given a price p, the demand $D_h(p)$ of the consumer $h \in H$ is defined as the set of solutions to the problem

$$\max_{x \in \mathbb{R}_+^K}(u_h(x) - p(x)). \tag{12}$$

The supply $S_l(p)$ of the producer $l \in L$ is the set of solutions to

$$\max_{x \in \mathbb{R}_+^K}(p(x) - c_l(x)). \tag{13}$$

A tuple $((x_h)_{h\in H}, (y_l)_{l\in L}, p)$ is said to be an *equilibrium of the economy* $\widehat{\mathcal{E}} = \{(u_h, W_h),\ h \in H,\ c_l,\ l \in L\}$ *with transferable utilities*, if $x_h \in D_h(p)$ for each $h \in H$ and $y_l \in S_l(p)$ for each $l \in L$, and the equality $\sum_{h\in H} x_h = \sum_{h\in H} W_h + \sum_{l\in L} y_l$ holds true.

We recall that an equilibrium in an economy with transferable utilities can be obtained as a solution to an optimization problem. To be specific, we consider the following aggregate utility and aggregate cost functions. The aggregate utility function U is given as the co-convolution of the functions u_h with $h \in H$, i.e.,

$$U(x) = \sup\{\sum_{h\in H} u_h(x_h) \mid \sum_{h\in H} x_h = x,\ x_h \in \mathbb{R}_+^K\}, \quad x \in \mathbb{R}_+^K.$$

The aggregate cost function C is given as the convolution of the functions c_l with $l \in L$, i.e.,

$$C(y) = \min\{\sum_{l\in L} c_l(y_l) \mid \sum_{l\in L} y_l = y,\ y_l \in \mathbb{R}^K_+\}, \quad y \in \mathbb{R}^K_+.$$

Proposition 3 *The aggregate functions C and U have the following properties:*

$$S(p) = \sum_{l\in L} S_l(p), \tag{14}$$

and

$$D(p) = \sum_{h\in H} D_h(p), \tag{15}$$

where $S(p) := \mathrm{Argmax}(p(y) - C(y))$, $D(p) := \mathrm{Argmax}(U(x) - p(x))$, $S_l(p) := \mathrm{Argmax}(p(y) - c_l(y))$, and $D_h(p) := \mathrm{Argmax}(u_h(x) - p(x))$.

Proof. Let us prove (14); (15) is proven similarly. By the definition of convolution, the epigraph of C is equal to the closure of the sum of the epigraphs of c_l. Since $c_l \geq 0$, its epigraph is a subset of the positive orthant $\mathbb{R}^K_+ \times \mathbb{R}_+$. Since the sum of closed convex subsets of $\mathbb{R}^K_+ \times \mathbb{R}_+$ is closed (see, for example, [21], Section 9), we have

$$\mathrm{epi}(C) = \sum_l \mathrm{epi}(c_l). \tag{16}$$

The identity (14) follows from (16) by a standard argument. Q.E.D.

An equilibrium in the economy $\widehat{\mathcal{E}}$ exists if and only if there exists a solution to the optimization problem

$$\max_{y\in\mathbb{R}^K_+} U(W + y) - C(y), \tag{17}$$

where $W = \sum_{h\in H} W_h$. Indeed, since equilibria are Pareto optimal, any equilibrium allocation gives a solution to (17). Conversely, let y^* be a solution to (17). Then we have

$$C(y) - C(y^*) \geq U(W + y) - U(W + y^*)$$

for any $y \in \mathbb{R}^K_+$. The function on the left hand side is concave, whereas the function on the right is convex. Therefore there exists a separating linear functional, and we can take any such functional to construct an equilibrium price. By (14) and (15), we can disaggregate both the vector $W + y^*$ into consumers' optimal solutions and the vector y^* into producers' optimal solutions.

A solution to (17) exists, when, for example, the individual utility functions are bounded and $C(y) \to +\infty$ with $||y|| \to +\infty$. In such a case, the concave function $U(W+y)-C(y)$ goes to $-\infty$ with $||y|| \to +\infty$. Existence of a solution is guaranteed

also when the cost functions have bounded effective domains[4]. Note that a pure exchange economy is obtained by setting $c(\mathbf{0}) = 0$ and $c(Y) = +\infty$ for $Y \neq 0$.

We now turn to a discrete economy by considering utility functions of the form $U(X, m) = u(X) + m$ with some function $u : \mathbb{Z}_+^K \to \mathbb{R}$, where $u(0) = 0$. Just as in the divisible case, the demand of the consumer h at a price p is defined as the set of solutions to the optimization problem

$$\max_{X \in \mathbb{Z}_+^K} (u_h(X) - p(X)). \tag{18}$$

A tuple $((X_h)_{h \in H}, (Y_l)_{l \in L}, p)$ is said to be an *equilibrium of an economy* $\mathcal{E}^{tu} = \{(u_h, W_h),\ h \in H,\ c_l,\ l \in L\}$ *with transferable utilities*, if X_h is a solution to (18) for each $h \in H$, Y_l is a solution to (1) for each $l \in L$, and the balance (2) holds.

An equilibrium of the convexified economy $\mathrm{co}(\mathcal{E}^{tu})$ exists if the utility functions are pseudoconcave, if the cost functions are pseudoconvex, and if (17) has a solution for their convexified counterparts. Thus, by our conditional Theorem 1, we have the following existence theorem for a discrete economy with transferable utilities.

Theorem 3 *Let $\mathcal{E}^{tu}$ be a discrete economy with transferable utilities. Suppose, for some class $\mathcal{D}$ of discrete convexity, that the utility functions u_h ($h \in H$) are $\mathcal{D}$-concave and the cost functions c_l ($l \in L$) are $\mathcal{D}$-convex with bounded effective domains. Then a competitive equilibrium exists in $\mathcal{E}^{tu}$.*

Remark. Theorem 3 states the existence of an equilibrium in terms of individual utility functions and cost functions. Since the disaggregation of a solution to (17) yields an equilibrium in models with transferable utilities, a necessary and sufficient condition for the existence of an equilibrium in $\mathcal{E}^{tu}$ can be formulated in aggregate terms (see, for example, [5] or [3]). An equilibrium for the economy $\mathcal{E}^{tu}$ exists if and only if there exists a Pareto optimal allocation of indivisible goods in the economy $\mathcal{E}^{tu}$, which would be Pareto optimal in the convexified economy $\mathrm{co}(\mathcal{E}^{tu})$. However, such necessary and sufficient condition, formulated in aggregate terms, is of little interest. □

So far, we have established the existence of an equilibrium for a discrete economy modulo a hypothetical class $\mathcal{D}$ of discrete-convex sets. What remains is to provide an interesting class of discrete-convex sets. In the next section, we shall consider the class of discrete convexity associated with integral generalized polymatroids, and show that in all known models of economies with indivisibilities and money, in which equilibria exist, the demand sets belong to this class.

[4] The effective domain of a function is the subset of the domain of definition where the function does not equal $+\infty$.

5 Discrete-convex sets and functions

5.1 Integral polyhedra and discrete convexity

A class of discrete convexity can be constructed from a well-behaved class of integral polyhedra. A polyhedron is the intersection of a finite number of closed halfspaces. A polyhedron $P \subset \mathbb{R}^K$ is said to be an *integral polyhedron* if $P = \mathrm{co}(P(\mathbb{Z}))$, where $P(\mathbb{Z}) := P \cap \mathbb{Z}^K$.

Assume that we are given a class $\mathcal{P}$ of polyhedra with the following properties:

DCP1. Any polyhedron $P \in \mathcal{P}$ is integral,

DCP2. For any polyhedra P, $Q \in \mathcal{P}$, we have $P \pm Q \in \mathcal{P}$ and

$$(P \pm Q)(\mathbb{Z}) = P(\mathbb{Z}) \pm Q(\mathbb{Z}). \tag{19}$$

A class of polyhedra $\mathcal{P}$ satisfying properties DCP1 and DCP2 is said to be a *polyhedral class of discrete convexity*. For such $\mathcal{P}$, the class $\mathcal{D}(\mathcal{P})$ of subsets of $\mathbb{Z}^K$ defined by $\mathcal{D}(\mathcal{P}) = \{P(\mathbb{Z}) \mid P \in \mathcal{P}\}$ satisfies DC1 and DC2. That is, $\mathcal{D}(\mathcal{P})$ forms a class of discrete convexity. We say that the class $\mathcal{D}(\mathcal{P})$ is *associated* with the polyhedral class $\mathcal{P}$.

We may always assume that a polyhedral class of discrete convexity $\mathcal{P}$ contains all singleton sets of integer points. For such a class $\mathcal{P}$ of polyhedra, (19) is equivalent to the following property, which is more convenient to verify:

$$\text{For any } P, Q \in \mathcal{P},\ P \cap Q \text{ contains an integer point if nonempty.} \tag{20}$$

The equivalence (20) $\Leftrightarrow$ (19) can be shown easily as follows: Let $X \in \mathbb{Z}^K$ be an integer point of $P - Q$, that is, $X = p - q$, $p \in P$, $q \in Q$. Then the intersection of P and $Q + X$ is nonempty and, because of (20), $P \cap (Q + X)$ contains an integral point. Therefore, we can choose p and q in $\mathbb{Z}^K$. This shows the implication (20) $\Rightarrow$ (19). Conversely, if $P \cap Q \neq \emptyset$, then $0 \in P - Q$. According to (19), 0 is obtained as the difference of two integer points of P and Q, i.e., $P \cap Q$ contains an integer point.

In dimension one, an integral polyhedron is nothing but a segment with integral endpoints, and the class of all integral polyhedra is a polyhedral class of discrete convexity. This is not the case in higher dimensions, as is easily seen.

Example 3 (*Hexagons*). Consider a class $\mathcal{H}$ of polyhedra in $\mathbb{R}^2$, which consists of polyhedra defined by the inequalities $a_1 \leq x_1 \leq b_1$, $a_2 \leq x_2 \leq b_2$, $c \leq x_1 + x_2 \leq d$,

where a_1, a_2, b_1, b_2, c and d are integers. It is easy to check that such hexagons (possibly degenerate to polyhedra with a smaller number of edges) has integral vertices. Since the intersection of hexagons of $\mathcal{H}$ is a hexagon of $\mathcal{H}$, we conclude, by (20), that $\mathcal{H}$ is a class of discrete convexity. □

We observe in Example 3 that the edges of the hexagons are parallel to one of e_1, e_2 or $e_1 - e_2$, where e_i denotes the ith unit vector. These vectors have the following property: any pair of the vectors forms a basis of the integer lattice $\mathbb{Z}^2$. As one can see from Example 1, if a class of integral polyhedra of $\mathbb{R}^2$ contains polyhedra which have two edges, one parallel to $e_1 - e_2$ and the other to $e_1 + e_2$, then it cannot be a class of discrete convexity. In fact, the pair of vectors $e_1 - e_2$ and $e_1 + e_2$ does not form a basis of $\mathbb{Z}^2$: for example, the point e_1 cannot be represented a linear combination of the vectors $e_1 - e_2$ and $e_1 + e_2$ with integer coefficients.

More generally, the property that "for a given collection of integral polytopes, any linearly independent subset of $|K|$ primitive vectors, which are parallel to edges of these polytopes form a basis of the abelian group $\mathbb{Z}^K$" turns out to be crucial for the collection to be a polyhedral class of discrete convexity.

Let us introduce the following notion: A collection $\mathcal{R}$ of vectors of $\mathbb{R}^K$ is said to be a *unimodular system* if, for any subset $R \subset \mathcal{R}$, the abelian group $\mathbb{Z}(R) = \{\sum_i a_i r_i \,|\, r_i \in R,\, a_i \in \mathbb{Z}\}$ coincides with the abelian subgroup $\mathbb{R}(R) \cap \mathbb{Z}^K$, where $\mathbb{R}(R) = \{\sum_i a_i r_i \,|\, r_i \in R,\, a_i \in \mathbb{R}\}$. We now give a precise statement (see [4] for a proof).

Theorem 4 *Let $\mathcal{P}$ be a collection of integral pointed polyhedra of $\mathbb{R}^K$ closed with respect to taking faces. Let $\mathcal{R}(\mathcal{P})$ denote the set of primitive vectors in $\mathbb{Z}^K$, which are parallel to edges of polyhedra of $\mathcal{P}$[5]. Then $\mathcal{P}$ is a polyhedral class of discrete convexity if and only if $\mathcal{R}(\mathcal{P})$ is a unimodular system.*

Recall that a polyhedron is said to be *pointed* if it has at least one vertex. Of course, a polytope is a pointed polyhedron.

Remark. In our model, the convex hulls of demands and supplies, being subsets of $\mathbb{R}^K_+$, are pointed polyhedra. Hence the classes of discrete convexity relevant to economic applications are those associated with pointed polyhedra. We can also assume that the set $\mathcal{R}(\mathcal{P})$ contains the standard basis $\{e_1, \ldots, e_{|K|}\}$ of $\mathbb{Z}^K$. In such a case, $\mathcal{R}(\mathcal{P})$ is a unimodular system if and only if every $|K|$ linearly independent vectors $r_1, \ldots, r_{|K|}$ of $\mathcal{R}(\mathcal{P})$ is a basis of the abelian group $\mathbb{Z}^K$ (see, for example [22], Chapter 19). □

[5] A vector r belongs to $\mathcal{R}(\mathcal{P})$ if and only if there is a polyhedron $P \in \mathcal{P}$ which has an edge of the form $[x, x + ar]$ or $x + \mathbb{R}_+ r$ with some $x \in \mathbb{Z}^K$ and $a \in \mathbb{N}$.

Theorem 4 shows a recipe to construct polyhedral classes of discrete convexity: Take a unimodular system $\mathcal{R}$ and consider all the integral polytopes whose edges are parallel to the vectors of $\mathcal{R}$. We now give an interesting and famous example of a unimodular system.

Example 4. The set $\mathbb{A}_K := \{\pm e_i, e_i - e_j \mid i, j \in K\}$ of vectors of $\mathbb{Z}^K$ is a unimodular system. Since $\mathbb{A}_K$ contains the standard basis, we need to show that any subset of $|K|$ linearly independent vectors of $\mathbb{A}_K$ forms a basis of $\mathbb{Z}^K$ (see the previous Remark).
Let $B \subset \mathbb{A}_K$ be a basis of $\mathbb{R}^K$. We are to show that B is a basis of $\mathbb{Z}^K$. One of the $\pm e_i$, $i \in K$, belongs to B, since otherwise B is a subset of the hyperplane $\sum_{i \in K} x_i = 0$ and B cannot be a basis of $\mathbb{R}^K$, where $\{x_i \mid i \in K\}$ denotes the dual basis to $\{e_i \mid i \in K\}$. Let $e_1 \in B$. If none of the vectors $\pm(e_i - e_1)$ belongs to B, then the set $B \setminus \{e_1\}$ is contained in the subspace $x_1 = 0$. By induction, $B \setminus \{e_1\}$ forms a basis of $\mathbb{Z}^{K \setminus \{1\}}$. Hence B is a basis of $\mathbb{Z}^K$. If $e_j - e_1$ belongs to B with some j, then, the replacement of $e_j - e_1$ with $e_j = e_1 + (e_j - e_1)$ in B yields a new basis B'. Obviously, B is a base of $\mathbb{Z}^K$ if and only if so is B'. Repeating this, we finally obtain B' that is free from the vectors $\pm(e_i - e_1)$. Then, B' is a basis of $\mathbb{Z}^K$, and, hence, so is B. □

Denote by $\mathcal{P}t(\mathbb{A}_K)$ the class of pointed polyhedra of $\mathbb{R}^K$ whose edges are parallel to vectors of $\mathbb{A}_K$, and denote by $\mathcal{IGP}$ its subclass of integer polyhedra. The class of polyhedra $\mathcal{P}t(\mathbb{A}_K)$ coincides with the class of *generalized polymatroids* (*g*-polymatroids) well known in discrete mathematics. Generalized polymatroids were introduced by Frank ([8]) as a generalization of polymatroids of Edmonds ([6]). This class of polyhedra, moreover, is equivalent to the class of cores of convex cooperative games, explored by Shapley ([23]). We mention that in discrete mathematics generalized polymatroids are investigated mainly in terms of systems of linear inequalities having "submodularity" property (in an appropriate sense) (see [9, 10]). However, this viewpoint is not used in the sequel. A subset of $\mathbb{Z}^K$ is called here a *PM-set* if it belongs to the class of discrete convexity associated with integral *g*-polymatroids.

A number of fundamental operations can be defined for integral *g*-polymatroids. The sum of integral *g*-polymatroids is again an integral *g*-polymatroid; the class of integral *g*-polymatroids contains segments $[0, r]$ for $r \in \mathbb{A}_K$, the sum of such segments, the simplex Δ and all its faces, the sum of these faces. We can also construct products; if P and Q are integral *g*-polymatroids in $\mathbb{R}^N$ and $\mathbb{R}^M$, respectively, then $P \times Q$ is an integral *g*-polymatroid in $\mathbb{R}^N \times \mathbb{R}^M$. We can use projections along a set of coordinates and, more generally, homomorphisms of lattices of integers of special types ([4]) to construct new integral *g*-polymatroids that are the images under such homomorphisms.

We now describe a few classes of easily recognizable g-polymatroids given by specific systems of inequalities. Recall, that a family $\mathcal{T}$ of subsets of a set K is called *laminar* if, for any A, $B \in \mathcal{T}$, there holds either $A \subseteq B$, or $B \subseteq A$, or $A \cap B = \emptyset$. Note that we may always assume that a laminar family $\mathcal{T}$ contains all singletons $\{k\}$, $k \in K$, and the whole set K.

Example 5. Let $\mathcal{T}$ be the collection of singletons and the elements of a chain $\mathcal{C} = \{K \supset C_1 \supset C_2 \supset \cdots \supset C_m\}$ with $m \leq |K|$. Then $\mathcal{T}$ is a laminar family. $\Box$

Proposition 4 *Let $\mathcal{T}$ be a laminar family. A polyhedron defined by the inequalities*

$$a_A \leq x(A) \leq b_A, \quad A \in \mathcal{T},$$

is a generalized polymatroid. If a_A and b_A are integers for every $A \in \mathcal{T}$, this polyhedron is an integral g-polymatroid.

Proof. We show that the edges of a polyhedron defined by the inequalities $a_A \leq x(A) \leq b_A$ for $A \in \mathcal{T}$ are parallel to the vectors of $\mathbb{A}_K$. For this, consider $(n-1)$ linearly independent functionals $\mathbf{1}_{S_i}$ with $S_i \in \mathcal{T}$ $(i = 1, \ldots, n-1)$ that defines an edge, where $n = |K|$ and $\mathbf{1}_{S_i}(x) = \sum_{l \in S_i} x^l$ for $x = (x^l)$.

There are two different cases: $K \neq \bigcup_{i=1}^{n-1} S_i$ and $K = \bigcup_{i=1}^{n-1} S_i$. In the first case, there exists exactly one element, say $k \in K$, outside $\bigcup S_i$, and the common kernel of $\mathbf{1}_{S_i}$, $i = 1, \ldots, n-1$, is generated by e_k. In the second case, since $\mathcal{T}$ is laminar and the kernel is one-dimensional, one can show that there exists a unique "nonseparable pair" $k, l \in K$, i.e., there is a unique pair of elements $k, l \in K$ such that there is no set in the collection S_i, $i = 1, \ldots, n-1$, which separates k and l ($k \in S_i$, $l \notin S_i$, or vice versa). Therefore, the kernel is proportional to $e_k - e_l$.

Since the edges of the polyhedron defined by $a_A \leq x(A) \leq b_A$, $A \in \mathcal{T}$, are parallel to kernels of $(n-1)$ linearly independent functionals $\mathbf{1}_{S_i}$, $S_i \in \mathcal{T}$, $i = 1, \ldots, n-1$, and, since these kernels are generated by vectors of $\mathbb{A}_K$, this polyhedron is a g-polymatroid. Similarly, one can show that if a_A and b_A are integers for all $A \in \mathcal{T}$, then any vertex of this polyhedron is an integer vector. Hence it is an integral g-polymatroid (for details, see [4], or see [9] for an alternative proof). Q.E.D.

Given a laminar family $\mathcal{T}$, denote by $\mathcal{P}(\mathcal{T})$ the class of polyhedra of the form

$$a_A \leq x(A) \leq b_A, \quad A \in \mathcal{T},$$

where a_A and b_A are integers for each $A \in \mathcal{T}$. Obviously, the intersection of polyhedra in $\mathcal{P}(\mathcal{T})$ remains in the class. The sum of polyhedra in $\mathcal{P}(\mathcal{T})$ is not necessarily a member of $\mathcal{P}(\mathcal{T})$, but it is always an integral g-polymatroid.

Remark. Demand sets in Gale's model (see Example 2) are PM-sets. In fact, the polytope given by a system of inequalities, $x(K) \leq 1$ and $x^k \geq 0$ for $k \in K$, is the simplex $\Delta_K = \mathrm{co}(\{0\} \cup \{\mathbf{1}_k \mid k \in K\})$. The simplex Δ_K belongs to $\mathcal{P}(\mathcal{T})$, where $\mathcal{T}$ is the trivial laminar set consisting of the collection of all singletons and the whole set K. All faces of Δ_K are defined by similar systems of inequalities. Thus, the demand sets, being the integer points of the faces of the simplex Δ_K, are PM-sets. □

5.2 $\mathcal{IGP}$-convex functions

A function $f : \mathbb{Z}^K \to \mathbb{R} \cup \{+\infty\}$ is said to be $\mathcal{IGP}$-convex if it fits Definition 4 for the class of PM-sets. Murota and Shioura ([19]) considered such class of functions under the name of $\mathrm{M}^\natural$-convex functions; see, also [17, 18].

We mention that, since $\mathcal{IGP}$ is a (polyhedral) class of discrete convexity, the convolution of a number of $\mathcal{IGP}$-convex functions is $\mathcal{IGP}$-convex. Also note that the sum of $\mathcal{IGP}$-convex functions is not $\mathcal{IGP}$-convex in general, since the class of integral g-polymatroids fails to be closed with respect to intersection. However, we have the following.

Proposition 5 *Let $\mathcal{T}$ be a laminar family of subsets of K. For every $A \in \mathcal{T}$, let $f_A : \mathbb{Z} \to \mathbb{R} \cup \{+\infty\}$ be a pseudoconvex function. Then the function $f : \mathbb{Z}^K \to \mathbb{R} \cup \{+\infty\}$ defined by*

$$f(X) = \sum_{A \in \mathcal{T}} f_A(X(A)), \quad X \in \mathbb{Z}^K, \tag{21}$$

is $\mathcal{IGP}$-convex.

Proof. Recall that $f^* : (\mathbb{R}^K)^* \to \mathbb{R}$ defined by $f^*(p) := \sup_{x \in \mathbb{R}^K}(p(x) - f(x))$ is called the conjugate function or the Legendre–Young–Fenchel transform of f. Let f and g be closed convex functions such that $\mathrm{epi}(f^*) + \mathrm{epi}(g^*)$ is a closed set. Then, it is not difficult to see that $\partial^*(f+g)(p) = \partial^* f(p_1) \cap \partial^* g(p_2)$ for some $p_1, p_2 \in (\mathbb{R}^K)^*$ such that $p = p_1 + p_2$, where $\partial^* f(p) = \{x \in \mathbb{R}^K \mid p(y) - p(x) \leq f(y) - f(x) \, \forall y \in \mathbb{R}^K\}$.

The effective domain of the conjugate function $(f_A)^*$ is located on the line $\mathbb{R}(\mathbf{1}_A)$. Therefore, the sum of the epigraphs $\mathrm{epi}(f_A^*) + \mathrm{epi}(f_B^*)$ is closed for any $A \neq B$. Hence, an area of affinity of the function $\sum_{A \in \mathcal{T}} f_A(X(A))$ is the intersection of strips of the form $a_A \leq x(A) \leq b_A$ for $A \in \mathcal{T}$. The area of affinity is a polyhedron in $\mathcal{P}(\mathcal{T})$, and hence is an integral g-polymatroid. Therefore, f is an $\mathcal{IGP}$-convex function. Q.E.D.

By Proposition 5 and Example 5, we have the following.

Corollary 1 *Let $\mathcal{T}$ be the collection of singletons and elements of a chain $\mathcal{C} = \{K = C_1 \supset C_2 \supset \cdots \supset C_m\}$. Let $f_k : \mathbb{Z} \to \mathbb{R} \cup \{+\infty\}$ with $k \in K$, and $g_i : \mathbb{Z} \to \mathbb{R} \cup \{+\infty\}$ with $i = 1, \ldots, m$, be pseudoconvex functions. Then the function*

$$f(X) = \sum_{k \in K} f_k(X^k) + \sum_{i=1}^{m} g_i(\sum_{k \in C_i} X^k), \quad X = (X^k) \in \mathbb{Z}^K, \tag{22}$$

is $\mathcal{IGP}$-convex.

Corollary 1 above is illustrated by the following examples.

Example 6. Consider a separable convex function, i.e., a function of the form

$$f(X) = \sum_{k \in K} f_k(X^k),$$

where f_k $(k \in K)$ are convex functions on $\mathbb{R}$. This is a special case of (22) with $\mathcal{T}$ consisting only of singletons. Hence separable convex functions are $\mathcal{IGP}$-convex, and Theorem 2 along with this class of functions is precisely a multidimensional generalization of Henry's result ([13]). □

Example 7. Consider a quasi-separable convex function, which is defined as

$$f(X) = \sum_{k \in K} f_k(X^k) + f_0(\sum_{k \in K} X^k), \quad X = (X^k) \in \mathbb{Z}^K,$$

where f_0 and f_k for $k \in K$ are convex functions on $\mathbb{R}$. This is again a special case of (22) with $\mathcal{T}$ consisting of singletons and the whole set K. Theorem 2 along with this type of $\mathcal{IGP}$-convex functions extends the existence result in [2]. □

The indifference levels of the utility functions appearing in [20, 24] also fit in the form of (22). In fact, the essential feature of Quinzii's model is that consumers never have use for more than one item of any of the goods. Formally, the consumer utility function takes the form

$$u(Z, m) = \max_{k \in \mathrm{supp}(Z)} u(k, m)$$

for $Z \in \{0, 1\}^K$ and $m > 0$, where $\mathrm{supp}(Z) = \{k \in K \mid Z^k > 0\}$. The associated indifference levels are of the form

$$q^m(Z) = \min_{k \in \mathrm{supp}(Z)} q^m(k).$$

In the following Example 8, we show that these indifference levels are $\mathcal{IGP}$-convex, and, moreover, also fit (22). This is a rationale underlying Quinzii's ([20]) existence result.

Example 8. Consider a function $U : 2^K \to \mathbb{R}$ of the form $U(A) = \min_{k \in A} u(k)$ with some function $u : K \to \mathbb{R}$. We show that such a function is a special case of (22) in Corollary 1. We extend U to $\mathbb{Z}_+^K$ as

$$U(X) = \min_{k \in \mathrm{supp}(X)} u(k), \qquad X \in \mathbb{Z}_+^K, \tag{23}$$

where $U(0) = \max_{k \in K} u(k)$ by convention, and $\mathrm{supp}(X) = \{k \in K \mid X^k > 0\}$. We index the elements of K in the decreasing order of the values of u, i.e., $u(1) \geq \ldots \geq u(n)$ with $n = |K|$, and set $d_1 = u(1)$ and $d_k = u(k) - u(k-1)$ for $k = 2, \ldots, n$. Using a pseudoconcave function $\theta : \mathbb{Z} \to \mathbb{R} \cup \{-\infty\}$ defined by

$$\theta(t) = \begin{cases} -\infty, & t < 0; \\ 0, & t = 0; \\ 1, & t \geq 1 \end{cases}$$

we can represent $U(X)$ as

$$U(X) = d_1 + d_2\theta(X^2 + \cdots + X^n) + \cdots + d_{n-1}\theta(X^{n-1} + X^n) + d_n\theta(X^n). \tag{24}$$

In fact, (24) is good for $X = 0$, and for $X \neq 0$, if k denotes the maximal element in $\mathrm{supp}(X)$, then the left hand side of (24) is $u(k)$, while the right hand side yields $d_1 + \cdots + d_k = u(k)$. Since, $d_k \leq 0$ (for $k > 1$), the functions $d_k\theta$ for $k > 1$ are pseudoconvex. Therefore, the function represented as (24) is in the form of (22) for $\mathcal{T} = \{\{1, \ldots, n\}, \{2, \ldots, n\}, \ldots, \{n\}\}$. Thus, a function of the form (23) is $\mathcal{IGP}$-convex. □

Example 9. Let K be partitioned, namely, $K = \bigcup_{s \in S} K_s$ and $K_s \cap K_{s'} = \emptyset$ for $s \neq s'$. For each $s \in S$, take a class of discrete convexity $\mathcal{K}_s$ in $\mathbb{Z}^{K_s}$ as well as a $\mathcal{K}_s$-convex function f_s. Then, since the cartesian product of classes of discrete convexity is a class of discrete convexity, the function defined by

$$f(X) = \sum_{s \in S} f_s(X|_{K_s}),$$

where $X|_{K_s} \in \mathbb{Z}^{K_s}$ is the projection of X on K_s, is a $(\prod_{s \in S} \mathcal{K}_s)$-convex function. □

Example 9 and Theorem 2 together ensure that an exchange economy has an equilibrium if the indifference levels of the agents' preference are of the form $q^m(X) = \sum_{s \in S} \min_{k \in \mathrm{supp}(X) \cap K_s} q^m(k)$.

Finally, it is mentioned that the generalization of Gale's model by van der Laan *et al.* ([15]) is a case where the demands are products of PM-sets. Such demands are obtained as the products of integer points in some faces of different standard unit simplices, $\prod_s \Delta_{K_s}$. Hence, the existence of an equilibrium in this model can be obtained as a consequence of Theorem 1 (see Example 2).

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Appendix

Proof of Proposition 2.

Put $W = \sum_{h\in H} W_h$. Let $Z^* \in \mathbb{Z}^K_+$ be such that, for any $Z \geq Z^*$, the aggregate cost of producing Z is larger than the total amount of money in the economy, i.e., $C(Z) > \sum_{h\in H} w_h$ for any $Z \geq Z^*$, where $C(Z) = \min\{\sum_{l\in L} c_l(Z_l) \mid Z = \sum_{l\in L} Z_l,\ Z_l \in \mathbb{Z}^K\}$. Such Z^* exists by Assumption 4.

We slightly modify the preferences of consumers. For each $h \in H$ denote by T_h the amount of money such that $(0, T_h) \sim_h (W + Z^*, \sum_{h\in H} w_h)$. Then the function $q_h^{T_h}$ defines the indifference level of the preference $\preceq_h$ which passes through the point $(W + Z^*, \sum_{h\in H} w_h)$, or, equivalently, through the point $(0, T_h)$. Define the modified preference $\widetilde{\preceq}_h$ by setting its indifference levels as follows: For any m with $m_h^0 \leq m \leq T_h$ the indifference level of $\widetilde{\preceq}_h$ passing through the point $(0, m)$ coincides with the indifference level of $\preceq_h$ passing through the same point. For any $m > T_h$ define $\widetilde{q}_h^m$ to be a parallel translation of $q_h^{T_h}$, i.e.,

$$\widetilde{q}_h^m(X) = q_h^{T_h}(X) + m - T_h, \quad X \in \mathbb{Z}^K. \tag{25}$$

Let $\widetilde{\mathcal{E}} = \{(\widetilde{\preceq}_h, (W_h, w_h)_{h\in H};\ (c_l)_{l\in L}\}$ be the modified economy. We claim that the equilibria of $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ coincide. In fact, by Assumption 1, every consumer in $\mathcal{E}$ has a positive amount of money, less than $\sum_{h\in H} w_h$, at an equilibrium (see (3)). Now according to (2) and (3), at any equilibrium of $\mathcal{E}$, each consumer has a vector of indivisibles bounded by $W + Z^* = \sum_{h\in H} W_h + Z^*$. However, for any $X \leq W + Z^*$ and $m \leq \sum_{h\in H} w_h$, the indifference levels of the preferences $\preceq_h$ and $\widetilde{\preceq}_h$ passing through (X, m) coincide. Therefore, the sets of equilibria of $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ coincide. Thus, we may assume that (25) is already satisfied in the initial economy $\mathcal{E}$.

We are to prove the existence of an equilibrium for the convexified economy $\mathrm{co}(\mathcal{E})$.

Set $Q = \sum_{h\in H} T_h$. Take the price cube

$$\mathbf{Q} = \{p \in \mathbb{R}^K_+ \,|\, 0 \leq p_k \leq Q,\ \forall k \in K\},$$

and define the price correspondence $\mathbf{P} : \mathbf{Q} \Rightarrow \mathbf{Q}$ through the following auxiliary construction.

For every $h \in H$, set

$$m_h(p) := \inf\{m \ : \ q_h^m(X) \geq \beta_h(p) - p(X),\ X \in \mathbb{Z}_+^K,\ X \leq W + Z^*\}. \qquad (26)$$

That is $m_h(p)$ is such that the indifference level $q_h^{m_h(p)}(\cdot)$ "touches" the budget set $B_h(p)$. Define now the indirect utility functions

$$u_h^p(X) = m_h(p) - q_h^{m_h(p)}(X), \quad X \in \mathbb{Z}_+^K.$$

According to Assumption 3, these functions are pseudoconcave and, because of (25), the functions u_h^p are bounded by T_h for any $p \in \mathbf{Q}$.

Let the aggregate utility function U^p be given by the co-convolution

$$U^p(x) = \max\{\sum_{h\in H} \widehat{u}_h^p(x_h) \mid \sum_{h\in H} x_h = x\}, \quad x \in \mathbb{R}_+^K,$$

where $\widehat{u}$ denotes the concavification of a function u, i.e., $\widehat{u} = -\mathrm{co}(-u)$ with $\mathrm{co}(\cdot)$ denoting the convexification of a function. Also let the aggregate cost function C be given by the convolution

$$C(y) = \min\{\sum_{l\in L} \mathrm{co}(c_l)(y_l) \mid \sum_{l\in L} y_l = y\}, \quad y \in \mathbb{R}_+^K.$$

Let y^p be a solution to a maximization problem

$$\max_{y\in\mathbb{R}_+^K} U^p(W + y) - C(y).$$

Such a solution exists, since $C(y) \to +\infty$ with $||y|| \to \infty$ and U^p is bounded by $Q = \sum_h T_h$ (since each u_h^p is bounded by T_h).

Put $M = \max_{y\in\mathbb{R}_+^K} U^p(W + y) - C(y)$. Then, U^p is a concave function, C is a convex function and $C(x) + M \geq U^p(W + x)$ holds for any $x \in \mathbb{R}_+^K$. Therefore, there exists a separating affine function of the form $M + p'$ with some linear functional p', i.e., we have

$$C(x) + M \geq p'(x) + M \geq U^p(W + x), \qquad x \in \mathbb{R}_+^K. \qquad (27)$$

For $p \in \mathbf{Q}$, let $\mathbf{P}(p)$ be the set of separating linear functionals p' satisfying (27).

We claim that $\mathbf{P}(p) \subset \mathbf{Q}$, i.e., $\mathbf{P} : \mathbf{Q} \Rightarrow \mathbf{Q}$. First note that the monotonicity of C and U^p implies that $p' \geq 0$, and hence $\mathbf{P}(p) \subset (\mathbb{R}^K)_+^*$. For any $p' \in \mathbf{P}(p)$, it

holds that $U^p(W+y^p) - p'(W+y^p) \geq U^p(W+y^p) - C(y^p) \geq 0$. Therefore, we have $Q \geq U^p(W+y^p) \geq p'(W+y^p)$. Since W is an integral vector with positive coordinates, we have $W \geq \mathbf{1}$, which implies that $p'_k < Q$ for all $k \in K$. Thus we have proven the claim.

The set $\mathbf{P}(p)$, being the set of separating linear functionals, is convex and compact. Since U^p is continuous with respect to p (as a consequence of the continuity of each u_h^p with respect to p), the correspondence $\mathbf{P}$ is closed. By the Kakutani theorem, $\mathbf{P}$ has a fixed point, say $p^* \in \mathbf{P}(p^*)$. Let us verify that p^* is an equilibrium price for $\mathrm{co}(\mathcal{E})$.

Obviously, $y^{p^*} < Z^*$. Since $q_h^{m_h(p^*)}$ touches the budget set $B_h(p^*)$, the following equality is satisfied:

$$\mathrm{Argmax}_{x \in \mathbb{R}^K_+}(u_h^{p^*}(x) - p^*(x)) = \mathrm{co}(D_h(p^*)).$$

(Recall that $D_h(p)$ denotes the set of all $X_h \in \mathbb{Z}^K_+$ such that $(X_h, \beta_h(p) - p(X_h))$ is a best element in the budget set $B_h(p)$ with respect to the preference $\preceq_h$.) We have $\mathrm{Argmax}(p^*(y) - C(y)) = \sum_{l \in L} \mathrm{co}(S_l(p^*))$ by (14). By (15) and (14) we have

$$W \in \sum_{h \in H} \mathrm{co}(D_h(p^*)) - \sum_{l \in L} \mathrm{co}(S_l(p^*)). \tag{28}$$

This implies that p^* is an equilibrium price for the convexified economy $\mathrm{co}(\widetilde{\mathcal{E}})$. Q.E.D.

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