ELLIPTIC LIE ALGEBRAS (INHOMOGENEOUS CASES)

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Abstract. Associated to each elliptic root system, we introduce a Lie algebra as an amalgamation of an affine Kac-Moody algebra and a generalization of a Heisenberg algebra. The derived algebra of the amalgamation algebra is shown to be the universal central extension of a certain twisted loop algebra of two variables.

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1. Introduction

We first explain briefly the geometric background of the present article (see [S2] and [S3] for detail). Kyoji Saito found that the set of vanishing cycles for an even-dimensional hypersurface isolated singularity form a *generalized root system*, that is, the set satisfies all the axioms of (ordinal) root system but the (positive) definiteness condition. He has constructed a general theory of the generalized root systems in [S1] and [S4]. In the theory, the set of vanishing cycles for elliptic, cuspidal, or unimodular exceptional singularities is the generalized root systems of signature $(l, 2, 0)$, $(l+1, 1, 1)$, and $(l+2, 0, 2)$, respectively, and which are called *elliptic*, *cuspidal*, and *e-hyperbolic root systems*, respectively, all of which have Witt index 2. Then it was a basic question whether one can attach a reasonable Lie algebra to such a generalized root system such that we can develop arguments similar to Grothendieck-Brieskorn-Slodowy theory, or in another words, we can construct a universal deformation of the corresponding singularity by using of the Lie algebra. For an elliptic root system, which corresponds to an elliptic singularity, we have partly answered to the half of the question in [SY].

Let us survey [SY] here. We focused on the fact in [S1] that each elliptic root system has an *elliptic simple root basis* nevertheless there exists no longer a geometric analogue of the Weyl chamber. We support the basis mainly because of the following two facts: (i) elliptic root systems are classified by the intersection matrices of the bases, (ii) the Coxeter element defined by the basis describes the degrees of a certain invariant ring of the elliptic Weyl group. The most significant difference between the elliptic basis and the ordinary basis is that there exists at least one pair of elements of the basis such that the inner product of them is strictly positive. Before going into Lie algebras, let us recall the fact: Borcherds defined in [B] a Lie algebra $\mathfrak{g}(Q)$ attached to any even lattice Q as a quotient of the vertex algebra attached to Q. Let R be a simply-laced[1] (hence homogeneous[2]) elliptic root system R, we

[1] All the edges are simple bonds (or double dotted bonds). See Appendix 7.3.

[2] All the roots have the same norm. There is two reduce marked elliptic root systems $A_1^{(1,1)}$ and $A_1^{(1,1)*}$ which are homogeneous but not simply-laced (these have bonds labeled ∞).

introduced a Lie algebra $\mathfrak{g}(R)$ as a subalgebra of Borchers' algebra $\mathfrak{g}(Q(R))$, where $Q(R)$ is the root lattice of R. We have shown the followings:

(a) The $\mathfrak{g}(R)$ has a root space decomposition. The set of positive-normed roots coincides with R. Each root space of positive-normed root is one-dimensional.
(b) The $\mathfrak{g}(R)$ has a presentation by a finite number of generators and a finite number of relations as follows: the generators consist of $\mathfrak{sl}_2$-triplets corresponding to the elements of the elliptic simple root basis, and the relations are the Serre relations and certain additional relations, which are caused by existence of pairs of elements of the basis whose inner products are positive.
(c) The $\mathfrak{g}(R)$ is isomorphic to the universal central extension of $\mathfrak{g}_{\mathrm{f}} \otimes \mathbb{C}[s^{\pm 1}, t^{\pm 1}]$, the tensor algebra of a simple Lie algebra $\mathfrak{g}_{\mathrm{f}}$ and the ring of Laurent polynomials of two variables $\mathbb{C}[s^{\pm 1}, t^{\pm 1}]$.
(d) The $\mathfrak{g}(R)$ has a presentation as an amalgamation of an affine Kac-Moody algebra and a Heisenberg algebra.

In this article, we introduce Lie algebras $\mathfrak{g}(R)$ to ALL the elliptic root systems R in a uniform way in Sect. 3 (more correctly, we introduce an algebra $\mathfrak{g}(R, G)$ for each reduced marked elliptic root system (R, G), which is defined in Sect. 2.3). We believe that we can use the Borcherds' algebra (as in [SY]) to introduce a Lie algebra for an inhomogeneous root system, but we do not adopt the Borcherds' algebra in this article, to avoid a difficulty of generalizing the theory of vertex algebra. We adopt the way in (d) as above, that is, we define the Lie algebra $\mathfrak{g}(R)$ as an amalgamation of an affine Kac-Moody algebra and a generalization of a Heisenberg algebra. We can show that the $\mathfrak{g}(R)$ has a presentation as (b) above, but their descriptions are fairly complicated. So we omit to describe $\mathfrak{g}(R)$ by finitely many generators and relations defined by the basis.

The main purpose of this article is find out an explicit structure of $\mathfrak{g}(R)$. We pay attention to the fact (c) above. (We shall refer Sect. 4.3.) Our question is: Is there some algebra $\tilde{\mathfrak{g}}$ which is similar to $\mathfrak{g}_{\mathrm{f}} \otimes \mathbb{C}[s^{\pm 1}, t^{\pm 1}]$ and whose universal central extension is $\mathfrak{g}(R)$? We will answer the question, which we survey as follows: Let $X_l^{(t_1,t_2)}$ be the type of R (more correctly, the type of (R, G)). There exist a simple Lie algebra $\mathfrak{g}_{\mathrm{f}}$ and a $\mathbb{Z}_{t_1,t_2} := \mathbb{Z}/t_1\mathbb{Z} \times \mathbb{Z}/t_2\mathbb{Z}$-grading $\mathfrak{g}_{\mathrm{f}} = \bigoplus_{(\bar{i},\bar{j}) \in \mathbb{Z}_{t_1,t_2}} (\mathfrak{g}_{\mathrm{f}})_{(\bar{i},\bar{j})}$ such that the derived algebra of $\mathfrak{g}(R)$ is the universal central extension of the subalgebra algebra $\bigoplus_{(i,j) \in \mathbb{Z}^2} (\mathfrak{g}_{\mathrm{f}})_{(\bar{i},\bar{j})} \otimes s^i t^j$, which we call a *twisted loop algebra*, of $\mathfrak{g}_{\mathrm{f}} \otimes \mathbb{C}[s^{\pm 1}, t^{\pm 1}]$. In Sect. 5, we exhibit explicitly all the twisted loop algebras associated to all the generalized root systems case

by case. Further, we determine explicitly the central extension of the twisted loop algebras in all the cases in goup I, II, and III in Sect. 2.3, and submit conjectures in all the cases in group IV (the author is very embarrassed to admit that he has not comlete the proof of the cases in group IV at this time).

Pollman has already introduced in [P] certain Lie algebras to all the elliptic root systems. But our algebras are more natural in the above sense. Note that her algebras are a quotient of $\mathfrak{g}(R)$.

All the elliptic root systems seem to be related to certain gradings of simple Lie algebras and seem to describe some inner structures of simple Lie algebras. Unfortunately there is not any systematic explanation about these phenomena, which the author would like to submit as an open problem.

The followings are the contens of this article. In Sect. 2, we recall the concept of the generalized root system, especially the concept of the reduced marked elliptic root systems. In Sect. 3, we introduce Lie algebras $\mathfrak{g}(R,G)$ to all the reduced marked elliptic root systems (R,G) and state the main theorems Theorem 1 and Theorem 3. Sect. 4 is a collection of preparations to the later sections. In Sect. 5, we introduce twisted loop algebra based on the group in the table in Sect. 2.3, to describe the algebras $\mathfrak{g}(R,G)$. In Sect. 6, we prove the main theorems. In Appendix 7, we exhibit all the reduced affine root systems and all the reduced marked elliptic root systems.

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2. Elliptic Root System

Here we recall the concept of the generalized root system, in particular, the elliptic root systems from [S1]. For the affine root systems, see also [H] or [K]. The main aim of this section is to state Fact 3.

2.1. **Generalized Root Systems.**

Let F be a finite dimensional real vector space equipped with a symmetric bilinear form I. A vector $\alpha \in F$ is called *non-isotropic* if $I(\alpha,\alpha) \neq 0$. Each non-isotropic vector α defines the *dual* $\alpha^\vee := 2\alpha/I(\alpha,\alpha)$ in F and the *reflection* w_α in the orthogonal group $O(F,I)$, where $w_\alpha u := u - I(u,\alpha^\vee)\alpha$ for all $u \in F$. Note that $I(\alpha,\alpha^\vee) = 2$ and $w_\alpha^2 = \mathrm{id}_F$, $w_\alpha(\alpha) = -\alpha$.

Let R be a subset of F such that all the elements of R are non-isotropic. We denote the subgroup generated by $\{w_\alpha \mid \alpha \in R\}$ in $O(F,I)$ by $W(R)$ and let $Q(R)$ be the sub-additive-group in F generated by R.

Definition 1. The set R is called a *generalized root system*, *root system* for short, belonging to (F,I) if R satisfies all of the followings:

(i) The sub-additive-group $Q(R)$ is a *full lattice* of F, that is, the natural mapping $Q(R) \otimes_{\mathbb{Z}} \mathbb{R} \to F$ is an isomorphism,
(ii) $I(\alpha^\vee, \beta) \in \mathbb{Z}$ for all $\alpha, \beta \in R$,
(iii) $w_\alpha \beta \in R$ for all $\alpha, \beta \in R$,
(iv) If $R = R_1 \cup R_2$ and $I(R_1, R_2) = 0$, then either R_1 or R_2 is empty.

Then, we call $Q(R)$ the *root lattice* and $W(R)$ the *Weyl group*. A root system R is called *reduced* if $\mathbb{R}\alpha \cap R = \{\pm\alpha\}$ for all $\alpha \in R$.

The *signature* of the root system R is the signature of the ambient space (F,I), which is the triplet of integers (μ_+, μ_0, μ_-), where μ_+ (resp. μ_-) is the maximum dimension of the positive (resp. negative) definite subspaces of (F,I) and μ_0 is the dimension of the radical $\operatorname{rad} F := \{v \in F \mid I(v,F) = \{0\}\}$. Note that Witt index $= \min(\mu_+, \mu_-) + \mu_0$. —

The root systems of signature $(l,0,0)$, which we call the *finite root systems*, coincide with the ordinal root systems as in [Bo].

2.2. **Affine Root Systems.**

Let R_{a} be a reduced root system of signature $(l,1,0)$ for some $l \in \mathbb{Z}_{\geq 0}$, which is an *affine root system* in the theory of Kac-Moody algebra in [K], belonging to $(F_{\mathrm{a}}, I_{\mathrm{a}})$. We call R_{a} *reduced affine* in this article. It is known that a reduced affine root system R_{a} has a simple root basis Γ_{a}, which we recall as follows.

We fix a generator b of the sublattice $Q(R_{\mathrm{a}}) \cap \operatorname{rad} F_{\mathrm{a}}$ of rank 1 :

$$Q(R_{\mathrm{a}}) \cap \operatorname{rad} F_{\mathrm{a}} = \mathbb{Z}b, \tag{1}$$

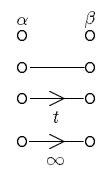
(b is unique up to sign) and define the half space $(F_{\mathrm{a}}^*)_{>0} := \{f \in F_{\mathrm{a}}^* \mid \langle f, b\rangle > 0\}$. For $\alpha \in R_{\mathrm{a}}$, put $H_\alpha := \{f \in F_{\mathrm{a}}^* \mid \langle f, \alpha\rangle = 0\}$, the reflection hyperplane in the dual space F_{a}^*. The hyperplanes $\{H_\alpha\}_{\alpha \in R_{\mathrm{a}}}$ divide the half space $(F_{\mathrm{a}})_{>0}^*$ into *chambers*, which are connected components of $(F_{\mathrm{a}})_{>0}^* \setminus \cup_{\alpha \in R_{\mathrm{a}}} H_\alpha$, since they are locally finite in the half space. Fix a chamber C and put $\Gamma_{\mathrm{a}} := \{\alpha \in R_{\mathrm{a}} \mid H_\alpha$ is a wall of C and $\langle \alpha, C\rangle > 0\}$. We call Γ_{a} an *affine simple root basis*. The basis Γ_{a} has the following properties:

Fact 1. ([K], *etc.*)

(i) $Q(R_{\mathrm{a}}) = \bigoplus_{\alpha \in \Gamma_{\mathrm{a}}} \mathbb{Z}\alpha$,
(ii) $W(R_{\mathrm{a}}) = W(\Gamma_{\mathrm{a}})$, *where* $W(\Gamma_{\mathrm{a}}) := \langle w_\alpha \mid \alpha \in \Gamma_{\mathrm{a}}\rangle$,

(iii) $R_{\mathrm{a}} = W(\Gamma_{\mathrm{a}})\Gamma_{\mathrm{a}}$, *where* $W(\Gamma_{\mathrm{a}})\Gamma_{\mathrm{a}} := \{w\alpha \mid w \in W(\Gamma_{\mathrm{a}}), \alpha \in \Gamma_{\mathrm{a}}\}$.

It is well known that the basis Γ_{a} define the *reduced affine Dynkin diagram*, which is a one-dimensional simplicial complex such that (i) the set of vertices are bijective to Γ_{a}, (ii) the two vertices $\alpha, \beta \in \Gamma_{\mathrm{a}}$ are joined by an (labeled) edge whose shape (and label) is(are) determined by the values $I(\alpha, \beta^\vee)$ and $I(\alpha^\vee, \beta)$. The followings are all of the shapes of (labeled) edges:



(no edge) if $I(\alpha, \beta) = 0$,

(simple bond) if $I(\alpha, \beta^\vee) = I(\alpha^\vee, \beta) = -1$,

if $I(\alpha, \beta^\vee) = -t$, $I(\alpha^\vee, \beta) = -1$ for $t = 2, 3, 4$

if $I(\alpha, \beta^\vee) = I(\alpha^\vee, \beta) = -2$.

Fact 2. (e.g. [K]) *The reduced affine Dynkin diagram is determined by the isomorphism class of reduced affine root systems. Conversely, the reduced affine Dynkin diagram determines uniquely the isomorphism class of reduced affine root systems.*

There is the list of all the reduced affine Dynkin diagrams in Appendix 7.2, where we give a *type* to each isomorphism class of reduced affine root systems.

Let us define the projection

$$\pi : F_{\mathrm{a}} \to F_{\mathrm{f}} := F_{\mathrm{a}}/\mathbb{R}b. \tag{2}$$

Then the image $R_{\mathrm{f}} := \pi(R_{\mathrm{a}})$ is a finite root system by (1). It is well known that there exist positive integers $\{n_\alpha\}_{\alpha \in \Gamma_{\mathrm{a}}}$ such that

$$\sum_{\alpha \in \Gamma_{\mathrm{a}}} n_\alpha \alpha = b \tag{3}$$

and there exists an element $\alpha_0 \in \Gamma_{\mathrm{a}}$ with $n_{\alpha_0} = 1$ satisfying that the image $\pi(\Gamma_{\mathrm{a}} \setminus \{\alpha_0\})$ is a simple root basis of R_{f}. We fix the α_0. Put $\Gamma_{\mathrm{f}} := \Gamma_{\mathrm{a}} \setminus \{\alpha_0\}$. By the choice, $Q(R_{\mathrm{a}})$ splits over $\mathbb{Z}$ as follows:

$$Q(R_{\mathrm{a}}) = \sum_{\alpha \in \Gamma_{\mathrm{f}}} \mathbb{Z}\alpha \oplus \mathbb{Z}b. \tag{4}$$

Let us identify F_{a} with $F_{\mathrm{f}} \oplus \mathbb{R}b$:

$$F_{\mathrm{a}} = F_{\mathrm{f}} \oplus \mathbb{R}b. \tag{5}$$

The splitting (4) and the identification (5) above induces the following identifications

$$\begin{array}{rccl} F_{\mathrm{a}} \supset & \sum_{\alpha \in \Gamma_{\mathrm{f}}} \mathbb{Z}\alpha & = & Q(R_{\mathrm{f}}) \subset F_{\mathrm{f}}, \\ F_{\mathrm{a}} \supset & Q(R_{\mathrm{a}}) & = & Q(R_{\mathrm{f}}) \oplus \mathbb{Z}b \subset F_{\mathrm{f}} \oplus \mathbb{R}b. \end{array} \tag{6}$$

Let $X_l^{(t)}$ be the type of R. The last identification induce the following presentation of R:

$$R = \{\alpha + ib \mid \alpha \in S_i,\ i \in \mathbb{Z}\}, \tag{7}$$

where $\{S_i\}$ are subsets of R_{f}. In Appendix 7.2, we exhibit $\{S_i\}$ explicitly.

2.3. **Elliptic Root Systems.**

The root systems of signature $(l, 2, 0)$ for some $l \in \mathbb{Z}_{\geq 0}$ are called the *elliptic root systems*. Let R be an elliptic root system belonging to (F, I). To study the R, we choose a *marking* G, which is a one-dimensional subspace of $\operatorname{rad} F$ such that the additive group $G \cap Q(R)$ is a full lattice of G. Fix a generator a of $G \cap Q(R)$:

$$G \cap Q(R) = \mathbb{Z}a \tag{8}$$

(a is unique up to sign). We consider the pair (R, G), which we call a *marked elliptic root system*. Let π_G be the projection

$$\pi : F \to F_{\mathrm{a}} := F/G. \tag{9}$$

Then the image $R_{\mathrm{a}} := \pi_G(R)$, which we call the *quotient root system of* (R, G), is an affine root system belonging to $(F_{\mathrm{a}}, I_{\mathrm{a}})$, where I_{a} is the form induced from I. Let us assume that R_{a} is reduced. Let $\Gamma(R_{\mathrm{a}})$ $(\subset R_{\mathrm{a}})$ be an affine simple root basis of R_{a}. We choose a set $\Gamma_{\mathrm{a}} \subset R$ such that the projection π_G induces a bijection $\pi_G|\Gamma_{\mathrm{a}} \xrightarrow{\sim} \Gamma(R_{\mathrm{a}})$. By the choice, $Q(R)$ splits over $\mathbb{Z}$ as follows:

$$Q(R) = \sum_{\alpha \in \Gamma_{\mathrm{a}}} \mathbb{Z}\alpha \oplus \mathbb{Z}a. \tag{10}$$

Let us identify F with $F_{\mathrm{a}} \oplus G$:

$$F = F_{\mathrm{a}} \oplus G. \tag{11}$$

The splitting (10) and the identification (11) above induces the following identifications

$$\begin{array}{lllll} F \supset & \sum_{\alpha \in \Gamma_{\mathrm{a}}} \mathbb{Z}\alpha & = & Q(R_{\mathrm{a}}) & \subset F_{\mathrm{a}}, \\ F \supset & Q(R) & = & Q(R_{\mathrm{a}}) \oplus \mathbb{Z}a & \subset F_{\mathrm{a}} \oplus G \\ & & \overset{(4)}{=} & Q(R_{\mathrm{f}}) \oplus \mathbb{Z}b \oplus \mathbb{Z}a & \subset F_{\mathrm{f}} \oplus \mathbb{R}b \oplus \mathbb{R}a. \end{array} \tag{12}$$

The last identification induces the following presentation of R:

$$R = \{\alpha + ib + ja \mid \alpha \in S_{(i,j)},\ (i,j) \in \mathbb{Z}^2\}, \tag{13}$$

where $\{S_{(i,j)}\}$ are subsets of R_{f}. We exhibit $\{S_{(i,j)}\}$ explicitely in Appendix 7.3.

On the other hand, let us focus on the splitting (10). We put

$$K(\alpha) := \{m \in \mathbb{Z} \mid \alpha + ma \in R\} \tag{14}$$

for all $\alpha \in R$. It is shown that $K(\alpha)$ is a non-zero ideal of $\mathbb{Z}$ (see [S1, Sect. 6.1]). We define the *counting* $k : R \to \mathbb{Z}$ by

$$k(\alpha) := \inf\{m \mid m \in K(\alpha),\ m > 0\}. \tag{15}$$

Definition 2. Let (R, G) be a marked elliptic root system and Γ_{a}, k as above. We consider the pair $(\Gamma_{\mathrm{a}}, k|\Gamma_{\mathrm{a}})$, where Γ_{a} is the affine Dynkin diagram which we identify with the affine simple root basis of the quotient root system R_{a} (see Fact 2) and $k|\Gamma_{\mathrm{a}}$ is the restriction of the counting in (15) on Γ_{a}. Let us call the pair $(\Gamma_{\mathrm{a}}, k|\Gamma_{\mathrm{a}})$ the *pre-basis* of (R, G). Let us call the marked elliptic root system (R, G) whose quoetient root system R_{a} is reduced a *reduced marked elliptic root system*, *r.m.e.r.s.* for short.

Fact 3. ([S1]) *Let (R, G) and (R', G') be two r.m.e.r.s.s. The following* (i) *and* (ii) *are equivarent*:

(i) *(R, G) and (R', G') are isomorphic.*
(ii) *Their pre-bases are euqal.*

We give a *type* to each isomorphic class of r.m.e.r.s.s, or pre-basis, as in Appendix 7.3. The followings are all the types of r.m.e.r.s.s:

group	types
I	$A_l^{(1,1)}(l \geq 1),\ B_l^{(1,1)}(l \geq 3),\ C_l^{(1,1)}(l \geq 2),\ D_l^{(1,1)}(l \geq 4),$ $E_l^{(1,1)}(l = 6, 7, 8),\ F_4^{(1,1)},\ G_2^{(1,1)}.$
II	$B_l^{(1,2)}(l \geq 3),\ C_l^{(2,1)}(l \geq 3),\ B_l^{(2,1)}(l \geq 2),\ C_l^{(1,2)}(l \geq 2),$ $BC_l^{(2,1)}(l \geq 2),\ F_4^{(1,2)}, F_4^{(2,1)},\ G_2^{(1,3)}, G_2^{(3,1)}.$
III	$B_l^{(2,2)}(l \geq 2),\ B_l^{(2,2)*}(l \geq 2),\ C_l^{(1,1)*}(l \geq 2),\ C_l^{(2,2)}(l \geq 3),$ $F_4^{(2,2)},\ G_2^{(3,3)}.$
IV	$BC_l^{(2,4)}(l \geq 1),\ BC_l^{(2,2)}(1)(l \geq 2),\ BC_l^{(2,2)}(2)(l \geq 2).$

Let us denote a general type of r.m.e.r.s.s by $X_l^{(t_1,t_2)}$, even though there exist types such as $X_l^{(t_1,t_2)*}$ and $BC_l^{(2,2)}(i)$ $(i = 1, 2)$.

Remark. The data in the pre-basis can be coded into a more sophisticated concept, the *elliptic simple root basis* Γ as in [S1]. The Γ is defined to be a set $\Gamma_{\mathrm{a}} \cup \Gamma^*_{\mathrm{max}}$, where Γ_{max} is a subset of Γ_{a} and $\Gamma^*_{\mathrm{max}} := \{\alpha + k(\alpha)a \mid \alpha \in \Gamma_{\mathrm{max}}\}$. The Γ defines the coxeter element $c := \prod_{\alpha \in \Gamma} w_\alpha \in W(R)$. The c is of finite order on F and the eigenvalues of c are deeply related to the degrees of generators of an invariant ring of $W(R)$.

3. Elliptic Lie Algebra

Here we introduce a Lie algebra to each reduced marked elliptic root system in a uniform way, and make the main statement, Theorem 1, which describes a structure of the algebra. The Lie algebra is introduced as an amalgamation of a Kac-Moody Lie algebra and a generalization of Heisenberg algebra. We will see that a basic constructing block of the algebra is $\hat{\mathfrak{sl}}_2$.

3.1. **Definition.**

Let (R, G) be a reduced marked elliptic root system belonging to (F, I). Let $(\Gamma_{\mathrm{a}}, k|\Gamma_{\mathrm{a}})$ be the pre-basis (see Definition 2). Recall the splitting in (12), and also F_{f}, b and a. We complexify these spaces

$$\begin{array}{rcl} (F_{\mathrm{f}})_{\mathbb{C}} & := & F_{\mathrm{f}} \otimes_{\mathbb{R}} \mathbb{C}, \\ F_{\mathbb{C}} & := & F \otimes_{\mathbb{R}} \mathbb{C} = (F_{\mathrm{f}})_{\mathbb{C}} \oplus \mathbb{C}b \oplus \mathbb{C}a, \end{array} \tag{16}$$

and extend $F_{\mathbb{C}}$ to the non-degenerate space

$$\tilde{F}_{\mathbb{C}} := (F_{\mathrm{f}})_{\mathbb{C}} \oplus \mathbb{C}b \oplus \mathbb{C}a \oplus \mathbb{C}\check{b} \oplus \mathbb{C}\check{a}, \tag{17}$$

where the form on the space is defined by the following table:

	$(F_{\mathrm{f}})_{\mathbb{C}}$	b	$\check{b}$	a	$\check{a}$
$(F_{\mathrm{f}})_{\mathbb{C}}$	$I\|_{(F_{\mathrm{f}})_{\mathbb{C}}}$	0	0	0	0
b	0	0	1	0	0
$\check{b}$	0	1	0	0	0
a	0	0	0	0	1
$\check{a}$	0	0	0	1	0

We again denote the form by I. We denote the dual space by

$$H := \tilde{F}_{\mathbb{C}}^* = \mathrm{Hom}_{\mathbb{C}}(\tilde{F}_{\mathbb{C}}, \mathbb{C}) \tag{18}$$

and define the linear isomorphism

$$\begin{array}{rcccl} h & : & \tilde{F}_{\mathbb{C}} & \to & H \\ & & v & \mapsto & I(v, \cdot). \end{array} \tag{19}$$

We identify the dual space H^* of H with $\tilde{F}_{\mathbb{C}}$ canonically and consider $Q(R)$ and R as subsets of H^*:

$$H^* = \tilde{F}_{\mathbb{C}} \supset Q(R) \supset R.$$

Finally, let us put

$$\Lambda_{\check{b}} := h_b, \quad \Lambda_{\check{a}} := h_a. \tag{20}$$

The following fact is easy to see.

$$h_{(u+mb+ma)^\vee} = h_{u^\vee} + \frac{2m}{I(u,u)}\Lambda_{\check{b}} + \frac{2n}{I(u,u)}\Lambda_{\check{a}}, \tag{21}$$

where $u \in (F_{\mathrm{f}})_{\mathbb{C}}$ and $m, n \in \mathbb{Z}$.

Definition 3. Let us define the Lie algebra $\mathfrak{g}(R,G)$ associated to a reduced marked elliptic root system (R,G) of type $X_l^{(t_1,t_2)}$ by the following presentation described by the pre-basis $(\Gamma_{\mathrm{a}}, k|\Gamma_{\mathrm{a}})$.

generators: H, $\{E^\alpha \mid \alpha \in \pm\Gamma_{\mathrm{a}}\}$, $\{H^{ia}_{\alpha^\vee} \mid i \in \mathbb{Z},\ \alpha \in \pm\Gamma_{\mathrm{a}}\}$.

relations:

$$\text{(i)} \quad \begin{cases} H \text{ is abelian,} \\ [h, H^{ia}_{\alpha^\vee}] = \langle h, ia\rangle H^{ia}_{\alpha^\vee} \quad \text{for } \alpha \in \pm\Gamma_{\mathrm{a}},\ i \in \mathbb{Z},\ h \in H, \\ [h, E^\alpha] = \langle h, \alpha\rangle E^\alpha \quad \text{for } \alpha \in \pm\Gamma_{\mathrm{a}},\ h \in H, \end{cases}$$

$$\text{(ii)} \quad \begin{cases} [E^\alpha, E^{-\alpha}] = -h_{\alpha^\vee} \quad \text{for } \alpha \in \Gamma_{\mathrm{a}}, \\ [E^\alpha, E^{-\beta}] = 0 \quad \text{for } \alpha, \beta \in \Gamma_{\mathrm{a}},\ \alpha \neq \beta, \\ (\operatorname{ad} E^\alpha)^{1-I(\alpha^\vee,\beta)} E^\beta = 0 \quad \text{for } \alpha, \beta \in \Gamma_{\mathrm{a}},\ \alpha \neq \beta, \\ (\operatorname{ad} E^{-\alpha})^{1-I(\alpha^\vee,\beta)} E^{-\beta} = 0 \quad \text{for } \alpha, \beta \in \Gamma_{\mathrm{a}},\ \alpha \neq \beta, \end{cases}$$

$$\text{(iii)} \quad \begin{cases} H^0_{\alpha^\vee} = h_{\alpha^\vee} \quad \text{for } \alpha \in \pm\Gamma_{\mathrm{a}}, \\ H^{ia}_{-\alpha^\vee} = -H^{ia}_{\alpha^\vee} \quad \text{for } i \in \mathbb{Z},\ \alpha \pm \Gamma_{\mathrm{a}}, \\ H^{ia}_{\alpha^\vee} = 0 \quad \text{for } i \notin k(\alpha)\mathbb{Z},\ \alpha \in \Gamma_{\mathrm{a}}, \\ [H^{ia}_{\alpha^\vee}, H^{ja}_{\beta^\vee}] = I(\alpha^\vee, \beta^\vee) i \delta_{i+j} \Lambda_{\tilde{a}} \quad \text{for } \alpha, \beta \in \pm\Gamma_{\mathrm{a}}, \\ \qquad\qquad\qquad\qquad\qquad\qquad i \in k(\alpha)\mathbb{Z},\ j \in k(\beta)\mathbb{Z}, \end{cases}$$

$$\text{(iv)} \quad \begin{cases} \frac{1}{2}\left[H^{ia}_{\alpha^\vee}, \frac{1}{2}[H^{ja}_{\alpha^\vee}, E^\alpha]\right] = \frac{1}{2}[H^{(i+j)a}_{\alpha^\vee}, E^\alpha], \quad \text{for } \alpha \in \pm\Gamma_{\mathrm{a}}, \\ \qquad\qquad\qquad\qquad\qquad\qquad i, j \in k(\alpha)\mathbb{Z}, \\ \left[\frac{1}{2}[H^{ia}_{\alpha^\vee}, E^\alpha], E^{-\alpha}\right] = -H^{ia}_{\alpha^\vee} \quad \text{for } \alpha \in \pm\Gamma_{\mathrm{a}}, \\ [H^{ia}_{\alpha^\vee}, E^\beta] = \frac{1}{2} I(\alpha^\vee, \beta)[H^{ia}_{\beta^\vee}, E^\beta] \quad \text{for } \alpha, \beta \in \pm\Gamma_{\mathrm{a}}, \\ \qquad\qquad\qquad\qquad\qquad\qquad i \in k(\alpha)\mathbb{Z}, \\ \left[[H^{ia}_{\alpha^\vee}, E^\alpha], E^\beta\right] = 0 \quad \text{for } \alpha, \beta \in \pm\Gamma_{\mathrm{a}}, I(\alpha,\beta) \geq 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad i \in k(\alpha)\mathbb{Z}. \end{cases}$$

Remark 1. The relations (i) allow $\mathfrak{g}(R,G)$ to have the root space decomposition with respect to H (see corollaries of Lemma 1 below). The relations (ii) is the ones of a Kac-Moody algebra (see Claim 3). The relations (iii) is the ones of a generalization of a Heisenberg algebra. The relations (iv) amalgamate the Kac-Moody algebra and the generalized Heisenberg algebra.

Remark 2. Let (R,G) be a homogeneous reduced elliptic root system (see the foot note of Introduction for *homogeneous*). Then $\mathfrak{g}(R,G)$ is isomorphic to $\tilde{\mathfrak{g}}(R)$ or $\tilde{\mathfrak{h}}^{\mathbb{Z}}_{\mathrm{af}} * \mathfrak{g}_{\mathrm{af}}$ in [SY]. Note that we drop the tilde in this article.

Remark 3. There exists a Lie-homomorphism from $\mathfrak{g}(R,G)$ to the algebra introduced by Pollman [P] to (R,G).

3.2. **Constructiong Blocks.**

We study the relations (i)–(iv) in Definition 3 a little more precisely. Let us put the subalgebra S_α ($\alpha \in \Gamma_{\mathrm{a}}$) of $\mathfrak{g}(R,G)$ by

$$S_\alpha := \langle H_{\alpha^\vee}^{ik(\alpha)a} \ (i \in \mathbb{Z}),\ E^\alpha,\ E^{-\alpha},\ \Lambda_a \rangle. \tag{22}$$

This is a primitive block of $\mathfrak{g}(R,G)$ in the sense that the set $\cup_{\alpha\in\Gamma_{\mathrm{a}}} S_\alpha$ generates $\mathfrak{g}(R,G)$ and we have the following facts. We shall consult [K] for the Kac-Moody algebras.

We shoul state the following fact first.

Claim 1. *S_α is isomorphic to $\hat{\mathfrak{sl}}_2$, the Kac-Moody algebra of type $A_1^{(1)}$, for all $\alpha \in \Gamma_{\mathrm{a}}$.*

Next, let us consider how to be glued the primitive blocks. To see this, we put the subalgebra $S_{\alpha,\beta}$ of $\mathfrak{g}(R,G)$ by

$$S_{\alpha,\beta} := \langle S_\alpha, S_\beta \rangle$$

for different two elements α and β in Γ_{a}. Then we have the following fact.

Claim 2. *Let $\alpha,\ \beta \in \Gamma_{\mathrm{a}}$ such that $I(\alpha^\vee,\beta) = -1$ and $I(\alpha,\beta^\vee) = -t \in \{-1,-2,-3\}$. Put*

$$X := \begin{cases} A & \text{if} \quad t=1, \\ B & \text{if} \quad t=2, \\ G & \text{if} \quad t=3. \end{cases}$$

Then $S_{\alpha,\beta}$ is isomorphic to the Kac-Moody Lie algebra of type $X_2^{(1)}$.

Finally, let us put the subalgebra $\mathfrak{g}_{\mathrm{af}}$ of $\mathfrak{g}(R,G)$ by

$$\mathfrak{g}_{\mathrm{af}} := \langle E^\alpha \ (\alpha \in \pm\Gamma_{\mathrm{a}}),\ \Lambda_b \rangle.$$

Then we have

Claim 3. *$\mathfrak{g}_{\mathrm{af}}$ is isomorphic to the Kac-Moody algebra of type $X_l^{(t_1)}$.*

3.3. **Root Space Decomposition.**

Let us consider a root space decomposition of $\mathfrak{g}(R,G)$ with respect to H. We can use the same techniques as in [SY] with some trivial exchanges. Here we just sketch briefly the argument in [SY].

The fact below is the most crucial.

Lemma 1. *$\mathfrak{g}(R,G)$ is spanned as a vector space by the following:*

(i) *H,*

(ii) $[B_1, B_2, \dots, B_k]$, *where* k *is a positive integer and each* B_i *is either* E^{α} $(\alpha \in \Gamma_{\mathrm{a}})$ *or* $H_{\alpha^\vee}^{ia}$ $(i \in \mathbb{Z},\ \alpha \in \pm\Gamma_{\mathrm{a}})$,
(iii) $[B_1, B_2, \dots, B_k]$, *where* k *is a positive integer and each* B_i *is either* $E^{-\alpha}$ $(\alpha \in \Gamma_{\mathrm{a}})$ *or* $H_{\alpha^\vee}^{ia}$ $(i \in \mathbb{Z},\ \alpha \in \pm\Gamma_{\mathrm{a}})$,

where we define $[B_1, B_2, \dots, B_k] := \big[[B_1, B_2, \dots, B_{k-1}], B_k\big]$ *and iteratively.*

According to the fact, it is natural to put $Q^+ := \bigoplus_{\alpha\in\Gamma_{\mathrm{a}}} \mathbb{Z}_{\geq 0}\alpha \oplus \mathbb{Z}a$ and $Q^- := -Q^+$. Then we have $Q^+ \cap Q^- = \mathbb{Z}a$. Let us denote the set of roots of $\mathfrak{g}(R,G)$ by Δ. The above fact implies

$$\Delta \subset Q^+ \cup Q^-.$$

Hence we have the followings, where we denote by $(\mathfrak{g}(R,G))_\alpha$ the root space of a root $\alpha \in \Delta$.

Corollary 1. (i) $(\mathfrak{g}(R,G))_0 = H$,
(ii) $(\mathfrak{g}(R,G))_{\alpha+ia} = \mathbb{C}[E^{\alpha}, H_{\alpha^\vee}^{ia}]$ *for all* $\alpha \in \pm\Gamma_{\mathrm{a}}$ *and* $i \in \mathbb{Z}$,
(iii) $(\mathfrak{g}(R,G))_{j\alpha+ia} = 0$ *for all* $\alpha \in \pm\Gamma_{\mathrm{a}}$, $i, j \in \mathbb{Z}$ *such that* $|j| \geq 2$.

Corollary 2. $\dim \big(\mathfrak{g}(R,G)\big)_\alpha = 1$ *for all* $\alpha \in R$.

Corollary 3. $\Delta \subset R \cup \operatorname{rad} Q(R)$.

Note that $\operatorname{rad} Q(R) = \mathbb{Z}b \oplus \mathbb{Z}a$. We will see that $\Delta = R \cup \operatorname{rad} Q(R)$, later.

3.4. **Main Result.**

Before stating our main result, Let us recall some notation (from [MEY]). Let $\mathfrak{g}$ be a Lie algebra. The ideal $\mathfrak{g}' := [\mathfrak{g}, \mathfrak{g}]$ of $\mathfrak{g}$ is called the *derived algebra* of $\mathfrak{g}$. We call $\mathfrak{g}$ *perfect* if the derived algebra of $\mathfrak{g}$ is $\mathfrak{g}$ itself. A *central extension* of $\mathfrak{g}$ is a Lie algebra $\hat{\mathfrak{g}}$ and a surjective homomorphism $\pi : \hat{\mathfrak{g}} \to \mathfrak{g}$ whose kernel lies in the center of $\hat{\mathfrak{g}}$. The pair $(\hat{\mathfrak{g}}, \pi)$ is a *covering* of $\mathfrak{g}$ if in addition $\hat{\mathfrak{g}}$ is perfect. A covering $(\hat{\mathfrak{g}}, \pi)$ is a *universal covering algebra* of $\mathfrak{g}$ if for every central extension $(\mathfrak{a}, \varpi)$ of $\mathfrak{g}$ there is a unique homomorphism $\rho : \hat{\mathfrak{g}} \to \mathfrak{a}$ for which $\varpi\rho = \pi$. It is known that every perfect Lie algebra has a universal covering algebra. We use the following fact, which can be shown easily.

Claim 4. *Let* $(\mathfrak{a}, \varpi)$ *be a covering of* $\mathfrak{g}$, *and let* $f : \mathfrak{a} \to \mathfrak{a}$ *be a Lie-homomorphism. If* $\varpi f = \varpi$, *then* $f = \mathrm{id}_{\mathfrak{a}}$.

$$\begin{array}{ccc} \mathfrak{a} & \xrightarrow{f} & \mathfrak{a} \\ \varpi \downarrow & \circlearrowleft & \downarrow \varpi \\ \mathfrak{g} & = & \mathfrak{g} \end{array}$$

Now we return to the algebra $\mathfrak{g}(R,G)$. Let $\mathfrak{g}'(R,G)$ be the derived algebra of $\mathfrak{g}(R,G)$. Then $\mathfrak{g}'(R,G)$ is perfect and codimension 2 in $\mathfrak{g}(R,G)$. More explicitly, if we describe $\mathfrak{g}(R,G)$ as

$$\mathfrak{g}(R,G) = H \oplus \bigoplus_{\alpha \in H^* \setminus \{0\}} \mathfrak{g}(R,G)_\alpha,$$

then

$$\mathfrak{g}'(R,G) = H' \oplus \bigoplus_{\alpha \in H^* \setminus \{0\}} \mathfrak{g}(R,G)_\alpha,$$

where $H' := h(F_{\mathbb{C}})$ (see (16) and (19)).

The main result of this article is to find an explicit description of each algebra $\mathfrak{g}(R,G)$. We describe it by the use of the algebras $\tilde{\mathfrak{g}}(X_l^{(t_1,t_2)})$ which are introduced in Sect. 5, together with their universal covering algebras $\hat{\mathfrak{g}}(X_l^{(t_1,t_2)}) \xrightarrow{\pi} \tilde{\mathfrak{g}}(X_l^{(t_1,t_2)})$.

Now we state our main theorem.

Theorem 1. *Let (R,G) be a reduced marked elliptic root system of type $X_l^{(t_1,t_2)}$. Then $\mathfrak{g}'(R,G)$, the derived algebra of $\mathfrak{g}(R,G)$, is the universal covering algebra of $\tilde{\mathfrak{g}}(X_l^{(t_1,t_2)})$, hence isomorphic to $\hat{\mathfrak{g}}(X_l^{(t_1,t_2)})$.*

Remark. This statement is still a conjecture for the types $BC_l^{(2,4)}$, $BC_l^{(2,2)}(1)$ and $BC_l^{(2,2)}(2)$.

Proof. Here we only sketch the proof. For a little more detail, see Sect. 6.2.

1. First note that $\mathfrak{g}'(R,G)$, $\hat{\mathfrak{g}} := \hat{\mathfrak{g}}(X_l^{(t_1,t_2)})$ (see (29)), and $\tilde{\mathfrak{g}} := \tilde{\mathfrak{g}}(X_l^{(t_1,t_2)})$ (see (30)) have naturally $Q(R)$-grading as algebras (see Sect. 4.3). Let $\pi : \hat{\mathfrak{g}} \to \tilde{\mathfrak{g}}$ be the natural projection (see (31)). Then π is compatible with the $Q(R)$-grading structures.

2. We construct a surjective homomorphism $\Phi : \mathfrak{g}'(R,G) \to \hat{\mathfrak{g}}(X_l^{(t_1,t_2)})$ which is compatible with the $Q(R)$-grading structures. This Φ is defined by the generators (see Sect. 6.2 for an example). We check well-definedness by seeing all the images of defining relations of $\mathfrak{g}(R,G)$ vanish. Let us denote the homogeneous spaces of degree $\alpha \in Q(R)$ by $\mathfrak{g}'(R,G)_\alpha$, $\hat{\mathfrak{g}}_\alpha$, and $\tilde{\mathfrak{g}}_\alpha$, respectively. Note that we have

$$\dim \mathfrak{g}'(R,G)_\alpha = \dim \hat{\mathfrak{g}}_\alpha = \dim \tilde{\mathfrak{g}}_\alpha = 1 \tag{23}$$

for all $\alpha \in R$ (see Cor. 2). Hence we know the restriction Φ_α of Φ on $\mathfrak{g}'(R,G)_\alpha$ is bijective.

3. Put $\Theta := \pi\Phi$

$$\begin{array}{ccc} \mathfrak{g}'(R,G) & \xrightarrow{\Phi} & \hat{\mathfrak{g}} \\ \Theta \downarrow & \circlearrowleft & \downarrow \pi \\ \tilde{\mathfrak{g}} & = & \tilde{\mathfrak{g}} \end{array}$$

and put $\mathfrak{z} := \ker\Theta$. We have the following exact sequence:

$$0 \ \to \ \mathfrak{z} \ \to \ \mathfrak{g}'(R,G) \ \stackrel{\Theta}{\to} \ \tilde{\mathfrak{g}} \ \to 0.$$

Let us see that $(\mathfrak{g}'(R,G),\Theta)$ is a central extension of $\tilde{\mathfrak{g}}$ as follows. Because Θ is compativle with the $Q(R)$-grading structures, $\ker\Theta$ also has the $Q(R)$-grading structure $\ker\Theta = \bigoplus_{\mu\in Q(R)}(\ker\Theta)_\mu$ (see (23)). Let us see $(\ker\Theta)_\mu \subset \mathfrak{z}$ for all $\mu \in \operatorname{rad} Q(R)$ as follows: we have

$$\left[[E^\alpha, H^{ia}_{\alpha^\vee}], (\ker\Phi)_\mu\right] \subset (\ker\Theta)_{\alpha+na+\mu} = 0$$

for all $\alpha \in \pm\Gamma_{\mathrm{a}}$ and $i \in \mathbb{Z}$, and we can see that the set $\{[E^\alpha, H^{ia}_{\alpha^\vee}] \mid \alpha \in \pm\Gamma_{\mathrm{a}},\ i \in \mathbb{Z}\}$ generates $\mathfrak{g}'(R,G)$.

4. We will see that $(\hat{\mathfrak{g}},\pi)$ is the universal covering algebra of $\tilde{\mathfrak{g}}$ in Sect. 5, Theorem 3. Hence there exists $\Psi : \hat{\mathfrak{g}} \to \mathfrak{g}'(R,G)$ such that $\Theta\Psi = \pi$ for the central extension $(\mathfrak{g}'(R,G),\Theta)$.

$$\begin{array}{ccccc} \mathfrak{g}'(R,G) & \stackrel{\Phi}{\longrightarrow} & \hat{\mathfrak{g}} & \stackrel{\Psi}{\longrightarrow} & \mathfrak{g}'(R,G) \\ \Theta\downarrow & \circlearrowleft & \downarrow\pi & \circlearrowleft & \downarrow\Theta \\ \tilde{\mathfrak{g}} & = & \tilde{\mathfrak{g}} & = & \tilde{\mathfrak{g}} \end{array}$$

Using Claim 4, we can see $\Psi\Phi = \mathrm{id}_{\mathfrak{g}'(R,G)}$ and $\Phi\Psi = \mathrm{id}_{\hat{\mathfrak{g}}}$. □

4. Preparations

This section is a collection of preperations which are used in the next section. We prepere some pecular notation and some well known facts.

In this section and next section, all the algebras and spaces are considered over $\mathbb{C}$, and $A \otimes B := A \otimes_{\mathbb{C}} B$ for two spaces A and B.

4.1. **Simple Lie Algebra.**

Let $\mathfrak{g}(X_l)$ the simple Lie algebra of type X_l, where $X_l = A_l$ $(l \in \mathbb{Z}_{l\geq 1})$, B_l $(l \in \mathbb{Z}_{l\geq 2})$, C_l $(l \in \mathbb{Z}_{\geq 3})$, D_l $(l \in \mathbb{Z}_{\geq 4})$, E_l $(l = 6,7,8)$, F_4, or G_2. We fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}(X_l)$ and consider the root space decomposition of $\mathfrak{g}(X_l)$ with respect to $\mathfrak{h}$. We denote by $\mathfrak{g}_\alpha$ the one-dimensional root space of a root α. Let us fix a simple root basis $\{\alpha_1,\dots,\alpha_l\} \subset \mathfrak{h}^*$ and Chevalley generators $\{e_i, h_i, f_i \mid i = 1,\dots,l\}$ of $\mathfrak{g}(X_l)$ satisfying that $e_i \in \mathfrak{g}_{\alpha_i}$, $f_i \in \mathfrak{g}_{-\alpha_i}$ and $[e_i,f_i] = h_i$ for all $i = 1,\dots,l$. Let α_0 be the lowest root with respect to the basis. We also fix a standard $\mathfrak{sl}_2$-triplet $\{e_0,h_0,f_0\}$ (that is, $[h_0,e_0] = 2e_0$, $[h_0,f_0] = -2f_0$, and $[e_0,f_0] = h_0$) satisfying that $e_0 \in \mathfrak{g}_{\alpha_0}$, $f_0 \in \mathfrak{g}_{-\alpha_0}$, and $h_0 = [e_0,f_0]$. Note that the triplet is unique up to non-zero scalar multiple. Let us call the set $\{\alpha_0,\dots,\alpha_l\}$ the *extended simple basis*, and let us call the set $\{e_i,f_i \mid i = 0,\dots,l\}$ the *extended Chevalley generators*. Let us denote by $(\cdot,\cdot)$ the Killing form (normalized properly) on $\mathfrak{g}$. We can assume that these indices are compatible with ones in Appendix

7.2, that is: the diagram of the extended simple basis $\{\alpha_0, \dots, \alpha_l\}$, which is determined by the values $\left(\frac{2\alpha_i}{(\alpha_i,\alpha_i)}, \alpha_j\right)$ and $\left(\alpha_i, \frac{2\alpha_j}{(\alpha_j,\alpha_j)}\right)$ for $i, j = 0, \dots, l$ in the same way as in Sect. 2.2, is the affine Dynkin diagram of type $X_l^{(1)}$ in Appendix 7.2.

Remark. Our usage of the symbols α_0, e_0, f_0 and h_0 here is different from the standard one. In this article, we cansider all these objects are inside a simple Lie algebra $\mathfrak{g}$.

For a subset $S = \{s_1, \dots, s_i\}$ of $\mathfrak{g}$, we put

$\langle S \rangle = \langle s_1, \dots, s_i \rangle :=$ the subalgebra of $\mathfrak{g}$ generated by $s_1, \dots, s_i$.

Let us recall Schur's Lemma here:

Fact 4. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra and V be a finite dimensional irreducible $\mathfrak{g}$-module. Let $f : V \to V$ be an $\mathfrak{g}$-endomorphism. Then there exists a constant $c \in \mathbb{C}$ such that $f = c\,\mathrm{id}_V$.*

Let us recall the first cohomology of $\mathfrak{g}$-module. Let $\mathfrak{g}$ be a Lie algebra and let V be a $\mathfrak{g}$-module. Here we recall briefly the first cohomology $H^1(\mathfrak{g}, V)$ of $\mathfrak{g}$ with coefficient in V.

$$H^1(\mathfrak{g}, V) := \ker d_1 / \operatorname{im} d_0,$$

where

$$\operatorname{im} d_0 := \left\{ f \in \mathrm{Hom}_{\mathbb{C}}(\mathfrak{g}, V) \;\middle|\; \begin{array}{l} \text{There exist } v_0 \in V \text{ such that} \\ f(x) = x \cdot v_0 \text{ for all } x \in \mathfrak{g} \end{array} \right\},$$

$\ker d_1 := \{f \in \mathrm{Hom}_{\mathbb{C}}(\mathfrak{g}, V) \mid f([x,y]) = x \cdot f(y) - y \cdot f(x) \text{ for all } x, y \in \mathfrak{g}\}$.

We recall Whitehead's lemma from [HS]:

Fact 5. *Let $\mathfrak{g}$ be a simple Lie algebra and V be a finite-dimensional $\mathfrak{g}$-module. Then $H^1(\mathfrak{g}, V) = 0$.*

4.2. **Algebra A and Space Ω_A/dA.**

Put $\mathbb{Z}_t := \mathbb{Z}/t\mathbb{Z}$ for a positive integer t. Let t_1 and t_2 be some two positive integres. We define the natural projection and $\mathbb{Z}_{t_1,t_2}$ as follows.

$$\begin{array}{rccc} {}^{\bar{}}: & \mathbb{Z}^2 & \to & \mathbb{Z}_{t_1,t_2} := \mathbb{Z}_{t_1} \times \mathbb{Z}_{t_2} \\ & (i,j) & \mapsto & \overline{(i,j)} := (\bar{i}, \bar{j}). \end{array} \tag{24}$$

We put $o = (0,0)$ (to avoid confusion, we use o instead of 0) and $\mathbb{Z}'_{t_1,t_2} := \mathbb{Z}_{t_1,t_2} \setminus \{\bar{o}\}$.

Let A be the ring of Laurent polynomials of two variables s and t,

$$A := \mathbb{C}[s^{\pm 1}, t^{\pm 1}].$$

We can induce $\mathbb{Z}^2$-grading on A by putting

$$\deg x^i y^j := (i,j) \in \mathbb{Z}^2.$$

Let Ω_A be the module of Köhler differentials attached to the commutative ring A and $d : A \to \Omega_A$ be the canonical differentiation. Let $^- : \Omega_A \to \Omega_A/dA$, the projection. For example, we have

$$\bar{0} = \overline{d(s^i t^j)} = \overline{s^i t^j (i\frac{ds}{s} + j\frac{dt}{t})} \tag{25}$$

for all i, $j \in \mathbb{Z}$. This module has a $\mathbb{Z}^2$-grading structure by putting

$$\deg \overline{d(s^i t^j) s^k t^l} := (i+k, j+l) \in \mathbb{Z}^2.$$

Fact 6. *The homogeneous space of degree (i,j) is*

$$\begin{array}{cc} \mathbb{C}\overline{s^i t^j \frac{ds}{s}} = \mathbb{C}\overline{s^i t^j \frac{dt}{t}}, & \text{if} \quad i \neq 0 \text{ and } j \neq 0, \\ \mathbb{C}\overline{s^i \frac{dt}{t}} & \text{if} \quad i \neq 0 \text{ and } j = 0, \\ \mathbb{C}\overline{t^j \frac{ds}{s}} & \text{if} \quad i = 0 \text{ and } j \neq 0, \\ \mathbb{C}\overline{\frac{ds}{s}} \oplus \mathbb{C}\overline{\frac{dt}{t}} & \text{if} \quad i = 0 \text{ and } j = 0. \end{array}$$

Hence the dimension of the homogeneous space of degree (i,j) is 1 unless $(i,j) = (0,0)$, this case is two-dimensional.

Proof. It is easily shown using (25). □

The $\mathbb{Z}^2$-gradings on A and Ω_A/dA naturally induces the $\mathbb{Z}_{t_1,t_2}$-gradings on them for any given two positive integers t_1 and t_2. We put

$$A_{\bar{\alpha}} := \{a \in A \mid a \text{ is homogeneous and } \deg a = \bar{\alpha}\},$$

$$(\Omega_A/dA)_{\bar{\alpha}} := \{a \in \Omega_A/dA \mid a \text{ is homogeneous and } \deg a = \bar{\alpha}\}$$

for all $\bar{\alpha} \in \mathbb{Z}_{t_1,t_2}$.

Let us define the following operators $s\frac{\partial}{\partial s}$ and $t\frac{\partial}{\partial t}$ on Ω_A/dA by

$$s\frac{\partial}{\partial s}\overline{d(s^i t^j)s^k t^l} := (i+k)\overline{d(s^i t^j)s^k t^l}, \tag{26}$$

$$t\frac{\partial}{\partial t}\overline{d(s^i t^j)s^k t^l} := (j+l)\overline{d(s^i t^j)s^k t^l}. \tag{27}$$

4.3. **Tensor Algebra and Root Space Decomposition.**

Let $\mathfrak{g}$ be a Lie algebra and B be a commutative algebra. Then we define a Lie algebra structure on $\mathfrak{g} \otimes B$ by putting

$$[x \otimes a, y \otimes b] := [x,y] \otimes ab,$$

where $x, y \in \mathfrak{g}$ and $[x,y]$ is the bracket in $\mathfrak{g}$, and $a, b \in B$.

Let $\mathfrak{g}$ be a simple Lie algebra and $\mathfrak{h}$ be a Cartan subalgebra. To consider a root space decomposition of the tensor algebra $\mathfrak{g} \otimes A$, we consider an extension of $\tilde{\mathfrak{g}} := \mathfrak{g} \otimes A$:

$$\hat{\mathfrak{g}} := \mathfrak{g} \otimes A \oplus \mathbb{C}s\frac{\partial}{\partial s} \oplus \mathbb{C}t\frac{\partial}{\partial t},$$

where we put

$$\begin{array}{rcl} [s\frac{\partial}{\partial s}, x \otimes s^i t^j] & := & i(x \otimes s^i t^j), \\ [t\frac{\partial}{\partial t}, x \otimes s^i t^j] & := & j(x \otimes s^i t^j) \end{array}$$

for $x \in \mathfrak{g}$ and i, $j \in \mathbb{Z}$, and

$$[\mathbb{C}s\frac{\partial}{\partial s} + \mathbb{C}t\frac{\partial}{\partial t}, \mathbb{C}s\frac{\partial}{\partial s} + \mathbb{C}t\frac{\partial}{\partial t}] := 0.$$

Put an abelian subalgebra

$$H := \mathfrak{h} \oplus \mathbb{C}s\frac{\partial}{\partial s} \oplus t\frac{\partial}{\partial t}$$

and its dual space

$$F_{\mathbb{C}} := H^*.$$

Let us define $\mathfrak{h}^* \subset F_{\mathbb{C}}$ and b, $a \in F_{\mathbb{C}}$ by the following table.

$F_{\mathbb{C}} \setminus H$	$\mathfrak{h}$	$s\frac{\partial}{\partial s}$	$t\frac{\partial}{\partial t}$
$\mathfrak{h}^*$	$\langle \cdot, \cdot \rangle$	0	0
b	0	1	0
a	0	0	1

Let $R_{\mathrm{f}} \subset \mathfrak{h}^*$ be the set of roots of $\mathfrak{g}$ and put $R_{\mathrm{f}}^{+0} := R_{\mathrm{f}} \cup \{0\}$. It is easy to see that the algebra $\hat{\mathfrak{g}}$ has a root space decomposition with respect to H and the set of roots are

$$\Delta := \{\alpha + mb + na \mid \alpha \in R_{\mathrm{f}}^{+0},\ m, n \in \mathbb{Z}\}.$$

Let us denote the root space of a root $\mu \in \Delta$ by $\hat{\mathfrak{g}}_\mu$. We have $\hat{\mathfrak{g}}_0 = H$, and $x \otimes s^i t^j \in \hat{\mathfrak{g}}_{\alpha + ib + ja}$ for $x \in \mathfrak{g}_\alpha$ $(\alpha \in \mathfrak{h}^*)$, i, $j \in \mathbb{Z}$.

We equip a $Q(R_{\mathrm{f}}) \oplus \mathbb{Z}b \oplus \mathbb{Z}a$-grading on $\tilde{\mathfrak{g}}$ by putting the homogeneous space of $\alpha + mb_n a \in Q(R_{\mathrm{f}}) \oplus \mathbb{Z}b \oplus \mathbb{Z}a$ to be the root space (of $\hat{\mathfrak{g}}$) of weight $\alpha + mb + na$ if $\alpha + mb + na \neq 0$, and the homogeneous space of 0 to be $\mathfrak{h}$.

4.4. **Central Extension.**

Let $\mathfrak{g}$ be a Lie algebra and W be a vector space. We define

$$Z^2(\mathfrak{g}, W) := \left\{ f : \mathfrak{g} \times \mathfrak{g} \to W \;\middle|\; \begin{array}{l} f \text{ is a bilinear form satisfying (i) and (ii):} \\ \text{(i)} \quad f(x,y) = -f(y,x), \\ \text{(ii)} \quad f([x,y],z) + f([y,z],x) + f([z,x],y) = 0 \end{array} \right\}$$

For each $f \in Z^2(\mathfrak{g}, W)$, we can define a Lie algebra structure on $\mathfrak{g} \oplus W$ in which W is central as follows. We define the bracket on $\mathfrak{g} \oplus W$ by

$$[a + x, b + y] := [a, b] + f(x, y),$$

where $a, b \in \mathfrak{g}$ and $[a, b]$ is the bracket in $\mathfrak{g}$ for $x, y \in W$. We say $\mathfrak{g} \oplus W$ is the *central extension of* $\mathfrak{g}$ *by* (W, f).

Let $\mathfrak{g}$ be a simple Lie algebra and let $\tilde{\mathfrak{g}} := \mathfrak{g} \otimes A$. Let us define a bilinear map ϕ by

$$\begin{array}{rccc} \phi: & (\mathfrak{g} \otimes A) \times (\mathfrak{g} \otimes A) & \to & \Omega_A/dA, \\ & (x \otimes s^i t^j, y \otimes s^k t^l) & \mapsto & (x, y)\overline{d(s^i t^j) s^k t^l}, \end{array} \tag{28}$$

where $(\cdot, \cdot)$ is the Killing form on $\mathfrak{g}$. The following fact is easy to see.

Fact 7. $\phi \in Z^2(\mathfrak{g} \otimes A, \Omega_A/dA)$.

According to the fact, we can consider the central extension of $\mathfrak{g}$ by $(\Omega_A/dA, \phi)$. We equip a $Q(R_{\mathrm{f}}) \oplus \mathbb{Z}b \oplus \mathbb{Z}a$-grading on the central extension by putting the homogeneous space of degree $\alpha + mb + na$ to be the direct sum of the homogeneous space of degree $\alpha + mb + na$ of $\tilde{\mathfrak{g}}$ and the homogeneous space of degree $mb + na$ of Ω_A/dA.

5. Twisted Loop Algebras

In this section, we introduce explicitly the algebra $\tilde{\mathfrak{g}}(X_l^{(t_1,t_2)})$ for each reduced marked elliptic root system of type $X_l^{(t_1,t_2)}$ and determin the universal covering algebra (which we recall in Sect. 3.4) $\hat{\mathfrak{g}}(X_l^{(t_1,t_2)})$ of $\tilde{\mathfrak{g}}(X_l^{(t_1,t_2)})$ for the reduced marked elliptic root systems belonging to group I, II, or III (see the table in Sect. 2.3). For group IV, we submit certain algebras which we conjecture to be the universal covering algebras. In each case, we shall refer Appendix 7.3 for an explicit description of the reduced marked elliptic root system. In the following, our proofs are sometimes, unfortunately, just direct calculations inside certain simple Lie algebras.

The following is general setting for Sect. 5.1–5.4, where we introduce algebras individually. In each case, we have two paragraphs (A) and (B).

Let (R, G) be an reduced marked elliptic root system of type $X_l^{(t_1,t_2)}$. We consider the $\mathbb{Z}_{t_1,t_2}$-gradings on A and on Ω_A/dA (see Sect. 4.2).

In (A), we introduce the algebra $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}(X_l^{(t_1,t_2)})$, using a $\mathbb{Z}_{t_1,t_2}$-grading on a Lie algebra $\mathfrak{g}$ as follows. We first choose an algebra $\mathfrak{g}$ and a simple subalgebra $\mathfrak{g}_{\bar{\circ}}$ of $\mathfrak{g}$. We consider $\mathfrak{g}$ as a $\mathfrak{g}_{\bar{\circ}}$-module by the adjoint action. Next, we define $\mathfrak{g}_{\bar{\circ}}$-submodules $\mathfrak{g}_{\bar{\alpha}}$ of $\mathfrak{g}$ for all $\bar{\alpha} \in \mathbb{Z}'_{t_1,t_2}$. Then we have the following fact in all the cases (even for group IV).

Theorem 2. *The set of* $\mathfrak{g}_{\bar{\alpha}}$*-modules* $\{\mathfrak{g}_{\bar{\alpha}} \mid \bar{\alpha} \in \mathbb{Z}_{t_1,t_2}\}$ *induces a* $\mathbb{Z}_{t_1,t_2}$*-grading structure on the Lie algebra* $\mathfrak{g}$*. That is,*

(i) $\mathfrak{g} = \bigoplus_{\bar{\alpha}\in\mathbb{Z}_{t_1,t_2}} \mathfrak{g}_{\bar{\alpha}}$,

(ii) $[\mathfrak{g}_{\bar{\alpha}}, \mathfrak{g}_{\bar{\beta}}] \subset \mathfrak{g}_{\bar{\alpha}+\bar{\beta}}$ *for all* $\bar{\alpha}$, $\bar{\beta} \in \mathbb{Z}_{t_1,t_2}$.

Proof. We can show this by case-by-case calculations. □

Now we define $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}(X_l^{(t_1,t_2)})$ as a subalgebra of $\mathfrak{g} \otimes A$ (see Sect. 4.3) by

$$\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}(X_l^{(t_1,t_2)}) := \bigoplus_{(i,j)\in\mathbb{Z}^2} \mathfrak{g}_{(\bar{i},\bar{j})} \otimes s^i t^j. \tag{29}$$

In (B), we introduce a space W and a form $\varphi \in Z^2(\mathfrak{g}, W)$, and define the algebra

$$\hat{\mathfrak{g}} = \hat{\mathfrak{g}}(X_l^{(t_1,t_2)}) := \text{the central extension of } \tilde{\mathfrak{g}} \text{ by } (W,\varphi) \tag{30}$$

(see Sect. 4.4) and the natural projection

$$\pi \ : \ \hat{\mathfrak{g}} = \tilde{\mathfrak{g}} \oplus W \ \to \tilde{\mathfrak{g}}. \tag{31}$$

The following fact (conjecture for group IV) is one of the main results of this article.

Theorem 3. $(\hat{\mathfrak{g}}, \pi)$ *is the universal covering algebra of* $\tilde{\mathfrak{g}}$*.*

Proof. We can show this using a proper transformation of Wilson's argument in [W]. See Sect. 6.1 for more detail. □

In the following subsections, we will use the form $\phi \in Z^2(\mathfrak{g}\otimes A, \Omega_A/dA)$ defined in (28) to define φ, and we will use the following fact on the Killing form $(\cdot,\cdot)$ on $\mathfrak{g}$, which is again true in all the cases (even for group IV), to define the space W.

Claim 5. (i) $(\mathfrak{g}_{\bar{\alpha}}, \mathfrak{g}_{\bar{\beta}}) = 0$ *unless* $\bar{\alpha} + \bar{\beta} = \bar{o}$ *for* $\bar{\alpha}$, $\bar{\beta} \in \mathbb{Z}_{t_1,t_2}$.

(ii) *Let* $\bar{\alpha}$, $\bar{\beta} \in \mathbb{Z}_{t_1,t_2}$ *such that* $\bar{\alpha} + \bar{\beta} = \bar{o}$*. Then the restriction of* $(\cdot,\cdot)$ *on* $\mathfrak{g}_{\bar{\alpha}} \times \mathfrak{g}_{\bar{\beta}}$ *is non-degenerate.*

Proof. We can show (i) by case-by-case calculations. (ii) almost follows (i). □

5.1. **Construction I.**

Here we consider the types in group I, that is, the types $X_l^{(t_1,t_2)}$ where both $t_1 = 1$ and $t_2 = 1$.

(A) Let $\mathfrak{g} := \mathfrak{g}(X_l)$ and $\mathfrak{g}_{\bar{o}} := \mathfrak{g}$. ($\mathbb{Z}_{t_1,t_2} = \{\bar{o}\}$.)

(B) Put $W := \Omega_A/dA$ and $\varphi := \phi$ (ϕ is in (28)).

Now we shall recall the definitions of $\tilde{\mathfrak{g}}$ and $\hat{\mathfrak{g}}$ in (29) and (30), and recall that we have Claim 5 and Theorem 3 and Theorem 1.

Remark. In these cases, $\hat{\mathfrak{g}}$ in (30) is the troidal algebra of two variables in the sence [MEY].

5.2. **Construction II.**

Here we consider the types in group II, that is, the types $X_l^{(t_1,t_2)}$ where either $t_1 = 1$ or $t_2 = 1$ and $t_1 t_2 > 1$. These cases can be handled by a slite exchange of the arguments in Kac [K] and Wilson [W].

(A) In the following, we consider Dynkin-diagram automorphisms σ of simple Lie algebras $\mathfrak{g}$. Let κ be the order of σ and put

$$\mathfrak{g}_{\bar{i}} := \{x \in \mathfrak{g} \mid \sigma(x) = \eta^i x\}$$

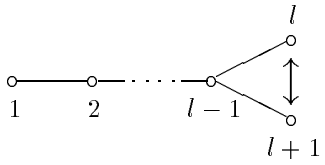
for $\bar{i} \in \mathbb{Z}_\kappa$, where $\eta := e^{2\pi\sqrt{-1}/\kappa}$, the κ-th primitive root of 1. We put

Case $t_1 = 1$: $\mathfrak{g}_{(\bar{0},\bar{i})} := \mathfrak{g}_{\bar{i}}$ for all $\bar{i} \in \mathbb{Z}_{t_2}$ $(t_2 = \kappa)$,

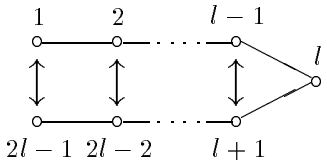
Case $t_2 = 1$: $\mathfrak{g}_{(\bar{i},\bar{0})} := \mathfrak{g}_{\bar{i}}$ for all $\bar{i} \in \mathbb{Z}_{t_1}$ $(t_1 = \kappa)$.

Let us define $\mathfrak{g}$ and σ ease by case-by-case as follows.

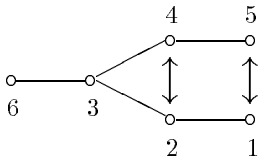
Case $B_l^{(2,1)}$ and $C_l^{(1,2)}$. $\mathfrak{g} = \mathfrak{g}(D_{l+1})$, σ :



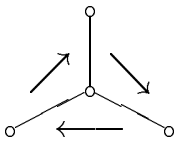
Case $C_l^{(2,1)}$ and $B_l^{(1,2)}$. $\mathfrak{g} = \mathfrak{g}(A_{2l-1})$, σ :



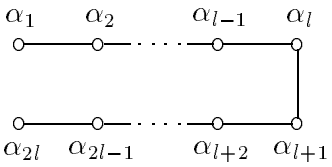
Case $F_4^{(2,1)}$ and $F_4^{(1,2)}$. $\mathfrak{g} = \mathfrak{g}(E_6)$, σ :



Case $G_2^{(3,1)}$ and $G_2^{(1,3)}$. $\mathfrak{g} = \mathfrak{g}(D_4)$, σ :



Case $BC_l^{(2,1)}$. $\mathfrak{g} = \mathfrak{g}(A_{2l})$, σ :



Now we shall recall the definitions of $\tilde{\mathfrak{g}}$ in (29).

(B) We put $W := \phi(\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}})$ and $\varphi := \phi$ (ϕ is in (28)). Note that $\phi(\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}) = (\Omega_A/dA)_{\bar{o}}$ according to Claim 5.

Now we shall recall the definitions of $\hat{\mathfrak{g}}$ in (30), and recall that we have Theorem 3 and Theorem 1.

Remark. Let (R, G), $X_l^{(t_1,t_2)}$, $\mathfrak{g}$, σ, κ and $\mathfrak{g}_{\bar{i}}$ ($\bar{i} \in \mathbb{Z}_\kappa$) as above. Let us assume $t_2 = 1$. Then Wilson has proved in [W] that the universal covering algebra of the subalgebra $\bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_{\bar{i}} \otimes s^i$ of $\mathfrak{g} \otimes \mathbb{C}[s^{\pm 1}]$ is isomorphic to the Kac-Moody algebra of type $X_l^{(t_1)}$ in [K].

5.3. **Construction III.**

Here we consider the types in group III, that is, the cases $X_l^{(t_1,t_2)}$ where $t_1 > 1$ and $t_2 > 1$.

(A) In the following case-by-case arguments, we introduce the following objects. First, we introduce a simple Lie algebra $\mathfrak{g}$ and a subalgebra $\mathfrak{g}_{\bar{o}}$, which is simple, of $\mathfrak{g}$. We consider $\mathfrak{g}$ as a $\mathfrak{g}_{\bar{o}}$-module by the adjoint action. Next, we define $\mathfrak{g}_{\bar{o}}$-modules $\mathfrak{g}'_{\bar{\alpha}}$ for all $\bar{\alpha} \in \mathbb{Z}'_{t_1,t_2}$. In all cases, surprisingly, we have the following facts.

Claim 6. *The submodule $\mathfrak{g}'_{\bar{\alpha}}$ is irreducible for all $\bar{\alpha} \in \mathbb{Z}'_{t_1,t_2}$.*

Proof. We can show this by case-by-case calculations. □

Claim 7. *There exist uniquely (up to scalar multiple) an element $z_{\bar{\alpha}} \in \mathfrak{g}$ (which may be 0) for all $\bar{\alpha} \in \mathbb{Z}'_{t_1,t_2}$ satisfying the following* (i) *and* (ii) :

(i) $[\mathfrak{g}_{\bar{o}}, z_{\bar{\alpha}}] = 0$ *for all* $\bar{\alpha} \in \mathbb{Z}_{t_1,t_2}$, $\bar{\alpha} \neq \bar{o}$,

(ii) $[\mathfrak{g}'_{\bar{\alpha}}, \mathfrak{g}'_{\bar{\beta}}] \subset \mathfrak{g}'_{\bar{\gamma}} \oplus \mathbb{C} z_{\bar{\gamma}}$ *for all* $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma} \in \mathbb{Z}_{t_1,t_2} \setminus \{\bar{o}\}$ *such that* $\bar{\alpha} + \bar{\beta} = \bar{\gamma}$.

Proof. We can show this by case-by-case calculations. □

Now we put

$$\mathfrak{g}_{\bar{\alpha}} := \mathfrak{g}'_{\bar{\alpha}} \oplus \mathbb{C}z_{\bar{\alpha}}$$

for $\bar{\alpha} \in \mathbb{Z}'_{t_1,t_2}$.

Let $\{z_{\bar{\alpha}} \mid \bar{\alpha} \in \mathbb{Z}'_{t_1,t_2}\}$ as above. Then we have the following facts.

Claim 8. (i) $[\mathfrak{g}'_{\bar{\alpha}}, z_{\bar{\alpha}}] = 0$,
(ii) $(\mathfrak{g}'_{\bar{\alpha}}, z_{\bar{\alpha}}) = 0$.

Proof. We can show this by case-by-case calculations. □

Claim 9. *Let $\bar{\alpha} \in \mathbb{Z}'_{t_1,t_2}$. Then $\mathfrak{g}_{\bar{\alpha}}$ and $\mathfrak{g}_{-\bar{\alpha}}$ are $\mathfrak{g}_{\bar{o}}$-isomorphic. In particular, $z_{\bar{\alpha}} \neq 0$ if and only if $z_{-\bar{\alpha}} \neq 0$.*

Proof. We can show this by case-by-case calculations. □

Claim 10. *Let $\bar{\alpha}$, $\bar{\beta} \in \mathbb{Z}'_{t_1,t_2}$ such that $\bar{\alpha} + \bar{\beta} \neq \bar{o}$ and $\bar{\alpha} \neq \bar{\beta}$. Put $\bar{\gamma} := -\bar{\alpha} - \bar{\beta}$. If $z_{\bar{\gamma}} \neq 0$, then we have the followings.*

(i) *The map*

$$\begin{array}{rccc} f_{\bar{\gamma}}: & \mathfrak{g}_{\bar{\alpha}} & \to & \mathfrak{g}_{-\bar{\beta}}, \\ & u & \mapsto & [z_{\bar{\gamma}}, u] \end{array}$$

is a $\mathfrak{g}_{\bar{o}}$-isomorphism. In particular, $f_{\bar{\gamma}}(\mathfrak{g}'_{\bar{\alpha}}) = \mathfrak{g}'_{\bar{\beta}}$ and $f_{\bar{\gamma}}(\mathbb{C}z_{\bar{\alpha}}) = \mathbb{C}z_{\bar{\beta}}$.

(ii) *The map*

$$\begin{array}{rccc} g_{\bar{\gamma}}: & \mathfrak{g}_{\bar{\alpha}} & \to & \mathrm{Hom}_{\mathbb{C}}(\mathfrak{g}_{\bar{\beta}}, \mathbb{C}), \\ & u & \mapsto & ([z_{\bar{\gamma}}, u], \cdot) \end{array}$$

is a $\mathfrak{g}_{\bar{o}}$-homomorphism, where $\mathfrak{g}_{\bar{o}}$ acts on $\mathrm{Hom}_{\mathbb{C}}(\mathfrak{g}_{\bar{\beta}}, \mathbb{C})$ by

$$(u \cdot h)(v) := -h(u, v)$$

for $u \in \mathfrak{g}_{\bar{\alpha}}$, $v \in \mathfrak{g}_{\bar{\beta}}$, and $h \in \mathrm{Hom}_{\mathbb{C}}(\mathfrak{g}_{\bar{\beta}}, \mathbb{C})$.

Proof. We can show these by case-by-case calculations. □

Now we shall recall the definitions of $\tilde{\mathfrak{g}}$ in (29).

(B) Let us first define a bilinear map $\psi : \tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}} \to A$ as follows. We define the bilinear form

$$\begin{array}{rccc} \bar{\psi}: & \mathfrak{g}_{\bar{\alpha}} \times \mathfrak{g}_{\bar{\beta}} & \to & \mathbb{C}, \\ & (x, y) & \mapsto & \bar{\psi}(x, y) \end{array} \tag{32}$$

for $\alpha := (i, j)$, $\beta := (k, l) \in \mathbb{Z}^2$ (we put $\gamma := -\alpha - \beta$ below) first by the following (i) and (ii):

(i) Case $\bar{\alpha}, \bar{\beta} \in \mathbb{Z}'_{t_1,t_2}$ such that $\bar{\alpha}+\bar{\beta} \neq \bar{o}$ and $\bar{\alpha} \neq \bar{\beta}$:

$$\bar{\psi}(x,y) := ([z_{\bar{\gamma}}, x], y) = -(x, [z_{\bar{\gamma}}, y]),$$

where $(\cdot,\cdot)$ is the Killing form of $\mathfrak{g}$ and $\gamma := -\alpha - \beta$. (Note that if $z_{\bar{\gamma}} = 0$, then $\bar{\psi}(\mathfrak{g}_{\bar{\alpha}}, \mathfrak{g}_{\bar{\beta}}) = 0$.)

(ii) The other cases : $\bar{\psi}(\mathfrak{g}_{\bar{\alpha}}, \mathfrak{g}_{\bar{\beta}}) := 0$.

Using the $\bar{\psi}$, we define the bilinear form by

$$\begin{array}{rccc} \psi & : & \tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}} & \to & \Omega_A/dA, \\ & & (x \otimes s^i t^j, y \otimes s^k t^l) & \mapsto & \bar{\psi}(x,y)\overline{d(s^i t^j) s^k t^l}. \end{array}$$

Then we see the following.

Claim 11. $\psi \in Z^2(\tilde{\mathfrak{g}}, \Omega_A/dA)$.

Let us put $W_1 := \phi(\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}})$ and $W_2 := \psi(\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}})$. According to Claim 5, we have $W_1 = (\Omega_A/dA)_{\bar{o}}$. We will describe Z_2 explicitly in each case. Finally, let us put $W := W_1 \oplus W_2$ and $\varphi := \phi \oplus \psi$ (ϕ is in (28)).

We will describe explicitly the map $\bar{\psi}$ by a table like as in (34). In the table, we used the following abbreviation

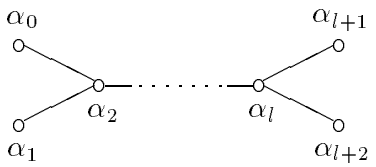
$$(z_{\overline{ij}}\cdot,\cdot) = (z_{\overline{ij}}u, v) := ([z_{(\bar{i},\bar{j})}, u], v) \tag{33}$$

and the value of $\bar{\psi}(u,v)$ is defined to be the $\big((\bar{i},\bar{j}),(\bar{k},\bar{l})\big)$- entry of the table for $u \in \mathfrak{g}_{(\bar{i},\bar{j})}$ and $v \in \mathfrak{g}_{(\bar{k},\bar{l})}$. We also use the symbol * in the $\big((\bar{i},\bar{j}),(\bar{k},\bar{l})\big)$-entry to indicate $(\bar{i},\bar{j})+(\bar{k},\bar{l}) = (\bar{0},\bar{0})$, that is, the Killing form $(\cdot,\cdot)$ is nondegenerate on $\mathfrak{g}_{(\bar{i},\bar{j})} \times \mathfrak{g}_{(\bar{k},\bar{l})}$ (see Claim 5).

Now we shall recall the definitions of $\hat{\mathfrak{g}}$ in (30) and recall that we have Theorem 3 and Theorem 1.

Before we go into each case, we shall recall the extended simple basis and extended Chevalley generators in Sect. 4.1.

5.3.1. *Case* $B_l^{(2,2)}$. Let $\mathfrak{g} := \mathfrak{g}(D_{l+2})$ and $\{\alpha_0, \cdots, \alpha_{l+2}\}$ be an extended simple basis of $\mathfrak{g}$ whose diagram is as follows:



Let $\{e_i, h_i, f_i \mid i = 0, \ldots, l+2\}$ be an extended Chevalley generators. Put the subalgebra $\mathfrak{g}_{\bar{o}}$ of $\mathfrak{g}$ by

$$\mathfrak{g}_{\bar{o}} := \langle \square_i \ (i = 2, \ldots, l),\ \square_{l+1} + \square_{l+2} \mid \square = e, f \rangle.$$

Then it is easy to see that $\mathfrak{g}_{\bar{o}} \simeq \mathfrak{g}(B_l)$. Put

$$\begin{aligned} \mathfrak{g}'_{(\bar{1},\bar{0})} &:= \mathfrak{g}_{\bar{o}}(e_0 + e_1), \\ \mathfrak{g}'_{(\bar{0},\bar{1})} &:= \mathfrak{g}_{\bar{o}}(e_0 - e_1), \\ \mathfrak{g}'_{(\bar{1},\bar{1})} &:= \mathfrak{g}_{\bar{o}}(h_{l+1} - h_{l+2}). \end{aligned}$$

In this case, $z_{\bar{\alpha}} \neq 0$ for all $\bar{\alpha} \in \mathbb{Z}_2^2$, and the explicit description of $\bar{\psi}$ in (32) is

(34)

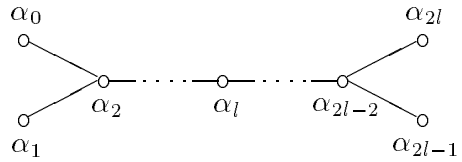
	$(\bar{0},\bar{0})$	$(\bar{1},\bar{0})$	$(\bar{0},\bar{1})$	$(\bar{1},\bar{1})$
$(\bar{0},\bar{0})$	0^*	0	0	0
$(\bar{1},\bar{0})$	0	0^*	$(z_{\bar{1}\bar{1}}\cdot,\cdot)$	$(z_{\bar{0}\bar{1}}\cdot,\cdot)$
$(\bar{0},\bar{1})$	0	$(z_{\bar{1}\bar{1}}\cdot,\cdot)$	0^*	$(z_{\bar{1}\bar{0}}\cdot,\cdot)$
$(\bar{1},\bar{1})$	0	$(z_{\bar{0}\bar{1}}\cdot,\cdot)$	$(z_{\bar{1}\bar{0}}\cdot,\cdot)$	0^*

(recall (33)). Using this $\bar{\psi}$, we have

$$W_2 = \bigoplus_{\bar{\alpha} \in \mathbb{Z}'_{2,2}} (\Omega_A/dA)_{\bar{\alpha}} \simeq \bigoplus_{\bar{\alpha} \in \mathbb{Z}'_{2,2}} A_{\bar{\alpha}} \tag{35}$$

as vector spaces.

5.3.2. *Case* $C_l^{(2,2)}$. Let $\mathfrak{g} := \mathfrak{g}(D_{2l})$ and $\{\alpha_0, \cdots, \alpha_{2l}\}$ be an extended simple root basis of $\mathfrak{g}$ whose diagram is as follows:



Let $\{e_i, h_i, f_i \mid i = 0, \ldots, 2l\}$ be an extended Chevalley generators. Put the subalgebra $\mathfrak{g}_{\bar{o}}$ of $\mathfrak{g}$ by

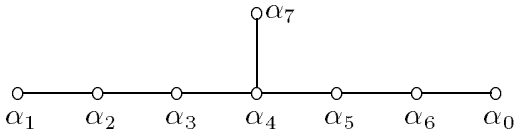
$$\mathfrak{g}_{\bar{o}} := \langle \square_i + \square_{2l-i} \ (i = 1, \ldots, l-1), \ \square_l \mid \square = e, f \rangle$$

Then it is easy to see that $\mathfrak{g}_{\bar{o}} \simeq \mathfrak{g}(D_{2l})$. Put

$$\begin{aligned} \mathfrak{g}'_{(\bar{1},\bar{0})} &:= \mathfrak{g}_{\bar{o}}(e_0 + e_{2l}), \\ \mathfrak{g}'_{(\bar{0},\bar{1})} &:= \mathfrak{g}_{\bar{o}}(e_0 - e_{2l}), \\ \mathfrak{g}'_{(\bar{1},\bar{1})} &:= \mathfrak{g}_{\bar{o}}(h_1 - h_{2l-1}). \end{aligned}$$

In this case, $z_{\bar{\alpha}} \neq 0$ for all $\bar{\alpha} \in \mathbb{Z}_2^2$, and $\bar{\psi}$ and W_2 have the same descriptions as in (34) and (35).

5.3.3. *Case* $F_4^{(2,2)}$. Let $\mathfrak{g} := \mathfrak{g}(E_7)$ and $\{\alpha_0, \cdots, \alpha_7\}$ be an extended simple root basis of $\mathfrak{g}$ whose diagram is as follows:



Let $\{e_i, h_i, f_i \mid i = 0, \ldots, 7\}$ be an extended Chevalley generators. Put the subalgebra $\mathfrak{g}_{\bar{\sigma}}$ of $\mathfrak{g}$ by

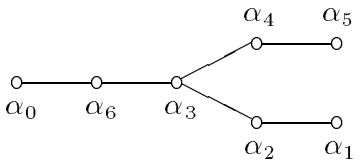
$$\mathfrak{g}_{\bar{\sigma}} := \langle \Box_2 + \Box_6,\ \Box_3 + \Box_5,\ \Box_4,\ \Box_7 \mid \Box = e, f \rangle$$

Then it is easy to see that $\mathfrak{g}_{\bar{\sigma}} \simeq \mathfrak{g}(F_4)$. Put

$$\begin{aligned} \mathfrak{g}'_{(\bar{1},\bar{0})} &:= \mathfrak{g}_{\bar{\sigma}}(e_0 + e_1), \\ \mathfrak{g}'_{(\bar{0},\bar{1})} &:= \mathfrak{g}_{\bar{\sigma}}(e_0 - e_1), \\ \mathfrak{g}'_{(\bar{1},\bar{1})} &:= \mathfrak{g}_{\bar{\sigma}}(h_2 - h_6). \end{aligned}$$

In this case, $z_{\bar{\alpha}} \neq 0$ for all $\bar{\alpha} \in \mathbb{Z}_2^2$, and $\bar{\psi}$ and W_2 have the same descriptions as in (34) and (35).

5.3.4. *Case* $G_2^{(3,3)}$. Let $\mathfrak{g} := \mathfrak{g}(E_6)$ and $\{\alpha_0, \cdots, \alpha_6\}$ be an extended simple root basis of $\mathfrak{g}$ whose diagram is as follows:



Let $\{e_i, h_i, f_i \mid i = 0, \ldots, 6\}$ be an extended Chevalley generators. Put the subalgebra $\mathfrak{g}_{\bar{\sigma}}$ of $\mathfrak{g}$ by

$$\mathfrak{g}_{\bar{\sigma}} := \langle \Box_2 + \Box_4 + \Box_6,\ \Box_3 \mid \Box = e, f \rangle$$

Then it is easy to see that $\mathfrak{g}_{\bar{\sigma}} \simeq \mathfrak{g}(G_2)$. Let $\eta := e^{2\pi\sqrt{-1}/3}$ be the 3rd primitive root of 1. Put

$$\begin{aligned} \mathfrak{g}'_{(\bar{1},\bar{0})} &:= \mathfrak{g}_{\bar{\sigma}}(e_0 + \eta e_1 + \eta^2 e_5), \\ \mathfrak{g}'_{(\bar{2},\bar{0})} &:= \mathfrak{g}_{\bar{\sigma}}(f_0 + \eta^2 f_1 + \eta f_5), \\ \mathfrak{g}'_{(\bar{0},\bar{1})} &:= \mathfrak{g}_{\bar{\sigma}}(e_0 + e_1 + e_5), \\ \mathfrak{g}'_{(\bar{1},\bar{1})} &:= \mathfrak{g}_{\bar{\sigma}}(f_0 + \eta f_1 + \eta^2 f_5), \\ \mathfrak{g}'_{(\bar{2},\bar{1})} &:= \mathfrak{g}_{\bar{\sigma}}(h_2 + \eta^2 h_4 + \eta h_6), \\ \mathfrak{g}'_{(\bar{0},\bar{2})} &:= \mathfrak{g}_{\bar{\sigma}}(f_0 + f_1 + f_5), \\ \mathfrak{g}'_{(\bar{1},\bar{2})} &:= \mathfrak{g}_{\bar{\sigma}}(h_2 + \eta h_4 + \eta^2 h_6), \\ \mathfrak{g}'_{(\bar{2},\bar{2})} &:= \mathfrak{g}_{\bar{\sigma}}(e_0 + \eta^2 e_1 + \eta e_5). \end{aligned}$$

In this case, $z_{\bar{\alpha}} \neq 0$ for all $\bar{\alpha} \in \mathbb{Z}_3^2$, and the explicit description of $\bar{\psi}$ in (32) is

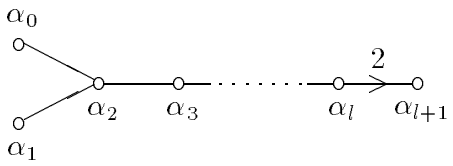
	$(\bar{0},\bar{0})$	$(\bar{1},\bar{0})$	$(\bar{2},\bar{0})$	$(\bar{0},\bar{1})$	$(\bar{1},\bar{1})$	$(\bar{2},\bar{1})$	$(\bar{0},\bar{2})$	$(\bar{1},\bar{2})$	$(\bar{2},\bar{2})$
$(\bar{0},\bar{0})$	0^*	0	0	0	0	0	0	0	0
$(\bar{1},\bar{0})$	0	0	0^*	$(z_{\bar{2}\bar{2}}\cdot,\cdot)$	$(z_{\bar{1}\bar{2}}\cdot,\cdot)$	$(z_{\bar{0}\bar{2}}\cdot,\cdot)$	$(z_{\bar{2}\bar{1}}\cdot,\cdot)$	$(z_{\bar{1}\bar{1}}\cdot,\cdot)$	$(z_{\bar{0}\bar{1}}\cdot,\cdot)$
$(\bar{2},\bar{0})$	0	0^*	0	$(z_{\bar{1}\bar{2}}\cdot,\cdot)$	$(z_{\bar{0}\bar{2}}\cdot,\cdot)$	$(z_{\bar{2}\bar{2}}\cdot,\cdot)$	$(z_{\bar{1}\bar{1}}\cdot,\cdot)$	$(z_{\bar{0}\bar{1}}\cdot,\cdot)$	$(z_{\bar{2}\bar{1}}\cdot,\cdot)$
$(\bar{0},\bar{1})$	0	$(z_{\bar{2}\bar{2}}\cdot,\cdot)$	$(z_{\bar{1}\bar{2}}\cdot,\cdot)$	0	$(z_{\bar{2}\bar{1}}\cdot,\cdot)$	$(z_{\bar{1}\bar{1}}\cdot,\cdot)$	0^*	$(z_{\bar{2}\bar{0}}\cdot,\cdot)$	$(z_{\bar{1}\bar{0}}\cdot,\cdot)$
$(\bar{1},\bar{1})$	0	$(z_{\bar{1}\bar{2}}\cdot,\cdot)$	$(z_{\bar{0}\bar{2}}\cdot,\cdot)$	$(z_{\bar{2}\bar{1}}\cdot,\cdot)$	0	$(z_{\bar{0}\bar{1}}\cdot,\cdot)$	$(z_{\bar{2}\bar{0}}\cdot,\cdot)$	$(z_{\bar{1}\bar{0}}\cdot,\cdot)$	0^*
$(\bar{2},\bar{1})$	0	$(z_{\bar{0}\bar{2}}\cdot,\cdot)$	$(z_{\bar{2}\bar{2}}\cdot,\cdot)$	$(z_{\bar{1}\bar{1}}\cdot,\cdot)$	$(z_{\bar{0}\bar{1}}\cdot,\cdot)$	0	$(z_{\bar{1}\bar{0}}\cdot,\cdot)$	0^*	$(z_{\bar{2}\bar{0}}\cdot,\cdot)$
$(\bar{0},\bar{2})$	0	$(z_{\bar{2}\bar{1}}\cdot,\cdot)$	$(z_{\bar{1}\bar{1}}\cdot,\cdot)$	0^*	$(z_{\bar{2}\bar{0}}\cdot,\cdot)$	$(z_{\bar{1}\bar{0}}\cdot,\cdot)$	0	$(z_{\bar{2}\bar{2}}\cdot,\cdot)$	$(z_{\bar{1}\bar{2}}\cdot,\cdot)$
$(\bar{1},\bar{2})$	0	$(z_{\bar{1}\bar{1}}\cdot,\cdot)$	$(z_{\bar{0}\bar{1}}\cdot,\cdot)$	$(z_{\bar{2}\bar{0}}\cdot,\cdot)$	$(z_{\bar{1}\bar{0}}\cdot,\cdot)$	0^*	$(z_{\bar{2}\bar{2}}\cdot,\cdot)$	0	$(z_{\bar{0}\bar{2}}\cdot,\cdot)$
$(\bar{2},\bar{2})$	0	$(z_{\bar{0}\bar{1}}\cdot,\cdot)$	$(z_{\bar{2}\bar{1}}\cdot,\cdot)$	$(z_{\bar{1}\bar{0}}\cdot,\cdot)$	0^*	$(z_{\bar{2}\bar{0}}\cdot,\cdot)$	$(z_{\bar{1}\bar{2}}\cdot,\cdot)$	$(z_{\bar{0}\bar{2}}\cdot,\cdot)$	0

(recall (33)). Using this $\bar{\psi}$, we have

$$W_2 = \bigoplus_{\bar{\alpha}\in\mathbb{Z}'_{3,3}} (\Omega_A/dA)_{\bar{\alpha}} \simeq \bigoplus_{\bar{\alpha}\in\mathbb{Z}'_{3,3}} A_{\bar{\alpha}}$$

as vector spaces.

5.3.5. *Case* $B_l^{(2,2)*}$. Let $\mathfrak{g} := (B_{l+1})$ and $\{\alpha_1, \cdots, \alpha_{l+1}\}$ be an extended simple root basis of $\mathfrak{g}$ whose diagram is as follows:



Let $\{e_i, h_i, f_i \mid i = 0, \ldots, l+1\}$ be an extended Chevalley generators. Put the subalgebra $\mathfrak{g}_{\bar{o}}$ of $\mathfrak{g}$ by

$$\mathfrak{g}_{\bar{o}} := \langle \Box_i \ (i = 2, \ldots, l+1) \mid \Box = e, f \rangle$$

Then it is easy to see that $\mathfrak{g}_{\bar{o}} \simeq \mathfrak{g}(B_l)$. Put

$$\begin{aligned} \mathfrak{g}'_{(\bar{1},\bar{0})} &:= \mathfrak{g}_{\bar{o}}(e_0 + e_1), \\ \mathfrak{g}'_{(\bar{0},\bar{1})} &:= \mathfrak{g}_{\bar{o}}(e_0 - e_1), \\ \mathfrak{g}'_{(\bar{1},\bar{1})} &:= \mathfrak{g}_{\bar{o}}(h_0 - h_1) = \mathbb{C}(h_0 - h_1). \end{aligned}$$

In this case, $z_{(\bar{1},\bar{1})} \neq 0$ and $z_{(\bar{1},\bar{0})} = z_{(\bar{0},\bar{1})} = 0$, and the explicit description of $\bar{\psi}$ in (32) is

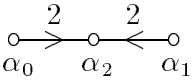
(36)

	$(\bar{0},\bar{0})$	$(\bar{1},\bar{0})$	$(\bar{0},\bar{1})$	$(\bar{1},\bar{1})$
$(\bar{0},\bar{0})$	0^*	0	0	0
$(\bar{1},\bar{0})$	0	0^*	$(z_{\bar{1}\bar{1}}\cdot,\cdot)$	0
$(\bar{0},\bar{1})$	0	$(z_{\bar{1}\bar{1}}\cdot,\cdot)$	0^*	0
$(\bar{1},\bar{1})$	0	0	0	0^*

(recall (33)). Using this $\bar{\psi}$, we have

$$W_2 = (\Omega_A/dA)_{(\bar{1},\bar{1})} \simeq A_{(\bar{1},\bar{1})}. \tag{37}$$

5.3.6. *Case* $A_1^{(1,1)*}$. Let $\mathfrak{g} := \mathfrak{g}(B_2)$ and $\{\alpha_0, \alpha_1, \alpha_2\}$ be an extended simple root basis of $\mathfrak{g}$ whose diagram is as follows:



Let $\{e_i, h_i, f_i \mid i = 0, 1, 2\}$ be an extended Chevalley generators. Put the subalgebra $\mathfrak{g}_{\bar{o}}$ of $\mathfrak{g}$ by

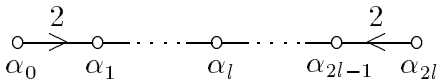
$$\mathfrak{g}_{\bar{o}} := \langle e_2, f_2 \rangle$$

Then it is easy to see that $\mathfrak{g}_{\bar{o}} \simeq \mathfrak{g}(A_1)$. Put

$$\begin{aligned} \mathfrak{g}'_{(\bar{1},\bar{0})} &:= \mathfrak{g}_{\bar{o}}(e_0 + e_1), \\ \mathfrak{g}'_{(\bar{0},\bar{1})} &:= \mathfrak{g}_{\bar{o}}(e_0 - e_1), \\ \mathfrak{g}'_{(\bar{1},\bar{1})} &:= \mathfrak{g}_{\bar{o}}(h_0 - h_1) = \mathbb{C}(h_0 - h_1). \end{aligned}$$

In this case, $z_{(\bar{1},\bar{1})} \neq 0$ and $z_{(\bar{1},\bar{0})} = z_{(\bar{0},\bar{1})} = 0$, and $\bar{\psi}$ and W_2 have the same description as in (36) and (37).

5.3.7. *Case* $C_l^{(1,1)*}$. Let $\mathfrak{g} := \mathfrak{g}(C_{2l})$. Let $\{\alpha_0, \cdots, \alpha_{2l}\}$ be an extended simple root basis of $\mathfrak{g}$ whose diagram is as follows:



Let $\{e_i, h_i, f_i \mid i = 1, \ldots, 2l\}$ be an extended Chevalley generators. Put the subalgebra $\mathfrak{g}_{\bar{o}}$ of $\mathfrak{g}$ by

$$\mathfrak{g}_{\bar{o}} := \langle \square_i + \square_{2l-i} \ (i = 1, \ldots, l-1),\ \square_l \mid \square = e, f \rangle$$

Then it is easy to see that $\mathfrak{g}_{\bar{o}} \simeq \mathfrak{g}(C_l)$. Put

$$\begin{aligned} \mathfrak{g}'_{(\bar{1},\bar{0})} &:= \mathfrak{g}_{\bar{o}}(e_0 + e_{2l}), \\ \mathfrak{g}'_{(\bar{0},\bar{1})} &:= \mathfrak{g}_{\bar{o}}(e_0 - e_{2l}), \\ \mathfrak{g}'_{(\bar{1},\bar{1})} &:= \mathfrak{g}_{\bar{o}}(h_0 - h_{2l}). \end{aligned}$$

In this case, $z_{(\bar{1},\bar{1})} \neq 0$ and $z_{(\bar{1},\bar{0})} = z_{(\bar{0},\bar{1})} = 0$, and $\bar{\psi}$ and W_2 have the same description as in (36) and (37).

5.4. **Construction IV.**

Here we consider the types in group IV, that is, the cases $BC_l^{(2,4)}$, $BC_l^{(2,2)}(1)$ and $BC_l^{(2,2)}(2)$. Unfortunately, the author has NOT yet completed the proofs of Theorem 3 and hence Theorem 1 for these cases. He has completed to check Theorem 2, Claim 6, Claim 7, Claim 8, Claim 9, and Claim 12 (ψ is defined below). So he submits the statements of Theorem 3 and hence Theorem 1 as CONJECTUREs.

(A) The setting is the same as one of Construction III which we repeat as follows. First we introduce a Lie algebra $\mathfrak{g}$ (which might be defined as a subalgebra of a Lie algebra $\dot{\mathfrak{g}}$) and a subalgebra $\mathfrak{g}_{\bar{o}}$, which is simple, of $\mathfrak{g}$. We consider $\mathfrak{g}$ as a $\mathfrak{g}_{\bar{o}}$-module by the adjoint action. Next, we define $\mathfrak{g}_{\bar{o}}$-modules $\mathfrak{g}'_{\bar{\alpha}}$ for all $\bar{\alpha} \in \mathbb{Z}'_{t_1,t_2}$. In all cases, surprisingly again, we have Claim 6, Claim 7, Claim 8, and Claim 9. Now we shall recall the definitions of $\tilde{\mathfrak{g}}$ in (29).

(B) Let us first define a bilinear map $\psi : \tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}} \to A$ as follows. We define the bilinear form $\bar{\psi}$ explicitly case-by-case below using abbreviation $(z_{\overline{ij}}\cdot,\cdot)$ as in (33) and * explained just below of (33). Using the $\bar{\psi}$, we define the bilinear form by

$$\begin{array}{rccc} \psi & : & \tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}} & \to & \Omega_A/dA, \\ & & (x \otimes s^i t^j, y \otimes s^k t^l) & \mapsto & \bar{\psi}(x,y)\overline{d(s^i t^j)s^k t^l}. \end{array}$$

Then we see the following.

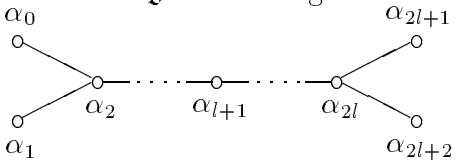
Claim 12. $\psi \in Z^2(\tilde{\mathfrak{g}}, \Omega_A/dA)$.

Let us put $W_1 := \phi(\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}})$ and $W_2 := \psi(\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}})$. According to Claim 5, we have $W_1 = (\Omega_A/dA)_{\bar{o}}$. We will describe Z_2 explicitly in each case. Finally, let us put $W := W_1 \oplus W_2$ and $\varphi := \phi \oplus \psi$ (ϕ is in (28)).

Now we shall recall the definitions of $\hat{\mathfrak{g}}$ in (30), and recall that we have Theorem 3 and Theorem 1, which are yet CONJECTUREs in group IV.

Before we go into each case, we shall recall the extended simple basis and extended Chevalley generators in Sect. 4.1.

5.4.1. *Case* $BC_l^{(2,4)}$. Let $\mathfrak{g} := \mathfrak{g}(D_{2l+2})$. Let $\{\alpha_0, \cdots, \alpha_{2l+2}\}$ be an extended simple root basis of $\mathfrak{g}$ whose diagram is as follows:



Let $\{e_i, h_i, f_i \mid i = 1, \dots, 2l+2\}$ be an extended Chevalley generators. Put the subalgebra $\mathfrak{g}_{\bar{o}}$ of $\mathfrak{g}$ by

$$\begin{aligned}\mathfrak{g}_{\bar{o}} := \quad & \langle \square_0 + \square_1 + \square_{2l+1} + \square_{2l+2}, \\ & \square_{l+1-i} + \square_{l+1+i} (i = 1, \dots, l-1) \mid \square = e, f \rangle\end{aligned}$$

Then it is easy to see that $\mathfrak{g}_{\bar{o}} \simeq \mathfrak{g}(B_l)$. Put

$$\begin{aligned}\mathfrak{g}'_{(\bar{1},\bar{0})} &:= \mathfrak{g}_{\bar{o}}(e_{l+1}), \\ \mathfrak{g}'_{(\bar{0},\bar{1})} &:= \mathfrak{g}_{\bar{o}}(h_0 + \eta h_{2l+1} + \eta^2 h_1 + \eta^3 h_{2l+2}), \\ \mathfrak{g}'_{(\bar{0},\bar{2})} &:= \mathfrak{g}_{\bar{o}}(h_0 + \eta^2 h_{2l+1} + \eta^4 h_1 + \eta^6 h_{2l+2}), \\ \mathfrak{g}'_{(\bar{0},\bar{3})} &:= \mathfrak{g}_{\bar{o}}(h_0 + \eta^3 h_{2l+1} + \eta^6 h_1 + \eta^9 h_{2l+2}),\end{aligned}$$

where $\eta := e^{2\pi\sqrt{-1}/4}$ is the 4th primitive root of 1, and put

$$\mathfrak{g}'_{(\bar{1},\bar{i})} := [\mathfrak{g}'_{(\bar{1},\bar{0})}, \mathfrak{g}'_{(\bar{0},\bar{i})}]$$

for all $\bar{i} = \bar{1}$, $\bar{2}$, and $\bar{3}$. In this case, $z_{(\bar{1},\bar{0})}$ and $z_{(\bar{1},\bar{2})}$ are not zero and the other $\{z_{(\bar{i},\bar{j})}\}$ are 0. Let us define $\psi : \tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}} \to \Omega_A/dA$ as in (32) using the following $\bar{\psi}$:

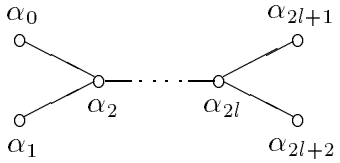
	$(\bar{0},\bar{0})$	$(\bar{1},\bar{0})$	$(\bar{0},\bar{1})$	$(\bar{1},\bar{1})$	$(\bar{0},\bar{2})$	$(\bar{1},\bar{2})$	$(\bar{0},\bar{3})$	$(\bar{1},\bar{3})$
$(\bar{0},\bar{0})$	0^*	0	0	0	0	0	0	0
$(\bar{1},\bar{0})$	0	0^*	0	0	0	0	0	0
$(\bar{0},\bar{1})$	0	0	0	$(z_{\bar{1}\bar{2}}\cdot,\cdot)$	0	0	0^*	$(z_{\bar{1}\bar{0}}\cdot,\cdot)$
$(\bar{1},\bar{1})$	0	0	$(z_{\bar{1}\bar{2}}\cdot,\cdot)$	0	0	0	$(z_{\bar{1}\bar{0}}\cdot,\cdot)$	0^*
$(\bar{0},\bar{2})$	0	0	0	0	0^*	0	0	0
$(\bar{1},\bar{2})$	0	0	0	0	0	0^*	0	0
$(\bar{0},\bar{3})$	0	0	0^*	$(z_{\bar{1}\bar{0}}\cdot,\cdot)$	0	0	0	$(z_{\bar{1}\bar{2}}\cdot,\cdot)$
$(\bar{1},\bar{3})$	0	0	$(z_{\bar{1}\bar{0}}\cdot,\cdot)$	0^*	0	0	$(z_{\bar{1}\bar{2}}\cdot,\cdot)$	0

(recall (33)). Using this $\bar{\psi}$, we have

$$W_2 = (\Omega_A/dA)_{(\bar{1},\bar{0})} \oplus (\Omega_A/dA)_{(\bar{1},\bar{2})} \simeq A_{(\bar{1},\bar{0})} \oplus A_{(\bar{1},\bar{2})}$$

as vector spaces.

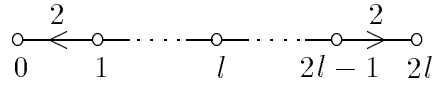
5.4.2. *Case* $BC_l^{(2,2)}(1)$. Let $\dot{\mathfrak{g}} := \mathfrak{g}(D_{2l+2})$. Let $\{\alpha_0, \cdots, \alpha_{2l+2}\}$ be an extended simple root basis of $\dot{\mathfrak{g}}$ whose diagram is as follows:



Let $\{e_i, h_i, f_i \mid i = 1, \dots, 2l+2\}$ be an extended Chevalley generators. Let us define $\{\tilde{e}_i, \tilde{h}_i, \tilde{f}_i \mid i = 0, \dots, 2l\}$ by

$$\begin{array}{rcl} \tilde{\square}_0 & := & \square_0 + \square_1, \\ \tilde{\square}_i & := & \square_{i+1} \ (i = 1, \dots, 2l-1), \\ \tilde{\square}_{2l} & := & \square_{2l+1} + \square_{2l+2} \end{array}$$

where $\square = e$, h, and f. Then the diagram of these new generators is the following:



Let us define the algebra $\mathfrak{g}$ to be the subalgebra of $\dot{\mathfrak{g}}$:

$$\mathfrak{g} := \langle \tilde{e}_i, \tilde{f}_k \mid i = 0, \dots, 2l \rangle.$$

Put the subalgebra $\mathfrak{g}_{\bar{o}}$ of $\mathfrak{g}$ by

$$\mathfrak{g}_{\bar{o}} := \langle \tilde{\square}_i \, (i = 0, \dots, 2l) \mid \square = e, f \rangle.$$

Then it is easy to see that $\mathfrak{g}_{\bar{o}} \simeq \mathfrak{g}(B_l)$. We put

$$\begin{array}{rcl} \mathfrak{g}'_{(\bar{1},\bar{0})} & := & \mathfrak{g}_{\bar{o}}(\tilde{e}_l), \\ \mathfrak{g}'_{(\bar{0},\bar{1})} & := & \mathfrak{g}_{\bar{o}}(\tilde{h}_0 - \tilde{h}_{2l}), \\ \mathfrak{g}'_{(\bar{1},\bar{1})} & := & [\mathfrak{g}'_{(\bar{1},\bar{0})}, \mathfrak{g}'_{(\bar{0},\bar{1})}]. \end{array}$$

In this case, $z_{(\bar{1},\bar{0})}$ is not zero and the other $\{z_{(\bar{i},\bar{j})}\}$ are 0. Let us define $\psi : \tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}} \to \Omega_A/dA$ as in (32) using the following $\bar{\psi}$:

(38)

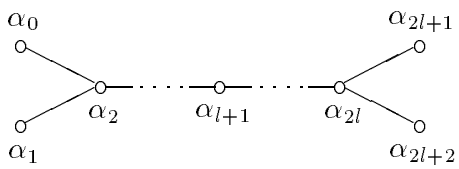
	$(\bar{0},\bar{0})$	$(\bar{1},\bar{0})$	$(\bar{0},\bar{1})$	$(\bar{1},\bar{1})$
$(\bar{0},\bar{0})$	0^*	0	0	0
$(\bar{1},\bar{0})$	0	0^*	0	0
$(\bar{0},\bar{1})$	0	0	0^*	$(z_{\bar{1}\bar{0}}\cdot,\cdot)$
$(\bar{1},\bar{1})$	0	0	$(z_{\bar{1}\bar{0}}\cdot,\cdot)$	0^*

(recall (33)). Using this $\bar{\psi}$, we have

$$W_2 = (\Omega_A/dA)_{(\bar{1},\bar{0})} \simeq A_{(\bar{1},\bar{0})} \tag{39}$$

as vector spaces.

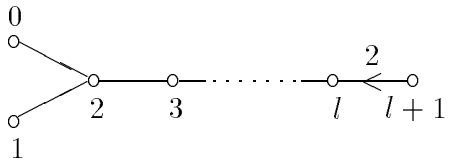
5.4.3. *Case* $BC_l^{(2,2)}(2)$. Let $\dot{\mathfrak{g}} := \mathfrak{g}(D_{2l+2})$. Let $\{\alpha_0, \cdots, \alpha_{2l+2}\}$ be an extended simple root basis of $\dot{\mathfrak{g}}$ whose diagram is as follows:



Let $\{e_i, h_i, f_i \mid i = 1, \dots, 2l+2\}$ be an extended Chevalley generators. Let us define $\{\tilde{e}_i, \tilde{h}_i, \tilde{f}_i \mid i = 0, \dots, 2l+1\}$ by

$$\begin{aligned} \tilde{\Box}_i &:= \Box_i + \Box_{2l+2-i} \ (i = 0, \dots, l), \\ \tilde{\Box}_{l+1} &:= \Box_{l+1} \end{aligned}$$

where $\Box = e$, h, and f. Then the diagram of these new generators is the following:



Let us define the algebra $\mathfrak{g}$ to be the subalgebra of $\dot{\mathfrak{g}}$:

$$\mathfrak{g} := \langle \tilde{e}_i,\ \tilde{f}_k \mid i = 0, \dots, l+1 \rangle.$$

Put the subalgebra $\mathfrak{g}_{\bar{\sigma}}$ of $\mathfrak{g}$ by

$$\mathfrak{g}_{\bar{\sigma}} := \langle \tilde{\Box}_0 + \tilde{\Box}_1,\ \tilde{\Box}_i \, (i = 2, \dots, l) \mid \Box = e, f \rangle.$$

Then it is easy to see that $\mathfrak{g}_{\bar{\sigma}} \simeq \mathfrak{g}(B_l)$. We put

$$\begin{aligned} \mathfrak{g}'_{(\bar{1},\bar{0})} &:= \mathfrak{g}_{\bar{\sigma}}(\tilde{e}_{l+1}), \\ \mathfrak{g}'_{(\bar{0},\bar{1})} &:= \mathfrak{g}_{\bar{\sigma}}(\tilde{h}_0 - \tilde{h}_1), \\ \mathfrak{g}'_{(\bar{1},\bar{1})} &:= [\mathfrak{g}'_{(\bar{1},\bar{0})}, \mathfrak{g}'_{(\bar{0},\bar{1})}]. \end{aligned}$$

In this case, $z_{(\bar{1},\bar{0})}$ is not zero and the other $\{z_{(\bar{i},\bar{j})}\}$ are 0. Let us define $\psi : \tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}} \to \Omega_A / dA$ as in (32) using the $\bar{\psi}$ as in (38). Then W_2 is as in (39).

6. Proofs

Here we proof Theorem 3 in Sect. 6.1 and Theorem 1 in Sect. 6.2 only for the case $B_l^{(2,2)}$. But in Sect. 6.1, we prove as general as possible. Note that there are some cases in which we need to modify the following proof properly.

6.1. **Proof of Theorem 3.**

We follow Wilson's proof of in [W]. Let (R, G) be a reducet elliptic root system of type $B_l^{(2,2)}$. Recall the $\mathfrak{g}$, $\tilde{\mathfrak{g}}$ and $(\hat{\mathfrak{g}}, \pi)$.

Let $(\mathfrak{m}, \varpi)$ be a central extension of $\tilde{\mathfrak{g}}$ and $\mathfrak{z}$ be the kernel of π

$$\mathfrak{z} := \ker \varpi.$$

We have the following exact sequence:

$$0 \ \to \ \mathfrak{z} \ \to \ \mathfrak{m} \ \xrightarrow{\varpi} \ \tilde{\mathfrak{g}} \ \to \ 0.$$

Let us consider the sections of ϖ:

$$S := \{i : \tilde{\mathfrak{g}} \to \mathfrak{m} \mid i \text{ is a linear map such that } \varpi\, i = \mathrm{id}\}.$$

For each $i \in S$, we have the splitting $\mathfrak{m} = i(\tilde{\mathfrak{g}}) \oplus \mathfrak{z}$ as a vector space. It is enough to construct a Lie-homomorphism $\rho : \hat{\mathfrak{g}} \to \mathfrak{m}$ satisfying $\varpi\rho = \pi$. We construct ρ by choosing good $i \in S$ and constructing $\chi : W \to \mathfrak{z}$ as follows.

6.1.1. We define a bilinear map

$$i^*_{\alpha,\beta} \ : \ \mathfrak{g}_{\bar{\alpha}} \times \mathfrak{g}_{\bar{\beta}} \ \to \ \mathfrak{z}$$

for all $i \in S$ and $\alpha = (p, l)$, $\beta = (q, m) \in \mathbb{Z}^2$ by

$$i^*_{\alpha,\beta}(u, v) := \big[i(u \otimes s^p t^l), i(u \otimes s^q t^m)\big] - i([u, v] \otimes s^{p+q} t^{l+m}),$$

where $u \in \mathfrak{g}_{\bar{\alpha}}$ and $v \in \mathfrak{g}_{\bar{\beta}}$. The followings are easy to see:

$$\text{(A)} \qquad i^*_{\alpha,\beta}(u, v) = -i^*_{\beta,\alpha}(v, u),$$

$$\text{(J)} \qquad i^*_{\alpha+\beta,\gamma}([u, v], w) + i^*_{\beta+\gamma,\alpha}([v, w], u) + i^*_{\gamma+\alpha,\beta}([w, u], v) = 0$$

for all α, β, $\gamma \in \mathbb{Z}^2$ and $u \in \mathfrak{g}_{\bar{\alpha}}$, $v \in \mathfrak{g}_{\bar{\beta}}$, $w \in \mathfrak{g}_{\bar{\gamma}}$.

We put

$$S_o := \{i \in S \mid i^*_{\alpha,o} = 0 \text{ for all } \alpha \in \mathbb{Z}^2\},$$

where $o := (0, 0) \in \mathbb{Z}^2$.

Claim 13. *$S_o \neq \emptyset$.*

Proof. Take an element $i \in S$. Let us construct $j \in S_o$ from i as follows. First we recall that $\mathrm{Hom}_{\mathbb{C}}(\mathfrak{g}_{\bar{\alpha}}, \mathfrak{z})$ ($\alpha \in \mathbb{Z}^2$) is a $\mathfrak{g}_0$-module by

$$\begin{array}{ccc} \mathfrak{g}_0 \times \mathrm{Hom}_{\mathbb{C}}(\mathfrak{g}_{\bar{\alpha}}, \mathfrak{z}) & \to & \mathrm{Hom}_{\mathbb{C}}(\mathfrak{g}_{\bar{\alpha}}, \mathfrak{z}), \\ (g, f) & \mapsto & g \cdot f, \end{array}$$

where $(g \cdot f)(x) := -f([g, x])$.

Putting $\beta = \gamma = 0$ in (J) and using (A), we have

$$(40) \qquad i^*_{\alpha,0}([u, v], w) - i^*_{\alpha,0}(u, [v, w]) + i^*_{\alpha,0}([w, u], v) = 0.$$

Let us define f^α for all $\alpha \in \mathbb{Z}^2$ by

$$\begin{array}{ccccc} f^\alpha & : & \mathfrak{g}_0 & \to & \mathrm{Hom}_{\mathbb{C}}(\mathfrak{g}_{\bar{\alpha}}, \mathfrak{z}) \\ & & v & \mapsto & f^\alpha_v, \end{array}$$

where $f^\alpha_v(u) := i^*_\alpha(u, v)$. By (40), we have

$$(41) \qquad f^\alpha_{[v,w]}(u) = f^\alpha_v([w, u]) - f^\alpha_w([v, u]).$$

The equation (41) implies that $f^\alpha \in \ker d_0$ (see Sect. 4.1). Now we use the following fact, which is shown by using Whitehead's lemma as in Fact 5. See also [MEY].

Fact 8. $H^1(\mathfrak{g}_{\bar{o}}, \mathrm{Hom}_{\mathbb{C}}(\mathfrak{g}_{\bar{\alpha}}, \mathfrak{z})) = 0$.

According to the fact, $\ker d_0 = \mathrm{im}\, d_1$ and hence there exists

$$g^{\alpha} \in \mathrm{Hom}_{\mathbb{C}}(\mathfrak{g}_{\bar{o}}, \mathrm{Hom}_{\mathbb{C}}(\mathfrak{g}_{\bar{\alpha}}, \mathfrak{z}))$$

such that

$$f_v^{\alpha}(u) = (v \cdot g^{\alpha})(u) = -g^{\alpha}([v,u])$$

for all $u \in \mathfrak{g}_{\bar{\alpha}}$, $v \in \mathfrak{g}_{\bar{o}}$, and $\alpha \in \mathbb{Z}^2$. Using the $\{g^{\alpha}\}$, we define $g \in S$ by

$$g(u \otimes s^p t^l) := g^{\alpha}(u)$$

for all $\alpha = (p,l) \in \mathbb{Z}^2$ and $u \in \mathfrak{g}_{\bar{\alpha}}$. Then we can see $j = i - g \in S_o$ by some calculations (in [W]). □

6.1.2. From now on, let us assume $i \in S_o$. Let us study $i^*_{\alpha,\beta} : \mathfrak{g}_{\bar{\alpha}} \times \mathfrak{g}_{\bar{\beta}} \to \mathfrak{z}$ for all $\alpha = (p,l)$ and $\beta = (q,m) \in \mathbb{Z}^2$. The following fact plays a role.

Claim 14. *The map $i^*_{\alpha,\beta}$ is a $\mathfrak{g}_{\bar{o}}$-invariant bilinear form on $\mathfrak{g}_{\bar{\alpha}} \times \mathfrak{g}_{\bar{\beta}}$ for all α, $\beta \in \mathbb{Z}^2$. That is,*

$$i^*_{\alpha,\beta}([u,x],v) = i^*_{\alpha,\beta}(u,[x,v])$$

for all $u \in \mathfrak{g}_{\bar{\alpha}}$, $v \in \mathfrak{g}_{\bar{\beta}}$, $x \in \mathfrak{g}_{\bar{o}}$.

Proof. We can show this easily useing (A) and (J). □

Lemma 2. *The form $i^*_{\alpha,\beta}$ has the following expression. Recall that $(\cdot,\cdot)$ is the Killing form on $\mathfrak{g}$.*

[1] *Case $(\bar{\alpha},\bar{\beta}) = (\bar{o},\bar{o})$. There exists an element $w_{\alpha,\beta} \in \mathfrak{z}$ such that $i^*_{\alpha,\beta}(u,v) = (u,v)w_{\alpha,\beta}$ for all $u \in \mathfrak{g}_{\bar{\alpha}}$ and $v \in \mathfrak{g}_{\bar{\beta}}$.*

[2] *Case $(\bar{\alpha},\bar{\beta})$ where either $\bar{\alpha} = \bar{o}$ or $\bar{\beta} = \bar{o}$ and both $\bar{\alpha} \neq \bar{o}$ and $\bar{\beta} \neq \bar{o}$. $i^*_{\alpha,\beta} \equiv 0$.*

[3] *Case $(\bar{\alpha},\bar{\beta})$ where both $\bar{\alpha} \neq \bar{o}$ and $\bar{\beta} \neq \bar{o}$ and $\bar{\alpha} + \bar{\beta} = \bar{o}$. There exists $w_{\alpha,\beta} \in \mathfrak{z}$ such that $i^*_{\alpha,\beta}(u,v) = (u,v)w_{\alpha,\beta}$ for all u, $v \in \mathfrak{g}_{\bar{\alpha}}$.*

[4] *Case $(\bar{\alpha},\bar{\beta})$ where both $\bar{\alpha} \neq \bar{o}$ and $\bar{\beta} \neq \bar{o}$ and $\bar{\alpha}+\bar{\beta} \neq \bar{o}$ and $\bar{\alpha} \neq \bar{\beta}$. There exists $w_{\alpha,\beta} \in \mathfrak{z}$ such that $i^*_{\alpha,\beta}(u,v) = ([z_{\bar{\gamma}},u],v)w_{\alpha,\beta}$ for all $u \in \mathfrak{g}_{\bar{\alpha}}$, $v \in \mathfrak{g}_{\bar{\beta}}$, where $\gamma := -\alpha - \beta$.*

Refer the following table.

	$\mathfrak{g}_{(\bar{0},\bar{0})}$	$\mathfrak{g}_{(\bar{1},\bar{0})}$	$\mathfrak{g}_{(\bar{0},\bar{1})}$	$\mathfrak{g}_{(\bar{1},\bar{1})}$
$\mathfrak{g}_{(\bar{0},\bar{0})}$	[1]	[2]	[2]	[2]
$\mathfrak{g}_{(\bar{1},\bar{0})}$	[2]	[3]	[4]	[4]
$\mathfrak{g}_{(\bar{0},\bar{1})}$	[2]	[4]	[3]	[4]
$\mathfrak{g}_{(\bar{1},\bar{1})}$	[2]	[4]	[4]	[3]

(In $G_2^{(2,2)}$ case, we need to check another case: both $\bar{\alpha} \neq \bar{o}$ and $\bar{\beta} \neq \bar{o}$ and $\bar{\alpha} + \bar{\beta} \neq \bar{o}$ and $\bar{\alpha} = \bar{\beta}$. But we can handle with this case with a little more contrivance.)

Before studying each case of [1]–[4], let us make two preparations below.

1. Let $\{w_\lambda\}_{\lambda \in \Lambda}$ be a $\mathbb{C}$-basis of $\mathfrak{z}$. We define $\mathfrak{g}_{\bar{o}}$-invariant bilinear forms

$$i^{*\lambda}_{\alpha,\beta} \ : \ \mathfrak{g}_{\bar{\alpha}} \times \mathfrak{g}_{\bar{\beta}} \ \to \ \mathbb{C} \tag{42}$$

for all $\lambda \in \Lambda$ by

$$i^*_{\alpha,\beta}(u,v) = \sum_{\lambda \in \Lambda} i^{*\lambda}_{\alpha,\beta}(u,v) w_\lambda$$

for all $u \in \mathfrak{g}_{\bar{\alpha}}$, $v \in \mathfrak{g}_{\bar{\beta}}$. Let us assume $i^*_{\alpha,\beta} \not\equiv 0$. There exists $\lambda_0 \in \Lambda$ such that $i^{*\lambda_0}_{\alpha,\beta} \not\equiv 0$. We put $g := i^{*\lambda_0}_{\alpha,\beta}$. This g defines $\mathfrak{g}_{\bar{o}}$-homomorphism

$$\begin{array}{rccc} \xi_g : & \mathfrak{g}_{\bar{\alpha}} & \to & \mathrm{Hom}_{\mathbb{C}}(\mathfrak{g}_{\bar{\beta}}, \mathbb{C}), \\ & u & \mapsto & g(u, \cdot). \end{array} \tag{43}$$

Note that if $\mathfrak{g}_{\bar{\alpha}}$ is an irreducible $\mathfrak{g}_{\bar{o}}$-module, then we have $\ker \xi_g = \{0\}$ and hence ξ_g is injective.

2. We also note the following fact, which is similar to Claim 8.

Claim 15. *For $\bar{\alpha}$, $\bar{\beta} \in \mathbb{Z}_{t_1,t_2}$, let f be a $\mathfrak{g}_{\bar{o}}$-invariant bilinear map $f : \mathfrak{g}_{\bar{\alpha}} \times \mathfrak{g}_{\bar{\beta}} \to V$, where V is a vector space. Then $f(\mathfrak{g}'_{\bar{\alpha}}, z_{\bar{\beta}}) = f(z_{\bar{\alpha}}, \mathfrak{g}'_{\bar{\beta}}) = 0$. In particular, $i^*_{\alpha,\beta}(\mathfrak{g}'_{\bar{\alpha}}, z_{\bar{\beta}}) = i^*_{\alpha,\beta}(z_{\bar{\alpha}}, \mathfrak{g}'_{\bar{\beta}}) = 0$.*

Proof. We can show this by calculations. Note that if $z_{\bar{\alpha}} = 0$ or $z_{\bar{\beta}} = 0$, this fact is obvious. □

Proof. Let us prove Lemma 2.

[1]. We can assume $i^*_{\alpha,\beta} \not\equiv 0$. Let $\lambda_0 \in \Lambda$ such that $i^{*\lambda_0}_{\alpha,\beta} \not\equiv 0$. Put $g := i^{*\lambda_0}_{\alpha,\beta}$. This g defines $\mathfrak{g}_{\bar{o}}$-homomorphism as in (43). We have $\ker \xi_g = \{0\}$, because $\mathfrak{g}_{\bar{o}}$ is irreducible $\mathfrak{g}_{\bar{o}}$-module. Since the dimensions of $\mathfrak{g}_{\bar{o}}$ and of $\mathrm{Hom}_{\mathbb{C}}(\mathfrak{g}_{\bar{o}}, \mathbb{C})$ are equal, we know ξ_g is an $\mathfrak{g}_{\bar{o}}$-isomorphism. On the other hands, the following map defined by the Killing form

$$\begin{array}{rccc} \xi : & \mathfrak{g}_{\bar{o}} & \to & \mathrm{Hom}_{\mathbb{C}}(\mathfrak{g}_{\bar{o}}, \mathbb{C}), \\ & u & \mapsto & (u, \cdot) \end{array}$$

is also an $\mathfrak{g}_{\bar{o}}$-isomorphism by the same reason. Due to Schur's lemma (Fact refschurslemma), we know there exists $c \in \mathbb{C}$ such that $g(u,v) = c(u,v)$ for all u, $v \in \mathfrak{g}_{\bar{o}}$. Combine all $i^{*\lambda}_{\alpha,\beta}$ ($\lambda \in \Lambda$), we show [1].

[2]. If $i^*_{\alpha,\beta} \not\equiv 0$, then we can embed $\mathfrak{g}_{\bar{o}}$ into $\mathfrak{g}_{\bar{\alpha}}$ or $\mathfrak{g}_{\bar{\beta}}$, and it is contradiction.

[3]. We consider the natural projection

$$p_{\bar\gamma} \ : \ \mathfrak{g}_{\bar\gamma} = \ \mathfrak{g}'_{\bar\gamma} \oplus \mathbb{C}z_{\bar\gamma} \ \to \ \mathbb{C}z_{\bar\gamma}, \quad u+v \ \mapsto \ v \tag{44}$$

for all $\bar\gamma \in \mathbb{Z}_{t_1,t_2}$ such that $\bar\gamma \neq \bar o$. Let us show the following fact first.

Claim 16. *Let $\bar\alpha$ and $\bar\beta$ as above and Let f be a $\mathfrak{g}_{\bar o}$-invarinat bilinear form*

$$f \ : \ \mathfrak{g}_{\bar\alpha} \times \mathfrak{g}_{\bar\beta} \ \to \ \mathbb{C}.$$

Then there exist two constants c_1, $c_2 \in \mathbb{C}$ such that

$$f(u,v) = c_1(u,v) + c_2(p_{\bar\alpha}(u), p_{\bar\beta}(v))$$

for all $u \in \mathfrak{g}_{\bar\alpha}$, $v \in \mathfrak{g}_{\bar\beta}$.

Proof. First recall Claim 15. Let $\bar f$ be the restriction of f on $\mathfrak{g}'_{\bar\alpha} \times \mathfrak{g}'_{\bar\beta}$. Then the map

$$\xi_{\bar f} : \ \mathfrak{g}'_{\bar\alpha} \ \to \ (\mathfrak{g}'_{\bar\beta})^*, \quad u \ \mapsto \ f(u,\cdot)$$

is a $\mathfrak{g}_{\bar o}$-homomorphim. Since $\mathfrak{g}'_{\bar\alpha}$ is an irreducible $\mathfrak{g}_{\bar o}$-module (see Claim 6), we have $\ker \xi_{\bar f} = \{0\}$ or $\mathfrak{g}'_{\bar\alpha}$. If $\ker \xi_{\bar f} = \mathfrak{g}'_{\bar\alpha}$, we have $\bar f \equiv 0$. Let us assume that $\ker \xi_{\bar f} = \{0\}$. Then $\xi_{\bar f}$ is an isomorphism. On the other hand, the map

$$\xi : \ \mathfrak{g}'_{\bar\alpha} \ \to \ (\mathfrak{g}'_{\bar\beta})^*, \quad u \ \mapsto \ (u,\cdot)$$

is also $\mathfrak{g}_{\bar o}$-module isomorphism (see Claim 5 and Claim 8). Due to Schur's lemma (Fact 4), there exists a constant $c_1 \in \mathbb{C}$ such that $\bar f(u,v) = c_1(u,v)$ for all u, $v \in \mathfrak{g}'_{\bar\alpha}$. Using Claim 8, we complete the proof of Claim 16. □

Corollary 4. *Let α, $\beta \in \mathbb{Z}^2$ such that $\bar\alpha + \bar\beta = \bar o$ and both $\bar\alpha \neq \bar o$ and $\bar\beta \neq \bar o$. Then there exist $w_{\alpha,\beta}$, $x_{\alpha,\beta} \in \mathfrak{z}$ such that*

$$i^*_{\alpha,\beta}(u,v) = (u,v)w_{\alpha,\beta} + \big(p_{\bar\alpha}(u), p_{\bar\beta}(v)\big)\, x_{\alpha,\beta}$$

for all u, $v \in \mathfrak{g}_{\bar\alpha}$.

Proof. See Claim 16. Use the decomposition of $i^*_{\alpha,\beta}$ in (42). □

Even though $\mathfrak{g}_{\bar\alpha}$ is not irreducible, we can show the similar fact as the irreudcible cases, which shows [2].

Lemma 3. *Let α and β as above. Then there exist $w_{\alpha,\beta} \in \mathfrak{z}$ such that*

$$i^*_{\alpha,\beta}(u,v) = (u,v)w_{\alpha,\beta}$$

for all u, $v \in \mathfrak{g}_{\bar\alpha}$.

Proof. We show $x_{\alpha,\beta} = 0$ in Cor. 4 using notation there. It is easy to see that if $z_{\bar{\alpha}} = 0$ or $z_{\bar{\beta}} = 0$, then this lemma is true. So, let us assume $z_{\bar{\alpha}} \neq 0$ and $z_{\bar{\beta}} \neq 0$. We use the following facts.

Claim 17. *Let α and β as above. Let α_1, $\alpha_2 \in \mathbb{Z}^2 \setminus \{(0,0)\}$ such that $\alpha_1 + \alpha_2 = \alpha$. If $z_{\bar{\beta}} \neq 0$, then we have followings:*

(i) *There exist $a_1 \in \mathfrak{g}'_{\bar{\alpha_1}}$ and $a_2 \in \mathfrak{g}'_{\bar{\alpha_2}}$ such that $([a_1, a_2], z_{\bar{\beta}}) \neq 0$.*
(ii) *There exist $b_1 \in \mathfrak{g}'_{\bar{\alpha_1}}$, $\in \mathfrak{g}'_{\bar{\alpha_2}}$ and $b \in \mathfrak{g}'_{\bar{\beta}}$ such that $([b_1, b_2], b) \neq 0$.*

Proof. We can show this by calculations. □

Applying (i) above, let us substitute $u = a_1$, $v = a_2$, $w = z_\beta$ in (J). Then we have

$$i^*_{\alpha_1+\alpha_2,\beta}([a_1, a_2], z_{\bar{\beta}}) + i^*_{\alpha_2+\beta,\alpha_1}([a_2, z_{\bar{\beta}}], a_1) + i^*_{\beta+\alpha_1,\alpha_2}([z_{\bar{\beta}}, a_1], a_2) = 0.$$

Using Cor. 4 and Claim 15, we have

$$(w_{\alpha,\beta} + x_{\alpha,\beta}) + w_{\alpha_2+\beta,\alpha_1} + w_{\beta+\alpha_1,\alpha_2} = 0. \tag{45}$$

Similarly, applying (ii) above, let us substitute $u = b_1$, $v = b_2$, $w = b$ in (J). Then we have

$$i^*_{\alpha_1+\alpha_2,\beta}([a_1, a_2], b) + i^*_{\alpha_2+\beta,\alpha_1}([a_2, b], a_1) + i^*_{\beta+\alpha_1,\alpha_2}([b, a_1], a_2) = 0.$$

Again using Cor. 4 and Claim 15, we have

$$w_{\alpha,\beta} + w_{\alpha_2+\beta,\alpha_1} + w_{\beta+\alpha_1,\alpha_2} = 0. \tag{46}$$

Combining (45) and (46), we have $x_{\alpha,\beta} = 0$. □

[4]. Let us put $\gamma := -\alpha - \beta$. Recall the projections $\{p_{\bar{\alpha}}\}$ in (44). Similarly to Claim 16 and its corollary, we can show the following facts:

Claim 18. *Let f be a $\mathfrak{g}_{\bar{\sigma}}$-invarinat bilinear form*

$$f \ : \ \mathfrak{g}_{\bar{\alpha}} \times \mathfrak{g}_{\bar{\beta}} \ \to \ \mathbb{C}.$$

Then there exist two constants c_1, $c_2 \in \mathbb{C}$ such that

$$f(u, v) = c_1([z_{\bar{\gamma}}, u], v) + c_2 \left(p_{-\bar{\beta}}([z_{\bar{\gamma}}, u]), p_{\bar{\beta}}(v)\right)$$

for all $u \in \mathfrak{g}_{\bar{\alpha}}$, $v \in \mathfrak{g}_{\bar{\beta}}$.

Corollary 5. *There exist $w_{\alpha,\beta}$, $x_{\alpha,\beta} \in \mathfrak{z}$ such that*

$$i^*_{\alpha,\beta}(u, v) = ([z_{\bar{\gamma}}, u], v)w_{\alpha,\beta} + \left(p_{\bar{\alpha}}([z_{\bar{\gamma}}, u]), p_{\bar{\beta}}(v)\right) x_{\alpha,\beta}$$

for all $u \in \mathfrak{g}_{\bar{\alpha}}$, $v \in \mathfrak{g}_{\bar{\beta}}$.

Similar to Lemma 3, we again have the following fact, which prove [4], even if $\mathfrak{g}_{\bar{\alpha}}$ and $\mathfrak{g}_{\bar{\beta}}$ may not be irreducible.

Lemma 4. *There exists $w_{\alpha,\beta} \in \mathfrak{z}$ such that*

$$i^*_{\alpha,\beta}(u,v) = ([z_{\bar\gamma}, u], v)w_{\alpha,\beta}$$

for all $u \in \mathfrak{g}_{\bar\alpha}$, $v \in \mathfrak{g}_{\bar\beta}$.

Proof. Similarly to the proof of Lemma 3, we show $x_{\alpha,\beta} = 0$ in Cor. 5 using notation there. If either $z_{\bar\alpha} = 0$ or $z_{\bar\beta} = 0$, we can see this fact using Claim 15. The follwing is a corollary of Claim 17.

Claim 19. *Let α, β and γ as above. Let α_1, $\alpha_2 \in \mathbb{Z}^2 \setminus \{\bar o\}$ such that $\alpha_1 + \alpha_2 = \alpha$. If $z_{\bar\beta} \neq 0$ and $z_{\bar\gamma} \neq 0$, we have the followings.*

(i) *There exist $a_1 \in \mathfrak{g}'_{\bar\alpha_1}$ and $a_2 \in \mathfrak{g}'_{\bar\alpha_2}$ such that $([z_{\bar\gamma}, [a_1, a_2]], z_{\bar\beta}) \neq 0$.*
(ii) *There exist $b_1 \in \mathfrak{g}'_{\bar\alpha_1}$, $b_2 \in \mathfrak{g}'_{\bar\alpha_2}$ and $b \in \mathfrak{g}'_{\bar\beta}$ such that $([z_{\bar\gamma}, [b_1, b_2]], b) \neq 0$.*

Proof. We can show this by the use of Claim 10. □

Applying the fact to (J), we can see $x_{\alpha,\beta} = 0$ similarly to the proof of Lemma 3. □

We have done all the cases [1]–[4] and finished the proof of Lemma 2. □

6.1.3. Let us construct the map $\chi : \hat{\mathfrak{g}} \to \mathfrak{m}$ as follows. We define $w^{(1)}_{\alpha,\beta}$ and $w^{(2)}_{\alpha,\beta} \in \mathfrak{z}$ for all α, $\beta \in \mathbb{Z}^2$ as follows:

$$w^{(1)}_{\alpha,\beta} := \begin{cases} w_{\alpha,\beta} & \text{if} \quad \bar\alpha + \bar\beta = \bar o, \\ 0 & \text{otherwise,} \end{cases}$$

$$w^{(2)}_{\alpha,\beta} := \begin{cases} w_{\alpha,\beta} & \text{if} \quad \bar\alpha + \bar\beta \neq \bar o, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$w_{\alpha,\beta} = w^{(1)}_{\alpha,\beta} + w^{(2)}_{\alpha,\beta}.$$

Note that $w^{(2)}_{o,\alpha} = w^{(2)}_{\alpha,o} = 0$ for all $\alpha \in \mathbb{Z}^2$ (Lemma 2 [2]).

Claim 20. *Let $i = 1$ or 2. We have the followings for all α, β, $\gamma \in \mathbb{Z}^2$.*

(i) $w^{(i)}_{\alpha,0} = 0$,
(ii) $w^{(i)}_{\alpha,\beta} = -w^{(i)}_{\beta,\alpha}$,
(iii) $w^{(i)}_{\alpha+\beta,\gamma} + w^{(i)}_{\beta+\gamma,\alpha} + w^{(1)}_{\gamma,\alpha+\beta} = 0$.

Proof. (i) Recall $i \in S_o$. (ii) Recall (A). (iii) Recall (J). If $i = 2$, then use the following fact: Let $x \in \mathfrak{g}_{\bar\alpha}$, $y \in \mathfrak{g}_{\bar\beta}$, and $x \in \mathfrak{g}_{\bar\gamma}$. Put $\bar\delta := -\bar\alpha - \bar\beta - \bar\gamma$. Let $w \in \mathfrak{g}_{\bar\delta}$. Then

$$\begin{aligned} & ([w,[x,y]],z) + ([w,[y,z]],x) + ([w,[z,x]],y) \\ = \; & ([[x,y],z],w) + ([[y,z],x],w) + ([[z,x],y],w) = 0. \end{aligned}$$

□

Recall $W_1 := \phi(\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}})$ and $W_2 := \psi(\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}})$. According to Claim 20, we have the following fact.

Corollary 6. *Let $i = 1$ or 2. There exists a linear map*

$$\begin{array}{rccc} \chi_i : & \dfrac{W_i}{d(s^p t^l) s^q t^m} & \to & \mathfrak{z}, \\ & & \mapsto & w^{(i)}_{(p,l),(q,m)}. \end{array}$$

Let us define $\chi : W_1 \oplus W_2 (=: W) \to \mathfrak{z}$ by $\chi := \chi_1 \oplus \chi_2$ and define

$$\begin{array}{rccc} \rho & : & \hat{\mathfrak{g}} = \tilde{\mathfrak{g}} \oplus W & \to & \mathfrak{m} \\ & & x \otimes s^i t^j + w & \mapsto & i(x \otimes s^i t^j) + \chi(w), \end{array}$$

where $w \in W$ and $i, j \in \mathbb{Z}$.

Claim 21. *The map ρ is a Lie-homomorphism satisfying $\varpi\rho = \pi$.*

Proof. Let $\alpha = (i, j)$, $\beta = (k, l) \in \mathbb{Z}^2$. If $\bar{\alpha} + \bar{\beta} = \bar{o}$, then we have in $\hat{\mathfrak{g}}$:

$$[x \otimes s^i t^j, y \otimes s^k t^l] = [x, y] \otimes s^{i+k} t^{j+l} + (x, y)\overline{d(s^i t^j) s^k t^l}$$

and in $\mathfrak{m}$:

$$[i(x \otimes s^i t^j), i(y \otimes s^k t^l)] = i([x, y] \otimes s^{i+k} t^{j+l}) + (x, y) w_{\alpha,\beta}.$$

If $\bar{\alpha} + \bar{\beta} \neq \bar{o}$, then, putting $\gamma := -\alpha - \beta$, we have in $\hat{\mathfrak{g}}$:

$$[x \otimes s^i t^j, y \otimes s^k t^l] = [x, y] \otimes s^{i+k} t^{j+l} + ([z_{\bar{\gamma}}, x], y)\overline{d(s^i t^j) s^k t^l}$$

and in $\mathfrak{m}$:

$$[i(x \otimes s^i t^j), i(y \otimes s^k t^l)] = i([x, y] \otimes s^{i+k} t^{j+l}) + ([z_{\bar{\gamma}}, x], y) w_{\alpha,\beta}.$$

□

6.2. **Proof of Theorem 1.**

Here we show a proof for case $B_l^{(2,2)}$ going along the arguments below Theorem 1.

1. Let us recall Sect. 3.3 first. The algebra $\mathfrak{g}(R, G)$ has a root space decomposition with respect to H. Let Δ be the set of roots. Then we know $\Delta \subset Q(R)$. We equip $\mathfrak{g}(R, G)$ the $Q(R)$-grading by putting the root space $\mathfrak{g}(R, G)_\alpha$ of $\alpha \in \Delta$ by the homogeneous space of degree α. Then $\mathfrak{g}'(R, G)$ inherits the $Q(R)$-grading. Noticing the identification $Q(R_{\mathrm{f}}) \oplus \mathbb{Z}b \oplus \mathbb{Z}a = Q(R)$ in (12), we know $\tilde{\mathfrak{g}}$ and $\hat{\mathfrak{g}}$ have the $Q(R)$-grading as in Sect. 4.2 and Sect. 4.3. The projection $\pi : \hat{\mathfrak{g}} \to \tilde{\mathfrak{g}}$ is defined in (31). It is easy to see the π is compatible with the $Q(R)$-grading structures.

2. Let us define the map

$$\tilde{\Phi} \ : \ \mathfrak{g}(R, G) \ \to \ \hat{\mathfrak{g}} \oplus \mathbb{C}s\frac{\partial}{\partial s} \oplus \mathbb{C}t\frac{\partial}{\partial t}$$

by the following:

$$\left\{\begin{array}{ccc} E^{\alpha_0} & \mapsto & (e_0+e_1)\otimes s, \\ h_{\alpha_0^\vee} & \mapsto & (h_0+h_1)+(h_0+h_1,h_0+h_1)\overline{\frac{ds}{s}}, \\ E^{-\alpha_0} & \mapsto & -(f_0+f_1)\otimes s^{-1}, \end{array}\right.$$

$$\left\{\begin{array}{ccc} E^{\alpha_j} & \mapsto & e_{j+1}, \\ h_{\alpha_j^\vee} & \mapsto & h_{j+1}, \\ E^{-\alpha_j} & \mapsto & -f_{j+1}, \end{array}\right\} \text{ for } j=1,\dots,l-1,$$

$$\left\{\begin{array}{ccc} E^{\alpha_l} & \mapsto & (e_{l+1}+e_{l+2}), \\ h_{\alpha_l^\vee} & \mapsto & (h_{l+1}+h_{l+2}), \\ E^{-\alpha_l} & \mapsto & -(f_{l+1}+f_{l+2}), \end{array}\right.$$

$$H^{ia}_{\alpha_0^\vee} \mapsto \left\{\begin{array}{c} (h_0+h_1)\otimes t^i+(h_0+h_1,h_0+h_1)\overline{s^{-1}d(st^i)} \\ \text{if} \quad i\in 2\mathbb{Z}, \\ (h_0-h_1)\otimes t^i+([z_{(\bar{1},\bar{1})},h_0+h_1],h_0+h_1)\overline{s^{-1}d(st^i)} \\ \text{if} \quad i\in 2\mathbb{Z}+1, \end{array}\right.$$

$$H^{ia}_{\alpha_j^\vee} \mapsto h_{\alpha_{j+1}^\vee}\otimes t^i \quad (j=1,\dots,l-1),$$

$$H^{ia}_{\alpha_l^\vee} \mapsto \left\{\begin{array}{ll} (h_{l+1}+h_{l+2})\otimes t^i & \text{if} \quad i\in 2\mathbb{Z}, \\ (h_{l+1}-h_{l+2})\otimes t^i & \text{if} \quad i\in 2\mathbb{Z}+1, \end{array}\right.$$

$$\Lambda_b \mapsto s\frac{\partial}{\partial s},$$

$$\Lambda_a \mapsto t\frac{\partial}{\partial t}.$$

The well-definedness is shown by checking all the images of the defining relations of $\mathfrak{g}(R,G)$ to vanish, which is shown by calculations using the following facts:

Fact 9. (i) $(e_0+e_1,f_0+f_1)=(h_0+h_1,h_0+h_1)$.

(ii) *There exists a constant* $C\in\mathbb{R}$ *such that*

$$CI(\alpha_i^\vee,\alpha_j^\vee)=\big(\Phi(h_{\alpha_i^\vee}),\Phi(h_{\alpha_i^\vee})\big)$$

for all $i,j=0,\dots,l$.

We can show the following fact.

Fact 10. $\Lambda_{\tilde{b}}\mapsto 2C\overline{\frac{ds}{s}}$ *and* $\Lambda_{\tilde{a}}\mapsto C\overline{\frac{dt}{t}}$, *where* C *is as in* Fact 9 (ii).

Proof. We can see the fact by using the following equations:

$$(h_0+h_1)+2(h_2+\cdots+h_l)+(h_{l+1}+h_{l+2})=0,$$

$$\alpha_0+\cdots+\alpha_l=b,$$

$$-\alpha_0^\vee=2(\alpha_1^\vee+\cdots+\alpha_{l-1}^\vee)+\alpha_l^\vee.$$

□

Let us define Φ as to be the restriction of $\tilde{\Phi}$ on $\mathfrak{g}'(R, G)$. Using the map Φ, the proof is completed by following the arguments 3 and 4 in the bottom of Theorem 1.

7. Appendix

Here we exhibit explicitly all the reduced affine root systems and reduced marked elliplic root systems. Sect. 7.1 is a general setting of Sect. 7.2 and Sect. 7.3.

7.1. **Finite Root Systems.**

Let R_{f} be a finite, or ordinal, root system belonging to $(F_{\mathrm{f}}, I_{\mathrm{f}})$ of type X_l. Then it is well known that X_l is either A_l $(l \geq 1)$, B_l $(l \geq 2)$, C_l $(l \geq 3)$, D_l $(l \geq 4)$, E_l $(l = 6, 7, 8)$, F_4, or G_2. See [Bo] for an explicit description of a finite root system. Let us put the quadratic form

$$q_{\mathrm{f}}(u) := I_{\mathrm{f}}(u, u)/2$$

for all $u \in F_{\mathrm{f}}$. It is known that the set $\{q_{\mathrm{f}}(\alpha) \mid \alpha \in R_{\mathrm{f}}\}$ is finite. Let us put

$$\begin{aligned} l_{\max} &:= \max\{q_{\mathrm{f}}(\alpha) \mid \alpha \in R_{\mathrm{f}}\}, \\ l_{\min} &:= \min\{q_{\mathrm{f}}(\alpha) \mid \alpha \in R_{\mathrm{f}}\}, \end{aligned}$$

and

$$\begin{aligned} S_{\mathrm{f}} &:= \{\alpha \in R_{\mathrm{f}} \mid q_{\mathrm{f}}(\alpha) = l_{\min}\}, \\ L_{\mathrm{f}} &:= \{\alpha \in R_{\mathrm{f}} \mid q_{\mathrm{f}}(\alpha) = l_{\max}\}, \\ M_{\mathrm{f}} &:= \{\alpha \in R_{\mathrm{f}} \mid l_{\min} < q_{\mathrm{f}}(\alpha) < l_{\max}\}. \end{aligned} \tag{47}$$

The roots in S_{f} are called *short*, and the roots in L_{f} are called *long*. We obviously have $R_{\mathrm{f}} = S_{\mathrm{f}} \cup M_{\mathrm{f}} \cup L_{\mathrm{f}}$. Note that M_{f} is empty unless $X = BC$.

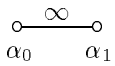
7.2. **Reduced Affine Root Systems.**

Here we exhibit all the reduced root systems. Let R be a reduced root system of type $X_l^{(t)}$. We know that R has the expression in (7). More precisely, it is known that S_i in (7) is determined by modulo $t\,\mathbb{Z}$. That is, R has the following expression:

$$R = \{\alpha + ib \mid \alpha \in (R_{\mathrm{f}})_{\bar{i}},\ i \in \mathbb{Z}_t\},$$

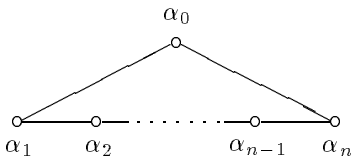
where the sets $\{(R_{\mathrm{f}})_{\bar{i}}\}$ are certain subsets of R_{f}, the finite root system of type X. In the following, we list the type, the diagram, and the subsets $\{(R_{\mathrm{f}})_{\bar{i}} \mid i \in \mathbb{Z}_t\}$. See (47) for S_{f} and M_{f}.

$A_1^{(1)}$



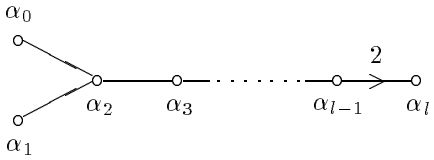
$(R_{\mathrm{f}})_{\bar{0}} := R_{\mathrm{f}}.$

$A_l^{(1)}$
$(l \geq 2)$



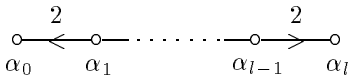
$(R_{\mathrm{f}})_{\bar{0}} := R_{\mathrm{f}}.$

$B_l^{(1)}$
$(l \geq 3)$



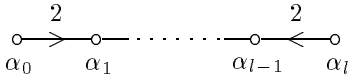
$(R_{\mathrm{f}})_{\bar{0}} := R_{\mathrm{f}}.$

$B_l^{(2)}$
$(l \geq 2)$



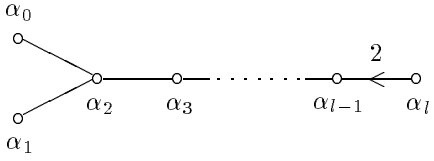
$(R_{\mathrm{f}})_{\bar{0}} := R_{\mathrm{f}}, \quad (R_{\mathrm{f}})_{\bar{1}} = S_{\mathrm{f}}.$

$C_l^{(1)}$
$(l \geq 2)$



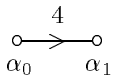
$(R_{\mathrm{f}})_{\bar{0}} := R_{\mathrm{f}}.$

$C_l^{(2)}$
$(l \geq 3)$



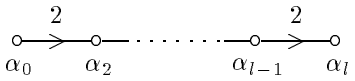
$$(R_{\mathrm{f}})_{\bar{0}} := R_{\mathrm{f}}, \quad (R_{\mathrm{f}})_{\bar{1}} = S_{\mathrm{f}}.$$

$BC_1^{(2)}$



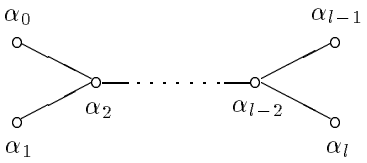
$$(R_{\mathrm{f}})_{\bar{0}} := S_{\mathrm{f}}, \quad (R_{\mathrm{f}})_{\bar{1}} = R_{\mathrm{f}}.$$

$BC_l^{(2)}$
$(l \geq 2)$



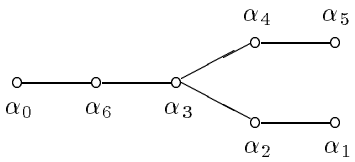
$$(R_{\mathrm{f}})_{\bar{0}} := S_{\mathrm{f}} \cup M_{\mathrm{f}}, \quad (R_{\mathrm{f}})_{\bar{1}} = R_{\mathrm{f}}.$$

$D_l^{(1)}$
$(l \geq 4)$



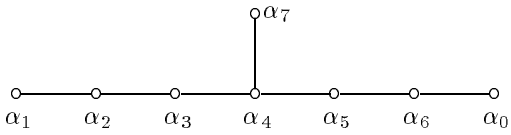
$$(R_{\mathrm{f}})_{\bar{0}} := R_{\mathrm{f}}.$$

$E_6^{(1)}$



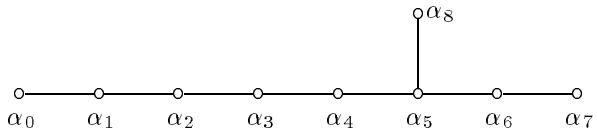
$$(R_{\mathrm{f}})_{\bar{0}} := R_{\mathrm{f}}.$$

$E_7^{(1)}$



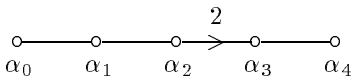
$$(R_{\mathrm{f}})_{\bar{0}} := R_{\mathrm{f}}.$$

$E_8^{(1)}$



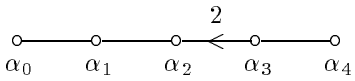
$$(R_{\mathrm{f}})_{\bar{0}} := R_{\mathrm{f}}.$$

$F_4^{(1)}$



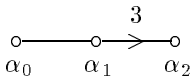
$$(R_{\mathrm{f}})_{\bar{0}} := R_{\mathrm{f}}.$$

$F_4^{(2)}$



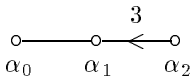
$$(R_{\mathrm{f}})_{\bar{0}} := R_{\mathrm{f}}, \quad (R_{\mathrm{f}})_{\bar{1}} = S_{\mathrm{f}}.$$

$G_2^{(1)}$



$$(R_{\mathrm{f}})_{\bar{0}} := R_{\mathrm{f}}.$$

$G_2^{(3)}$



$$(R_{\mathrm{f}})_{\bar{0}} := R_{\mathrm{f}}, \quad (R_{\mathrm{f}})_{\bar{1}} = (R_{\mathrm{f}})_{\bar{2}} = S_{\mathrm{f}}.$$

7.3. **Reduced Marked Elliptic Root Systems.**

Here we exhibit all the reduced marked elliptic root systems. Let (R, G) be a reduced elliptic root system of type $X_l^{(t_1,t_2)}$ and let $(\Gamma_{\mathrm{a}}, k|\Gamma_{\mathrm{a}})$ be its pre-basis. We know that R has the expression in (13). More

precisely, It is shown that $S_{(i,j)}$ in (13) is determined by modulo $t_1\,\mathbb{Z} \times t_2\,\mathbb{Z}$. That is, R has the following expression:

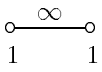
$$R = \{\alpha + ib + ja \mid \alpha \in (R_{\mathrm{f}})_{(\bar{i},\bar{j})},\ (\bar{i},\bar{j}) \in \mathbb{Z}_{t_1,t_2}\}.$$

(see Sect. 4.2), where the sets $\{(R_{\mathrm{f}})_{(\bar{i},\bar{j})} \mid (\bar{i},\bar{j}) \in \mathbb{Z}_{t_1,t_2}\}$ are certain subsets of R_{f}, the finite root system of type X. In the following, we list the type, the diagram of Γ_{a} in which $k|\Gamma_{\mathrm{a}}$ is coded, and the subsets $\{(R_{\mathrm{f}})_{\bar{i}} \mid i \in \mathbb{Z}_t\}$. See (47) for S_{f} and M_{f}. The following 1, 2 are the ways in which we describe.

1. In each diagram, each vertex α is labeled by $k(\alpha)$, the value of the counting k at α.

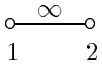
2. In each table, we exhibit $(R_{\mathrm{f}})_{(\bar{i},\bar{j})}$ as the $(\bar{i},\bar{j})$-entry. Note that the size of column (vertical length of the table) is t_1 and the size of row (horizontal length of the table) is t_2, (For example, in the table of case $B_l^{(1,2)}$, $t_1 = 1$, $t_2 = 2$, $(R_{\mathrm{f}})_{\bar{o}} = R_{\mathrm{f}}$, and $(R_{\mathrm{f}})_{(\bar{0},\bar{1})} = S_{\mathrm{f}}$.)

$A_1^{(1,1)}$



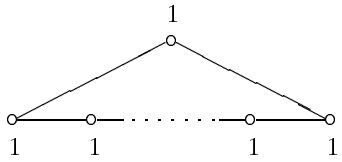
	$\bar{0}$
$\bar{0}$	R_{f}

$A_1^{(1,1)*}$



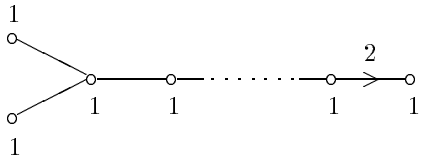
	$\bar{0}$	$\bar{1}$
$\bar{0}$	R_{f}	R_{f}
$\bar{1}$	R_{f}	$\emptyset$

$A_l^{(1,1)}$
$(l \geq 2)$



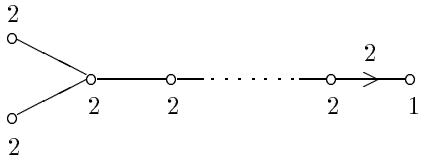
	$\bar{0}$
$\bar{0}$	R_{f}

$B_l^{(1,1)}$
$(l \geq 3)$



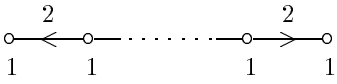
	$\bar{0}$
$\bar{0}$	R_{f}

$B_l^{(1,2)}$
$(l \geq 3)$



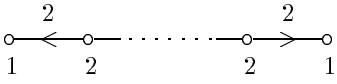
	$\bar{0}$	$\bar{1}$
$\bar{0}$	R_{f}	S_{f}

$B_l^{(2,1)}$
$(l \geq 2)$



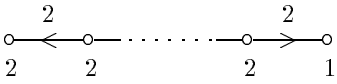
	$\bar{0}$
$\bar{0}$	R_{f}
$\bar{1}$	S_{f}

$B_l^{(2,2)}$
$(l \geq 2)$



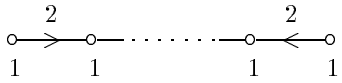
	$\bar{0}$	$\bar{1}$
$\bar{0}$	R_{f}	S_{f}
$\bar{1}$	S_{f}	S_{f}

$B_l^{(2,2)*}$
$(l \geq 2)$



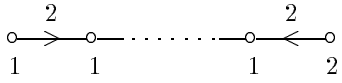
	$\bar{0}$	$\bar{1}$
$\bar{0}$	R_{f}	S_{f}
$\bar{1}$	S_{f}	$\emptyset$

$C_l^{(1,1)}$
$(l \geq 2)$



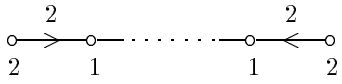
	$\bar{0}$
$\bar{0}$	R_{f}

$C_l^{(1,1)*}$
$(l \geq 2)$



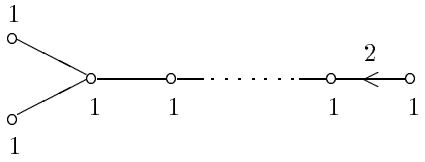
	$\bar{0}$	$\bar{1}$
$\bar{0}$	R_{f}	R_{f}
$\bar{1}$	R_{f}	S_{f}

$C_l^{(1,2)}$
$(l \geq 2)$



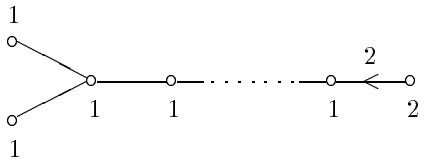
	$\bar{0}$	$\bar{1}$
$\bar{0}$	R_{f}	S_{f}

$C_l^{(2,1)}$
$(l \geq 3)$



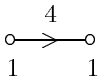
	$\bar{0}$
$\bar{0}$	R_{f}
$\bar{1}$	S_{f}

$C_l^{(2,2)}$
$(l \geq 3)$



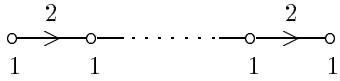
	$\bar{0}$	$\bar{1}$
$\bar{0}$	R_{f}	S_{f}
$\bar{1}$	S_{f}	S_{f}

$BC_1^{(2,1)}$



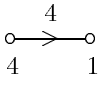
	$\bar{0}$
$\bar{0}$	S_{f}
$\bar{1}$	R_{f}

$BC_l^{(2,1)}$
$(l \geq 2)$



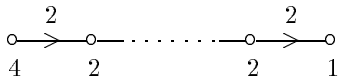
	$\bar{0}$
$\bar{0}$	$S_{\mathrm{f}} \cup M_{\mathrm{f}}$
$\bar{1}$	R_{f}

$BC_1^{(2,4)}$



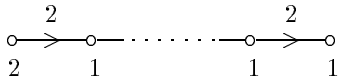
	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$
$\bar{0}$	S_{f}	S_{f}	S_{f}	S_{f}
$\bar{1}$	R_{f}	S_{f}	S_{f}	S_{f}

$BC_l^{(2,4)}$
$(l \geq 2)$



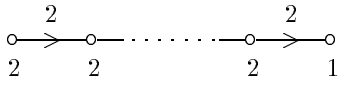
	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$
$\bar{0}$	$S_{\mathrm{f}} \cup M_{\mathrm{f}}$	S_{f}	$S_{\mathrm{f}} \cup M_{\mathrm{f}}$	S_{f}
$\bar{1}$	R_{f}	S_{f}	$S_{\mathrm{f}} \cup M_{\mathrm{f}}$	S_{f}

$BC_l^{(2,2)}(1)$
$(l \geq 2)$



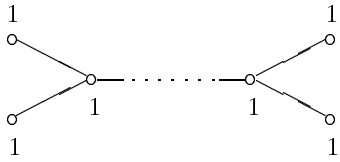
	$\bar{0}$	$\bar{1}$
$\bar{0}$	$S_{\mathrm{f}} \cup M_{\mathrm{f}}$	$S_{\mathrm{f}} \cup M_{\mathrm{f}}$
$\bar{1}$	R_{f}	$S_{\mathrm{f}} \cup M_{\mathrm{f}}$

$BC_l^{(2,2)}(2)$
$(l \geq 2)$



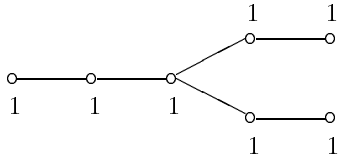
	$\bar{0}$	$\bar{1}$
$\bar{0}$	$S_{\mathrm{f}} \cup M_{\mathrm{f}}$	S_{f}
$\bar{1}$	R_{f}	S_{f}

$D_l^{(1,1)}$
$(l \geq 4)$



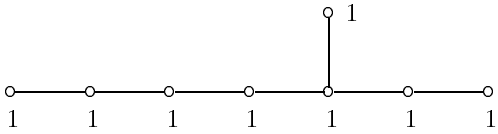
	$\bar{0}$
$\bar{0}$	R_{f}

$E_6^{(1,1)}$



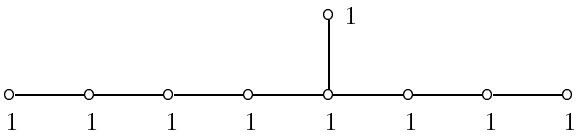
	$\bar{0}$
$\bar{0}$	R_{f}

$E_7^{(1,1)}$



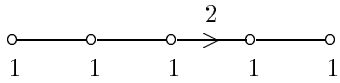
	$\bar{0}$
0	R_{f}

$E_8^{(1,1)}$



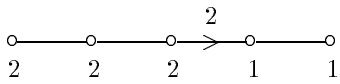
	$\bar{0}$
$\bar{0}$	R_{f}

$F_4^{(1,1)}$



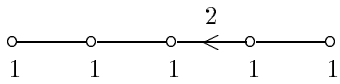
	$\bar{0}$
$\bar{0}$	R_{f}

$F_4^{(1,2)}$



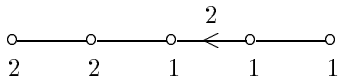
	$\bar{0}$	$\bar{1}$
$\bar{0}$	R_{f}	S_{f}
$\bar{1}$	S_{f}	S_{f}

$F_4^{(2,1)}$



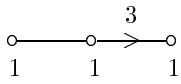
	$\bar{0}$
$\bar{0}$	R_{f}

$F_4^{(2,2)}$



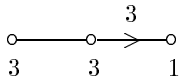
	$\bar{0}$	$\bar{1}$
$\bar{0}$	R_{f}	S_{f}
$\bar{1}$	S_{f}	S_{f}

$G_2^{(1,1)}$



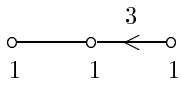
	$\bar{0}$
$\bar{0}$	R_{f}

$G_2^{(1,3)}$



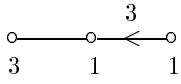
	$\bar{0}$	$\bar{1}$	$\bar{2}$
$\bar{0}$	R_{f}	S_{f}	S_{f}

$G_2^{(3,1)}$



	$\bar{0}$
$\bar{0}$	R_{f}
$\bar{1}$	S_{f}
$\bar{2}$	S_{f}

$G_2^{(3,3)}$



	$\bar{0}$	$\bar{1}$	$\bar{2}$
$\bar{0}$	R_{f}	S_{f}	S_{f}
$\bar{1}$	S_{f}	S_{f}	S_{f}
$\bar{2}$	S_{f}	S_{f}	S_{f}

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