RIMS Preprint No. 1307, Kyoto University, December 2000

New Characterizations of M-convex Functions and Connections to Mathematical Economics †

Kazuo Murota and Akihisa Tamura

December 12, 2000

Abstract

The concept of M-convex functions plays a central role in "discrete convex analysis," a unified framework of discrete optimization recently developed by Murota and others. This paper gives two new characterizations of $M/M^\natural$-convex functions generalizing Gul and Stacchetti's results on the equivalence among the single improvement condition, the gross substitutes condition and the no complementarities condition for set functions (utility functions on $\{0,1\}$ vectors) as well as Fujishige and Yang's observation on the connection to M-convexity. We also discuss implications of our results in an exchange economy with indivisible goods.

Kazuo Murota
Research Institute for Mathematical Sciences,
Kyoto University, Kyoto 606-8502, Japan.
phone: +81-75-753-7221, facsimile: +81-75-753-7272
e-mail: murota@kurims.kyoto-u.ac.jp.

Akihisa Tamura
Research Institute for Mathematical Sciences,
Kyoto University, Kyoto 606-8502, Japan.
phone: +81-75-753-7236, facsimile: +81-75-753-7272
e-mail: tamura@kurims.kyoto-u.ac.jp.

† This work is supported by a Grant-in-Aid of the Ministry of Education, Science, Sports and Culture of Japan.

1 Introduction

"Discrete convex analysis," which was recently proposed by Murota [15, 16], is a unified framework of discrete optimization with reference to existing studies on submodular functions [5, 14, 8], generalized polymatroids [6, 7, 8], valuated matroids [3, 4, 18] and convex analysis [25]. The concepts of M/M$^\natural$-convex functions play a central role in discrete convex analysis and have been investigated by Murota [15, 16, 17], Fujishige and Murota [9], Murota and Shioura [21, 22, 23] and Shioura [26].

On the other hand, in mathematical economics, an economy with indivisible goods has been investigated by many authors. Among others, Kelso and Crawford [13] showed the nonemptyness of the core of the matching model between firms and workers with money, under the gross substitutes condition (GS). Gul and Stacchetti [12] introduced other conditions, called the single improvement condition (SI) and the no complementarities condition (NC), which are equivalent to (GS), for set functions (utility functions on $\{0,1\}$ vectors). Recently, Fujishige and Yang [10, 11] observed the connection between (GS) and M$^\natural$-convexity for set functions. An initial encounter of mathematical economics with M$^\natural$-convexity was brought about in the papers of Danilov, Koshevoy and Murota [1, 2]. Their results imply the existence of a competitive equilibrium in an exchange economy with indivisible goods if the utility function of each agent is quasilinear in money and its reservation value function is M$^\natural$-concave.

The present work is motivated by recent results by Gul and Stacchetti [12] and Fujishige and Yang [10, 11]. This paper gives two new characterizations of M$^\natural$-convex functions by using two novel conditions labeled (M$^\natural$-GS) and (M$^\natural$-SI) in this paper. For set functions, (NC) is equivalent to the M$^\natural$-concavity, and (GS) and (SI) are associated with the concave versions of (M$^\natural$-GS) and (M$^\natural$-SI), respectively. Thus, our results are extensions of the equivalence among (NC), (GS) and (SI). Combination of the results in [1, 2] with our characterization of M$^\natural$-concavity yields another sufficient condition for the existence of a competitive equilibrium in an exchange economy with indivisible goods.

This paper is organized as follows. Section 2 reviews definitions and known results on M/M$^\natural$-convex functions. Section 3 states our results and Section 5 proves these. Section 4 discusses implications of our results in an exchange economy with indivisible goods.

2 Preliminaries on M/M$^\natural$-convexity

In this section, we review several definitions and known results on M/M$^\natural$-convex functions.

Let V be a nonempty finite set, and $\mathbf{Z}$ and $\mathbf{R}$ be the sets of integers and reals, respectively. We define the support, positive support and negative support of $z = (z(v) : v \in V) \in \mathbf{Z}^V$ by

$$\begin{aligned} \mathrm{supp}(z) &= \{v \in V \mid z(v) \neq 0\}, \\ \mathrm{supp}^+(z) &= \{v \in V \mid z(v) > 0\}, \\ \mathrm{supp}^-(z) &= \{v \in V \mid z(v) < 0\}. \end{aligned}$$

For each $S \subseteq V$, we denote by χ_S the characteristic vector of S defined by

$$\chi_S(v) = \begin{cases} 1 & \text{if } v \in S \\ 0 & \text{otherwise} \end{cases}$$

and write simply χ_u instead of $\chi_{\{u\}}$ for each $u \in V$. For $p \in \mathbf{R}^V$ and $f : \mathbf{Z}^V \to \mathbf{R} \cup \{\pm\infty\}$, we define functions $\langle p, x\rangle$ and $f[p](x)$ by

$$\langle p, x\rangle = \sum_{v \in V} p(v)x(v) \quad \text{and} \quad f[p](x) = f(x) + \langle p, x\rangle \quad (x \in \mathbf{Z}^V),$$

and define

$$\mathrm{dom} f = \{x \in \mathbf{Z}^V \mid -\infty < f(x) < +\infty\}, \tag{2.1}$$

called the *effective domain* of f.

A function $f : \mathbf{Z} \to \mathbf{R} \cup \{+\infty\}$ with $\mathrm{dom} f \neq \emptyset$ is called *M-convex* [16] if it satisfies

(M-EXC) for $x, y \in \mathrm{dom} f$ and $u \in \mathrm{supp}^+(x - y)$, there exists $v \in \mathrm{supp}^-(x - y)$ such that

$$f(x) + f(y) \geq f(x - \chi_u + \chi_v) + f(y + \chi_u - \chi_v).$$

A function f with $\mathrm{dom} f \subseteq \{0, 1\}^V$ is M-convex if and only if $-f$ is a valuated matroid in the sense of [3, 4]. We note that (M-EXC) is also represented as: for $x, y \in \mathrm{dom} f$,

$$f(x) + f(y) \geq \max_{u \in \mathrm{supp}^+(x-y)} \min_{v \in \mathrm{supp}^-(x-y)} [\, f(x - \chi_u + \chi_v) + f(y + \chi_u - \chi_v) \,],$$

where the maximum and the minimum over an empty set are $-\infty$ and $+\infty$, respectively.

The concept of M$^\natural$-convexity is a variant of M-convexity. Let 0 denote a new element not in V and define $\hat{V} = \{0\} \cup V$. A function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$

with $\mathrm{dom}f \neq \emptyset$ is called *$M^\natural$-convex* [21] if it is expressed in terms of an M-convex function $\hat{f} : \mathbf{Z}^{\hat{V}} \to \mathbf{R} \cup \{+\infty\}$ as

$$f(x) = \hat{f}(x_0, x) \quad \text{with } x_0 = -x(V), \tag{2.2}$$

where $x(V) = \sum_{v \in V} x(v)$. Namely, an $\mathrm{M}^\natural$-convex function is a function obtained as the projection of an M-convex function. Conversely, an $\mathrm{M}^\natural$-convex function f determines the corresponding M-convex function $\hat{f}$ by

$$\hat{f}(x_0, x) = \begin{cases} f(x) & \text{if } x_0 = -x(V) \\ +\infty & \text{otherwise} \end{cases} \tag{2.3}$$

up to a translation of $\mathrm{dom}f$ in the direction of 0. An $\mathrm{M}^\natural$-convex function can also be defined by using an exchange property.

Theorem 2.1 ([21]) *A function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ with $\mathrm{dom}f \neq \emptyset$ is $M^\natural$-convex if and only if it satisfies*

($\mathrm{M}^\natural$-EXC) *for $x, y \in \mathrm{dom}f$ and $u \in \mathrm{supp}^+(x - y)$,*

$$\begin{aligned} f(x) + f(y) \geq \; & \min[\, f(x - \chi_u) + f(y + \chi_u), \\ & \min_{v \in \mathrm{supp}^-(x-y)} \{f(x - \chi_u + \chi_v) + f(y + \chi_u - \chi_v)\} \,]. \end{aligned}$$

We note that ($\mathrm{M}^\natural$-EXC) is also represented as: for $x, y \in \mathrm{dom}f$,

$$f(x) + f(y) \geq \max_{u \in \mathrm{supp}^+(x-y)} \min_{v \in \mathrm{supp}^-(x-y) \cup \{0\}} [\, f(x - \chi_u + \chi_v) + f(y + \chi_u - \chi_v) \,]$$

where we assume χ_0 is the zero vector on V.

Whereas the concept of $\mathrm{M}^\natural$-convexity is equivalent to that of M-convexity as above, Theorem 2.1 and the definition of M-convexity imply that an M-convex function is $\mathrm{M}^\natural$-convex. That is, we have

$$f : (\mathrm{M}^\natural\text{-EXC}) \iff \hat{f} : (\text{M-EXC}) \implies \hat{f} : (\mathrm{M}^\natural\text{-EXC}). \tag{2.4}$$

For a function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$, its *convex extension* $\bar{f}$ is defined by

$$\bar{f}(z) = \sup_{p \in \mathbf{R}^V, \alpha \in \mathbf{R}} \{\langle p, z \rangle + \alpha \mid \langle p, y \rangle + \alpha \leq f(y) \; (y \in \mathbf{Z}^V)\} \quad (z \in \mathbf{R}^V).$$

We say that f is *convex-extendible* if $f(x) = \bar{f}(x)$ for any $x \in \mathbf{Z}^V$.

Theorem 2.2 ([23]) *An $M^\natural$-convex function is convex-extendible.*

It is known that the effective domain of an $\mathrm{M}/\mathrm{M}^\natural$-convex function has a special structure called $\mathrm{M}/\mathrm{M}^\natural$-convex set. A nonempty set $B \subseteq \mathbf{Z}^V$ is said to be an *M-convex set* if it satisfies

(B-EXC) for $x, y \in B$ and $u \in \mathrm{supp}^+(x-y)$, there exists $v \in \mathrm{supp}^-(x-y)$ such that $x - \chi_u + \chi_v \in B$ and $y + \chi_u - \chi_v \in B$,

and B is said to be an *$M^\natural$-convex set* if it satisfies

($\mathrm{B}^\natural$-EXC) for $x, y \in B$ and $u \in \mathrm{supp}^+(x-y)$, there exists $v \in \mathrm{supp}^-(x-y) \cup \{0\}$ such that $x - \chi_u + \chi_v \in B$ and $y + \chi_u - \chi_v \in B$.

An M-convex set ($\mathrm{M}^\natural$-convex set, respectively) is a synonym of the set of integer points of the base polyhedron of an integral submodular system (a generalized polymatroid, respectively). See [8] for submodular systems and generalized polymatroids. The following characterization of the base polyhedra of submodular systems was first announced by Tomizawa [27] without proof (proofs of the "only if" part and "if" part can be found in [8] and Appendix of [11], respectively).

Theorem 2.3 *A pointed polyhedron $P \subseteq \mathbf{R}^V$ is a base polyhedron if and only if for each extreme point x in P the tangent cone of P at x is generated by a set of vectors taken from $\chi_u - \chi_v$ $(u, v \in V, u \neq v)$.*

We can characterize M-convex sets by using a weaker condition than (B-EXC).

Theorem 2.4 (see, e.g. [19, 20]) (B-EXC) *is equivalent to*

(B-EXC$_-$) *for $x, y \in B$ and $u \in \mathrm{supp}^+(x-y)$, there exists $v \in \mathrm{supp}^-(x-y)$ such that $x - \chi_u + \chi_v \in B$.*

Characterizations of $\mathrm{M}/\mathrm{M}^\natural$-convex functions by $\mathrm{M}/\mathrm{M}^\natural$-convex sets are known.

Theorem 2.5 ([15, 16]) *For $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ such that* dom$f$ *is nonempty and bounded,*

(1) *f is M-convex if and only if* $\arg\min f[p]$ *is M-convex for any $p \in \mathbf{R}^V$,*

(2) *f is $M^\natural$-convex if and only if* $\arg\min f[p]$ *is $M^\natural$-convex for any $p \in \mathbf{R}^V$.*

Several equivalent conditions to (M-EXC) are known [19, 20]. Here we introduce a local exchange property stated by

($\mathrm{M\text{-}EXC}_{\mathrm{loc}}$) for $x, y \in \mathrm{dom}f$ with $\|x-y\|_1 = 4$, there exist $u \in \mathrm{supp}^+(x-y)$ and $v \in \mathrm{supp}^-(x-y)$ such that

$$f(x) + f(y) \geq f(x - \chi_u + \chi_v) + f(y + \chi_u - \chi_v)$$

where $\|z\|_1$ denotes the ℓ_1 norm $\sum_{v \in V} |z(v)|$.

Theorem 2.6 ([19, 20]) *For $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ with* dom$f \neq \emptyset$, *f satisfies* (M-EXC) *if and only if* dom$f$ *is an M-convex set and f satisfies* ($\mathrm{M\text{-}EXC}_{\mathrm{loc}}$).

3 New characterizations of M/M$^\natural$-convexity

We give our new characterizations of M/M$^\natural$-convex functions. For this, we introduce four properties of functions:

(M-SI) for $p \in \mathbf{R}^V$ and $x, y \in \mathrm{dom} f$ with $f[p](x) > f[p](y)$,

$$f[p](x) > \min_{u \in \mathrm{supp}^+(x-y)} \min_{v \in \mathrm{supp}^-(x-y)} f[p](x - \chi_u + \chi_v),$$

(M$^\natural$-SI) for $p \in \mathbf{R}^V$ and $x, y \in \mathrm{dom} f$ with $f[p](x) > f[p](y)$,

$$f[p](x) > \min_{u \in \mathrm{supp}^+(x-y) \cup \{0\}} \min_{v \in \mathrm{supp}^-(x-y) \cup \{0\}} f[p](x - \chi_u + \chi_v),$$

(M$^\natural$-GS$_{\mathrm{W}}$) for $p, q \in \mathbf{R}^V$ and $x \in \mathrm{dom} f$ such that $x \in \arg\min f[p]$, $p \leq q$ and $\arg\min f[q] \neq \emptyset$, there exists y such that

$$\begin{aligned} &y \in \arg\min f[q], \\ &y(v) \geq x(v) \quad \text{if } p(v) = q(v), \end{aligned}$$

(M$^\natural$-GS) for $(p_0, p), (q_0, q) \in \mathbf{R}^{\{0\} \cup V}$ and $x \in \mathrm{dom} f$ such that $x \in \arg\min f[p - p_0 \chi_V]$, $(p_0, p) \leq (q_0, q)$ and $\arg\min f[q - q_0 \chi_V] \neq \emptyset$, there exists y such that

$$\begin{aligned} &y \in \arg\min f[q - q_0 \chi_V], \\ &y(v) \geq x(v) \quad \text{if } p(v) = q(v), \\ &y(V) \leq x(V) \quad \text{if } p_0 = q_0. \end{aligned}$$

Assume that two functions $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ and $\hat{f} : \mathbf{Z}^{\{0\} \cup V} \to \mathbf{R} \cup \{+\infty\}$ are transformed to each other by (2.2) and (2.3). By easy calculations, we have the following relations:

$$f : (\mathrm{M}^\natural\text{-SI}) \iff \hat{f} : (\mathrm{M}\text{-SI}) \implies \hat{f} : (\mathrm{M}^\natural\text{-SI}), \tag{3.1}$$

$$f : (\mathrm{M}^\natural\text{-GS}) \iff \hat{f} : (\mathrm{M}^\natural\text{-GS}_{\mathrm{W}}), \tag{3.2}$$

$$f : (\mathrm{M}^\natural\text{-GS}) \implies f : (\mathrm{M}^\natural\text{-GS}_{\mathrm{W}}), \tag{3.3}$$

$$\hat{f} : (\mathrm{M}^\natural\text{-GS}) \iff \hat{f} : (\mathrm{M}^\natural\text{-GS}_{\mathrm{W}}), \tag{3.4}$$

where (3.4) means that if the effective domain lies on the hyperplane $\{x \in \mathbf{R}^V \mid x(V) = c\}$ for some $c \in \mathbf{Z}$, then (M$^\natural$-GS) and (M$^\natural$-GS$_{\mathrm{W}}$) are equivalent.

Our main results are the following theorems.

Theorem 3.1 *For $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ with $\mathrm{dom} f \neq \emptyset$,*

(1) *f is M-convex if and only if it satisfies* (M-SI),

(2) *f is $M^\natural$-convex if and only if it satisfies* ($M^\natural$-SI).

Theorem 3.2

(1) *An M-convex function f satisfies* ($M^\natural$-GS_W).

(2) *An $M^\natural$-convex function f satisfies* ($M^\natural$-GS).

The converse of Theorem 3.2 also holds under some conditions.

Theorem 3.3 *For $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ such that f is convex-extendible and* dom$f$ *is a nonempty bounded set, the following statements hold:*

(1) *if f satisfies* ($M^\natural$-GS_W) *and* dom$f \subseteq \{x \in \mathbf{R}^V \mid x(V) = c\}$ *for some $c \in \mathbf{Z}$, then f is M-convex,*

(2) *if f satisfies* ($M^\natural$-GS), *then f is $M^\natural$-convex.*

Combining Theorems 3.2 and 3.3, we obtain characterizations of M/$M^\natural$-convex functions using ($M^\natural$-GS_W) and ($M^\natural$-GS).

Theorem 3.4 *For $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ such that f is convex-extendible and* dom$f$ *is a nonempty bounded set on the hyperplane $H = \{x \in \mathbf{R}^V \mid x(V) = c\}$ for some $c \in \mathbf{Z}$, f is M-convex if and only if f satisfies* ($M^\natural$-GS_W).

Theorem 3.5 *For $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ such that f is convex-extendible and* dom$f$ *is nonempty and bounded, f is $M^\natural$-convex if and only if f satisfies* ($M^\natural$-GS).

For functions defined on hypercubes, we can simplify assumptions in Theorem 3.5 and replace ($M^\natural$-SI) in Theorem 3.1 with a weaker condition.

Theorem 3.6 *For $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ with* dom$f = \{0,1\}^V$, *the following statements hold:*

(1) *if f satisfies* ($M^\natural$-GS_W) *then f also satisfies* ($M^\natural$-GS),

(2) *if f satisfies* ($M^\natural$-GS_W) *then f is $M^\natural$-convex.*

(3) *f satisfies* ($M^\natural$-SI) *if and only if it does* ($M^\natural$-SI_W), *where*

($M^\natural$-SI_W) *for $p \in \mathbf{R}^V$ and $x \in$* dom$f$ *with $x \notin \arg\min f[p]$, there exist $u, v \in V \cup \{0\}$ such that $f[p](x) > f[p](x - \chi_u + \chi_v)$.*

Combination of Theorems 3.1, 3.2, 3.6 and (3.3) yield the following result as an immediate consequence.

Corollary 3.7 *For* $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ *with* $\mathrm{dom} f = \{0,1\}^V$, *we have*

$$(\mathrm{M}^\natural\text{-EXC}) \iff (\mathrm{M}^\natural\text{-SI}) \iff (\mathrm{M}^\natural\text{-SI}_\mathrm{W}) \iff (\mathrm{M}^\natural\text{-GS}) \iff (\mathrm{M}^\natural\text{-GS}_\mathrm{W}).$$

Remark 3.8 The converse of (3.3) does not hold. For instance, we can easily verify that the function defined by

$$f(x) = \begin{cases} 0 & \text{if } x = (0,0) \text{ or } (1,0) \text{ or } (0,1) \\ -1 & \text{if } x = (1,-1) \\ +\infty & \text{otherwise} \end{cases}$$

satisfies ($\mathrm{M}^\natural$-GS_W) and not ($\mathrm{M}^\natural$-GS). Since f is convex-extendible and $\mathrm{dom} f$ is nonempty bounded, this example also shows that ($\mathrm{M}^\natural$-GS) in Theorem 3.5 is not replaced with ($\mathrm{M}^\natural$-GS_W). ∎

Remark 3.9 The equivalence between ($\mathrm{M}^\natural$-SI) and ($\mathrm{M}^\natural$-SI_W) does not hold in general. For example, let us consider the function defined by

$$f(x) = \begin{cases} 0 & \text{if } x = (1,0) \\ 1 & \text{if } x = (1,-1) \text{ or } (0,1) \\ 3 & \text{if } x = (-1,1) \text{ or } (0,-1) \\ 4 & \text{if } x = (-1,0) \\ 10 & \text{if } x = (0,0) \\ +\infty & \text{otherwise.} \end{cases}$$

By an exhaustive check, f satisfies ($\mathrm{M}^\natural$-SI_W). However, ($\mathrm{M}^\natural$-SI) for $p = (0,0), x = (-1,0), y = (1,0)$ does not hold. ∎

Remark 3.10 Condition ($\mathrm{M}^\natural$-GS_W) is related to greedy algorithms. For the sake of simplicity, let us consider the case where $f : \{0,1\}^V \to \{0,+\infty\}$ and $\mathrm{dom} f$ is nonempty and lower monotone (i.e., $x \leq y \in \mathrm{dom} f \Longrightarrow x \in \mathrm{dom} f$), that is, where $\mathrm{dom} f$ is associated with an independence system on V. It is well-known that $\mathrm{dom} f$ is (associated with) the family of independent sets of a matroid if and only if the ordinary greedy algorithm finds an optimal solution of the problem $\max\{\langle p, x\rangle \mid x \in \mathrm{dom} f\}$ for any $p \in \mathbf{R}^V$ (more specifically, for any integral nonnegative weight vector p). From the procedure of the greedy algorithm, if $\mathrm{dom} f$ is a matroid, then we have the property

$$\begin{aligned} &p \geq q,\ \chi_X \in \arg\max_x \langle p, x\rangle \\ &\quad \Longrightarrow \exists \chi_Y \in \arg\max_x \langle q, x\rangle : \{i \in X \mid p(i) = q(i)\} \subseteq Y \end{aligned}$$

which is a restatement of ($\mathrm{M}^\natural$-GS_W) in the above case. ∎

Remark 3.11 The relationship between (M-EXC) and (M-SI) is extended to the class of quasi M-convex functions in [24]. ∎

4 Implications in mathematical economics

In this section, we discuss some implications of our results in the context of mathematical economics. Specifically, we explain that the conditions ($M^\natural$-GS) and ($M^\natural$-SI) treated in Section 3 are natural generalizations of the gross substitutes condition and the single improvement condition for utility functions (set functions) investigated by Kelso and Crawford [13] and Gul and Stacchetti [12]. The role of $M^\natural$-convex functions for the existence of a competitive equilibrium in an exchange economy with indivisible goods is also discussed.

First we describe only briefly a competitive exchange economy with indivisible goods and money, in which money is treated as a divisible good. Let H be a finite set of agents and K be a finite set of indivisible goods. Each agent $h \in H$ has an initial endowment of indivisible goods and money $(x_h^\circ, m_h^\circ) \in \mathbf{Z}_+^K \times \mathbf{R}_+$, where $x_h^\circ(k)$ denotes the number of goods $k \in K$ and m_h° the amount of money in his/her initial endowment. Here $\mathbf{Z}_+$ and $\mathbf{R}_+$ denote the sets of nonnegative integers and reals, respectively. Given a price vector $p \in \mathbf{R}_+^K$, each agent independently wants to maximize his/her profit or utility by buying or selling several goods. We assume that agent h's utility is represented by a quasilinear (or transferable) utility function $\bar{U}_h : \mathbf{Z}_+^K \times \mathbf{R}_+ \to \mathbf{R} \cup \{-\infty\}$ defined by

$$\bar{U}_h(x, m) = U_h(x) + m \qquad ((x, m) \in \mathbf{Z}_+^K \times \mathbf{R}_+)$$

where $U_h : \mathbf{Z}^K \to \mathbf{R} \cup \{-\infty\}$ with $\mathrm{dom}U_h \subseteq \mathbf{Z}_+^K$, and $\mathrm{dom}U_h$ is defined by (2.1). Function U_h is called a reservation value function. It is natural to assume that $\mathrm{dom}U_h$ is bounded because none can have an infinite number of indivisible goods. Under a certain condition [1], the maximization of $\bar{U}_h$ subject to the budget constraint can be reduced to the maximization of $U_h(x) - \langle p, x\rangle$. Thus, we can define h's demand function (correspondence) $D_h : \mathbf{R}_+^K \to 2^{\mathbf{Z}_+^K}$ by

$$D_h(p) = \arg\max_{x \in \mathbf{Z}^K}[U_h(x) - \langle p, x\rangle] \qquad (p \in \mathbf{R}_+^K)$$

and h's demand set by $D_h(p) \subseteq \mathbf{Z}_+^K$. A tuple $((x_h : h \in H), p)$, where $x_h \in \mathbf{Z}_+^K$ for $h \in H$ and $p \in \mathbf{R}_+^K$, is called an *equilibrium* or a *competitive equilibrium* if the following conditions hold:

$$x_h \in D_h(p) \qquad (h \in H),$$
$$\sum_{h \in H} x_h^\circ = \sum_{h \in H} x_h.$$

That is, each agent can achieve what he/she wishes to achieve, and the balance of supply and demand holds.

[1] For example, U_h : concave-extendible, $\mathrm{dom}U_h$: bounded, m_h° : sufficiently large.

Since a utility function is usually assumed as to be concave in mathematical economics, a "discrete concave" function is natural in this context. Here we adopt an $\mathrm{M}^\natural$-concave function, which is the negative of an $\mathrm{M}^\natural$-convex function defined in Section 2. We first rewrite our results in Section 3 in terms of $\mathrm{M}^\natural$-concave functions.

A function $U : \mathbf{Z}^K \to \mathbf{R} \cup \{-\infty\}$ with $\mathrm{dom}U \neq \emptyset$ is called an *$M^\natural$-concave function* if it satisfies

($-\mathrm{M}^\natural$-EXC) for $x, y \in \mathrm{dom}U$ and $i \in \mathrm{supp}^+(x-y)$,

$$U(x) + U(y) \leq \max_{j \in \mathrm{supp}^-(x-y) \cup \{0\}} \{U(x - \chi_i + \chi_j) + U(y + \chi_i - \chi_j)\}.$$

It is known [23] that an $\mathrm{M}^\natural$-concave function U is submodular, i.e.,

$$U(x) + U(y) \geq U(x \vee y) + U(x \wedge y) \qquad (x, y \in \mathrm{dom}U),$$

where vectors $x \vee y$ and $x \wedge y$ are defined by

$$(x \vee y)(v) = \max\{x(v), y(v)\}, \quad (x \wedge y)(v) = \min\{x(v), y(v)\} \qquad (v \in V).$$

We also consider the concave versions of the other properties ($\mathrm{M}^\natural$-$\mathrm{GS_W}$), ($\mathrm{M}^\natural$-GS), ($\mathrm{M}^\natural$-SI) and ($\mathrm{M}^\natural$-$\mathrm{SI_W}$) considered in Section 3:

($-\mathrm{M}^\natural$-$\mathrm{GS_W}$) for $p, q \in \mathbf{R}^K$ and $x \in \mathrm{dom}U$ such that $x \in \arg\max U[-p]$, $p \leq q$ and $\arg\max U[-q] \neq \emptyset$, there exists y such that

$$\begin{aligned} &y \in \arg\max U[-q], \\ &y(i) \geq x(i) \quad \text{if } p(i) = q(i), \end{aligned}$$

($-\mathrm{M}^\natural$-GS) for $(p_0, p), (q_0, q) \in \mathbf{R}^{\{0\} \cup K}$ and $x \in \mathrm{dom}U$ such that $x \in \arg\max U[-p + p_0\chi_K]$, $(p_0, p) \leq (q_0, q)$ and $\arg\max U[-q + q_0\chi_K] \neq \emptyset$, there exists y such that

$$\begin{aligned} &y \in \arg\max U[-q + q_0\chi_K], \\ &y(i) \geq x(i) \quad \text{if } p(i) = q(i), \\ &y(K) \leq x(K) \quad \text{if } p_0 = q_0, \end{aligned}$$

($-\mathrm{M}^\natural$-SI) for $p \in \mathbf{R}^K$ and $x, y \in \mathrm{dom}U$ with $U[-p](x) < U[-p](y)$,

$$U[-p](x) < \max_{i \in \mathrm{supp}^+(x-y) \cup \{0\}} \max_{j \in \mathrm{supp}^-(x-y) \cup \{0\}} U[-p](x - \chi_i + \chi_j).$$

($-\mathrm{M}^\natural$-$\mathrm{SI_W}$) for $p \in \mathbf{R}^K$ and $x \in \mathrm{dom}U$ with $x \notin \arg\max U[-p]$, there exist $i, j \in K \cup \{0\}$ such that $U[-p](x) < U[-p](x - \chi_i + \chi_j)$.

We are primarily interested in the case where $\mathrm{dom}U$ is bounded. Then $\arg\max U[-q]$ is nonempty for any $q \in \mathbf{R}^K$.

Theorem 4.1 *For a function $U : \mathbf{Z}^K \to \mathbf{R} \cup \{-\infty\}$ such that $\mathrm{dom}U$ is nonempty and bounded, the following statements are equivalent:*

(1) *U satisfies* ($-$M$^\natural$-EXC)*, that is, U is $M^\natural$-concave,*

(2) *U satisfies* ($-$M$^\natural$-GS) *and is concave-extendible (i.e., $-U$ is convex-extendible),*

(3) *U satisfies* ($-$M$^\natural$-SI)*.*

Corollary 4.2 *For a function $U : \mathbf{Z}^K \to \mathbf{R} \cup \{-\infty\}$ with $\mathrm{dom}U = \{0,1\}^K$, we have*

$$(-\mathrm{M}^\natural\text{-EXC}) \Longleftrightarrow (-\mathrm{M}^\natural\text{-SI}) \Longleftrightarrow (-\mathrm{M}^\natural\text{-SI}_\mathrm{W}) \Longleftrightarrow (-\mathrm{M}^\natural\text{-GS}) \Longleftrightarrow (-\mathrm{M}^\natural\text{-GS}_\mathrm{W}).$$

The conditions ($-$M$^\natural$-GS) and ($-$M$^\natural$-SI) above are generalizations of the gross substitutes condition (GS) and the single improvement condition (SI) considered in [13] and [12]. We now describe these conditions and discuss the relationship between the results of [12] and Theorem 4.1 (Corollary 4.2).

The gross substitutes condition was originally proposed for set functions by Kelso and Crawford [13]. A function $\tilde{U} : 2^K \to \mathbf{R}$ is said to satisfy the *gross substitutes condition* if

(GS) for $p, q \in \mathbf{R}^K$ with $p \leq q$ and $X \in \arg\max \tilde{U}[-p]$, there exists $Y \in \arg\max \tilde{U}[-q]$ such that $\{i \in X \mid p(i) = q(i)\} \subseteq Y$.

Furthermore, Gul and Stacchetti [12] proposed two other conditions (SI) and (NC). A function $\tilde{U} : 2^K \to \mathbf{R}$ is said to satisfy the *single improvement condition* if

(SI) for $p \in \mathbf{R}^K$ and $X \notin \arg\max \tilde{U}[-p]$, there exists $Y \subseteq K$ such that $|X \backslash Y| \leq 1$, $|Y \setminus X| \leq 1$ and $\tilde{U}[-p](X) < \tilde{U}[-p](Y)$.

A function $\tilde{U} : 2^K \to \mathbf{R}$ has *no complementarities* if

(NC) for $p \in \mathbf{R}^K$, $X, Y \in \arg\max \tilde{U}[-p]$ and $I \subseteq X \setminus Y$, there exists $J \subseteq Y \setminus X$ such that $(X \setminus I) \cup J \in \arg\max \tilde{U}[-p]$.

They proved the equivalence among those conditions, i.e.,

$$(\mathrm{GS}) \Longleftrightarrow (\mathrm{SI}) \Longleftrightarrow (\mathrm{NC}). \tag{4.1}$$

The equivalence (4.1) can be interpreted as a special case of our result. We consider a function $U : \mathbf{Z}^K \to \mathbf{R} \cup \{-\infty\}$ with $\mathrm{dom}U = \{0,1\}^K$, which can be identified with a set function $\tilde{U} : 2^K \to \mathbf{R}$ given by $\tilde{U}(X) = U(\chi_X)$ $(X \subseteq K)$.

Conditions ($-$M$^\natural$-GS$_{\rm W}$) and ($-$M$^\natural$-SI$_{\rm W}$) for U correspond to (GS) and (SI) for the associated set function $\tilde{U}$, respectively. By Corollary 4.2, we have

$$\tilde{U} : (\mathrm{GS}) \iff U : \mathrm{M}^\natural\text{-concave} \iff \tilde{U} : (\mathrm{SI}). \tag{4.2}$$

On the other hand, we have

$$\begin{aligned} \tilde{U} : (\mathrm{NC}) &\iff \arg\max U[-p] : \mathrm{M}^\natural\text{-convex set for any } p \in \mathbf{R}^K \\ &\iff U : \mathrm{M}^\natural\text{-concave}, \end{aligned} \tag{4.3}$$

where the first equivalence follows from the multiple basis exchange property of matroids [28, Theorem 4.3.1] and the second one from the concave version of (2) of Theorem 2.5. Hence we obtain (4.1). We note that the equivalence between (SI) and M$^\natural$-concavity for set functions was first pointed out by Fujishige and Yang [10, 11].

Moreover, Gul and Stacchetti [12] proposed a property, called *strong no complementarities* for a set function $\tilde{U} : 2^K \to \mathbf{R}$ given by

(SNC) for $X, Y \subseteq K$ and $I \subseteq X \setminus Y$, there exists $J \subseteq Y \setminus X$ such that

$$\tilde{U}(X) + \tilde{U}(Y) \leq \tilde{U}((X \setminus I) \cup J) + \tilde{U}((Y \setminus J) \cup I),$$

and observed that (SNC) is stronger than (NC). Condition (SNC) with the restriction $|I| = 1$ is essentially the same as the exchange property ($-$M$^\natural$-EXC) for the function U with $\mathrm{dom}U = \{0,1\}^K$ associated with $\tilde{U}$.

We finally return to an exchange economy with indivisible goods. A function $U : \mathbf{Z}^K \to \mathbf{R} \cup \{-\infty\}$ is said to be *monotone nondecreasing* if $x \leq y \Rightarrow U(x) \leq U(y)$ for any $x, y \in \mathrm{dom}U$. The following theorem is implied by the results of [1, 2].

Theorem 4.3 *In an exchange economy defined by indivisible goods set K, agent set H and reservation value functions $(U_h : h \in H)$, where $U_h : \mathbf{Z}^K \to \mathbf{R} \cup \{-\infty\}$ with $\mathrm{dom}U_h \subseteq \mathbf{Z}^K_+$, if each U_h is monotone nondecreasing and M$^\natural$-concave, then there exists a competitive equilibrium $((x_h : h \in H), p)$ for any initial total endowment $x^\circ \in \sum_{h \in H} \mathrm{dom}U_h$, where the summation means the Minkowski sum.*

Kelso and Crawford [13] and Gul and Stacchetti [12] showed the existence of an equilibrium under the condition (GS). By Corollary 4.2 as well as (4.2) and (4.3), we see that Theorem 4.3 above is a generalization of those results.

5 Proofs

In this section, we prove Theorems 3.1, 3.2, 3.3 and 3.6.

5.1 Proof of Theorem 3.1

By the relations (2.4) and (3.1), it is sufficient to prove the assertion (1) of Theorem 3.1, We first prove the "only if" part of (1).

Lemma 5.1 *An M-convex function f satisfies* (M-SI).

Proof. Here we verify the assertion in the case where $p = 0$, because if f is M-convex then $f[p]$ is obviously M-convex for any $p \in \mathbf{R}^V$. Let $x, y \in \mathrm{dom} f$ with $f(x) > f(y)$. From (M-EXC), for any $u_1 \in \mathrm{supp}^+(x - y)$, there exists $v_1 \in \mathrm{supp}^-(x - y)$ such that

$$f(y) \geq [f(x - \chi_{u_1} + \chi_{v_1}) - f(x)] + f(y_2)$$

where $y_2 = y + \chi_{u_1} - \chi_{v_1}$. By applying (M-EXC) again to x, y_2, there exist $u_2 \in \mathrm{supp}^+(x - y_2)$ and $v_2 \in \mathrm{supp}^-(x - y_2)$ such that

$$f(y_2) \geq [f(x - \chi_{u_2} + \chi_{v_2}) - f(x)] + f(y_3)$$

where $y_3 = y + \chi_{u_1} + \chi_{u_2} - \chi_{v_1} - \chi_{v_2}$. By repeating the above discussion $m = \|x - y\|_1/2$ times, we obtain that $y_m = y + \sum_{i=1}^{m}(\chi_{u_i} - \chi_{v_i}) = x$ and

$$f(x) > f(y) \geq f(x) + \sum_{i=1}^{m}[f(x - \chi_{u_i} + \chi_{v_i}) - f(x)].$$

Hence, $0 > f(x - \chi_{u_i} + \chi_{v_i}) - f(x)$ holds for some i. ∎

Proof of (1) of Theorem 3.1. By Lemma 5.1, the "only if" part holds. We will show the "if" part by using Theorem 2.6, that is, by proving that $B = \mathrm{dom} f$ is an M-convex set and f satisfies ($\mathrm{M\text{-}EXC}_{\mathrm{loc}}$).

We first show that B satisfies (B-EXC$_-$) (see Theorem 2.4). Let $x, y \in B$ and $u \in \mathrm{supp}^+(x - y)$. For a sufficiently large number $L > 0$, we define $p \in \mathbf{R}^V$ by

$$p(v) = \begin{cases} L^2 & \text{if } v = u \\ L & \text{if } v \in \mathrm{supp}^-(x - y) \\ 0 & \text{otherwise.} \end{cases}$$

Then, we have $f[p](x) > f[p](y)$. By (M-SI), we can select $w \in \mathrm{supp}^+(x - y)$ and $v \in \mathrm{supp}^-(x - y)$ such that

$$0 < f[p](x) - f[p](x - \chi_w + \chi_v) = f(x) - f(x - \chi_w + \chi_v) + p(w) - L.$$

Since L is sufficiently large, we must have $w = u$. Hence $x - \chi_u + \chi_v \in B$ which shows (B-EXC$_-$).

Suppose, to the contrary, that (M-EXC$_{\rm loc}$) does not hold for $x, y \in B$. Since $\|x-y\|_1 = 4$, we can write $y = x-\chi_{u_1}-\chi_{u_2}+\chi_{v_1}+\chi_{v_2}$ with $\{u_1,u_2\}\cap\{v_1,v_2\} = \emptyset$. From the assumption, we have

$$f(y) - f(x) < \min \left\{ \begin{array}{l} [f(x-\chi_{u_1}+\chi_{v_1})-f(x)] + [f(x-\chi_{u_2}+\chi_{v_2})-f(x)], \\ [f(x-\chi_{u_1}+\chi_{v_2})-f(x)] + [f(x-\chi_{u_2}+\chi_{v_1})-f(x)] \end{array} \right\}. \quad (5.1)$$

Here we consider the case where $u_1 \neq u_2$ and $v_1 \neq v_2$, because we can similarly prove the assertion in other cases. We define a bipartite graph $G = (V^+, V^-; E)$ with edge weight c by

$$\begin{aligned} & V^+ = \{u_1, u_2\}, \quad V^- = \{v_1, v_2\}, \\ & E = \{(u_i, v_j) \mid f(x-\chi_{u_i}+\chi_{v_j})-f(x) < +\infty\}, \\ & c(u_i, v_j) = f(x-\chi_{u_i}+\chi_{v_j})-f(x). \end{aligned}$$

Since B satisfies (B-EXC), G has a perfect matching. From the duality of linear programming, there exists $p \in \mathbf{R}^{V^+\cup V^-}$ such that $p(u_1) + p(u_2) - p(v_1) - p(v_2)$ is equal to the right hand side of (5.1) and

$$f(x-\chi_{u_i}+\chi_{v_j})-f(x) = c(u_i, v_j) \geq p(u_i) - p(v_j) \quad (i, j = 1, 2). \quad (5.2)$$

From (5.1) and (5.2), we have

$$f[p](x) > f[p](y), \qquad f[p](x) \leq f[p](x - \chi_{u_i} + \chi_{v_j}) \quad (i, j = 1, 2).$$

This, however, contradicts the assumption that f satisfies (M-SI). Hence, (M-EXC$_{\rm loc}$) holds. ∎

5.2 Proof of Theorem 3.2

Here we prove the assertion (1) of Theorem 3.2. The assertion (2) follows from (1), (2.4) and (3.2).

Proof of (1) of Theorem 3.2. Let $x \in \arg\min f[p]$ and $y \in \arg\min f[q]$, and suppose y minimizes $||y - x||_1$ among all vectors in $\arg\min f[q]$. Suppose, to the contrary, that there exists some $u \in V$ such that $p(u) = q(u)$ and $x(u) > y(u)$. We apply (M-EXC) to x, y, and $u \in \mathrm{supp}^+(x - y)$ to obtain some $v \in \mathrm{supp}^-(x - y)$ such that

$$f(x) + f(y) \geq f(x - \chi_u + \chi_v) + f(y + \chi_u - \chi_v). \quad (5.3)$$

Since $x \in \arg\min f[p]$ and $y \in \arg\min f[q]$, we have

$$f[p](x) \leq f[p](x - \chi_u + \chi_v), \qquad f[q](y) \leq f[q](y + \chi_u - \chi_v). \quad (5.4)$$

It follows from (5.4) that

$$f(x-\chi_u+\chi_v) \geq f(x)+p(u)-p(v), \quad f(y+\chi_u-\chi_v) \geq f(y)-q(u)+q(v). \tag{5.5}$$

From (5.3) and (5.5), we have

$$f(x)+f(y) \geq f(x)+f(y)+\{p(u)-q(u)\}+\{q(v)-p(v)\} \geq f(x)+f(y).$$

Thus, all inequalities in (5.3), (5.4) and (5.5) hold as equality. Hence we have that $y+\chi_u-\chi_v \in \arg\min f[q]$. This, however, contradicts the choice of y. ∎

5.3 Proof of Theorem 3.3

By (2.4) and (3.2), it is enough to show the assertion (1) of Theorem 3.3.

Proof of (1) of Theorem 3.3. We will prove the M-convexity of f by using (1) of Theorem 2.5. It is sufficient to show that $B = \arg\min f$ is an M-convex set, because if f satisfies ($\mathrm{M}^\natural$-GS_W) then $f[p]$ also does for any $p \in \mathbf{R}^V$.

Since an M-convex set coincides with the set of integer points of the base polyhedron of an integral submodular system, we will show that $B = \overline{B} \cap \mathbf{Z}^V$ and $\overline{B}$ is the base polyhedron of a submodular system, where $\overline{B}$ denotes the convex hull of B. The first condition $B = \overline{B} \cap \mathbf{Z}^V$ follows from the fact that f is convex-extendible. To prove the second condition that $\overline{B}$ is a base polyhedron, Theorem 2.3 is applied. Let E be any edge of $\overline{B}$. Since $B = \arg\min f$ and E is a face of $\overline{B}$, there exists $p \in \mathbf{R}^V$ such that $E \cap \mathbf{Z}^V = \arg\min f[p]$. For any distinct x and y of $E \cap \mathbf{Z}^V$, $|\mathrm{supp}^+(x-y)| \geq 1$ and $|\mathrm{supp}^-(x-y)| \geq 1$ hold, because $B \subseteq \{z \in \mathbf{R}^V \mid z(V) = c\}$. Fix $u \in \mathrm{supp}^+(x-y)$ and put $q = p + \varepsilon\chi_u$ for a sufficiently small positive number ε. From ($\mathrm{M}^\natural$-GS_W) for p, q and x, there exists $\bar{y} \in \arg\min f[q]$ such that

$$\bar{y}(v) \geq x(v) \qquad (\forall v \neq u). \tag{5.6}$$

Since $f[q](y) < f[q](x)$, we have $x \notin \arg\min f[q]$, and hence, $x \neq \bar{y}$. Since $\mathrm{dom}f$ is bounded and ε is sufficiently small, $\bar{y} \in \arg\min f[p]$ (i.e., $\bar{y} \in E \cap \mathbf{Z}^V$) must hold. Then $x-y$ is a multiple of $x-\bar{y}$. By (5.6), we have $|\mathrm{supp}^+(x-y)| = |\mathrm{supp}^+(x-\bar{y})| = 1$. Analogously, we can prove that $|\mathrm{supp}^-(x-y)| = 1$. Put $\mathrm{supp}^-(x-y) = \{v\}$. From the fact that $x(V) = y(V)$, $x-y$ is a multiple of $\chi_u - \chi_v$. Hence, B is an M-convex set. ∎

5.4 Proof of Theorem 3.6

The assertions (1) and (3) of Theorem 3.6 are proven in Lemmas 5.3 and 5.4, respectively. If $\mathrm{dom}f = \{0,1\}^V$, f is convex-extendible. Thus, the assertion (2) of Theorem 3.6 follows from Theorem 3.5 and (1) of Theorem 3.6.

Lemma 5.2 *We assume that* $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ *with* $\mathrm{dom} f = \{0,1\}^V$ *satisfies* ($\mathrm{M}^\natural$-GS_W). *Then for* $p, q \in \mathbf{R}^V$ *with* $p \leq q$ *and* $x \in \arg\min f[p]$, *there exists* $y \in \arg\min f[q]$ *such that* $y(V) \leq x(V)$ *and* $y(v) \geq x(v)$ *for any* $v \in V$ *with* $p(v) = q(v)$.

Proof. It is sufficient to show the assertion in the case where $q = p + \lambda\chi_u$ with $\lambda > 0$ and $u \in V$. If $x \in \arg\min f[q]$ then we can take x for y. In the sequel, we assume that $x \notin \arg\min f[q]$. Since f satisfies ($\mathrm{M}^\natural$-GS_W), there exists y such that

$$y \in \arg\min f[q] \quad \text{and} \quad y(v) \geq x(v) \qquad (\forall v \neq u). \tag{5.7}$$

We note that

$$y(u) = 0 \quad \text{and} \quad x(u) = 1 \tag{5.8}$$

must hold because $x \notin \arg\min f[q]$, $y \in \arg\min f[q]$ and $f[q](z) = f[p](z) + \lambda z(u)$ for any z. In addition, we assume that

$$y \text{ minimizes } |\mathrm{supp}^+(y - x)| \text{ among the candidates for (5.7).} \tag{5.9}$$

We claim that $|\mathrm{supp}^+(y - x)| \leq 1$, which, combined with (5.8), implies $y(V) \leq x(V)$. To prove the claim we suppose, to the contrary, that $|\mathrm{supp}^+(y - x)| \geq 2$. By (5.8), there exists $\bar{\lambda} \in [0, \lambda)$ such that $x, y \in \arg\min f[q_1]$ where $q_1 = p + \bar{\lambda}\chi_u$. Since $\mathrm{dom} f = \{0,1\}^V$, for a sufficiently large number $L > 0$, we have $x, y \in \arg\min f[q_2]$ where $q_2 = q_1 + L\chi_R - L\chi_S$, $R = \{v \in V \mid x(v) = y(v) = 0\}$ and $S = \{v \in V \mid x(v) = y(v) = 1\}$. Furthermore, for a sufficiently small positive number ε, let $q_3 = q_2 + |\mathrm{supp}^+(y - x)|\varepsilon\chi_u + \varepsilon\chi_{\mathrm{supp}^+(y-x)}$. We have $f[q_3](x) = f[q_3](y)$ because of (5.8). By (5.9) and the definition of q_3, it is easy to show that

$$\{x, y\} = \arg\min f[q_3].$$

For a sufficiently small positive number $\bar{\varepsilon} < \varepsilon$ and any $w \in \mathrm{supp}^+(y - x)$, then we have

$$\{x\} = \arg\min f[q_3 + \bar{\varepsilon}\chi_w]. \tag{5.10}$$

Hence $|\mathrm{supp}^+(y - x)| \leq 1$ holds; since otherwise, (5.10) contradicts that f satisfies ($\mathrm{M}^\natural$-GS_W) (for q_3, $q_3 + \bar{\varepsilon}\chi_w$ and y). This establishes the claim. ∎

Lemma 5.3 *If a function* $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ *with* $\mathrm{dom} f = \{0,1\}^V$ *satisfies* ($\mathrm{M}^\natural$-GS_W), *then it also satisfies* ($\mathrm{M}^\natural$-GS).

Proof. It is enough to show that f satisfies ($\mathrm{M}^\natural$-GS) in the cases where

Case 1: $q_0 = p_0 + \lambda$ and $q = p$ with $\lambda > 0$, or

Case 2: $q_0 = p_0$ and $q = p + \lambda\chi_u$ with $\lambda > 0$ and $u \in V$.

Case 2 has been treated in Lemma 5.2. We will consider Case 1 below.

Let $x \in \arg\min f[p - p_0\chi_V]$. If $x \in \arg\min f[p - q_0\chi_V]$ then we can take x for y. We assume that $x \notin \arg\min f[p - q_0\chi_V]$ and that

$$\bar{\lambda} = \max\{\mu \in [0, \lambda) \mid x \in \arg\min f[p - (p_0 + \mu)\chi_V]\}.$$

We prove below the claim:

Claim: there exists $y \in \arg\min f[p - (p_0 + \bar{\lambda})\chi_V]$ such that $y(V) > x(V)$ and $y(v) \geq x(v)$ for $v \in V$.

By using the claim while increasing $\bar{\lambda}$ as a parameter, we can prove the assertion in Case 1 from the strict monotonicity of $y(V)$.

The proof of the above claim follows. Let $q_1 = p - (p_0 + \bar{\lambda})\chi_V$. Since $x \notin \arg\min f[p - (p_0 + \lambda')\chi_V]$ for any $\lambda' > \lambda$ and $\mathrm{dom} f$ is bounded, there exists y such that

$$y \in \arg\min f[q_1] \quad \text{and} \quad y(V) > x(V). \tag{5.11}$$

In addition, we assume that

$$y \text{ minimizes } |\mathrm{supp}^+(x - y)| \text{ among the candidates for (5.11).} \tag{5.12}$$

The claim follows from $\mathrm{supp}^+(x - y) = \emptyset$, which can be shown as follows. Suppose, to the contrary, that

$$\mathrm{supp}^+(x - y) \neq \emptyset. \tag{5.13}$$

Since $\mathrm{dom} f = \{0, 1\}^V$, for a sufficiently large number $L > 0$, we have $x, y \in \arg\min f[q_2]$ where $q_2 = q_1 + L\chi_R - L\chi_S$, $R = \{v \in V \mid x(v) = y(v) = 0\}$ and $S = \{v \in V \mid x(v) = y(v) = 1\}$. Let ε be a sufficiently small positive number and put

$$q_3 = q_2 + \varepsilon\chi_{\mathrm{supp}^-(x-y)} - \frac{|\mathrm{supp}^-(x - y)|\varepsilon}{y(V) - x(V)}\chi_V.$$

Then, we have $f[q_3](x) = f[q_3](y)$. We first show that $\{x, y\} = \arg\min f[q_3]$. Since ε is sufficiently small, all minimizers of $f[q_3]$ must belong to $\arg\min f[q_2]$. Let z be any element, if any, of $\arg\min f[q_2] \setminus \{x, y\}$. By (5.12), we can easily show that

$$z(V) < y(V) \quad \text{and} \quad \mathrm{supp}^+(x - z) = \mathrm{supp}^+(x - y). \tag{5.14}$$

By (5.14), we have $y(V) - z(V) = |\mathrm{supp}^-(x - y)| - |\mathrm{supp}^-(x - z)|$. Thus we obtain

$$\begin{aligned} & f[q_3](z) - f[q_3](y) \\ = \; & \left\{|\mathrm{supp}^-(x - z)| - |\mathrm{supp}^-(x - y)|\right\}\varepsilon - \frac{|\mathrm{supp}^-(x - y)|(z(V) - y(V))\varepsilon}{y(V) - x(V)} \\ = \; & \frac{|\mathrm{supp}^+(x - y)|\,\{y(V) - z(V)\}\,\varepsilon}{y(V) - x(V)} > 0, \end{aligned}$$

where the second equality follows from $y(V)-x(V) = |\mathrm{supp}^-(x-y)| - |\mathrm{supp}^+(x-y)|$ and the last inequality from (5.11), (5.13) and (5.14). This shows $\{x,y\} = \arg\min f[q_3]$. For a sufficiently small positive number $\bar{\varepsilon} < \varepsilon$ and $w \in \mathrm{supp}^-(x-y)$, we have

$$\{x\} = \arg\min f[q_3 + \bar{\varepsilon}\chi_w]. \tag{5.15}$$

On the other hand, (5.11) and (5.13) imply

$$|\mathrm{supp}^-(x-y)| \geq 2. \tag{5.16}$$

However, (5.15) and (5.16) contradict that f satisfies ($\mathrm{M}^\natural$-GS_W) (for q_3, $q_3 + \bar{\varepsilon}\chi_w$ and y). Hence, we must have $\mathrm{supp}^+(x-y) = \emptyset$, which in turn establishes the claim. ∎

Lemma 5.4 *For a function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ with $\mathrm{dom} f = \{0,1\}^V$, f satisfies* ($\mathrm{M}^\natural$-SI) *if and only if it does* ($\mathrm{M}^\natural$-SI_W).

Proof. The "only if" part is trivial. Let $x, y \in \mathrm{dom} f$ with $f[p](x) > f[p](y)$. For a sufficiently large number $L > 0$, we put $q = p + L\chi_R - L\chi_S$ where $R = \{v \in V \mid x(v) = y(v) = 0\}$ and $S = \{v \in V \mid x(v) = y(v) = 1\}$. Obviously, $f[q](x) > f[q](y)$ holds. Since L is sufficiently large, by ($\mathrm{M}^\natural$-SI_W) for $f[q]$ and x, there exists $u \in \mathrm{supp}^+(x-y) \cup \{0\}$ and $v \in \mathrm{supp}^-(x-y) \cup \{0\}$ such that $f[q](x) > f[q](x - \chi_u + \chi_v)$, that is, $f[p](x) > f[p](x - \chi_u + \chi_v)$. ∎

Acknowledgments

We thank Akiyoshi Shioura for his helpful comments.

References

[1] V. Danilov, G. Koshevoy, and K. Murota, Equilibria in economies with indivisible goods and money, RIMS Preprint 1204, Kyoto University, 1998.

[2] V. Danilov, G. Koshevoy, and K. Murota, Discrete convexity and equilibria in economies with indivisible goods and money, *Math. Social Sci.* to appear.

[3] A. W. M. Dress and W. Wenzel, Valuated matroid: A new look at the greedy algorithm, *Appl. Math. Lett.* **3** (1990) 33–35.

[4] A. W. M. Dress and W. Wenzel, Valuated matroids, *Adv. Math.* **93** (1992) 214–250.

[5] J. Edmonds, Submodular functions, matroids and certain polyhedra, in: R. Guy, H. Hanani, N. Sauer and J. Schönheim, eds., *Combinatorial Structures and Their Applications* (Gordon and Breach, New York, 1970) 69–87.

[6] A. Frank, Generalized polymatroids, in: A. Hajnal, L. Lovász and V. T. Sós, eds., *Finite and Infinite Sets, I* (North-Holland, Amsterdam, 1984) 285–294.

[7] A. Frank and É. Tardos, Generalized polymatroids and submodular flows, *Math. Programming* **42** (1988) 489–563.

[8] S. Fujishige, *Submodular Functions and Optimization*, Annals of Discrete Mathematics 47 (North-Holland, Amsterdam, 1991).

[9] S. Fujishige and K. Murota, Notes on L-/M-convex functions and the separation theorems, *Math. Programming* **88** (2000) 129–146.

[10] S. Fujishige and Z. Yang, Indivisibilities, money, and equilibrium, Discrete Mathematics and Systems Science Research Report, No. 00-08, Division of Systems Science, Graduate School of Engineering Science, Osaka University, 2000.

[11] S. Fujishige and Z. Yang, A note on Kelso and Crawford's gross substitutes condition, preprint, 2000.

[12] F. Gul and E. Stacchetti, Walrasian equilibrium with gross substitutes, *J. Econom. Theory* **87** (1999) 95–124.

[13] A. S. Kelso, Jr. and V. P. Crawford. Job matching, coalition formation, and gross substitutes, *Econometrica* **50** (1982) 1483–1504.

[14] L. Lovász, Submodular functions and convexity, in: A. Bachem, M. Grötschel and B. Korte, eds., *Mathematical Programming – The State of the Art*, (Springer-Verlag, Berlin, 1983) 235–257.

[15] K. Murota, Convexity and Steinitz's exchange property, *Adv. Math.* **124** (1996) 272–311.

[16] K. Murota, Discrete convex analysis, *Math. Programming* **83** (1998) 313–371.

[17] K. Murota, Discrete convex analysis — Exposition on conjugacy and duality, in: L. Lovász, A. Gyarfas, G. O. H. Katona, A. Recski and L. Szekely, eds., *Graph Theory and Combinatorial Biology* (The Janos Bolyai Mathematical Society, Budapest, 1999) 253–278.

[18] K. Murota, *Matrices and Matroids for Systems Analysis* (Springer-Verlag, Berlin, 2000).

[19] K. Murota, *Discrete Convex Analysis* (in Japanese) (Kyoritsu-Shuppan, Tokyo) to appear.

[20] K. Murota, *Discrete Convex Analysis*, in preparation.

[21] K. Murota and A. Shioura, M-convex function on generalized polymatroid, *Math. Oper. Res.* **24** (1999) 95–105.

[22] K. Murota and A. Shioura, Extension of M-convexity and L-convexity to polyhedral convex functions, *Adv. in Appl. Math.* **25** (2000) 352–427.

[23] K. Murota and A. Shioura, Relationship of M-/L-convex functions with discrete convex functions by Miller and by Favati–Tardella, *Discrete Appl. Math.* to appear.

[24] K. Murota and A. Shioura, Quasi M-convex and L-convex functions: Quasi-convexity in discrete optimization, RIMS Preprint 1306, Kyoto University, 2000.

[25] R. T. Rockafellar, *Convex Analysis* (Princeton University Press, Princeton, 1970).

[26] A. Shioura, Minimization of an M-convex function, *Discrete Appl. Math.* **84** (1998) 215–220.

[27] N. Tomizawa, Theory of hyperspace XVI — On the structures of hedrons (in Japanese), Papers of the Technical Groups on Circuits and Systems, Institute of Electronics and Communication Engineers of Japan, CAS82-174, 2000.

[28] N. White (ed.), *Theory of Matroids* (Cambridge University Press, London, 1986).