Backward global solutions characterizing annihilation dynamics of travelling fronts

Hiroki Yagisita
Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan.

Abstract. We consider a reaction-diffusion equation $u_t = u_{xx} + f(u)$, where f has exactly three zeros 0, α and 1 $(0 < \alpha < 1)$, $f_u(0) < 0$, $f_u(1) < 0$ and $\int_0^1 f(u)du \geq 0$. Then, the equation has a travelling wave solution $u(x,t) = \phi(x - ct)$ with $\phi(-\infty) = 0$ and $\phi(+\infty) = 1$. Known results suggest that for an initial state $u_0(x)$ with $\underline{\lim}_{x\to\pm\infty} u_0(x) > \alpha$ having two interfaces at a large distance, $u(x,t)$ approaches a pair of travelling wave solutions $\phi(x - p_1(t)) + \phi(-x + p_2(t))$ for a long time, and then the travelling fronts eventually disappear by colliding with each other. While our results establish this process, they show that *there is a (backward) global solution* $\psi(x,t)$ and that *the annihilation process is approximated by a solution* $\psi(x - x_0, t - t_0)$.

Keywords: bistable reaction-diffusion equation, entire solution, travelling wave, collision, collapse, invariant manifold.

1 Introduction

In this paper, we consider the scalar bistable reaction-diffusion equation

$$\begin{cases} u_t = u_{xx} + f(u), \quad t > 0, \quad x \in \mathbf{R}, \\ u(0) = u_0 \in BU(\mathbf{R}), \end{cases} \tag{1.1}$$

where $BU(\mathbf{R})$ is the space of bounded uniformly continuous functions from $\mathbf{R}$ to $\mathbf{R}$ with the supremum norm, and the reaction term f satisfies the following conditions:

1 $f \in C^2(\mathbf{R})$,

2 f has exactly three zeros 0, α and 1 $(0 < \alpha < 1)$,

3 $f_u(0) < 0$, $f_u(1) < 0$,

4 $\int_0^1 f(u)du \geq 0$.

It is known [e.g. 4, Section 4.4] that the reaction-diffusion equation (1.1) has a unique (except for translation) travelling wave solution $u(x,t) = \phi(x - ct)$, where (ϕ, c) satisfies

$$\phi''(z) + c\phi'(z) + f(\phi(z)) = 0 \tag{1.2}$$

with $\phi(-\infty) = 0$ and $\phi(+\infty) = 1$. Then $c \leq 0$ holds from $\int_0^1 f(u)du \geq 0$. We normalize the definition of ϕ by requiring $\phi(0) = 1/2$.

This solution is linearly stable except for neutral translational perturbations. Specifically, the following is known [e.g. 10, Section 5.4].

Theorem A *(1) The operator* $-(\frac{\partial^2}{\partial z^2} + c\frac{\partial}{\partial z} + f_u(\phi(z))) : BU(\mathbf{R}) \to BU(\mathbf{R})$ *is a sectorial one with a simple eigenvalue* 0. *The remainder of the spectrum has real part greater than some positive constant.*

(2) There exist δ, C *and* $\gamma > 0$ *such that for any* $u_0 \in BU(\mathbf{R})$ *with* $\|u_0(x) - \phi(x)\|_{C^0} \leq \delta$, *there exists* $x_0 \in \mathbf{R}$ *satisfying*

$$\|u(x,t) - \phi(x - x_0 - ct)\|_{C^0} \leq Ce^{-\gamma t}\|u_0(x) - \phi(x)\|_{C^0}$$

for all $t \geq 0$.

Moreover, Fife and McLeod [6] showed the following theorem, which gives a global stability result for the travelling wave solution $\phi(x-ct)$.

Theorem B *If* $\overline{\lim}_{x\to-\infty}u_0(x)<\alpha$ *and* $\underline{\lim}_{x\to+\infty}u_0(x)>\alpha$ *hold, then*

$$\inf_{x_0\in\mathbf{R}}\|u(x,t)-\phi(x-x_0)\|_{C^0}\to 0 \quad as \quad t\to+\infty$$

holds.

Also, Fife and McLeod [6] showed the following, which means that the pair of the travelling wave solutions going to $x=\pm\infty$ has strong attractivity.

Theorem C *Suppose that* $c<0$, $\overline{\lim}_{x\to\pm\infty}u_0(x)<\alpha$, $u_0(x)\geq\eta$ $(|x|<L)$ *for some* $\eta>\alpha$ *and* $u_0(x)\geq\zeta$ $(|x|<\infty)$ *for some* $\zeta>-\infty$ *hold. If* L *is large enough depending on* η *and* ζ, *then* $u(x,t)$ *approaches (uniformly in* x *and exponentially in* t*) a pair of diverging travelling wave solutions*

$$\phi(x-x_1-ct)+\phi(-x-x_2-ct)-1.$$

On the other hand, when $\underline{\lim}_{x\to\pm\infty}u_0(x)>\alpha$ holds, the following is known [e.g. 5].

Proposition D *If* $\underline{\lim}_{x\to\pm\infty}u_0(x)>\alpha$ *holds, then* $\lim_{t\to+\infty}\|u(x,t)-1\|_{C^0}=0$ *holds.*

For an initial state $u_0(x)$ with $\underline{\lim}_{x\to\pm\infty}u_0(x)>\alpha$ having two interfaces at a large distance, Theorems A, B and C suggest that $u(x,t)$ approaches a pair of travelling wave solutions

$$\phi(x-p_1(t))+\phi(-x+p_2(t))$$

for a long time. Then, Proposition D suggests that the travelling fronts eventually disappear by colliding with each other. While our main results (Theorem 1.1 and Corollary 1.4) establish this process, they show that *there is a (backward) global solution* $\psi(x,t)$ and that *the annihilation process is approximated by a solution* $\psi(x-x_0,t-t_0)$.[1]

Theorem 1.1 *There exists a solution* $\psi\in C(\mathbf{R},BU(\mathbf{R}))$ *of* $u_t=u_{xx}+f(u)$ *satisfying* $\lim_{t\to+\infty}\|\psi(t)-1\|_{C^0(\mathbf{R})}=0$, $\psi(-x,t)=\psi(x,t)$ *and the following.*

[1] For mathematical studies on motion and collapse of fronts in (1.1) from other aspects, we can refer to [e.g. 1, 2, 3, 7, 8, 9, 11, 12].

(1) There exists $p \in C^1(\mathbf{R})$ *such that*

$$p(-\infty) = +\infty, \ \dot{p}(-\infty) = c$$

and

$$\lim_{t \to -\infty} \|\psi(x,t) - (\phi(x - p(t)) + \phi(-x - p(t)))\|_{C^0(\mathbf{R})} = 0$$

hold.

(2) There exist $\delta > 0$, $C > 0$ *and* $\gamma > 0$ *such that for any* $t_0 \in \mathbf{R}$ *and* $u_0 \in BU(\mathbf{R})$ *satisfying* $\|u_0 - \psi(t_0)\|_{C^0(\mathbf{R})} \leq \delta$, *there exist* x_0, $t_0' \in \mathbf{R}$ *and a solution* $u \in C([0, +\infty), BU(\mathbf{R}))$ *of* $u_t = u_{xx} + f(u)$ *with* $u(0) = u_0$ *such that*

$$\|u(x,t) - \psi(x - x_0, t - t_0')\|_{C^0(\mathbf{R})} \leq Ce^{-\gamma t}\|u_0(x) - \psi(x, t_0)\|_{C^0(\mathbf{R})}$$

holds for all $t \geq 0$.

Theorem 1.1 leads to the following.

Corollary 1.2 *For any* $T \in [-\infty, +\infty)$ *and solution* $\bar{\psi} \in C((T, +\infty), BU(\mathbf{R}))$ *of* $u_t = u_{xx} + f(u)$, *if there exist* $\{p_n\}_{n=1}^{\infty}$, $\{q_n\}_{n=1}^{\infty} \subset \mathbf{R}$ *and* $\{T_n\}_{n=1}^{\infty} \subset (T, +\infty)$ *such that*

$$\lim_{n \to \infty} (p_n - q_n) = +\infty$$

and

$$\lim_{n \to \infty} \|\bar{\psi}(x, T_n) - (\phi(x - p_n) + \phi(-x + q_n))\|_{C^0(\mathbf{R})} = 0 \tag{1.3}$$

hold, then $T = -\infty$ *holds and there exist* x_0 *and* $t_0 \in \mathbf{R}$ *satisfying*

$$\psi(x,t) = \bar{\psi}(x + x_0, t + t_0).$$

Proof. By Theorem 1.1 (1), there exists $\{t_n'\}_{n=1}^{\infty} \subset \mathbf{R}$ with $\lim_{n \to \infty} t_n' = -\infty$ such that

$$\lim_{n \to \infty} \|\psi(x, t_n') - (\phi(x - \frac{p_n - q_n}{2}) + \phi(-x - \frac{p_n - q_n}{2}))\|_{C^0(\mathbf{R})} = 0$$

holds. Hence, from (1.3),

$$\lim_{n \to \infty} \|\bar{\psi}(x + \frac{p_n + q_n}{2}, T_n) - \psi(x, t_n')\|_{C^0(\mathbf{R})} = 0$$

holds. By Theorem 1.1 (2), if $n \in \{1,2,\cdots\}$ is sufficiently large, then there exist x_n and $t_n \in \mathbf{R}$ such that

$$\|\bar{\psi}(x, t+T_n) - \psi(x-x_n, t+T_n-t_n)\|_{C^0(\mathbf{R})}$$

$$\leq Ce^{-\gamma t}\|\bar{\psi}(x+\frac{p_n+q_n}{2}, T_n) - \psi(x, t_n')\|_{C^0(\mathbf{R})}$$

holds for all $t \geq 0$. Therefore, we obtain

$$\lim_{n\to\infty} \sup_{t \geq T_n - t_n} \|\bar{\psi}(x+x_n, t+t_n) - \psi(x,t)\|_{C^0(\mathbf{R})} = 0. \tag{1.4}$$

Hence, from (1.3),

$$\lim_{n\to\infty} \|\psi(x, T_n - t_n) - (\phi(x-(p_n-x_n)) + \phi(-x+(q_n-x_n)))\|_{C^0(\mathbf{R})} = 0$$

holds. Because $\lim_{n\to\infty}((p_n-x_n)-(q_n-x_n)) = +\infty$ also holds, by Theorem 1.1 (1), we obtain $\lim_{n\to\infty}(T_n - t_n) = -\infty$.

Now, we show that there exists $\bar{t}_0 \in \mathbf{R}$ such that $\lim_{n\to\infty} t_n = \bar{t}_0$ holds. Assume that there exist $\{N_n\}_{n=1}^{\infty}$ and $\{M_n\}_{n=1}^{\infty} \subset \{1,2,\cdots\}$ such that $\lim_{n\to\infty} N_n = \lim_{n\to\infty} M_n = \infty$ and $\inf_{n=1,2,\cdots}(t_{N_n} - t_{M_n}) > 0$ hold. Then, by (1.4),

$$\lim_{n\to\infty} \|\psi(x,t) - \psi(x + x_{N_n} - x_{M_n}, t + t_{N_n} - t_{M_n})\|_{C^0(\mathbf{R})} = 0$$

holds for all $t \in \mathbf{R}$. This is contradiction with $\inf_{n=1,2,\cdots}(t_{N_n} - t_{M_n}) > 0$. Hence, $\lim_{n\to\infty} t_n = \bar{t}_0 \in \mathbf{R}$ holds.

Because $\lim_{n\to\infty}(T_n - t_n) = -\infty$ and $\lim_{n\to\infty} t_n = \bar{t}_0 \in \mathbf{R}$ hold, we obtain $T = \lim_{n\to\infty} T_n = -\infty$. Also, by (1.4),

$$\lim_{(n,m)\to(\infty,\infty)} \|\psi(x, t-\bar{t}_0) - \psi(x + x_n - x_m, t - \bar{t}_0)\|_{C^0(\mathbf{R})} = 0$$

holds for all $t \in \mathbf{R}$. Hence, we have $\lim_{(n,m)\to(\infty,\infty)} |x_n - x_m| = 0$. There exists $\bar{x}_0 \in \mathbf{R}$ such that $\lim_{n\to\infty} x_n = \bar{x}_0$ holds. Therefore, by (1.4), we obtain $\bar{\psi}(x+\bar{x}_0, t+\bar{t}_0) = \psi(x,t)$. q.e.d.

Definition 1 *For $l > 0$, $\delta \in (0, \min\{\alpha, 1-\alpha\})$ and $L > 0$, a closed subset $\Xi_{l,\delta,L}$ of $BU(\mathbf{R})$ is defined by*

$$\Xi_{l,\delta,L} = \{u \in BU(\mathbf{R}) | \, 0 \leq u(x) \leq \alpha - \delta \ (|x| < l - L),$$

$$0 \le u(x) \le 1 \ (l - L \le |x| \le l + L), \ \alpha + \delta \le u(x) \le 1 \ (l + L < |x|)\}.$$

For $\bar{l} > 0$, $\bar{\delta} \in (0, \min\{\alpha, 1-\alpha\})$ *and* $\bar{L} > 0$, *a closed subset* $\Pi_{\bar{l},\bar{\delta},\bar{L}}$ *of* $BU(\mathbf{R})$ *is defined by*

$$\Pi_{\bar{l},\bar{\delta},\bar{L}} = \bigcup_{l \ge \bar{l}} \Xi_{l,\bar{\delta},\bar{L}}.$$

Proposition 1.3 *For any* $\bar{\delta}_0 \in (0, \min\{\alpha, 1-\alpha\})$, $\bar{L}_0 > 0$ *and* $\varepsilon > 0$, *there exist* $\bar{l}_0 > 0$, $L > 0$ *and* $T > 0$ *such that for any* $l \ge \bar{l}_0$ *and* $u_0 \in \Xi_{l,\bar{\delta}_0,\bar{L}_0}$, *there exist* x_1, $x_2 \in [l - L, l + L]$ *and a solution* $u \in C([0, +\infty), BU(\mathbf{R}))$ *of* $u_t = u_{xx} + f(u)$ *with* $u(0) = u_0$ *such that*

$$\|u(x,T) - (\phi(x - x_1 - cT) + \phi(-x - x_2 - cT))\|_{C^0(\mathbf{R})} < \varepsilon$$

holds.

Theorem 1.1 and Proposition 1.3 lead to the following.

Corollary 1.4 *For any* $\bar{\delta}_0 \in (0, \min\{\alpha, 1-\alpha\})$, $\bar{L}_0 > 0$, $T_0 \in \mathbf{R}$ *and* $\varepsilon > 0$, *there exists* $\bar{l}_0 > 0$ *such that for any* $u_0 \in \Pi_{\bar{l}_0,\bar{\delta}_0,\bar{L}_0}$, *there exist* $x_0 \in \mathbf{R}$, $t_0 \ge -T_0$ *and a solution* $u \in C([0, +\infty), BU(\mathbf{R}))$ *of* $u_t = u_{xx} + f(u)$ *with* $u(0) = u_0$ *such that*

$$\sup_{t \ge T_0} \|u(x + x_0, t + t_0) - \psi(x,t)\|_{C^0(\mathbf{R})} < \varepsilon$$

holds.

Proof. We first show that there exist $M > 0$ and $\varepsilon' \in (0, \varepsilon)$ such that for any p, q and $t \in \mathbf{R}$, if

$$p + q \ge M$$

and

$$\|\psi(x,t) - (\phi(x-p) + \phi(-x-q))\|_{C^0(\mathbf{R})} < \left(1 + \frac{1}{2C}\right)\varepsilon' \tag{1.5}$$

hold, then $t \le T_0$ holds. Assume that there exist $\{p_n\}_{n=1}^{\infty}$, $\{q_n\}_{n=1}^{\infty} \subset \mathbf{R}$ and $\{t_n\}_{n=1}^{\infty} \subset (T_0, +\infty)$ such that

$$\lim_{n \to \infty} (p_n + q_n) = +\infty$$

and

$$\lim_{n\to\infty} \|\psi(x,t_n) - (\phi(x-p_n) + \phi(-x-q_n))\|_{C^0(\mathbf{R})} = 0$$

hold. Then, from Corollary 1.2, $T_0 = -\infty$ holds. This is contradiction for $T_0 \in \mathbf{R}$.

By Proposition 1.3, there exist L, T and $\bar{l}_0' > 0$ such that for any $l \geq \bar{l}_0'$ and $u_0 \in \Xi_{l,\bar{\delta}_0,\bar{L}_0}$, there exist x_1 and $x_2 \geq l - (L - cT)$ such that

$$\|u(x,T) - (\phi(x-x_1) + \phi(-x-x_2))\|_{C^0(\mathbf{R})} < \min\left\{\frac{\varepsilon'}{2C}, \frac{\delta}{2}\right\} \tag{1.6}$$

holds. Then, let $\bar{l}_0 > 0$ be sufficiently large. Because $\frac{x_1+x_2}{2} > 0$ is sufficiently large, by Theorem 1.1 (1), there exists $t_0' \in \mathbf{R}$ such that

$$\|\psi(x,t_0') - (\phi(x - \frac{x_1+x_2}{2}) + \phi(-x - \frac{x_1+x_2}{2}))\|_{C^0(\mathbf{R})} < \min\left\{\frac{\varepsilon'}{2C}, \frac{\delta}{2}\right\}$$

holds. Therefore, we have

$$\|u(x + \frac{x_1-x_2}{2}, T) - \psi(x,t_0')\|_{C^0(\mathbf{R})} < \min\{\varepsilon'/C, \delta\}.$$

Hence, by Theorem 1.1 (2), there exist x_0 and $t_0 \in \mathbf{R}$ such that

$$\sup_{t\geq T} \|u(x,t) - \psi(x-x_0, t-t_0)\|_{C^0(\mathbf{R})} < \varepsilon' \tag{1.7}$$

holds. Hence, from (1.6), we have

$$\|\psi(x,T-t_0) - (\phi(x-(x_1-x_0)) + \phi(-x-(x_2+x_0)))\|_{C^0(\mathbf{R})} < \left(1 + \frac{1}{2C}\right)\varepsilon'.$$

Because $(x_1-x_0)+(x_2+x_0)$ is sufficiently large and (1.5) holds, $T - t_0 \leq T_0$ holds. Hence, from (1.7), $\sup_{t\geq T_0} \|u(x+x_0, t+t_0) - \psi(x,t)\|_{C^0(\mathbf{R})} < \varepsilon$ holds. q.e.d.

This paper is organized as follows. In Section 2, we define an extended system $(\mathrm{ES})_{l_0,r_0}$ of $u_t = u_{xx} + f(u)$, and state that there exists an invariant manifold S for $(\mathrm{ES})_{l_0,r_0}$ with asymptotic stability (Theorem 2.3). Then, by using Theorem 2.3, we prove Theorem 1.1 (1) and (2) in Sections 3 and 4, respectively. The proof of Theorem 2.3 is given in Section 5. Proposision 1.3 is proved in Section 6.

2 An extended system and an invariant manifold

In this section, we define a semilinear parabolic system $(\mathrm{ES})_{l,r}$ for l and $r > 0$ (Definition 4), show that the system $(\mathrm{ES})_{l,r}$ is an extension of the semilinear parabolic equation $u_t = u_{xx} + f(u)$ in a sence (Proposition 2.2) and state that there exists an invariant manifold S with asymptotic stability in $(\mathrm{ES})_{l_0,r_0}$ for some l_0 and $r_0 > 0$ (Theorem 2.3).

In virtue of (1.2) and Theorem A (1), the following definition is allowed.

Definition 2 *A Banach space X and a sectorial operator A in X with $Re\ \sigma(A) > 0$ are defined by*

$$X = \{u \in BU(\mathbf{R}) |\, \langle u(y), e^{cy}\phi'(y)\rangle := \int_{-\infty}^{+\infty} u(y)e^{cy}\phi'(y)dy = 0\}^2,$$

$$\|(g,h)\|_X = \|(g,h)\|_{C^0} := \|g\|_{C^0} + \|h\|_{C^0}$$

and

$$D(A) = X \cap \{u \in C^2(\mathbf{R}) |\, u, u_y, u_{yy} \in BU(\mathbf{R})\}^2,$$

$$A(g,h)$$

$$= -(g_{yy}(y) + cg_y(y) + f_u(\phi(y))g(y), h_{yy}(y) + ch_y(y) + f_u(\phi(y))h(y)),$$

respectively. A Banach space X' and an open subset U of X' are defined by

$$X' = D(A^{3/4}),\ \|(g,h)\|_{X'} = \|A^{3/4}(g,h)\|_X$$

and

$$U = \{(g,h) \in X' |$$

$$\langle \phi'(y) + g_y(y), e^{cy}\phi'(y)\rangle \neq 0,\ \langle \phi'(y) + h_y(y), e^{cy}\phi'(y)\rangle \neq 0\},$$

respectively.

A Banach space Y and a sectorial operator B in Y with $Re\ \sigma(B) > 0$ are defined by

$$Y = BU(\mathbf{R}),\ \|u\|_Y = \|u\|_{C^0}$$

and

$$D(B) = \{u \in C^2(\mathbf{R}) |\, u, u_x, u_{xx} \in BU(\mathbf{R})\},$$

$$Bu = -(u_{xx} + f_u(1)u),$$

respectively. Banach spaces Y' *and* Y'' *are defined by*

$$Y' = D(B^{3/4}),\ \|u\|_{Y'} = \|B^{3/4}u\|_Y$$

and

$$Y'' = D(B^{7/8}),\ \|u\|_{Y''} = \|B^{7/8}u\|_Y,$$

respectively.

Definition 3 *For* $l > 0$*, a map* $f_l \in C^1(X' \times \mathbf{R}^2, Y)$ *is defined by*

$$f_l[g,h,p,q](y) = f(\phi(y) + g(y)) - (f(\phi(y)) + f_u(\phi(y))g(y))$$
$$-\frac{2}{l}\frac{1}{1+e^{(y+p)/l}}(\phi'(y) + g_y(y)) + \frac{1}{l^2}\frac{1}{1+e^{(y+p)/l}}(\phi(y) + g(y))$$
$$+\frac{1}{1+e^{-(y+p)/l}}f(\phi(y) + \phi(-y-p-q) + g(y) + h(-y-p-q))$$
$$-f(\frac{1}{1+e^{-(y+p)/l}}(\phi(y) + \phi(-y-p-q) + g(y) + h(-y-p-q))).$$

For $l > 0$*, maps* $F_l^1, F_l^2 \in C^1(U \times \mathbf{R}^2, Y)$ *and* $F_l \in C^1(U \times \mathbf{R}^2, X)$ *are defined by*

$$F_l^1[g,h,p,q](y)$$
$$= f_l[g,h,p,q](y) - \frac{\langle f_l[g,h,p,q](y), e^{cy}\phi'(y)\rangle}{\langle \phi'(y) + g_y(y), e^{cy}\phi'(y)\rangle}(\phi'(y) + g_y(y)),$$
$$F_l^2[g,h,p,q] = F_l^1[h,g,q,p]$$

and

$$F_l = (F_l^1, F_l^2),$$

respectively.

For $l > 0$*, maps* $G_l^1, G_l^2 \in C^1(U \times \mathbf{R}^2)$ *and* $G_l \in C^1(U \times \mathbf{R}^2, \mathbf{R}^2)$ *are defined by*

$$G_l^1[g,h,p,q] = -\frac{\langle f_l[g,h,p,q](y), e^{cy}\phi'(y)\rangle}{\langle \phi'(y) + g_y(y), e^{cy}\phi'(y)\rangle},$$
$$G_l^2[g,h,p,q] = G_l^1[h,g,q,p]$$

and

$$G_l = (G_l^1, G_l^2),$$

respectively.

The following lemma is easily seen.

Lemma 2.1 *There exists a cutoff function* $\chi(x) \in C^\infty(\mathbf{R})$ *satisfying* $\chi(x) = 0$ $(x < 1/2)$, $\chi(x) = 1$ $(x > 1)$ *and* $0 \le \chi'(x) \le 4$.

The following defines some semilinear parabolic systems.

Definition 4 *For* $l > 0$ *and* $r > 0$, *semilinear parabolic systems* $(\mathrm{ES})_l$, $(\mathrm{ES})_{l,r}$ *in* $U \times \mathbf{R}^2$ *and* (RD) *in* Y *are difined by*

$$(\mathrm{ES})_l \quad \begin{cases} \frac{d}{dt}(g,h) + A(g,h) = F_l[g,h,p,q], \\ \frac{d}{dt}(p,q) = (c,c) + G_l[g,h,p,q], \\ \qquad\qquad (g,h,p,q) \in U \times \mathbf{R}^2, \end{cases}$$

$$(\mathrm{ES})_{l,r} \quad \begin{cases} \frac{d}{dt}(g,h) + A(g,h) = \chi(\frac{p}{lr})\chi(\frac{q}{lr})F_l[g,h,p,q], \\ \frac{d}{dt}(p,q) = (c,c) + \chi(\frac{p}{lr})\chi(\frac{q}{lr})G_l[g,h,p,q], \\ \qquad\qquad (g,h,p,q) \in U \times \mathbf{R}^2 \end{cases}$$

and

$$(\mathrm{RD}) \quad \frac{d}{dt}u + Bu = f(u) - f_u(1)u, \ u \in Y,$$

respectively.

Definition 5 *For* $l > 0$, *a closed subset* Ω_l *of* $U \times \mathbf{R}^2$ *is definede by*

$$\Omega_l = \{(g,h,p,q) \in U \times \mathbf{R}^2 |$$

$$\frac{1}{1+e^{x/l}}(\phi(x-p) + g(x-p)) = \frac{1}{1+e^{-x/l}}(\phi(-x-q) + h(-x-q))\}.$$

For p *and* $q \in \mathbf{R}$, *a map* $\Theta_{p,q} \in C^1(U, Y')$ *is defined by*

$$\Theta_{p,q}[g,h](x) = \phi(x-p) + \phi(-x-q) + g(x-p) + h(-x-q).$$

The following states that the system $(\mathrm{ES})_l$ is an extension of the equation (RD) in a sence.

Proposition 2.2 *Suppose that* $l > 0$, $T > 0$, $(g,h,p,q) \in C([0,T), U \times \mathbf{R}^2)$ *is a solution of* $(\mathrm{ES})_l$ *and* $(g,h,p,q)(0) \in \Omega_l$. *Then,* $(g,h,p,q) \in C([0,T), \Omega_l)$ *holds and* $\Theta_{p,q}[g,h] \in C([0,T), Y')$ *is a solution of* (RD).

Proof. Let $v \in C([0,T),Y)$ be defined by

$$\begin{aligned} & v(x,t) \\ &= \frac{1}{1+e^{x/l}}(\phi(x-p(t))+g(x-p(t),t)) \\ &-\frac{1}{1+e^{-x/l}}(\phi(-x-q(t))+h(-x-q(t),t)). \end{aligned}$$

Then,

$$\frac{1}{1+e^{-x/l}}\Theta(x,t)+v(x,t)=\phi(x-p(t))+g(x-p(t),t) \tag{2.1}$$

and

$$\frac{1}{1+e^{x/l}}\Theta(x,t)-v(x,t)=\phi(-x-q(t))+h(-x-q(t),t) \tag{2.2}$$

hold with $\Theta(x,t) := \Theta_{p,q}[g,h](x,t)$. We have

$$\begin{aligned} &\frac{1}{1+e^{-x/l}}\Theta_x(x,t)+\frac{1}{l}\frac{1}{1+e^{-x/l}}\frac{1}{1+e^{x/l}}\Theta(x,t)+v_x(x,t) \\ &= \phi'(x-p(t))+g_y(x-p(t),t), \end{aligned} \tag{2.3}$$

$$\begin{aligned} &\frac{1}{1+e^{x/l}}\Theta_x(x,t)-\frac{1}{l}\frac{1}{1+e^{-x/l}}\frac{1}{1+e^{x/l}}\Theta(x,t)-v_x(x,t) \\ &= -(\phi'(-x-q(t))+h_y(-x-q(t),t)), \end{aligned} \tag{2.4}$$

$$\begin{aligned} &\frac{1}{1+e^{-x/l}}\Theta_{xx}(x,t)+\frac{2}{l}\frac{1}{1+e^{-x/l}}\frac{1}{1+e^{x/l}}\Theta_x(x,t) \\ &-\frac{1}{l^2}\frac{1}{1+e^{-x/l}}\frac{1}{1+e^{x/l}}\left(\frac{1}{1+e^{-x/l}}-\frac{1}{1+e^{x/l}}\right)\Theta(x,t)+v_{xx}(x,t) \\ &= \phi''(x-p(t))+g_{yy}(x-p(t),t) \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} &\frac{1}{1+e^{x/l}}\Theta_{xx}(x,t)-\frac{2}{l}\frac{1}{1+e^{-x/l}}\frac{1}{1+e^{x/l}}\Theta_x(x,t) \\ &+\frac{1}{l^2}\frac{1}{1+e^{-x/l}}\frac{1}{1+e^{x/l}}\left(\frac{1}{1+e^{-x/l}}-\frac{1}{1+e^{x/l}}\right)\Theta(x,t)-v_{xx}(x,t) \\ &= \phi''(-x-q(t))+h_{yy}(-x-q(t),t). \end{aligned} \tag{2.6}$$

From (1.2) and (2.1), we get

$$\frac{1}{1+e^{-x/l}}\Theta_t(x,t)+v_t(x,t)$$

$$=-c\phi'(x-p(t))+g_{yy}(x-p(t),t)+f_u(\phi(x-p(t)))g(x-p(t),t)$$

$$+f_l[g,h,p,q](x-p(t))$$

$$=\phi''(x-p(t))+g_{yy}(x-p(t),t)$$

$$+f(\phi(x-p(t)))+f_u(\phi(x-p(t)))g(x-p(t),t)+f_l[g,h,p,q](x-p(t))$$

$$=\phi''(x-p(t))+g_{yy}(x-p(t),t)$$

$$-\frac{2}{l}\frac{1}{1+e^{x/l}}(\phi'(x-p(t))+g_y(x-p(t),t))$$

$$+\frac{1}{l^2}\frac{1}{1+e^{x/l}}(\phi(x-p(t))+g(x-p(t),t))$$

$$+f(\phi(x-p(t))+g(x-p(t),t))$$

$$+\frac{1}{1+e^{-x/l}}f(\phi(x-p(t))+\phi(-x-q(t))+g(x-p(t),t)+h(-x-q(t),t))$$

$$-f(\frac{1}{1+e^{-x/l}}(\phi(x-p(t))+\phi(-x-q(t))+g(x-p(t),t)+h(-x-q(t),t))).$$

Hence, from (2.1), (2.2), (2.3) and (2.5), we obtain

$$\frac{1}{1+e^{-x/l}}\Theta_t(x,t)+v_t(x,t) \tag{2.7}$$

$$=\frac{1}{1+e^{-x/l}}\Theta_{xx}(x,t)+v_{xx}(x,t)-\frac{2}{l}\frac{1}{1+e^{x/l}}v_x(x,t)+\frac{1}{l^2}\frac{1}{1+e^{x/l}}v(x,t)$$

$$+f(\frac{1}{1+e^{-x/l}}\Theta(x,t)+v(x,t))-f(\frac{1}{1+e^{-x/l}}\Theta(x,t))+\frac{1}{1+e^{-x/l}}f(\Theta(x,t)).$$

From (1.2) and (2.2), we get

$$\frac{1}{1+e^{x/l}}\Theta_t(x,t)-v_t(x,t)$$

$$=-c\phi'(-x-q(t))+h_{yy}(-x-q(t),t)+f_u(\phi(-x-q(t)))h(-x-q(t),t)$$

$$+f_l[h,g,q,p](-x-q(t))$$

$$= \phi''(-x-q(t)) + h_{yy}(-x-q(t),t)$$
$$+f(\phi(-x-q(t))) + f_u(\phi(-x-q(t)))h(-x-q(t),t) + f_l[h,g,q,p](-x-q(t))$$
$$= \phi''(-x-q(t)) + h_{yy}(-x-q(t),t)$$
$$-\frac{2}{l}\frac{1}{1+e^{-x/l}}(\phi'(-x-q(t)) + h_y(-x-q(t),t))$$
$$+\frac{1}{l^2}\frac{1}{1+e^{-x/l}}(\phi(-x-q(t)) + h(-x-q(t),t))$$
$$+f(\phi(-x-q(t)) + h(-x-q(t),t))$$
$$+\frac{1}{1+e^{x/l}}f(\phi(x-p(t)) + \phi(-x-q(t)) + g(x-p(t),t) + h(-x-q(t),t))$$
$$-f(\frac{1}{1+e^{x/l}}(\phi(x-p(t)) + \phi(-x-q(t)) + g(x-p(t),t) + h(-x-q(t),t))).$$

Hence, from (2.1), (2.2), (2.4) and (2.6), we obtain

$$\frac{1}{1+e^{x/l}}\Theta_t(x,t) - v_t(x,t) \tag{2.8}$$
$$= \frac{1}{1+e^{x/l}}\Theta_{xx}(x,t) - v_{xx}(x,t) - \frac{2}{l}\frac{1}{1+e^{-x/l}}v_x(x,t) - \frac{1}{l^2}\frac{1}{1+e^{-x/l}}v(x,t)$$
$$+f(\frac{1}{1+e^{x/l}}\Theta(x,t) - v(x,t)) - f(\frac{1}{1+e^{x/l}}\Theta(x,t)) + \frac{1}{1+e^{x/l}}f(\Theta(x,t)).$$

From (2.7) and (2.8), on $t \in (0,T)$,

$$\Theta_t$$
$$= \Theta_{xx} + f(\Theta) - \frac{2}{l}v_x - \frac{1}{l^2}\left(\frac{1}{1+e^{-x/l}} - \frac{1}{1+e^{x/l}}\right)v$$
$$+f(\frac{1}{1+e^{-x/l}}\Theta + v) - f(\frac{1}{1+e^{-x/l}}\Theta) + f(\frac{1}{1+e^{x/l}}\Theta - v) - f(\frac{1}{1+e^{x/l}}\Theta)$$

and

$$v_t$$
$$= v_{xx} + \frac{2}{l}\left(\frac{1}{1+e^{-x/l}} - \frac{1}{1+e^{x/l}}\right)v_x + \frac{1}{l^2}\left(\frac{1}{(1+e^{-x/l})^2} + \frac{1}{(1+e^{x/l})^2}\right)v$$
$$+\frac{1}{1+e^{x/l}}\left(f(\frac{1}{1+e^{-x/l}}\Theta(x,t) + v(x,t)) - f(\frac{1}{1+e^{-x/l}}\Theta(x,t))\right)$$

$$-\frac{1}{1+e^{-x/l}}\left(f(\frac{1}{1+e^{x/l}}\Theta(x,t)-v(x,t))-f(\frac{1}{1+e^{x/l}}\Theta(x,t))\right)$$

hold. Therefore, from $v(x,0)=0$, we obtain

$$v(x,t)=0$$

on $t\in[0,T)$. Hence,

$$\Theta_t=\Theta_{xx}+f(\Theta)$$

holds on $t\in(0,T)$. q.e.d.

We now state the main technical result of the paper. The proof of this result is given in Section 5.

Theorem 2.3 *There exist $D_0>0$, $L_0>0$, $l_0>0$ and $r_0>0$ satisfying the following.*

There uniquely exists an invariant manifold

$$S=\left\{(g,h,p,q)\in U\times\mathbf{R}^2|\ (g,h)=\Sigma(p,q)\ \left(=(\Sigma^1(p,q),\Sigma^2(p,q))\right)\right\}$$

for $(\mathrm{ES})_{l_0,r_0}$ *with* $\sup_{(p,q)\in\mathbf{R}^2}\|\Sigma(p,q)\|_{X'}\le D_0$. *Further,*

$$\Sigma^2(p,q)=\Sigma^1(q,p),$$

$$\|\Sigma(p_1,q_1)-\Sigma(p_2,q_2)\|_{X'}\le\|(p_1,q_1)-(p_2,q_2)\|_{\mathbf{R}^2},$$

$$\|\tilde{G}(p_1,q_1)-\tilde{G}(p_2,q_2)\|_X\le L_0,$$

and

$$\lim_{(p,q)\to(+\infty,+\infty)}(\|\Sigma(p,q)\|_{X'}+\|\tilde{F}(p,q)\|_X+\|\tilde{G}(p,q)\|_{\mathbf{R}^2})=0$$

hold, where $\tilde{F}\in C(\mathbf{R}^2,X)$ *and* $\tilde{G}\in C(\mathbf{R}^2,\mathbf{R}^2)$ *be defined by* $\tilde{F}(p,q)=\chi(\frac{p}{l_0r_0})\chi(\frac{q}{l_0r_0})F_{l_0}[\Sigma(p,q),p,q]$ *and* $\tilde{G}(p,q)=\chi(\frac{p}{l_0r_0})\chi(\frac{q}{l_0r_0})G_{l_0}[\Sigma(p,q),p,q]$, *respectively. Also, there exist $C_0>0$ and $\gamma_0>0$ such that for any $(g_0,h_0,p_0,q_0)\in X'\times\mathbf{R}^2$ satisfying $\|(g_0,h_0)\|_{X'}\le 2D_0$, there exist solutions $(g,h,p,q)\in C([0,+\infty),U\times\mathbf{R}^2)$ and $(\bar{g},\bar{h},\bar{p},\bar{q})\in C([0,+\infty),S)$ of $(\mathrm{ES})_{l_0,r_0}$ such that*

$$(g,h,p,q)(0)=(g_0,h_0,p_0,q_0)$$

and for any $t\ge 0$,

$$\|(g,h,p,q)(t)-(\bar{g},\bar{h},\bar{p},\bar{q})(t)\|_{X'\times\mathbf{R}^2}\le C_0e^{-\gamma_0t}\|(g_0,h_0)-\Sigma(p_0,q_0)\|_{X'}$$

hold.

Definition 6 *A local invariant manifold* $\tilde{S}$ *for* $(\mathrm{ES})_{l_0}$ *is defined by*

$$\tilde{S} = \{(g, h, p, q) \in S |\ p > l_0 r_0,\, q > l_0 r_0\}.$$

3 Existence of backward global solutions

In this section, we show Theorem 1.1 (1).

Lemma 3.1 *There exist* $\delta_0 > 0$ *and* $\omega \in C^1(\{h \in BU(\mathbf{R}) | \, \|h\|_{C^0} < \delta_0\})$ *such that*

$$\|\omega\|_{C^1} < +\infty,$$
$$\langle (\phi(y) + h(y)) - \phi(y - \omega[h]), e^{cy}\phi'(y - \omega[h])\rangle = 0,$$
$$\langle (\phi(y) + h(y)) - \phi(y - \omega[h]), e^{cy}\phi''(y - \omega[h])\rangle$$
$$\neq \langle \phi'(y - \omega[h]), e^{cy}\phi'(y - \omega[h])\rangle$$

and

$$\langle h(y), e^{cy}\phi'(y)\rangle = 0 \implies \omega[h] = 0$$

hold.

Proof. By (1.2), there exist C and $\sigma > 0$ such that $(1 + e^{cz})(|\phi'(z)| + |\phi''(z)|) \leq Ce^{-\sigma|z|}$ holds for all $z \in \mathbf{R}$. Hence, we define $f \in C^1(BU(\mathbf{R}) \times \mathbf{R})$ by $f(h, x) = \int_{-\infty}^{+\infty} ((\phi(z) + h(z)) - \phi(z - x))\, e^{cz}\phi'(z - x)dz$.

Then, $f(h, 0) = 0$ holds when $\langle h(y), e^{cy}\phi'(y)\rangle = 0$ holds. Hence, because $f_x(h, x) = \langle \phi'(y - x), e^{cy}\phi'(y - x)\rangle - \langle (\phi(y) + h(y)) - \phi(y - x), e^{cy}\phi''(y - x)\rangle$ also holds, this lemma follows from the implicit function theorem. q.e.d.

Definition 7 *Open subsets* V^1, V^2 *of* $\mathbf{R} \times Y''$ *and* V *of* $\mathbf{R}^2 \times Y''$ *are defined by*

$$V^1 = \left\{ (\tilde{p}, u) \in \mathbf{R} \times Y'' | \, \left\| \frac{1}{1 + e^{-x/l_0}} u(x) - \phi(x - \tilde{p}) \right\|_{C^0} < \delta_0 \right\},$$
$$V^2 = \left\{ (\tilde{q}, u) \in \mathbf{R} \times Y'' | \, \left\| \frac{1}{1 + e^{x/l_0}} u(x) - \phi(-x - \tilde{q}) \right\|_{C^0} < \delta_0 \right\}$$

and

$$V = \{(\tilde{p}, \tilde{q}, u) \in \mathbf{R}^2 \times Y'' | \, (\tilde{p}, u) \in V^1, (\tilde{q}, u) \in V^2\},$$

respectively.

Maps $P \in C^1(V^1)$ *and* $Q \in C^1(V^2)$ *are defined by*

$$P[\tilde{p}, u] = \tilde{p} + \omega \left[\frac{1}{1 + e^{-(y+\tilde{p})/l_0}} u(y + \tilde{p}) - \phi(y) \right]$$

and

$$Q[\tilde{q},u] = \tilde{q} + \omega\left[\frac{1}{1+e^{-(y+\tilde{q})/l_0}}u(-y-\tilde{q}) - \phi(y)\right],$$

respectively.

For p and $q \in \mathbf{R}$, maps Φ_p and $\Psi_q \in C^1(Y'', Y')$ are defined by

$$\Phi_p[u](y) = \frac{1}{1+e^{-(y+p)/l_0}}u(y+p) - \phi(y)$$

and

$$\Psi_q[u](y) = \frac{1}{1+e^{-(y+q)/l_0}}u(-y-q) - \phi(y),$$

respectively.

Lemma 3.2

$$(\Phi_{P[\tilde{p},u]}[u], \Psi_{Q[\tilde{q},u]}[u], P[\tilde{p},u], Q[\tilde{q},u]) \in \Omega_{l_0}$$

and

$$\Theta_{P[\tilde{p},u],Q[\tilde{q},u]}[\Phi_{P[\tilde{p},u]}[u], \Psi_{Q[\tilde{q},u]}[u]] = u$$

hold for all $(\tilde{p}, \tilde{q}, u) \in V$.

Proof. Because $\omega[\frac{1}{1+e^{-(y+\tilde{p})/l_0}}u(y+\tilde{p}) - \phi(y)] = P[\tilde{p},u] - \tilde{p}$ holds, from Lemma 3.1,

$$\langle \frac{1}{1+e^{-(y+\tilde{p})/l_0}}u(y+\tilde{p}) - \phi(y-(P[\tilde{p},u]-\tilde{p})), e^{cy}\phi'(y-(P[\tilde{p},u]-\tilde{p}))\rangle$$

$$= 0$$

and

$$\langle \frac{1}{1+e^{-(y+\tilde{p})/l_0}}u(y+\tilde{p}) - \phi(y-(P[\tilde{p},u]-\tilde{p})), e^{cy}\phi''(y-(P[\tilde{p},u]-\tilde{p}))\rangle$$

$$\neq \langle \phi'(y-(P[\tilde{p},u]-\tilde{p})), e^{cy}\phi'(y-(P[\tilde{p},u]-\tilde{p}))\rangle$$

hold. Hence,

$$\langle \Phi_{P[\tilde{p},u]}[u](y), e^{cy}\phi'(y)\rangle = 0$$

and

$$\langle \Phi_{P[\tilde{p},u]}[u](y), e^{cy}\phi''(y)\rangle \neq \langle \phi'(y), e^{cy}\phi'(y)\rangle$$

hold. We have $\langle \phi'(y) + (\Phi_{P[\tilde{p},u]}[u])_y(y), e^{cy}\phi'(y)\rangle \neq 0$. Because we also have $\langle \phi'(y) + (\Psi_{Q[\tilde{q},u]}[u])_y(y), e^{cy}\phi'(y)\rangle \neq 0$, $(\Phi_{P[\tilde{p},u]}[u], \Psi_{Q[\tilde{q},u]}[u]) \in U$ holds. Then, we can immediately get this lemma. q.e.d.

From Proposition D, the following lemma immediately follows.

Lemma 3.3 *There exist N and $\delta > 0$ such that for any $u_0 \in Y$ satisfying $\inf_{x_0 \geq N} \|u_0(x) - (\phi(x - x_0) + \phi(-x - x_0))\|_Y \leq \delta$, there exists a solution $u \in C([0, +\infty), Y)$ of* (RD) *with $u(0) = u_0$ and $\lim_{t \to +\infty} \|u(t) - 1\|_Y = 0$.*

Proposition 3.4 *There exist $N_0 > l_0 r_0$ and $p \in C^1((-\infty, 0])$ such that $(\Sigma(p,p), p, p) \in C((-\infty, 0], \tilde{S})$ is a solution of $(\mathrm{ES})_{l_0}$ with $p(0) = N_0$. Further, $p(-\infty) = +\infty$ and $(\Sigma(p(t), p(t)), p(t), p(t)) \in \Omega_{l_0}$ hold for all $t \leq 0$.*

Proof. We remember $\lim_{p' \to +\infty} \|\Sigma(p', p')\|_{X'} = 0$ from Theorem 2.3. Let $N_0 > l_0 r_0$ be a constant such that $\phi(-p') \leq 1/8$ and $\|\Sigma(p', p')\|_{C^0} \leq 1/8$ hold for all $p' \geq N_0$. Then, by $\Theta_{p',p'}[\Sigma(p', p')](0) \leq 1/2$,

$$\|1 - \Theta_{p',p'}[\Sigma(p', p')]\|_{C^0} \geq 1/2 \tag{3.1}$$

holds for all $p' \geq N_0$.

Let $N > 0$ be a constant satisfying Lemma 3.3 and $(x_0, x_0, \phi(x - x_0) + \phi(-x - x_0)) \in V$ for $x_0 \geq N$. For $x_0 \geq N$, by $u_{x_0,0}$, $p_{x_0,0}$ and $g_{x_0,0}$, we denote $\phi(x - x_0) + \phi(-x - x_0)$, $P[x_0, u_{x_0,0}]$ $(= Q[x_0, u_{x_0,0}])$ and $\Phi_{p_{x_0,0}}[u_{x_0,0}]$ $(= \Psi_{p_{x_0,0}}[u_{x_0,0}])$, respectively. Then, $\lim_{x_0 \to +\infty}(|p_{x_0,0} - x_0| + \|g_{x_0,0}\|_{C^2}) = 0$ holds. Hence, by Theorem 2.3, if x_0 is sufficiently large, then there exist solutions $(g_{x_0}, g_{x_0}, p_{x_0}, p_{x_0})$ and $(\Sigma(\bar{p}_{x_0}, \bar{p}_{x_0}), \bar{p}_{x_0}, \bar{p}_{x_0}) \in C([0, +\infty), U \times \mathbf{R}^2)$ of $(\mathrm{ES})_{l_0, r_0}$ such that

$$(g_{x_0}, g_{x_0}, p_{x_0}, p_{x_0})(0) = (g_{x_0,0}, g_{x_0,0}, p_{x_0,0}, p_{x_0,0})$$

and

$$\lim_{t \to +\infty} \|(g_{x_0}, g_{x_0}, p_{x_0}, p_{x_0})(t) - (\Sigma(\bar{p}_{x_0}, \bar{p}_{x_0}), \bar{p}_{x_0}, \bar{p}_{x_0})(t)\|_{X' \times \mathbf{R}^2} = 0 \tag{3.2}$$

hold. Because of $\lim_{x_0 \to +\infty}(\|(g_{x_0,0}, g_{x_0,0})\|_{X'} + \|\Sigma(p_{x_0,0}, p_{x_0,0})\|_{X'}) = 0$, we also have

$$\lim_{x_0 \to +\infty} \sup_{t \geq 0} \|(g_{x_0}, g_{x_0}, p_{x_0}, p_{x_0})(t) - (\Sigma(\bar{p}_{x_0}, \bar{p}_{x_0}), \bar{p}_{x_0}, \bar{p}_{x_0})(t)\|_{X' \times \mathbf{R}^2} = 0. \tag{3.3}$$

Fix any constant $\delta \geq 0$. By contradiction, we now show that if x_0 is sufficiently large, then there exists $T_{\delta,x_0} > 0$ satisfying $p_{x_0}(T_{\delta,x_0}) = N_0 + \delta$ and $p_{x_0}(t) > N_0 + \delta$ for $t \in [0, T_{\delta,x_0})$. Assume that x_0 is sufficiently large and $p_{x_0}(t) > N_0 + \delta$ holds for all $t \in [0, +\infty)$. Then, from $N_0 > l_0 r_0$, $(g_{x_0}, g_{x_0}, p_{x_0}, p_{x_0})$ is a solutions of $(\mathrm{ES})_{l_0}$ on $t \in [0, +\infty)$. Hence, because $(g_{x_0,0}, g_{x_0,0}, p_{x_0,0}, p_{x_0,0}) \in \Omega_{l_0}$ holds from Lemma 3.2, by Proposition 2.2, $(g_{x_0}, g_{x_0}, p_{x_0}, p_{x_0})(t) \in \Omega_{l_0}$ holds and $\Theta_{(p_{x_0}, p_{x_0})(t)}[(g_{x_0}, g_{x_0})(t)]$ is a solution of (RD) on $t \in [0, +\infty)$. Because $\Theta_{(p_{x_0,0}, p_{x_0,0})}[(g_{x_0,0}, g_{x_0,0})] = u_{x_0,0}$ also holds from Lemma 3.2, by Lemma 3.3, $\lim_{t \to +\infty} \|\Theta_{(p_{x_0}, p_{x_0})(t)}[(g_{x_0}, g_{x_0})(t)] - 1\|_{C^0} = 0$ holds. This is contradiction by (3.1) and (3.2). Therefore, if x_0 is sufficiently large, then there exists $T_{\delta,x_0} > 0$ satisfying $p_{x_0}(T_{\delta,x_0}) = N_0 + \delta$ and $p_{x_0}(t) > N_0 + \delta$ for $t \in [0, T_{\delta,x_0})$.

Because of $(g_{x_0,0}, g_{x_0,0}, p_{x_0,0}, p_{x_0,0}) \in \Omega_{l_0}$ and $p_{x_0}(t) > l_0 r_0$ for $t \in [0, T_{\delta,x_0}]$, by Proposition 2.2, if x_0 is sufficiently large, then $(g_{x_0}, g_{x_0}, p_{x_0}, p_{x_0})(T_{\delta,x_0}) \in \Omega_{l_0}$ holds. Also, by (3.3) and $p_{x_0}(T_{\delta,x_0}) = N_0 + \delta$, we get $\lim_{x_0 \to +\infty} \|(g_{x_0}, g_{x_0}, p_{x_0}, p_{x_0})(T_{\delta,x_0}) - (\Sigma(N_0 + \delta, N_0 + \delta), N_0 + \delta, N_0 + \delta)\|_{X' \times \mathbf{R}^2} = 0$. Therefore, $(\Sigma(N_0 + \delta, N_0 + \delta), N_0 + \delta, N_0 + \delta) \in \Omega_{l_0}$ holds for all $\delta \geq 0$.

We conclude this proof by showing that if $\delta > 0$ is sufficiently large, then there exist $T_\delta > 0$ and a solution $(\Sigma(p_\delta, p_\delta), p_\delta, p_\delta) \in C([0, +\infty), S)$ of $(\mathrm{ES})_{l_0, r_0}$ with $p_\delta(0) = N_0 + \delta$ and $p_\delta(T_\delta) = N_0$. Assume that $\delta > 0$ is sufficiently large and $p_\delta(t) > N_0$ holds on $t \in [0, +\infty)$ for the solution $(\Sigma(p_\delta, p_\delta), p_\delta, p_\delta) \in C([0, +\infty), S)$ of $(\mathrm{ES})_{l_0, r_0}$ with $p_\delta(0) = N_0 + \delta$. Then, $(\Sigma(p_\delta, p_\delta), p_\delta, p_\delta)$ is a solutions of $(\mathrm{ES})_{l_0}$ on $t \in [0, +\infty)$. Hence, by Proposition 2.2 and $(\Sigma(p_\delta(0), p_\delta(0)), p_\delta(0), p_\delta(0)) \in \Omega_{l_0}$, $\Theta_{(p_\delta(t), p_\delta(t))}[\Sigma(p_\delta(t), p_\delta(t))]$ is a solution of (RD) on $t \in [0, +\infty)$. By Lemma 3.3 and $\lim_{\delta \to +\infty} \|\Sigma(p_\delta(0), p_\delta(0))\|_{X'} = 0$, $\lim_{t \to +\infty} \|\Theta_{(p_\delta(t), p_\delta(t))}[\Sigma(p_\delta(t), p_\delta(t))] - 1\|_{C^0} = 0$ holds. This is contradiction with (3.1) and $p_\delta(t) > N_0$ on $t \in [0, +\infty)$. q.e.d.

Definition 8 *For $p \geq N_0$, $\tilde{\Theta}_p \in Y'$ is defined by $\tilde{\Theta}_p = \Theta_{p,p}[\Sigma(p, p)]$.*

In virtue of Propositions 2.2 and 3.4, the following definition is allowed.

Definition 9 *A solution $\psi \in C(\mathbf{R}, Y)$ of (RD) is defined satisfying $\psi(0) = \tilde{\Theta}_{N_0}$ and $\psi(t) = \tilde{\Theta}_{p(t)}$ on $t \leq 0$.*

Now, we prove Theorem 1.1 (1).

Proof of Theorem 1.1 (1).

Because of $\Sigma^2(p,q) = \Sigma^1(q,p)$ from Theorem 2.3, we have $\psi(-x,t) = \psi(x,t)$. From Proposition 3.4, $p(-\infty) = +\infty$ holds. Hence, because of $\lim_{p\to+\infty}(\|\Sigma(p,p)\|_{X'}+\|\tilde{G}(p,p)\|_{\mathbf{R}^2}) = 0$ from Theorem 2.3, we obtain $\lim_{t\to-\infty} \|\psi(x,t) - (\phi(x-p(t)) + \phi(-x-p(t)))\|_Y = 0$ and $\dot{p}(-\infty) = c$. By Lemma 3.3, $\lim_{t\to+\infty} \|\psi(t) - 1\|_Y = 0$ also holds. q.e.d.

4 Uniformly asymptotic stability

In this section, we prove Theorem 1.1 (2).

Proposition 4.1 *Let $T > 0$ and $(\tilde{p}, \tilde{q}, u) \in C([0,T), V)$ be given. Suppose that u is a solution of* (RD)*. Then,*

$$(\Phi_{P[\tilde{p},u]}[u], \Psi_{Q[\tilde{q},u]}[u], P[\tilde{p},u], Q[\tilde{q},u]) \in C([0,T), \Omega_{l_0})$$

is a solution of $(\mathrm{ES})_{l_0}$ *and*

$$\Theta_{P[\tilde{p},u],Q[\tilde{q},u]}[\Phi_{P[\tilde{p},u]}[u], \Psi_{Q[\tilde{q},u]}[u]] = u$$

holds on $t \in [0,T)$.

Proof. From Lemma 3.2, $(\Phi_{P[\tilde{p},u]}[u], \Psi_{Q[\tilde{q},u]}[u], P[\tilde{p},u], Q[\tilde{q},u]) \in \Omega_{l_0}$ and $\Theta_{P[\tilde{p},u],Q[\tilde{q},u]}[\Phi_{P[\tilde{p},u]}[u], \Psi_{Q[\tilde{q},u]}[u]] = u$ hold on $t \in [0,T)$. We have $P[\tilde{p},u] \in C([0,T))$. Hence, because $B^{3/4}(\frac{1}{1+e^{-x/l_0}}u(x,t)) \in C([0,T), BU(\mathbf{R}))$ holds, we have $B^{3/4}(\Phi_{P[\tilde{p},u]}[u]) \in C([0,T), BU(\mathbf{R}))$. Because we also have $Q[\tilde{q},u] \in C([0,T))$ and $B^{3/4}(\Psi_{Q[\tilde{q},u]}[u]) \in C([0,T), BU(\mathbf{R}))$, we get

$$(\Phi_{P[\tilde{p},u]}[u], \Psi_{Q[\tilde{q},u]}[u], P[\tilde{p},u], Q[\tilde{q},u]) \in C([0,T), \Omega_{l_0}).$$

We denote $\frac{1}{1+e^{-x/l_0}}u(x,t)$, $\Phi_{P[\tilde{p}(t),u(t)]}[u(t)](y)$ and $P[\tilde{p}(t),u(t)]$ by $v(x,t)$, $g(y,t)$ and $p(t)$, respectively. Then, $\omega[v(y+\tilde{p}(t),t) - \phi(y)] = p(t) - \tilde{p}(t)$ holds. From Lemma 3.1, we have

$$\langle v(x,t) - \phi(x-p(t)), e^{cx}\phi'(x-p(t))\rangle = 0$$

and

$$\begin{aligned}&\langle v(x,t) - \phi(x-p(t)), e^{cx}\phi''(x-p(t))\rangle \\ &\neq \langle \phi'(x-p(t)), e^{cx}\phi'(x-p(t))\rangle .\end{aligned}$$

Hence, by the implicit function theorem, $p \in C^1((0,T), \mathbf{R})$ holds. Hence,

$$g_t(y,t) = v_t(y+p(t),t) + \dot{p}(t)v_x(y+p(t),t) \tag{4.1}$$

holds for all $t \in (0,T)$. From $\langle g(y,t), e^{cy}\phi'(y)\rangle = 0$ and $\langle g(y,t), e^{cy}\phi''(y)\rangle \neq \langle \phi'(y), e^{cy}\phi'(y)\rangle$,

$$\langle g_t(y,t), e^{cy}\phi'(y)\rangle = 0 \tag{4.2}$$

and

$$\langle \phi'(y) + g_y(y,t), e^{cy}\phi'(y)\rangle \neq 0 \tag{4.3}$$

hold for all $t \in (0,T)$.

We also denote $\frac{1}{1+e^{x/l_0}}u(x,t)$ by $w(x,t)$. We get

$$v_t(x,t) = \frac{1}{1+e^{-x/l_0}}(u_{xx}(x,t) + f(u(x,t)))$$

$$= \frac{1}{1+e^{-x/l_0}}\left((1+e^{-x/l_0})v(x,t)\right)_{xx} + \frac{1}{1+e^{-x/l_0}}f(v(x,t)+w(x,t))$$

$$= v_{xx}(x,t) + f(v(x,t)) - \frac{2}{l_0}\frac{1}{1+e^{x/l_0}}v_x(x,t) + \frac{1}{l_0^2}\frac{1}{1+e^{x/l_0}}v(x,t)$$

$$+\frac{1}{1+e^{-x/l_0}}f(v(x,t)+w(x,t)) - f(\frac{1}{1+e^{-x/l_0}}(v(x,t)+w(x,t))).$$

Further, we denote $\Psi_{Q[\tilde{q},u]}[u]$ and $Q[\tilde{q},u]$ by h and q, respectively. Then, we have

$$v_t(y+p(t),t)$$

$$= \phi''(y) + g_{yy}(y,t) + f(\phi(y)+g(y,t))$$

$$-\frac{2}{l_0}\frac{1}{1+e^{(y+p(t))/l_0}}(\phi'(y)+g_y(y,t)) + \frac{1}{l_0^2}\frac{1}{1+e^{(y+p(t))/l_0}}(\phi(y)+g(y,t))$$

$$+\frac{1}{1+e^{-(y+p(t))/l_0}}f(\phi(y)+\phi(-y-p(t)-q(t))+g(y,t)+h(-y-p(t)-q(t),t))$$

$$-f(\frac{1}{1+e^{-(y+p(t))/l_0}}(\phi(y)+\phi(-y-p(t)-q(t))+g(y,t)+h(-y-p(t)-q(t),t)))$$

and

$$v_x(y+p(t),t) = \phi'(y) + g_y(y,t).$$

Hence, from (1.2) and (4.1), we obtain

$$g_t(y,t)$$

$$= g_{yy}(y,t) + cg_y(y,t) + f_u(\phi(y))g(y,t)$$

$$+(\dot{p}(t)-c)(\phi'(y)+g_y(y,t))$$

$$+f_{l_0}[(g,h,p,q)(t)](y).$$

From (4.2) and (4.3), we have $\dot{p}(t) = c - \frac{\langle f_{l_0}[(g,h,p,q)(t)](y), e^{cy}\phi'(y)\rangle}{\langle \phi'(y)+g_y(y,t), e^{cy}\phi'(y)\rangle}$. Therefore, we have $g_t(y,t) = g_{yy}(y,t) + cg_y(y,t) + f_u(\phi(y))g(y,t) + F_l^1[(g,h,p,q)(t)](y)$ and $\dot{p}(t) = c + G_l^1[(g,h,p,q)(t)]$. Because we also have $h_t(y,t) = h_{yy}(y,t) + ch_y(y,t) + f_u(\phi(y))h(y,t) + F_l^2[(g,h,p,q)(t)](y)$ and $\dot{q}(t) = c + G_l^2[(g,h,p,q)(t)]$, $(\Phi_{P[\tilde{p},u]}[u], \Psi_{Q[\tilde{q},u]}[u], P[\tilde{p},u], Q[\tilde{q},u])$ is a solution of $(\mathrm{ES})_{l_0}$ on $t \in [0,T)$.

q.e.d.

Lemma 4.2 *For any $\varepsilon > 0$, there exists $M > N_0$ such that*

$$(p + x_0, p - x_0, \tilde{\Theta}_p(x - x_0)) \in V,$$

$$\|(P[p + x_0, \tilde{\Theta}_p(x - x_0)], Q[p - x_0, \tilde{\Theta}_p(x - x_0)]) - (p + x_0, p - x_0)\|_{\mathbf{R}^2} < \varepsilon$$

and

$$(\Phi_{P[p+x_0,\tilde{\Theta}_p(x-x_0)]}[\tilde{\Theta}_p(x - x_0)], \Psi_{Q[p-x_0,\tilde{\Theta}_p(x-x_0)]}[\tilde{\Theta}_p(x - x_0)], P[p + x_0, \tilde{\Theta}_p(x - x_0)], Q[p - x_0, \tilde{\Theta}_p(x - x_0)]) \in \tilde{S}$$

hold for all $x_0 \in \mathbf{R}$ and $p \geq M + |x_0|$.

Proof. Let $\varepsilon > 0$ be given. Then, there exists $\delta > 0$ such that for any $\tilde{p}$, $\tilde{q} \in \mathbf{R}$ and $u \in Y''$ satisfying $\|\frac{1}{1+e^{-x/l_0}}u(x) - \phi(x - \tilde{p})\|_{C^0} < \delta$ and $\|\frac{1}{1+e^{x/l_0}}u(x) - \phi(-x - \tilde{q})\|_{C^0} < \delta$,

$$(\tilde{p}, \tilde{q}, u) \in V$$

and

$$\|(P[\tilde{p}, u], Q[\tilde{q}, u]) - (\tilde{p}, \tilde{q})\|_{\mathbf{R}^2} < \min\{\varepsilon, 1\}$$

hold. Suppose that $M > N_0$ is sufficiently large. Then, because $\lim_{p\to+\infty} \|\Sigma(p,p)\|_{X'} = 0$ holds from Theorem 2.3, we have $\|\frac{1}{1+e^{-x/l_0}}\tilde{\Theta}_p(x - x_0) - \phi(x - (p + x_0))\|_{C^0} < \delta$ and $\|\frac{1}{1+e^{x/l_0}}\tilde{\Theta}_p(x - x_0) - \phi(-x - (p - x_0))\|_{C^0} < \delta$. Therefore, we obtain

$$(p + x_0, p - x_0, \tilde{\Theta}_p(x - x_0)) \in V,$$

$$\|(P[p + x_0, \tilde{\Theta}_p(x - x_0)], Q[p - x_0, \tilde{\Theta}_p(x - x_0)]) - (p + x_0, p - x_0)\|_{\mathbf{R}^2} < \varepsilon,$$

$$P[p + x_0, \tilde{\Theta}_p(x - x_0)] > l_0 r_0$$

and

$$Q[p - x_0, \tilde{\Theta}_p(x - x_0)] > l_0 r_0.$$

There exists $\bar{p} \in C^1((-\infty, 0], [p, +\infty))$ with $\bar{p}(0) = p$ such that $\tilde{\Theta}_{\bar{p}} \in C((-\infty, 0], Y')$ is a solution of (RD). Then, $\tilde{\Theta}_{\bar{p}}(x - x_0) \in C((-\infty, 0], Y')$ is also a solution of (RD). Because $\bar{p} \geq M + |x_0|$ holds, we obtain

$$(\bar{p} + x_0, \bar{p} - x_0, \tilde{\Theta}_{\bar{p}}(x - x_0)) \in V,$$

$$P[\bar{p} + x_0, \tilde{\Theta}_{\bar{p}}(x - x_0)] > l_0 r_0$$

and

$$Q[\bar{p} - x_0, \tilde{\Theta}_{\bar{p}}(x - x_0)] > l_0 r_0.$$

Therefore, by Proposition 4.1,

$$(\Phi_{P[\bar{p}+x_0, \tilde{\Theta}_{\bar{p}}(x-x_0)]}[\tilde{\Theta}_{\bar{p}}(x - x_0)], \Psi_{Q[\bar{p}-x_0, \tilde{\Theta}_{\bar{p}}(x-x_0)]}[\tilde{\Theta}_{\bar{p}}(x - x_0)],$$

$$P[\bar{p} + x_0, \tilde{\Theta}_{\bar{p}}(x - x_0)], Q[\bar{p} - x_0, \tilde{\Theta}_{\bar{p}}(x - x_0)])$$

is a solution of $(\mathrm{ES})_{l_0, r_0}$ on $t \in (-\infty, 0]$.

Also, because of $\lim_{p \to +\infty}(\|\Sigma(p, p)\|_{X'} + \|\tilde{F}(p, p)\|_X) = 0$ from Theorem 2.3, $\lim_{p \to +\infty}(\|\Sigma^1(p, p)\|_{Y''} + \|\Sigma^2(p, p)\|_{Y''}) = 0$ holds. Hence, because $M > N_0$ is sufficiently large, $\|\Sigma^1(\bar{p}, \bar{p})\|_{Y''}$ and $\|\Sigma^2(\bar{p}, \bar{p})\|_{Y''}$ are sufficently small. Hence, because $|P[\bar{p} + x_0, \tilde{\Theta}_{\bar{p}}(x - x_0)] - (\bar{p} + x_0)|$ and $|Q[\bar{p} - x_0, \tilde{\Theta}_{\bar{p}}(x - x_0)] - (\bar{p} - x_0)|$ are also sufficiently small,

$$\|(\Phi_{P[\bar{p}+x_0, \tilde{\Theta}_{\bar{p}}(x-x_0)]}[\tilde{\Theta}_{\bar{p}}(x - x_0)], \Psi_{Q[\bar{p}-x_0, \tilde{\Theta}_{\bar{p}}(x-x_0)]}[\tilde{\Theta}_{\bar{p}}(x - x_0)])\|_{X'}$$

$$\leq 2D_0$$

holds for all $t \leq 0$. Therefore, by Thorem 2.3, for any $T > 0$, there exists $(p_T, q_T) \in \mathbf{R}^2$ satisfying

$$\|(\Phi_{P[p+x_0, \tilde{\Theta}_p(x-x_0)]}[\tilde{\Theta}_p(x - x_0)], \Psi_{Q[p-x_0, \tilde{\Theta}_p(x-x_0)]}[\tilde{\Theta}_p(x - x_0)]) - \Sigma(p_T, q_T)\|_{X'}$$

$$+\|(P[p + x_0, \tilde{\Theta}_p(x - x_0)], Q[p - x_0, \tilde{\Theta}_p(x - x_0)]) - (p_T, q_T)\|_{\mathbf{R}^2}$$

$$\leq 6C_0 D_0 e^{-\gamma_0 T}.$$

Hence, from

$$\lim_{T\to+\infty}(p_T,q_T)=(P[p+x_0,\tilde{\Theta}_p(x-x_0)],Q[p-x_0,\tilde{\Theta}_p(x-x_0)])$$

and

$$\lim_{T\to+\infty}\Sigma(p_T,q_T)$$
$$=(\Phi_{P[p+x_0,\tilde{\Theta}_p(x-x_0)]}[\tilde{\Theta}_p(x-x_0)],\Psi_{Q[p-x_0,\tilde{\Theta}_p(x-x_0)]}[\tilde{\Theta}_p(x-x_0)]),$$

we get

$$\Sigma(P[p+x_0,\tilde{\Theta}_p(x-x_0)],Q[p-x_0,\tilde{\Theta}_p(x-x_0)])$$
$$=(\Phi_{P[p+x_0,\tilde{\Theta}_p(x-x_0)]}[\tilde{\Theta}_p(x-x_0)],\Psi_{Q[p-x_0,\tilde{\Theta}_p(x-x_0)]}[\tilde{\Theta}_p(x-x_0)]).$$

q.e.d.

Lemma 4.3 *For any $\varepsilon>0$, there exists $M>0$ such that for any p_0 and $q_0\geq M$, there exist p_0' and $x_0\in\mathbf{R}$ such that*

$$|p_0'-\frac{p_0+q_0}{2}|\leq\varepsilon,\ |x_0-\frac{p_0-q_0}{2}|\leq\varepsilon,$$

$$(p_0'+x_0,p_0'-x_0,\tilde{\Theta}_{p_0'}(x-x_0))\in V$$

and

$$(p_0,q_0)=(P[p_0'+x_0,\tilde{\Theta}_{p_0'}(x-x_0)],Q[p_0'-x_0,\tilde{\Theta}_{p_0'}(x-x_0)])$$

hold.

Proof. Fix any $\varepsilon>0$. Then, let $M>0$ be sufficiently large.

If $(p_0',x_0)\in[\frac{p_0+q_0}{2}-\varepsilon,\frac{p_0+q_0}{2}+\varepsilon]\times[\frac{p_0-q_0}{2}-\varepsilon,\frac{p_0-q_0}{2}+\varepsilon]$ holds, then $p_0'-|x_0|\geq\left(\frac{p_0+q_0}{2}-\varepsilon\right)-\left(\frac{|p_0-q_0|}{2}+\varepsilon\right)\geq M-2\varepsilon$ holds. Hence, by Lemma 4.2, $|P[p_0'+x_0,\tilde{\Theta}_{p_0'}(x-x_0)]-(p_0'+x_0)|<\varepsilon$ and $|Q[p_0'-x_0,\tilde{\Theta}_{p_0'}(x-x_0)]-(p_0'-x_0)|<\varepsilon$ hold. We have

$$\begin{pmatrix}p_1'\\x_1\end{pmatrix}$$
$$:=\begin{pmatrix}\frac{p_0+q_0}{2}\\\frac{p_0-q_0}{2}\end{pmatrix}-\frac{1}{2}\begin{pmatrix}1&1\\1&-1\end{pmatrix}\begin{pmatrix}P[p_0'+x_0,\tilde{\Theta}_{p_0'}(x-x_0)]-(p_0'+x_0)\\Q[p_0'-x_0,\tilde{\Theta}_{p_0'}(x-x_0)]-(p_0'-x_0)\end{pmatrix}$$
$$\in[\frac{p_0+q_0}{2}-\varepsilon,\frac{p_0+q_0}{2}+\varepsilon]\times[\frac{p_0-q_0}{2}-\varepsilon,\frac{p_0-q_0}{2}+\varepsilon].$$

Therefore, by Brouwer's fixed point theorem, there exists $(p_0', x_0) \in [\frac{p_0+q_0}{2} - \varepsilon, \frac{p_0+q_0}{2} + \varepsilon] \times [\frac{p_0-q_0}{2} - \varepsilon, \frac{p_0-q_0}{2} + \varepsilon]$ satisfying

$$\begin{pmatrix} p_0' \\ x_0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{p_0+q_0}{2} \\ \frac{p_0-q_0}{2} \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} P[p_0' + x_0, \tilde{\Theta}_{p_0'}(x - x_0)] - (p_0' + x_0) \\ Q[p_0' - x_0, \tilde{\Theta}_{p_0'}(x - x_0)] - (p_0' - x_0) \end{pmatrix}.$$

Hence, there exists $(p_0', x_0) \in [\frac{p_0+q_0}{2} - \varepsilon, \frac{p_0+q_0}{2} + \varepsilon] \times [\frac{p_0-q_0}{2} - \varepsilon, \frac{p_0-q_0}{2} + \varepsilon]$ satisfying

$$(p_0, q_0) = (P[p_0' + x_0, \tilde{\Theta}_{p_0'}(x - x_0)], Q[p_0' - x_0, \tilde{\Theta}_{p_0'}(x - x_0)]).$$

q.e.d.

Lemma 4.4 *There exists $M > N_0$ such that for any $N \geq M$, $\bar{p}_0 \geq N + 1/2$ and $\bar{q}_0 \geq N + 1/2$ with $|\bar{p}_0 - \bar{q}_0| \leq 1$, there exist $x_0 \in (-1, 1)$, $T'' > 0$, solutions $\tilde{\Theta}_{\bar{p}''} \in C((-\infty, T''], Y'')$ of* (RD) *with $\bar{p}''(T'') = N$ and $(\Sigma(\bar{p}, \bar{q}), \bar{p}, \bar{q}) \in C((-\infty, T''], \tilde{S})$ of* $(\mathrm{ES})_{l_0}$ *with $(\bar{p}(0), \bar{q}(0)) = (\bar{p}_0, \bar{q}_0)$ such that*

$$\bar{p} > l_0 r_0 + 1,\ \bar{q} > l_0 r_0 + 1$$

and

$$\tilde{\Theta}_{\bar{p}''}(x - x_0) = \Theta_{\bar{p},\bar{q}}[\Sigma(\bar{p}, \bar{q})](x)$$

hold on $t \in (-\infty, T'']$.

Proof. Let $M > N_0$ be sufficiently large. By Lemma 4.3, there exist $\bar{p}_0'' > N$ and $x_0 \in (-1, 1)$ such that

$$(\bar{p}_0, \bar{q}_0) = (P[\bar{p}_0'' + x_0, \tilde{\Theta}_{\bar{p}_0''}(x - x_0)], Q[\bar{p}_0'' - x_0, \tilde{\Theta}_{\bar{p}_0''}(x - x_0)])$$

holds. Also, there exist $T'' > 0$ and a solution $\tilde{\Theta}_{\bar{p}''} \in C((-\infty, T''], Y'')$ of (RD) with $\bar{p}''(0) = \bar{p}_0''$ and $\bar{p}''(T'') = N$. From Lemma 4.2,

$$(\bar{p}'' + x_0, \bar{p}'' - x_0, \tilde{\Theta}_{\bar{p}''}(x - x_0)) \in V,$$

$$P[\bar{p}'' + x_0, \tilde{\Theta}_{\bar{p}''}(x - x_0)] > l_0 r_0 + 1,$$

$$Q[\bar{p}'' - x_0, \tilde{\Theta}_{\bar{p}''}(x - x_0)] > l_0 r_0 + 1$$

and

$$(\Phi_{P[\bar{p}''+x_0,\tilde{\Theta}_{\bar{p}''}(x-x_0)]}[\tilde{\Theta}_{\bar{p}''}(x-x_0)], \Psi_{Q[\bar{p}''-x_0,\tilde{\Theta}_{\bar{p}''}(x-x_0)]}[\tilde{\Theta}_{\bar{p}''}(x-x_0)],$$

$$P[\bar{p}''+x_0,\tilde{\Theta}_{\bar{p}''}(x-x_0)], Q[\bar{p}''-x_0,\tilde{\Theta}_{\bar{p}''}(x-x_0)]) \in \tilde{S}$$

hold for all $t \leq T''$. We denote $(P[\bar{p}''+x_0,\tilde{\Theta}_{\bar{p}''}(x-x_0)], Q[\bar{p}''-x_0,\tilde{\Theta}_{\bar{p}''}(x-x_0)]) \in C((-\infty,T''],\mathbf{R}^2)$ by $(\bar{p},\bar{q})$. Then, from Proposition 4.1, $(\Sigma(\bar{p},\bar{q}),\bar{p},\bar{q}) \in C((-\infty,T''],\tilde{S})$ is a solution of $(\mathrm{ES})_{l_0}$ and

$$\Theta_{\bar{p},\bar{q}}[\Sigma(\bar{p},\bar{q})](x) = \tilde{\Theta}_{\bar{p}''}(x-x_0)$$

holds for all $t \leq T''$. q.e.d.

Lemma 4.5 *There exists $C>0$ such that $\|\Phi_{p_1}[\tilde{\Theta}_p]-\Phi_{p_2}[\tilde{\Theta}_p]\|_{Y'} \leq C|p_1-p_2|$ and $\|\Psi_{q_1}[\tilde{\Theta}_p]-\Psi_{q_2}[\tilde{\Theta}_p]\|_{Y'} \leq C|q_1-q_2|$ hold for all $p_1,p_2,q_1,q_2>0$ and $p \geq N_0$.*

Proof. The solution $\psi \in C(\mathbf{R},BU(\mathbf{R}))$ of (RD) satisfies $\sup_{t\in\mathbf{R}}\|\psi\|_{C^0} < +\infty$ and $\sup_{t\in\mathbf{R}}\|f(\psi)\|_{C^0} < +\infty$. Hence, by (1.4) of [6], we have $\sup_{t\in\mathbf{R}}\|\psi\|_{C^1} < +\infty$. Also, we have $\sup_{t\in\mathbf{R}}\|f(\psi)\|_{C^1} < +\infty$. Hence, by (1.5) of [6], $\sup_{t\in\mathbf{R}}\|\psi\|_{C^2} < +\infty$ holds. Also, $\sup_{t\in\mathbf{R}}\|f(\psi)\|_{C^2} < +\infty$ holds. Hence, because $\sup_{t\in\mathbf{R}}\|B\psi\|_Y < +\infty$ and $\sup_{t\in\mathbf{R}}\|B(f(\psi))\|_Y < +\infty$ hold, we obtain $\sup_{t\in\mathbf{R}}\|B^{1+3/4}\psi\|_Y < +\infty$. Hence, $\sup_{t\in\mathbf{R}}\|\psi\|_{C^3} < +\infty$ holds. There exists $C'>0$ such that $\|\Phi_{p_1}[\psi(t)]-\Phi_{p_2}[\psi(t)]\|_{C^2} \leq C'|p_1-p_2|$ and $\|\Psi_{q_1}[\psi(t)]-\Psi_{q_2}[\psi(t)]\|_{C^2} \leq C'|q_1-q_2|$ hold for all $p_1,p_2,q_1,q_2>0$ and $t \in \mathbf{R}$. q.e.d.

Lemma 4.6 *There exist $M>0$, $\delta>0$ and $C>0$ such that for any $p \geq M$ and $u \in Y''$ with $\|u-\tilde{\Theta}_p\|_{Y''} \leq \delta$,*

$$(p,p,u) \in V$$

and

$$\|(\Phi_{P[p,u]}[u],\Psi_{Q[p,u]}[u]) - \Sigma(P[p,u],Q[p,u])\|_{X'} \leq C\|u-\tilde{\Theta}_p\|_{Y''}$$

hold.

Proof. Let $M > 0$ and $\delta > 0$ be sufficiently large and small, respectively. Because of $\lim_{p\to+\infty} \|\Sigma(p,p)\|_{X'} = 0$ from Theorem 2.3, $\|\frac{1}{1+e^{-x/l_0}}u(x) - \phi(x-p)\|_{C^0}$ and $\|\frac{1}{1+e^{x/l_0}}u(x) - \phi(-x-p)\|_{C^0}$ are sufficiently small. Hence, $(p,p,u) \in V$ holds. Because of $(\Phi_{P[p,\tilde{\Theta}_p]}[\tilde{\Theta}_p], \Psi_{Q[p,\tilde{\Theta}_p]}[\tilde{\Theta}_p]) = \Sigma(P[p,\tilde{\Theta}_p], Q[p,\tilde{\Theta}_p])$ from Lemma 4.2, we have

$$\begin{aligned}
&\|(\Phi_{P[p,u]}[u], \Psi_{Q[p,u]}[u]) - \Sigma(P[p,u], Q[p,u])\|_{X'} \\
&\leq \|(\Phi_{P[p,u]}[u], \Psi_{Q[p,u]}[u]) - (\Phi_{P[p,\tilde{\Theta}_p]}[\tilde{\Theta}_p], \Psi_{Q[p,\tilde{\Theta}_p]}[\tilde{\Theta}_p])\|_{X'} \\
&\quad + \|\Sigma(P[p,\tilde{\Theta}_p], Q[p,\tilde{\Theta}_p]) - \Sigma(P[p,u], Q[p,u])\|_{X'} \\
&\leq \|(\Phi_{P[p,u]}[u], \Psi_{Q[p,u]}[u]) - (\Phi_{P[p,u]}[\tilde{\Theta}_p], \Psi_{Q[p,u]}[\tilde{\Theta}_p])\|_{Y'^2} \\
&\quad + \|(\Phi_{P[p,u]}[\tilde{\Theta}_p], \Psi_{Q[p,u]}[\tilde{\Theta}_p]) - (\Phi_{P[p,\tilde{\Theta}_p]}[\tilde{\Theta}_p], \Psi_{Q[p,\tilde{\Theta}_p]}[\tilde{\Theta}_p])\|_{Y'^2} \\
&\quad + \|(P[p,\tilde{\Theta}_p], Q[p,\tilde{\Theta}_p]) - (P[p,u], Q[p,u])\|_{\mathbf{R}^2}.
\end{aligned}$$

Hence, from Lemma 4.5, there exists $C > 0$ such that

$$\begin{aligned}
&\|(\Phi_{P[p,u]}[u], \Psi_{Q[p,u]}[u]) - \Sigma(P[p,u], Q[p,u])\|_{X'} \\
&\leq \|(\Phi_{P[p,u]}[u], \Psi_{Q[p,u]}[u]) - (\Phi_{P[p,u]}[\tilde{\Theta}_p], \Psi_{Q[p,u]}[\tilde{\Theta}_p])\|_{Y'^2} \\
&\quad + C^{1/2}\|(P[p,\tilde{\Theta}_p], Q[p,\tilde{\Theta}_p]) - (P[p,u], Q[p,u])\|_{\mathbf{R}^2} \\
&\leq C\|u - \tilde{\Theta}_p\|_{Y''}
\end{aligned}$$

holds. q.e.d.

Lemma 4.7 *There exists $M > N_0$ such that $(p,p,\tilde{\Theta}_p) \in V$ and $P[p,\tilde{\Theta}_p] = Q[p,\tilde{\Theta}_p] = p$ hold for all $p \geq M$.*

Proof. Let $M > N_0$ be sufficiently large. Then, because of $(\Sigma(p,p),p,p) \in \Omega_{l_0}$ from Proposition 3.4,

$$\frac{1}{1+e^{-x/l_0}}\tilde{\Theta}_p(x) - \phi(x-p) = (\Sigma^1(p,p))(x-p)$$

and

$$\frac{1}{1+e^{x/l_0}}\tilde{\Theta}_p(x) - \phi(-x-p) = (\Sigma^2(p,p))(-x-p)$$

hold. Hence, because of $\lim_{p\to+\infty} \|\Sigma(p,p)\|_{X'} = 0$ from Theorem 2.3, we have $(p,p,\tilde{\Theta}_p) \in V$. From Lemma 3.1, we also have $P[p,\tilde{\Theta}_p] = Q[p,\tilde{\Theta}_p] = p$.

q.e.d.

Proposition 4.8 *There exist* $N_0' > N_0$, $\delta_0' > 0$ *and* $C_0' > 0$ *such that for any* $p_0 \geq N_0' + 2$ *and* $u_0 \in Y$ *with* $\|u_0 - \tilde{\Theta}_{p_0}\|_Y \leq \delta_0'$, *there exist* $x_0 \in \mathbf{R}$, $T > 0$, *solutions* $\tilde{\Theta}_{\bar{p}}$ *and* $u \in C([0,T], Y)$ *of* (RD) *with* $\bar{p}(T) = N_0'$ *and* $u(0) = u_0$ *such that*

$$\|u(x,t) - \tilde{\Theta}_{\bar{p}(t)}(x - x_0)\|_Y \leq C_0' e^{-\gamma_0 t} \|u_0(x) - \tilde{\Theta}_{p_0}(x)\|_Y$$

holds for all $t \in [0,T]$.

Proof. [Step 1] In this step, we show the following.

There exist $N_0' > N_0$, $\delta_0'' > 0$ *and* $C_0'' > 0$ *such that for any* $p_0'' \geq N_0' + 1$ *and* $u_0'' \in Y''$ *with* $\|u_0'' - \tilde{\Theta}_{p_0''}\|_{Y''} \leq \delta_0''$, *there exist* $x_0 \in (-1,1)$, $T'' > 0$, *solutions* $\tilde{\Theta}_{\bar{p}''} \in C((-\infty, T''], Y'')$ *and* $u'' \in C([0,T''], Y'')$ *of* (RD) *with* $\bar{p}''(T'') = N_0'$ *and* $u''(0) = u_0''$ *such that*

$$\|u''(x,t) - \tilde{\Theta}_{\bar{p}''(t)}(x - x_0)\|_Y \leq C_0'' e^{-\gamma_0 t} \|u_0''(x) - \tilde{\Theta}_{p_0''}(x)\|_{Y''}$$

holds for all $t \in [0,T'']$. *Further, there exists a solution* $(\Sigma(\bar{p},\bar{q}), \bar{p}, \bar{q}) \in C((-\infty, 0], \tilde{S})$ *of* $(\mathrm{ES})_{l_0}$ *satisfying* $\|(\bar{p}(0), \bar{q}(0)) - (p_0'', p_0'')\|_{\mathbf{R}^2} \leq C_0'' \|u_0'' - \tilde{\Theta}_{p_0''}\|_{Y''}$ *and* $\tilde{\Theta}_{\bar{p}''}(x - x_0) = (\Theta_{\bar{p},\bar{q}}[\Sigma(\bar{p},\bar{q})])(x)$ *on* $t \in (-\infty, 0]$.

Let $K > 0$ be sufficiently large and fixed. Then, let $N_0' > N_0$ and $\delta_0'' > 0$ be sufficiently large and small, respectively. Because $\lim_{(p,q)\to(+\infty,+\infty)} \|\Sigma(p, q)\|_{X'} = 0$ holds from Theorem 2.3, by Lemma 4.6, $\|(\Phi_{P[p_0'',u_0'']}[u_0''], \Psi_{Q[p_0'',u_0'']}[u_0'']\|_{X'} \leq 2D_0$ holds. We denote $(\Phi_{P[p_0'',u_0'']}[u_0''], \Psi_{Q[p_0'',u_0'']}[u_0''], P[p_0'', u_0''], Q[p_0'', u_0''])$ by (g_0, h_0, p_0, q_0). By Theorem 2.3, there exist solutions $(g,h,p,q) \in C([0, +\infty), U \times \mathbf{R}^2)$ and $(\Sigma(\bar{p},\bar{q}), \bar{p}, \bar{q}) \in C(\mathbf{R}, S)$ of $(\mathrm{ES})_{l_0,r_0}$ such that

$$(g,h,p,q)(0) = (g_0, h_0, p_0, q_0)$$

and for any $t \geq 0$,

$$\|(g,h,p,q)(t) - (\Sigma(\bar{p},\bar{q}), \bar{p}, \bar{q})(t)\|_{X' \times \mathbf{R}^2}$$
$$\leq C_0 e^{-\gamma_0 t} \|(g_0, h_0) - \Sigma(p_0, q_0)\|_{X'}$$

hold. By Lemma 4.6,

$$\|(g,h,p,q)(t) - (\Sigma(\bar{p},\bar{q}), \bar{p}, \bar{q})(t)\|_{X' \times \mathbf{R}^2} \tag{4.4}$$
$$\leq K e^{-\gamma_0 t} \|u_0'' - \tilde{\Theta}_{p_0''}\|_{Y''}$$

holds for all $t \geq 0$.

From Lemma 4.7, we have

$$\begin{aligned}
&\|(p_0'' - \bar{p}(0), p_0'' - \bar{q}(0)\|_{\mathbf{R}^2} \\
&= \|(P[p_0'', \tilde{\Theta}_{p_0''}] - \bar{p}(0), Q[p_0'', \tilde{\Theta}_{p_0''}] - \bar{q}(0)\|_{\mathbf{R}^2} \\
&\leq \|(P[p_0'', \tilde{\Theta}_{p_0''}] - p_0, Q[p_0'', \tilde{\Theta}_{p_0''}] - q_0)\|_{\mathbf{R}^2} \\
&\quad + \|(p_0 - \bar{p}(0), q_0 - \bar{q}(0)\|_{\mathbf{R}^2} \\
&= \|(P[p_0'', \tilde{\Theta}_{p_0''}] - P[p_0'', u_0''], Q[p_0'', \tilde{\Theta}_{p_0''}] - Q[p_0'', u_0'']\|_{\mathbf{R}^2} \\
&\quad + \|(p_0 - \bar{p}(0), q_0 - \bar{q}(0)\|_{\mathbf{R}^2}.
\end{aligned}$$

Hence, by (4.4), we obtain

$$\|(p_0'' - \bar{p}(0), p_0'' - \bar{q}(0)\|_{\mathbf{R}^2} \leq 2K \|u_0'' - \tilde{\Theta}_{p_0''}\|_{Y''}. \tag{4.5}$$

Because $\|(p_0'' - \bar{p}(0), p_0'' - \bar{q}(0)\|_{\mathbf{R}^2} \leq 2K\,\delta_0''$ and $p_0'' \geq N_0' + 1$ hold, we have $\bar{p}(0) \geq N_0' + 1/2$, $\bar{q}(0) \geq N_0' + 1/2$ and $|\bar{p}(0) - \bar{q}(0)| \leq 1$. Hence, by Lemma 4.4, there exist $x_0 \in (-1, 1)$, $T'' > 0$ and a solution $\tilde{\Theta}_{\bar{p}''} \in C((-\infty, T''], Y'')$ of (RD) with $\bar{p}''(T'') = N_0'$ such that

$$\bar{p} > l_0 r_0 + 1,\ \bar{q} > l_0 r_0 + 1$$

and

$$\Theta_{\bar{p},\bar{q}}[\Sigma(\bar{p}, \bar{q})](x) = \tilde{\Theta}_{\bar{p}''}(x - x_0) \tag{4.6}$$

hold on $t \in (-\infty, T'']$. From (4.4), $p(t) > l_0 r_0$ and $q(t) > l_0 r_0$ hold for all $t \in [0, T'']$. Therefore, (g, h, p, q) is a solution of $(\mathrm{ES})_{l_0}$ on $t \in [0, T'']$. Hence, because $(g_0, h_0, p_0, q_0) \in \Omega_{l_0}$ and $\Theta_{p_0,q_0}[g_0, h_0] = u_0''$ hold from Lemma 3.2, by Proposition 2.2,

$$\Theta_{p,q}[g, h] = u''$$

holds on $t \in [0, T'']$. Hence, by (4.4), (4.5) and (4.6), there exists $C_0'' > 0$ such that

$$\|(\bar{p}(0), \bar{q}(0)) - (p_0'', p_0'')\|_{\mathbf{R}^2} \leq C_0'' \|u_0'' - \tilde{\Theta}_{p_0''}\|_{Y''}$$

and for any $t \in [0, T'']$,

$$\|u''(x, t) - \tilde{\Theta}_{\bar{p}''(t)}(x - x_0)\|_{C^0} \leq C_0'' e^{-\gamma_0 t} \|u_0''(x) - \tilde{\Theta}_{p_0''}(x)\|_{Y''}$$

hold.

[Step 2] In this step, we prove Proposition 4.8.

We denote $(\sup_{p\geq N_0'+1}|c+G_l^1[\Sigma(p,p),p,p]|)^{-1}>0$ by t_0. Let $K>0$ be sufficiently large and fixed. Let $\delta_0'>0$ be sufficiently small. Then, there exists a solution $\tilde{\Theta}_p\in C([0,t_0],Y'')$ of (RD) with $p(0)=p_0$ such that

$$p(t_0)\geq N_0'+1$$

and

$$\|u(t_0)-\tilde{\Theta}_{p(t_0)}\|_{Y''}\leq K\|u_0-\tilde{\Theta}_{p_0}\|_Y$$

hold. Hence, by Step 1, there exist $x_0\in(-1,1)$, $T>t_0$ and a solution $\tilde{\Theta}_{\bar{p}}\in C([0,T],Y)$ of (RD) with $\bar{p}(T)=N_0'$ such that

$$\|u(x,t)-\tilde{\Theta}_{\bar{p}(t)}(x-x_0)\|_Y\leq C_0''Ke^{-\gamma_0(t-t_0)}\|u_0(x)-\tilde{\Theta}_{p_0}(x)\|_Y \tag{4.7}$$

holds for all $t\in[t_0,T]$. Further, there exists a solution $(\Sigma(\bar{p},\bar{q}),\bar{p},\bar{q})\in C([0,t_0],\tilde{S})$ of $(\mathrm{ES})_{l_0}$ satisfying $\|(p(t_0),p(t_0))-(\bar{p}(t_0),\bar{q}(t_0))\|_{\mathbf{R}^2}\leq C_0''K\|u_0-\tilde{\Theta}_{p_0}\|_Y$ and $\tilde{\Theta}_{\bar{p}(t)}(x-x_0)=(\Theta_{\bar{p}(t),\bar{q}(t)}[\Sigma(\bar{p}(t),\bar{q}(t))])(x)$ for $t\in[0,t_0]$. Hence, we obtain

$$\begin{aligned}
&\sup_{t\in[0,t_0]}\|\tilde{\Theta}_{p(t)}(x)-\tilde{\Theta}_{\bar{p}(t)}(x-x_0)\|_Y\\
&=\sup_{t\in[0,t_0]}\|\Theta_{p(t),p(t)}[\Sigma(p(t),p(t))]-\Theta_{\bar{p}(t),\bar{q}(t)}[\Sigma(\bar{p}(t),\bar{q}(t))]\|_{C^0}\\
&\leq K\sup_{t\in[0,t_0]}\|(p(t),p(t))-(\bar{p}(t),\bar{q}(t))\|_{\mathbf{R}^2}\\
&\leq K^2\|(p(t_0),p(t_0))-(\bar{p}(t_0),\bar{q}(t_0))\|_{\mathbf{R}^2}\\
&\leq C_0''K^3\|u_0(x)-\tilde{\Theta}_{p_0}(x)\|_Y.
\end{aligned}$$

From (4.7), with $C_0'=C_0''(K+K^3)e^{\gamma_0 t_0}$,

$$\|u(x,t)-\tilde{\Theta}_{\bar{p}(t)}(x-x_0)\|_Y\leq C_0'e^{-\gamma_0 t}\|u_0(x)-\tilde{\Theta}_{p_0}(x)\|_Y$$

holds for all $t\in[0,T]$. q.e.d.

Now, we prove Theorem 1.1 (2).

Proof of Theorem 1.1 (2).

Because of $\lim_{t\to+\infty}\|\psi(t)-1\|_Y=0$ and $Re\,\sigma(B)>0$, there exist T_1, δ_1, C_1 and $\gamma_1>0$ such that for any $t_0\geq T_1$ and $u_0\in Y$ with $\|u_0-\psi(t_0)\|_Y\leq\delta_1$, there exists a solution $u\in C([0,+\infty),Y)$ of (RD) with $u(0)=u_0$ such that

$$\|u(t)-\psi(t+t_0)\|_Y\leq C_1e^{-\gamma_1 t}\|u_0-\psi(t_0)\|_Y \tag{4.8}$$

holds for all $t\geq 0$. We define $T_2<0$ such that $\psi(T_2)=\tilde{\Theta}_{N_0'+2}$ holds. Then, there exist δ_2 and $C_2>0$ such that for any $t_0\in[T_2,T_1]$ and $u_0\in Y$ with $\|u_0-\psi(t_0)\|_Y\leq\delta_2$, there exists a solution $u\in C([0,+\infty),Y)$ of (RD) with $u(0)=u_0$ such that

$$\|u(t)-\psi(t+t_0)\|_Y\leq C_2\|u_0-\psi(t_0)\|_Y$$

holds for all $t\in[0,T_1-t_0]$. Hence, by (4.8), for any $t_0\geq T_2$ and $u_0\in Y$ with $\|u_0-\psi(t_0)\|_Y\leq\min\{\delta_1/C_2,\delta_2\}$, there exists a solution $u\in C([0,+\infty),Y)$ of (RD) with $u(0)=u_0$ such that

$$\|u(t)-\psi(t+t_0)\|_Y\leq C_1C_2e^{\gamma_1(T_1-T_2)}e^{-\gamma_1 t}\|u_0-\psi(t_0)\|_Y$$

holds for all $t\geq 0$. Hence, by Proposition 4.8, with $\delta=\min\{\frac{\delta_1}{C_2C_0'},\frac{\delta_2}{C_0'},\delta_0'\}$, $C=C_1C_2C_0'e^{\gamma_1(T_1-T_2)}$ and $\gamma=\min\{\gamma_1,\gamma_0\}$, for any $t_0\in\mathbf{R}$ and $u_0\in Y$ with $\|u_0-\psi(t_0)\|_Y\leq\delta$, there exist x_0, $t_0'\in\mathbf{R}$ and a solution $u\in C([0,+\infty),Y)$ of (RD) with $u(0)=u_0$ such that

$$\|u(x,t)-\psi(x-x_0,t-t_0')\|_Y\leq Ce^{-\gamma t}\|u_0(x)-\psi(x,t_0)\|_Y$$

holds for all $t\geq 0$. q.e.d.

5 Proof of Theorem 2.3

According to Section 6.1 of [10], we prove Theorem 2.3.

Lemma 5.1 *There exist $C_1 > 0$ and a monotone increasing function $\epsilon_1 : (0, +\infty) \to (0, +\infty)$ with $\lim_{s \to +0} \epsilon_1(s) = +0$ such that for any $l \geq 1$, $p > 0$, $q > 0$ and $(g, h) \in X'$ satisfying $\|(g, h)\|_{X'} \leq 1$,*

$$\|f_l[g, h, p, q]\|_Y \leq \epsilon_1(\max\{l/p, l/q\}) + \epsilon_1(1/l)\|(g, h)\|_{X'} + C_1\|(g, h)\|_{X'}^2$$

holds.

Proof. Because $f(a + b) - (f(a) + f_u(a)b) = \left(\int_0^1 f_{uu}(a + \sigma b)(1 - \sigma)d\sigma\right) b^2$ holds, we have

$$\begin{aligned} (5.1) \qquad & \|f(\phi(y) + g(y)) - (f(\phi(y)) + f_u(\phi(y))g(y))\|_{C^0} \\ & \leq \left(\sup_{u \in [-1,2]} |f_{uu}(u)|\right) \|g\|_{C^0}^2. \end{aligned}$$

When $y \leq -\frac{1}{2}p$ holds,

$$\left|-\frac{2}{l}\frac{1}{1 + e^{(y+p)/l}}\phi'(y) + \frac{1}{l^2}\frac{1}{1 + e^{(y+p)/l}}\phi(y)\right| \leq 2 \max_{x \leq -\frac{1}{2}p} |\phi'(x)| + \max_{x \leq -\frac{1}{2}p} |\phi(x)|$$

holds. When $-\frac{1}{2}p \leq y$ holds,

$$\left|-\frac{2}{l}\frac{1}{1 + e^{(y+p)/l}}\phi'(y) + \frac{1}{l^2}\frac{1}{1 + e^{(y+p)/l}}\phi(y)\right| \leq 3\|\phi\|_{C^1} e^{-\frac{1}{2}\frac{p}{l}}$$

holds. Therefore, we have

$$\begin{aligned} (5.2) \qquad & \left\|-\frac{2}{l}\frac{1}{1 + e^{(y+p)/l}}\phi'(y) + \frac{1}{l^2}\frac{1}{1 + e^{(y+p)/l}}\phi(y)\right\|_{C^0} \\ & \leq 2 \max_{x \leq -\frac{1}{2}p} |\phi'(x)| + \max_{x \leq -\frac{1}{2}p} |\phi(x)| + 3\|\phi\|_{C^1} e^{-\frac{1}{2}\frac{p}{l}}. \end{aligned}$$

We get

$$(5.3) \qquad \left\|-\frac{2}{l}\frac{1}{1 + e^{(y+p)/l}}g_y(y) + \frac{1}{l^2}\frac{1}{1 + e^{(y+p)/l}}g(y)\right\|_{C^0} \leq \frac{3}{l}\|g\|_{C^1}.$$

We have $f(\frac{1}{1+e^{-(y+p)/l}}a) = \left(\int_0^1 f_u(\sigma\frac{1}{1+e^{-(y+p)/l}}a)d\sigma\right)\frac{1}{1+e^{-(y+p)/l}}a$. Hence, when $y \leq -p-\frac{1}{2}q$ holds,

$$|\frac{1}{1+e^{-(y+p)/l}}f(\phi(y)+\phi(-y-p-q)+g(y)+h(-y-p-q))$$

$$-f(\frac{1}{1+e^{-(y+p)/l}}(\phi(y)+\phi(-y-p-q)+g(y)+h(-y-p-q)))|$$

$$\leq \left(\sup_{u\in[-2,4]}|f(u)|+4\sup_{u\in[-2,4]}|f_u(u)|\right)e^{-\frac{1}{2}\frac{q}{l}}$$

holds. Because $f(a) = f_u(0)a+\left(\int_0^1 f_{uu}(\sigma a)(1-\sigma)d\sigma\right)a^2$ and $f(\frac{1}{1+e^{-(y+p)/l}}a)$ $= f_u(0)\frac{1}{1+e^{-(y+p)/l}}a+\left(\int_0^1 f_{uu}(\sigma\frac{1}{1+e^{-(y+p)/l}}a)(1-\sigma)d\sigma\right)\frac{1}{(1+e^{-(y+p)/l})^2}a^2$ hold, we have $\frac{1}{1+e^{-(y+p)/l}}f(a)-f(\frac{1}{1+e^{-(y+p)/l}}a) = \left(\int_0^1 f_{uu}(\sigma a)(1-\sigma)d\sigma\right)\frac{1}{1+e^{-(y+p)/l}}a^2 -$ $\left(\int_0^1 f_{uu}(\sigma\frac{1}{1+e^{-(y+p)/l}}a)(1-\sigma)d\sigma\right)\frac{1}{(1+e^{-(y+p)/l})^2}a^2$. Hence, when $-p-\frac{1}{2}q \leq y \leq -\frac{1}{2}p$ holds,

$$|\frac{1}{1+e^{-(y+p)/l}}f(\phi(y)+\phi(-y-p-q)+g(y)+h(-y-p-q))$$

$$-f(\frac{1}{1+e^{-(y+p)/l}}(\phi(y)+\phi(-y-p-q)+g(y)+h(-y-p-q)))|$$

$$\leq 2\left(\sup_{u\in[-2,4]}|f_{uu}(u)|\right)(\max_{x\leq-\frac{1}{2}p}|\phi(x)|+\max_{x\leq-\frac{1}{2}q}|\phi(x)|+\|g\|_{C^0}+\|h\|_{C^0})^2$$

holds. Because of $f(\frac{1}{1+e^{-(y+p)/l}}a)-f(a) = \left(\int_0^1 f_u(a+\sigma\left(\frac{1}{1+e^{-(y+p)/l}}-1\right)a)d\sigma\right)$ $\left(\frac{1}{1+e^{-(y+p)/l}}-1\right)a$, we have $\frac{1}{1+e^{-(y+p)/l}}f(a)-f(\frac{1}{1+e^{-(y+p)/l}}a) = \left(\frac{1}{1+e^{-(y+p)/l}}-1\right)$ $\left(f(a)-a\left(\int_0^1 f_u(a+\sigma\left(\frac{1}{1+e^{-(y+p)/l}}-1\right)a)d\sigma\right)\right)$. Hence, when $-\frac{1}{2}p \leq y$ holds,

$$|\frac{1}{1+e^{-(y+p)/l}}f(\phi(y)+\phi(-y-p-q)+g(y)+h(-y-p-q))$$

$$-f(\frac{1}{1+e^{-(y+p)/l}}(\phi(y)+\phi(-y-p-q)+g(y)+h(-y-p-q)))|$$

$$\leq \left(\sup_{u\in[-2,4]}|f(u)|+4\sup_{u\in[-2,4]}|f_u(u)|\right)e^{-\frac{1}{2}\frac{p}{l}}$$

holds. Therefore, we obtain

$$
\begin{aligned}
(5.4)\quad & \| \frac{1}{1+e^{-(y+p)/l}} f(\phi(y)+\phi(-y-p-q)+g(y)+h(-y-p-q)) \\
& -f(\frac{1}{1+e^{-(y+p)/l}}(\phi(y)+\phi(-y-p-q)+g(y)+h(-y-p-q)))\|_{C^0} \\
& \leq \left(\sup_{u\in[-2,4]} |f(u)| + 4 \sup_{u\in[-2,4]} |f_u(u)| \right) (e^{-\frac{1}{2}\frac{p}{l}} + e^{-\frac{1}{2}\frac{q}{l}}) \\
& +2 \left(\sup_{u\in[-2,4]} |f_{uu}(u)| \right) (\max_{x\leq -\frac{1}{2}p} |\phi(x)| + \max_{x\leq -\frac{1}{2}q} |\phi(x)| + \|g\|_{C^0} + \|h\|_{C^0})^2.
\end{aligned}
$$

From (5.1), (5.2), (5.3) and (5.4), we obtain

$$
\begin{aligned}
& \|f_l[g,h,p,q]\|_{C^0} \\
& \leq \left(\sup_{u\in[-1,2]} |f_{uu}(u)| \right) \|g\|^2_{C^0} \\
& +2 \max_{x\leq -\frac{1}{2}p} |\phi'(x)| + \max_{x\leq -\frac{1}{2}p} |\phi(x)| + 3\|\phi\|_{C^1} e^{-\frac{1}{2}\frac{p}{l}} + \frac{3}{l}\|g\|_{C^1} \\
& + \left(\sup_{u\in[-2,4]} |f(u)| + 4 \sup_{u\in[-2,4]} |f_u(u)| \right) (e^{-\frac{1}{2}\frac{p}{l}} + e^{-\frac{1}{2}\frac{q}{l}}) \\
& +2 \left(\sup_{u\in[-2,4]} |f_{uu}(u)| \right) (\max_{x\leq -\frac{1}{2}p} |\phi(x)| + \max_{x\leq -\frac{1}{2}q} |\phi(x)| + \|g\|_{C^0} + \|h\|_{C^0})^2 \\
& \leq 2 \max_{x\leq -\frac{1}{2}p} |\phi'(x)| + \max_{x\leq -\frac{1}{2}p} |\phi(x)| \\
& +2 \left(\sup_{u\in[-2,4]} |f_{uu}(u)| \right) (2\|\phi\|_{C^0} + 4)(\max_{x\leq -\frac{1}{2}p} |\phi(x)| + \max_{x\leq -\frac{1}{2}q} |\phi(x)|) \\
& +3\|\phi\|_{C^1} e^{-\frac{1}{2}\frac{p}{l}} + \left(\sup_{u\in[-2,4]} |f(u)| + 4 \sup_{u\in[-2,4]} |f_u(u)| \right) (e^{-\frac{1}{2}\frac{p}{l}} + e^{-\frac{1}{2}\frac{q}{l}}) \\
& + \frac{3}{l}\|g\|_{C^1} + 3 \left(\sup_{u\in[-2,4]} |f_{uu}(u)| \right) (\|g\|_{C^0} + \|h\|_{C^0})^2.
\end{aligned}
$$

q.e.d.

Lemma 5.2 *There exists a monotone increasing function* $\epsilon_2 : (0, +\infty) \to (0, +\infty)$ *with* $\lim_{s \to +0} \epsilon_2(s) = +0$ *such that for any* $l \geq 1$, $p > 0$, $q > 0$ *and* $(g, h) \in X'$ *satisfying* $\|(g, h)\|_{X'} \leq 1$,

$$\|(\partial f_l)[g, h, p, q]\|_{L(X' \times \mathbf{R}^2, Y)} \leq \epsilon_2(\max\{l/p, l/q, 1/l, \|(g, h)\|_{X'}\})$$

holds, where ∂f_l *denotes Frechet derivative of* f_l.

Proof. When $y \leq -p - \frac{1}{2}q$ holds,

$$|\frac{1}{1 + e^{-(y+p)/l}}(f_u(\phi(y) + \phi(-y - p - q) + g(y) + h(-y - p - q))$$

$$-f_u(\frac{1}{1 + e^{-(y+p)/l}}(\phi(y) + \phi(-y - p - q) + g(y) + h(-y - p - q))))|$$

$$\leq 2 \left(\sup_{u \in [-2,4]} |f_u(u)| \right) e^{-\frac{1}{2}\frac{q}{l}}$$

holds. We have $f_u(\frac{1}{1+e^{-(y+p)/l}}a) - f_u(a) = \left(\int_0^1 f_{uu}(a + \sigma\left(\frac{1}{1+e^{-(y+p)/l}} - 1\right)a)d\sigma\right) \left(\frac{1}{1+e^{-(y+p)/l}} - 1\right)a$. Hence, when $-p - \frac{1}{2}q \leq y \leq -\frac{1}{2}p$ holds,

$$|\frac{1}{1 + e^{-(y+p)/l}}(f_u(\phi(y) + \phi(-y - p - q) + g(y) + h(-y - p - q))$$

$$-f_u(\frac{1}{1 + e^{-(y+p)/l}}(\phi(y) + \phi(-y - p - q) + g(y) + h(-y - p - q))))|$$

$$\leq \left(\sup_{u \in [-2,4]} |f_{uu}(u)| \right) (\max_{x \leq -\frac{1}{2}p} |\phi(x)| + \max_{x \leq -\frac{1}{2}q} |\phi(x)| + \|g\|_{C^0} + \|h\|_{C^0})$$

holds. Also, when $-\frac{1}{2}p \leq y$ holds,

$$|\frac{1}{1 + e^{-(y+p)/l}}(f_u(\phi(y) + \phi(-y - p - q) + g(y) + h(-y - p - q))$$

$$-f_u(\frac{1}{1 + e^{-(y+p)/l}}(\phi(y) + \phi(-y - p - q) + g(y) + h(-y - p - q))))|$$

$$\leq 4 \left(\sup_{u \in [-2,4]} |f_{uu}(u)| \right) e^{-\frac{1}{2}\frac{p}{l}}$$

holds. Therefore, we obtain

$$
\begin{aligned}
(5.5)\quad & \Big\| \frac{1}{1+e^{-(y+p)/l}} (f_u(\phi(y)+\phi(-y-p-q)+g(y)+h(-y-p-q)) \\
& -f_u(\frac{1}{1+e^{-(y+p)/l}}(\phi(y)+\phi(-y-p-q)+g(y)+h(-y-p-q))))\Big\|_{C^0} \\
& \leq 2\left(\sup_{u\in[-2,4]}|f_u(u)|\right) e^{-\frac{1}{2}\frac{q}{l}} \\
& + \left(\sup_{u\in[-2,4]}|f_{uu}(u)|\right)(4e^{-\frac{1}{2}\frac{p}{l}} + \max_{x\leq -\frac{1}{2}p}|\phi(x)| + \max_{x\leq -\frac{1}{2}q}|\phi(x)| + \|g\|_{C^0} + \|h\|_{C^0}).
\end{aligned}
$$

Because we have

$$
\begin{aligned}
& (((\partial f_l)[g,h,p,q])(\Delta g,\Delta h,\Delta p,\Delta q))(y) \\
& = (f_u(\phi(y)+g(y)) - f_u(\phi(y)))(\Delta g)(y) \\
& -\frac{2}{l}\frac{1}{1+e^{(y+p)/l}}(\Delta g)_y(y) + \frac{1}{l^2}\frac{1}{1+e^{(y+p)/l}}(\Delta g)(y) \\
& +\frac{1}{1+e^{-(y+p)/l}}(f_u(\phi(y)+\phi(-y-p-q)+g(y)+h(-y-p-q)) \\
& -f_u(\frac{1}{1+e^{-(y+p)/l}}(\phi(y)+\phi(-y-p-q)+g(y)+h(-y-p-q)))) \\
& ((\Delta g)(y)+(\Delta h)(-y-p-q)) \\
& +\frac{1}{1+e^{-(y+p)/l}}\frac{1}{1+e^{(y+p)/l}}\left(\frac{2}{l^2}(\phi'(y)+g_y(y)) - \frac{1}{l^3}(\phi(y)+g(y))\right)\Delta p \\
& +\frac{1}{l}\frac{1}{1+e^{-(y+p)/l}}\frac{1}{1+e^{(y+p)/l}} \\
& f(\phi(y)+\phi(-y-p-q)+g(y)+h(-y-p-q))\Delta p \\
& -\frac{1}{l}\frac{1}{1+e^{-(y+p)/l}}\frac{1}{1+e^{(y+p)/l}} \\
& f_u(\frac{1}{1+e^{-(y+p)/l}}(\phi(y)+\phi(-y-p-q)+g(y)+h(-y-p-q))) \\
& (\phi(y)+\phi(-y-p-q)+g(y)+h(-y-p-q))\Delta p
\end{aligned}
$$

$$-\frac{1}{1+e^{-(y+p)/l}}(f_u(\phi(y)+\phi(-y-p-q)+g(y)+h(-y-p-q))$$

$$-f_u(\frac{1}{1+e^{-(y+p)/l}}(\phi(y)+\phi(-y-p-q)+g(y)+h(-y-p-q))))$$

$$(\phi'(-y-p-q)+h_y(-y-p-q))(\Delta p+\Delta q),$$

by (5.5), we can easily get this lemma. q.e.d.

From Lemmas 5.1 and 5.2, the following lemma immediately follows.

Lemma 5.3 *There exist $c_3>0$, $C_3>0$ and a monotone increasing function $\epsilon_3:(0,+\infty)\to(0,+\infty)$ with $\lim_{s\to+0}\epsilon_3(s)=+0$ satisfying the following.*

(1) For any $l\geq 1$, $r\geq 1$ and $(g,h,p,q)\in X'\times\mathbf{R}^2$ with $\|(g,h)\|_{X'}\leq c_3$,

$$\|F_{l,r}[g,h,p,q]\|_X\leq\epsilon_3(1/r)+\epsilon_3(1/l)\|(g,h)\|_{X'}+C_3\|(g,h)\|_{X'}^2$$

and

$$\|(\partial F_{l,r})[g,h,p,q]\|_{L(X'\times\mathbf{R}^2,X)}+\|(\partial G_{l,r})[g,h,p,q]\|_{L(X'\times\mathbf{R}^2,\mathbf{R}^2)}$$

$$\leq\epsilon_3(\max\{1/r,1/l,\|(g,h)\|_{X'}\})$$

hold, where $F_{l,r}[g,h,p,q]$ and $G_{l,r}[g,h,p,q]$ denote $\chi(\frac{p}{lr})\chi(\frac{q}{lr})F_l[g,h,p,q]$ and $\chi(\frac{p}{lr})\chi(\frac{q}{lr})G_l[g,h,p,q]$, respectively.

(2) For any $l\geq 1$, $p>0$, $q>0$ and $(g,h)\in X'$ with $\|(g,h)\|_{X'}\leq c_3$,

$$\|F_l[g,h,p,q]\|_X+\|G_l[g,h,p,q]\|_{\mathbf{R}^2}$$

$$\leq\epsilon_3(\max\{l/p,l/q,\|(g,h)\|_{X'}\})$$

holds.

We obtain the following by Lemma 5.3 (1).

Proposition 5.4 *For any $\Delta\in(0,1]$, there exists $l_1\geq 1$ such that for any $d>0$, there exists $r_1\geq 1$ such that for any $l\geq l_1$ and $r\geq r_1$, there exists an invariant manifold*

$$S=\{(g,h,p,q)\in U\times\mathbf{R}^2\,|\,(g,h)=\sigma(p,q)\}$$

for $(\mathrm{ES})_{l,r}$ satisfying $\|\sigma(p,q)\|_{X'}\leq d$ and $\|\sigma(p_1,q_1)-\sigma(p_2,q_2)\|_{X'}\leq\Delta\|(p_1,q_1)-(p_2,q_2)\|_{\mathbf{R}^2}$.

Proof. Let M and $\beta > 0$ be constants such that $\|e^{-At}\|_{L(X,X)} \leq Me^{-\beta t}$ and $\|A^{3/4}e^{-At}\|_{L(X,X)} \leq Mt^{-3/4}e^{-\beta t}$ hold for all $t > 0$.

Let $\Delta \in (0,1]$ be given, and $l_1 \geq 1$ be a constant such that

$$\epsilon_3(1/l_1) \leq \frac{\beta}{2(1+\Delta)},$$

$$\epsilon_3(1/l_1)M\left(\int_0^\infty u^{-3/4}e^{-\frac{\beta}{2}u}du\right) \leq \frac{\Delta}{1+\Delta}$$

and

$$\epsilon_3(1/l_1)M\left(\int_0^\infty u^{-3/4}e^{-\beta u}du\right) \leq \frac{1}{8}$$

hold.

Let $d > 0$ be given, $D > 0$ be a constant defined by

$$D = \min\left\{d, \frac{c_3}{2}, \frac{1}{2l_1}, \frac{1}{32C_3M\left(\int_0^\infty u^{-3/4}e^{-\beta u}du\right)}\right\},$$

and $r_1 \geq l_1$ be a constant such that

$$\epsilon_3(1/r_1)M\left(\int_0^\infty u^{-3/4}e^{-\beta u}du\right) \leq \frac{D}{8}$$

holds.

Then, by Lemma 5.3 (1), the hypotheses of Theorem 6.1.2 of [10] for $(\mathrm{ES})_{l,r}$ are satisfied under the notations $Y = \mathbf{R}^2$, $\alpha = 3/4$, $U = \{(g,h) \in X' | \, \|(g,h)\|_{X'} < 2D\}$, $\lambda = \epsilon_3(1/l_1)$, $N = \frac{D}{2M\left(\int_0^\infty u^{-3/4}e^{-\beta u}du\right)}$, $M_1 = 1$, $M_2 = \frac{\beta}{2(1+\Delta)}$ and $\mu = \frac{\beta}{2(1+\Delta)}$. Hence, this proposition follows from Theorem 6.1.2 of [10]. q.e.d.

We also obtain the following by Lemma 5.3 (1).

Proposition 5.5 *There exist $D_1 > 0$, $\delta_1 > 0$, $K_1 > 0$, $\gamma_1 > 0$, $L_1 \geq 1$ and $R_1 \geq 1$ such that for any $l \geq L_1$ and $r \geq R_1$, there exists an invariant manifold*

$$S = \{(g,h,p,q) \in U \times \mathbf{R}^2 | \, (g,h) = \sigma(p,q)\}$$

for $(\mathrm{ES})_{l,r}$ satisfying $\|\sigma(p,q)\|_{X'} \leq D_1$, $\|\sigma(p_1,q_1) - \sigma(p_2,q_2)\|_{X'} \leq \|(p_1,q_1) - (p_2,q_2)\|_{\mathbf{R}^2}$ and the following. For any $(g_0,h_0,p_0,q_0) \in X' \times \mathbf{R}^2$ with $\|(g_0,h_0)\|_{X'}$

$\leq \delta_1$, *there exist solutions* $(g,h,p,q) \in C([0,+\infty), U \times \mathbf{R}^2)$ *and* $(\bar{g},\bar{h},\bar{p},\bar{q}) \in C([0,+\infty), S)$ *of* $(\mathrm{ES})_{l,r}$ *such that*

$$(g,h,p,q)(0) = (g_0,h_0,p_0,q_0)$$

and for any $t \geq 0$,

$$\|(g,h,p,q)(t) - (\bar{g},\bar{h},\bar{p},\bar{q})(t)\|_{X' \times \mathbf{R}^2} \leq K_1 e^{-\gamma_1 t} \|(g_0,h_0) - \sigma(p_0,q_0)\|_{X'}$$

hold.

Proof. Let $K > 0$ be a constant such that Theorem 6.1.4 of [10] holds for $\alpha = 3/4$, and M and $\beta > 0$ be constants such that $\|e^{-At}\|_{L(X,X)} \leq M e^{-\beta t}$ and $\|A^{3/4} e^{-At}\|_{L(X,X)} \leq M t^{-3/4} e^{-\beta t}$ hold for all $t > 0$.

Let $L_1 \geq 1$ be a constant such that

$$\epsilon_3(1/L_1) \leq \frac{\beta}{4},$$

$$\epsilon_3(1/L_1) M \left(\int_0^\infty u^{-3/4} e^{-\frac{\beta}{2}u} du \right) \leq \frac{1}{4}$$

and

$$\epsilon_3(1/L_1)(2+KM) M \left(\int_0^\infty u^{-3/4} e^{-\beta u} du \right) \leq \frac{1}{4}$$

hold, $D_1 > 0$ be a constant defined by

$$D_1 = \min \left\{ \frac{c_3}{2}, \frac{1}{2L_1}, \frac{1}{8C_3(2+KM)^2 M \left(\int_0^\infty u^{-3/4} e^{-\beta u} du\right)} \right\},$$

and $R_1 \geq L_1$ be a constant such that

$$\epsilon_3(1/R_1) M \left(\int_0^\infty u^{-3/4} e^{-\beta u} du \right) \leq \frac{D_1}{8}$$

holds.

Then, by Lemma 5.3 (1), the hypotheses of Theorem 6.1.4 of [10] for $(\mathrm{ES})_{l,r}$ are satisfied under the notations $Y = \mathbf{R}^2$, $\alpha = 3/4$, $U = \{(g,h) \in X' | \, \|(g,h)\|_{X'} < (2+KM)D_1\}$, $\lambda = \epsilon_3(1/L_1)$, $N = \frac{D_1}{2M\left(\int_0^\infty u^{-3/4} e^{-\beta u} du\right)}$, $M_1 = 1$, $M_2 = \frac{\beta}{4}$, $\mu = \frac{\beta}{4}$, $\Delta = 1$ and $D = D_1$. Hence, there exists an invariant manifold

$$S = \{(g,h,p,q) \in U \times \mathbf{R}^2 | \, (g,h) = \sigma(p,q)\}$$

for $(\mathrm{ES})_{l,r}$ with $\|\sigma(p,q)\|_{X'} \leq D_1$ and $\|\sigma(p_1,q_1)-\sigma(p_2,q_2)\|_{X'} \leq \|(p_1,q_1)-(p_2,q_2)\|_{\mathbf{R}^2}$ satisfying the following. If $(g,h,p,q)(t)$ is a solution of $(\mathrm{ES})_{l,r}$ on $0 \leq t < T$ with $\|(g,h)(t)\|_{X'} < (2+KM)D_1$, then

$$\|(g,h)(t)-\sigma((p,q)(t))\|_{X'} \leq KMe^{-\gamma_1 t}\|(g,h)(0)-\sigma((p,q)(0))\|_{X'}$$

holds on $0 \leq t < T$ with $\gamma_1 = \left(1-8\left(\epsilon_3(1/L_1)M\left(\int_0^\infty u^{-3/4}e^{-\frac{\beta}{2}u}du\right)\right)^4\right)\beta > \frac{1}{2}\beta$. Therefore, according to Proof of Corollary 6.1.5 of [10], with $\delta_1 = \frac{D_1}{2KM}$ and $K_1 = \left(1+\frac{\beta}{2\gamma_1-\beta}KM\right)KM$, we can see that for any $(g_0,h_0,p_0,q_0) \in X' \times \mathbf{R}^2$ with $\|(g_0,h_0)\|_{X'} \leq \delta_1$, there exist solutions $(g,h,p,q) \in C([0,+\infty),U \times \mathbf{R}^2)$ and $(\bar{g},\bar{h},\bar{p},\bar{q}) \in C([0,+\infty),S)$ of $(\mathrm{ES})_{l,r}$ such that

$$(g,h,p,q)(0) = (g_0,h_0,p_0,q_0)$$

and for any $t \geq 0$,

$$\|(g,h,p,q)(t)-(\bar{g},\bar{h},\bar{p},\bar{q})(t)\|_{X' \times \mathbf{R}^2} \leq K_1 e^{-\gamma_1 t}\|(g_0,h_0)-\sigma(p_0,q_0)\|_{X'}$$

hold. q.e.d.

Now, we prove Theorem 2.3.

Proof of Theorem 2.3.

Let D_1, δ_1, K_1, $\gamma_1 > 0$, L_1 and $R_1 \geq 1$ be constants satisfying Proposition 5.5, and l_1 and $r_1 \geq 1$ be constants such that Proposition 5.4 holds for $\Delta = 1$ and $d = \delta_1/2$.

We define positive constants D_0, l_0 and r_0 by $D_0 = d$, $l_0 = \max\{l_1, L_1\}$ and $r_0 = \max\{r_1, R_1\}$, respectively. Then, by Proposition 5.4, there exists an invariant manifold

$$S = \{(g,h,p,q) \in U \times \mathbf{R}^2 \,|\, (g,h) = \Sigma(p,q)\}$$

for $(\mathrm{ES})_{l_0,r_0}$ with $\|\Sigma(p,q)\|_{X'} \leq D_0$.

Also, by Proposition 5.5, there exists an invariant manifold

$$S' = \{(g,h,p,q) \in U \times \mathbf{R}^2 \,|\, (g,h) = \sigma(p,q)\}$$

for $(\mathrm{ES})_{l_0,r_0}$ with $\|\sigma(p,q)\|_{X'} \leq D_1$ and $\|\sigma(p_1,q_1)-\sigma(p_2,q_2)\|_{X'} \leq \|(p_1,q_1)-(p_2,q_2)\|_{\mathbf{R}^2}$ satisfying the following. For any $(g_0,h_0,p_0,q_0) \in X' \times \mathbf{R}^2$ with

$\|(g_0,h_0)\|_{X'} \le 2D_0$, there exist solutions $(g,h,p,q) \in C([0,+\infty), U\times\mathbf{R}^2)$ and $(\bar g,\bar h,\bar p,\bar q)\in C([0,+\infty),S')$ of $(\mathrm{ES})_{l_0,r_0}$ such that

$$(g,h,p,q)(0)=(g_0,h_0,p_0,q_0)$$

and for any $t\ge 0$,

$$\|(g,h,p,q)(t)-(\bar g,\bar h,\bar p,\bar q)(t)\|_{X'\times\mathbf{R}^2}\le K_1e^{-\gamma_1 t}\|(g_0,h_0)-\sigma(p_0,q_0)\|_{X'}$$

hold. Fix any point $(p_0,q_0)\in\mathbf{R}^2$, and let (p,q) denote the solution in S with $(p,q)(0)=(p_0,q_0)$. Then, because of $\|\Sigma(p,q)\|_{X'}\le 2D_0$, for any $t<0$, there exists $(\bar p_0,\bar q_0)\in\mathbf{R}^2$ such that

$$\|(\Sigma(p_0,q_0),p_0,q_0)-(\sigma(\bar p_0,\bar q_0),\bar p_0,\bar q_0)\|_{X'\times\mathbf{R}^2}\le (D_0+D_1)K_1e^{\gamma_1 t}$$

holds. Hence, because $\lim_{t\to-\infty}(\bar p_0,\bar q_0)=(p_0,q_0)$ and $\lim_{t\to-\infty}\sigma(\bar p_0,\bar q_0)=\Sigma(p_0,q_0)$ holds, we have $\Sigma(p_0,q_0)=\sigma(p_0,q_0)$. That is $S=S'$. Therefore, there uniquely exists an invariant manifold

$$S=\{(g,h,p,q)\in U\times\mathbf{R}^2|\,(g,h)=\Sigma(p,q)\,(=(\Sigma^1(p,q),\Sigma^2(p,q)))\}$$

for $(\mathrm{ES})_{l_0,r_0}$ with $\|\Sigma(p,q)\|_{X'}\le D_0$. From symmetry of $(\mathrm{ES})_{l_0,r_0}$ and uniqueness of S, $\Sigma^2(p,q)=\Sigma^1(q,p)$ holds.

Fix any $\varepsilon>0$. By Proposition 5.4, there exist $r_\varepsilon\ge r_0$ and an invariant manifold

$$S_\varepsilon=\{(g,h,p,q)\in U\times\mathbf{R}^2|\,(g,h)=\Sigma_\varepsilon(p,q)\}$$

for $(\mathrm{ES})_{l_0,r_\varepsilon}$ with $\|\Sigma_\varepsilon(p,q)\|_{X'}\le\min\{\varepsilon/2,2D_0\}$. Let p_0 and $q_0>l_0r_\varepsilon$ be sufficiently large. Then, there exists $(\bar p_0,\bar q_0)\in\mathbf{R}^2$ such that

$$\|(\Sigma_\varepsilon(p_0,q_0),p_0,q_0)-(\Sigma(\bar p_0,\bar q_0),\bar p_0,\bar q_0)\|_{X'\times\mathbf{R}^2}$$

$$\le(D_0+D_1)K_1e^{-\gamma_1\frac{\min\{p_0,q_0\}-l_0r_\varepsilon}{2\sup_{p>l_0r_\varepsilon,q>l_0r_\varepsilon}\|(c,c)+G_{l_0}[\Sigma_\varepsilon(p,q),p,q]\|_{\mathbf{R}^2}}}$$

holds. Hence, because p_0 and q_0 are sufficiently large, $\|(\Sigma_\varepsilon(p_0,q_0),p_0,q_0)-(\Sigma(\bar p_0,\bar q_0),\bar p_0,\bar q_0)\|_{X'\times\mathbf{R}^2}\le\varepsilon/8$ holds. We have $\|\Sigma(p_0,q_0)\|_{X'}\le\|\Sigma_\varepsilon(p_0,q_0)\|_{X'}$ $+\|\Sigma(\bar p_0,\bar q_0)-\Sigma_\varepsilon(p_0,q_0)\|_{X'}+\|\Sigma(p_0,q_0)-\Sigma(\bar p_0,\bar q_0)\|_{X'}\le\varepsilon/2+\varepsilon/8+\varepsilon/8<\varepsilon$. Therefore, we obtain $\lim_{(p,q)\to(+\infty,+\infty)}\|\Sigma(p,q)\|_{X'}=0$. Hence, by Lemma 5.3 (2), $\lim_{(p,q)\to(+\infty,+\infty)}(\|\tilde F(p,q)\|_X+\|\tilde G(p,q)\|_{\mathbf{R}^2})=0$ also holds. q.e.d.

6 Characterizing annihilation dynamics

In this section, we give the proof of Proposition 1.3. This section is independent of Sections 2, 3, 4 and 5.

Lemma 6.1 *For any $\delta_0 \in (0, \alpha)$, there exist P_1, D_1, k_1 and $\lambda_1 > 0$ such that for any $\overline{p}(t)$ and $\overline{\sigma}(t)$ satisfying*

$$\begin{cases} \dot{\overline{p}} = c - D_1(e^{-\lambda_1 \overline{p}} + \overline{\sigma}), & \overline{p} \geq P_1, \\ \dot{\overline{\sigma}} = -k_1 \overline{\sigma}, & 0 \leq \overline{\sigma} \leq \alpha - \delta_0, \end{cases} \tag{6.1}$$

the function $\overline{u}(x,t)$ defined by

$$\overline{u}(x,t) = \phi(|x| - \overline{p}(t)) + e^{-\lambda_1(|x| + \overline{p}(t))} + \overline{\sigma}(t) \tag{6.2}$$

is a super-solution of $u_t = u_{xx} + f(u)$ in $\mathbf{R}$.

Proof. [Step 1] In this step, we show the following.

There exist δ_1 and $k_1 > 0$ such that

$$f(a) - f(a+b) \geq k_1 b \tag{6.3}$$

holds for all $a \in [0, \delta_1] \cup [1 - \delta_1, 1]$ and $b \in [0, (\alpha - \delta_0) + \delta_1]$.

Take a constant $\delta > 0$ such that $\inf_{u \in [0,\delta] \cup [1-\delta, 1]}(-f_u(u)) > 0$ holds. Then, let $\delta_1 \in (0, \min\{\delta, \delta_0/4\}]$ be a constant satisfying $\inf_{u \in [0,\delta_1]} f(u) > \sup_{u \in [\delta, \alpha - \delta_0/2]} f(u)$.

Because of $\inf_{u \in [0,\delta]}(-f_u(u)) > 0$, $f(a) - f(a+b) > 0$ holds when $0 \leq a < a + b \leq \delta$ holds. Because of $\inf_{u \in [0,\delta_1]} f(u) > \sup_{u \in [\delta, \alpha - \delta_0/2]} f(u)$, $f(a) - f(a+b) > 0$ holds when $0 \leq a \leq \delta_1$ and $\delta \leq a + b$ hold. Because of $\inf_{u \in [1-\delta, 1]}(-f_u(u)) > 0$, $f(a) - f(a+b) > 0$ holds when $1 - \delta_1 \leq a < a + b \leq 1$ holds. Because of $f(a) \geq 0$ and $f(a+b) < 0$, $f(a) - f(a+b) > 0$ holds when $1 - \delta_1 \leq a \leq 1$ and $1 < a + b$ hold.

Therefore, $\frac{f(a) - f(a+b)}{b} > 0$ holds when $b > 0$ holds. Because $-f_u(a) > 0$ also holds, we have

$$k_1 := \inf_{a \in [0,\delta_1] \cup [1-\delta_1, 1],\ b \in (0, (\alpha - \delta_0) + \delta_1]} \frac{f(a) - f(a+b)}{b} > 0.$$

Then, (6.3) holds.

[Step 2] In this step, we prove Lemma 6.1.

Let $\lambda_1 > 0$ be a constant satisfying

$$\lambda_1(\lambda_1 + c) \leq k_1 \tag{6.4}$$

and

$$\lim_{p \to +\infty} \phi'(-p)e^{\lambda_1 p} = 0.$$

Let $P_1 > 0$ be a constant such that

$$\phi'(-p) \leq \lambda_1 e^{-\lambda_1 p} \tag{6.5}$$

and

$$e^{-\lambda_1 p} \leq \delta_1 \tag{6.6}$$

hold for all $p \geq P_1$. Let $L_1 > 0$ be a constant satisfying

$$\phi(z) \in [0, \delta_1] \cup [1 - \delta_1, 1] \tag{6.7}$$

for $|z| \geq L_1$, and $D_1 > 0$ be defined by

$$D_1 = \frac{\sup_{0 \leq u \leq 1+\alpha-\delta_0+\delta_1} f_u(u) + k_1}{\inf_{|z| \leq L_1} \phi'(z)}. \tag{6.8}$$

Then, on $x > 0$, because

$$\begin{aligned} &-\overline{u}_{xx} + \overline{u}_t - f(\overline{u}) \\ &= -\phi'' - \dot{\overline{p}}\phi' - f(\overline{u}) - \lambda_1(\lambda_1 + \dot{\overline{p}})e^{-\lambda_1(x+\overline{p})} + \dot{\overline{\sigma}} \\ &= (c - \dot{\overline{p}})\phi' + f(\phi) - f(\overline{u}) - \lambda_1(\lambda_1 + \dot{\overline{p}})e^{-\lambda_1(x+\overline{p})} + \dot{\overline{\sigma}} \end{aligned}$$

holds from (6.2), by (6.1) and (6.4), we get

$$\begin{aligned} &-\overline{u}_{xx} + \overline{u}_t - f(\overline{u}) \\ &\geq f(\phi) - f(\overline{u}) - k_1(e^{-\lambda_1(x+\overline{p})} + \overline{\sigma}). \end{aligned}$$

Hence, by (6.1), (6.2), (6.3), (6.6) and (6.7),

$$-\overline{u}_{xx} + \overline{u}_t - f(\overline{u}) \geq 0 \tag{6.9}$$

holds in $x \in \{x \in \mathbf{R} | \, x > 0, |x - \overline{p}| \geq L_1\}$.

On $x > 0$, because

$$-\overline{u}_{xx} + \overline{u}_t - f(\overline{u})$$
$$= (c - \dot{\overline{p}})\phi' + f(\phi) - f(\overline{u}) - \lambda_1(\lambda_1 + \dot{\overline{p}})e^{-\lambda_1(x+\overline{p})} + \dot{\overline{\sigma}}$$

holds, by (6.1) and (6.4), we also get

$$-\overline{u}_{xx} + \overline{u}_t - f(\overline{u})$$
$$\geq D_1\phi'(e^{-\lambda_1\overline{p}} + \overline{\sigma}) + f(\phi) - f(\overline{u}) - k_1(e^{-\lambda_1(x+\overline{p})} + \overline{\sigma}).$$

Hence, because

$$-\overline{u}_{xx} + \overline{u}_t - f(\overline{u})$$
$$\geq D_1\phi'(e^{-\lambda_1\overline{p}} + \overline{\sigma}) - \left(\sup_{0\leq u\leq 1+\alpha-\delta_0+\delta_1} f_u(u) + k_1\right)(e^{-\lambda_1(x+\overline{p})} + \overline{\sigma})$$

holds from (6.1), (6.2) and (6.6), by (6.8),

$$-\overline{u}_{xx} + \overline{u}_t - f(\overline{u}) \geq 0 \tag{6.10}$$

holds in $x \in \{x \in \mathbf{R} | \, x > 0, |x - \overline{p}| \leq L_1\}$.

Also, we have

$$\lim_{x\downarrow+0} \frac{\overline{u}(x,t) - \overline{u}(0,t)}{x} = \phi'(-\overline{p}) - \lambda_1 e^{-\lambda_1\overline{p}} \leq 0$$

by (6.1) and (6.5). Hence, by (6.9) and (6.10), $\overline{u}(x,t)$ is a super-solution of $u_t = u_{xx} + f(u)$ in $\mathbf{R}$. q.e.d.

Lemma 6.2 *For any $\delta_0 \in (0, 1-\alpha)$, there exist D_2 and $k_2 > 0$ such that for any $\underline{p}(t)$ and $\underline{\sigma}(t)$ satisfying*

$$\begin{cases} \dot{\underline{p}} = c + D_2\underline{\sigma}, \\ \dot{\underline{\sigma}} = -k_2\underline{\sigma}, \qquad 0 \leq \underline{\sigma} \leq (1-\alpha) - \delta_0, \end{cases} \tag{6.11}$$

the function $\underline{u}(x,t)$ defined by

$$\underline{u}(x,t) = \phi(|x| - \underline{p}(t)) - \underline{\sigma}(t) \tag{6.12}$$

is a sub-solution of $u_t = u_{xx} + f(u)$ in $\mathbf{R}$.

Proof. [Step 1] In this step, we show the following.

There exist δ_2 *and* $k_2 > 0$ *such that*

$$f(a-b) - f(a) \geq k_2 b \tag{6.13}$$

holds for all $a \in [0, \delta_2] \cup [1-\delta_2, 1]$ *and* $b \in [0, (1-\alpha) - \delta_0]$.

Take a constant $\delta > 0$ such that $\inf_{u \in [0,\delta] \cup [1-\delta,1]}(-f_u(u)) > 0$ holds. Then, let $\delta_2 \in (0, \min\{\delta, \delta_0/2\}]$ be a constant satisfying $\sup_{u \in [1-\delta_2,1]} f(u) < \inf_{u \in [\alpha+\delta_0/2, 1-\delta]} f(u)$.

Because of $\inf_{u \in [1-\delta,1]}(-f_u(u)) > 0$, $f(a-b) - f(a) > 0$ holds when $1 - \delta \leq a - b < a \leq 1$ holds. Because of $\sup_{u \in [1-\delta_2,1]} f(u) < \inf_{u \in [\alpha+\delta_0/2, 1-\delta]} f(u)$, $f(a-b) - f(a) > 0$ holds when $1 - \delta_2 \leq a \leq 1$ and $a - b \leq 1 - \delta$ hold. Because of $\inf_{u \in [0,\delta]}(-f_u(u)) > 0$, $f(a-b) - f(a) > 0$ holds when $0 \leq a - b < a \leq \delta$ holds. Because of $f(a) \leq 0$ and $f(a-b) > 0$, $f(a-b) - f(a) > 0$ holds when $0 \leq a \leq \delta_2$ and $a - b < 0$ hold.

Therefore, $\frac{f(a-b)-f(a)}{b} > 0$ holds when $b > 0$ holds. Because $-f_u(a) > 0$ also holds, we have

$$k_2 := \inf_{a \in [0,\delta_2] \cup [1-\delta_2,1],\ b \in (0,(1-\alpha)-\delta_0]} \frac{f(a-b) - f(a)}{b} > 0.$$

Then, (6.13) holds.

[Step 2] In this step, we prove Lemma 6.2.

Let $L_2 > 0$ be a constant satisfying

$$\phi(z) \in [0, \delta_2] \cup [1-\delta_2, 1] \tag{6.14}$$

for $|z| \geq L_2$, and $D_2 > 0$ be defined by

$$D_2 = \frac{\sup_{-(1-\alpha)+\delta_0 \leq u \leq 1} f_u(u) + k_2}{\inf_{|z| \leq L_2} \phi'(z)}. \tag{6.15}$$

Then, on $x > 0$, because

$$\begin{aligned} &\underline{u}_{xx} - \underline{u}_t + f(\underline{u}) \\ &= \phi'' + \dot{\underline{p}}\phi' + f(\underline{u}) + \dot{\underline{\sigma}} \\ &= (\dot{\underline{p}} - c)\phi' + f(\underline{u}) - f(\phi) + \dot{\underline{\sigma}} \end{aligned}$$

holds from (6.12), by (6.11), we get

$$\begin{aligned} &\underline{u}_{xx} - \underline{u}_t + f(\underline{u}) \\ &\geq f(\underline{u}) - f(\phi) - k_2\underline{\sigma}. \end{aligned}$$

Hence, by (6.11), (6.12), (6.13) and (6.14),

$$\underline{u}_{xx} - \underline{u}_t + f(\underline{u}) \geq 0 \tag{6.16}$$

holds in $x \in \{x \in \mathbf{R} | \, x > 0, |x - \underline{p}| \geq L_2\}$.

On $x > 0$, because

$$\begin{aligned} &\underline{u}_{xx} - \underline{u}_t + f(\underline{u}) \\ &= (\dot{\underline{p}} - c)\phi' + f(\underline{u}) - f(\phi) + \dot{\underline{\sigma}} \end{aligned}$$

holds, by (6.11), we get

$$\begin{aligned} &\underline{u}_{xx} - \underline{u}_t + f(\underline{u}) \\ &\geq D_2\phi'\underline{\sigma} + f(\underline{u}) - f(\phi) - k_2\underline{\sigma}. \end{aligned}$$

Hence, because we have

$$\begin{aligned} &\underline{u}_{xx} - \underline{u}_t + f(\underline{u}) \\ &\geq D_2\phi'\underline{\sigma} - \left(\sup_{-(1-\alpha)+\delta_0 \leq u \leq 1} f_u(u) + k_2 \right) \underline{\sigma} \end{aligned}$$

from (6.11) and (6.12), by (6.15),

$$\underline{u}_{xx} - \underline{u}_t + f(\underline{u}) \geq 0 \tag{6.17}$$

holds in $x \in \{x \in \mathbf{R} | \, x > 0, |x - \underline{p}| \leq L_2\}$.

Also, we have

$$\lim_{x \downarrow +0} \frac{\underline{u}(x,t) - \underline{u}(0,t)}{x} = \phi'(-\underline{p}) \geq 0$$

from (6.11). Hence, by (6.16) and (6.17), $\underline{u}(x,t)$ is a sub-solution of $u_t = u_{xx} + f(u)$ in $\mathbf{R}$. q.e.d.

Definition 10 *For $L > 0$, a closed subset Λ_L of $BU(\mathbf{R})$ is defined by*

$$\Lambda_L = \{u \in BU(\mathbf{R}) | \, \phi(x - L) \leq u(x) \leq \phi(x + L)\}.$$

We obtain the following by Lemmas 6.1 and 6.2.

Lemma 6.3 *Let $\bar{\delta}_0 \in (0, \min\{\alpha, 1-\alpha\})$ and $\bar{L}_0 > 0$ be given. Then, there exists $L > 0$ satisfying the following.*

For any $\varepsilon' > 0$, there exists $\bar{l}_0 > 0$ such that for any $l \geq \bar{l}_0$ and $u_0 \in \Xi_{l,\bar{\delta}_0,\bar{L}_0}$, there exists a solution $u \in C([0,+\infty), BU(\mathbf{R}))$ of $u_t = u_{xx} + f(u)$ with $u(0) = u_0$ such that

$$\inf_{v_0 \in \Lambda_L} \left\| \chi(x+1) u(x, \bar{l}_0^{1/2}) - v_0(x - (l + c\bar{l}_0^{1/2})) \right\|_{C^0(\mathbf{R})} < \varepsilon'$$

and

$$\sup_{(x,t) \in [-1,1] \times [\bar{l}_0^{1/2}, 2\bar{l}_0^{1/2}]} (|u(x,t)| + |u_x(x,t)|) < \varepsilon'$$

hold.

Proof. We take constants $y_1 > y_2$ such that $\phi(y_1) = 1 - \alpha + \bar{\delta}_0$ and $\phi(y_2) = 1 - \alpha - \bar{\delta}_0$ hold. Let $L > 0$ be defined by $L = \max\{\bar{L}_0 + y_1 + \frac{D_1}{k_1}(\alpha - \bar{\delta}_0) + 1, \bar{L}_0 - y_2 + \frac{D_2}{k_2}(1 - \alpha - \bar{\delta}_0)\}$.

Then,

$$\begin{aligned} \phi(|x| - (l + \bar{L}_0 - y_2)) - (1 - \alpha - \bar{\delta}_0) &\leq u_0(x) \\ &\leq \phi(|x| - (l - \bar{L}_0 - y_1)) + (\alpha - \bar{\delta}_0) \end{aligned} \tag{6.18}$$

holds for all $u_0 \in \Xi_{l,\bar{\delta}_0,\bar{L}_0}$. Let $(\overline{p}, \overline{\sigma})$ and $(\underline{p}, \underline{\sigma}) \in C^1([0, 2\bar{l}_0^{1/2}], \mathbf{R}^2)$ be solutions of

$$\begin{cases} \dot{\overline{p}} = c - D_1(e^{-\lambda_1 \overline{p}} + \overline{\sigma}), & \overline{p}(0) = l - \bar{L}_0 - y_1, \\ \dot{\overline{\sigma}} = -k_1 \overline{\sigma}, & \overline{\sigma}(0) = \alpha - \bar{\delta}_0 \end{cases}$$

and

$$\begin{cases} \dot{\underline{p}} = c + D_2 \underline{\sigma}, & \underline{p}(0) = l + \bar{L}_0 - y_2, \\ \dot{\underline{\sigma}} = -k_2 \underline{\sigma}, & \underline{\sigma}(0) = 1 - \alpha - \bar{\delta}_0, \end{cases}$$

respectively. If $\bar{l}_0 > 0$ is sufficiently large, then

$$\overline{p}(t) \geq l/2,$$

$$\overline{p}(t) \geq l + ct - \left(\bar{L}_0 + y_1 + \frac{D_1}{k_1}(\alpha - \bar{\delta}_0) + 1 \right)$$

and

$$\underline{p}(t) \leq l + ct + \left(\bar{L}_0 - y_2 + \frac{D_2}{k_2}(1 - \alpha - \bar{\delta}_0) \right)$$

hold on $t \in [0, 2\bar{l}_0^{1/2}]$. Hence, by Lemmas 6.1, 6.2 and (6.18), when $\bar{l}_0 > 0$ is sufficiently large,

$$\phi(|x| - (l + ct + L)) - (1 - \alpha - \bar{\delta}_0)e^{-k_2 t} \le u(x,t)$$

$$\le \phi(|x| - (l + ct - L)) + e^{-\frac{\lambda_1}{2}l} + (\alpha - \bar{\delta}_0)e^{-k_1 t}$$

holds on $t \in [0, 2\bar{l}_0^{1/2}]$ for all $u_0 \in \Xi_{l,\bar{\delta}_0,\bar{L}_0}$. Therefore, for any $\varepsilon > 0$, if $\bar{l}_0 > 0$ is sufficiently large, then

$$\phi(|x| - (l + c\bar{l}_0^{1/2} + L)) - \varepsilon \le u(x, \bar{l}_0^{1/2})$$

$$\le \phi(|x| - (l + c\bar{l}_0^{1/2} - L)) + \varepsilon$$

and

$$\sup_{(x,t)\in[-2,2]\times[\bar{l}_0^{1/2}-1, 2\bar{l}_0^{1/2}]} |u(x,t)| < \varepsilon$$

hold for all $u_0 \in \Xi_{l,\bar{\delta}_0,\bar{L}_0}$. Hence, by (1.4) of [6], there exists $\bar{l}_0 > 0$ such that

$$\inf_{v_0\in\Lambda_L} \left\| \chi(x+1)u(x,\bar{l}_0^{1/2}) - v_0(x - (l + c\bar{l}_0^{1/2})) \right\|_{C^0(\mathbf{R})} < \varepsilon'$$

and

$$\sup_{(x,t)\in[-1,1]\times[\bar{l}_0^{1/2}, 2\bar{l}_0^{1/2}]} (|u(x,t)| + |u_x(x,t)|) < \varepsilon'$$

hold for all $u_0 \in \Xi_{l,\bar{\delta}_0,\bar{L}_0}$. q.e.d.

We obtain the following by Theorems A and B.

Lemma 6.4 *For any L and $\varepsilon > 0$, there exist T' and $\delta > 0$ such that for any $g \in C([0,T'], BU(\mathbf{R}))$ with $\sup_{t\in[0,T']} \|g(t)\|_{C^0} < \delta$ and solution $v \in C([0,T'], BU(\mathbf{R}))$ of $v_t = v_{xx} + f(v) + g(t)$ with $\inf_{u_0\in\Lambda_L} \|v(0) - u_0\|_{C^0} < \delta$,*

$$\inf_{x_0\in[-L,L]} \|v(x,T') - \phi(x - x_0 - cT')\|_Y < \varepsilon/2$$

holds.

Proof. [Step 1] In this step, we show the following.

For any L and $\varepsilon > 0$, there exists $T' > 0$ satisfying the following. For any $u_0 \in \Lambda_L$, there exists a solution $u \in C([0, +\infty), BU(\mathbf{R}))$ of $u_t = u_{xx} + f(u)$ with $u(0) = u_0$ such that

$$\inf_{x_0 \in [-L, L]} \|u(x, T') - \phi(x - x_0 - cT')\|_Y < \varepsilon/4$$

holds.

Fix any $\delta > 0$. By Theorem B, for any $u_0 \in \Lambda_L$, there exists $\bar{T}_{u_0} > 0$ such that

$$\inf_{x_0 \in \mathbf{R}} \|u(x, \bar{T}_{u_0}) - \phi(x - x_0)\|_Y < \delta/2$$

holds. Hence, for any $v_0 \in \Lambda_L$, there exists a open neighborhood W_{v_0} of v_0 such that

$$\inf_{x_0 \in \mathbf{R}} \|u(x, \bar{T}_{v_0}) - \phi(x - x_0)\|_Y < \delta$$

holds for all $u_0 \in W_{v_0}$. Because Λ_L is compact in Y, there exist v_1, v_2, $\cdots$ and $v_n \in \Lambda_L$ such that $\Lambda_L \subset \bigcup_{i=1}^n W_{v_i}$ holds. Therefore, for any $u_0 \in \Lambda_L$, there exists $i \in \{1, 2, \cdots, n\}$ such that

$$\inf_{x_0 \in \mathbf{R}} \|u(x, \bar{T}_{v_i}) - \phi(x - x_0)\|_Y < \delta$$

holds. Hence, because $\delta > 0$ is any constant, by Theorem A (2), with $T' = \max_{i=1,\cdots,n} \bar{T}_{v_i} > 0$,

$$\inf_{x_0 \in \mathbf{R}} \|u(x, T') - \phi(x - x_0)\|_Y < \varepsilon/4$$

holds for all $u_0 \in \Lambda_L$. Because $\phi(x - L - cT') \le u(x, T') \le \phi(x + L - cT')$ holds for all $u_0 \in \Lambda_L$, we have

$$\inf_{x_0 \in [-L, L]} \|u(x, T') - \phi(x - x_0 - cT')\|_Y < \varepsilon/4.$$

[Step 2] In this step, we prove Lemma 6.4.

Because $(v - u)_t + B(v - u) = -f_u(1)(v - u) + f(v) - f(u) + g(t)$ holds,

$$\|v(t) - u(t)\|_Y$$

$$\le \sup_{0 \le s \le T'} \|e^{-Bs}\|_{L(Y,Y)}$$

$$\left((1+T')\delta + 2\left(\sup_{-\delta\le u\le 1+\delta}|f_u(u)|\right)\int_0^t \|v(s)-u(s)\|_Y ds\right)$$

holds on $t\in[0,T']$. Hence, by Gronwall's inequlity, there exists $\delta>0$ such that

$$\|v(x,T')-u(x,T')\|_Y < \varepsilon/4$$

holds. By Step 1, we get

$$\inf_{x_0\in[-L,L]}\|v(x,T')-\phi(x-x_0-cT')\|_Y < \varepsilon/2.$$

q.e.d.

Now, we prove Proposition 1.3.

Proof of Proposition 1.3.

Let $L>0$ be a constant such that Lemma 6.3 holds for $\bar\delta_0$ and $\bar L_0$, and T' and $\delta>0$ be constants such that Lemma 6.4 holds for L and ε. Then, let $\bar l_0>0$ be sufficiently large.

Fix any $l\ge\bar l_0$ and $u_0\in\Xi_{l,\bar\delta_0,\bar L_0}$. We define $v\in C([0,+\infty),BU(\mathbf{R}))$ and $g\in C((0,+\infty),BU(\mathbf{R}))$ by

$$v(x,t)=\chi(x+1)u(x,t)$$

and

$$\begin{aligned} &g(x,t)\\ &=\chi(x+1)u_{xx}(x,t)-(\chi(x+1)u(x,t))_{xx}\\ &+\chi(x+1)f(u(x,t))-f(\chi(x+1)u(x,t)), \end{aligned}$$

respectively. Then, $v_t=v_{xx}+f(v)+g(t)$ holds on $t\in(0,+\infty)$. Further, by Lemma 6.3, we have $\inf_{u_0\in\Lambda_L}\left\|v(x+(l+c\bar l_0^{1/2}),\bar l_0^{1/2})-u_0(x)\right\|_{C^0(\mathbf{R})}<\delta$ and $\sup_{t\in[\bar l_0^{1/2},\bar l_0^{1/2}+T']}\|g(x,t)\|_{C^0(\mathbf{R})}<\delta$. Therefore, by Lemma 6.4, with $T=\bar l_0^{1/2}+T'$, there exists $x_1\in[l-L,l+L]$ such that

$$\|\chi(x+1)u(x,T)-\phi(x-x_1-cT)\|_{C^0(\mathbf{R})}<\varepsilon/2$$

holds. Also, there exists $x_2\in[l-L,l+L]$ such that

$$\|\chi(x+1)u(-x,T)-\phi(x-x_2-cT)\|_{C^0(\mathbf{R})}<\varepsilon/2$$

holds. Because $\bar{l}_0 > 0$ is sufficiently large, $x_1 + cT$ and $x_2 + cT > 0$ are sufficiently large. Therefore,

$$\|u(x,T) - (\phi(x - x_1 - cT) + \phi(-x - x_2 - cT))\|_{C^0(\mathbf{R})} < \varepsilon$$

holds. q.e.d.

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