VERTEX OPERATOR ALGEBRA ARISING FROM THE MINIMAL SERIES $M(3,p)$ AND MONOMIAL BASIS

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Dedicated to Professor B. M. McCoy on the occasion of his sixtieth birthday

ABSTRACT. We study a vertex operator algebra (VOA) V related to the $M(3,p)$ Virasoro minimal series. This VOA reduces in the simplest case $p = 4$ to the level two integrable vacuum module of $\widehat{sl}_2$. On V there is an action of a commutative current $a(z)$, which is an analog of the current $e(z)$ of $\widehat{sl}_2$. Our main concern is the subspace W generated by this action from the highest weight vector of V. Using the Fourier components of $a(z)$, we present a monomial basis of W and a semi-infinite monomial basis of V. We also give a Gordon type formula for their characters.

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1. Introduction

Let us recall some basic facts about integrable $\widehat{\mathfrak{sl}}_2$–modules. Fix in $\mathfrak{sl}_2$ the standard basis $\{e,h,f\}$, and let $e(z)$, $h(z)$, $f(z)$ be the corresponding currents of $\widehat{\mathfrak{sl}}_2$:

$$e(z)=\sum_{i\in\mathbb{Z}} e_i z^{-i-1}, \quad h(z)=\sum_{i\in\mathbb{Z}} h_i z^{-i-1}, \quad f(z)=\sum_{i\in\mathbb{Z}} f_i z^{-i-1}.$$

It is well known that the operators $e(z)^{k+1}$, $f(z)^{k+1}$ act as zero on any level k integrable representation of $\widehat{\mathfrak{sl}}_2$. Conversely, the relations $[e(z),e(z')]=0$ and $e(z)^{k+1}=0$ allow us to describe integrable representations as follows.

Let us start from a general setting. Consider the polynomial ring A in generators $\{a_i\}_{i\in\mathbb{Z}}$, and let $a(z)=\sum_{i\in\mathbb{Z}} a_i z^{-i-1}$. A is a graded algebra with the assignment $\deg a_i=-i$. We say that an A-module W belongs to the category $\mathcal{O}$ if it is $\mathbb{Z}$-graded : $W=\oplus_{j\in\mathbb{Z}} W_j$, and $W_j=0$ for sufficiently small j. On a module W in the category $\mathcal{O}$, certain infinite combinations of the generators a_i have a well-defined action. For example, each coefficient of $a(z)^m$ $(m=1,2,\cdots)$ has a meaning as an operator on W. Denote by $\tilde{A}$ the algebra obtained by adjoining to A the coefficients of arbitrary polynomials in $a(z)$ and its derivatives $\partial a(z)$, $\partial^2 a(z),\cdots$ $(\partial=\partial/\partial z)$. For each non-negative integer k, let J_{k+1} be the ideal generated by the coefficients of $a(z)^{k+1}$. We define the quotient algebra $A_{k+1}=\tilde{A}/J_{k+1}$. Note that $a(z)$ considered in A_{k+1} satisfies the same relations as $e(z)$. It is also convenient to add to A_{k+1} the elements D and D^{-1} with the relations $Da_iD^{-1}=a_{i+2}$, $DD^{-1}=D^{-1}D=1$.

In [2], it was shown that the algebra A_{k+1} (extended by D) has a remarkable class of irreducible representations $\pi_{\alpha,\beta}$, where α, β are non–negative integers such that $\alpha+\beta=k$. The space $\pi_{\alpha,\beta}$ has a basis consisting of semi–infinite monomials $\prod_{i\in\mathbb{Z}} a_i^{b_i}$, where the exponents $\{b_i\}_{i\in\mathbb{Z}}$ run over sequences of non–negative integers satisfying the following conditions.

(a): $b_i+b_{i+1}\le k$,
(b): there exists an n_0 such that $b_n=0$ for $n\le n_0$,
(c): there exists an n_1 such that for $n\ge n_1$

$$b_n=\begin{cases}\alpha & (n \text{ odd}),\\ \beta & (n \text{ even}).\end{cases}$$

Actually, under the identification $a(z)=e(z)$, $\pi_{\alpha,\beta}$ is nothing but the irreducible integrable representation of $\widehat{\mathfrak{sl}}_2$ with the highest weight (α,β).

At this point, it is natural to ask the following questions. What will happen if we replace the relation $a(z)^{k+1} = 0$ by something else? For which kind of relations is it possible to find an algebra similar to A_{k+1} and its irreducible representations which have bases formed by semi–infinite monomials? These questions are rather complicated, and for 'generic' relations we cannot hope for such a construction. Therefore, it is important to study concrete examples (cf. [1],[3]).

Simplest examples arise when we try to generalize the case of $\widehat{\mathfrak{sl}}_2$ with $k = 1$. Namely, fix a positive integer m and consider the current $a(z)$ with the relations

$$a(z)^2 = 0, \big(\partial a(z)\big)^2 = 0, \cdots, \big(\partial^m a(z)\big)^2 = 0.$$

Such an $a(z)$ can be realized as $a(z) = \phi_{\sqrt{2m}}(z)$, where $\phi_\beta(z) =: e^{\beta\varphi(z)} :$ stands for the vertex operator with momentum β. The current $a(z)$ together with $\partial\varphi(z)$ and $a^*(z) = \phi_{-\sqrt{2m}}(z)$ form a vertex operator algebra (VOA). They are analogous to the currents $e(z)$, $h(z)$, $f(z)$ of $\widehat{\mathfrak{sl}}_2$. The $a(z)$ (resp. $a^*(z)$) are commutative, $\partial\varphi(z)$ generates the Heisenberg algebra, and $[\partial\varphi(z), a(w)]$ (resp. $[\partial\varphi(z), a^*(w)]$) have the same form as in the case of $\widehat{\mathfrak{sl}}_2$. A major difference exists in the bracket $[a(z), a^*(w)]$ which becomes a differential polynomial in $\partial\varphi(z)$.

This VOA has $2m$ irreducible representations enumerated by an integer l with $0 \le l < 2m$. These representations have a basis consisting of monomials $\prod_{i\in\mathbb{Z}} a_i^{b_i}$, where

(a'): $b_i + b_{i+1} + \cdots + b_{i+2m-1} \le 1$,
(b'): There exists an n_0 such that $b_n = 0$ for $n \le n_0$,
(c'): There exists an n_1 such that for $n \ge n_1$

$$b_n = \begin{cases} 1 & (n \equiv l \mod 2m), \\ 0 & (\text{otherwise}). \end{cases}$$

Now let us try to generalize the case of $\widehat{\mathfrak{sl}}_2$ with $k = 2$. The simplest idea is to replace the relation $a(z)^3 = 0$ by two relations $a(z)^3 = 0$, $a(z)\big(\partial a(z)\big)^2 = 0$. It is possible to construct a space which admits an action of $a(z)$ and has a basis formed by semi-infinite monomials $\prod_{i\in\mathbb{Z}} a_i^{b_i}$. The exponents $\{b_i\}$ satisfy conditions similar to (a'), (b') and (c') above, wherein the most interesting property (a') is now replaced by the condition $b_i + b_{i+1} + b_{i+2} \le 2$.

The corresponding VOA can be constructed explicitly. To do that, recall first the following well–known construction of $\widehat{\mathfrak{sl}}_2$ with $k = 2$. Consider the VOA obtained as the tensor product of the Virasoro minimal theory $(3, 4)$ (Ising model) and the lattice vertex algebra generated

by $\phi_{\pm 1}(z)$. Let $\psi(z)$ be the $(2,1)$ primary field of the $(3,4)$ theory (Ising fermion). Then the formulas

$$a(z) = \psi(z)\phi_1(z),$$
$$a^*(z) = \psi(z)\phi_{-1}(z),$$

give a realization of the currents $e(z), f(z)$ of $\widehat{\mathfrak{sl}}_2$ at level 2. In particular $a(z)^3 = a^*(z)^3 = 0$. In the setting above, let us now replace the $(3,4)$ theory by the $(3,5)$ theory. We also replace the Ising fermion by the $(2,1)$ primary field of the latter, and the vertex operators $\phi_{\pm 1}(z)$ by $\phi_{\pm\sqrt{3/2}}(z)$. It turns out that the resulting current $a(z)$ satisfies the desired relations $a(z)^3 = 0$ and $a(z)\big(\partial a(z)\big)^2 = 0$.

As a next step we can try a larger set of cubic relations, for example $a(z)^3 = 0$, $a(z)\big(\partial a(z)\big)^2 = 0$ and $a(z)\partial a(z)\partial^2 a(z) = 0$. But even this case is rather hard to study. In this paper we try to see what will happen if we replace the $(3,4)$ or the $(3,5)$ theory in the previous construction by some other Virasoro minimal model. Though the situation in general is obscure, the above construction goes through for the $(3,p)$ theory. For example, for the $(3,7)$ theory, we get an abelian current $a(z)$ satisfying 5 cubic relations.

Our main results are the following.

(i) For the $(3,p)$ theory we construct the current $a(z)$ and describe the set of cubic relations.
(ii) We find a semi–infinite monomial basis of the resulting VOA.
(iii) We obtain a Gordon–type (fermionic) formula for the characters of VOA and its natural subspace which we call 'principal subspace'.

The text is organized as follows. In Section 2, after preparing the notation, we introduce the VOA and the principal subspace, and state the main results. Using the 'functional model' [2], the study of the characters of these spaces is reduced to that of certain spaces of symmetric polynomials which arise as correlation functions (matrix elements of products of currents). In Section 3 we determine the structure of three point functions. We use these results in Section 4 to give an upper bound for the characters. By comparing it with known fermionic formulas for the Virasoro minimal characters [5], we find that this bound is in fact exact. Section 5 is devoted to a combinatorics to obtain the monomial basis.

2. Construction of Vertex Operator Algebras

2.1. **Notation.** The subject of the present article is a vertex operator algebra given as a tensor product of two conformal field theories —

the $(3,p)$ Virasoro minimal series and a free bosonic theory. First we review a few basic facts about these and fix the notation.

Let $p \geq 4$ be an integer not divisible by 3. The $(3,p)$ minimal series representations of the Virasoro algebra is characterized by the central charge

$$c' = 1 - \frac{2(p-3)^2}{p}.$$

Let $T'(z) = \sum_{n\in\mathbb{Z}} L'_n z^{-n-2}$ denote the current of the Virasoro algebra. Let further $M_{r,s} = M_{r,s}(3,p)$ ($r = 1,2$, $1 \leq s \leq p-1$) be the irreducible Virasoro module with central charge c' and highest weight

$$\Delta_{r,s} = \frac{(pr-3s)^2 - (p-3)^2}{12p}, \tag{2.1}$$

and let $|r,s\rangle \in M_{r,s}$ be the corresponding highest weight vector. We have $M_{2,s} \simeq M_{1,p-s}$.

To each $M_{r,s}$ there corresponds the (r,s) primary field $\psi_{r,s}(z)$ of conformal dimension (2.1). We shall particularly be concerned with the cases $(r,s) = (1,1),(2,1)$. The corresponding primary fields are the identity $\psi_{1,1}(z) = I$, and the $(2,1)$ primary field $\psi(z) = \psi_{2,1}(z)$. The latter is characterized by the operator product expansion

$$T'(z)\psi(w) = \frac{\Delta_{2,1}}{(z-w)^2}\psi(w) + \frac{1}{z-w}\partial\psi(w) + O(1),$$

where $\Delta_{2,1} = (p-2)/4$. Viewed as an operator $M_{1+i,1} \to M_{2-i,1}$ ($i = 0,1$), $\psi(z)$ has a Fourier mode expansion

$$\psi(z) = \sum_{n\in\mathbb{Z}} \psi_n z^{-n-\frac{p-2}{2}i}.$$

The modes are indexed so that ψ_n has degree $-n$: $[L'_0, \psi_n] = -n\psi_n$. We normalize $\psi(z)$ by $\psi_0|1,1\rangle = |2,1\rangle$, $\psi_0|2,1\rangle = |1,1\rangle$.

Next let $h(z) = \sum_{n\in\mathbb{Z}} h_n z^{-n-1}$ denote the current of the Heisenberg algebra

$$[h_m, h_n] = m\delta_{m+n,0}.$$

For a complex number γ, let

$$\mathcal{F}_\gamma = \mathbb{C}[h_{-1}, h_{-2}, \cdots]|\gamma\rangle$$

be the Fock space with the highest weight vector $|\gamma\rangle$ satisfying

$$h_n|\gamma\rangle = 0 \ (n > 0), \qquad h_0|\gamma\rangle = \gamma|\gamma\rangle.$$

The Virasoro algebra acts on $\mathcal{F}_\gamma$ through the current

$$T''_\lambda(z) = \frac{1}{2} : h(z)^2 : + \lambda\partial h(z) \tag{2.2}$$

with the central charge $c''_\lambda = 1-12\lambda^2$. Here λ is a complex number, and the normal ordering rule : $h_m h_n := h_m h_n$ $(m \le n)$ is implied. Finally let $\phi_\alpha(z) : \mathcal{F}_\gamma \to \mathcal{F}_{\gamma+\alpha}$ denote the chiral vertex operator

$$\phi_\alpha(z) = \exp\left(-\alpha \sum_{n<0} \frac{h_n}{n} z^{-n}\right) e^{\alpha Q} z^{\alpha h_0} \exp\left(-\alpha \sum_{n>0} \frac{h_n}{n} z^{-n}\right),$$

where $e^{\alpha Q}$ is the isomorphism of vector spaces $\mathcal{F}_\gamma \xrightarrow{\sim} \mathcal{F}_{\gamma+\alpha}$ such that $[e^{\alpha Q}, h_n] = 0$ $(n \ne 0)$, $e^{\alpha Q}|\gamma\rangle = |\gamma+\alpha\rangle$.

In the next subsection we consider the Virasoro current

$$T(z) = T'(z) + T''_\lambda(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2} \tag{2.3}$$

which acts on $M_{r,s} \otimes \mathcal{F}_\gamma$ with the central charge $c = c' + c''_\lambda$.

2.2. **Vertex operator algebra.** The $(2,1)$ field $\psi(z)$ obeys a particularly simple fusion rule, which is expressed as an operator product expansion

$$\begin{aligned}
&(z-w)^{(p-2)/2}\psi(z)\psi(w) \\
&\quad = I + \frac{p-2}{2c'}(z-w)^2 T'(w) + O((z-w)^3) \\
&\quad = (w-z)^{(p-2)/2}\psi(w)\psi(z).
\end{aligned} \tag{2.4}$$

This means that each matrix element of $(z-w)^{(p-2)/2}\psi(z)\psi(w)$ can be continued analytically to $\mathbb{C}^\times \times \mathbb{C}^\times$, coincides with that of $(w-z)^{(p-2)/2}\psi(w)\psi(z)$, and has the expansion (2.4) as $z \to w$.

Let us modify $\psi(z)$ to get rid of the factor $(z-w)^{(p-2)/2}$. Consider the currents

$$a(z) = \psi(z)\phi_\beta(z), \tag{2.5}$$

$$a^*(z) = \psi(z)\phi_{-\beta}(z), \tag{2.6}$$

where

$$\beta = \sqrt{\frac{p-2}{2}}.$$

Note that

$$\phi_\beta(z)\phi_{\pm\beta}(w) = (z-w)^{\pm(p-2)/2} : \phi_\beta(z)\phi_{\pm\beta}(w) : . \tag{2.7}$$

The currents (2.5), (2.6) act on the space $V = \bigoplus_{n\in\mathbb{Z}} V_n$ with

$$V_n = \begin{cases} M_{1,1} \otimes \mathcal{F}_{n\beta} & (n \text{ even}), \\ M_{2,1} \otimes \mathcal{F}_{n\beta} & (n \text{ odd}). \end{cases} \tag{2.8}$$

Here $\psi(z)$ acts on the first tensor factor while $\phi_{\pm\beta}(z)$ acts on the second. They have Fourier mode expansions in integral powers of z,

$$a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1},$$
$$a^*(z) = \sum_{n\in\mathbb{Z}} a^*_n z^{-n-p+3},$$

where $a_n, a^*_n \in \mathrm{End}(V)$. In the case $p = 4$, V is isomorphic to the level 2 integrable vacuum module of $\widehat{\mathfrak{sl}_2}$ under the identification $a(z) = e(z)$, $a^*(z) = f(z)$.

For convenience we choose

$$\lambda = \beta - \beta^{-1}$$

in (2.2), so that $a(z)$ has conformal dimension 1 with respect to $T(z)$ given by (2.3). On each V_n we have an action of $T(z)$ and $h(z)$. Under this joint action of the Virasoro and the Heisenberg algebras, V_n is irreducible and is generated by the vector

$$v_n = \begin{cases} |1,1\rangle \otimes |n\beta\rangle & (n \text{ even}), \\ |2,1\rangle \otimes |n\beta\rangle & (n \text{ odd}). \end{cases}$$

We call v_n extremal vectors. They satisfy

$$L_m v_n = 0, \quad h_m v_n = 0 \qquad (m > 0), \tag{2.9}$$

$$a_m v_{2k+i} = 0 \qquad (m \geq -(p-2)k), \tag{2.10}$$
$$a^*_m v_{2k+i+1} = 0 \qquad (m \geq (p-2)k + 2),$$

$$a_{-1-(p-2)k} v_{2k+i} = v_{2k+i+1}, \quad a^*_{1+(p-2)k} v_{2k+i+1} = v_{2k+i}, \tag{2.11}$$

where $k \in \mathbb{Z}$ and $i = 0, 1$. In addition, the extremal vector v_0 satisfies

$$L_0 v_0 = L_{-1} v_0 = 0. \tag{2.12}$$

Recall that two formal series $A(z), B(z) \in \mathrm{End}(V)[[z, z^{-1}]]$ are said to be mutually local if $(z-w)^N A(z)B(w) = \varepsilon_{AB}(z-w)^N B(w)A(z)$ holds for some integer N and $\varepsilon_{AB} = \pm 1$. The currents $T(z)$, $h(z)$, $a(z)$, $a^*(z)$ are pairwise mutually local in this sense.

Proposition 2.1. *The space V has the structure of a vertex operator algebra.*

Proof. V is spanned by vectors Pv_0, with P running over the monomials of $\{L_n, h_n, a_n, a^*_n\}_{n\in\mathbb{Z}}$. Clearly $L_{-2}v_0, h_{-1}v_0, a_{-1}v_0, a^*_{-p+3}v_0$ are linearly independent. Hence the statement follows from the generalities on vertex operator algebras (see e.g. [4], p.110). □

We have the following operator product expansions as $z \to w$.

$$
\begin{aligned}
T(z)T(w) &= \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial T(w) + O(1),\\
T(z)h(w) &= \frac{-2\lambda}{(z-w)^3} + \frac{1}{(z-w)^2}h(w) + \frac{1}{z-w}\partial h(w) + O(1),\\
h(z)h(w) &= \frac{1}{(z-w)^2} + O(1),\\
T(z)a(w) &= \frac{1}{(z-w)^2}a(w) + \frac{1}{z-w}\partial a(w) + O(1),\\
T(z)a^*(w) &= \frac{p-3}{(z-w)^2}a^*(w) + \frac{1}{z-w}\partial a^*(w) + O(1),\\
h(z)a(w) &= \frac{\beta}{z-w}a(w) + O(1),\\
h(z)a^*(w) &= \frac{-\beta}{z-w}a^*(w) + O(1),\\
a(z)a(w) &= O(1),\\
a^*(z)a^*(w) &= O(1),\\
a(z)a^*(w) &= \frac{1}{(z-w)^{p-2}}\left(I + O(z-w)\right).
\end{aligned}
$$

In the below we consider the algebra generated by $a(z)$. There is no singularity in the expansions of $a(z)a(w)$. To be more precise, we have

$$
a(z)a(w) = \left(I + \frac{p-2}{2c'}(z-w)^2T'(w) + O((z-w)^3)\right) : \phi_\beta(z)\phi_\beta(w) : .
$$

Note also that $[L_0, a_n] = -na_n$.

The operators L_0 and $H = h_0/\beta$ give rise to a $\mathbb{Z}\times\mathbb{Z}$-gradation $V = \bigoplus_{d,n\in\mathbb{Z}} V_{d,n}$, where

$$
V_{d,n} = \{v \in V \mid L_0 v = dv,\ Hv = nv\}.
$$

Quite generally, for any bi-graded vector space $U = \bigoplus_{d,n\in\mathbb{Z}} U_{d,n}$, we call

$$
\mathrm{ch}_{q,z}U = \mathrm{tr}_U\left(q^{L_0}z^H\right) = \sum_{d,n}(\dim\ U_{d,n})\, q^d z^n
$$

the character of U. When there is no fear of confusion, we call

$$
\mathrm{ch}_q U_n = \sum_d (\dim U_{d,n}) q^d
$$

the character of $U_n = \oplus_{d\in\mathbb{Z}} U_{d,n}$ as well.

2.3. **Principal subspace.** From now on we will focus attention to the current $a(z)$. For us it is important to note the following commutativity property.

Proposition 2.2. *For any $n, m \in \mathbb{Z}$ we have*

$$[a_n, a_m] = 0. \tag{2.13}$$

Proof. This is an immediate consequence of the definition (2.5) and the operator product expansion (2.4) and (2.7). □

Thus the linear span $\mathfrak{n} = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}a_n$ is an abelian Lie subalgebra of $\mathfrak{gl}(V)$. Let us consider the subspace of V generated over $\mathfrak{n}$ by an extremal vector v_n,

$$V^n = U(\mathfrak{n})\, v_n.$$

We have a sequence of embeddings

$$\cdots \longrightarrow V^4 \longrightarrow V^2 \longrightarrow V^0 \longrightarrow V^{-2} \longrightarrow \cdots . \tag{2.14}$$

On the other hand, the operator $\tau = e^{2\beta Q}$ has the property

$$\tau a_n \tau^{-1} = a_{n-p+2}. \tag{2.15}$$

In view of (2.10), (2.11), we have an isomorphism of vector spaces

$$\tau \ : \ V^n \xrightarrow{\sim} V^{n+2}.$$

Following [2] we call

$$W = V^0 = \mathbb{C}[a_{-1}, a_{-2}, \cdots]v_0$$

the principal subspace.

Proposition 2.3. *The space V coincides with the inductive limit of the sequence* (2.14), $\cup_N V^{-N} = \varinjlim_N V^{-N}$.

Proof. Set $\tilde{V} = \cup_N V^{-N}$. We show that $\tilde{V}$ is invariant under the action of h_n, L_n for all n. The assertion $\tilde{V} = V$ will then follow from the fact that $v_m \in \tilde{V}$ and the irreducibility of V_m.

Let us verify that $h_n \tilde{V} \subset \tilde{V}$ $(n \geq N)$ holds for any N. The case $N = 0$ is clear from (2.9) and

$$[h_n, a(z)] = \beta z^n a(z). \tag{2.16}$$

Suppose the statement is true for some N. Acting on v_m with the relation

$$a(z)^2 =: \phi_\beta(z)^2 :, \tag{2.17}$$

we find that $h_{N-1} v_m \in \tilde{V}$ for all m. Together with (2.16), this in turn implies that $h_{N-1}\tilde{V} \subset \tilde{V}$.

Similarly we can show that $L_n\tilde{V} \subset \tilde{V}$ $(n \geq N)$ for any N, by using

$$[L_n, a(z)] = z^n(z\partial + n + 1)a(z), \tag{2.18}$$

$$a(z)\partial^2 a(z) =: \phi_\beta(z)\partial^2\phi_\beta(z) : + \frac{p-2}{c'} : \phi_\beta(z)^2 : T'(z). \tag{2.19}$$

□

2.4. **Main results.** Since a_n's are commutative, the principal subspace W is presented as a quotient of the polynomial ring $A = \mathbb{C}[\xi_{-1}, \xi_{-2}, \cdots]$ in indeterminates $\{\xi_{-n}\}_{n\geq 1}$,

$$W \simeq A/\mathfrak{a},$$

where $\mathfrak{a}$ is an ideal of A. The ring A is bi-graded, $A = \bigoplus_{d,n\in\mathbb{Z}} A_{d,n}$, by assigning the bi-degree $(d, 1)$ to ξ_{-d}. Note that $A_{d,n} = 0$ if $d < n$. We set $A_n = \bigoplus_{d\in\mathbb{Z}} A_{d,n}$. The following propositions give some typical elements of $\mathfrak{a}_3 = \mathfrak{a} \cap A_3$.

Proposition 2.4. *We have*

$$a(z)^2\partial^\nu a(z) = 0 \qquad (0 \leq \nu \leq p-3). \tag{2.20}$$

Equivalently, the sum

$$\sum_{\substack{i,j,k>0\\ i+j+k=d}} (k-1)(k-2)\cdots(k-\nu)\xi_{-i}\xi_{-j}\xi_{-k}$$

belongs to $\mathfrak{a}_3$ *for any* d *and* ν $(0 \leq \nu \leq p-3)$.

Proof. This is an immediate consequence of

$$a(z)^2 a(w) = (z-w)^{p-2}\psi(w) : \phi_\beta(z)^2\phi_\beta(w) :, \tag{2.21}$$

which follows from (2.17) and (2.7). □

In general, $\mathfrak{a}$ contains more complicated elements. For example, if $p \geq 7$, the relation given below does not follow from (2.20) and their derivatives.

Proposition 2.5. *We have*

$$\begin{aligned} &p(p-4)a(z)^2\partial^{p-1}a(z) + 4(p-3)^2 a(z)\partial a(z)\partial^{p-2}a(z) \\ &+2(2(p-3)^2 - p)a(z)\partial^2 a(z)\partial^{p-3}a(z) = 0. \end{aligned}$$

Proof. Using (2.21) and (2.19), we expand $a(z)^2a(w)$, $a(z)\partial a(z)a(w)$ and $a(z)\partial^2 a(z)a(w)$ as $w \to z$. Comparing the coefficients of $(z-w)^{p-3}$, we obtain the assertion. □

Our first result states that all the relations among $\{a_n\}_{n\in\mathbb{Z}}$ are generated from cubic ones.

Theorem 2.6. *The ideal* $\mathfrak{a}$ *is generated by* $\mathfrak{a}_3 = \mathfrak{a} \cap A_3$.

Proof of Theorem 2.6 will be given in Sections 3 and 4. In the course we shall also find the character of $\mathfrak{a}_3$:

Proposition 2.7.

$$\mathrm{ch}_q \mathfrak{a}_3 = q^3 \frac{(1-q^{p-2})(1-q^{p-1})}{(1-q)(1-q^2)(1-q^3)}. \tag{2.22}$$

Our second result concerns the monomial basis of W and V. By the definition, W is spanned by 'monomial vectors'

$$a_{-\lambda_1} \cdots a_{-\lambda_n} v_0 \tag{2.23}$$

where $n \geq 0$ and $\lambda_1 \geq \cdots \geq \lambda_n \geq 1$. We say that (2.23) is admissible if

$$\lambda_i - \lambda_{i+2} \geq p-2 \qquad (i = 1, \cdots, n-2). \tag{2.24}$$

Theorem 2.8. *Admissible monomial vectors* (2.23) *constitute a basis of* W.

Theorem 2.8 will be proved in Section 5. This theorem leads to the following 'semi-infinite' description of the space V (cf.[2, 3]). Note that the vector v_0 can be formally rewritten as

$$\begin{aligned} v_0 &= a^2_{p-3} v_{-2} = a^2_{p-3} a^2_{2p-5} v_{-4} = \cdots \\ &= a^2_{p-3} a^2_{2p-5} a^2_{3p-7} \cdots v_{-\infty}. \end{aligned}$$

Hence, in view of Proposition 2.3, any vector $v \in V$ can be written as a linear combination of the expressions

$$v = a^{\alpha_{-m-1}}_{-m-1} a^{\alpha_{-m}}_{-m} a^{\alpha_{-m+1}}_{-m+1} \cdots v_{-\infty},$$

where $\{\alpha_j\}_{j=-m-1}^{\infty}$ runs over sequences of nonnegative integers subject to the conditions

$$\alpha_j + \alpha_{j-1} + \cdots + \alpha_{j-p+3} \leq 2 \qquad \text{for all } j,$$

$$\alpha_j = \begin{cases} 2 & (j \equiv -1 \mod p-2) \\ 0 & (\text{otherwise}) \end{cases} \qquad \text{for sufficiently large } j\ .$$

3. Three Point Functions

3.1. **Functional model.** In order to study the structure of W, we make use of the 'functional model' introduced in [2] for the restricted dual space W^*. Let us recall this description. Let $\Lambda_n = \mathbb{C}[x_1, \cdots, x_n]^{S_n}$ denote the space of symmetric polynomials in n variables, where S_n stands for the symmetric group on n letters. Let $\Lambda_{d,n}$ be the subspace

consisting of polynomials of homogeneous degree d. The dual vector space of $A_{d,n}$ can be viewed as the space $\Lambda_{d-n,n}$ through the coupling

$$\begin{aligned}&\langle f(x_1,\cdots,x_n),\xi_{-\lambda_1}\cdots\xi_{-\lambda_n}\rangle\\&=\mathrm{Res}_{x_1=\cdots=x_n=0}f(x_1,\cdots,x_n)x_1^{-\lambda_1}\cdots x_n^{-\lambda_n}dx_1\cdots dx_n.\end{aligned}$$

The dual space of $W_n \simeq A_n/\mathfrak{a}_n$ is the annihilator of $\mathfrak{a}_n$,

$$I_n = \{f \in \Lambda_n \mid \langle f, \xi\rangle = 0 \ \forall \xi \in \mathfrak{a}_n\}.$$

Since the restriction map $V_n^* \to W_n^*$ is surjective, the space I_n coincides with the set of symmetric polynomials

$$f_\psi(x_1,\cdots,x_n) = \psi\left(a(x_1)\cdots a(x_n)v_0\right)$$

where ψ runs over V_n^*. In view of the condition (2.9) and(2.12) and the commutation relations (2.16), (2.18), the right action of the Heisenberg and the Virasoro algebras on V_n^* translate as

$$f_{\psi\cdot h_m}(x_1,\cdots,x_n) = \beta P_m \cdot f_\psi(x_1,\cdots,x_n) \qquad (m>0), \tag{3.1}$$

$$\begin{aligned}f_{\psi\cdot L_m}(x_1,\cdots,x_n) &= l_m(f_\psi)(x_1,\cdots,x_n)\\&\quad +(m+1)P_m\cdot f_\psi(x_1,\cdots,x_n) \qquad (m\geq 1),\end{aligned} \tag{3.2}$$

where

$$P_m = \sum_{j=1}^n x_j^m, \quad l_m = \sum_{j=1}^n x_j^{m+1}\frac{\partial}{\partial x_j}.$$

Hence I_n is an ideal of the polynomial ring Λ_n, and is invariant under the 'half' $\bigoplus_{m\geq -1}\mathbb{C}l_m$ of the Virasoro algebra.

Let ψ_n be the composition $V_n \to \mathbb{C}v_n \to \mathbb{C}$, where the first map is the projection along the subspace $\oplus_{d>\deg v_n} V_{d,n}$. Set $\varphi_n = f_{\psi_n}$.

Proposition 3.1. *The ideal I_n is generated by polynomials of the form $l_{m_1}\cdots l_{m_k}(\varphi_n)$ $(m_1,\cdots,m_k \geq -1)$.*

Proof. By the Poincaré-Birkhoff-Witt theorem, any $\psi \in V^*$ is a linear combination of elements of the form $\psi_n L_{m_k}\cdots L_{m_1}H_{n_l}\cdots H_{n_1}$. The assertion follows from (3.1), (3.2) and the relations $\psi_n L_0 = \deg v_n \psi_n$ and $\psi_n L_{-1} = 0$. □

The polynomial φ_n is a product of matrix elements of $\psi(z)$ and $\phi_\beta(z)$,

$$\begin{aligned}&\varphi_n(x_1,\cdots,x_n)\\&=\langle 1+i,1|\psi(x_1)\cdots\psi(x_n)|1,1\rangle\langle n\beta|\phi_\beta(x_1)\cdots\phi_\beta(x_n)|0\rangle,\end{aligned} \tag{3.3}$$

where $\langle r,s|$ denotes the highest weight vector of the right module $M_{r,s}^*$, and $i=0,1$ according to whether n is even or odd. The second factor is simply $\prod_{i<j}(x_i-x_j)^{\beta^2}$.

3.2. **3 point functions.** Let us study the ideal I_3 more closely. For $n = 3$, the first factor of (3.3) is the four point function of the $(2,1)$ field with one point taken to infinity. As is well known, it satisfies a hypergeometric differential equation. We thus obtain the following explicit expression for the polynomial φ_3.

Proposition 3.2.

$$\begin{aligned}(3.4)\qquad &\varphi_3(x_1,x_2,x_3)\\ &= (x_1-x_2)^{p-2}F\left(1-\frac{p}{3},2-p,2-\frac{2p}{3};\frac{x_3-x_2}{x_1-x_2}\right),\end{aligned}$$

where $F(\alpha,\beta,\gamma;z)$ denotes the Gauss hypergeometric function.

Though not apparent, the right hand side of (3.4) is symmetric in (x_1,x_2,x_3).

Proof. Since $\psi_3 L_{-1} = 0$ and $\psi_3 L_0 = (p+1)\psi_3$ (see (2.11)), we have $l_{-1}\varphi_3 = 0$, $l_0\varphi_3 = (p-2)\varphi_3$. Hence we can write

$$(3.5)\qquad \varphi_3(x_1,x_2,x_3) = (x_1-x_2)^{p-2}g\left(\frac{x_3-x_2}{x_1-x_2}\right)$$

with some polynomial $g(z)$ satisfying $g(0) = 1$. On the other hand, the null vector condition for $\langle 2,1| \in M^*_{2,1}$ reads $\psi_3(pL'_2 - 3{L'_1}^2) = 0$. Rewriting this equation by using $L'_n = L_n - L''_n$ and (3.1),(3.2), we obtain

$$\begin{aligned}(3.6)\qquad &\big(2p\,l_2 - 6\,l_1^2 + 6(p-2)P_1 l_1\\ &\qquad -(2p-3)(p-2)P_1^2 + 3(p-2)P_2\big)\varphi_3 = 0.\end{aligned}$$

This implies that, up to a constant multiple, $g(z)$ is the unique polynomial solution of the hypergeometric equation

$$z(1-z)\frac{d^2g}{dz^2} + \left(2-\frac{2p}{3}+\frac{4}{3}(p-3)z\right)\frac{dg}{dz} - \frac{1}{3}(p-2)(p-3)g = 0.$$

□

Proposition 3.3. *There exist symmetric polynomials A_1, B_2, Q_{m-1}, R_m in (x_1,x_2,x_3) such that*

$$(3.7)\qquad l_1^2(\varphi_3) = A_1 l_1(\varphi_3) + B_2\varphi_3,$$

$$(3.8)\qquad l_m(\varphi_3) = Q_{m-1}l_1(\varphi_3) + R_m\varphi_3, \qquad (m \geq 2).$$

Proof. Eq.(3.7) is a consequence of (3.6) and (3.8) with $m = 2$. Let us verify (3.8). Set $z = (x_3 - x_2)/(x_1 - x_2)$ and note that

$$l_m(z) = Q_{m-1} l_1(z),$$
$$Q_{m-1} = \mathrm{Sym} \frac{x_1^{m+1} x_2}{(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)},$$

where Sym stands for symmetrization. Using (3.5), we obtain

$$l_m(\varphi_3) - Q_{m-1} l_1(\varphi_3) = (p-2) R_m \varphi_3,$$

where

$$R_m = \frac{x_1^{m+1} - x_2^{m+1}}{x_1 - x_2} - (x_1 + x_2) Q_{m-1}.$$

It is easy to see that R_m is symmetric. □

Proposition 3.4. *The ideal I_3 is generated by φ_3 and $l_1(\varphi_3)$,*

$$I_3 = \Lambda_3 \varphi_3 + \Lambda_3 l_1(\varphi_3).$$

Moreover

$$\Lambda_3 \varphi_3 \cap \Lambda_3 l_1(\varphi_3) = \Lambda_3(\varphi_3 \times l_1(\varphi_3)).$$

Proof. The first assertion follows from Proposition 3.3. To see the second, it suffices to show that φ_3 and $l_1(\varphi_3)$ do not have a non-trivial common factor. Let

$$C_n^\nu(z) = \frac{\Gamma(n+2\nu)}{n!\Gamma(2\nu)} F(2\nu + n, -n, \nu + \frac{1}{2}; \frac{1-z}{2})$$

denote the Gegenbauer polynomial. The recursion relations

$$(n+2) C_{n+2}^\nu(z) = 2(n+\nu+1) z C_{n+1}^\nu(z) - (n+2\nu) C_n^\nu(z),$$
$$(1-z^2) \frac{d}{dz} C_n^\nu(z) = -nz C_n^\nu(z) + (n+2\nu-1) C_{n-1}^\nu(z),$$

together with $C_0^\nu(z) = 1$ imply that $C_n^\nu(z)$ does not have multiple zeroes provided $2\nu \notin \mathbb{Z}$. Our assertion follows from this and (3.4). □

Proposition 3.4 enables us to determine the character of I_3 as

$$\mathrm{ch}_q I_3 = \frac{q^{p-2} + q^{p-1} - q^{2p-3}}{(1-q)(1-q^2)(1-q^3)}. \tag{3.9}$$

By duality we obtain the character (2.22) of $\mathfrak{a}_3$.

3.3. **Nested structure.** It turns out that there is a nested structure relating the ideal I_3 for p with that for $p-3$. Here and after, we use the superscript $[p]$ to indicate the dependence on p of various quantities, e.g., $W = W^{[p]}$, $I_3 = I_3^{[p]}$, $\varphi_3 = \varphi_3^{[p]}$. In the cases $p = 4, 5$, we define $I_3^{[p-3]} = \Lambda_3$, $\varphi_3^{[p-3]} = 1$.

In what follows we set

$$D_n = \prod_{1 \le i < j \le n} (x_i - x_j)^2.$$

Proposition 3.5. *We have an exact sequence*

$$0 \longrightarrow D_3\, I_3^{[p-3]} \xrightarrow{\ \iota\ } I_3^{[p]} \xrightarrow{\ \pi\ } (x-y)^{p-2}\mathbb{C}[x,y] \longrightarrow 0, \tag{3.10}$$

where ι is the inclusion into Λ_3, and π is the restriction map sending $g(x_1, x_2, x_3)$ to $g(x, y, y)$.

Proof. First we show that the image of ι is contained in $I_3^{[p]}$. In view of Proposition 3.4 and

$$D_3 l_1(\varphi_3^{[p-3]}) = l_1(D_3\varphi_3^{[p-3]}) - D_3^{-1} l_1(D_3) \cdot D_3\varphi_3^{[p-3]},$$

it suffices to show that $D_3\varphi_3^{[p-3]}$ belongs to $I_3^{[p]}$.

From (3.4) we have

$$\begin{aligned} l_1(\varphi_3) &= (p-2)(x_1-x_2)^{p-1} \\ &\quad \times \Big(\frac{x_2+x_3}{x_1-x_2} F(\alpha, 3\alpha-1, 2\alpha; z) + (1-z) F(\alpha, 3\alpha, 2\alpha; z) \Big), \end{aligned} \tag{3.11}$$

where $\alpha = 1 - p/3$ and $z = (x_3 - x_2)/(x_1 - x_2)$. Using an identity

$$\begin{aligned} &\frac{3(3\alpha+1)}{2(2\alpha+1)} z^2 (1-z) F(\alpha+1, 3\alpha+2, 2\alpha+2; z) \\ &= (z-2) F(\alpha, 3\alpha-1, 2\alpha; z) + 2(1-z+z^2) F(\alpha, 3\alpha, 2\alpha; z), \end{aligned}$$

which can be verified via series expansions, we find

$$k D_3\, \varphi_3^{[p-3]} = (P_1^2 - 3P_2) l_1(\varphi_3^{[p]}) + (p-2)(3P_3 - P_1P_2)\varphi_3^{[p]},$$

where $P_j = x_1^j + x_2^j + x_3^j$ and $k = -9(p-2)(p-4)/2(2p-9)$.

It is clear that $\pi \circ \iota = 0$. By (3.4) and (3.11) we have $\pi(\varphi_3^{[p]}) = (x-y)^{p-2}$ and $\pi(l_1(\varphi_3^{[p]})) = (p-2)(x+y)(x-y)^{p-2}$. Hence π maps $I_3^{[p]}$ to $(x-y)^{p-2}\mathbb{C}[x,y]$. Let $(x-y)^{p-2}J$ be the image. Then $1, x+y \in J$, and J is invariant under multiplication by $x+2y$ and $2xy+y^2$. It follows that $J = \mathbb{C}[x,y]$.

Finally we have from (3.9)

$$\begin{aligned}\mathrm{ch}_q I_3^{[p]} - q^6 \mathrm{ch}_q I_3^{[p-3]} &= q^{p-2}\frac{1}{(1-q)^2} \\ &= \mathrm{ch}_q\big((x-y)^{p-2}\mathbb{C}[x,y]\big),\end{aligned}$$

which shows the exactness of (3.10) at the middle. □

4. Gordon Type Filtration

4.1. **Gordon type filtration.** The goal of this section is to prove Theorem 2.6.

Let $\widetilde{\mathfrak{a}}^{[p]}$ be the ideal of A generated by $\mathfrak{a}_3^{[p]}$. Let $\widetilde{I}^{[p]} \subset \Lambda$ ($\Lambda = \bigcup_{n\geq 0}\Lambda_n$) be the annihilator of $\widetilde{\mathfrak{a}}^{[p]}$. As before, for $p = 4,5$ we define $\widetilde{I}^{[p-3]}$ to be Λ. We have

$$\widetilde{\mathfrak{a}}^{[p]} \subset \mathfrak{a}^{[p]}, \qquad \widetilde{I}^{[p]} \supset I^{[p]}. \tag{4.1}$$

We shall show that in fact the equality holds.

Proposition 4.1. *Let $m \geq 1$, $f \in \widetilde{I}_n^{[p]}$. Then*

$$f(x_1, \cdots, x_{n-2m}, y_1, y_1, \cdots, y_m, y_m)$$

is divisible by

$$\prod_{\substack{1\leq i\leq n-2m \\ 1\leq k\leq m}} (x_i - y_k)^{p-2} \prod_{1\leq k<l\leq m} (y_k - y_l)^{2(p-2)}.$$

Proof. When $m = 1$, this is a consequence of Proposition 3.5 and the definition of $\widetilde{I}_n^{[p]}$. The general case follows from the case $m = 1$ by further specializing arguments. □

Consider the map

$$\Lambda_n \longrightarrow \Lambda_{n-2} \otimes \mathbb{C}[y]$$

which sends $f(x_1, \cdots, x_n)$ to $f(x_1, \cdots, x_{n-2}, y, y)$. The kernel of this map is $D_n\Lambda_n$.

Proposition 4.2. *For $n \geq 3$ we have*

$$\widetilde{I}_n^{[p]} \cap D_n\Lambda_n = D_n\widetilde{I}_n^{[p-3]}.$$

Proof. In the case $n = 3$, we have shown this relation in Proposition 3.3. Suppose $n \geq 4$. Then, for $g \in \Lambda_n$, $D_n g$ belongs to $\widetilde{I}_n^{[p]}$ if and only if the coupling $\langle D_n g, \xi\eta\rangle$ vanishes for any $\xi \in \mathfrak{a}_3^{[p]}$ and $\eta \in A_{n-3}$. The latter means that $D_n g \in \mathfrak{a}_3^{[p]}$ as a polynomial in (x_1, x_2, x_3). Hence, by Proposition 3.3, this holds if and only if the product hg with $h =$

$\prod_{1\le i\le 3}\prod_{4\le j\le n}(x_i-x_j)^2$ belongs to the ideal $I_3^{[p-3]}$ as a polynomial in (x_1,x_2,x_3). Suppose g is homogeneous of degree d, and let

$$g=\sum_k g'_k(x_1,x_2,x_3)g''_{d-k}(x_4,\cdots,x_n),$$
$$h=\sum_k h'_k(x_1,x_2,x_3)h''_{6(n-3)-k}(x_4,\cdots,x_n),$$

be the decomposition into homogeneous components in (x_1,x_2,x_3) with the indicated degree. Using $h'_0\neq 0$, we see by induction on k that the above condition holds if and only if $g'_k\in\widetilde{I}_3^{[p-3]}=I_3^{[p-3]}$ for all k. The last condition is equivalent to $g\in\widetilde{I}_n^{[p-3]}$. □

We are now in a position to estimate the character of $\widetilde{I}^{[p]}$. For each n, consider the following filtration

$$F_{-1}=\{0\}\subset F_0\subset\cdots\subset F_{m-1}\subset F_m\subset\cdots\subset\widetilde{I}_n^{[p]}, \tag{4.2}$$

where the subspace F_m $(0\le m\le n/2-1)$ is defined by

$$F_m=\{f\in\widetilde{I}_n^{[p]}\mid f\big|_{x_{n-2m-1}=x_{n-2m},\cdots,x_{n-1}=x_n}=0\}.$$

Then we have the map

$$\begin{aligned}&F_m/F_{m-1}\longrightarrow\Lambda_{n-2m}\otimes\Lambda_m,\\ &f(x_1,\cdots,x_n)\mapsto f(x_1,\cdots,x_{n-2m},y_1,y_1,\cdots,y_m,y_m),\end{aligned} \tag{4.3}$$

which is injective by the definition. Combining Propositions 4.1 with 4.2 we find

Proposition 4.3. *The image of the map* (4.3) *is contained in the space*

$$\begin{aligned}&\prod_{1\le i<j\le n-2m}(x_i-x_j)^2\prod_{\substack{1\le i\le n-2m\\ 1\le k\le m}}(x_i-y_k)^{p-2}\prod_{1\le k<l\le m}(y_k-y_l)^{2(p-2)}\\ &\times\widetilde{I}_{n-2m}^{[p-3]}\otimes\Lambda_m.\end{aligned} \tag{4.4}$$

For two formal series $\chi=\sum_{d,n}c_{d,n}q^dz^n$ and $\chi'=\sum_{d,n}c'_{d,n}q^dz^n$, we write $\chi\le\chi'$ if $c_{d,n}\le c'_{d,n}$ holds for all d,n. Proposition 4.3 entails that

$$\begin{aligned}\mathrm{ch}_{q,z}I^{[p]}&\le\mathrm{ch}_{q,z}\widetilde{I}^{[p]}\\ &\le\sum_{n,m}\frac{q^{n(n-1)+(p-2)nm+(p-2)m(m-1)}}{(q)_m}z^{n+2m}\sum_d(\dim\widetilde{I}_{d,n}^{[p-3]})\,q^d,\end{aligned}$$

where $(q)_m=\prod_{j=1}^m(1-q^j)$. Iterating the above inequality s times where

$$s=\text{ the integer part of }\frac{p}{3}, \tag{4.5}$$

we are led to the estimate

$$\mathrm{ch}_{q,z} I^{[p]} \leq \sum_{m_0,m_1,\cdots,m_s\geq 0} \frac{q^{{}^t\mathbf{m}B\mathbf{m}+{}^t\mathbf{A}\mathbf{m}}}{(q)_{m_0}(q)_{m_1}\cdots(q)_{m_s}}(qz)^{m_0+2m_1+\cdots+2m_s}. \tag{4.6}$$

Here we have set

$${}^t\mathbf{m} = (m_0, m_1, \cdots, m_s), \tag{4.7}$$

$$B = \begin{pmatrix} s & \frac{p+s-3}{2} & \frac{p+s-4}{2} & \cdots & \frac{p-2}{2} \\ \frac{p+s-3}{2} & p+s-3 & p+s-4 & \cdots & p-2 \\ \frac{p+s-4}{2} & p+s-4 & p+s-4 & \cdots & p-2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{p-2}{2} & p-2 & p-2 & \cdots & p-2 \end{pmatrix}, \tag{4.8}$$

$${}^t\mathbf{A} = (-s, -p+s+1, \cdots, -p+2). \tag{4.9}$$

Since $I^{[p]}_{d-n,n} \simeq W^*_{d,n}$, (4.6) gives an upper bound for the character $\mathrm{ch}_{q,z} W^{[p]} = \mathrm{ch}_{q,qz} I^{[p]}$.

4.2. **Comparison with known characters.** In this subsection we fix p and s as in (4.5), and suppress the dependence on p from the notation. In [5], a large family of 'fermionic' formulas have been obtained for the characters of the Virasoro minimal series $M_{r,s}(p',p)$. The following formula is included as a special case of their result.

$$\mathrm{ch}_{q,z} M_{i+1,1}(3,p) = \sum_{\substack{m_0,m_1,\cdots,m_{s-1}\geq 0 \\ m_0\equiv i \bmod 2}} \frac{q^{{}^t\overline{\mathbf{m}}\overline{B}^{(i)}\overline{\mathbf{m}}+{}^t\overline{\mathbf{A}}^{(i)}\overline{\mathbf{m}}}}{(q)_{m_0}(q)_{m_1}\cdots(q)_{m_{s-1}}},$$

where $i = 0, 1$, and [1)]

$${}^t\overline{\mathbf{m}} = (m_0, m_1, \cdots, m_{s-1}),$$

$$\overline{B}^{(i)} = \begin{pmatrix} \frac{s-i+1}{4} & \frac{s-1}{2} & \frac{s-2}{2} & \cdots & \frac{1}{2} \\ \frac{s-1}{2} & s-1 & s-2 & \cdots & 1 \\ \frac{s-2}{2} & s-2 & s-2 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2} & 1 & 1 & \cdots & 1 \end{pmatrix},$$

$${}^t\overline{\mathbf{A}}^{(i)} = (\frac{s+i-1}{2}, s-1, s-2, \cdots, 1).$$

[1)] In eq.(2.21), [5], for the last diagonal entry of B one should apply the rule of the third line rather than the fourth in the present case. We thank B. M. McCoy for communicating this to us.

As an immediate consequence we obtain the following expression for the character of the space V in (2.8):

$$\mathrm{ch}_{q,z}V = \frac{1}{(q)_\infty}\sum_{m_0,m_1,\cdots,m_s\geq 0}\frac{q^{{}^t\mathbf{m}B\mathbf{m}+{}^t\mathbf{A}\mathbf{m}}}{(q)_{m_0}(q)_{m_1}\cdots(q)_{m_{s-1}}}(qz)^{m_0+2m_1+\cdots+2m_s}, \tag{4.10}$$

where the matrix B and the vector $\mathbf{A}$ are the same as those given in (4.8), (4.9).

Let us compare the formula (4.10) with the estimate (4.6). Recall the isomorphism $\tau : V^n \to V^{n+2}$ in (2.15). Since

$$\tau H = (H-2)\tau,$$
$$\tau(L_0 + (p-2)H) = (L_0 - 2)\tau,$$

we have

$$\begin{aligned}\mathrm{ch}_{q,z}V^{-2l} &= \mathrm{tr}_{V^0}\left(\tau^l q^{L_0} z^H \tau^{-l}\right)\\ &= q^{(p-2)l(l+1)}(qz)^{-2l}\times \mathrm{ch}_{q,q^{-(p-2)l}z}W.\end{aligned}$$

Combining this with (4.6) we find

$$\mathrm{ch}_{q,z}V^{-2l} \leq \sum_{m_0,m_1,\cdots,m_s\geq 0}\frac{q^{{}^t\mathbf{m}B\mathbf{m}+{}^t\mathbf{A}\mathbf{m}}}{(q)_{m_0}(q)_{m_1}\cdots(q)_{m_s+l}}(qz)^{m_0+2m_1+\cdots+2m_s}. \tag{4.11}$$

As $l\to\infty$, the right hand side tends to the character (4.10) obtained from the known formulas. Since $V = \varinjlim_N V^{-N}$, we conclude that the equality holds in the intermediate steps (4.6), (4.11). In summary, we have shown that

Proposition 4.4. *The map* (4.3) *is an isomorphism. We have* $\widetilde{I}^{[p]} = I^{[p]}$.

The proof of Theorem 2.6 is now complete.

5. Monomial Basis

5.1. **Spanning set.** The aim of this section is to prove Theorem 2.8. In the present subsection we show first that the monomial vectors (2.23) span the space $W^{[p]}$.

Let $\lambda = (\lambda_1,\cdots,\lambda_n)$ $(\lambda_1\geq\cdots\geq\lambda_n\geq 0)$ be a partition of length at most n. We set $|\lambda| = \lambda_1+\cdots+\lambda_n$. The set of monomial symmetric functions (S_n^λ being the stabilizer of λ)

$$m_\lambda = \sum_{\sigma\in S_n/S_n^\lambda} x_{\sigma(1)}^{\lambda_1}\cdots x_{\sigma(n)}^{\lambda_n}\in\Lambda_n$$

and the set of monomials

$$\xi_\lambda = \xi_{-\lambda_1-1}\cdots\xi_{-\lambda_n-1} \in A_n$$

are the dual bases to each other. We introduce a total ordering among partitions of length at most n by the lexicographic ordering with respect to $(|\lambda|, \lambda_1, \cdots, \lambda_{n-1})$. For a symmetric polynomial $f = \sum_\lambda c_\lambda m_\lambda \in \Lambda_n$, let μ be the largest element in $\{\lambda \mid c_\lambda \neq 0\}$. We call m_μ the leading monomial of f.

Proposition 5.1. *Let $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ be admissible in the sense of (2.24). Then there exists an element $f \in I_3^{[p]}$ whose leading monomial is m_λ.*

Proof. First consider the case $\lambda = (p-2, j, 0)$ $(0 \le j \le p-2)$. Let

$$f_j(x_1, x_2, x_3) = (l_1 - (p-2)P_1)^j \varphi_3(x_1, x_2, x_3). \tag{5.1}$$

Using the recursion relation

$$\Big((z-1)\frac{d}{dz} + \beta\Big)F(\alpha, \beta, \gamma; z) = \frac{(\gamma-\alpha)\beta}{\gamma}F(\alpha, \beta+1, \gamma+1; z),$$

we find

$$\begin{aligned} f_j(x_1, x_2, 0) &= (-1)^j 2^{-j} \prod_{i=0}^{j-1}(p-2-i) \\ &\times x_1^{p-2} x_2^j F\left(1 - \frac{p}{3}, 2-p+j, 2-\frac{2p}{3}+j; \frac{x_2}{x_1}\right). \end{aligned}$$

This shows that the leading monomial of f_j is $m_{(p-2,j,0)}$.

The leading monomial of $(x_1+x_2+x_3)^k(x_1x_2+x_1x_3+x_2x_3)^l(x_1x_2x_3)^m f_j$ is $m_{(p-2+k+l+m, j+l+m, m)}$. The general case is covered by an appropriate choice of k, l, m. □

Proposition 5.2. *For $n = 3$, admissible monomials constitute a basis of $W_3^{[p]}$.*

Proof. For $d \ge 0$, let $\{\lambda^{(1)}, \cdots, \lambda^{(N_d)}\}$ be the set of admissible partitions satisfying $|\lambda| = d$. By the proposition above, there exist polynomials $f^{(1)}, \cdots, f^{(N_d)} \in I_3^{[p]}$ such that $f^{(j)} = \sum_{i=1}^{N_d} c_{ij} m_{\lambda^{(i)}}$, where $(c_{ij})_{1 \le i,j \le N_d}$ is a triangular matrix with nonzero diagonal entries. This implies that the image of the set $X_d = \{\xi_{\lambda^{(i)}}\}_{1 \le i \le N_d}$ under the canonical map $A \to$

$A/\mathfrak{a}$ gives a linearly independent set. On the other hand, we have

$$\begin{aligned}\sum_{d\geq 0} N_d q^d &= \sum_{\substack{0\leq a\leq b\leq c\\ c-a\geq p-2}} q^{a+b+c}\\ &= \frac{q^{p-2}+q^{p-1}-q^{2p-3}}{(1-q)(1-q^2)(1-q^3)}\\ &= \mathrm{ch}_q I_3^{[p]}.\end{aligned}$$

Therefore X_d is a basis of $I_{d,3}^{[p]}$. □

Proposition 5.3. *The set of admissible monomials span* $W^{[p]}$.

Proof. Let $\xi_\lambda = \xi_{-\lambda_1-1}\cdots\xi_{-\lambda_n-1}$ be an arbitrary monomial. We use the same letter to denote its image in $A/\mathfrak{a}$. Suppose there is a successive triple $(i-1,i,i+1)$ of indices for which $(\lambda_{i-1},\lambda_i,\lambda_{i+1})$ is not admissible. By Proposition 5.2, modulo the ideal $\mathfrak{a}_3$ the factor $\xi_{-\lambda_{i-1}-1}\xi_{-\lambda_i-1}\xi_{-\lambda_{i+1}-1}$ can be written as a linear combination of monomials $\xi_{-\mu_{i-1}-1}\xi_{-\mu_i-1}\xi_{-\mu_{i+1}-1}$ such that

$$\begin{aligned}&\mu_{i-1}-\mu_{i+1}\geq p-2>\lambda_{i-1}-\lambda_{i+1},\\ &\mu_{i-1}+\mu_i+\mu_{i+1}=\lambda_{i-1}+\lambda_i+\lambda_{i+1}.\end{aligned}$$

With this replacement, ξ_λ becomes a linear combination of monomials $\xi_{\tilde{\lambda}}$. We claim that for each such term we have $\sum_{i=1}^n i\lambda_i > \sum_{i=1}^n i\tilde{\lambda}_i$. Since for a given d there are only a finite number of partitions with $|\lambda|=d$, this procedure terminates after a finite number of steps. The result is a linear combination of admissible monomials.

There are four cases to consider.

$$\begin{aligned}&1):\mu_{i-1}>\lambda_{i-1},\ \mu_i<\lambda_i,\ \mu_{i+1}>\lambda_{i+1},\\ &2):\mu_{i-1}>\lambda_{i-1},\ \mu_i<\lambda_i,\ \mu_{i+1}\leq\lambda_{i+1},\\ &3):\mu_{i-1}>\lambda_{i-1},\ \mu_i\geq\lambda_i,\ \mu_{i+1}\leq\lambda_{i+1},\\ &4):\mu_{i-1}\leq\lambda_{i-1},\ \mu_i>\lambda_i,\ \mu_{i+1}<\lambda_{i+1}.\end{aligned}$$

Let us consider the case 1). Let k be the number such that $\lambda_{k-1}\geq\mu_{i-1}>\lambda_k$. Then $\tilde{\lambda}$ has the form $\tilde{\lambda}_k=\mu_{i-1}$, $\tilde{\lambda}_j=\lambda_{j-1}$ $(k<j<i)$, $\tilde{\lambda}_j=\mu_j$ $(j=i,i+1)$ and $\tilde{\lambda}_j=\lambda_j$ for the rest. Then

$$\begin{aligned}\sum_{i=1}^n i\lambda_i-\sum_{i=1}^n i\tilde{\lambda}_i &= \sum_{j=k}^{i-2}(\mu_{i-1}-\lambda_j)+\mu_{i-1}-\mu_{i+1}-\lambda_{i-1}+\lambda_{i+1}\\ &> 0.\end{aligned}$$

The other cases can be checked similarly. □

5.2. **Counting admissible monomials.** To conclude the proof of Theorem 2.8, it remains to show that the number of admissible monomials of a given degree (d,n) coincides with $\dim W^{[p]}_{d,n}$. For convenience we rewrite λ_i+1 in the previous subsection as λ_i. Given $N \geq 1$, let us consider $\lambda = (\lambda_1, \cdots, \lambda_n)$ satisfying

$$\begin{aligned} &N \geq \lambda_1 \geq \cdots \geq \lambda_n \geq 1, \\ &\lambda_i - \lambda_{i+2} \geq p-2 \qquad (1 \leq i \leq n-2). \end{aligned}$$

Let $\mathcal{C}^{[p]}_{N,n}$ denote the set of such partitions. For $n=0$, $\mathcal{C}^{[p]}_{N,0}$ is a single-element set consisting of the empty partition. We set $\mathcal{C}^{[p]}_{N,d,n} = \{\lambda \in \mathcal{C}^{[p]}_{N,n} \mid |\lambda| = d\}$. For a set X, we denote its cardinality by $\sharp X$. In this subsection we derive a recursion relation relating $\sharp\mathcal{C}^{[p]}_{N,d,n}$ to $\sharp\mathcal{C}^{[p-3]}_{L,d',l}$.

In the sequel we assume that $p \geq 4$. Following a similar construction in [6], we introduce three kinds of transformations $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$ for elements of $\mathcal{C}^{[p]}_{N,n}$. The first two are given as follows.

$$\begin{aligned} \mathcal{B}_1 \ &: \ \mathcal{C}^{[p-3]}_{L,l} \to \mathcal{C}^{[p]}_{L+2l-2,l}, \\ &\quad (\lambda_1, \cdots, \lambda_{l-1}, \lambda_l) \ \mapsto \ (\lambda_1 + 2(l-1), \cdots, \lambda_{l-1}+2, \lambda_l), \\ \mathcal{B}_2 \ &: \ \mathcal{C}^{[p]}_{N,n} \to \mathcal{C}^{[p]}_{N+p-2,n+2}, \\ &\quad (\nu_1, \cdots, \nu_n) \ \mapsto \ (\nu_1 + p-2, \cdots, \nu_n + p-2, 1, 1). \end{aligned}$$

The third transformation $\mathcal{B}_3$ is defined only on a subset of $\mathcal{C}^{[p]}_{N,n}$. Given an element $\nu = (\nu_1, \cdots, \nu_n) \in \mathcal{C}^{[p]}_{N,n}$, consider the set S_ν of indices i $(1 \leq i \leq n)$ for which one of the following conditions hold.

$$\begin{aligned} &(a_i): \ \ \nu_i = \nu_{i+1}, \\ &(b_i): \ \ \nu_i = \nu_{i+1}+1 \text{ and } \nu_{i-1} - \nu_{i+1} \geq p-1, \\ &(c_i): \ \ \nu_{i-1} > \nu_i > \nu_{i+1} \text{ and } \nu_{i-1} - \nu_{i+1} = p-2. \end{aligned}$$

Here we set $\nu_0 = +\infty$ and $\nu_{l+1} = -\infty$. Then S_ν is empty if and only if ν has the form $\mathcal{B}_1(\lambda)$ for some $\lambda \in \mathcal{C}^{[p-3]}_{L,l}$. For $i \in S_\nu$, set

$$\nu'^{(i)}_j = \begin{cases} \nu_j + \delta_{i+1,j} & \text{in the case } (b_i), \\ \nu_j + \delta_{i,j} & \text{otherwise.} \end{cases}$$

Let $i_0 = \max S_\nu$. We define $\mathcal{B}_3(\nu)$ by the following rule: If $\nu'^{(i_0)} \in \mathcal{C}^{[p]}_{N,n}$, then $\mathcal{B}_3(\nu) = \nu'^{(i_0)}$. If $\nu'^{(i_0)} \notin \mathcal{C}^{[p]}_{N,n}$, $i_0 - 1 \in S_\nu$ and $\nu'^{(i_0-1)} \in \mathcal{C}^{[p]}_{N,n}$, then $\mathcal{B}_3(\nu) = \nu'^{(i_0-1)}$. In all other cases $\mathcal{B}_3(\nu)$ is not defined. We have also

the inverse transformation of $\mathcal{B}_3$. Notation being as above, let

$$\nu''^{(i)}_j = \begin{cases} \nu_j - \delta_{i+1,j} & \text{in the case } (a_i), \\ \nu_j - \delta_{i,j} & \text{otherwise.} \end{cases}$$

If $\nu''^{(i_0)} \in \mathcal{C}^{[p]}_{N,n}$, then we define $\mathcal{B}^*_3(\nu) = \nu''^{(i_0)}$. Otherwise $\mathcal{B}^*_3(\nu)$ is not defined. By case checking one can show

Lemma 5.4. *If $\mathcal{B}_3(\nu)$ is defined, then $\mathcal{B}^*_3\mathcal{B}_3(\nu)$ is defined and equals ν. If $\mathcal{B}^*_3(\nu)$ is defined, then $\mathcal{B}_3\mathcal{B}^*_3(\nu)$ is defined and equals ν.*

The following assertions can be easily verified.

Lemma 5.5. *If $S_\nu \neq \emptyset$ and $n \geq 2$, then there exist unique $k \geq 0$ and $\overline{\nu} \in \mathcal{C}^{[p]}_{N-p+2,n-2}$ such that $\nu = \mathcal{B}^k_3\mathcal{B}_2(\overline{\nu})$.*

In fact, $k = \max\{j \mid \mathcal{B}^{*j}_3(\nu) \text{ is defined}\}$.

Lemma 5.6. *If $\nu = \mathcal{B}_2\mathcal{B}_1(\lambda)$ with $\lambda \in \mathcal{C}^{[p-3]}_{L,l}$, then $\mathcal{B}^k_3(\nu)$ is defined for $0 \leq k \leq 2L - (p-6)(l-2) + 2$.*

Lemma 5.7. *If $\widetilde{\nu} = \mathcal{B}^k_3\mathcal{B}_2(\nu)$ is defined for $\nu \in \mathcal{C}^{[p]}_{N,n}$, then $\mathcal{B}^{\widetilde{k}}_3\mathcal{B}_2(\widetilde{\nu})$ is defined if and only if $0 \leq \widetilde{k} \leq k$.*

In general, consider the composition

$$\nu = (\mathcal{B}^{\mu_m}_3\mathcal{B}_2)\cdots(\mathcal{B}^{\mu_1}_3\mathcal{B}_2)\mathcal{B}_1(\lambda) \tag{5.2}$$

where $\lambda \in \mathcal{C}^{[p-3]}_{L,l}$. It is defined for

$$0 \leq \mu_m \leq \cdots \leq \mu_1 \leq 2L - (p-6)(l-2) + 2$$

and

$$\nu \in \mathcal{C}^{[p]}_{L+2(l-1)+(p-2)m,l+2m}.$$

Let $\mathcal{P}_{M,m}$ denote the set of partitions $\mu = (\mu_1, \cdots, \mu_m)$ satisfying $M \geq \mu_1 \geq \cdots \geq \mu_m \geq 0$. From the above considerations we obtain a map

$$\bigcup_{\substack{l,m\geq 0 \\ l+2m=n}} \mathcal{C}^{[p-3]}_{N-2(l-1)-(p-2)m,l} \times \mathcal{P}_{2(N-1)-(p-2)(n-2),m} \longrightarrow \mathcal{C}^{[p]}_{N,n}, \tag{5.3}$$

$$(\lambda, \mu) \mapsto \nu,$$

where ν is given by (5.2).

Conversely, using Lemma 5.5–5.7, we can put any $\nu \in \mathcal{C}^{[p]}_{N,n}$ into the form (5.2) in a unique manner. Hence we have shown

Proposition 5.8. *The map* (5.3) *is a bijection.*

Let us rewrite this result in terms of the generating function

$$\widetilde{\chi}^{[p]}_{N,n}(q) = \sum_{\nu \in \mathcal{C}^{[p]}_{N,n}} q^{|\nu|}. \tag{5.4}$$

Proposition 5.9.

$$\widetilde{\chi}^{[p]}_{N,n}(q) = \sum_{\substack{l,m\geq 0\\ l+2m=n}} q^{l(l-1)+(p-2)lm+(p-2)m(m-1)+2m} \times \begin{bmatrix} 2(N-1)-(p-2)(n-2)+m \\ m \end{bmatrix} \widetilde{\chi}^{[p-3]}_{N-2(l-1)-(p-2)m,l}(q), \tag{5.5}$$

$$\widetilde{\chi}^{[p]}_{N,n}(q) = q^n \begin{bmatrix} N-1+n \\ n \end{bmatrix} \qquad \text{for } p = 1, 2. \tag{5.6}$$

Proof. In the bijection (5.3), we have the relation

$$|\nu| = |\lambda| + |\mu| + l(l-1) + (p-2)lm + (p-2)m(m-1) + 2m.$$

Eq.(5.5) is a consequence of this. Since

$$\sum_{\mu \in \mathcal{P}_{M,m}} q^{|\mu|} = \begin{bmatrix} M+m \\ m \end{bmatrix},$$

and $\mathcal{C}^{[p]}_{N,n} = \mathcal{P}_{N,n} \backslash \mathcal{P}_{N,n-1}$ for $p = 1, 2$, we have for $p = 1, 2$

$$\begin{aligned} \widetilde{\chi}^{[p]}_{N,n}(q) &= \begin{bmatrix} N+n \\ n \end{bmatrix} - \begin{bmatrix} N-1+n \\ n-1 \end{bmatrix} \\ &= q^n \begin{bmatrix} N-1+n \\ n \end{bmatrix}. \end{aligned}$$

□

5.3. **Proof of Theorem 2.8.** Recall the filtration (4.2). In Proposition 4.4 we have shown the isomorphism

$$\begin{aligned} &F_m/F_{m-1} \\ &\simeq \prod_{1\leq i<j\leq l} (x_i - x_j)^2 \prod_{\substack{1\leq i\leq l\\ 1\leq k\leq m}} (x_i - y_k)^{p-2} \prod_{1\leq k<h\leq m} (y_k - y_h)^{2(p-2)} \\ &\times I_l^{[p-3]} \otimes \Lambda_m, \end{aligned} \tag{5.7}$$

where $l = n - 2m$. Therefore the character $\chi^{[p]}_n(q) = \mathrm{ch}_q W^{[p]}_n$ satisfies the recursion relation

$$\chi^{[p]}_n(q) = \sum_{\substack{l,m\geq 0\\ l+2m=n}} \frac{q^{l(l-1)+(p-2)lm+(p-2)m(m-1)+2m}}{(q)_m} \chi^{[p-3]}_l(q), \tag{5.8}$$

$$\chi^{[p]}_n(q) = \frac{q^n}{(q)_n} \qquad \text{for } p = 1, 2. \tag{5.9}$$

Let us compare this result with the limit $N \to \infty$ of (5.4)

$$\widetilde{\chi}_n^{[p]}(q) = \lim_{N\to\infty} \widetilde{\chi}_{N,n}^{[p]}(q).$$

Proposition 5.10.

$$\chi_n^{[p]}(q) = \widetilde{\chi}_n^{[p]}(q).$$

Proof. Proposition 5.3 shows that $\chi_n^{[p]}(q) \le \widetilde{\chi}_n^{[p]}(q)$. On the other hand, letting $N \to \infty$ in (5.5),(5.6), we find that $\widetilde{\chi}_n^{[p]}(q)$ satisfies the same recursion relations (5.8),(5.9) as $\chi_n^{[p]}(q)$. The assertion is proved. □

This completes the proof of Theorem 2.8.

5.4. **Finitization.** For $N \ge 1$, consider the space

$$W^{N,[p]} = W^{[p]}/\mathbb{C}[a_{-N-1}, a_{-N-2}, \cdots]W^{[p]}.$$

This is a finite dimensional space. Let Λ_n^N denote the subspace of Λ_n consisting of symmetric polynomials in n variables whose degree in each variable is at most N. Then the dual vector space of $W_n^{N,[p]}$ is

$$I_n^{N-1,[p]} = I_n^{[p]} \cap \Lambda_n^{N-1}.$$

In this section we show a 'finitized' version of Proposition 5.2, which states that admissible monomials with $\lambda_1 \le N$ constitute a basis of $W_n^{N,[p]}$ (see Theorem 5.13 below).

First let us give a refinement of Proposition 3.5.

Proposition 5.11. *Let*

$$J^{N-1} = \{g \in \mathbb{C}[x,y] \mid \deg_x g \le N-p+1,\ \deg_y g \le 2N-p\}.$$

Then the sequence

(5.10)

$$0 \longrightarrow D_3 I_3^{N-5,[p-3]} \xrightarrow{\iota} I_3^{N-1,[p]} \xrightarrow{\pi} (x-y)^{p-2} J^{N-1} \longrightarrow 0$$

is exact. Here the maps ι, π are as given in (3.10).

Proof. The only nontrivial part is the surjectivity of π. We prove it by induction on $M = N - p + 1 \ge 0$.

Consider the case $M = 0$. Let $f_j(x_1, x_2, x_3)$ be the polynomial (5.1). Since $(l_1 - (p-2)P_1)\Lambda_3^{p-2} \subset \Lambda_3^{p-2}$, we have $f_j \in I_3^{p-2}$. Using $\pi(\varphi_3) = (x-y)^{p-2}$ we find, for $0 \le j \le p-2$,

$$\begin{aligned} \pi(f_j) &= \left(x^2\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y} - (p-2)(x+2y)\right)^j ((x-y)^{p-2}) \\ &= \frac{(p-2)!}{(p-2-j)!}(x-y)^{p-2}(-y)^j. \end{aligned}$$

Since $J^{p-2} = \mathrm{span}\{y^j \mid 0 \le j \le p-2\}$, π is surjective.

Suppose the statement is true for M. Then there exist polynomials $f_{k,l} \in I_3^{N-1,[p]}$ $(0 \le k \le M, 0 \le l \le 2M+p-2)$ such that $\pi(f_{k,l}) = (x-y)^{p-2}x^k y^l$. Multiplying $f_{k,l}$ with elementary symmetric polynomials, we see that $(x+2y)x^k y^l$, $(2xy+y^2)x^k y^l$ and $xy^2 \cdot x^k y^l$ belong to $\pi(I_3^{N,[p]})$ for $0 \le k \le M$, $0 \le l \le 2M+p-2$. Their linear span contains the basis $\{x^k y^l \mid 0 \le k \le M+1,\ 0 \le l \le 2M+p\}$ of J^N. □

Let

$$\mathcal{A}'^{N,[p]}_n = \{\xi_{-\lambda_1} \cdots \xi_{-\lambda_n} \mid \lambda = (\lambda_1, \cdots, \lambda_n) \in \mathcal{C}^{[p]}_{N,n}\},$$

$$\mathcal{A}''^{N,[p]}_n = \{\xi_{-\lambda_1} \cdots \xi_{-\lambda_n} \mid \lambda_1 > N,\ \lambda_1 \ge \cdots \ge \lambda_n \ge 1, \lambda_i - \lambda_{i+2} \ge p-2\}.$$

By Theorem 2.8, the image of $\mathcal{A}^{[p]}_n = \mathcal{A}'^{N,[p]}_n \cup \mathcal{A}''^{N,[p]}_n$ is a basis of $W^{[p]}_n$. Let $\pi^N_n : A_n \to W^{N,[p]}_n$ denote the canonical projection. Then $W^{N,[p]}_n$ is spanned by $\pi^N_n(\mathcal{A}'^{N,[p]}_n)$.

Proposition 5.12. *The set $\pi^N_3(\mathcal{A}'^{N,[p]}_3)$ gives a basis of $W^{N,[p]}_3$.*

Proof. Set

$$\chi^{N,[p]}_n(q) = \mathrm{ch}_q W^{N,[p]}_n.$$

Proposition 5.11 yields the following recursion relation.

$$\chi^{N,[p]}_3(q) = q^{p+1}\frac{1-q^{N-p+2}}{1-q}\frac{1-q^{2N-p+1}}{1-q} + q^6\chi^{N-4,[p-3]}_3(q),$$

$$\chi^{N,[p]}_3(q) = q^3\begin{bmatrix} N+2 \\ 3 \end{bmatrix} \qquad \text{for } p = 1,2.$$

This is the same recursion relation (5.5),(5.6) satisfied by $\widetilde{\chi}^{[p]}_{N,3}(q)$. Therefore we have $\chi^{N,[p]}_3(q) = \widetilde{\chi}^{[p]}_{N,3}(q)$, and the proposition follows. □

Theorem 5.13. *For all $N \ge 1$ and $n, d \ge 0$ we have*

$$\dim(W^{N,[p]})_{d,n} = \sharp\, \mathcal{C}^{[p]}_{N,d,n}.$$

Proof. The image of the set $\mathcal{A}''^{N,[p]}_n$ in the subspace $(\mathbb{C}[a_{-N-1}, a_{-N-2}, \cdots]W^{[p]})_n$ is linearly independent. The theorem is equivalent to the statement that it is also a spanning set.

Let $\lambda = (\lambda_1, \cdots \lambda_n)$ be a partition with $\lambda_1 > N$, and consider the process given in the proof of Proposition 5.3 of reducing the monomial $\xi_{-\lambda_1} \cdots \xi_{-\lambda_n}$. Each step of the reduction consists in replacing a successive non-admissible triple $\xi_{-\lambda_{i-1}}\xi_{-\lambda_i}\xi_{-\lambda_{i+1}}$ by a linear combination of admissible ones. We claim that the resulting monomials $\xi_{-\mu_1} \cdots \xi_{-\mu_n}$ all belong to the subspace $(\mathbb{C}[a_{-N-1}, a_{-N-2}, \cdots]W^{[p]})_n$. In fact, it is

clear in the case $\lambda_{i-1} \leq N$. In the case $\lambda_{i-1} > N$, it is a consequence of Proposition 5.12. Repeating this process a finite number of times we arrive at a linear combination of admissible monomials in $(\mathbb{C}[a_{-N-1}, a_{-N-2}, \cdots]W^{[p]})_n$. The proof is over. □

Solving the the recursion relation (5.5), (5.6) for $\widetilde{\chi}_{N,n}(q)$ we are led to the following formula.

Theorem 5.14.

$$\mathrm{ch}_{q,z} W^{N,[p]} = \sum_{m_0,m_1,\cdots,m_s \geq 0} q^{{}^t\mathbf{m}B\mathbf{m}+{}^t\mathbf{A}\mathbf{m}} (qz)^{m_0+2\sum_{i=1}^s m_i} \prod_{i=0}^{s} \begin{bmatrix} M_i \\ m_i \end{bmatrix},$$

where $\mathbf{m} = {}^t(m_0, m_1, \cdots, m_s)$, B *and* $\mathbf{A}$ *are as in* (4.7)–(4.9), *and*

$$M_i = \begin{cases} N_0 - 1 & (i = 0), \\ 2(N_i - 1) - (p - 2 - 3(s-i))(l_{i+1} - 2) + m_i & (1 \leq i \leq s), \end{cases}$$

$$N_i = N - \sum_{i<j\leq s} (l_j - 1 + (p - 2 - 3s + 3j)m_j),$$

$$l_i = m_0 + 2 \sum_{1\leq j<i} m_j.$$

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