ON CLASSIFICATION OF $\mathbb{Q}$-FANO 3-FOLDS OF GORENSTEIN INDEX 2. II

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Abstract. In the previous paper, we obtained a list of numerical possibilities of $\mathbb{Q}$-Fano 3-folds X with $\operatorname{Pic} X = \mathbb{Z}(-2K_X)$ and $h^0(-K_X) \geq 4$ containing index 2 points P such that $(X, P) \simeq (\{xy + z^2 + u^a = 0\}/\mathbb{Z}_2(1,1,1,0), o)$ for some $a \in \mathbb{N}$. Moreover we showed that such an X is birational to a simpler Mori fiber space. In this paper, we prove their existence except for a few cases by constructing a Mori fiber space with desired properties and reconstructing X from it.

Contents

Notation and Conventions.

$\mathbb{N}$: The set of positive integers.
$\sim$: Linear equivalence.
$\equiv$: Numerical equivalence.
$\mathbb{F}_n$: Segre-del Pezzo scroll of degree n.
$\mathbb{F}_{n,0}$: Surface obtained by contracting the negative section of $\mathbb{F}_n$.
Q_3: Smooth quadric 3-fold.
ODP: Ordinary double point, i.e., singularity analytically isomorphic to $\{xy + z^2 + u^2 = 0 \subset \mathbb{C}^4\}$.
QODP: Singularity analytically isomorphic to $\{xy+z^2+u^2 = 0 \subset \mathbb{C}^4/\mathbb{Z}_2(1,1,1,0)\}$.
B_i $(1 \leq i \leq 5)$: Factorial Gorenstein terminal Fano 3-fold of Fano index 2, and with Picard number 1 and $(-K)^3 = 8i$, where K is the canonical divisor.
A_{2g-2} $(1 \leq g \leq 12$ and $g \neq 11)$: Factorial Gorenstein terminal Fano 3-fold of Fano index 1, and with Picard number 1 and genus g.
Abuse of notation: We use the same notation for transforms of curves by birational maps as original ones.

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0. Introduction

In this paper we work over $\mathbb{C}$, the complex number field.

Definition 0.0 ($\mathbb{Q}$-Fano variety)**.** Let X be a normal projective variety. X is said to be a *terminal* (resp. *canonical*, *klt*, etc.) $\mathbb{Q}$-*Fano variety* if X has only terminal (resp. canonical, Kawamata log terminal, etc.) singularities and $-K_X$ is ample. By replacing 'ample' with 'nef and big', *terminal* (resp. *canonical*, *klt*, etc.) *weak* $\mathbb{Q}$-*Fano varieties* are similarly defined. If X has only terminal singularities, then we say that X is a $\mathbb{Q}$-*Fano variety* for short and if X has only Gorenstein terminal (resp. canonical, klt, etc.) singularities, we say that X is a *Gorenstein terminal* (resp. *canonical*, *klt*, etc.) *Fano variety.*

Let $I(X) := \min\{I | IK_X$ is a Cartier divisor$\}$ and we call $I(X)$ the *Gorenstein index* of X.

Write $I(X)(-K_X) \equiv r(X)H(X)$, where $H(X)$ is a primitive Cartier divisor and $r(X) \in \mathbb{N}$. (Note that $H(X)$ is unique since $\operatorname{Pic} X$ is torsion free.) Then we call $\frac{r(X)}{I(X)}$ the *Fano index* of X and denote it by $F(X)$.

In the previous paper [Taka00], we formulate a generalization of Takeuchi's method [Take89] for the classification of smooth Fano 3-folds and use it for a partial classification of $\mathbb{Q}$-Fano 3-folds X with the following properties.

Main Assumption 0.1. (1) The Picard number of X is 1,
(2) the Gorenstein index of X is 2,
(3) the Fano index of X is $\frac{1}{2}$,
(4) $h^0(-K_X) \geq 4$, and
(5) there exists an index 2 point P such that
$$(X, P) \simeq (\{xy + z^2 + u^a = 0\}/\mathbb{Z}_2(1, 1, 1, 0), o)$$
for some $a \in \mathbb{N}$.

Let $f \colon Y \to X$ be the weighted blow-up at P with weight $\frac{1}{2}(1, 1, 1, 2)$. In the previous paper [Taka00], we proved that Y is a weak $\mathbb{Q}$-Fano 3-fold and obtained the following diagram.

$$\begin{tikzcd} & Y \arrow[r, dashed] \arrow[dl, "f"'] & Y' \arrow[dr, "f'"] & \\ X & & & X', \end{tikzcd}$$

where
(1) $Y \dashrightarrow Y'$ is an isomorphism and f' is a crepant divisorial contraction or
(2) $Y \dashrightarrow Y'$ is a flop or a composite of a flop and a flip, and f' is an extremal contraction which is not isomorphic in codimension 1.

We use the following notation.

Notation 0.2.
- $\tilde{E} :=$ the strict transform of E on Y',
- $n := 2((-K_Y)^3 - (-K_{Y'})^3)$,
- $e := E^3 - \tilde{E}^3 - 4n$,
- Rational numbers z and u are defined as follows. In case f' is birational, the f'-exceptional divisor E' satisfies $E' \equiv z(-K_{Y'}) - u\tilde{E}$. Otherwise the pull-back L of an ample generator of $\operatorname{Pic} X'$ satisfies $L \equiv z(-K_{Y'}) - u\tilde{E}$,
- $h := h^0(-K_X)$, and
- N is the number of $\frac{1}{2}(1, 1, 1)$-singularities obtained by deforming non-Gorenstein points of X locally.

The following is the main theorem of [Taka00]:

Theorem 0.3. *Let X be as in Main Assumption 0.1. Consider the diagram*

$$\begin{tikzcd} & Y \arrow[r, dashed] \arrow[dl, "f"'] & Y' \arrow[dr, "f'"] & \\ X & & & X', \end{tikzcd}$$

as above. Then the possibilities of X are classified as in Tables 1–5 and Tables $1'$–$5'$ with the notation of 0.2. In particular we have $(-K_X)^3 \leq 15$ and $h^0(-K_X) \leq 10$.

Table 1. f' is of $(2,1)$-type. I

No.	h	$(-K_X)^3$	N	e	n	z	$\deg C$	$g(C)$	X'
1.1	6	7	2	7	0	4	7	8	[5]
1.2	6	15/2	3	7	0	2	3	0	$[2], I(X')=2$
1.3	6	15/2	3	6	1	4	6	3	[5]
1.4	7	17/2	1	6	0	3	9	9	$\mathbb{P}^3$
1.5	7	9	2	6	0	2	6	3	[3]
1.6	7	9	2	5	1	3	8	5	$\mathbb{P}^3$
1.7	7	19/2	3	5	1	2	5	0	[3]
1.8	7	19/2	3	4	2	3	7	1	$\mathbb{P}^3$
1.9	8	21/2	1	6	0	1	3	0	B_3
1.10	8	21/2	1	5	0	2	9	6	Q_3
1.11	8	11	2	4	1	2	8	3	Q_3
1.12	9	25/2	1	5	0	1	5	1	B_4
1.13	10	29/2	1	4	0	1	7	2	B_5
1.14	10	15	2	3	1	1	6	0	B_5

Table 1′. f' is of $(2,1)$-type. I

h	$(-K_X)^3$	N	e	n	z	$\deg C$	$g(C)$	X'
8	23/2	3	3	2	2	7	0	Q_3

Notation and Remarks for Table 1 and Table 1′.

$C := f'(E')$,

$\deg C := (H(X') \cdot C)$ *(see Definition 0.0 for the definition of $H(X')$),*

$g(C) :=$ *the genus of C in case X has only $\frac{1}{2}(1,1,1)$-singularities,*

see [San96] *for the definition of* $[i]$,

$u = z + 1$.

Table 2. f' is of $(2,1)$-type. II

No.	$(-K_X)^3$	N	e	$\deg C$	X'
2.1	7/2	3	10	1	A_6
2.2	4	4	8	2	A_8
2.3	9/2	5	6	3	A_{10}
2.4	5	6	4	4	A_{12}

Table 2′. f' is of $(2,1)$-type. II

$(-K_X)^3$	N	e	$\deg C$	X'
11/2	7	2	5	A_{14}

Notation and Remarks for Table 2 and Table 2′.

$$C := f'(E'),$$
$$\deg C := (-K_{X'} \cdot C),$$
$$z = u = 1,$$
$$h = 4 \text{ and } n = 0.$$

Table 3. f' is $(2,0)$-type or crepant divisorial.

No.	h	$(-K_X)^3$	N	e	n	type of f'
3.1	4	5/2	1	15	0	$(2,0)_4$
3.1′	4	5/2	1	/	/	crep. div.
3.2	4	3	2	12	0	$(2,0)_8$
3.3	4	4	4	9	3	$(2,0)_1$
3.4	4	9/2	5	8	3	$(2,0)_5$

Remarks for Table 3.

$z = u = 1$,

(No. 3.1) X' also belongs to this class,

(No. 3.1′) X' is a Fano 3-fold of $(-K_{X'})^3 = 2$ and with a canonical singularity along the image of f'-exceptional divisor,

(No. 3,2) $X' \simeq A_4$ with one Gorenstein terminal singularity,

(No. 3,3) X' is smooth, isomorphic to A_{10},

(No. 3.4) X' is smooth, isomorphic to A_{16}.

Table 4. f' is of $(3,2)$-type.

No.	h	$(-K_X)^3$	N	e	n	$\deg\Delta$
4.1	5	11/2	3	8	0	8
4.2	5	6	4	7	1	6
4.3	6	13/2	1	7	0	7
4.4	6	7	2	6	1	6
4.5	6	15/2	3	5	2	5
4.6	6	8	4	4	3	4
4.7	6	17/2	5	3	4	3
4.8	10	29/2	1	6	0	0

Table 4′. f' is of $(3,2)$-type.

h	$(-K_X)^3$	N	e	n	$\deg\Delta$
5	13/2	5	6	2	4
5	7	6	5	3	2
5	15/2	7	4	4	0
6	9	6	2	5	2
6	19/2	7	1	6	1

Notation and Remarks for Table 4 and 4′.

$\Delta :=$ *the discriminant divisor of* f',

$\deg \Delta$ *is measured by the ample generator of* $\operatorname{Pic} X'$,

in case $h = 5$, $z = u = 2$ *and* $X' \simeq \mathbb{F}_{2,0}$,

in case $h = 6$, $z = u = 1$ *and* $X' \simeq \mathbb{P}^2$,

in case $h = 10$, $z = 1, u = 2$ *and* $X' \simeq \mathbb{P}^2$.

Table 5. f' *is of* $(3,1)$*-type.*

No.	h	$(-K_X)^3$	N	e	n	$\deg F$
5.1	4	9/2	5	9	0	6
5.2	5	9/2	1	9	0	3
5.3	5	5	2	8	1	4
5.4	5	11/2	3	7	2	5
5.5	5	6	4	6	3	6

Table 5′. f' *is of* $(3,1)$*-type.*

h	$(-K_X)^3$	N	e	n	$\deg F$
4	5	6	8	1	8

Notation and Remarks for Table 5 and 5′.

$F :=$ *a general fiber of* f',

in case $h = 4$, $z = u = 2$,

in case $h = 5$, $z = u = 1$.

Based on these lists, we derive some geometric properties of such a $\mathbb{Q}$-Fano 3-fold X in Sections 1–3.

Miles Reid conjectured that every $\mathbb{Q}$-Fano 3-fold has an effective anti-canonical divisor with only canonical singularities. The conjecture is affirmative in case of Gorenstein canonical Fano 3-folds [Sho79b] and [Reid83]. In §1, we prove the following:

Theorem 0.4 (See Corollary 1.2)**.** *Assume that for any index* 2 *point* P*, there is an isomorphism*

$$(X, P) \simeq (\{xy + z^2 + u^a = 0\}/\mathbb{Z}_2(1,1,1,0), o)$$

for some $a \in \mathbb{N}$*. Then* $|-K_X|$ *has a member with only canonical singularities.*

In §2, we study deformation theoretic properties of X and obtain the following:

Theorem 0.5 (See Corollaries 2.2 and 2.3)**.** *Let* X *be a* $\mathbb{Q}$*-factorial* $\mathbb{Q}$*-Fano* 3*-fold with (1)–(4) of Main Assumption 0.1. Let* $N := \operatorname{aw}(X)$*. Then the following hold.*

(1) X *can be deformed to a* $\mathbb{Q}$*-factorial* $\mathbb{Q}$*-Fano* 3*-fold* X' *with (1)–(4) in Main Assumption 0.1 and with only QODP's or* $\frac{1}{2}(1,1,1)$*-singularities as its singularities.*

(2) If $N > 1$ *(resp.* $N = 1$*),* X *can be transformed to a* $\mathbb{Q}$*-factorial* $\mathbb{Q}$*-Fano* 3*-fold* $\tilde{Z}'$ *with (1)–(4) of Main Assumption 0.1 and with only QODP's or* $\frac{1}{2}(1,1,1)$*-singularities as its singularities and* $h^0(-K_{\tilde{Z}'}) = h$ *and* $\operatorname{aw}(\tilde{Z}') = N - 1$ *(resp.*

a smooth Fano 3-fold $\tilde{Z}'$ with $\rho(\tilde{Z}') = 1$, $F(\tilde{Z}') = 1$ and $h^0(-K_{\tilde{Z}'}) = h$) as follows.

$$\begin{tikzcd} & & \tilde{Y} \arrow[dl, "\tilde{f}"'] \arrow[dr, "\tilde{g}"] & & \\ X \arrow[r, "\text{def}"] & \tilde{X} & & \tilde{Z} \arrow[r, "\text{def}"] & \tilde{Z}', \end{tikzcd}$$

where $* \xrightarrow{\text{def}} **$ means that $**$ is a small deformation of $*$,
$\tilde{X}$ is a $\mathbb{Q}$-Fano 3-fold with the properties (1)–(4) in Main Assumption 0.1 and with only ODP's, QODP's or $\frac{1}{2}(1,1,1)$-singularities as its singularities,
$\tilde{f}\colon \tilde{Y} \to \tilde{X}$ is similarly chosen to f in Theorem 0.3, and
$\tilde{g}\colon \tilde{Y} \to \tilde{Z}$ be the anti-canonical model.

(2) is an analogue to Reid's fantasy about Calabi-Yau 3-folds [Reid87a].
In §3, we prove the following:

Theorem 0.6 (See Corollary 3.1)**.** *If any index 2 point is a $\frac{1}{2}(1,1,1)$-singularity, X can be embedded into a weighted projective space $\mathbb{P}(1^h, 2^N)$, where $h := h^0(-K_X)$ and N is the number of $\frac{1}{2}(1,1,1)$-singularities on X.*

We hope to determine the defining equation of X explicitely in some weighted projective space containing $\mathbb{P}(1^h, 2^N)$ as S. Mukai did in case of Fano 3-folds (see [Muk89], [Muk92] and [Muk95]).

In §4, as announced in [Taka00], we prove the existence of $\mathbb{Q}$-Fano 3-folds with Main Assumption 0.1. The main results are Thereoms 0.10–0.21. Proposition 0.8 gives a sufficient condition for the reconstruction of X.

Assumption 0.7. In Theorems 0.10–0.21 (A), we assume that X has only $\frac{1}{2}(1,1,1)$-singularities and fix f as in Theorem 0.3 (and then X is classified as in Tables 1–5). We use the notation of Tables 1–5 freely.

Proposition 0.8. *Let Y' be a projective 3-fold with only $\frac{1}{2}(1,1,1)$-singularities and n is a non-negative integer. Assume the following conditions.*

(1) $\rho(Y') = 2$.
(2) In case $n > 0$, there are smooth rational curves l_i $(0 \le i \le n-1)$ such that
  (2-1) l_i are numerically equivalent.
  (2-2) l_i are mutually disjoint and are contained in $\operatorname{Reg} Y'$.
  (2-3) $\operatorname{Bs}|-K_{Y'}|$ is the union of l_i and $\frac{1}{2}(1,1,1)$-singularities.
  (2-4) $-K_{Y'} \cdot l_i = -1$, or
  in case $n = 0$, $-K_{Y'}$ is nef and big.
(3) $(-K_{Y'})^3 + \frac{n}{2} > 0$.
(4) There is an irreducible divisor $\tilde{E}$ such that $\tilde{E} \cdot l_i = -1$ in case $n > 0$, $(-K_{Y'})^2 \tilde{E} = 1 - n$ and $(-K_{Y'})\tilde{E}^2 = -2 - 2n$.
(5) In case $n = 0$, there exists an extremal ray R of Y' such that $\tilde{E} \cdot R < 0$.

Then the following hold.

(i) There is a birational map $Y' \dashrightarrow Y$ which is one flop or a composite of one anti-flip and one flop.
(ii) There is an extremal contraction $f\colon Y \to X$ of $(2,0)_4$-type or $(2,0)_{10}$-type whose exceptional divisor is the strict transform of $\tilde{E}$.
(iii) X is a $\mathbb{Q}$-Fano 3-fold with only $\frac{1}{2}(1,1,1)$-singularities or QODP's.

Remark 0.9. In Theorems 0.10–0.21 (A), we assume that X has only $\frac{1}{2}(1,1,1)$-singularities. However, X reconstructed in Theorems 0.10–0.21 (B) by using Proposition 0.8 has possibly worse singularity than $\frac{1}{2}(1,1,1)$-singularities.

Theorem 0.10 (Table 1). *(A) Let X be a $\mathbb{Q}$-Fano 3-fold as in Table 1. Then*

(1) images of n flipped curves $l_i\,(0 \le i \le n-1)$ are $(z+2)$-secant lines of C with respect to $\frac{1}{z+1}(-K_{X'})$,

(2) $l_i \subset \operatorname{Reg} Y'$ and l_i are mutually disjoint, and

(3) $\operatorname{Bs}|-K_{X'}-C|$ is the union of C, l_i and $\frac{1}{2}(1,1,1)$-singularities.

(B) Conversely let X' be a $\mathbb{Q}$-Fano 3-fold as in Table 1 and $C \subset X'$ a smooth curve of degree and genus given in the same row. Let n and z be integers given in the same row. Assume that

(1) C has $(z+2)$-secant lines $l_i\,(0 \le i \le n-1)$ with respect to $\frac{1}{z+1}(-K_{X'})$ such that $l_i \subset \operatorname{Reg} X'$ and l_i are mutually disjoint,

(2) $\operatorname{Bs}|-K_{X'}-C|$ is the union of C, l_i and $\frac{1}{2}(1,1,1)$-singularities, and

(3) There exists a surface $S \equiv \frac{z}{z+1}(-K_{X'})$ containing C.

Let $f\colon Y' \to X'$ be the blow-up of X' along C and E' f'-exceptional divisor. Then the following hold.

(i) S is irreducible and $C \not\subset \operatorname{Sing} S$.

(ii) Y', l_i and the strict transform $\tilde{E}$ of S on Y' satisfy the conditions of Proposition 0.8. Let X be a $\mathbb{Q}$-Fano 3-fold obtained as in Proposition 0.8. Then X belong to the same row in Table 1 as X'.

(C) In any case of Table 1, there exists an example of (X', C, l_i) as in (B) and hence that of a $\mathbb{Q}$-Fano 3-fold X .

Theorem 0.11 (Table 2). *(A) Let X be a $\mathbb{Q}$-Fano 3-fold of No. 2.1 (resp. 2.2). Then*

(1) $X' \simeq A_6$ (resp. A_8) and has 2 (resp. 3) singularities P_i on C such that (X', P_i) are isomorphic to $(\{xy+zw=0\}, o)$ in $\mathbb{C}^4$ or $(\{xy+z^2+w^3=0\}, o)$ in $\mathbb{C}^4$, and

(2) C is a smooth rational curve such that $(-K_{X'} \cdot C) = 1$ (resp. 2).

(B) Conversely let (X', C, P_i) be as in (A). Then the following hold.

(i) There exists a divisorial contraction $f'\colon Y' \to X'$ of $(2,1)$-type whose center is C (note that by [Taka00, Proposition 2.2 (4c)]*, Y has only $\frac{1}{2}(1,1,1)$-singularities).*

(ii) There is a unique member $\tilde{E}$ of $|-K_{Y'}-E'|$, where E' is f'-exceptional divisor.

(iii) Y' and $\tilde{E}$ satisfy the conditions of Proposition 0.8. Let X be a $\mathbb{Q}$-Fano 3-fold obtained as in Proposition 0.8. Then X is of No. 2.1 (resp. No. 2.2).

(C) There exists an example of (X', C) as in (A) for No. 2.1 (resp. No. 2.2) and hence there exists a $\mathbb{Q}$-Fano 3-fold of No. 2.1 (resp. No. 2.2).

Remark 0.12. Examples are not known for No. 2.3 or 2.4.

Theorem 0.13 (Table 3). *(No. 3.1 or 3.1$'$) Let X be a $\mathbb{Q}$-Fano 3-fold of No. 3.1 or 3.1$'$. Then $X \simeq ((5) \subset \mathbb{P}(1^4, 2))$. X' is also of No. 3.1 if so is X.*

(No. 3.2) Let X be a $\mathbb{Q}$-Fano 3-fold of No. 3.2. Then $X \simeq ((3,4) \subset \mathbb{P}(1^4, 2^2))$ and $X' \simeq A_4$ with one Gorenstein terminal singularity.

(No. 3.3) (A) Let X be a $\mathbb{Q}$-Fano 3-fold of No. 3.3. Then

(1) X' is smooth and isomorphic to A_{10}, and

(2) there exist exactly three lines through Q, which is the image of the f'-exceptional divisor E'.

(B) Conversely let X' be a smooth 3-fold isomorphic to A_{10} such that there exists a point Q where exactly three lines $l_i\,(i=0,1,2)$ pass through. Let $f'\colon Y' \to X'$ be the blow-up at Q and E' the exceptional divisor. Then the following hold.

(i) $\operatorname{Bs}|-K_{Y'}| = l_0 \cup l_1 \cup l_2$.

(ii) There is a unique member $\tilde{E}$ of $|-K_{Y'}-E'|$.

(iii) Y', l_i *and* $\tilde{E}$ *satisfy the conditions of Proposition 0.8. Let* X *be a* $\mathbb{Q}$*-Fano* 3*-fold obtained as in Proposition 0.8. Then* X *is of No. 3.3.*

(C) There exists an example of (X', Q) *as in (B) and hence there exists a* $\mathbb{Q}$*-Fano* 3*-fold of No. 3.3.*

Remark 0.14. Examples are not known for No. 3.4.

Theorem 0.15 (Table 4, I.)**.** *(A) Let* X *be a* $\mathbb{Q}$*-Fano* 3*-fold such that* $n \geq 1$ *and* $X' \simeq \mathbb{F}_{0,2}$*. Let* $\mu_1 \colon W_1 \to Y'$ *be the blow-up along a flipped curve* l_0*. Then there is a flop* $W_1 \dashrightarrow {W_1}'$ *over* X' *and an extremal contraction of* $(2,1)$*-type* $\nu_1 \colon {W_1}' \to {Y_1}'$ *over* X'*. Let* m_1 *be the image of* ν_1*-exceptional divisor. Then the following hold.*

(1) $n = 1$.

(2) m_1 *is a smooth rational curve with* $(-K_{{Y_1}'} \cdot m_1) = 8$ *such that* $m_1 \subset \operatorname{Reg} {Y_1}'$, ${f_1}'|_{m_1}$ *is an isomorphism and* $\operatorname{Bs}|-K_{{Y_1}'} - m_1|$ *is the union of* m_1 *and* $\frac{1}{2}(1,1,1)$*-singularities.*

(3) ${Y_1}'$ *is a* $\mathbb{Q}$*-Fano* 3*-fold with* $(-K_{{Y_1}'})^3 = 17$ *and a unique flipping ray.*

(4) Let ${Y_1}' \dashrightarrow {Y_2}'$ *be the flip. Then* ${Y_2}'$ *is a smooth divisor in* $\mathbb{P} := \mathbb{P}(\mathscr{O}_{\mathbb{P}^1}^{\oplus 2} \oplus \mathscr{O}_{\mathbb{P}^1}(1)^{\oplus 2})$ *over* ${X_2}' \simeq \mathbb{P}^1$ *and linearly equivalent to* $2H + M$*, where* H *is the tautological divisor of* $\mathbb{P}$ *and* M *is a fiber of the natural projection* $\mathbb{P} \to {X_2}'$*.* ${Y_2}'$ *has two disjoint sections which are connected components of the intersection of* ${Y_2}'$ *and the subvariety* V *of* $\mathbb{P}$ *associated to the surjection* $\mathscr{O}_{\mathbb{P}^1}^{\oplus 2} \oplus \mathscr{O}_{\mathbb{P}^1}(1)^{\oplus 2} \to \mathscr{O}_{\mathbb{P}^1}^{\oplus 2}$*.*

Consequently we obtain the following diagram.

$$\begin{tikzcd}
& W_1 \arrow[dl, "\mu_1"'] \arrow[r, dashed] & {W_1}' \arrow[dr, "\nu_1"] & & & \\
Y' \arrow[dr, "f'"] & & & {Y_1}' \arrow[dl, "{f_1}'"'] \arrow[r, dashed] & {Y_2}' \arrow[dr, "{f_2}'"] & \\
& X' \arrow[r, equal] & X' & & & {X_2}',
\end{tikzcd}$$

where ${f_i}'$ *are the natural projections.*

(B) Conversely let ${Y_2}'$ *be as in (A) (3). Then there is an anti-flip* ${Y_2}' \dashrightarrow {Y_1}'$*. Moreover let* $m_1 \subset {Y_1}'$ *be as in (A) (1) and* $\nu_1 \colon {W_1}' \to {Y_1}'$ *the blow-up along* m_1*. Then there is a flop* ${W_1}' \dashrightarrow W_1$ *over* X' *and an extremal contraction of* $(2,1)$*-type* $\mu_1 \colon W_1 \to Y'$ *over* X'*. Let* l_0 *be the image of* μ_1*-exceptional divisor and* L *the pull-back of a line of* X' *on* Y'*. Then the following hold.*

(i) There is a unique member $\tilde{E}$ *of* $|-K_{Y'} - D|$ *and* $\tilde{E}$ *is irreducible, where* D *is a Weil divisor such that* $2D \sim L$*.*

(ii) Y', l_0 *and* $\tilde{E}$ *satisfy the conditions of Proposition 0.8. Let* X *be a* $\mathbb{Q}$*-Fano* 3*-fold obtained as in Proposition 0.8. Then* X *is of No. 4.2.*

(C) There exists an example of $({Y_2}', m_1)$ *as in (B) and hence there exists a* $\mathbb{Q}$*-Fano* 3*-fold of No. 4.2.*

Remark 0.16. Examples are not known for No. 4.1.

Theorem 0.17 (Table 4, II)**.** *(A) Let* X *be a* $\mathbb{Q}$*-Fano* 3*-fold of No. 4.3. Then the following hold.* Y' *is a weak Fano* 3*-fold with* $\rho(Y') = 2$, $(-K_{Y'})^3 = 6$ *and a conic bundle structure over* $\mathbb{P}^2$*.* $\tilde{E}$ *is an irreducible divisor which is generically a* 2*-section such that* $(-K_{Y'})^2 \tilde{E} = 1$ *and* $(-K_{Y'}) \tilde{E}^2 = -2$*.*

(B) *Conversely let* $(Y', \tilde{E}')$ *be as in (A). Then they satisfy the conditions of Proposition 0.8. Let* X *be a* $\mathbb{Q}$*-Fano* 3*-fold obtained as in Proposition 0.8. Then* X *is of No. 4.3.*

(C) *There exists an example of* $(Y', \tilde{E}')$ *as in (A) and hence there exists a* $\mathbb{Q}$*-Fano* 3*-fold of No. 4.3.*

Theorem 0.18 (Table 4, III)**.** *(A) Let* X *be a* $\mathbb{Q}$*-Fano* 3*-fold of No. 4.4–No. 4.7. Let* $\mu_1 \colon W_1 \to Y'$ *be the blow-up along a flipped curve* l_0*. Then there is a flop* $W_1 \dashrightarrow {W_1}'$ *over* X' *and an extremal contraction of* $(2,1)$*-type* $\nu_1 \colon {W_1}' \to {Y_1}'$ *over* X'*. Let* m_1 *be the image of* ν_1*-exceptional divisor.*

(A-1) Assume that X *is a* $\mathbb{Q}$*-Fano* 3*-fold of No. 4.4. Then* ${Y_1}' \simeq ((2,2) \subset \mathbb{P}^2 \times \mathbb{P}^2)$ *or a double cover of* $((1,1) \subset \mathbb{P}^2 \times \mathbb{P}^2)$ *ramified along a smooth anti-canonical divisor. Let* $p_i\,(i = 1, 2)$ *be the two structure morphism of conic bundles. Then* m_1 *is a smooth rational curve such that* $p_i|_{m_1}$ *are isomorphisms and* $p_i(m_1)$ *are line.*

(A-2) Assume that X *is a* $\mathbb{Q}$*-Fano* 3*-fold of No. 4.5–No. 4.7. Then* ${Y_1}'$ *is a weak Fano* 3*-fold. Let* l_1 *be the transform of a flipped curve other than* l_0*. Let* $\mu_2 \colon W_2 \to {Y_1}'$ *be the blow-up along* l_1*. Then there is a flop* $W_2 \dashrightarrow {W_2}'$ *over* X' *and an extremal contraction of* $(2,1)$*-type* $\nu_2 : {W_2}' \to {Y_2}'$ *over* X'*.* ${Y_2}'$ *is the blow-up of* ${X_2}' \simeq B_{n+1}$ *along a curve* γ *. Moreover* γ*,* m_1 *and the image of* ν_2*-exceptional divisor* m_2 *are normal rational curves of degree* $n-1$ *intersecting the common* $n-2$ *points simply.*

Consequently we obtain the following diagram.

(No. 4.4)

$$\begin{tikzcd}
& W_1 \arrow[r, dashed] \arrow[dl, "\mu_1"'] & {W_1}' \arrow[dr, "\nu_1"] & \\
Y' \arrow[dr, "f'"'] & & & {Y_1}' \arrow[dl, "{f_1}'"] \\
& X' \arrow[r, equal] & X', &
\end{tikzcd}$$

where ${f_1}'$ *is the natural projection.*

(No. 4.5–No. 4.7)

$$\begin{tikzcd}
& W_1 \arrow[r, dashed] \arrow[dl, "\mu_1"'] & {W_1}' \arrow[dr, "\nu_1"] & & W_2 \arrow[r, dashed] \arrow[dl, "\mu_2"'] & {W_2}' \arrow[dr, "\nu_2"] & & \\
Y' \arrow[dr, "f'"'] & & & {Y_1}' \arrow[dl, "{f_1}'"'] \arrow[dr, "{f_1}'"] & & & {Y_2}' \arrow[dl, "{f_2}'"'] \arrow[dr, "h"] & \\
& X' \arrow[r, equal] & X' & & X' \arrow[r, equal] & X' & & {X_2}',
\end{tikzcd}$$

where ${f_i}'$ *are the natural projections and* h *is the blow-up along* γ*.*

(B) *(B-1) Conversely let* $({Y_1}', m_1)$ *be as in (A1). Let* $\nu_1 \colon {W_1}' \to {Y_1}'$ *the blow-up along* m_1*. Then there is a flop* ${W_1}' \dashrightarrow W_1$ *over* X' *and an extremal contraction of* $(2,1)$*-type* $\mu_1 \colon W_1 \to Y'$ *over* X'*. Let* l_0 *be the image of* μ_1*-exceptional divisor and* L *the pull-back of a line of* X' *on* Y'*.*

(B-2) Conversely let $({X_2}', m_1, m_2, \gamma)$ *be as in (A2). Let* $h \colon {Y_2}' \to {X_2}'$ *be the blow-up of* ${X_2}'$ *along* γ *and* $\nu_2 \colon {W_2}' \to {Y_2}'$ *the blow-up along* m_2*. Then there is a flop* ${W_2}' \dashrightarrow W_2$ *over* X' *and an extremal contraction of* $(2,1)$*-type* $\mu_2 \colon W_2 \to {Y_1}'$ *over* X'*. Let* l_1 *be the image of* μ_2*-exceptional*

divisor. Let $\nu_1 \colon W_1{}' \to Y_1{}'$ *the blow-up along* m_1*. Then there is a flop* $W_1{}' \dashrightarrow W_1$ *over* X' *and an extremal contraction of* $(2,1)$*-type* $\mu_1 \colon W_1 \to Y'$ *over* X'*. Let* l_0 *be the image of* μ_1*-exceptional divisor and* L *the pull-back of a line of* X' *on* Y'*.*

Then the following hold.

(i) There is a unique member $\tilde{E}$ *of* $|-K_{Y'} - L|$ *and* $\tilde{E}$ *is irreducible,*

(ii) Y'*,* l_i *and* $\tilde{E}$ *satisfy the conditions of Proposition 0.8. Let* X *be a* $\mathbb{Q}$*-Fano* 3*-fold obtained as in Proposition 0.8. Then* X *is of No. 4.4 in the case (B1), or No. 4.5–No. 4.7 in the case (B2).*

(C) There exists an example of $(Y_2{}', m_1)$ *as in (A1) (resp.* $(X_2{}', m_1, m_2, \gamma)$ *as in (A2)) and hence there exists a* $\mathbb{Q}$*-Fano* 3*-fold of No. 4.4 (resp. No. 4.5–No. 4.7).*

Theorem 0.19 (Table 4, IV)**.** *(A) Let* X *be a* $\mathbb{Q}$*-Fano* 3*-fold of No. 4.8. Then the following hold.* Y' *is a weak Fano* 3*-fold with* $(-K_{Y'})^3 = 14$ *and a* $\mathbb{P}^1$*-bundle structure over* $\mathbb{P}^2$*.* $\tilde{E}$ *is an irreducible divisor which is generically a* 2*-section such that* $(-K_{Y'})^2\tilde{E} = 1$ *and* $(-K_{Y'})\tilde{E}^2 = -2$*.*

(B) Conversely let $(Y', \tilde{E}')$ *be as in (A). Then they satisfy the conditions of Proposition 0.8. Let* X *be a* $\mathbb{Q}$*-Fano* 3*-fold obtained as in Proposition 0.8. Then* X *is of No. 4.8.*

(C) There exists an example of $(Y', \tilde{E}')$ *as in (A) and hence there exists a* $\mathbb{Q}$*-Fano* 3*-fold of No. 4.8.*

Theorem 0.20 (Table 5, I)**.** *(No. 5.2) Let* X *be a* $\mathbb{Q}$*-Fano* 3*-fold of No. 5.2. Then* $X \simeq ((3,3) \subset \mathbb{P}(1^5, 2))$*. Moreover by* [Take99]*,* Y' *is embedded in* $\mathbb{P}(\mathscr{O}_{\mathbb{P}_1}^{\oplus 3} \oplus \mathscr{O}_{\mathbb{P}_1}(1))$ *as a divisor linearly equivalent to* $3H + F$*, where* H *is the tautological divisor and* F *is a fiber.*

(No. 5.3) (A) Let X *be a* $\mathbb{Q}$*-Fano* 3*-fold of No. 5.3. Then the following hold.* Y' *is a smooth complete intersection of two members of* $|2H - F|$ *of* $\mathbb{P}(\mathcal{E})$*, where* $\mathcal{E} := \mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(1)^{\oplus 3} \oplus \mathscr{O}_{\mathbb{P}^1}(2)$*,* H *is the tautological divisor of* $\mathbb{P}(\mathcal{E})$*,* F *is a fiber of the natural projection* $\mathbb{P}(\mathcal{E}) \to \mathbb{P}^1$*.*

(B) Conversely Y' *is given as in (A). Then the following hold.*

(i) $\rho(Y') = 2$*.*

(ii) $\operatorname{Bs}|-K_{X'}| = l_0$*, where* l_0 *is the section associated to the surjection* $\mathcal{E} \to \mathscr{O}_{\mathbb{P}^1}$*.*

(iii) There exists a unique member $\tilde{E}$ *of* $|(H-2F)|_{Y'}|$ *and* $\tilde{E}$ *is irreducible.*

(iv) Y'*,* l_0 *and* $\tilde{E}$ *satisfy the conditions of Proposition 0.8. Let* X *be a* $\mathbb{Q}$*-Fano* 3*-fold obtained as in Proposition 0.8. Then* X *is of No. 5.3.*

(C) There exists an example of Y' *as in (B) and hence there exists a* $\mathbb{Q}$*-Fano* 3*-fold of No. 5.3.*

Theorem 0.21 (Table 5, II)**.** *(A) Let* X *be a* $\mathbb{Q}$*-Fano* 3*-fold of No. 5.4 or No. 5.5. Let* $\mu_1 \colon W_1 \to Y'$ *be the blow-up along a flipped curve* l_0*. Then there is a flop* $W_1 \dashrightarrow W_1{}'$ *over* X' *and an extremal contraction of* $(2,1)$*-type* $\nu_1 \colon W_1{}' \to Y_1{}'$ *over* X'*. Let* $F_1{}''$ *be the strict transform of* μ_1*-exceptional divisor on* $Y_1{}'$ *and* m_1 *the image of* ν_1*-exceptional divisor.*

(A-1) Assume that X *is a* $\mathbb{Q}$*-Fano* 3*-fold of No. 5.4. Then*

(1) $Y_1{}' \simeq \mathbb{P}(\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(1)^{\oplus 2})$*,*

(2) $F_1{}''$ *is a surface linearly equivalent to* $2H + L_1$*, where* H *is the tautological divisor of* $Y_1{}'$ *and* L_1 *is a fiber of the natural projection* $f_1{}' \colon Y_1{}' \to X' \simeq \mathbb{P}^1$*, and*

(3) m_1 *is a curve on* $F_1{}''$ *with* $g(m_1) = 9$ *and* $(-K_{Y_1{}'} \cdot m_1) = 33$ *such that* $\operatorname{Bs}|-K_{Y_1{}'} - m_1| = m_1 \cup l_1$ *and* m_1 *and* l_1 *intersect at*

one point simply, where l_1 is the section of ${f_1}'$ associated to the surjection $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2} \to \mathcal{O}_{\mathbb{P}^1}$.

(A-2) Assume that X is a $\mathbb{Q}$-Fano 3-fold of No. 5.5. Then the following hold.

(1) ${Y_1}'$ is a smooth divisor in $\mathbb{P} := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2})$ linear equivalent to $2H$, where H is the tautological divisor. Let V be the subvariety of $\mathbb{P}$ associated to the surjection $\mathcal{O}_{\mathbb{P}^1}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2} \to \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}$. Then $V \cap {Y_1}'$ is a disjoint union of two sections l_i of ${f_1}'$.

(2) ${F_1}''$ is a surface linearly equivalent to $2H|_{{Y_1}'} + L_1$, where L_1 is a fiber of the natural morphism ${f_1}' \colon {Y_1}' \to X'$.

(3) m_1 is a curve on ${F_1}''$ with $g(m_1) = 3$ and $(-K_{{Y_1}'} \cdot m_1) = 16$ such that $\mathrm{Bs}\,|-K_{{Y_1}'} - m_1| = m_1 \cup l_1 \cup l_2$ and m_1 and l_i $(i = 1, 2)$ intersect at one point simply.

Consequently we obtain the following diagram.

$$\begin{tikzcd}
 & W_1 \arrow[r, dashed] \arrow[dl, "\mu_1"'] & {W_1}' \arrow[dr, "\nu_1"] & \\
Y' \arrow[dr, "f'"] & & & {Y_1}' \arrow[dl, "{f_1}'"'] \\
 & X' \arrow[r, equal] & X' &
\end{tikzcd}$$

(B) Conversely let $({Y_1}', {F_1}'', m_1, l_i\,(i \geq 1))$ be as in (A1) for No. 5.4 (resp. (A2) for No. 5.5). Let $\nu_1 \colon {W_1}' \to {Y_1}'$ be the blow-up along m_1. Then there is a flop ${W_1}' \dashrightarrow W_1$ over X' and an extremal contraction of $(2,1)$-type $\mu_1 \colon W_1 \to Y'$ over X'. Let l_0 be the image of μ_1-exceptional divisor and L a fiber of the natural projection $f' \colon Y' \to X'$. Then the following hold.

(i) There is a unique member $\tilde{E}$ of $|-K_{Y'} - L|$ and $\tilde{E}$ is irreducible.

(ii) Y', l_i and $\tilde{E}$ satisfy the conditions of Proposition 0.8. Let X be a $\mathbb{Q}$-Fano 3-fold obtained as in Proposition 0.8. Then X is of No. 5.4 (resp. No. 5.5).

(C) There exists an example of $({Y_1}', {F_1}'', m_1, l_i)$ be as in (A1) for No. 5.4 (resp. (A2) for No. 5.5) and hence there exists a $\mathbb{Q}$-Fano 3-fold of No. 5.4 (resp. No. 5.5).

Remark 0.22. Examples are not known for No. 5.1.

Moreover in Section 5, we deny the possibilities of $\mathbb{Q}$-Fano 3-folds in Tables $1'$–$5'$ in Theorem 0.3.

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1. On existence of an anti-canonical divisor with only canonical singularities for a $\mathbb{Q}$-Fano 3-fold as in Theorem 0.3

Theorem 1.0. *Let X be as in Theorem 0.3 and P an index 2 point satisfying (5) of Main Assumption 0.1. Let $f \colon Y \to X$ be the weighted blow-up with weights*

$\frac{1}{2}(1,1,1,2)$ *and* E *the exceptional divisor. If* $n \geq 2$ *for* P*, then let* Q *be the unique index* 2 *point on* E*. Then the following hold.*

(1) $H^0(\mathscr{O}_Y(-K_Y)) \to H^0(\mathscr{O}_E(-K_Y|_E))$ *is surjective, Moreover if* $n \geq 2$*, then* $\mathrm{Bs}|-K_Y| = \{Q\}$ *near* E *or if* $n = 1$*, then* $\mathrm{Bs}|-K_Y| = \emptyset$ *near* E*.*

(2) $\mathrm{Bs}|-K_X| = \{P\}$ *near* P*.*

Proof. Consider the exact sequence

$$0 \to \mathscr{O}_Y(-K_Y - E) \to \mathscr{O}_Y(-K_Y) \to \mathscr{O}_E(-K_Y|_E).$$

To see that the map $H^0(\mathscr{O}_Y(-K_Y)) \to H^0(\mathscr{O}_E(-K_Y|_E))$ is surjective, it suffices to prove $h^0(\mathscr{O}_Y(-K_Y - E)) = h - 3$ by [Taka00, Proposition 2.3]. Note that this is equivalent to $h^0(\mathscr{O}_{Y'}(-K_{Y'} - \tilde{E})) = h - 3$ (we use the notation of Theorem 0.3). We can prove this using the data of Tables 1–5 and $1'$–$5'$ in Theorem 0.3 as follows.

Tables 1 and 1′: We have $-K_{Y'} - \tilde{E} \sim f'^*D$, where D is a primitive ample Weil divisor (we can easily see that the linear equivalent class of D is unique). Hence $h^0(-K_{Y'} - \tilde{E}) = h^0(D)$. $h^0(D) = h - 3$ is easy to see.

Tables 2, 2′ and 3: We have $-K_{Y'} - \tilde{E} \sim E'$ whence $h^0(-K_{Y'} - \tilde{E}) = 1 = h - 3$.

Table 4 and 4′: Since $-K_{Y'} - \tilde{E} - K_{Y'}$ is nef and big, we can compute $h^0(-K_{Y'} - \tilde{E})$ by Riemann-Roch theorem and we are done. But if $h = 5, 6$, then we can prove this more directly as follows. If $h = 5$, then $h^0(-K_{Y'} - \tilde{E}) = h^0({f'}_*\mathscr{O}_{Y'}(-K_{Y'} - \tilde{E})) = h^0(\mathscr{O}_{X'}(l)) = 2 = h - 3$, where l is a ruling of X'. Similarly if $h = 6$, then $h^0(-K_{Y'} - \tilde{E}) = h^0(L) = 3 = h - 3$.

Table 5 and 5′: Since $-K_{Y'} - \tilde{E} - K_{Y'}$ is nef and big, we can compute $h^0(-K_{Y'} - \tilde{E})$ by Riemann-Roch theorem and we are done.

Note that

$$(Y, Q) \simeq (\{xy + z^2 + u^{a-1} = 0\}/\mathbb{Z}_2(1,1,1,0), o).$$

Hence inductively we can construct the sequence of the weighted blow-ups with weights $\frac{1}{2}(1,1,1,2)$ at index 2 points on exceptional divisors and denote it by

$$Z_a \xrightarrow{f_a} Z_{a-1} \xrightarrow{f_{a-1}} \cdots \xrightarrow{f_2} Z_1 := Y$$

and set $f_1 := f$. Let F_i be the f_i-exceptional divisor and ${F_i}'$ its strict transform on Z_{i+1}. We prove that $H^0(\mathscr{O}_{Z_i}(-K_{Z_i})) \to H^0(\mathscr{O}_{F_i}(-K_{Z_i}|_{F_i}))$ is surjective. For $i = 1$, we proved the claim as above. Assume that the assertion holds for $i - 1$. Then by $H^0(\mathscr{O}_{Z_i}(-K_{Z_i})) \simeq H^0(\mathscr{O}_{Z_{i-1}}(-K_{Z_{i-1}}))$ and $H^0(\mathscr{O}_{F_{i-1}}(-K_{Z_{i-1}}|_{F_{i-1}})) \simeq H^0(\mathscr{O}_{{F_{i-1}}'}(-K_{Z_i}|_{{F_{i-1}}'}))$, $H^0(\mathscr{O}_{Z_i}(-K_{Z_i})) \to H^0(\mathscr{O}_{{F_{i-1}}'}(-K_{Z_i}|_{{F_{i-1}}'}))$ is also surjective. Note that $H^0(\mathscr{O}_{{F_{i-1}}'}(-K_{Z_i}|_{{F_{i-1}}'})) \simeq H^0(\mathscr{O}_{{F_{i-1}}' \cap F_i}(-K_{Z_i}|_{{F_{i-1}}' \cap F_i}))$ and $H^0(\mathscr{O}_{F_i}(-K_{Z_i}|_{F_i})) \simeq H^0(\mathscr{O}_{{F_{i-1}}' \cap F_i}(-K_{Z_i}|_{{F_{i-1}}' \cap F_i}))$. Hence $H^0(\mathscr{O}_{Z_i}(-K_{Z_i})) \to H^0(\mathscr{O}_{F_i}(-K_{Z_i}|_{F_i}))$ is surjective. Hence we know that for the exceptional set F for $Z_a \to X$, $\mathrm{Bs}|-K_{Z_a}| \cap F = \emptyset$. Since $H^0(\mathscr{O}_X(-K_X)) \simeq H^0(\mathscr{O}_Y(-K_Y)) \simeq H^0(\mathscr{O}_{Z_a}(-K_{Z_a}))$, we finish the proof of Theorem 1.0. □

Proposition 1.1. *Let* X *be a klt weak* $\mathbb{Q}$*-Fano* 3*-fold satisfying the following conditions.*

(1) $|-K_X| \neq \emptyset$*.*

(2) There are a finite number of non-Gorenstein points on X*.*

(3) There is a member of $|-K_X|$ *which is normal near non-Gorenstein points.*

Then $|-K_X|$ *has a member which is normal and has only canonical singularities outside non-Gorenstein points of* X*.*

Proof. The proof is almost the same as one of [Amb99, Main Theorem] or [Mel99, Theorem 1]. So we only give an outline of the proof. Let $U := \{x | x \text{ is a Gorenstein point of } X\}$. Let S be a general member of $|-K_X|$. Let $\gamma := \max\{t | K_X + tS|_U$ is log canonical$\}$. It suffices to prove that if there is an element of $\mathrm{CLC}(K_X + \gamma S|_U)$ contained in $\mathrm{Bs}|-K_X|$, it is $S|_U$. Assume the contrary and let Z be a minimal element of $\mathrm{CLC}(K_X + \gamma S|_U)$ contained in $\mathrm{Bs}|-K_X|$. By the assumption (3), Z is a complete variety. Hence by using [Taka00, Theorem 1.0] (KKV vanishing theorem), we know that it suffices to prove $H^0(\mathscr{O}_Z(-K_X|_Z)) \neq 0$. It is done by Adjunction Theorem and a non-vanishing argument. □

Corollary 1.2. *Let X be a $\mathbb{Q}$-Fano 3-fold with Main Assumption 0.1 and assume moreover that any index 2 point satisfies (5). Then $|-K_X|$ has a member with only canonical singularities.*

Proof. Fix an index 2 point P and the weighted blow-up f as in Theorem 1.0. By Theorem 1.0 and Proposition 1.1, we can find a member $S \in |-K_Y|$ such that S is normal and has only canonical singularities outside index 2 points of Y. But by Theorem 1.0 again, S has only canonical singularities near E. So since $f|_S$ is crepant, $f(S)$ has only canonical singularity outside index 2 points of X except P. Since P is any index 2, we can find a member of $|-K_X|$ with only canonical singularities. □

2. On deformations of $\mathbb{Q}$-Fano 3-folds as in Theorem 0.3

Our starting point in this section is the following theorem proved by T. Minagawa:

Theorem 2.0 (T. Minagawa). *Let X be a $\mathbb{Q}$-Fano 3-fold (resp. weak $\mathbb{Q}$-Fano 3-fold) with $I(X) = 2$. Assume that there exists a smooth member of $|-2K_X|$. Then there exists a flat family $\mathfrak{f}\colon \mathfrak{X} \to (\Delta, 0)$ over a 1-dimensional disc $(\Delta, 0)$ such that $X \simeq \mathfrak{f}^{-1}(0)$ and $\mathfrak{f}^{-1}(t)$ is a $\mathbb{Q}$-Fano 3-fold (resp. a weak $\mathbb{Q}$-Fano 3-fold) with only ODP's, QODP's or $\frac{1}{2}(1,1,1)$-singularities as its singularities for $t \in \Delta \setminus \{0\}$.*

Proof. See [Min99a, Theorem 2.4]. □

Theorem 2.1. *Let X be a (not necessarily $\mathbb{Q}$-factorial) weak $\mathbb{Q}$-Fano 3-fold with $I(X) = 2$. Assume that*

(1_X) $h^0(-K_X) \geq 4$,
(2_X) near an index 2 point P, $\mathrm{Bs}|-K_X| = \{P\}$, and
(3_X) there is no divisor contracted to a point by the morphism defined by $|-mK_X|$ for $m \gg 0$.

Then X can be deformed to a weak $\mathbb{Q}$-Fano 3-fold with only $\frac{1}{2}(1,1,1)$-singularities as its singularities.

Proof. By (1_X) and [Taka00, Theorem 4.1], $|-2K_X|$ is free. So by Theorem 2.0, we may assume that X has ODP's, $\frac{1}{2}(1,1,1)$-singularities or QODP's as its singularities. Let $f\colon Y \to X$ be the composite of the weighted blow-ups at all QODP's of X (as in Theorem 0.3), $g\colon Z \to Y$ the composite of the blow-ups at all $\frac{1}{2}(1,1,1)$-singularities of Y and $h := g \circ f$. Then by the choice of h and (2_X), $-K_Z$ is nef. Moreover by (1_X), we have $h^0(-K_Z) = h^0(-K_X) \geq 4$. Hence by Riemann-Roch theorem, $(-K_Z)^3 > 0$. So Z is a Gorenstein weak Fano 3-fold. We verify that the assumption (3_Z) holds. Assume that there is a divisor S on Z which is contracted to a point by the morphism defined by $|-mK_Z|$. By the choice of h, S is not h-exceptional since an h-exceptional divisor contains a curve negative for K_Z. If $E \cap S \neq \emptyset$ for a prime h-exceptional divisor E, then $E \cap S$ is a curve since E is a Cartier divisor. By the nature of S, $E \cap S$ is numerically trivial for $-K_Z$.

So since E contains a curve which is numerically trivial for $-K_Z$, E must be the strict transform of an f-exceptional divisor E' and moreover S must intersect also the g-exceptional divisor F of the blow-up at the $\frac{1}{2}(1,1,1)$-singularity on E'. Then $S \cap F$ is numerically positive for $-K_Z$, a contradiction. Hence S is disjoint from h-exceptional divisors. However, $h(S)$ is contracted to a point by the morphism defined by $|-mK_X|$, a contradiction. Hence Z satisfies (3_Z).

Step 1. smoothing ODP's. We prove that Z is smoothable by the same method as [Nam97]. Note that the following claim (which is [Nam97, Proposition 2]) holds for a weak Fano 3-fold Z satisfying (3_Z) without any change in his proof.

Claim. *Let D be a member of $|-K_Z|$ with only canonical singularities. Then* $\operatorname{Pic} Z \to \operatorname{Pic} D$ *is an injection.*

Let $\mathfrak{Z} \to \Delta$ be a 1-parameter smoothing of Z. Then by [KM92, Proposition 11.4], we obtain the deformation $\mathfrak{Y} \to \Delta$ of Y which satisfies the commutative diagram

$$\begin{tikzcd} \mathfrak{Z} \arrow[rr] \arrow[dr] & & \mathfrak{Y} \arrow[dl] \\ & \Delta. & \end{tikzcd}$$

Then $\mathfrak{Z}_t \to \mathfrak{Y}_t$ is a composite of $(2,0)_4$ type contractions for $t \in \Delta$ since a contraction of type $(2,0)_4$ is stable under a deformation by [Kod63]. Hence Y_t has only $\frac{1}{2}(1,1,1)$-singularities as its singularities. Similarly we can prove that this can be blown down to a smoothing of ODP's of X using [KM92, the proof of Theorem 12.3.1].

Step 2. deforming QODP's to $\frac{1}{2}(1,1,1)$-singularities. By induction, we only have to deform one QODP to two $\frac{1}{2}(1,1,1)$-singularities. Let P be a QODP of X and E (resp. F) the strict transform of f-exceptional divisor over P (resp. g-exceptional divisor over P). Then $E \simeq \mathbb{F}_4$ and there exists a primitive crepant birational morphism $p \colon Y \to W$ which contracts E to a smooth rational curve. Then by [Min99b, Proposition 3.5 (i)], there is a small deformation

$$\begin{tikzcd} \mathfrak{Y} \arrow[rr] \arrow[dr] & & \mathfrak{W} \arrow[dl] \\ & \Delta & \end{tikzcd}$$

of p such that $\mathfrak{Y}_t \to \mathfrak{W}_t$ is an isomorphism for $t \neq 0$. By the same argument as Step 1, we have a birational morphism $\mathfrak{Y}_t \to \mathfrak{X}_t$, where $\mathfrak{X}_t$ is a deformation of X. If there exists a QODP which specializes to P, then $\mathfrak{Y}_t \to \mathfrak{W}_t$ must be a contraction of the same type as p, a contradiction. Hence P is deformed to two $\frac{1}{2}(1,1,1)$-singularities. □

Corollary 2.2. *Let X be a $\mathbb{Q}$-factorial $\mathbb{Q}$-Fano 3-fold with (1)–(4) in Main Assumption 0.1. Then X can be deformed to a $\mathbb{Q}$-factorial $\mathbb{Q}$-Fano 3-fold X' with (1)–(4) in Main Assumption 0.1 and with only $\frac{1}{2}(1,1,1)$-singularities as its singularities.*

Proof. The proof is similar to the next corollary. □

Corollary 2.3. *Let X be a $\mathbb{Q}$-factorial $\mathbb{Q}$-Fano 3-fold with (1)–(4) in Main Assumption 0.1. Let $N := \operatorname{aw}(X)$. Then if $N > 1$ (resp. $N = 1$), X can be transformed to a $\mathbb{Q}$-factorial $\mathbb{Q}$-Fano 3-fold $\tilde{Z}'$ with (1)–(4) in Main Assumption 0.1 and with only $\frac{1}{2}(1,1,1)$-singularities as its singularities and $h^0(-K_{\tilde{Z}'}) = h$ and $\operatorname{aw}(\tilde{Z}') = N-1$ (resp. a smooth Fano 3-fold $\tilde{Z}'$ with $\rho(\tilde{Z}') = 1$, $F(\tilde{Z}') = 1$ and*

$h^0(-K_{\tilde{Z}'}) = h$) as follows.

$$\begin{tikzcd}
& & \tilde{Y} \arrow[dl, "\tilde{f}"'] \arrow[dr, "\tilde{g}"] & & \\
X \arrow[r, "\mathrm{def}"] & \tilde{X} & & \tilde{Z} \arrow[r, "\mathrm{def}"] & \tilde{Z}',
\end{tikzcd}$$

where $* \xrightarrow{\mathrm{def}} **$ means that $**$ is a small deformation of $*$,

$\tilde{X}$ is a $\mathbb{Q}$-Fano 3-fold with Main Assumption 0.1 and with only ODP's, QODP's or $\frac{1}{2}(1,1,1)$-singularities as its singularities,

$\tilde{f}\colon \tilde{Y} \to \tilde{X}$ is similarly chosen to f in Theorem 0.3, and $\tilde{g}\colon \tilde{Y} \to \tilde{Z}$ be the anti-canonical model.

Proof. By Theorem 2.0, there is a deformation $X \xrightarrow{\mathrm{def}} \tilde{X}$ as stated above. Note that we may assume that the $\mathbb{Q}$-factoriality is preserved by [KM92, Theorem 12.1.10]. Moreover by Tables 1–5 and $1'$–$5'$ of Theorem 0.3, and the result of [San95], [San96], $\tilde{X}$ satisfies Main Assumption 0.1. By Theorem 0.3, we obtain $\tilde{Y}$ and $\tilde{Z}$ as above and $\rho(\tilde{Z}) = 1$. If $N = 1$ and $h = 4$, then $\tilde{Z}$ may have canonical singularities but in this case $\Phi_{|-K_X|}$ is a double cover of $\mathbb{P}^3$ by [Muk95, Theorem 6.5 and Proposition 7.8] and so $\tilde{Z}$ has a smoothing $\tilde{Z}'$. Except this case, we apply Theorem 2.1 for $\tilde{Z}$. we only have to check that $\tilde{Z}$ satisfies the assumption $(2_{\tilde{Z}})$. By Theorem 1.0, $(2_{\tilde{Y}})$ holds and so does $(2_{\tilde{Z}})$. Hence $\tilde{Z}$ can be deformed to a $\mathbb{Q}$-Fano 3-fold $\tilde{Z}'$ with only QODP's or $\frac{1}{2}(1,1,1)$-singularities as its singularities. Next we show $\tilde{Z}'$ has the properties as stated above. By [KM92, the proof of Corollary 12.3.4], we have $\rho(\tilde{Z}') = 1$. If $N > 1$, $F(\tilde{Z}') = \frac{1}{2}$ by Tables 1–5 and $1'$–$5'$ of Theorem 0.3 and [San95], [San96]. If $N = 1$, we have clearly $F(\tilde{Z}') = 1$. Hence we are done. □

The following is similar to Shokurov's theorem [Sho79a]:

Corollary 2.4. *Let X be a $\mathbb{Q}$-factorial $\mathbb{Q}$-Fano 3-fold with (1)–(4) of Main Assumption 0.1. Then for any index 2 point P, there exists a smooth rational curve l through P such that $-K_X \cdot l = \frac{1}{2}$.*

Proof. First we treat the case that any index 2 point is of type as in Main Assumption 0.1 (5). By Tables 1–5 and $1'$–$5'$ in Theorem 0.3, e is positive or f' is a crepant divisorial contraction for any choice of an index 2 point P. Let $g\colon Y \to Z$ be the anti-canonical model. Let l' be a flopping curve if g is a flopping contraction or a general fiber of E' if g is a crepant divisorial contraction. Then in the former case, by [Taka00, Lemma 4.3], (resp. in the latter case, by the proof of Theorem 0.3 (see [Taka00]) in Case 5), we have $g(E) \simeq E$ whence $E.l' = 1$. Hence $l := f(l')$ is what we want.

Next we treat the general case. Let $\mathfrak{f}\colon \mathfrak{X} \to \Delta$ be a flat family as in Theorem 2.0. By [KM92, Corollary 12.3.4], $\rho(\mathfrak{X}_t) = 1$ and moreover by Tables 1–5 and $1'$–$5'$, [San95] and [San96], $\mathfrak{X}_t (t \neq 0)$ satisfies Main Assumption 0.1. Let P be an index 2 point on X and P_t an index 2 point on $\mathfrak{X}_t$ which specializes to P. By the first part of this proof, there is a curve l_t on $\mathfrak{X}_t (t \neq 0)$ such that $l_t \simeq \mathbb{P}^1$, $P_t \in l_t$ and $-K_{\mathfrak{X}_t} \cdot l_t = \frac{1}{2}$. Since there are only countably many components of relative Hilbert scheme $\operatorname{Hilb}(\mathfrak{X}/\Delta)$, we may assume that they form a flat family over Δ. Moreover by the properness of a component of relative Hilbert scheme, this family extends over 0. Let l be its fiber over 0. Then l is what we want. □

3. Embedding $\mathbb{Q}$-Fano 3-folds as in Theorem 0.3

The next result is a first step for the classification of Mukai's type [Muk95, Theorem 1.10].

Theorem 3.0. *Let X be a (not necessarily $\mathbb{Q}$-factorial) canonical $\mathbb{Q}$-Fano 3-fold of $I(X)=2$. Assume that X has the following properties.*

(1) $|-K_X|$ is indecomposable, i.e., $|-K_X|$ contains no member which is a sum of two movable Weil divisors.
(2) $|-K_X|$ has no base curve containing an index 2 point.
(3) $|-K_X|$ has a member with only canonical singularities.
(4) $h^0(X,\mathscr{O}(-K_X))\geq 4$.
(5) Any index 2 point of X is $\frac{1}{2}(1,1,1)$-singularity.

Then except the following two cases (a) and (b), X is embedded into $\mathbb{P}(1^h,2^N)$ and $-K_X$ is the restriction of $\mathscr{O}(1)$, where $h:=h^0(-K_X)$ and N is the number of $\frac{1}{2}(1,1,1)$-singularities.

(a) $\Phi_{|-K_X|}$ is a double cover of $\mathbb{P}^3$ branched along a sextic.
(b) $\Phi_{|-K_X|}$ is a double cover of a quadric hypersurface branched along the intersection with a quartic.

(Note that in case (a),

$$X\simeq((6)\subset\mathbb{P}(1^4,3)).$$

Note also that in case (b),

$$X\simeq((2,4)\subset\mathbb{P}(1^5,2))$$

but the number of weight 2 is not equal to the number of index 2 point.)

Moreover $\bigoplus_{m=0}^{\infty}H^0(X,-mK_X)$ is generated by elements of degree ≤ 2 and related by elements of degree ≤ 6.

If $h=4$ and $N=1$, then $X\simeq((5)\subset\mathbb{P}(1^4,2))$.
If $h=4$ and $N=2$, then $X\simeq((3,4)\subset\mathbb{P}(1^4,2^2))$.
If $h=5$ and $N=1$, then $X\simeq((3,3)\subset\mathbb{P}(1^5,2))$.

Proof. We prove this by induction on N.

In case $N=0$, the assertion follows from [Muk95, Theorem 6.5 and Proposition 7.8]. Next we prove that if the assertion holds in case X has $N-1$ $\frac{1}{2}(1,1,1)$-singularities, then so does it in case X has N $\frac{1}{2}(1,1,1)$-singularities. Let X be a $\mathbb{Q}$-Fano 3-fold satisfying the assumptions of this theorem and with N $\frac{1}{2}(1,1,1)$-singularities. Let $f\colon Y\to X$ be the blow-up at a $\frac{1}{2}(1,1,1)$-singularity. Let E be the exceptional divisor of f. Then Y is a weak $\mathbb{Q}$-Fano 3-fold by [Taka00, Proposition 4.2]. By the assumption (4), Y is not a $\mathbb{Q}$-Fano 3-fold. Let $g\colon Y\to Z$ be the anti-canonical model and $\overline{E}:=g(E)$.

Claim 1. *Z satisfies the assumption of this theorem and has $N-1$ $\frac{1}{2}(1,1,1)$-singularities.*

Proof. By $-K_Y=g^*(-K_Z)$, if $|-K_Z|$ is decomposable, $|-K_X|$ must be decomposable, a contradiction. Hence (1) is satisfied. By (2) for X, neither $|-K_Y|$ has a base curve containing an index 2 point. Hence any g-exceptional curve does not contain an index 2 point. So by $-K_Y=g^*(-K_Z)$, (2) is satisfied and (5) is also satisfied. Let D be a member of $|-K_X|$ with only canonical singularities. Then the strict transform D' of D on Y has the same property since $D'\to D$ is crepant. Since $D'\to g(D')$ is crepant, $g(D')$ has also the same property. Hence (3) is satisfied. By $-K_Y=g^*(-K_Z)$ and $h^0(-K_Y)=h^0(-K_X)$, we know that (4) is satisfied. □

Hence by the assumption of the induction, one of the following three cases occurs.
Case α. $Z \subset \mathbb{P}(1^h, 2^{N-1})$ and $-K_Z = \mathscr{O}_Z(1)$.
Case β. Z is of type (a).
Case γ. Z is of type (b).

Claim 2. $\mathrm{Bs}|-K_X|$ *coincides with* $\frac{1}{2}(1,1,1)$*-singularities as a set.*

Proof. If $N = 0$, the assertion follows from [Muk95, Theorem 6.5 and Proposition 7.8]. Hence by Claim 1, the assertion follows by induction with respect to the number of $\frac{1}{2}(1,1,1)$-singularities. □

Case α. We first show that $\overline{E} \simeq E$. By the proof of Claim 2, the assertion similar to Claim 2 holds for $\mathrm{Bs}|-K_Y|$. Hence $H^0(\mathscr{O}_Y(-K_Y)) \to H^0(\mathscr{O}_E(-K_Y)) \simeq H^0(\mathscr{O}_{\mathbb{P}^2}(1))$ is surjective. Hence $H^0(\mathscr{O}_Y(-mK_Y)) \to H^0(\mathscr{O}_E(-mK_Y))$ is also surjective for all $m \geq 0$ since $\oplus_{m\geq 0} H^0(\mathscr{O}_{\mathbb{P}^2}(m))$ is simply generated. So $\overline{E} \simeq E$ since g is defined by $|-mK_Y|$ for some $m > 0$.

We note here that there is an elementary transformation $\mathbb{P}(1^h, 2^N) \dashrightarrow \mathbb{P}(1^h, 2^{N-1})$ which is decomposed as follows. Let $\mathbb{P}$ be the projective bundle over $\mathbb{P}(1^h, 2^{N-1})$ whose associated vector bundle is $\mathscr{O} \oplus \mathscr{O}(-2)$ and T the effective tautological divisor (which is unique). Let a be the contraction morphism of T. Then the image of $\mathbb{P}$ by a is isomorphic to $\mathbb{P}(1^h, 2^N)$. Let $b\colon \mathbb{P} \to \mathbb{P}(1^h, 2^{N-1})$ be the natural projection. Then our elementary transformation is $b \circ a^{-1}$.

We seek a natural morphism $Y \to \mathbb{P}$. For this, we prove that there is a natural surjection $g^*(\mathscr{O}_Z \oplus \mathscr{O}_Z(-2)) \to \mathscr{O}_Y(E)$.

There is a natural injection $\mathscr{O}_Y(-E) \to \mathscr{O}_Y$ which represents $\mathscr{O}_Y(-E)$ as the ideal sheaf of E. By [Taka00, Theorem 4.1], there is a member $S \in |-2K_X|$ such that $f^*S \cap E = \emptyset$. Associated to S, there is an injection $\mathscr{O}_Y(-f^*S) \to \mathscr{O}_Y$. This gives an injection $\mathscr{O}_Y(-E) \to g^*\mathscr{O}_Z(2)$ since $g^*\mathscr{O}_Z(2) \simeq \mathscr{O}_Y(-2K_Y)$ and $-f^*(-2K_X) \sim -(-2K_Y) - E$. Hence we can define a injection $\mathscr{O}_Y(-E) \to g^*(\mathscr{O}_Z \oplus \mathscr{O}_Z(2))$. Since $f^*S \cap E = \emptyset$, the cokernel of this map is locally free and hence the dual of this map is a surjection. Let $\iota\colon Y \to \mathbb{P}$ be the morphism over $\mathbb{P}(1^h, 2^{N-1})$ associated to the surjection $d : g^*(\mathscr{O}_Z \oplus \mathscr{O}_Z(-2)) \to \mathscr{O}_Y(E)$ and $p' := p|_{\iota(Y)}$. Note that ι is finite since E is g-ample.

Claim 3. $\iota(Y)$ *is normal.*

Proof. First we see that $\iota(Y)$ is smooth near $\iota(E)$. Let y' be a point of $\iota(E)$ and $z = p'(y')$. Let $m := g^{-1}(z)_{\mathrm{red}}$. If m is 0-dimensional, then g and p' are isomorphisms over z since Z is normal whence ι is an isomorphism over y'. In particular $\iota(Y)$ is smooth at y' since so is Y at $\iota^{-1}(y')$. So we may assume that m is 1-dimensional. By the surjectivity of d, its restriction to m is also surjective. By $E \simeq \overline{E}$, $m \simeq \mathbb{P}^1$. Hence $\iota|_m$ is isomorphism whence ι is injective on m. Let $y := \iota^{-1}(y')$, $A := \mathscr{O}_{Y,y}$ and $B := \mathscr{O}_{Z,z}$. We will prove that the natural morphism $B[t] \to A$ is surjective, where t is a local parameter of ${p'}^{-1}(z)$ at y'. By this map $B[t] \to A$, t is sent to a local parameter of m and two local parameters of $\iota(E)$ at y' are sent to that of E at y. Hence $B[t] \to A$ is surjective. So ι is an isomorphism over y'.

Next we complete the proof of the claim. It suffices to prove that $\iota_*\mathscr{O}_Y = \mathscr{O}_{\iota(Y)}$. The natural morphism $\mathscr{O}_{\iota(Y)} \to \iota_*\mathscr{O}_Y$ is injective since the kernel is at most torsion sheaf. Let $\mathcal{C}$ be its cokernel. We prove that ${p'}_*\mathcal{C} = 0$. By the exact sequence

$$0 \to \mathscr{O}_{\iota(Y)} \to \iota_*\mathscr{O}_Y \to \mathcal{C} \to 0,$$

we have

$$0 \to {p'}_*\mathscr{O}_{\iota(Y)} \to {p'}_*\iota_*\mathscr{O}_Y \to {p'}_*\mathcal{C} \to R^1{p'}_*\mathscr{O}_{\iota(Y)}.$$

Since ${p'}_*\mathscr{O}_{\iota(Y)} \to p_*\iota_*\mathscr{O}_Y$ is an isomorphism by the normality of Z, it suffices to prove that $R^1p_*\mathscr{O}_{\iota(Y)} = 0$. Consider the exact sequence

$$0 \to \mathcal{I}_{\iota(Y)} \to \mathscr{O}_{\mathbb{P}} \to \mathscr{O}_{\iota(Y)} \to 0.$$

Since the dimension of a fiber of $p \le 1$, we have $R^2p_*\mathcal{I}_{\iota(Y)} = 0$. Since $\mathbb{P}$ is a $\mathbb{P}^1$-bundle, we have $R^1p_*\mathscr{O}_{\mathbb{P}} = 0$. Thus we obtain $R^1{p'}_*\mathscr{O}_{\iota(Y)} = 0$ whence ${p'}_*\mathcal{C} = 0$.

Since every fiber of $g\colon Y \to Z$ intersects $\iota(E)$ and $\iota(Y)$ is smooth at points of $\iota(E)$, any fiber is not contained in the singular locus of $\iota(Y)$. Let l be any 1-dimensional fiber of g. By the theorem on formal functions, we have $\mathcal{C} \otimes \mathscr{O}_l = 0$ because $\dim \operatorname{Supp} \mathcal{C} \otimes \mathscr{O}_l = 0$ (note that l is not contained in the singular locus of $\iota(Y)$) and ${p'}_*\mathcal{C} = 0$. Hence by Nakayama's lemma, $\mathcal{C} = 0$. □

Hence $\iota\colon Y \to \iota(Y)$ is finite and birational and $\iota(Y)$ is normal, it is an isomorphism by the Zariski's Main Theorem. Hence $X \simeq a(\iota(Y))$ is naturally embedded into $\mathbb{P}(1^h, 2^N)$ and $-K_X = \mathscr{O}(1)$.

For the next two cases, we directly prove that if $h = 4$ and $N = 1$, then $X \simeq ((5) \subset \mathbb{P}(1^4, 2))$ and if $h = 5$ and $N = 1$, then $X \simeq ((3,3) \subset \mathbb{P}(1^5, 2))$ below, which complete the induction.

Recall that $|-2K_X|$ is free by [Taka00, Theorem 4.1]. So we can take a smooth curve C which is the intersection of general members of $|-K_X|$ and $|-2K_X|$. Let $L := -K_X|_C$. Note that L is a Cartier divisor such that $K_C = 2L$. We describe $R(C, L) := \oplus_{m \ge 0} H^0(\mathscr{O}_C(mL))$ by using [Reid90, Theorem 3.4]. By a composite of blow-ups of $\frac{1}{2}(1,1,1)$-singularities and crepant contractions, we can reach a Gorenstein Fano 3-fold W. Let $W' \subset \mathbb{P}(1^h)$ be the image of $\Phi_{|-K_W|}$. Let $\pi\colon C \to C'$ be the restriction of the rational map $X \dashrightarrow W'$ to C.

Assume that W does not satisfy (a) or (b). Then $X \dashrightarrow W'$ is birational whence by choosing C generally, we may assume that π is a birational map. Assume that W satisfies (a) (resp. (b)). Then $W \to W'$ is a double cover of $\mathbb{P}^3$ (resp. a (possibly singular) quadric 3-fold). Since X does not satisfy (a) (resp. (b)), $(-K_X)^3 \ge \frac{5}{2}$ (resp. $(-K_X)^3 \ge \frac{9}{2}$) whence $L \cdot C \ge 5$ (resp. $L \cdot C \ge 9$). So by choosing C generally, we may assume that π is a birational map or a double cover of a plane curve of degree ≥ 3 (resp. a space curve of degree ≥ 5).

In any case, C' is not a normal rational curve in $\mathbb{P}(1^{h-1})$. Note that $C' = \Phi_{|L|}(C)$. Hence $H^0(\mathscr{O}_C(L)) \otimes H^0(\mathscr{O}_C(2L)) \to H^0(\mathscr{O}_C(3L))$ is surjective. (Note that $K_C = 2L$.) So by [ibid.], $R(C, L)$ is generated by elements of degree ≤ 2 and related by elements of degree ≤ 6,which in turn show that the same things hold for $\oplus_{m \ge 0} H^0(\mathscr{O}_X(-mK_X))$.

Assume that $h = 4$ and $N = 1$. Since $\deg L = 5$, π is birational. By the genus formula of a plane curve, we have $p_a(C') = 6$. On the other hand, $g(C) = 6$. So π is an isomorphism, which in turn show that $X \simeq ((5) \subset \mathbb{P}(1^4, 2))$.

Assume that $h = 4$ and $N = 2$. If there is a relation of degree 2 in $R(C, L)$, C' is a conic in $\mathbb{P}^2$, a contradiction. Hence there is no relation of degree 2 in $R(C, L)$. Then we find easily the relation of $R(C, L)$ by [ibid.] and conclude $C \simeq ((3,4) \subset \mathbb{P}(1^3, 2))$, which in turn shows that $X \simeq ((3,4) \subset \mathbb{P}(1^4, 2^2))$.

Assume that $h = 5$ and $N = 1$. Since $\deg L = 9$, π is birational. By easy computations, we have $h^0(\mathscr{O}(L)) = 4$, $h^0(\mathscr{O}(2L)) = 10$ and $h^0(\mathscr{O}(3L)) = 18$. Hence there are at least two relations of degree 3 among elements of degree 1. This means that C' is contained in two cubics in $\mathbb{P}^3$ whence $C' \simeq ((3,3) \subset \mathbb{P}^3)$. We have $g(C) = p_a(C') = 10$. Hence $C \simeq C'$, which in turn shows that $X \simeq ((3,3) \subset \mathbb{P}(1^5, 2))$.

Now we complete the proof of this theorem.

□

Remark 3.1. The assumption that $h^0(-K_X) \geq 4$ is necessary for Theorem 3.0 by the existence of the following.

$$X \simeq ((12) \subset \mathbb{P}(1^3, 4, 6))$$

which satisfies $h^0(-K_X) = 3$.

Corollary 3.2. *Let X be a $\mathbb{Q}$-Fano 3-fold with Main Assumption 0.1. Assume that any index 2 point of X is $\frac{1}{2}(1,1,1)$-singularity. Then X is embedded into $\mathbb{P}(1^h, 2^N)$ and $-K_X$ is the restriction of $\mathscr{O}(1)$, where $h := h^0(-K_X)$ and N is the number of $\frac{1}{2}(1,1,1)$-singularities. Moreover X is an intersection of weighted hypersurfaces of degree ≤ 6.*

If $h = 4$ and $N = 1$, then $X \simeq ((5) \subset \mathbb{P}(1^4, 2))$.
If $h = 4$ and $N = 2$, then $X \simeq ((3,4) \subset \mathbb{P}(1^4, 2^2))$.
If $h = 5$ and $N = 1$, then $X \simeq ((3,3) \subset \mathbb{P}(1^5, 2))$.

Proof. By Corollary 2.4, Theorem 1.0 and Corollary 1.2, we can see that the assumptions of Theorem 3.0 are satisfied for X. Hence we are done. □

Remark 3.3. There are two possibilities of $f' \colon Y' \to X'$ for $X \simeq ((5) \subset \mathbb{P}(1^4, 2))$. Both of them actually occurs. Indeed, let $f_5(x_0, x_1, x_2, x_3, y) = 0$ be a quintic in $\mathbb{P}(1^4, 2)$, where $\operatorname{wt} x_i = 1$ and $\operatorname{wt} y = 2$. If we take f_5 generally, then $\{f_5 = 0\}$ is an example for No. 3.1. If we take f_5 specially, for example, $f_5 \equiv x_0 y^2 + \sum_{i=0}^{3} {x_i}^5$, then $\{f_5 = 0\}$ is an example for No. 3.1$'$.

By Corollary 3.2, we can improve [Taka00, Theorem 4.1] and Theorem 1.0 for X as in Corollary 3.2 as follows:

Corollary 3.4. *Let X be a $\mathbb{Q}$-Fano 3-fold with Main Assumption 0.1. Assume that any non-Gorenstein point is $\frac{1}{2}(1,1,1)$-singularity. Then*

(1) $-2K_X$ is very ample,
(2) $\operatorname{Bs}|-K_X|$ is the union of $\frac{1}{2}(1,1,1)$-singularities and a general member of $|-K_X|$ has only ordinary double points, and
(3) Fix $f, Y, Y', \ldots$ etc. as in Theorem 0.3. Then $\operatorname{Bs}|-K_{Y'}|$ is the union of flipped curves and $\frac{1}{2}(1,1,1)$-singularities.

Proof. The proof of (1) and (2) are clear from Corollary 3.2. (3) follows from (2) and [Taka00, Proposition 2.1 (4)]. □

4. Construction of examples of $\mathbb{Q}$-Fano 3-folds as in Theorem 0.3

Note that the assertions for No. 3.1, No. 3.1$'$, No. 3.2 and No. 5.2 are proved in Corollary 3.2 and Remark 3.3.

Proof of Proposition 0.8. Assume that $n > 0$. Let R' be the ray of $\overline{\mathrm{NE}}(Y')$ generated by the numerical class of l_i. By (2-3) and (2-4), R' is extremal. By (2-2) and (2-3), a general member D of $|-K_{Y'}|$ is smooth along l_i and has only canonical singularities (see [MM85, Proposition 6.8]. By (2-4), $D \cdot l_i = -1$. Hence we have $\mathcal{N}_{l_i/Y'} \simeq \mathscr{O}_{\mathbb{P}^1}(-1) \oplus \mathscr{O}_{\mathbb{P}^1}(-2)$. Moreover R' is contractible since for another general member D' of $|-K_{Y'}|$, $(Y', D + \epsilon D')$ is klt for $0 < \epsilon \ll 1$ and $(-(K_{Y'} + D + \epsilon D') \cdot R') > 0$. Moreover the contraction associated to R' can be regarded as a log flipping contraction for $(Y', D + \epsilon D')$. Let $Y' \dashrightarrow {Y_0}'$ be the log flip. The log flip coincides the anti-flip for $K_{Y'}$ and by $\mathcal{N}_{l_i/Y'} \simeq \mathscr{O}_{\mathbb{P}^1}(-1) \oplus \mathscr{O}_{\mathbb{P}^1}(-2)$, ${Y_0}'$ has only $\frac{1}{2}(1,1,1)$-singularities. By (2-3) and [Taka00, Proposition 2.1 (4)], $\operatorname{Bs}|-K_{{Y_0}'}|$ is the union of $\frac{1}{2}(1,1,1)$-singularities. In particular $-K_{{Y_0}'}$ is nef.

In case $n = 0$, set ${Y_0}' := Y'$.

In any case $-K_{{Y_0}'}$ is nef and moreover by (3) and [Taka00, Proposition 2.1 (5)], we have $(-K_{{Y_0}'})^3 = (-K_{Y'})^3 + \frac{n}{2} > 0$. Hence $-K_{{Y_0}'}$ is big.

Let $\tilde{E}_0$ be the strict transform of $\tilde{E}$ on $Y_0{}'$. By (4), we can show that $(-K_{Y_0{}'})^2\tilde{E}_0 = 1$ and $(-K_{Y_0{}'})\tilde{E}_0{}^2 = -2$ by [Taka00, Lemma 3.2 (3)].

Since $-K_{Y_0{}'}$ is nef and big, we can construct a diagram

$$Y_0{}' \dashrightarrow \cdots \dashrightarrow Y_i{}' \dashrightarrow Y_{i+1}{}' \dashrightarrow \cdots \dashrightarrow Y := Y_k{}' \xrightarrow{f} X$$

as in [Taka00, Set up 3.3] starting from $Y_0{}'$ by setting $D = \tilde{E}_0$, where $Y_i{}' \dashrightarrow Y_{i+1}{}'$ is a flop or a flip for $i = 0$ and a flip for $i \geq 1$. Let $\tilde{E}_i$ (resp. E) be the strict transform of $\tilde{E}_0$ on $Y_i{}'$ (resp. Y). Let $R_i := R$ if $n = 0$ and $i = 0$ and the $K_{Y_i{}'}$-negative extremal ray otherwise. By [Taka00, Lemma 3.1], we have

$$(-K_Y)^2 E = 1 - \sum a_i d_i, \tag{4.1}$$

$$(-K_Y)E^2 = -2 - \sum a_i{}^2 d_i, \; \textit{and} \tag{4.2}$$

$$E^3 = \tilde{E}_0{}^3 - \sum a_i{}^3 d_i - e, \tag{4.3}$$

Claim 4.1. $\tilde{E}_i \cdot R_i < 0$. *In particular* a_i *are non-positive. Moreover* a_i *are integers.*

Proof. We can prove the assertion by induction. For $i = 0$, $\tilde{E}_0 \cdot R_0 < 0$ can be easily checked. Assume that the assertion holds for the numbers less than i. Then we have

$$a_j \leq 0 \, (j < i) \tag{4.4}$$

and moreover the other extremal ray than R_i is positive for $\tilde{E}_i$. Note that the linear system of a sufficient multiple of $-K_{Y_i{}'}$ is free outside a finite number of curves because the linear system of a sufficient multiple of $-K_{Y_1{}'}$ is free. So $-K_{Y_i{}'}|_{\tilde{E}_i}$ is numerically equivalent to an effective 1-cycle. Note that $-K_{Y_i{}'}\tilde{E}_i{}^2 \leq -K_{Y'}\tilde{E}^2 = -2$ by (4.2) and (4.4). Hence we have $\tilde{E}_i \cdot R_i < 0$.

Since $Y_0{}'$ has only at worst index 2 singularities, so is $Y_i{}'$. Hence $a_i = 2(\tilde{E}_i \cdot \gamma_i) \in \mathbb{Z}$ if $Y_i{}' \dashrightarrow Y_{i+1}{}'$ is a flip. □

By this claim, we know that f is a divisorial contraction whose exceptional divisor is E. If f is a crepant divisorial contraction, then $l = 0$. But $(-K_{Y'})^2\tilde{E} = 1$, a contradiction. Hence f is a K_Y-negative contraction. Assume that f is of $(2,1)$-type which contracts E to a curve C'. Then $(-K_X \cdot C') = (-K_Y + E)(-K_Y)E = -1 - \sum d_i a_i(a_i + 1) < 0$, a contradiction since X is a $\mathbb{Q}$-Fano 3-fold. So f is of $(2,0)$-type. Then we have $-K_Y E^2 \geq -2$ by [Taka00, Proposition 2.3]. On the other hand $-K_Y E^2 \leq -K_{Y'}\tilde{E}^2 = -2$. Hence there is no flip. So $(-K_Y)^2 E = (-K_{Y'})^2\tilde{E} = 1$ and hence again by [Taka00, Proposition 2.3], f is the blow-up at a $\frac{1}{2}(1,1,1)$-singularity or the weighted blow-up at a QODP with weights $\frac{1}{2}(1,1,1,2)$ (we use the coordinate as stated in the definition of QODP). □

Table 1.

Proof of Theorem 0.10 (A). (1) is easily checked. The former half of (2) follows from [Taka00, Proposition 2.1 (4)]. (3) follows from Corollary 3.4. We prove the latter half of (2). Assume the contrary. Then there is a non-trivial fiber l of f' intersecting two l_i's. l must be a flopping curve containing two $\frac{1}{2}(1,1,1)$-singularities on Y_1, where $Y' \dashrightarrow Y_1$ is the anti-flip, a contradiction to Corollary 3.4. □

Proof of Theorem 0.10 (B).

(i) Assume that S is reducible. The following argument is similar to [Muk93, Section 4]. The possibilities of an irreducible component T are classified in [Reid94]. We have genus formulae and degree formulae of curves on T and by virtue of these formulae, we obtain a contradiction except No. 1.6, 1.10 and 1.11. We treat only No. 1.6 here. In this case, there is a possibility that C is contained in a smooth quadric T such that C is a divisor of $(2,6)$-type. Note that $-K_{X'}|_T$ is of $(4,4)$-type. Hence $-K_{X'}|_T - C$ is not effective, a contradiction to the assumption (2).

Note that if S is irreducible and nonnormal, then $C' := \operatorname{Sing} S \simeq \mathbb{P}^1$ and $-K_S \cdot C' = 1$ by [Reid94]. Hence $C \not\subset \operatorname{Sing} S$.

(ii) We show that conditions of Proposition 0.8 are satisfied. By (i), $\tilde{E} = f'^*S - E'$ and $\tilde{E}$ is irreducible. On the other hand l_i are $(z+2)$-secant lines with respect to $\frac{1}{z+1}(-K_{X'})$. Thus by easy case by case calculations, we see that (2-4), (3) and (4) hold. Since $\operatorname{Bs}|-K_{X'} - C|$ is the union of C, l_i and $\frac{1}{2}(1,1,1)$-singularities, $\operatorname{Bs}|-K_{Y'}|$ is the union of l_i and $\frac{1}{2}(1,1,1)$-singularities. For (5) in case $n = 0$, we have only to take R as the extremal ray different from one associated to f'. Other conditions are clearly satisfied.

□

Proof of Theorem 0.10 (C). We construct an example of the data given in (B).

Let S be a smooth Cartier divisor in X' such that $S \equiv \frac{z}{z+1}(-K_{X'})$. We can take such an S (it is well-known if $I(X') = 1$. If $I(X') > 1$, see [San96, Remark 4.1]). S is a del Pezzo surface. We represent S as blow-up at r points of $\mathbb{P}^2$ in general position, where $r := e + n$. Let l_i $(0 \le i \le r-1)$ be its exceptional curves and l the total transform of a line in $\mathbb{P}^2$. Let $D := l + l_0 + \cdots + l_{n-1}$ and $C := -K_{X'}|_S - D$. Then we show that $|C|$ is free. By computing intersection numbers with (-1)-curves, we can check that C is nef in any case in the table. Let $M := C - K_S$. Check that $M^2 > 4$. Hence if $|C|$ is not free, there is an effective divisor l such that $M \cdot l = 1$ and $l^2 = 0$ whence $-K_S \cdot l = 1$ by Reider's theorem [RI]. But $l \cdot (K_S + l) = -1$ is a contradiction. So $|C|$ is free.

Hence we can take a smooth member from $|C|$. We denote it by C. l_i are $(z+2)$-secant lines of C which are mutually disjoint. Moreover since $h^1(\mathscr{O}_{X'}(-K_{X'} - S) = 0$, $\operatorname{Bs}|-K_{X'} - S|$ is the union of $\frac{1}{2}(1,1,1)$-singularities and $-K_{X'}|_S - C = D$, $\operatorname{Bs}|-K_{X'} - C|$ is the union of C, l_i and $\frac{1}{2}(1,1,1)$-singularities. □

Table 2.

Proof of Theorem 0.11 (A). The proof is almost clear. □

Proof of Theorem 0.11 (B).
For simplicity, we assume that P_i are ODP's.

(i) f' is constructed as follows (see also [Taka00, Proposition 2.2]). Let $\nu' \colon \tilde{X}' \to X'$ be the composite of the blow-ups at $P_0 \sim P_{N-2}$ and $F_i{}'$ the exceptional divisor over P_i. Let $\mu' \colon \hat{X}' \to \tilde{X}'$ be the blow-up along the strict transform $\tilde{C}$ of C and F' the μ'-exceptional divisor. We denote the strict transforms of the two fibers of $F_i \simeq \mathbb{P}^1 \times \mathbb{P}^1$ through $F_i \cap \tilde{C}$ by l_{ij} $(j = 1,2)$. Note that $-K_{\hat{X}'} \cdot l_{ij} = 0$. We can easily see that $|-K_{\hat{X}'}|$ is free by $P \cap X' = C$, where P is the linear subspace spanned by C and $-K_{\hat{X}'}$ is big. Hence l_{ij}'s are flopping curves on $\hat{X}'$ and we can see that the classes of l_{i1} and l_{i2} belong to the same ray. Let $\hat{X}' \dashrightarrow \hat{X}'^{+}$ be the flop. Then the strict transforms of $F_i{}'$'s on $\hat{X}'^{+}$ are $\mathbb{P}^2$'s and we can contract them to $\frac{1}{2}(1,1,1)$-singularities. Let $g' \colon \hat{X}'^{+} \to Y'$ be the contraction morphism, $f' \colon Y' \to X'$ the natural morphism.

(ii) We use the notation in the proof of (i) below. Since $|-K_{\hat{X}'^{+}}|$ is free, the assertion follows.

(iii) Let E' be the strict transform of F' and ${F'}^+$ the strict transform of F' on $\hat{X}'^+$. Then $-K_{\hat{X}'^+} - {F'}^+ = {g'}^*(-K_{Y'} - E')$. Moreover $h^0(-K_{\hat{X}'^+} - {F'}^+) = h^0(-K_{\hat{X}'} - F')$. Note that there is a smooth member of $|-K_{\tilde{X}'}|$ containing $\tilde{C}$ because $\mathrm{Bs}|-K_{\tilde{X}'} - \tilde{C}| = \tilde{C}$. Hence we have $\mathcal{N}_{\tilde{C}/\tilde{X}'} \simeq \mathscr{O}(-1) \oplus \mathscr{O}(-2)$. Hence $F' \simeq \mathbb{F}_1$ and $-K_{\hat{X}'}|_{F'} \sim C_0 + l$, where C_0 is the minimal section of F' and l is a fiber of F'. So we have $h^0(-K_{\hat{X}'}|_{F'}) = 3$ and $H^0(-K_{\hat{X}'}) \to H^0(-K_{\hat{X}'}|_{F'})$ is surjective since $|-K_{\hat{X}'}|$ is free. Thus $h^0(\mathscr{O}_{Y'}(-K_{Y'} - E') = 1$ since $h^0(-K_{\hat{X}'}) = 4$.

(iv) For (5), we only have to take R as the extremal ray different from one associated to f'. The assertion of (3) and (4) follows from (i) and (ii) except the irreducibility of $\tilde{E}$. Note that in the proof of Proposition 0.8, we need the irreducibility of $\tilde{E}$ only after the proof of Claim 4.1. Hence we can proceed as in the proof of Proposition 0.8 to the proof of Proposition 4.1 and know that f is a divisorial contraction. Let F be f-exceptional divisor and $\overline{E}'$ the strict transform of E' on Y. By setting $D = \overline{E}'$ and $D' = F$, we can apply the construction in [Taka00, Set up 3.2]. Moreover by taking f' as f in [Taka00, Set up 3.3] and applying [Taka00, Lemma 3.5], we can write $\overline{E}' = z(-K_Y) - uF$. By [Taka00, Lemma 3.6], $z, u \in \mathbb{N}$. Moreover we have $z \leq u$ by (ii) and [Taka00, Claim 3.8]. Note that $(-K_{Y'})^3 = \frac{N+3}{2}$, $(-K_{Y'})^2 E' = \frac{N+1}{2}$, $(-K_{Y'}){E'}^2 = \frac{N-5}{2}$ and ${E'}^3 = \frac{N-23}{2}$. Thus we can figure out the solutions of the equations in [Taka00, §3]. We see that $z = u = 1$, $Y' \dashrightarrow Y$ is a flop and f is an extremal contraction of $(2,0)_4$-type or $(2,0)_{10}$-type. As a consequence, F is the strict transform of $\tilde{E}$.

□

In the proof below, the hardest part is the proof of $\mathbb{Q}$-factoriality of X'.

Proof of Theorem 0.11 (C) for No. 2.1. Let C be a line in $\mathbb{P}^5$ and $P_1, P_2 \in C$ two points. Let $\nu\colon B \to \mathbb{P}^5$ be the composite of the blow-ups at P_1 and P_2 and $\mu\colon A \to B$ the blow-up along the transform C' of C. Let ${E_i}'$ be the strict transform of the exceptional divisor over P_i, F the exceptional divisor over C' and H the total transform of a hyperplane of $\mathbb{P}^5$. Let $G := 3H - 2{E_1}' - 2{E_2}' - F$.

Claim 1. *(1) $|G|$ is free.*

(2) For an irreducible curve on A, $G \cdot l = 0$ if and only if l is a fiber of $F \simeq \mathbb{P}^3 \times \mathbb{P}^1 \to \mathbb{P}^3$.

Proof. (1) Since $G = (H - {E_1}' - {E_2}' - F) + (H - {E_1}') + (H - {E_2}')$, and $|H - {E_1}' - {E_2}' - F|$, $|H - {E_1}'|$ and $|H - {E_2}'|$ are free, $|G|$ is free.

(2) First we show that $F \simeq \mathbb{P}^3 \times \mathbb{P}^1$. Note that $A \to \mathbb{P}^5$ is the composite of the blow-up along C and the blow-ups along the fiber over P_i's. Hence $F \simeq \mathbb{P}^3 \times \mathbb{P}^1$.

By the decomposition as in (1), if $G \cdot l = 0$, then $({E_i}' + F) \cdot l = 0$ for $i = 1, 2$. From this, it is easy to see that $l \subset F$. By easy calculations, we can see that $-F|_F \sim H_1 + H_2$, where H_1 (resp. H_2) is the pull-back of a hyperplane of $\mathbb{P}^3$ (resp. $\mathbb{P}^1$). Hence $G|_F \sim H_1$. So (2) is now clear.

□

In particular $|G|$ is not composed of a pencil and hence its general member G' is smooth.

Set $Q := 2H - {E_1}' - {E_2}' - F$. Similarly we see that $|Q|$ is free and general member Q' of $|Q|$ is smooth. Set $X'' := G' \cap Q'$. We can assume that X'' is smooth. Let $g\colon A \to \overline{A}$ be the Stein factorization of the morphism defined by $|G|$. We denote the image on $\overline{A}$ of an object $*$ on A by $\overline{*}$. Then by Claim 1, we can see that $\overline{Q'}$ has only terminal hypersurface singularities (along $\overline{Q'} \cap \overline{F}$) and $\overline{X''}$ is an ample divisor of $\overline{Q'}$. Hence by the Grothandieck-Lefschetz theorem [Gro68, p.135, 3.18], we have $\rho(\overline{X''}) = \rho(\overline{Q'})$. Since $g|_{X''}$ and $g|_{Q'}$ is primitive, we have $\rho(X'') = \rho(Q') = 4$. Denote the image of X'' on $\mathbb{P}^5$ by X'. Then we can see that X' is factorial and P_i's are ODP's of X'. □

Proof of Theorem 0.11 (C) for No. 2.2. We construct X' with only ODP's.

Claim 1. *Let V (resp. X') be a $(2,2)$-complete intersection in $\mathbb{P}^6$ (resp. a quadric section of V) with the following properties.*

(1) V (resp. X') contains a smooth conic C, and
(2) V (resp. X') has three ODP's $P_0 \sim P_2$ on C and outside P_i's, V (resp. X') is smooth.

Then X' is factorial.

Proof. We claim that V contains the plane P spanned by C. Let σ be the pencil which consists of quadrics in $\mathbb{P}^6$ containing V. Since P_i is an ODP on V, there is a quadric in σ which is singular at P_i. If there is a quadric in σ which is singular at all P_i's, then it is singular on P and hence V is singular along C, a contradiction. So σ is generated by two quadrics which are singular at some P_i. But such quadrics contains P and hence V contains P.

Let $\nu\colon \tilde{V} \to V$ be the composite of the blow-ups at $P_0 \sim P_2$ and F_i the exceptional divisor over P_i. Let $\tilde{X}'$ be the strict transform of X' on $\tilde{V}$ and H the total transform of a hyperplane section of V. Then $\tilde{X}' \sim 2H - F_0 - F_1 - F_2$. Note that $|H - F_i - F_j|$ is free outside the strict transform l_{ij} of the line through P_i and P_j and $|H - F_k|$ is free (note that l_{ij} is contained in V since $l_{ij} \subset P$). By this, we can easily see that $|\tilde{X}'|$ is free and $\tilde{X}'$ is numerically trivial only for l_{ij}'s $((i,j) = (0,1),(1,2),(2,0))$.

Let ϕ be the morphism defined by $|\tilde{X}'|$. Then ϕ-exceptional curves are l_{ij}'s. We prove that $\mathrm{Leff}(\tilde{V}, \tilde{X}')$ holds and $\tilde{X}'$ meets every effective divisor on $\tilde{V}$. By [H, p.165, Proposition 1.1] and the argument of [H, p.172, the proof of Theorem 1.5], it suffices to prove that $\mathrm{cd}\,(\tilde{V} - \tilde{X}') < 3$, i.e., for any coherent sheaf F on $\tilde{V} - \tilde{X}'$, $H^i(\tilde{V} - \tilde{X}', F) = 0$ for all $i \geq 3$. Let $\overline{V} := \phi(\tilde{V})$ and $\overline{X'} := \phi(\tilde{X}')$. Consider the Leray spectral sequence

$$E_2^{pq} = H^p(\overline{V} - \overline{X'}, R^q{\phi'}_*F) \Rightarrow E^{p+q} = H^{p+q}(\tilde{V} - \tilde{X}', F),$$

where $\phi' := \phi|_{\tilde{V} - \tilde{X}'}$. Since $\overline{V} - \overline{X'}$ is affine and the dimension of every fiber of $\phi \leq 1$, we have $E_2^{pq} = 0$ for $p \geq 1$ or $q \geq 2$ whence $E^{p+q} = 0$ for $p + q \geq 2$. So the assertion follows.

Moreover since $\tilde{X}'$ is nef and big, $H^i(\tilde{V}, \mathscr{O}(-n\tilde{X}')) = 0$ for $n \geq 1$ and $i = 1, 2$ by KKV vanishing theorem. Hence by the Grothandieck-Lefschetz theorem [G, p.135, 3.18] (or [H, p.178, Theorem 3.1]), we have $\mathrm{Pic}\,\tilde{X}' \simeq \mathrm{Pic}\,\tilde{V} \simeq \mathbb{Z}^4$. So $\rho(\tilde{X}'/X') = 3$ which imply that X' is factorial. □

We give a pair (V, X') satisfying the condition of Claim 1. Let C be a smooth conic in $\mathbb{P}^6$ and $P_0 \sim P_2$ three points on C. We can choose a coordinate of $\mathbb{P}^6$ such that $C = \{x_0x_1 + x_1x_2 + x_2x_0 = x_3 = x_4 = x_5 = x_6 = 0\}$ and $P_i = \{x_j = 0 \text{ for } j \neq i\}$.

Claim 2. *Let X' be a $(2,2,2)$-complete intersection in $\mathbb{P}^6$ satisfying the following conditions.*

(1) X' is factorial,
(2) X' contains a smooth conic C, and
(3) X' has three ODP's $P_0 \sim P_2$ on C and outside P_i's, X' is smooth.

Then X' is the intersection of three quadrics $Q_1 \sim Q_3$ of the following forms by permuting P_i's if necessary.

$$Q_1 := \{m_0x_0 + m_1x_1 + q_1 = 0\},$$
$$Q_2 := \{pm_1x_1 + m_2x_2 + q_2 = 0\}, \text{ and}$$
$$Q_3 := \{x_0x_1 + x_1x_2 + x_2x_0 + \sum_{i=3}^{6} l_ix_i = 0\},$$

where $p \in \mathbb{C}$, m_i (resp. q_i) is a linear form (resp. a quadratic form) of $x_3 \sim x_6$ and l_i is a linear form of $x_0 \sim x_6$.

Conversely if $X' = Q_1 \cap Q_2 \cap Q_3$, where Q_i is of the form as above and m_i, q_i and l_i are suitably general, then X' satisfies (1) $\sim$ (3).

Proof. Let γ be the net which consists of quadrics containing X'. γ contains a member Q_1 which is singular at P_2. Then Q_1 is of the form as above. If $m_1 = m_2 = 0$, then Q_1 is singular on the plane P spanned by C and hence X' is singular along C, a contradiction. Hence $m_1 \neq 0$ or $m_2 \neq 0$. By permuting P_1 and P_2 if necessary, we may assume that $m_1 \neq 0$. γ contains a member Q_2 which is singular at P_0. Q_2 is of the form as

$$\{m_1{}'x_1 + m_2x_2 + q_2 = 0\},$$

where $m_1{}'$ and m_2 (resp. q_2) are linear forms (resp. is a quadratic form) of $x_3 \sim x_6$. γ also contains a member Q' which is singular at P_1. If Q_1, Q_2 and Q' generate γ, then X' contains the plane P, a contradiction to the factoriality and $F(X') = 1$. Hence Q' is contained in the pencil generated by Q_1 and Q_2. So $m_1{}' = pm_1$ for some $p \in \mathbb{C}$ and

$$Q = \{-pm_0x_0 + m_2x_2 + (q_2 - pq_1) = 0\}.$$

Since X' does not contain P as noted above, γ contains a member Q_3 of the form as in the statement. Q_3 is not contained in the pencil generated by Q_1 and Q_2 and hence Q_i's generate γ.

Conversely let $X' := Q_1 \cap Q_2 \cap Q_3$, where Q_i is of the form as above and m_i, q_i and l_i are suitably general. We can easily check that X' satisfies (2) and (3). Set $V := Q_1 \cap Q_2$. We may assume that V satisfies the condition of Claim 1. Hence by Claim 1, X' is factorial. □

□

Table 3.

Proof of Theorem 0.13 (A). This is almost clear. □

Proof of Theorem 0.13 (B). (i) The assertion follows because X' is an intersection of quadrics.

(ii) By (i), the rank of the natural map $H^0(-K_{Y'}) \to H^0(\mathscr{O}(-K_{Y'}|_{E'}))$ is 3. Hence there is a unique member $\tilde{E}$ of $|-K_{Y'} - E'|$ since $h^0(-K_{Y'}) = 4$.

(iii) The conditions of Proposition 0.8 are easily checked except irreducibility of $\tilde{E}$. The proof of irreducibility of $\tilde{E}$ is similar to one of No. 2.2 so we omit it. □

Proof of Theorem 0.13 (C). We construct an example of the data given in (B). The Grassmannian $G(2,5)$ (parameterizing 2-dimensional subspaces of 5-dimensional vector space) can be embedded into $\mathbb{P}^9$ by the Plücker embedding. Its defining equations are $x_{ij}x_{kl} - x_{ik}x_{jl} + x_{jk}x_{il} = 0$ for all $1 \leq i < j < k < l \leq 5$, where x_{pq} ($1 \leq p < q \leq 5$) is a Plücker coordinate. Let Q be the point defined by $x_{pq} = 0$ for any $(p,q) \neq (1,2)$. Let m_1 (resp. m_2) be the line $\subset G(2,5)$ defined by $x_{pq} = 0$ for any $(p,q) \neq (1,2),(1,3)$ (resp. $(p,q) \neq (1,2),(2,4)$). Let m_3 be the line $\subset G(2,5)$ defined by the equations $x_{pq} = r_{pq}x_{12}$ for $(p,q) \neq (1,2)$ such that $r_{34} = r_{35} = r_{45} = 0, r_{13}r_{24} - r_{23}r_{14} = 0, r_{13}r_{25} - r_{23}r_{15} = 0, r_{14}r_{25} - r_{24}r_{15} = 0$ and $r_{15}r_{25} \neq 0$. Let H be the 3-plane spanned by m_1, m_2 and m_3. Then $G(2,5) \cap H = m_1 \cup m_2 \cup m_3$. Hence by [MM85, Proposition 6.8], there are two hyperplane H_1, H_2 and a quadric Q such that $X' := G(2,5) \cap H_1 \cap H_2 \cap Q$ is smooth and X' contains m_1, m_2 and m_3. Since the tangent space of X' at Q also contains all the lines on X' through Q, it is equal to H. Hence there are only three lines on X' through Q. $\square$

The proof of the following lemma is easy so we omit it.

Lemma 4.2. *Let $f\colon X \to Y$ be the blow-up of a smooth 3-fold Y along a smooth curve C on Y and E the exceptional divisor. Then*
(1) $E^3 = -\deg \mathscr{N}_{C/Y}$,
(2) $(-K_X)E^2 = 2g(C) - 2$,
(3) $(-K_X)^2E = (-K_Y \cdot C) + 2 - 2g(C)$,
(4) $(-K_X)^3 = (-K_Y)^3 - 2\{(-K_Y \cdot C) - g(C) + 1\}$.

Claim 4.3. *Consider the situation as in* [Taka00, Set up 3.2]. *Assume that*
(1) f is of $(3,2)$-type.
(2) $X \simeq \mathbb{P}^2$.
(3) There exists a degenerate fiber $l \subset \operatorname{Reg} Y$ of f.
Then $z, u \in \mathbb{N}$.

Proof. $z \in \mathbb{N}$ and the positivity of u are proved in [Taka00, Claim 3.6]. Note that there exists a smooth rational curve $m \subset \operatorname{Reg} Y$ such that $D \cdot m = 1$. Hence $u = z(-K_{Y'}) \cdot m - \overline{D'} \cdot m \in \mathbb{N}$. $\square$

Table 4, I.

Proof of Theorem 0.15 (A).

Claim 4.4. *$f'|_{l_0}$ is an isomorphism and $f'(l_0)$ does not pass the vertex v of X'.*

Proof. Let $L' \in |-K_{Y'} - \tilde{E}|$ be a general member. Note that $f'(L')$ is a ruling of X'. Then $L' \cdot l_0 = 1$ and l_0 does not pass through $\frac{1}{2}(1,1,1)$-singularities. Hence $f'|_{l_0}$ is birational. By $L' \cdot l_0 = 1$, $f'(l_0) \sim 2r$, where r is a ruling of X'. Thus $f'(l_0)$ is smooth and so $f'|_{l_0}$ is an isomorphism. If $v \in f'(l_0)$, then $f'(l_0)$ must be reducible, a contradiction. $\square$

Let F_1 be the μ_1-exceptional divisor and γ_i irreducible components intersecting l_0 of degenerate fibers of f'. Then $-K_{W_1}$ is relatively nef over X'. Let R_1 be the other extremal ray of W_1 over X' than that associated to μ_1. Then $\operatorname{supp} R_1 = \cup\gamma_i$ and R_1 is a flopping ray. Let $W_1 \dashrightarrow {W_1}'$ be the flop and ${R_1}'$ the other extremal ray of ${W_1}'$ over X' than the flopped ray. Let γ be the transform of a non-degenerate fiber intersecting l_0 and ${\gamma_i}^+$ the flopped curves. Since $G_1 \cdot {\gamma_i}^+ > 0$ and $G_1 \cdot \gamma < 0$, we have $G_1 \cdot {R_1}' < 0$, where G_1 is the strict transform of ${f'}^{-1}f'(l_0)$ on ${W_1}'$. So the contraction $\nu_1\colon {W_1}' \to {Y_1}'$ is an extremal contraction of $(2,1)$-type whose exceptional divisor is G_1. Let $l_i\,(i \geq 1)$ be flipped curves different from l_0. Since $\operatorname{Bs}|-K_{Y'}|$ is the union of $l_i\,(i \geq 0)$ and $\frac{1}{2}(1,1,1)$-singularities, $\operatorname{Bs}|-K_{{W_1}'}|$ is the union of $l_i\,(i \geq 1)$ and $\frac{1}{2}(1,1,1)$-singularities. Then by easy calculations based on Lemma 4.2, we have $(-K_{{Y_1}'} \cdot m_1) = 2n + 6$ and $(-K_{{Y_1}'})^3 = 4n + 13$.

Claim 4.5. *$l_i\,(i \geq 1)$ does not intersect γ_j on ${W_1}'$.*

Proof. If $l_i\,(i \geq 1)$ intersects γ_j on ${W_1}'$, then γ_j is a flopped curve on Y_1, where $Y' \dashrightarrow Y_1$ is the anti-flip. But γ_j passes through two $\frac{1}{2}(1,1,1)$-singularities, a contradiction to Corollary 3.3. $\square$

Hence we know that $-K_{Y_1'} \cdot l_i = 1\,(i \geq 1)$ and Y_1' is a weak Fano 3-fold. We can apply the construction in [Taka00, Set up 3.3] starting from ${f_1}'\colon {Y_1}' \to X'$ by setting $D = L_1$. By Claim 4.3, $z, u \in \mathbb{N}$.

Claim 4.6. $|-K_{Y_1'} - L_1| \neq \emptyset$.

Proof. By $h^0(-K_{W_1}) = 5$ and $h^0(-K_{W_1}|_{F_1}) = 3$, we have $|-K_{W_1} - F_1| \neq \emptyset$. Hence we have the assertion. □

Then we can figure out the solutions of the equations as in [Taka00, §3] and we know that

(i) the case $n = 2$ does not occur, and
(ii) in case $n = 1$, ${Y_1}'$ is a $\mathbb{Q}$-Fano 3-fold with the properties as stated in Theorem 0.15 (1), (2) and ${Y_2}'$ is a quadric bundle over ${X_2}' \simeq \mathbb{P}^1$, where ${Y_2}'$ is obtained from ${Y_1}'$ by the flip.

From now on assume that $n = 1$. ${Y_2}'$ can be embedded in a $\mathbb{P}^3$-bundle $\mathbb{P}(\mathscr{E})$ over $\mathbb{P}^1$, where $\mathscr{E} := \bigoplus_{i=0}^{3} \mathscr{O}(a_i)$ is a vector bundle of rank 4. We may assume that

$$a_0 = 0 \leq a_1 \leq a_2 \leq a_3. \tag{4.5}$$

Let H be the tautological divisor and M a fiber. In $\mathbb{P}(\mathscr{E})$, ${Y_2}'$ is linearly equivalent to $2H - aM$ for some $a \in \mathbb{Z}$. Since $-K_{Y_2'} = 2H|_{Y_2'} + (2 + a - \sum_{i=0}^{3} a_i){L_2}'$ and $(-K_{Y_2'}) = 16$, we have $(-K_{Y_2'})^3 = 16a - 8\sum_{i=0}^{3} a_i + 48 = 16$. So we obtain

$$\sum_{i=0}^{3} a_i = 2a + 4. \tag{4.6}$$

Note that $\tilde{L_1} \sim -K_{Y_2'} - L_2 = 2H|_{Y_1'} - (a+3)L_2$ by (4.6), where $\tilde{L_1}$ is the strict transform of L_1 on ${Y_2}'$. Let α^+ be a flipped curve for ${Y_1}' \dashrightarrow {Y_2}'$. Then since $\tilde{L_1} \cdot \alpha^+ = -2$ and $-K_{Y_2'} \cdot \alpha^+ = -1$, we have

$$2H \cdot \alpha^+ = a + 1 \geq 0. \tag{4.7}$$

Moreover since $-3M'$ is not effective, we have

$$h^0(2H - (a+3)M) \leq h^0((2H|_{Y_1'} - (a+3)L_2) = 4. \tag{4.8}$$

Thus we have $a = -1$ and $a_1 = 0$ and $a_2 = a_3 = 1$ by (4.14), (4.6), (4.7) and (4.8). □

Proof of Theorem 0.15 (B). Similarly to the proof of Proposition 0.8, we see that there is an anti-flip ${Y_2}' \dashrightarrow {Y_1}'$ whose flipped curves are connected components of $V \cap {Y_2}'$ and ${Y_1}'$ has only $\frac{1}{2}(1,1,1)$-singularities. Moreover we know that $L_1 := -K_{Y_1'} - \tilde{L_2}$ is nef, where $\tilde{L_2}$ is the strict transform of L_2. By the base point free theorem (see [KMM87]), L_1 is semi-ample. Since $(L_1)^3 = 0$ and $(L_1)^2 \not\equiv 0$, a sufficient multiple of L_1 defines a conic bundle structure ${f'}_1\colon {Y_1}' \to X'$. In particular ${Y_1}'$ is a $\mathbb{Q}$-Fano 3-fold since both of its extremal rays are $K_{Y_1'}$-negative. Since $-K_{Y_2'} - L_2 = 2(H|_{Y_2'} - L_2)$ and the transform of $H|_{Y_2'} - L_2$ on ${Y_1}'$ is not Cartier, we know that $X' \simeq \mathbb{F}_{2,0}$ whence we know that $|L_1|$ is actually free.

Let G_1 be ν_1-exceptional divisor and ${F_1}'$ the strict transform of ${f_1'}^{-1} f_1'(m_1)$. By a similar argument to (A), we know that there is a flop ${W_1}' \dashrightarrow W_1$ over X' and an extremal contraction $\mu_1\colon W_1 \to Y'$ over X' of $(2,1)$-type, whose exceptional divisor F_1 is the strict transform of ${F_1}'$ on W_1. Let $l_0 := \mu(F_1)$. Since $\operatorname{Bs}|-K_{Y_1'} - m_1|$ is the union of m_1 and $\frac{1}{2}(1,1,1)$-singularities, $\mathrm{Bs}\,|-K_{Y'}-l_0|$ is the union of l_0 and $\frac{1}{2}(1,1,1)$-singularities. It is easy to see that $-K_{Y'}\cdot l_0=-1$ by (A)(1) and Lemma 4.2. Hence $\mathrm{Bs}\,|-K_{Y'}|$ is the union of l_0 and $\frac{1}{2}(1,1,1)$-singularities.

(1) $D_2\in |H|_{Y_2'}-L_2|$ be a general smooth member and D_1 (resp. D) its strict transform on ${Y_1}'$ (resp. Y'). Since $-K_{D_2}=H|_{Y_2'}$, $-K_{D_2}$ is nef and big and numerically trivial only for two flipped curves on D_2. Thus D_1 is a del Pezzo surface with two ODP's at $\frac{1}{2}(1,1,1)$-singularities of ${Y_1}'$. Since $D_1 \dashrightarrow D$ is a composite of a blow-up at a smooth point and a blow-down at a smooth point, D is a weak del Pezzo surface with two ODP's. Consider the exact sequence

$$0\to \mathscr{O}_{Y'}(-K_{Y'}-D)\to \mathscr{O}_{Y'}(-K_{Y'})\to \mathscr{O}_D(-K_{Y'})\to 0.$$

By singular Riemann-Roch theorem on surfaces with only canonical singularities [Reid87b, (9.1) Theorem] and KKV vanishing theorem, we have $h^0(\mathscr{O}_D(-K_{Y'}))=5$. Note that $\mathrm{Bs}\,|-K_{Y'}|\cap D$ consists of one point and $|-K_{Y'}|_D|$ is free outside two ODP's. Thus $\dim \mathrm{Im}\,(H^0(\mathscr{O}_{Y'}(-K_{Y'}))\to H^0(\mathscr{O}_D(-K_{Y'}))=4$. On the other hand, we have $h^0(\mathscr{O}_{Y'}(-K_{Y'}))=5$ and so $|-K_{Y'}-D|$ has a unique member $\tilde{E}$. We prove that $\tilde{E}$ is irreducible. If $\tilde{E}$ is reducible, it is a union of an irreducible surface $\tilde{E}'$ which is a generically a 2-section for f' and surfaces which are mapped to curves on X'. Hence $h^0(-K_{Y'}-L)\neq 0$. Since $|D|$ is movable, $\tilde{E}|_D$ is effective for a general D. Hence by the exact sequence

$$0\to \mathscr{O}_{Y'}(-K_{Y'}-L)\to \mathscr{O}_{Y'}(-K_{Y'}-D)\to \mathscr{O}_D(-K_D)\to 0,$$

we have $h^0(-K_{Y'}-L)=0$, a contradiction.

(2) (2-3) is checked before the proof of (i). Other conditions are easily checked.

□

Proof of Theorem 0.15 (C). Let ${Y_2}'$ be a smooth divisor in $\mathbb{P}(\mathscr{O}_{\mathbb{P}^1}^{\oplus 2}\oplus\mathscr{O}_{\mathbb{P}^1}(1)^{\oplus 2})$ linearly equivalent to $2H+M$. Let $L_2:=M|_{Y_2'}$. Since ${Y_2}'$ is an ample divisor, $\rho({Y_2}')=2$ by [Gro68, p.135, 3.18]. By looking on the local charts, we can easily see that if we take ${Y_2}'$ generally, $\mathrm{Bs}\,|-K_{Y_2'}|=\mathrm{Bs}\,|H|_{Y_2'}-L_2|=V\cap {Y_2}'$, which is a disjoint union of two sections. We denote by L_1 a general member of $|L_1|$.

Claim 4.7. *L_1 is a del Pezzo surface of degree* 2.

Proof. Denote by $\tilde{L_1}$ the strict transform of L_1 on ${Y_2}'$. Since $-K_{\tilde{L_1}}=L_2|_{\tilde{L_1}}$, $|-K_{\tilde{L_1}}|$ is free and $(-K_{\tilde{L_1}})^2=0$. Since $\pi\colon \tilde{L_1}\to L_1$ is the blow-up at two points, L_1 is a weak del Pezzo surface of degree 2. Assume that there is a (-2)-curve δ on L_1. Then on $\tilde{L_1}$, δ is a component of a degenerate fiber of $\tilde{L_1}\to {X_2}'$ and does not intersect any π-exceptional curve. Then another component δ' of the fiber containing δ intersects both of two π-exceptional curves. Then, however, δ' satisfies $-K_{Y_1'}\cdot\delta'=0$ on ${Y_1}'$. But this contradicts the fact that ${Y_1}'$ is a $\mathbb{Q}$-Fano 3-fold. □

We can regard L_1 as a surface obtained by blowing up $\mathbb{P}^2$ at 7 points and let $e_i (i=1,\ldots,7)$ be the exceptional curves, where we may assume that e_1 is a section of ${f_1}'$ and $e_i\,(i\geq 2)$ are components of different 6 degenerate fibers. Let ${m_1}':=4l-3e_1-\sum\limits_{i=2}^{7}e_i$, where l is the pull-back of a line in $\mathbb{P}^2$.

Claim 4.8. *$|{m_1}'|$ is free.*

Proof. Since ${m_1}' = (3l - 2e_1 - \sum_{i=2}^{7} e_i) + (l - e_1)$, $|{m_1}'|$ is nef. Assume that $|{m_1}'|$ is not free. Since ${m_1}' - K_{\tilde{L_1}'}$ is nef and $({m_1}' - K_{\tilde{L_1}'})^2 > 4$, by [Reide88], there is an effective divisor Z such that

(i) $({m_1}' - K_{\tilde{L_1}'}) \cdot Z = 0$ and $Z^2 = -1$ or

(ii) $({m_1}' - K_{\tilde{L_1}'}) \cdot Z = 1$ and $Z^2 = 0$.

In case (i), we have $(-K_{\tilde{L_1}'}) \cdot Z = 0$ and $Z^2 = -1$ but this contradicts the genus formula. In case (ii), we have $(-K_{\tilde{L_1}'}) \cdot Z = 0$ and $Z^2 = 0$ by the genus formula and moreover $(-K_{\tilde{L_1}'}) \equiv aZ$ for some $a \in \mathbb{Q}$. By $(-K_{\tilde{L_1}'}) \cdot {m_1}' = 3$ and $Z \cdot {m_1}' = 1$, we have $a = 3$. On the other hand, by $(-K_{\tilde{L_1}'}).e_1 = 1$, $Z.e_1 = \frac{1}{3}$, a contradiction. Hence we have the assertion. □

Let $m_1 \in |{m_1}'|$ be a general smooth member.

Claim 4.9. $\mathrm{Bs}\,|-K_{{Y_1}'} - m_1| = m_1$.

Proof. Since $|-K_{{Y_2}'} - \tilde{L_1}| = |L_2|$, $\mathrm{Bs}\,|-K'_{Y_1} - m_1|$ is at most the union of L_1 and two flipping curves. Consider the exact sequence

$$0 \to \mathscr{O}_{{Y_1}'}(-K_{{Y_1}'} - L_1) \to \mathscr{O}_{{Y_1}'}(-K_{{Y_1}'}) \to \mathscr{O}_{L_1}(-K_{{Y_1}'}) \to 0.$$

Since $-2K_{{Y_1}'} - L_1$ is nef and big, $h^1(\mathscr{O}(-2K_{{Y_1}'} - L_1)) = 0$ by KKV vanishing. Hence $H^0(\mathscr{O}_{{Y_1}'}(-K_{{Y_1}'})) \to H^0(\mathscr{O}_{L_1}(-K_{{Y_1}'}))$ is surjective. Moreover we have $|-K'_{Y_1}|_{L_1} - m_1| = |l|$. We may assume that m_1 is disjoint from two flipping curves on ${Y_1}'$. Thus $\mathrm{Bs}\,|-K_{{Y_1}'} - m_1| = m_1$. □

It is easy to see that m_1 has other properties as in (A)(2). □

Table 4, II.

Proof of Theorem 0.17 (A) and (B). These are almost clear. □

Proof of Theorem 0.17 (C). We know that there is an example for this case by Corollary 2.3 and the existence of an example for No. 1.1 ($h = 6$ and $N = 2$) since only No. 4.3 satisfies $h = 6$ and $N = 1$. □

Table 4, III.

Proof of Theorem 0.18 (A). Let F_1 be μ_1-exceptional divisor. By an argument similar to the proof of Theorem 0.15 (A), we know that there is a flop $W_1 \dashrightarrow {W_1}'$ over X' and an extremal contraction $\nu_1 \colon {W_1}' \to {Y_1}'$ of $(2,1)$-type over X' whose exceptional divisor G_1 is the strict transform of ${f'}^{-1} f'(l_0)$ on ${W_1}'$. Let $l_i\,(i \geq 1)$ be flipped curves different from l_0. Since $\mathrm{Bs}\,|-K_{Y'}| = \bigcup_{i=0}^{n-1} l_i$, $\mathrm{Bs}\,|-K_{{W_1}'}| = \bigcup_{i=1}^{n-1} l_i$. Then by easy calculations, we have $(-K_{{W_1}'})^2 G_1 = n + 3$ and $(-K_{{W_1}'})^3 = 6$. By Lemma 4.2, $(-K_{{Y_1}'} \cdot m_1) = n + 1$ and $(-K_{{Y_1}'})^3 = 2n + 10$. Similarly to the proof of Claim 4.5, we know that any $l_i\,(i \geq 1)$ does not intersect flopping curves for $W_1 \dashrightarrow {W_1}'$. Hence we have $G_1 \cdot l_i = 1$ and $-K_{{Y_1}'} \cdot l_i = 0\,(i \geq 1)$. Thus ${Y_1}'$ is a weak Fano 3-fold.

Assume that X is a $\mathbb{Q}$-Fano 3-fold of No. 4.4. We can apply the construction in [Taka00, Set up 3.3] starting from ${f_1}' \colon {Y_1}' \to X'$ by setting $D = L_1$. By Claim 4.3, $z, u \in \mathbb{N}$ and similarly to the proof of Claim 4.6, we can see that $|-K_{{Y_1}'} - L_1| \neq \emptyset$. Then we can figure out the solutions of the equations as in [Taka00, §3] and we know that ${Y_1}'$ is a Fano 3-fold with $(-K_{{Y_1}'})^3 = 12$ and two conic bundle structures. By [MM81, Table 1], we know that ${Y_1}'$ and m_1 are as in the statement of Theorem 0.18 (A-1).

Assume that X is a $\mathbb{Q}$-Fano 3-fold of No. 4.5–No. 4.7. ${Y_1}'$ has a flopping ray since $n \geq 2$. We use the notation of Theorem 0.18 (A-2). Let F_2 be μ_2-exceptional divisor. By an argument similar to the proof of Theorem 0.15 (A), we know that there is a flop $W_2 \dashrightarrow {W_2}'$ over X' and an extremal contraction $\nu_2 \colon {W_2}' \to {Y_2}'$ of $(2,1)$-type over X' whose exceptional divisor G_2 is the strict transform of ${f_1}'^{-1}{f_1}'(l_1)$ on ${W_2}'$. Let $m_2 := \nu_2(G_2)$. Then by easy calculations based on Lemma 4.2, we have $(-K_{{Y_2}'} \cdot m_2) = n$ and $(-K_{{Y_2}'})^3 = 4n+10$. Note that $\mathrm{Bs}\,|-K_{{W_2}'}| = \bigcup_{i=2}^{n-2} l_i$ outside m_1.

Claim 4.10. *$l_1 \cap l_i = \emptyset\,(i \geq 2)$.*

Proof. If $l_1 \cap l_i \neq \emptyset$, then Y' has a non-degenerate fiber δ intersecting l_0, l_1 and l_2. Then $-K_{Y_1} \cdot \delta = \frac{1}{2}$ and there are three $\frac{1}{2}(1,1,1)$-singularities on it. If there is a member of $|-K_{Y_1}|$ which does not contain δ, then $-K_{Y_1} \cdot \delta \geq \frac{3}{2}$, a contradiction. So $\delta \subset \mathrm{Bs}\,|-K_{Y_1}|$. But this contradicts Corollary 3.3. □

Similarly to the proof of Claim 4.5, we know that any $l_i\,(i \geq 2)$ does not intersect flopping curves for $W_2 \dashrightarrow {W_2}'$. Hence we have $G_2 \cdot l_i = 1$ and $-K_{{Y_1}'} \cdot l_i = 1\,(i \geq 1)$. Hence ${Y_2}'$ is a weak Fano 3-fold. We can apply the construction in [Taka00, Set up 3.3] starting from ${f_1}' \colon {Y_1}' \to X'$ by setting $D = L_1$. By Claim 4.3, $z, u \in \mathbb{N}$ and similarly to the proof of Claim 4.6, we can see that $|-K_{{Y_1}'} - L_1| \neq \emptyset$. Then we can figure out the solutions of the equations as in [Taka00, §3] and we know that ${Y_2}'$ is actually a Fano 3-fold with $(-K_{{Y_2}'})^3 = 4n+10$. Since ${Y_2}'$ has a conic bundle structure, we know that ${Y_2}'$ and γ are as in the statement of Theorem 0.18 (A-2) by [MM81, Table 1].

Claim 4.11. *(i) γ, m_1 and m_2 intersect the common $n-2$ points simply.*
(ii) $l_i\,(i \geq 2)$ are fibers of h which intersect m_1 or m_2.
(iii) γ, m_1 and m_2 are smooth rational curves of degree $n-1$.

Proof. By Claim 4.10, (i) follows. By [MM83, Theorem 5.1 (1)], we have $-K_{{Y_2}'} = L_2 + M_2$, where L_2 (resp. M_2) is the pull-back of a hyperplane section of X' (resp. ${X_2}'$). Since $-K_{{Y_2}'} \cdot l_i = 1$ and $L_2 \cdot l_i = 1$, we have $M_2 \cdot l_i = 0$. Hence l_i are fibers of h. Moreover by $-K_{{Y_2}'} \cdot m_j = n$ and $L_2 \cdot l_i = 1$, we have $M_2 \cdot m_j = n-1$. Hence by $-K_{{Y_2}'} = 2M_2 - G_3$, where G_3 is h-exceptional divisor, we have $G_3 \cdot m_j = n-2$. Thus (ii) holds. (iii) follows from (i) and (ii). □

□

Proof of Theorem 0.18 (B). First we consider the case X is a $\mathbb{Q}$-Fano 3-fold of No. 4.4. Let G_1 be ν_1-exceptional divisor and ${F_1}'$ the strict transform of ${f_1'}^{-1}f_1'(m_1)$, where $f_1' \colon {Y_1}' \to X' \simeq \mathbb{P}^2$ is one of two structure morphisms of conic bundles. By an argument similar to the proof of 0.15 (A), we know that there is a flop ${W_1}' \dashrightarrow W_1$ over X' and an extremal contraction $\mu_1 \colon W_1 \to Y'$ of $(2,1)$-type over X' whose exceptional divisor F_1 is the strict transform of ${F_1}'$ on W_1.

(i) Let ${f_1}'' \colon {Y_1}' \to {X_1}' \simeq \mathbb{P}^2$ be the morphism of the other conic bundle structure. By [MM83, Theorem 5.1 (1)], we have $-K_{{Y_2}'} = L_1 + M_1$, where L_1 (resp. M_1) is the pull-back of a hyperplane section of X' (resp. ${X_1}'$). Hence ${f_1'}^{-1}f_1'(m_1) + {f_1''}^{-1}f_1''(m_1) \in |-K_{{Y_1}'}|$. Let $\tilde{E}''$ be the strict transform of ${f_1''}^{-1}f_1''(m_1)$ on ${W_1}'$. Then ${F_1}' + \tilde{E}'' + G_1 \in |-K_{{W_1}'}|$. Hence the strict transform $\tilde{E}$ of $\tilde{E}''$ on Y' satisfies $\tilde{E} \sim -K_{Y'} - L$. Conversely we can easily see that if $\tilde{E} \sim -K_{Y'} - L$ is effective, it is the strict transform of ${f_1''}^{-1}f_1''(m_1)$. Hence the uniqueness of $\tilde{E}$ follows. We prove that $\tilde{E}$ is irreducible. If $\tilde{E}$ is reducible, it is a union of an irreducible surfece $\tilde{E}'$ which is a generically a 2-section for f' and surfaces which are mapped to curves on X'. Hence $h^0(-K_{Y'}-2L)\neq 0$. Since $|L|$ is free, $\tilde{E}|_L$ is effective for a general L. Hence by the exact sequence

$$0\to \mathscr{O}_{Y'}(-K_{Y'}-2L)\to \mathscr{O}_{Y'}(-K_{Y'}-L)\to \mathscr{O}_L(-K_L)\to 0,$$

we have $h^0(-K_{Y'}-2L)=0$, a contradiction.

(ii) Since $\mathrm{Bs}\,|-K_{Y_1{}'}-m_1|=m_1$, $\mathrm{Bs}\,|-K_{Y'}-l_0|=l_0$. We can easily see that other conditions of Proposition 0.8 are satisfied.

Next we consider the case X is a $\mathbb{Q}$-Fano 3-fold of No. 4.5–No. 4.7. By [MM85, (7.7)], $Y_2{}'$ is a Fano 3-fold with a conic bundle structure $f_2'\colon Y_2{}'\to X'\simeq\mathbb{P}^2$. By the assumption, $m_1\cap m_2=\emptyset$ on $Y_2{}'$. Let $l_i\,(2\leq i\leq n-1)$ be fibers of h over $m_1\cap m_2\subset X_2{}'$.

Claim 4.12. *(a) There is a unique hyperplane section of $X_2{}'$ containing m_1 and m_2.*

(b) $\mathrm{Bs}\,|-K_{Y_2{}'}-m_1-m_2|=m_1\cup m_2\cup l_2\cup\cdots\cup l_{n-1}$.

Proof. (a) Let $h'\colon Y_2{}''\to X_2{}'$ be the blow-up of $X_2{}'$ along m_1 and $G_3{}'$ h'-exceptional divisor. Then by [MM85, (7.7)], $Y_2{}''$ is a Fano 3-fold with a conic bundle structure $f_2{}''\colon Y_2{}''\to X''$. By [MM83, Theorem 5.1 (1)], we have $-K_{Y_2{}''}=L_2{}'+M_2{}'$, where $L_2{}'$ (resp. $M_2{}'$) is the pull-back of a hyperplane section of X'' (resp. $X_2{}'$). Hence we have $L_2{}'=M_2{}'-G_3{}'$. So the image of ${f_2{}''}^{-1}f_2{}''(m_2)$ on $X_2{}'$ is a hyperplane section of $X_2{}'$ containing m_1 and m_2 and such a hyperplane section is obtained by this way.

(b) By [MM83, Theorem 5.1 (1)], we have $-K_{Y_2{}'}=L_2+M_2$, where L_2 (resp. M_2) is the pull-back of a hyperplane section of X' (resp. $X_2{}'$). Since $-K_{Y_2{}'}\cdot m_j=n$ and $M_2\cdot m_j=n-1$, we have $L_2\cdot m_j=1$. Let $L_{2,j}\,(j=1,2)$ be the member of $|L_2|$ containing m_j. Let $\tilde{E}''$ be the strict transform of a hyperplane section of $X_2{}'$ containing m_1 and m_2. Since $\mathrm{Bs}\,|M_2-m_j|=m_j\cup l_2\cup\cdots\cup l_{n-1}$, we have $\mathrm{Bs}\,|-K_{Y_2{}'}-m_1-m_2|\subset(L_{2,1}\cap L_{2,2}\cap\tilde{E}'')\cup(m_1\cup m_2\cup l_2\cup\cdots\cup l_{n-1})$. Since $L_{2,1}\cap L_{2,2}\cap m_i\subset L_{2,1}\cap L_{2,2}\cap\tilde{E}''$ and $L_{2,1}\cdot L_{2,2}\cdot\tilde{E}''=2$, we have $L_{2,1}\cap L_{2,2}\cap\tilde{E}''=L_{2,1}\cap L_{2,2}\cap(m_1\cup m_2)$. So we have the assertion.

□

Let G_2 be ν_2-exceptional divisor and $F_2{}'$ the strict transform of $L_{2,2}$, where f_2' is the structure morphism of conic bundle. By similar argument to the proof of Theorem 0.15 (A), we know that there is a flop $W_2{}'\dashrightarrow W_2$ over X' and an extremal contraction $\mu_2\colon W_2\to Y_1{}'$ of $(2,1)$-type over X' whose exceptional divisor F_2 is the strict transform of $F_2{}'$ on W_2. By Claim 4.12,

$$\mathrm{Bs}\,|-K_{Y_1{}'}-m_1-l_1|=m_1\cup l_1\cup\cdots\cup l_{n-1}\tag{4.9}$$

Let G_1 be ν_1-exceptional divisor and $F_1{}'$ the strict transform of ${f_1'}^{-1}f_1'(m_1)$, where $f_1'\colon Y_1{}'\to X'$ is the natural morphism. By a similar argument to above, we know that there is a flop $W_1{}'\dashrightarrow W_1$ over X' and an extremal contraction $\mu_1\colon W_1\to Y'$ of $(2,1)$-type over X' whose exceptional divisor F_1 is the strict transform of $F_1{}'$ on W_1.

(i) Since $L_{2,1}+\tilde{E}''\in|-K_{Y_2{}'}|$ (see the proof of Claim 4.12 for the notation), the strict transform $\tilde{E}$ of $\tilde{E}''$ on Y' satisfies $\tilde{E}\sim -K_{Y'}-L$, where L is the pull-back of a line of X'. Conversely we can easily see that if $\tilde{E}\sim -K_{Y'}-L$ is effective, it is the strict transform of $\tilde{E}''$. Hence the uniqueness of $\tilde{E}$ follows from Claim 4.12. The irreducibility of $\tilde{E}$ can be proved similarly to No. 4.4.

(ii) By 4.9, $\mathrm{Bs}\,|-K_{Y'}-l_0-l_1|=l_0\cup l_1\cup\cdots\cup l_{n-1}$. But since $l_0\cup l_1\subset\mathrm{Bs}\,|-K_{Y'}|$, we have $\mathrm{Bs}\,|-K_{Y'}|=l_0\cup l_1\cup\cdots\cup l_{n-1}$.

Claim 4.13. *l_i are mutually disjoint.*

Proof. It suffices to prove that $l_i\,(i \geq 2)$ do not intersect flopping curves for ${W_2}' \dashrightarrow W_2$ or ${W_1}' \dashrightarrow W_1$. This follows from Claim 4.12 because otherwise there exists a member of $|-K_{{W_1}'}|$ (resp. $|-K_{{W_2}'}|$) which intersects a flopping curve for ${W_2}' \dashrightarrow W_2$ (resp. ${W_1}' \dashrightarrow W_1$) but does not contain it, a contradiction. □

We can easily see that other conditions of Proposition 0.8 are satisfied.

□

Proof of Theorem 0.18 (C). For No. 4.4, the proof is almost clear. For No. 4.5–No. 4.7, we prove that on $B_d\,(d = 3, 4, 5)$, there exist three smooth rational curves γ, m_1 and m_2 of degree $d-2$ which intersect the common $d-3$ points simply. First note that there exists at least one such a curve on a smooth hyperplane section H of B_d. Call it γ. Since $\mathrm{Bs}\,|H-\gamma| = \gamma$ by [MM85, Proposition 6.8], There exists two other hyperplane sections H_1 and H_2 containing γ. It is easy to see that we can take m_i on H_i as desired. □

Table 4, IV.

Proof of Theorem 0.19 (A) and (B). These are almost clear. □

Proof of Theorem 0.19 (C). Let $Z_1 := \mathbb{P}(\mathscr{O}_{\mathbb{P}^2} \oplus \mathscr{O}_{\mathbb{P}^2}(-1))$, $\pi\colon Z_1 \to X' \simeq \mathbb{P}^2$ the natural projection and $\overline{E} \simeq \mathbb{P}^2$ the unique member of the tautological linear system. Let $P_1 \sim P_6$ be six points in a general position on X'. We denote by ${P_i}'$ the point on $\overline{E}$ corresponding to P_i. Let F_{ij} be the pull-back of a line of X' through P_i and P_j and l_{ij} the section of F_{ij} intersecting $\overline{E}$ at two points ${P_i}'$ and ${P_j}'$. We may assume l_{12}, l_{34} and l_{56} do not intersect mutually. Let $Z_1 \dashrightarrow Z_2$ be the elementary transformation along l_{12}, $Z_2 \dashrightarrow Z_3$ the elementary transformation along the strict transform of l_{34}, $Z_3 \dashrightarrow Y'$ the elementary transformation along the strict transform of l_{56}. Let $\tilde{E}$ be the strict transform of $\overline{E}$ and L the pull-back of a line on Y'. $\tilde{E}$ is obtained by blow-up $\overline{E}$ at ${P_1}' \sim {P_6}'$. Let m_i be the exceptional curve over ${P_i}'$. Then we can prove that $-K_{Y'} = L + 2\tilde{E}$ and $\tilde{E}|_{\tilde{E}} = 2l - \sum m_i$, where l is the total transform of a line of $\overline{E}$. Let $C_{i_1i_2i_3i_4i_5}$ be the (-1)-curve on $\tilde{E}$ linearly equivalent to $2l - m_{i_1} - m_{i_2} - m_{i_3} - m_{i_4} - m_{i_5}$. Let R be the other extremal ray of $\overline{\mathrm{NE}}(Y')$ than one generated by the class of fiber of $Y' \to X'$. We check the following.

(1) $-K_{Y'}$ is nef and big,
(2) R is generated by the class of $C_{i_1i_2i_3i_4i_5}$ and $\mathrm{Supp}\,R = \cup C_{i_1i_2i_3i_4i_5}$, and
(3) $-K_{Y'} \cdot R = 0$.

Since $\tilde{E} \cdot C_{i_1i_2i_3i_4i_5} = -1$ and $\rho(Y') = 2$, we have $\tilde{E} \cdot R < 0$. Hence $R \subset \mathrm{Im}(\overline{NE}(\tilde{E}) \to \overline{NE}(\tilde{Y'}))$. On the other hand it is easy to check that $-K_{Y'}|_{\tilde{E}}$ is nef and numerically trivial only for $C_{i_1i_2i_3i_4i_5}$'s. Hence we have $-K_{Y'} \cdot R = 0$ and (2) follows. (3) becomes also clear. For (1), the nefness is already checked. The bigness follows from a direct calculation. In fact we have $(-K_{Y'})^3 = 14$. □

Table 5, I.

Proof of Theorem 0.20 (A). The proof is similar to [Take96]. Since Y' is a smooth del Pezzo fibration whose fibers are del Pezzo surfaces of degree 4, Y' can be embedded in $\mathbb{P}^4$-bundle. Let $\mathcal{E} := \sum_{i=0}^{4} \oplus \mathscr{O}(a_i)$ be the associated vector bundle of rank 5, where we may choose $a_0 = 0$ and $a_0 \leq a_1 \leq a_2 \leq a_3 \leq a_4$. Let H be the tautological divisor and F a fiber. In $\mathbb{P}(\mathcal{E})$, Y' is a complete intersection of $V_1 \in |2H - aF|$ and $V_2 \in |2H - bF|$ for some a and b. We may assume that $a \geq b$. Since

$-K_{Y'} = H|_{Y'} + (a+b+2-\sum a_i)L$, we have $(-K_{Y'})^3 = 10(a+b) - 8\sum a_i + 24 = 4$. So we obtain

$$5(a+b) = 4\sum a_i - 10. \tag{A1}$$

Note that $\tilde{E} \sim -K_{Y'} - L = H - (\frac{a+b+6}{4})L$ by (A1). Let $c := \frac{a+b+6}{4}$. Since $H - cF - V_1$ and $(H-cF)|_{V_1} - Y'$ is not effective, we have $h^0(H-cF) \leq h^0((H-cF)|_{V_1}) \leq h^0((H-cF)|_{Y'}) = 1$. So we obtain

$$a_3 < c \text{ and } a_4 \leq c. \tag{A2}$$

This gives $\sum a_i \leq 4c - 3 = a + b + 3$ and hence by (A1), we obtain $a + b \leq 2$. On the other hand, there is a flipped curve $m_1{}'$ on $\tilde{E}$, which satisfies $\tilde{E} \cdot m_1{}' = -2$ and $F \cdot m_1{}' = 1$. Hence $H \cdot m_1{}' = \frac{a+b-2}{4}$. This is non-negative so we have $a + b \geq 2$. So we obtain $a + b = 2$ and by (A1), $\sum a_i = 5$. Together with (A2), we have $\mathcal{E} := \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^1}(2)$. If $a \geq 3$, then V_1 must be non-reduced or reducible by looking at a local coordinate. If $a = 2$, then $\tilde{E}$ must be singular along $m_1{}'$. Hence we have $a = b = 1$ and we are done. □

Proof of Theorem 0.20 (B).

(1) Let $\mu\colon Q \to \mathbb{P}(\mathcal{E})$ be the blow-up along l_0 and G the exceptional divisor. Since $N_{l_0/\mathbb{P}(\mathcal{E})} = \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^1}(2)$, G contains the subvariety W which corresponds to the surjection $\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^1}(2) \to \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 3}$. Note that $|V' := \mu^*(2H - F) - G|$ is free since $\mu^*(H-F) - G$ and μ^*H is free. We can easily prove that for an irreducible curve l, $V' \cdot l = 0$ if and only if l is a fiber of the natural projection $W \simeq \mathbb{P}^2 \times \mathbb{P}^1 \to \mathbb{P}^2$ and if $V' \cdot l = 0$, then $K_Q \cdot l = 0$. So the Stein factorization of $\Phi_{|V'|}$ (we call it $\nu\colon Q \to R$) is a crepant primitive birational contraction and R has only hypersurface singularities. Hence by the Grothandieck-Lefschetz theorem [Gro68, p.135, 3.18], a complete intersection of two members of $|\nu(V')|$ has the same Picard group as R. So for the strict transform $V_i{}'$ of V_i, $\rho(\nu(V_1{}') \cap \nu(V_2{}')) = 2$ and hence $\rho(Y') = 2$.
(2) Since $-K_{Y'} = H - F|_{Y'}$, $\mathrm{Bs}|-K_{Y'}| = l_0$.
(3) By $h^0(H - 2F) = 1$, it is easy to see that $h^0((H-2F)|_{Y'}) = 1$. The irreducibility of $\tilde{E}$ can be proved similarly to No. 4.4.
(4) This is easily proved.

□

Proof of Theorem 0.20 (C). It is easy to see that we can take Y' as in (A) by looking on the local charts. □

Table 5, II.

Proof of Theorem 0.21 (A). Let F_1 be μ_1-exceptional divisor. Since $-K_{Y'}$ is relatively very ample over X', $|-K_{W_1}|$ is relatively free and big over X'. Let R_1 be the extremal ray of W_1 over X' other than that associated to μ_1.

Claim 4.14. *(1) R_1 is a flopping ray and an irreducible curve whose numerical class generates R_1 is a transform of a curve γ on X with $-K_X \cdot \gamma = \frac{1}{2}$ passing the $\frac{1}{2}(1,1,1)$-singularity on $l_0{}^-$, where $l_0{}^- \subset X$ is the strict transform of flipping curve corresponding to l_0.*
(2) Let l_1 be a flipped curve different from l_0. Then γ does not intersect l_1 on W_1.

Proof. (1) By Theorem 0.3, there is a curve γ on X with $-K_X \cdot \gamma = \frac{1}{2}$ passing the $\frac{1}{2}(1,1,1)$-singularity on $l_0{}^-$. By [Taka00, Proposition 2.1 (4)], we have $-K_{Y'} \cdot \gamma = \frac{1}{2} + \alpha$ and $\tilde{E} \cdot \gamma = 2\alpha + \beta$ for a positive rational number $\alpha \in \frac{\mathbb{Z}}{2}$ and a non-negative rational number β. α (resp. β) describes the effect of the flip $Y_1 \dashrightarrow Y'$ (resp. the flop $Y \dashrightarrow Y_1$) (see [Taka00, §3] for the notation). By $L \sim -K_{Y'} - \tilde{E}$, we have $L \cdot \gamma = \frac{1}{2} - \alpha - \beta$. Since L is nef, we have $\alpha = \frac{1}{2}$ and $\beta = 0$. Moreover by $-K_{W_1} = \mu_1^*(-K_{Y'}) - F_1$, we have $-K_{W_1} \cdot \gamma = 0$. So the numerical class of γ generates R_1. Conversely let γ' be an irreducible curve whose numerical class generates R_1. When the fiber of f' containing γ' is anti-canonically embedded in a projective space, γ' is a line and hence $F_1 \cdot \gamma' = 1$. Since $\tilde{E}$ is smooth along l_0, we have $\mu_1^* L \sim -K_{W_1} - \tilde{E}'$, where $\tilde{E}'$ is the strict transform of $\tilde{E}$ on W_1. Hence we have $\tilde{E}' \cdot \gamma' = 0$ and $\tilde{E} \cdot \gamma' = 1$. By reversing the argument above, we can easily see that γ' is a curve as in the statement of Claim 4.14 on X. The finiteness of the number of γ''s follows from Theorem 0.3. In particular R_1 is a flopping ray.

(2) If γ intersects l_1, then γ is a flopped curve on Y_1, where $Y \dashrightarrow Y_1$ is the flop. But γ passes two $\frac{1}{2}(1,1,1)$-singularities, a contradiction to Corollary 3.3.

□

Let R'_1 be the extremal ray of ${W_1}'$ over X' other than that associated to the flop $W_1 \dashrightarrow {W_1}'$ and $\nu_1 \colon {W_1}' \to {Y_1}'$ is the associated contraction. Let L' be a general fiber of $W_1 \to X'$ and denote by L the image of L' on Y'. Then by Claim 4.14, we may assume that L' is a del Pezzo surface of degree 4 in case X is a $\mathbb{Q}$-Fano 3-fold of No. 5.4 (resp. degree 5 in case X is a $\mathbb{Q}$-Fano 3-fold of No. 5.5). We consider L' is anti-canonically embedded in a projective space.

Claim 4.15. *ν_1 is of $(2,1)$-type.*

Proof. It suffices to prove that ν_1 is not of $(3,2)$-type. Assume that ν_1 is of $(3,2)$-type. Then ${Y_1}'$ is a $\mathbb{P}^1$-bundle over X'. Let M be the pull-back of a general section of ${Y_1}' \to X'$ on ${W_1}'$ and M' (resp. M'') the transform of M on W_1 (resp. Y'). We may assume that $M'|_{L'}$ is a smooth conic. Note that $F_1|_{L'}$ is a line. Since L' is an intersection of quadrics, $M'|_{L'}$ intersects a line at most one point. Hence $(M''|_L)^2 \leq 1$, a contradiction to the fact that the image of $\operatorname{Pic} Y' \to \operatorname{Pic} L$ is generated by $-K_L$ and $(-K_L)^2 \geq 4$. □

Let G_1 be the ν_1-exceptional divisor and $m_1 := \nu_1(G_1)$. Let ${G_1}'$ (resp. ${G_1}''$) the strict transform of G_1 on W_1 (resp. Y'). Note that ${G_1}'|_{L'}$ is a union of lines intersecting $F_1|_{L'}$ at one point. Since the image of $\operatorname{Pic} Y' \to \operatorname{Pic} L$ is generated by $-K_L$, we know that in case X is a $\mathbb{Q}$-Fano 3-fold of No. 5.4 (resp. No. 5.5), $G'|_{L'}$ is a union of five (resp. three) lines and ${W_1}'$ is a $\mathbb{P}^2$-bundle (resp. a quadric bundle). Hence we can write ${G_1}'' = 2(-K_{Y'}) + aL$ (resp. ${G_1}'' = (-K_{Y'}) + aL$) for some $a \in \mathbb{Z}$). Note that ${G_1}' = \mu_1^* {G_1}'' - 5F_1$ (resp. ${G_1}' = \mu_1^* {G_1}'' - 3F_1$). Then by easy calculations, we have

$$(-K_{{W_1}'})^2 G = 4a + 5. \tag{4.10}$$

$$(-K_{{W_1}'}) G^2 = 10a - 14. \tag{4.11}$$

$$\text{(resp. } (-K_{{W_1}'})^2 G = 5a + 2. \tag{4.12}$$

$$(-K_{{W_1}'}) G^2 = 6a - 8). \tag{4.13}$$

Assume that X is a $\mathbb{Q}$-Fano 3-fold of No. 5.4. Since ${W_1}'$ is a $\mathbb{P}^2$-bundle, $(-K_{{W_1}'})^3 = 54$. So we have $a = 3$, $g(m_1) = 9$ and $(-K_{{Y_1}'} \cdot m_1) = 33$ by (4.10), (4.11) and Lemma 4.2. Let l_1 be flipped curves different from l_0. Since $\operatorname{Bs}|-K_{Y'}| = l_0 \cup l_1$, $\operatorname{Bs}|-K_{{W_1}'}| = l_1$. By Claim 4.14 (2), we have $G_1 \cdot l_1 = (2(-K_{Y'}) + 3L \cdot l_1) = 1$ whence $(-K_{{Y_1}'} \cdot l_1) = 0$. Thus ${Y_1}'$ is a weak Fano 3-fold.

Claim 4.16. *${Y_1}'$ has no crepant divisorial contraction.*

Proof. Assume the contrary. Then by the method of [Take99], we can easily see that ${Y_1}' \simeq \mathbb{P}(\mathscr{O}^{\oplus 2} \oplus \mathscr{O}(2))$. Let H be the tautological divisor, L_1 a fiber and T the subvariety associated to the surjection $\mathscr{O}^{\oplus 2} \oplus \mathscr{O}(2) \to \mathscr{O}^{\oplus 2}$. Then $T \simeq \mathbb{P}^1 \times \mathbb{P}^1$ and a horizontal fiber l satisfies $-K_{{Y_1}'} \cdot l = 0$. Since there exists only a finite number of curves intersecting $-K_{{W_1}'}$ negatively, $m_1 \not\subset T$. If there exists an exceptional curve m for ${W_1}' \dashrightarrow W_1$ contained in the strict transform T' of T on ${W_1}'$, it must be the transform of a fiber of $T \to X'$ intersecting m_1. Note that $T \cdot m_1 = (H - 2L_1) \cdot m_1 = 1$. Hence m and m_1 intersect at one point simply. Then, however, $-K_{{W_1}'} \cdot m = 2$, a contradiction. Hence if we take l generally, l does not intersect an exceptional curve for ${W_1}' \dashrightarrow W_1$. Thus $F_1 \cdot l = {F_1}'' \cdot l = (2H + L_1) \cdot l = 1$ and so $-K_{Y'} \cdot l = 1$. If l intersects l_1, then l must be a flopping curve on Y_1 containing two $\frac{1}{2}(1,1,1)$-singularities, where $Y' \dashrightarrow Y_1$ is the anti-flip. This contradicts Corollary 3.4. If l intersects F_1 at a point of the negative section of F_1, then l must be a flopping curve on Y_1. Hence by the finiteness of the number of flopping curves, we may assume that l intersects F_1 outside the negative section of F_1 by taking l generally. Then $-K_{Y_1} \cdot l = \frac{1}{2}$. Since $\tilde{E} \cdot l = (-K_{Y'} - L) \cdot l = 0$, we have $E \cdot l = 0$. Since the strict transform of $\tilde{E}$ on ${Y_1}'$ is linearly equivalent to $-K_{{Y_1}'} - L_1$, it is not equal to T. Hence we may assume that $l \cap E = \emptyset$. Thus we have $-K_Y \cdot l = \frac{1}{2}$. However this contradicts the finiteness of the number of such curves. □

Hence by the list of [Take99], ${Y_1}' \simeq \mathbb{P}(\mathscr{O} \oplus \mathscr{O}(1)^{\oplus 2})$. By ${G_1}' = \mu_1^* {G_1}'' - 5F_1$ and ${G_1}'' = 2(-K_{Y'}) + 3L$, we have $3{F_1}'' = 2(-K_{{Y_1}'}) + 3L_1$. So we obtain the description of ${F_1}''$.

Assume that X is a $\mathbb{Q}$-Fano 3-fold of No. 5.5. By (4.12), (4.13) and Lemma 4.2, we have $g(m_1) = 3a - 3$, $(-K_{{Y_1}'} \cdot m_1) = 11a - 6$ and $(-K_{{Y_1}'})^3 = 16a$. Since $g(m_1) \geq 0$, we have $a \geq 1$. Let l_i $(i \geq 1)$ be transforms of flipped curves different from l_0. By Claim 4.14 (2), we have $G_1 \cdot l_i = ((-K_{Y'}) + aL \cdot l_i) = a - 1$ whence $(-K_{{Y_1}'} \cdot l_i) = a - 2$.

Claim 4.17. $a \neq 1$.

Proof. Assume that $a = 1$. ${Y_2}'$ can be embedded in a $\mathbb{P}^3$-bundle $\mathbb{P}(\mathscr{E})$ over $\mathbb{P}^1$, where $\mathscr{E} := \bigoplus_{i=0}^{3} \mathscr{O}(a_i)$ is a vector bundle of rank 4. We may assume that

$$a_0 = 0 \leq a_1 \leq a_2 \leq a_3. \tag{4.14}$$

Let H be the tautological divisor and M a fiber. In $\mathbb{P}(\mathscr{E})$, ${Y_2}'$ is linearly equivalent to $2H - aM$ for some $a \in \mathbb{Z}$. Since $-K_{{Y_1}'} = 2H|_{{Y_1}'} + (2 + a - \sum_{i=0}^{3} a_i)L_1$ and $(-K_{{Y_1}'}) = 16$, we have $(-K_{{Y_1}'})^3 = 16a - 8\sum_{i=0}^{3} a_i + 48 = 16$. So we obtain

$$\sum_{i=0}^{3} a_i = 2a + 4. \tag{4.15}$$

Let $\tilde{E}'$ be the strict transform of $\tilde{E}$ on ${Y_1}'$. Note that $\tilde{E} \sim -K_{{Y_1}'} - L_1 = 2H|_{{Y_1}'} - (a+3)L_1$ by (4.15). Since $L_1 \cdot l_1 = 1$ and $-K_{{Y_1}'} \cdot l_1 = -1$, we have

$$2H \cdot l_1 = a + 1 \geq 0. \tag{4.16}$$

Moreover since $h^j(-3M) = 0\,(j = 0, 1)$, we have $h^0(2H - (a+3)M) = h^0((2H|_{{Y_1}'} - (a+3)L_1) = 1$. Hence

$$2a_3 = a + 3,\ a_2 < a_3. \tag{4.17}$$

By (4.17), we have $2(\sum_{i=0}^{3} a_i) \leq 2(a+1) + a + 3$ and then by (4.15), $a \leq -3$. But this contradicts (4.16). $\square$

Assume that $a \geq 2$. Since $\mathrm{Bs}\,|-K_{Y'}| = \bigcup_{i=0}^{n-1} l_i$, $\mathrm{Bs}\,|-K_{W_1{}'}| = \bigcup_{i=1}^{n-1} l_i$. Moreover we know that $(-K_{Y_1{}'} \cdot l_i) = a - 2 \geq 0$ by the assumption. So $Y_1{}'$ is a weak Fano 3-fold. If $Y_1{}'$ has a crepant divisorial contraction, then by the method of [Take99], we can easily see that $(-K_{Y_1{}'})^3 = 8$, a contradiction. Hence by the list of [Take99], we have $a = 2$. Moreover since $2F_1{}'' = -K_{Y_1{}'} + 2L_1$, $-K_{Y_1{}'}$ is divisible by 2. Thus $Y_1{}'$ is a smooth divisor in $\mathbb{P}(\mathscr{O}^{\oplus 2} \oplus \mathscr{O}(1)^{\oplus 2})$ linearly equivalent to $2H$, where H is the tautological divisor. By $G_1{}' = \mu_1^* G_1{}'' - 3F_1$ and $G_1{}'' = (-K_{Y'}) + 2L$, we have $2F_1{}'' = (-K_{Y_1{}'}) + 2L_1$. So we obtain the description of $F_1{}''$. $\square$

Proof of Theorem 0.21 (B). Let G_1 be ν_1-exceptional divisor and $F_1{}'$ the strict transform of $F_1{}''$. Since $\mathrm{Bs}\,|-K_{Y_1{}'} - m_1| = m_1 \cup l_1 \cup \cdots \cup l_{n-1}$, where l_i are all the flopping curves on $Y_1{}'$, $|-K_{W_1{}'}|$ is free outside l_i. By an argument similar to the proof of (A), we know that there is a flop $W_1{}' \dashrightarrow W_1$ over X' and an extremal contraction $\mu_1 \colon W_1 \to Y'$ of $(2,1)$-type over X' whose exceptional divisor F_1 is the strict transform of $F_1{}'$ on W_1. Since $\mathrm{Bs}\,|-K_{Y_1{}'} - m_1| = m_1 \cup l_1 \cup \cdots \cup l_{n-1}$, $\mathrm{Bs}\,|-K_{Y'} - l_0| = l_0 \cup \cdots \cup l_{n-1}$. It is easy to see that $-K_{Y'} \cdot l_i = -1$ by (A1)(3) or (A2)(3). Hence $\mathrm{Bs}\,|-K_{Y'}| = l_0 \cup \cdots \cup l_{n-1}$.

(i) Consider the exact sequence

$$0 \to \mathscr{O}_{Y'}(-K_{Y'} - L) \to \mathscr{O}_{Y'}(-K_{Y'}) \to \mathscr{O}_L(-K_L) \to 0,$$

where L is a general fiber of f'. First we treat No. 5.4. Since $\mathrm{Bs}\,|-K_{Y'}| \cap L = (L \cap l_0) \cup (L \cap l_1)$ consists of two points and $-K_L$ is very ample, the dimension of $\mathrm{Im}\,(H^0(\mathscr{O}_{Y'}(-K_{Y'})) \to H^0(\mathscr{O}_L(-K_L))$ is 4. On the other hand $h^0(\mathscr{O}_{Y'}(-K_{Y'})) = 5$. Hence we have $h^0(\mathscr{O}_{Y'}(-K_{Y'} - L)) = 1$.

Next we treat No. 5.5. Note that a general fiber of $W_1 \to X'$ is a del Pezzo surface of degree 5. Moreover $\mathrm{Bs}\,|-K_{Y'}| \cap L = (L \cap l_0) \cup (L \cap l_1) \cup (L \cap l_2)$ consists of three points and $-K_L$ is very ample. Thus the dimension of $\mathrm{Im}\,(H^0(\mathscr{O}_{Y'}(-K_{Y'})) \to H^0(\mathscr{O}_L(-K_L))$ is 4. On the other hand $h^0(\mathscr{O}_{Y'}(-K_{Y'})) = 5$. Hence we have $h^0(\mathscr{O}_{Y'}(-K_{Y'} - L)) = 1$.

Hence in any case, let $\tilde{E} \in |-K_{Y'} - L|$ be the unique member. The irreducibility of $\tilde{E}$ can be proved similarly to No. 4.4.

(ii) **Claim 4.18.** *l_i are mutually disjoint.*

Proof. It suffices to prove that $l_i\,(i \geq 2)$ do not intersect flopping curves for $W_1{}' \dashrightarrow W_1$. This follows from $\mathrm{Bs}\,|-K_{Y'}| = l_0 \cup \cdots \cup l_{n-1}$ because otherwise there exists a member of $|-K_{W_1{}'}|$ which intersects a flopping curve for $W_1{}' \dashrightarrow W_1$ but does not contain it, a contradiction. $\square$

(2-3) is checked before the proof of (i). We can easily see that other conditions are satisfied.

$\square$

Proof of Theorem 0.21 (C). We prove that there exists m_1 as in (A-1) (3) or (A-2) (3). The assertion about the base locus is put off till Claim 4.21.

First we treat No. 5.4. Let $F_1{}'' \in |2H + L_1|$ be a general member, where L_1 is a fiber of the natural projection $p : Y_1{}' := \mathbb{P}(\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(1)^{\oplus 2}) \to X' \simeq \mathbb{P}^1$. Then we show that $F_1{}''$ is a del Pezzo surface of degree 1. Note that $\Phi_{|H|}(Y_1{}') \subset \mathbb{P}^4$ is a singular quadric Q with one ODP. Q contain a smooth del Pezzo surface S of degree 1 embedded in $\mathbb{P}^4$ by $|-K_S + l|$, where $l \simeq \mathbb{P}^1$ such that $l^2 = 0$. We may assume that the transform of S on ${Y_1}'$ is isomorphic to S. We denote it also by S. Write $S = aH + bL_1$. Let δ be the section of p corresponding to $\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(1)^{\oplus 2} \to \mathscr{O}_{\mathbb{P}^1}$. Note that δ is the exceptional curve for ${Y_1}' \to Q$. Since $S \cdot \delta = 1$, we have $b = 1$. Moreover since $(-K_S)^2 = ((3-a)H - L_1)^2(aH + L_1) = 1$, we have $a = 1$. Hence $S \in |2H + L_1|$ whence by taking ${F_1}''$ generally ${F_1}''$ is a del Pezzo surface of degree 1.

We can regard ${F_1}''$ as a surface obtained by blowing up $\mathbb{P}^2$ at 8 points and let $e_i\,(i = 1, \dots, 8)$ be the exceptional curves, where we may assume that e_1 is a section of $p|_{{F_1}''}$ and $e_i\,(i \geq 2)$ are components of different 7 degenerate fibers. Let $\pi\colon \tilde{F_1}'' \to {F_1}''$ be the blow-up at ${F_1}'' \cap \delta$ and e_9 π-exceptional divisor (note that ${F_1}'' \cap \delta$ consists of one point). Since $-K_{{F_1}''} = (H - L_1)|_{{F_1}''}$ and $\operatorname{Bs}|H - L_1| = \delta$, $|-K_{\tilde{F_1}''}|$ is free. Let ${m_1}' := 11l - 6e_1 - 3\sum_{i=2}^{8} e_i - e_9$, where l is the pull-back of a line in $\mathbb{P}^2$.

Claim 4.19. *$|{m_1}'|$ is free.*

Proof. Since ${m_1}' = (3l - 2e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7) + (3l - 2e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_8) + (3l - 2e_1 - e_2 - e_3 - e_4 - e_5 - e_7 - e_8) + (2l - e_6 - e_7 - e_8 - e_9)$, $|{m_1}'|$ is nef. Assume that $|{m_1}'|$ is not free. Since ${m_1}' - K_{\tilde{F_1}''}$ is nef and $({m_1}' - K_{\tilde{F_1}''})^2 > 4$, we can apply [Reide88] and obtain a contradiction similarly to the proof of Claim 4.8. $\square$

Let $m_1 \in |{m_1}'|$ be a general smooth member and we also denote by m_1 the image of m_1 on ${F_1}''$, which is also smooth. It is easy to check that $g(m_1) = 9$ and $(-K_{{Y_1}'} \cdot m_1) = 33$.

Next we treat No. 5.5. Let $V \in |H + M_1|$ be a general member, where M_1 is a fiber of the natural projection $p\colon \mathbb{P}(\mathscr{O}^{\oplus 2} \oplus \mathscr{O}(1)^{\oplus 2}) \to X'$. Then we show that ${F_1}'' := V \cap {Y_1}''$ is a del Pezzo surface of degree 2 if we take ${Y_1}'' \in |2H|$ generally. Note first that $H|_V$ is ample since H is numerically trivial only for horizontal sections of the subvariety S corresponding to $\mathscr{O}^{\oplus 2} \oplus \mathscr{O}(1)^{\oplus 2} \to \mathscr{O}^{\oplus 2}$ and we may assume that V does not contain them. Moreover since $\deg p_*\mathscr{O}_V(H|_V) = 3$, we have $p_*\mathscr{O}_V(H|_V) \simeq \mathscr{O}(1)^{\oplus 3}$. Hence $V \simeq \mathbb{P}^2 \times \mathbb{P}^1$ and $H|_V$ is a divisor of $(1,1)$-type. Since $H^0(\mathscr{O}(2H)) \to H^0(\mathscr{O}_V(2H|_V))$ is surjective, it suffices to prove that a general divisor of $\mathbb{P}^2 \times \mathbb{P}^1$ of $(2,2)$-type is a del Pezzo surface. But this is clear.

We can regard ${F_1}''$ as a surface obtained by blowing up $\mathbb{P}^2$ at 7 points and let $e_i\,(i = 1, \dots, 7)$ be the exceptional curves, where we may assume that e_1 is a section of $p|_{{F_1}''}$ and $e_i\,(i \geq 2)$ are components of different 6 degenerate fibers. Let $\pi\colon \tilde{{F_1}''} \to {F_1}''$ be the blow-up at ${F_1}'' \cap S$ and $e_j (j = 8, 9)$ π-exceptional divisors (note that ${F_1}'' \cap S$ consists of two points). Since $-K_{{F_1}''} = (H - M_1)|_{{F_1}''}$ and $\operatorname{Bs}|H - M_1| = S$, $|-K_{\tilde{F_1}''}|$ is free. Let ${m_1}' := 7l - 4e_1 - 2\sum_{i=2}^{7} e_i - e_8 - e_9$, where l is the pull-back of a line in $\mathbb{P}^2$.

Claim 4.20. *$|{m_1}'|$ is free.*

Proof. Since ${m_1}' = 2(3l - 2e_1 - \sum_{i=2}^{7} e_i) + (l - e_8 - e_9)$, $|{m_1}'|$ is nef.Assume that $|{m_1}'|$ is not free. Since ${m_1}' - K_{\tilde{F_1}''}$ is nef and $({m_1}' - K_{\tilde{F_1}''})^2 > 4$, we can apply [Reide88] and obtain a contradiction similarly to the proof of Claim 4.8. $\square$

Let $m_1 \in |{m_1}'|$ be a general smooth member and we also denote by m_1 the image of m_1 on ${F_1}''$, which is also smooth. It is easy to check that $g(m_1) = 3$ and $(-K_{{Y_1}'} \cdot m_1) = 16$.

Claim 4.21. $\operatorname{Bs}|-K_{{Y_1}'} - m_1| = m_1 \cup l_1 \cup \dots \cup l_{n-1}$.

Proof. Consider the exact sequence

$$0 \to \mathscr{O}_{{Y_1}'}(-K_{{Y_1}'} - {F_1}'') \to \mathscr{O}_{{Y_1}'}(-K_{{Y_1}'}) \to \mathscr{O}_{{F_1}''}(-K_{{Y_1}'}) \to 0.$$

Since $h^1(\mathscr{O}(-K_{Y_1'} - F_1{''})) = 0$, $H^0(\mathscr{O}_{Y_1'}(-K_{Y_1'})) \to H^0(\mathscr{O}_{F_1''}(-K_{Y_1'}))$ is surjective. Note that the base locus of $|-K_{Y_1'} - F_1{''}| = |H|_{Y_1'} - L_1|$ is $l_1 \cup \cdots \cup l_{n-1}$. Since $|-K'_{Y_1}|_{F_1''} - m_1| = |l|$, we have the assertion. $\square$

$\square$

5. Excluding some possibilities

Next we exclude the cases in Tables 1′–5′. By Corollary 2.2, we exclude these cases assuming that X has only $\frac{1}{2}(1,1,1)$-singularities.

Proposition 5.1. *Assume that X has only $\frac{1}{2}(1,1,1)$-singularities. Let l be an irreducible component of a flopping curve. Then $\mathcal{N}_{l/Y} \simeq \mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(-2)$ or $\mathscr{O}_{\mathbb{P}^1}(-1) \oplus \mathscr{O}_{\mathbb{P}^1}(-1)$.*

Proof. Since $|-2K_X|$ is very ample by Corollary 3.3 and $\bar{l} := f(l)$ is a line with respect to $-2K_X$, the natural map $H^0(\mathscr{O}(-2K_Y) \otimes \mathcal{I}_l) \to \mathscr{O}(-2K_Y) \otimes \mathcal{I}_l$ is surjective, where $\mathcal{I}_l$ is the ideal sheaf of l. Hence there is a smooth member S of $|-2K_Y|$ containing l. The assertion follows from this easily. $\square$

Table 1′. $h = 8$ and $N = 3$.

We prove that this case does not occur. By Proposition 5.1, $\tilde{E}$ is normal and has only canonical singularities. Let $Y' \dashrightarrow Y_1$ be the anti-flip and $E_1 \subset Y_1$ the strict transform of $\tilde{E}$. Note that there is a natural morphism $\tilde{E} \overset{\mu_2}{\to} E_1 \overset{\mu_1}{\to} E$ and the exceptional locus of μ_1 is the union of flopped curves and the exceptional locus of μ_2 is the union of flipped curves. Set $\mu := \mu_1 \circ \mu_2$. Let $\mu_3 \colon \hat{E} \to \tilde{E}$ be the minimal resolution and $\nu := \mu \circ \mu_3$. Then ν is a composite of 5 times of blow-ups at points. Let F_i $(i = 1, \ldots, 5)$ be the total transforms of the exceptional curves of one point blow-ups and l the total transform of a line of E. As in the proof of (A) of Table 1, a flipping curve l_1 intersects E_1 transversely at a smooth point and $\tilde{E} \to E_1$ is a two point blow-up at smooth points. Hence we may assume that $\mu_2 \circ \mu_3(F_1) \sim \mu_2 \circ \mu_3(F_3)$ are flopped curves and $\mu_3(F_4)$ and $\mu_3(F_5)$ are flipped curves, which are (-1)-curves contained in the smooth locus of $\tilde{E}$.

Let γ be a smooth curve on E and $\hat{\gamma}$ (resp. $\tilde{\gamma}$) the strict transform of γ on $\hat{E}$ (resp. $\tilde{E}$). Then we can write $\hat{\gamma} = (\nu)^*\gamma - \sum_{i=i_1}^{i_\alpha} F_i - \sum_{j=j_1}^{j_\beta} F_i$, where $i_1 < i_2 \ldots i_\alpha \leq 3 < j_1 \cdots < j_\beta \leq 5$. Then we have

(1) $-K_{Y'} \cdot \tilde{\gamma} = -K_Y \cdot \gamma + \beta$, and
(2) $\tilde{E} \cdot \tilde{\gamma} = E \cdot \gamma + \alpha + 2\beta$.

In fact, (1) follows from [Taka00, Proposition 2.1 (4)], and (2) follows from (1) and $(-K_{\hat{E}}) \cdot \hat{\gamma} = (-K_E) \cdot \gamma - (\alpha + \beta)$.

By Riemann-Roch, we can see that $h^0(2l - \sum_{i=1}^5 F_i) > 0$. Let m be a member of $|2l - \sum_{i=1}^5 F_i|$ and $m' := \nu(m)$.

If m' is a reducible (possibly non-reduced) conic, let $m' = m_1 + m_2$ be the irreducible decomposition. We can express $\hat{m_1}$ and $\hat{m_2}$ as $\hat{\gamma}$. If there are at most 2 blow-ups on the strict transform of m_i, then $\hat{m_i}$ is not a (-2)-curve and hence it is not contracted by $f'|_{\tilde{E}}$. Since $f'|_{\tilde{E}}(m)$ is a line, one of m_i's must be contracted. Hence there are at least 3 blow-ups on one m_i, say m_1. We use α and β for m_1. Then we have $E' \cdot \tilde{m_1} = 8 - (3\alpha + 4\beta) \leq 8 - 3\alpha \leq -1$ and $-K_{X'} \cdot f'(m_1) = 9 - 3(\alpha + \beta) \leq 0$. Hence $\tilde{m_1}$ is a fiber of $E'$and $\alpha = 3$ and $\beta = 0$, i.e., $\hat{m_1} = \nu^* m_1 - F_1 - F_2 - F_3$.

Next we show that $\hat{m_2} = \nu^* m_2 - F_4 - F_5$. It suffices to prove that neither F_1, F_2 or F_3 contains F_4 or F_5. If otherwise, one of $\mu_3(F_4)$ and $\mu_3(F_5)$, say $\mu_3(F_4)$ intersects one of $\mu_3(F_1) \sim \mu_3(F_3)$. If a flipping curve intersects an irreducible component a of a flopped curve, then a become a fiber of E' on Y' by [Taka00, Proposition 2.2 (4)]. By $\hat{m_1} = \nu^* m_1 - F_1 - F_2 - F_3$, the centers of F_4 are not on the strict transform of

m_1. Hence the transform of a on E' intersects $\tilde{m_1}$, a contradiction to irreducibility of a fiber of E'. In particular we know that $m_1 \neq m_2$. So we have $\tilde{E} \cdot \tilde{m_2} = 2$ and $-K_{Y'} \cdot \tilde{m_2} = 3$. Hence $E' \cdot \tilde{m_2} = 0$ and $-K_{X'} \cdot f'(\tilde{m_2}) = 3$. The latter shows that $f'(\tilde{m_2})$ is a line on X' and hence $\tilde{m_2} \not\subset E'$. So the former shows that $E' \cap \tilde{m_2} = \emptyset$. In particular $\tilde{m_1} \cap \tilde{m_2} = \emptyset$. But this is a contradiction since there is no blow-up at the intersection of m_1 and m_2 by $m = \hat{m_1} + \hat{m_2}$.

If m' is a smooth conic, m' is the strict transform of m. We have $E' \cdot m' = -1$ and $-K_{X'} \cdot f'(m') = 3$. Hence $f'(m')$ is a line on X'. But $f'(m') = C$ since $m' \subset E'$, a contradiction.

Table 2′. $N = 7$.

We deny the possibility of $N = 7$ in Table 2′ in Theorem 0.3. But by the same method, we obtain some properties of No. 2.3 and No. 2.4 in Table 2 of Theorem 0.3.

Let P be a Gorenstein singularity on C. Let $g\colon Z \to X'$ be the blow-up of P and F the exceptional divisor. Since $(P \in X') \simeq (o \in ((xy + zw = 0) \subset \mathbb{C}^4))$ or $(o \in ((xy + z^2 + w^3 = 0) \subset \mathbb{C}^4))$ by [Taka00, Proposition 2.2] and X' is $\mathbb{Q}$-factorial, $\rho(Z) = 2$. Since $-K_{X'}$ is very ample and $-K_Z = g^*(-K_{X'}) - F$, $|-K_Z|$ is free and $(-K_Z)^3 > 0$. Hence Z is a weak Fano 3-fold. Starting by g, we consider the following diagram similar to one in the section 3 and do calculations similar to one there.

$$\begin{tikzcd} & Z \arrow[r, dashed] \arrow[dl, "f"'] & Z' \arrow[dr, "g'"] & \\ X' & & & X''. \end{tikzcd}$$

Let $\tilde{F}$ be the strict transform of F on Z'. Then by a similar way, we have $(-K_{Z'})^2\tilde{F} = 2$, $(-K_{Z'})(\tilde{F})^2 = -2$ and $(\tilde{F})^3 = 2 - e'$, where e' is a non-negative integer. Set $d := (-K_{Z'})^3$ and input these into (1-1)–(5-1). Then we obtain the following.

If $N = 5$, then $e' = 6$ and g' is a conic bundle over $\mathbb{P}^2$ with deg $\Delta' = 6$, where Δ' is the discriminant divisor for g'.

If $N = 6$, then $e' = 5$ and g' is of $(2, 1)$-type and $X'' \simeq \mathbb{P}^3$. Let F' be the exceptional divisor of g'. Then $F' \sim 3(-K_{Z'}) - 4\tilde{F}$. For the center C' of g', $\deg C' = 8$ and $p_a(C') = 6$.

If $N = 7$, then $e' = 4$ and g' is of $(2, 1)$-type and $X'' \simeq Q_3$. (Since X'' is $\mathbb{Q}$-factorial, X'' is a smooth quadric.) Let F' be the exceptional divisor of g'. Then $F' \sim 2(-K_{Z'}) - 3\tilde{F}$. For the center C' of g', $\deg C' = 8$ and $p_a(C') = 4$.

Assume that $N = 7$. Then Z' has 5 Gorenstein singular points on the strict transform of C. Since X'' is smooth, C' must have 5 singular points by [Cu, Theorem 4]. But by $p_a(C') = 4$, this is impossible.

Table 4′. $h = 5$ and $N = 6, 7$.

Let l be the fiber containing two $\frac{1}{2}(1,1,1)$-singularities and Q a $\frac{1}{2}(1,1,1)$-singularity. Let $g\colon Z \to Y'$ be the blow-up at Q. Then the transform l' of l is a flipping curve. Let $Z \dashrightarrow Z'$ be the flip. Then Z' has a conic bundle structure $g'\colon Z' \to W$ over $W \simeq \mathbb{F}_2$. There is a natural morphism $\mu\colon W \to X'$. Let Δ' be the strict transform of Δ on W. Note that Δ does not pass through the vertex of X' by [Taka00, Proposition 2.4 (3-1)].

If $\deg \Delta = 2$, then $\Delta' \sim C_0 + 2f$, where C_0 is the negative section and f is a fiber. Since Δ' is disjoint from C_0, $\Delta' \simeq \mathbb{P}^1$. This contradicts [MM85, Proposition 4.7 (1)]. Hence this case does not occur and moreover by Corollary 2.3, the case that $\deg \Delta = 0$ does not occur.

Table 4′. $h = 6$ and $N = 6, 7$.

If $\deg \Delta = 1$, then $\Delta \simeq \mathbb{P}^1$. Hence this case does not occur by [MM85, Proposition 4.7 (1)]. If $\deg \Delta = 2$, then by the same reason, Δ must be a reducible conic. But this contradicts [MM85, Proposition 4.7 (2)].
Table 5′. $h = 4$.

We deny the possibilities of Table 5′ in Theorem 0.3. But by the same method, we obtain some properties of No. 5.1 as below.

By Riemann-Roch theorem, we can see $h^0(-K_{Y'} - \tilde{E}) = 1$. Let $D \in |-K_{Y'} - \tilde{E}|$ be the unique member. Since $2D \sim F$, $2D$ is a multiple fiber and since the reduced part of any fiber is irreducible, D is irreducible. Since D is not Cartier at $\frac{1}{2}(1,1,1)$-singularities, all $\frac{1}{2}(1,1,1)$-singularities are contained in D. Let Q be a $\frac{1}{2}(1,1,1)$-singularity and $g\colon Z \to Y'$ be the blow-up at Q. Let G be the exceptional divisor and D' the strict transform of D on Z. Set $D' = g^*D - \delta G$. We can prove that $|-K_Z|$ is free outside the transforms of flipped curves. By considering extremal rays over X', we obtain a diagram

$$Z_0 := Z \dashrightarrow Z_1 \dashrightarrow \cdots \dashrightarrow Z_k := Z \xrightarrow{g'} Y''$$

similar to one in [Taka00, Lemma 3.2]. Let G_i (resp. D_i) be the strict transform of G (resp. D') and R_i the extremal ray in $\overline{\mathrm{NE}}(Z_i/X')$ which is other than the ray associated to g if $i = 0$ or K_{Z_i}-negative if $i \geq 1$. If R_0 is a crepant divisorial ray, then the exceptional divisor is D'. By $(-K_Z)^2 D' = (-K_Z)^2(g^*D - \delta G) = \deg D - \delta = 0$, $\delta \in \mathbb{N}$. But since D is not a Cartier divisor, δ cannot be an integer, a contradiction.

Claim 3. $D_i \cdot R_i < 0$.

Proof. The proof is similar to one of Claim 4.1. $\square$

Hence g' is $K_{Z'}$-negative divisorial contraction of D_k. By calculations similar to [Taka00, Lemma 3.1], we have

$$(-K_{Z'})^2 D_k = \deg D - \delta - \sum a_i d_i, \tag{1}$$

$$(-K_{Z'})(D_k)^2 = -2\delta^2 - \sum {a_i}^2 d_i, \text{ and} \tag{2}$$

$$(D_k)^3 = -4\delta^3 - \sum {a_i}^3 d_i - e, \tag{3}$$

where e, a_i and d_i are defined similarly to [Taka00, Lemma 3.1]. Note that a_i is a non-positive integer and $e \leq 0$. Assume that g' is of $(2,1)$-type. Then by

$$(-K_{Z'} + D_k)^2 D_k = \deg D - \delta(2\delta + 1)^2 - \sum d_i a_i (a_i + 1)^2 - e = 0. \tag{4}$$

On the other hand,

$$(-K_{Z'} - D_k)^2 D_k = -4(-K_{Z'})(D_k)^2 = 8\delta^2 + 4\sum d_i {a_i}^2 = 8(1-g) - 2m, \tag{5}$$

where g is the genus of $g'(D_k)$ and m is an natural number. Hence $\delta = \frac{1}{2}$. But this contradicts (4) since $\deg D = 3$ or 4.

Assume that f is of $(2,0)$-type. By [Taka00, Proposition 2.3], we have $(-K_{Z'})(D_k)^2 \geq -2$. So by (2), we have $\delta = \frac{1}{2}$ and $a_i = -1$ if $a_i \neq 0$. By [Taka00, Proposition 2.3], we have $e = 0$ (i.e., there is no flop while $Z \dashrightarrow Z'$) and if $\deg D = 3$, then g' is of $(2,0)_1$-type and $\sum d_i = \frac{3}{2}$ and if $\deg D = 4$, then g' is of $(2,0)_5$-type and $\sum d_i = 1$. In any case Y'' is smooth.

If $\deg D = 4$, then $Y'' \to X$ is a quadric bundle. Hence by [Mor82, Theorem (3.5)], we can write $-K_{Y''} \sim 2H + aF'$, where F' is a fiber, H is a divisor such that $H|_{F'}$ is a hyperplain section and a is an integer. Let H' be the transform of ${g'}^*H$ on Y. Then we have $-K_Y \sim 2H' + (2a-3)(-K_Y - E)$ (note that $D \sim -K_Y - \tilde{E}$). So there is a divisor $\overline{E}$ such that $E \sim 2\overline{E}$, a contradiction. Hence the case $\deg D = 4$ does not occur.

Assume that $\deg D = 3$ below. Since $n = 0$, $-K_Z$ is nef and big and so is $-K_{Y''}$. By calculations similar to [Taka00, Lemma 3.1], we can see that Y'' has a flopping ray and after the flop $Y'' \dashrightarrow Y''^{+}$, there is an extremal contraction $h \colon Y''^{+} \to W$ of $(2,0)_1$-type such that W is A_{18}. Note that the strict transform of G is obtained from G by blow-up three points on the line $l := D' \cap G$ and contracting the strict transform of l.

Corollary 5.2. *Let X be a $\mathbb{Q}$-factorial $\mathbb{Q}$-Fano 3-fold with (1)–(4) in Main Assumption 0.1. Then $(-K_X)^3$ and $\mathrm{aw}(X)$ are effectively bounded as in Theorem 0.3.*

Proof. By Theorem 0.3 and Theorem 2.0, we obtain the assertion since $(-K)^3$ and aw are invariant under a deformation. $\square$

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