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Computation of competitive equilibria of indivisible commodities via M-convex submodular flow problem †

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Abstract

This paper considers an economic model in which producers and consumers trade various indivisible commodities through a perfectly divisible commodity, money. On the basis of the recent developments in discrete mathematics (combinatorial optimization), we give an efficient algorithm to decide whether a competitive equilibrium exists or not, when cost functions of the producers are $M^\natural$-convex and utility functions of the consumers are $M^\natural$-concave and quasilinear in money, where $M^\natural$-convexity is closely related to the gross substitutes condition.

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1 Introduction

This paper addresses a computational aspect of competitive equilibria in an economy with indivisible commodities by fully utilizing the recent developments of "discrete convex analysis" in discrete mathematics. The key role is played by the concept of $\mathrm{M}^\natural$-concave functions[1], where $\mathrm{M}^\natural$-concavity is closely related to the gross substitutes condition proposed by Kelso and Crawford (1982) and investigated by Gul and Stacchetti (1999).

In mathematical economics, models with indivisible commodities have been investigated and theorems on the existence of equilibria have been established by many authors. Henry (1970) studied a model in which all the indivisible commodities are the same. Shapley and Shubik (1972), Shapley and Scarf (1974), Kaneko (1982), Quinzii (1984) and Gale (1984) proved the existence of equilibria in economies in which each agent initially owns one indivisible commodity and consumes at most one commodity. Kaneko (1982) and Kaneko and Yamamoto (1986) showed the existence of an equilibrium in a generalized assignment market in which each seller owns a number of units of an indivisible commodity of the same type and each buyer wants to buy no more than one indivisible commodity. Recently, Bikhchandani and Mamer (1997), van der Laan, Talman and Yang (1997), Beviá, Quinzii and Silva (1999), Gul and Stacchetti (1999) and Yang (2000) dealt with generalized models in which all the indivisible commodities are differentiated and agents can use several indivisible commodities. Danilov, Koshevoy and Murota (2001) considered a framework consistent with "discrete convex analysis." In the model, each agent can produce and consume several units of the same indivisible commodity. Their results imply the existence of an equilibrium in an exchange economy with indivisible commodities if the utility function of each agent is quasilinear in money and $\mathrm{M}^\natural$-concave (see also Section 3).

Algorithms for computing an equilibrium abound in the literature. A standard approach is to use fixed-point algorithms (e.g., see Scarf, 1982 and 1989). Kaneko and Yamamoto (1986) numerically calculated equilibria in housing markets by a fixed-point algorithm. Fixed-point algorithms are not sufficient for models with indivisible commodities in that they usually fail to find an integer-valued solution. Kelso and Crawford (1982) proposed an algorithm for a two-sided many-to-one matching model between firms and workers. The algorithm is guaranteed to terminate in finite time, but not in polynomial time. Bikhchandani and Mamer (1997) formulated an integer program for an exchange economy and proved that there exists an equilibrium if and only if the integer program and its linear programming relaxation have the same optimal solution. While an equilibrium can be found by

[1]The symbol $\natural$ in $\mathrm{M}^\natural$ should be read "natural."

solving the relaxation, the size of the relaxation is exponential in the number of commodities.

In discrete mathematics, "discrete convex analysis" was recently proposed by Murota (1996, 1998, 2001) as a unified framework of discrete optimization with reference to existing studies on submodular functions (e.g., Edmonds, 1970, Lovász, 1983, Fujishige, 1991), generalized polymatroids (e.g., Frank and Tardos, 1988, Fujishige, 1991), valuated matroids (e.g., Dress and Wenzel, 1992, Murota, 2000) and convex analysis (e.g., Rockafellar, 1970). The concepts of M-convex and $\mathrm{M}^\natural$-convex functions play a central role in discrete convex analysis and have been investigated by Murota (1996, 1998, 1999), Fujishige and Murota (2000), Fujishige and Yang (2000), Murota and Shioura (1999, 2000, 2001), Murota and Tamura (2001) and so on. Fujishige and Yang first pointed out the equivalence between $\mathrm{M}^\natural$-concavity and the gross substitutes condition for set functions, and Murota and Tamura generalized their result (see Section 3).

In this paper, we deal with an economic model in which producers and consumers trade various indivisible commodities through a perfectly divisible commodity, money. The producers have $\mathrm{M}^\natural$-convex cost functions and the consumers have $\mathrm{M}^\natural$-concave utility functions quasilinear in money. Our contribution is to propose an efficient algorithm for finding an equilibrium. The algorithm consists of two phases: the first phase computes productions and consumptions in an equilibrium by solving an M-convex submodular flow problem and the second phase finds an equilibrium price vector by solving a shortest path problem. Both the smallest and the largest equilibrium price vectors can be computed.

This paper is organized as follows. Section 2 explains our model. Section 3 reviews the definitions of M-convex and $\mathrm{M}^\natural$-convex functions and the M-convex submodular flow problem, and the relationship between $\mathrm{M}^\natural$-concavity and several properties relevant to mathematical economics. Section 4 is the main part which describes our algorithms.

2 The Model

First we introduce several notations. We denote by $\mathbf{R}$, $\mathbf{R}_+$, $\mathbf{Z}$ and $\mathbf{Z}_+$ the sets of all reals, all nonnegative reals, all integers and all nonnegative integers, respectively. Let K be a finite set. For a vector $q \in \mathbf{R}^K$ and a function $f : \mathbf{Z}^K \to \mathbf{R} \cup \{\pm\infty\}$, we define functions $\langle q, x\rangle$ and $f[q](x)$ by

$$\langle q, x\rangle = \sum_{k\in K} q(k)x(k) \quad \text{and} \quad f[q](x) = f(x) + \langle q, x\rangle \qquad (x \in \mathbf{Z}^K),$$

where $q(k)$ and $x(k)$ denote the k-th components of q and x, respectively, and define the sets of minimizers and maximizers of f and the effective domain of f by

$$\begin{aligned} \arg\min f &= \{x \in \mathbf{Z}^K \mid f(x) \leq f(y) \quad (\forall y \in \mathbf{Z}^K)\}, \\ \arg\max f &= \{x \in \mathbf{Z}^K \mid f(x) \geq f(y) \quad (\forall y \in \mathbf{Z}^K)\}, \\ \operatorname{dom} f &= \{x \in \mathbf{Z}^K \mid -\infty < f(x) < +\infty\}. \end{aligned}$$

For each $x \in \operatorname{dom} f$, the set defined by

$$\partial_{\mathbf{R}} f(x) = \{p \in \mathbf{R}^K \mid f(y) - f(x) \geq \langle p, y - x\rangle \quad (\forall y \in \mathbf{Z}^K)\}$$

is called the *subdifferential* of f at x, and an element p of $\partial_{\mathbf{R}} f(x)$ is called a *subgradient* of f at x. From the above definitions, we see

$$p \in \partial_{\mathbf{R}} f(x) \iff x \in \arg\min f[-p]. \tag{1}$$

We also define $\partial'_{\mathbf{R}}$ by $\partial'_{\mathbf{R}} f(x) = -\partial_{\mathbf{R}}(-f)(x)$. That is, $\partial'_{\mathbf{R}}$ denotes the concave version of $\partial_{\mathbf{R}}$.

The present paper studies an economy with a finite set L of producers, a finite set H of consumers, a finite set K of indivisible commodities and a perfectly divisible commodity, namely money. *Productions* of producers and *consumptions* of consumers are integer-valued vectors in $\mathbf{Z}^K$ representing the numbers of indivisible commodities that they consume and produce. Here producers' inputs are represented by negative numbers and their outputs by positive numbers, and conversely, consumers' inputs are represented by positive numbers and their outputs by negative numbers. In the model, for a given price vector $p = (p(k) : k \in K) \in \mathbf{R}^K$ of commodities, each producer l independently schedules a production in order to maximize l's profit, and each consumer h independently schedules a consumption to maximize h's utility under h's budget constraint, and all agents exchange commodities by buying or selling those through money.

We assume that producer l's profit is described by his/her *cost function* $C_l : \mathbf{Z}^K \to \mathbf{R} \cup \{+\infty\}$ whose value is expressed in units of money. That is, l's *profit function* $\pi_l : \mathbf{R}^K \to \mathbf{R}$ is defined by

$$\pi_l(p) = \max_{y \in \mathbf{Z}^K} \{\langle p, y\rangle - C_l(y)\} \qquad (p \in \mathbf{R}^K).$$

Producer l's *supply function (correspondence)* $S_l : \mathbf{R}^K \to 2^{\mathbf{Z}^K}$ represents the set of all productions which attain the maximum of l's profit for a given price vector, that is,

$$S_l(p) = \arg\max_{y \in \mathbf{Z}^K} \{\langle p, y\rangle - C_l(y)\} \qquad (p \in \mathbf{R}^K).$$

Each consumer $h \in H$ has an initial endowment of indivisible commodities and money which is represented by a vector $(x_h^\circ, m_h^\circ) \in \mathbf{Z}_+^K \times \mathbf{R}_+$, where $x_h^\circ(k)$ denotes the number of commodities $k \in K$ and m_h° the amount of money in his/her initial endowment. In the model, each consumer h shares in the profits of the producers and θ_{lh} denotes the share of the profit of producer l owned by consumer h. The numbers θ_{lh} are nonnegative and $\sum_{h \in H} \theta_{lh} = 1$ for all $l \in L$. Thus, consumer h gains an income which is expressed by a function $\beta_h : \mathbf{R}^K \to \mathbf{R}$ defined by

$$\beta_h(p) = \langle p, x_h^\circ \rangle + m_h^\circ + \sum_{l \in L} \theta_{lh} \pi_l(p) \qquad (p \in \mathbf{R}^K).$$

We assume that each consumer's utility is quasilinear in money. That is, consumer h's utility is represented by a quasilinear utility function $\bar{U}_h : \mathbf{Z}^K \times \mathbf{R} \to \mathbf{R} \cup \{-\infty\}$ defined by

$$\bar{U}_h(x, m) = U_h(x) + m \qquad ((x, m) \in \mathbf{Z}^K \times \mathbf{R})$$

where $U_h : \mathbf{Z}^K \to \mathbf{R} \cup \{-\infty\}$ whose value is expressed in units of money. It is natural to assume that $\operatorname{dom} U_h$ is bounded because none can consume an infinite number of indivisible commodities. We further assume that the amount of money m_h° in h's initial endowment is sufficiently large for any $h \in H$. Since consumer h's schedule maximizes $\bar{U}_h$ under the budget constraint, h's behavior is formulated in terms of an optimization problem

$$\begin{array}{ll} \text{Maximize} & U_h(x) + m \\ \text{subject to} & \langle p, x \rangle + m \leq \beta_h(p). \end{array}$$

Since $\operatorname{dom} U_h$ is bounded and m_h° is large, we can take $m = \beta_h(p) - \langle p, x \rangle$ to reduce the above problem to an unconstrained optimization problem

$$\text{Maximize} \quad U_h(x) - \langle p, x \rangle.$$

Thus, we can define h's *demand function (correspondence)* $D_h : \mathbf{R}^K \to 2^{\mathbf{Z}^K}$ by

$$D_h(p) = \arg\max_{x \in \mathbf{Z}^K} \{U_h(x) - \langle p, x \rangle\} \qquad (p \in \mathbf{R}^K).$$

A tuple $((x_h \mid h \in H), (y_l \mid l \in L), p)$, where $x_h \in \mathbf{Z}^K$, $y_l \in \mathbf{Z}^K$ and $p \in \mathbf{R}^K$, is called an *equilibrium* or a *competitive equilibrium* if the following conditions hold:

$$x_h \in D_h(p) \qquad (h \in H), \tag{2}$$

$$y_l \in S_l(p) \qquad (l \in L), \tag{3}$$

$$\sum_{h \in H} x_h = \sum_{h \in H} x_h^\circ + \sum_{l \in L} y_l, \tag{4}$$

$$p \geq 0. \tag{5}$$

That is, each agent achieves what he/she wishes to achieve, the balance of supply and demand holds and an equilibrium price vector is nonnegative. In order to

characterize equilibria, we adopt the aggregate cost function $\Psi : \mathbf{Z}^K \to \mathbf{R} \cup \{\pm\infty\}$ of the market defined by

$$\Psi(z) = \inf \left\{ \sum_{l \in L} C_l(y_l) - \sum_{h \in H} U_h(x_h) \mid \sum_{h \in H} x_h - \sum_{l \in L} y_l = z \right\} \qquad (z \in \mathbf{Z}^K). \tag{6}$$

Then we obtain the following characterization of equilibria.

Lemma 2.1 *Given an initial total endowment $x^\circ = \sum_{h \in H} x_h^\circ$, the following statements hold.*

(a) *There exists an equilibrium if and only if $(-\partial_{\mathbf{R}} \Psi(x^\circ)) \cap \mathbf{R}_+^K \neq \emptyset$.*

(b) *A price vector $p \in \mathbf{R}^K$ is an equilibrium price vector if and only if $p \in (-\partial_{\mathbf{R}} \Psi(x^\circ)) \cap \mathbf{R}_+^K$.*

(c) *If $((x_h \mid h \in H), (y_l \mid l \in L))$ satisfies (2), (3) and (4) for some p (which may fail to be nonnegative), then*

$$-\partial_{\mathbf{R}} \Psi(x^\circ) = \left(\bigcap_{h \in H} \partial'_{\mathbf{R}} U_h(x_h) \right) \cap \left(\bigcap_{l \in L} \partial_{\mathbf{R}} C_l(y_l) \right),$$

and if, in addition, $(-\partial_{\mathbf{R}} \Psi(x^\circ)) \cap \mathbf{R}_+^K \neq \emptyset$, then $((x_h \mid h \in H), (y_l \mid l \in L), p')$ is an equilibrium for any $p' \in (-\partial_{\mathbf{R}} \Psi(x^\circ)) \cap \mathbf{R}_+^K$.

Proof. From (6), we have

$$\Psi[p](z) = \inf \left\{ \sum_{l \in L} C_l[-p](y_l) - \sum_{h \in H} U_h[-p](x_h) \mid \sum_{h \in H} x_h - \sum_{l \in L} y_l = z \right\}, \tag{7}$$

and hence,

$$\inf_z \Psi[p](z) = \sum_{l \in L} \inf_{y_l} C_l[-p](y_l) - \sum_{h \in H} \sup_{x_h} U_h[-p](x_h). \tag{8}$$

We assume that $((x_h \mid h \in H), (y_l \mid l \in L))$ has (2), (3) and (4) for some p. Then,

$$\Psi[p](x^\circ) = \sum_{l \in L} C_l[-p](y_l) - \sum_{h \in H} U_h[-p](x_h) = \min \Psi[p], \tag{9}$$

where the second equality follows from (8). By (1), (2), (3) and (9),

$$-\partial_{\mathbf{R}} \Psi(x^\circ) = \left(\bigcap_{h \in H} \partial'_{\mathbf{R}} U_h(x_h) \right) \cap \left(\bigcap_{l \in L} \partial_{\mathbf{R}} C_l(y_l) \right)$$

i.e., the first assertion of (c) holds. If $((x_h \mid h \in H), (y_l \mid l \in L), p)$ is an equilibrium, i.e., if p is nonnegative, then we have $p \in (-\partial_{\mathbf{R}} \Psi(x^\circ)) \cap \mathbf{R}_+^K \neq \emptyset$. Furthermore, for any $p' \in (-\partial_{\mathbf{R}} \Psi(x^\circ)) \cap \mathbf{R}_+^K$, $((x_h \mid h \in H), (y_l \mid l \in L), p')$ satisfies (2) and (3), that is, this is an equilibrium.

We assume that $(-\partial_{\mathbf{R}}\Psi(x^\circ)) \cap \mathbf{R}^K_+ \neq \emptyset$. By (1), there exists $p \geq 0$ with $x^\circ \in \arg\min \Psi[p]$. Then, $(x_h \mid h \in H)$ and $(y_l \mid l \in L)$ attaining the minimum of the right-hand side of (7) for x° satisfy (9). By (8) we have $x_h \in \arg\max U_h[-p]$ $(h \in H)$ and $y_l \in \arg\min C_l[-p]$ $(l \in L)$. Since (4) holds for $(x_h \mid h \in H)$ and $(y_l \mid l \in L)$, $((x_h \mid h \in H), (y_l \mid l \in L), p)$ is an equilibrium. The above discussion proves (a) and (b). ∎

3 M-convexity

3.1 Definitions

Here we review several definitions and known results on M-convex and M$^\natural$-convex functions due to Murota (1998), and Murota and Shioura (1999), respectively.

Let V be a nonempty finite set. We define the positive support and negative support of $z = (z(v) : v \in V) \in \mathbf{Z}^V$ by

$$\mathrm{supp}^+(z) = \{v \in V \mid z(v) > 0\} \quad \text{and} \quad \mathrm{supp}^-(z) = \{v \in V \mid z(v) < 0\}.$$

For each $S \subseteq V$, we denote by χ_S the characteristic vector of S defined by

$$\chi_S(v) = \begin{cases} 1 & \text{if } v \in S \\ 0 & \text{otherwise} \end{cases}$$

and write simply χ_u instead of $\chi_{\{u\}}$ for each $u \in V$.

A function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ with $\mathrm{dom}\, f \neq \emptyset$ is called *M-convex* if it satisfies

(M-EXC) for $x, y \in \mathrm{dom}\, f$ and $u \in \mathrm{supp}^+(x-y)$, there exists $v \in \mathrm{supp}^-(x-y)$ such that

$$f(x) + f(y) \geq f(x - \chi_u + \chi_v) + f(y + \chi_u - \chi_v).$$

We note that (M-EXC) is also represented as: for $x, y \in \mathrm{dom}\, f$,

$$f(x) + f(y) \geq \max_{u \in \mathrm{supp}^+(x-y)} \min_{v \in \mathrm{supp}^-(x-y)} [\, f(x - \chi_u + \chi_v) + f(y + \chi_u - \chi_v)\,],$$

where the maximum and the minimum over an empty set are $-\infty$ and $+\infty$, respectively. From (M-EXC), the effective domain of an M-convex function lies on a hyperplane $\{x \in \mathbf{R}^V \mid x(V) = \text{constant}\}$, where $x(V) = \sum_{v \in V} x(v)$.

The concept of M$^\natural$-convexity is a variant of M-convexity. Let 0 denote a new element not in V and define $\hat{V} = \{0\} \cup V$. A function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ with $\mathrm{dom}\, f \neq \emptyset$ is called *M$^\natural$-convex* if it is expressed in terms of an M-convex function $\hat{f} : \mathbf{Z}^{\hat{V}} \to \mathbf{R} \cup \{+\infty\}$ as

$$f(x) = \hat{f}(x_0, x) \quad \text{with } x_0 = -x(V).$$

Namely, an $\mathrm{M}^\natural$-convex function is a function obtained as the projection of an M-convex function. Conversely, an $\mathrm{M}^\natural$-convex function f determines the corresponding M-convex function $\hat{f}$ by

$$\hat{f}(x_0, x) = \begin{cases} f(x) & \text{if } x_0 = -x(V) \\ +\infty & \text{otherwise} \end{cases}$$

up to a translation of $\mathrm{dom}\, f$ in the direction of 0. An $\mathrm{M}^\natural$-convex function can also be defined by using an exchange property.

Theorem 3.1 (Murota and Shioura, 1999) *A function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ with $\mathrm{dom}\, f \neq \emptyset$ is $M^\natural$-convex if and only if it satisfies*

($\mathrm{M}^\natural$-EXC) *for $x, y \in \mathrm{dom}\, f$ and $u \in \mathrm{supp}^+(x - y)$,*

$$f(x) + f(y) \geq \min[\, f(x - \chi_u) + f(y + \chi_u), \min_{v \in \mathrm{supp}^-(x-y)} \{f(x - \chi_u + \chi_v) + f(y + \chi_u - \chi_v)\}\,].$$

We note that ($\mathrm{M}^\natural$-EXC) is also represented as: for $x, y \in \mathrm{dom}\, f$,

$$f(x) + f(y) \geq \max_{u \in \mathrm{supp}^+(x-y)} \min_{v \in \mathrm{supp}^-(x-y) \cup \{0\}} [\, f(x - \chi_u + \chi_v) + f(y + \chi_u - \chi_v)\,]$$

where we assume χ_0 is the zero vector on V.

Whereas the concept of $\mathrm{M}^\natural$-convexity is equivalent to that of M-convexity as above, Theorem 3.1 and the definition of M-convexity imply that an M-convex function is $\mathrm{M}^\natural$-convex. That is, we have

$$f : (\mathrm{M}^\natural\text{-EXC}) \iff \hat{f} : (\text{M-EXC}) \implies \hat{f} : (\mathrm{M}^\natural\text{-EXC}).$$

The minimizers of an $\mathrm{M}^\natural$-convex function has a nice characterization which can be checked efficiently.

Theorem 3.2 (Murota, 1996, 1998) *For an $M^\natural$-convex function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$ and $x \in \mathrm{dom}\, f$,*

$$f(x) \leq f(y)\ \ (\forall y \in \mathbf{Z}^V) \quad \iff \quad f(x) \leq f(x - \chi_u + \chi_v)\ \ (\forall u, v \in \{0\} \cup V).$$

3.2 M-convex submodular flow problem

Here, we describe the M-convex submodular flow problem (MSFP) which is introduced by Murota (1999) as an extension of the minimum cost flow problem and the submodular flow problem.

An instance of the MSFP consists of a directed network $N = (V, A, \gamma, \underline{c}, \overline{c})$ and an M-convex function $f : \mathbf{Z}^V \to \mathbf{R} \cup \{+\infty\}$, where V is the vertex-set, A is the arc-set, $\gamma : A \to \mathbf{R}$ is the cost function, $\underline{c} : A \to \mathbf{Z} \cup \{-\infty\}$ and $\overline{c} : A \to \mathbf{Z} \cup \{+\infty\}$ are functions defining lower and upper capacities of arcs. For an arc $a = (u, v) \in A$, u and v are called the *initial vertex* and the *terminal vertex* of a respectively, and a is said to *leave* u and *enter* v. For each vertex v, let $\delta^+ v$ and $\delta^- v$ denote the sets of leaving arcs and entering arcs of v, respectively. A *flow* ξ in N is a function $\xi : A \to \mathbf{Z}$, where it should be noted that we consider integer-valued flow only. A function $\partial\xi : V \to \mathbf{Z}$ derived from ξ by

$$\partial\xi(v) = \sum\{\xi(a) \mid a \in \delta^+ v\} - \sum\{\xi(a) \mid a \in \delta^- v\} \qquad (v \in V)$$

is called the *boundary* of ξ. The *M-convex submodular flow problem (MSFP)* is an optimization problem formulated by

$$\begin{aligned}
&\text{Minimize} && \Gamma(\xi) = \sum_{a \in A} \gamma(a)\xi(a) + f(\partial\xi) && (10)\\
&\text{subject to} && \underline{c}(a) \le \xi(a) \le \overline{c}(a) \qquad (a \in A), && (11)\\
& && \partial\xi \in \operatorname{dom} f, && (12)\\
& && \xi \in \mathbf{Z}^A. && (13)
\end{aligned}$$

A flow ξ satisfying (11), (12) and (13) is called a *feasible flow*. A feasible flow is said to be *optimal* if it minimizes the objective function of (10) among the feasible flows. An optimal flow can be characterized by a potential $p : V \to \mathbf{R}$ as below.

Theorem 3.3 (Murota, 1999) *Let $N = (V, A, \gamma, \underline{c}, \overline{c})$ and f be an instance of the MSFP. For any feasible flow $\xi : A \to \mathbf{Z}$, the following conditions are equivalent:*

(OPT) *ξ is an optimal flow,*

(POT) *there exists a potential $p : V \to \mathbf{R}$ such that*

(i) $\gamma(a) + p(\partial^+ a) - p(\partial^- a) > 0 \implies \xi(a) = \underline{c}(a)$,
$\gamma(a) + p(\partial^+ a) - p(\partial^- a) < 0 \implies \xi(a) = \overline{c}(a)$,

(ii) $\partial\xi \in \arg\min f[-p]$.

Algorithms for the MSFP are given by Murota (1999) and by Iwata and Shigeno. Those algorithms find an optimal flow and an optimal potential, and the latter is a polynomial time algorithm in $|V|$ and $\log_2 C$, where C is a certain number satisfying

$$C \le \max_{a \in A} |\gamma(a)| + 2 \max_{x \in \operatorname{dom} f} |f(x)|.$$

3.3 $M^\natural$-concavity in mathematical economics

Since a utility function is usually assumed as to be concave in mathematical economics, a "discrete concave function" is natural in our context. Here we introduce nice features of an $M^\natural$-concave function from the point of view of mathematical economics.

An $M^\natural$-concave function is the negative of an $M^\natural$-convex function and is defined (characterized) by an exchange property. A function $U : \mathbf{Z}^K \to \mathbf{R} \cup \{-\infty\}$ with $\operatorname{dom} U \neq \emptyset$ is called an *$M^\natural$-concave function* if it satisfies

($-M^\natural$-EXC) for $x, y \in \operatorname{dom} U$ and $i \in \operatorname{supp}^+(x-y)$,

$$U(x) + U(y) \leq \max_{j \in \operatorname{supp}^-(x-y) \cup \{0\}} \{U(x - \chi_i + \chi_j) + U(y + \chi_i - \chi_j)\}.$$

For a function $f : \mathbf{Z}^K \to \mathbf{R} \cup \{-\infty\}$, its *concave closure* $\underline{f}$ is defined by

$$\underline{f}(z) = \inf_{p \in \mathbf{R}^V, \alpha \in \mathbf{R}} \{\langle p, z\rangle + \alpha \mid \langle p, y\rangle + \alpha \geq f(y) \quad (\forall y \in \mathbf{Z}^K)\} \qquad (z \in \mathbf{R}^K).$$

We say that f is *concave-extendible* if $f(x) = \underline{f}(x)$ for any $x \in \mathbf{Z}^K$. An $M^\natural$-concave function deserves its name in the following sense.

Lemma 3.4 (Murota, 1996) *An $M^\natural$-concave function is concave-extendible.*

A utility function generally has decreasing marginal returns, which is equivalent to submodularity in the discrete case. This is also the case with an $M^\natural$-concave function.

Lemma 3.5 (Murota and Shioura, 2001) *An $M^\natural$-concave function U is submodular, i.e.,*

$$U(x) + U(y) \geq U(x \vee y) + U(x \wedge y) \qquad (x, y \in \operatorname{dom} U),$$

where vectors $x \vee y$ and $x \wedge y$ are defined by

$$(x \vee y)(i) = \max\{x(i), y(i)\}, \quad (x \wedge y)(i) = \min\{x(i), y(i)\} \qquad (i \in K).$$

We now consider natural generalizations of the gross substitutes condition and the single improvement condition which were originally proposed for set functions by Kelso and Crawford (1982), and Gul and Stacchetti (1999), respectively.

($-M^\natural$-GS_W) For $p, q \in \mathbf{R}^K$ and $x \in \operatorname{dom} U$ such that $p \leq q$, $x \in \arg\max U[-p]$ and $\arg\max U[-q] \neq \emptyset$, there exists $y \in \arg\max U[-q]$ such that

$$y(i) \geq x(i) \quad \text{if } p(i) = q(i).$$

($-$M$^\natural$-GS) For (p_0, p), $(q_0, q) \in \mathbf{R}^{\{0\}\cup K}$ and $x \in \operatorname{dom} U$ such that $(p_0, p) \leq (q_0, q)$, $x \in \arg\max U[-p + p_0\mathbf{1}]$ and $\arg\max U[-q + q_0\mathbf{1}] \neq \emptyset$, there exists $y \in \arg\max U[-q + q_0\mathbf{1}]$ such that

$$\begin{aligned} y(i) &\geq x(i) \quad \text{if } p(i) = q(i), \\ y(K) &\leq x(K) \quad \text{if } p_0 = q_0, \end{aligned}$$

where $\mathbf{1}$ denotes the vector of all ones.

($-$M$^\natural$-SI$_{\mathrm{W}}$) For $p \in \mathbf{R}^K$ and $x \in \operatorname{dom} U$ with $x \notin \arg\max U[-p]$, there exist $i, j \in \{0\} \cup K$ such that $U[-p](x) < U[-p](x - \chi_i + \chi_j)$.

($-$M$^\natural$-SI) For $p \in \mathbf{R}^K$ and x, $y \in \operatorname{dom} U$ with $U[-p](x) < U[-p](y)$,

$$U[-p](x) < \max_{i \in \operatorname{supp}^+(x-y)\cup\{0\}} \max_{j \in \operatorname{supp}^-(x-y)\cup\{0\}} U[-p](x - \chi_i + \chi_j).$$

We are primarily interested in the case where $\operatorname{dom} U$ is bounded. Then, for any $q \in \mathbf{R}^K$, $\arg\max U[-q]$ is nonempty. Condition ($-$M$^\natural$-GS$_{\mathrm{W}}$) states that if the price vector is increased from p to q then the consumer still wants to consume at least the same numbers of the commodities whose prices remain the same. ($-$M$^\natural$-GS) says that when prices increase ($p \leq q$ and $p_0 = q_0$), the consumer wants a consumption such that the numbers of the commodities whose prices remain the same do not decrease and the total number of commodities does not increase. ($-$M$^\natural$-GS) also says that when all prices decrease by the same amount ($p = q$ and $p_0 < q_0$), the consumer wants at least the same number of each commodity. Obviously, the condition ($-$M$^\natural$-GS) is stronger than ($-$M$^\natural$-GS$_{\mathrm{W}}$). ($-$M$^\natural$-SI$_{\mathrm{W}}$) states that a nonoptimal consumption x can be improved strictly by either removing a commodity from x, adding a commodity to x, or doing both. Moreover, ($-$M$^\natural$-SI) is stronger than ($-$M$^\natural$-SI$_{\mathrm{W}}$) in the sense that the consumer can bring consumption x nearer to any better consumption y.

An M$^\natural$-concave function satisfies these conditions, and furthermore, it is characterized by these conditions.

Theorem 3.6 (Murota and Tamura, 2001) *For a function $U : \mathbf{Z}^K \to \mathbf{R} \cup \{-\infty\}$ such that $\operatorname{dom} U$ is nonempty and bounded, the following statements are equivalent:*

(a) *U satisfies ($-$M$^\natural$-EXC), that is, U is M$^\natural$-concave,*

(b) *U satisfies ($-$M$^\natural$-GS) and is concave-extendible,*

(c) *U satisfies ($-$M$^\natural$-SI).*

For a set function, all the above conditions are equivalent.

Theorem 3.7 (Murota and Tamura, 2001) *For $U : \{0,1\}^K \to \mathbf{R}$, we have*

$$(-\mathrm{M}^\natural\text{-EXC}) \Longleftrightarrow (-\mathrm{M}^\natural\text{-SI}) \Longleftrightarrow (-\mathrm{M}^\natural\text{-SI}_\mathrm{W}) \Longleftrightarrow (-\mathrm{M}^\natural\text{-GS}) \Longleftrightarrow (-\mathrm{M}^\natural\text{-GS}_\mathrm{W}).$$

Conditions $(-\mathrm{M}^\natural\text{-GS}_\mathrm{W})$ and $(-\mathrm{M}^\natural\text{-SI}_\mathrm{W})$ for functions on $\{0,1\}$-hypercubes correspond to the gross substitutes condition and the single improvement condition for set functions, respectively. We note that the equivalence between the gross substitutes condition and the single improvement condition for set functions was first pointed out by Gul and Stacchetti (1999), and that the equivalence between the single improvement condition and $\mathrm{M}^\natural$-concavity for set functions was by Fujishige and Yang (2000).

We finally return to equilibria of economic models with indivisible commodities. A function $U : \mathbf{Z}^K \to \mathbf{R} \cup \{-\infty\}$ is said to be *monotone nondecreasing* if $x \leq y \Rightarrow U(x) \leq U(y)$ for any $x, y \in \operatorname{dom} U$. Gul and Stacchetti (1999) showed the existence of an equilibrium in an exchange economy under the gross substitutes condition and the monotone nondecreasing condition. By Theorem 3.7, we see that Theorem 3.8 is a generalization of the result. Theorems 3.8 and 3.9 stated explicitly by Murota (2001) are implied by the results of Danilov, Koshevoy and Murota (2001).

Theorem 3.8 (Danilov et al., 2001, Murota, 2001) *In an exchange economy, where $L = \emptyset$, if each U_h is monotone nondecreasing and $M^\natural$-concave, then there exists an equilibrium $((x_h \mid h \in H), p)$ for any initial total endowment $x^\circ \in \sum_{h \in H} \operatorname{dom} U_h$, where the summation means the Minkowski sum.*

Theorem 3.9 (Danilov et al., 2001, Murota, 2001) *Suppose that each C_l is $M^\natural$-convex and that each U_h is $M^\natural$-concave in the model defined in Section 2. If the continuous model obtained by regarding all indivisible commodities as divisible has an equilibrium for an initial total endowment, then there exists an equilibrium $((x_h \mid h \in H), (y_l \mid l \in L), p)$ of indivisible commodities, where cost functions and utility functions in the continuous model are the convex extensions of C_l and concave extensions of U_h, respectively.*

The equilibrium price vectors form a well-behaved polyhedron, $\mathrm{L}^\natural$-convex polyhedron investigated by Fujishige and Murota (2000) and Murota and Shioura (2000). A polyhedron $P \subseteq \mathbf{R}^K$ is called an *$L^\natural$-convex polyhedron* if

$$p,\ q \in P \implies (p - \alpha\mathbf{1}) \vee q,\ p \wedge (q + \alpha\mathbf{1}) \in P \qquad (0 \leq \forall\alpha \in \mathbf{R}). \tag{14}$$

Theorem 3.10 (Murota, 2001) *Suppose that each C_l is $M^\natural$-convex and each U_h is $M^\natural$-concave in the model defined in Section 2 and that there exists an equilibrium for an initial total endowment x°. Then the set $P^*(x^\circ)$ of all the equilibrium price vectors is an $L^\natural$-convex polyhedron. This means in particular ($\alpha = 0$ in (14)) that*

$$p, q \in P^*(x^\circ) \implies p \vee q,\ p \wedge q \in P^*(x^\circ). \tag{15}$$

(15) implies the existence of the smallest equilibrium price vector, and furthermore, it implies the existence of the largest equilibrium price vector if $P^*(x^\circ)$ is bounded.

4 Computation of equilibria

This section is the main part of the paper. We show how to calculate an equilibrium in the economy in which each C_l is M$^\natural$-convex and each U_h is M$^\natural$-concave. We first formulate the problem of finding an equilibrium as an MSFP. Solution of the problem yields consumptions and productions satisfying (2), (3), (4) as well as a price vector which, however, may not be nonnegative. This is the first phase of our algorithm. The second phase finds an equilibrium price vector by solving a shortest path problem, which is explained at the end of this section. Our algorithm finds an equilibrium if one exists; otherwise either the MSFP or the shortest path problem has no solution. We can also modify the second phase to find the smallest or the largest equilibrium price vector.

For any vector $z \in \mathbf{R}^K$, we denote by $\hat{z}$ the vector $(-z(K), z) \in \mathbf{R}^{\{0\}\cup K}$ whose 0-th component is the negative of the sum of the others. For M$^\natural$-convex cost function $C_l : \mathbf{Z}^K \to \mathbf{R} \cup \{+\infty\}$, we define the corresponding M-convex function $\hat{C}_l : \mathbf{Z}^{\{0\}\cup K} \to \mathbf{R} \cup \{+\infty\}$ by

$$\hat{C}_l(y_0, y) = \begin{cases} C_l(y) & \text{if } y_0 = -y(K) \\ +\infty & \text{otherwise} \end{cases} \qquad (l \in L).$$

We also define the M$^\natural$-concave function $\hat{U}_h : \mathbf{Z}^{\{0\}\cup K} \to \mathbf{R} \cup \{-\infty\}$ associated with $U_h : \mathbf{Z}^K \to \mathbf{R} \cup \{-\infty\}$ by

$$\hat{U}_h(x_0, x) = \begin{cases} U_h(x) & \text{if } x_0 = -x(K) \\ -\infty & \text{otherwise} \end{cases} \qquad (h \in H).$$

In the same way as (6), we consider the aggregate cost function

$$\hat{\Psi}(\tilde{z}) = \inf \left\{ \sum_{l\in L} \hat{C}_l(\tilde{y}_l) - \sum_{h\in H} \hat{U}_h(\tilde{x}_h) \mid \sum_{h\in H} \tilde{x}_h - \sum_{l\in L} \tilde{y}_l = \tilde{z} \right\} \qquad (\tilde{z} \in \mathbf{Z}^{\{0\}\cup K}).$$

Ψ and $\hat{\Psi}$ have the following relation.

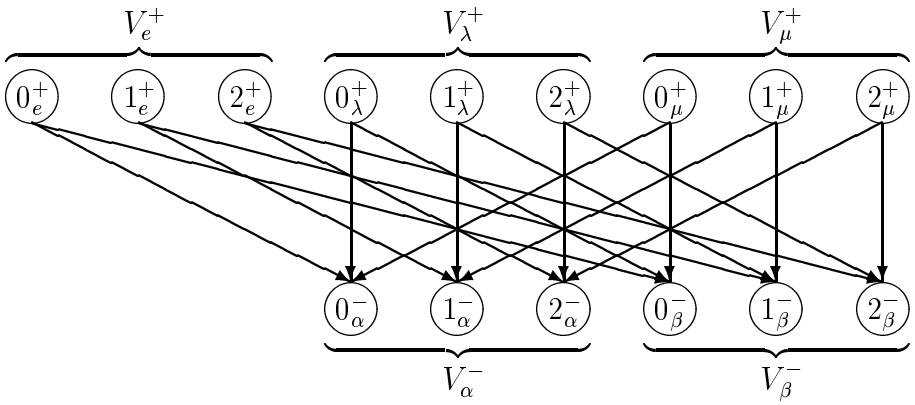

Figure 1: Graph of the M-convex submodular flow problem for the evaluation of M-convex function $\hat{\Psi}(\hat{x}^\circ)$ in the case where consumer-set $H = \{\alpha, \beta\}$, producer-set $L = \{\lambda, \mu\}$ and commodity-set $K = \{1, 2\}$.

Lemma 4.1 *For a vector* $\tilde{p} = (p_0, p) \in \mathbf{R}^{\{0\}\cup K}$ *and* $z \in \mathbf{Z}^K$,

$$p_0 = 0 \implies \hat{\Psi}[\tilde{p}](\hat{z}) = \Psi[p](z).$$

Proof. From the definitions of $\hat{C}_l$ and $\hat{U}_h$, we have

$$\hat{\Psi}[\tilde{p}](\hat{z}) = \Psi[p](z) - p_0 \sum_{k\in K} z(k),$$

which implies the assertion. ∎

The function $\hat{\Psi}$ is the integer infimal convolution of M-convex functions $\hat{C}_l$ ($l \in L$) and $-\hat{U}_h$ ($h \in H$). It is known that the integer infimal convolution of M-convex functions is also M-convex and can be evaluated by solving an instance of the MSFP (e.g., Murota, 1996). We will demonstrate how $\hat{\Psi}(\hat{x}^\circ)$ is evaluated.

The instance of the MSFP for the evaluation of $\hat{\Psi}(\hat{x}^\circ)$ is defined as follows. Consider a directed bipartite graph $G = (V^+, V^-, A)$ whose vertex-partition V^+, V^- and arc-set A are defined by

$$\begin{aligned}
V^+ &= V_e^+ \cup \left(\bigcup_{l\in L} V_l^+ \right), \\
V^- &= \bigcup_{h\in H} V_h^-, \\
V_e^+ &= \{k_e^+ \mid k \in \{0\} \cup K\}, \\
V_l^+ &= \{k_l^+ \mid k \in \{0\} \cup K\} \qquad (l \in L), \\
V_h^- &= \{k_h^- \mid k \in \{0\} \cup K\} \qquad (h \in H), \\
A &= \{(k_l^+, k_h^-) \mid l \in L, h \in H, k \in \{0\}\cup K\} \\
&\quad \cup \{(k_e^+, k_h^-) \mid h \in H, k \in \{0\}\cup K\}.
\end{aligned}$$

Note that V_e^+, V_l^+ and V_h^- are copies of $\{0\} \cup K$, respectively. In Figure 1, we draw the graph G for the case where $H = \{\alpha, \beta\}$, $L = \{\lambda, \mu\}$ and $K = \{1, 2\}$. For each arc $a \in A$, we put $\underline{c}(a) = -\infty$, $\overline{c}(a) = +\infty$ and $\gamma(a) = 0$. By using the indicator function $\delta_e : \mathbf{Z}^{V_e^+} \to \mathbf{R} \cup \{+\infty\}$ of $\{\hat{x}^\circ\}$ defined by

$$\delta_e(\tilde{w}) = \begin{cases} 0 & \text{if } \tilde{w} = \hat{x}^\circ \\ +\infty & \text{otherwise,} \end{cases}$$

we define a function $f : \mathbf{Z}^{V^+ \cup V^-} \to \mathbf{R} \cup \{+\infty\}$ by

$$f(\tilde{w}, (\tilde{y}_l \mid l \in L), (\tilde{x}_h \mid h \in H)) = \delta_e(\tilde{w}) + \sum_{l \in L} \hat{C}_l(\tilde{y}_l) - \sum_{h \in H} \hat{U}_h(-\tilde{x}_h),$$

where $\tilde{w} \in \mathbf{Z}^{V_e^+}$, $\tilde{y}_l \in \mathbf{Z}^{V_l^+}$ $(l \in L)$ and $\tilde{x}_h \in \mathbf{Z}^{V_h^-}$ $(h \in H)$. The function f is M-convex, since functions $\hat{C}_l$ $(l \in L)$ are M-convex and functions $\hat{U}_h$ $(h \in H)$ are M-concave. This is the instance of the MSFP that we use for the computation of $\hat{\Psi}(\hat{x}^\circ)$.

An optimal flow and an optimal potential of the MSFP have the following nice properties on which the evaluation of $\hat{\Psi}(\hat{x}^\circ)$ relies.

Lemma 4.2 *Assume that the above instance of MSFP has an optimal flow $\xi \in \mathbf{Z}^A$ and an optimal potential $\bar{p} \in \mathbf{R}^{V^+ \cup V^-}$. Let*

$$\tilde{x}_h^* = -\partial\xi|_{V_h^-} \ (h \in H), \qquad \tilde{y}_l^* = \partial\xi|_{V_l^+} \ (l \in L), \qquad \tilde{w}^* = \partial\xi|_{V_e^+}, \qquad \tilde{p} = \bar{p}|_{V_e^+},$$

where $\tilde{x}_h^* = -\partial\xi|_{V_h^-}$ means the restriction of $-\partial\xi$ to V_h^-, i.e., $\tilde{x}_h^*(k) = -\partial\xi(k)$ $(k \in V_h^-)$, etc., and we regard $\tilde{x}_h^*$, $\tilde{y}_l^*$, $\tilde{w}^*$ and $\tilde{p}$ as vectors on $\{0\} \cup K$. Then the following statements hold.

(a) $\tilde{w}^* = \hat{x}^\circ$, $\tilde{x}_h^*(0) = -\tilde{x}_h^*(K)$ $(h \in H)$ *and* $\tilde{y}_l^*(0) = -\tilde{y}_l^*(K)$ $(l \in L)$.

(b) $\hat{x}^\circ + \sum_{l \in L} \tilde{y}_l^* = \sum_{h \in H} \tilde{x}_h^*$.

(c) $\bar{p}(k_h^-) = \bar{p}(k_l^+) = \bar{p}(k_e^+)$ *for any* $k \in \{0\} \cup K$, $h \in H$ *and* $l \in L$.

(d) $\hat{x}^\circ \in \arg\min \hat{\Psi}[\tilde{p}]$.

(e) $\tilde{x}_h^* \in \arg\max \hat{U}_h[-\tilde{p}]$ $(h \in H)$ *and* $\tilde{y}_l^* \in \arg\min \hat{C}_l[-\tilde{p}]$ $(l \in L)$.

Proof. (a): This follows directly from the definitions of $\hat{C}_l$, $\hat{U}_h$ and δ_e.

(b): From the definition of G, for each $k \in \{0\} \cup K$, we have

$$\begin{aligned} \tilde{w}^*(k) + \sum_{l \in L} \tilde{y}_l^*(k) &= \sum_{h \in H} \xi(k_e^+, k_h^-) + \sum_{l \in L} \sum_{h \in H} \xi(k_l^+, k_h^-) \\ &= \sum_{h \in H} \left[\xi(k_e^+, k_h^-) + \sum_{l \in L} \xi(k_l^+, k_h^-) \right] \\ &= \sum_{h \in H} \tilde{x}_h^*(k). \end{aligned}$$

Then the assertion follows from $\tilde{w}^* = \hat{x}^\circ$.

(c): Since $\gamma(a) = 0$, $\underline{c}(a) = -\infty$ and $\overline{c}(a) = +\infty$ for any $a \in A$, condition (i) of (POT) of Theorem 3.3 implies

$$\bar{p}(\partial^+ a) - \bar{p}(\partial^- a) = 0 \qquad (a \in A).$$

This and the structure of G show the assertion.

(e): Assertion (c) implies

$$\begin{aligned} \min f[-\bar{p}] &= \min \left\{ \delta_e[-\tilde{p}](\tilde{w}) + \sum_{l \in L} \hat{C}_l[-\tilde{p}](\tilde{y}_l) - \sum_{h \in H} \hat{U}_h[-\tilde{p}](-\tilde{x}_h) \right\} \\ &= \sum_{l \in L} \min_{\tilde{y}} \hat{C}_l[-\tilde{p}](\tilde{y}) - \sum_{h \in H} \max_{\tilde{x}} \hat{U}_h[-\tilde{p}](\tilde{x}) - \langle \tilde{p}, \hat{x}^\circ \rangle \qquad (16) \\ &= \min \hat{\Psi}[\tilde{p}] - \langle \tilde{p}, \hat{x}^\circ \rangle. \qquad (17) \end{aligned}$$

We also have

$$f[-\bar{p}](\partial\xi) = \sum_{l \in L} \hat{C}_l[-\tilde{p}](\tilde{y}_l^*) - \sum_{h \in H} \hat{U}_h[-\tilde{p}](\tilde{x}_h^*) - \langle \tilde{p}, \hat{x}^\circ \rangle. \qquad (18)$$

Condition (ii) of (POT) of Theorem 3.3, (16) and (18) yield (e).

(d): Condition (ii) of (POT) of Theorem 3.3 and (17) say

$$f[-\bar{p}](\partial\xi) = \min \hat{\Psi}[\tilde{p}] - \langle \tilde{p}, \hat{x}^\circ \rangle.$$

On the other hand, assertion (b) guarantees

$$f(\partial\xi) = \min \left\{ \sum_{l \in L} \hat{C}_l(\tilde{y}_l) - \sum_{h \in H} \hat{U}_h(\tilde{x}_h) \mid \sum_{h \in H} \tilde{x}_h - \sum_{l \in L} \tilde{y}_l = \hat{x}^\circ \right\} = \hat{\Psi}(\hat{x}^\circ).$$

The above equalities show

$$\hat{\Psi}[\tilde{p}](\hat{x}^\circ) = f(\partial\xi) + \langle \tilde{p}, \hat{x}^\circ \rangle = f[-\bar{p}](\partial\xi) + \langle \bar{p}, \partial\xi \rangle + \langle \tilde{p}, \hat{x}^\circ \rangle = \min \hat{\Psi}[\tilde{p}],$$

where $\langle \bar{p}, \partial\xi \rangle = 0$ by (b) and (c). ∎

The following theorem guarantees that consumptions and productions satisfying (2), (3), (4) can be computed by solving the above instance of the MSFP, and that if the optimal potential is nonnegative then it serves as an equilibrium price vector.

Theorem 4.3 *Assume that the above instance of MSFP has an optimal flow* $\xi \in \mathbf{Z}^A$ *and an optimal potential* $\bar{p} \in \mathbf{R}^{V^+ \cup V^-}$ *with* $\bar{p}(0_e^+) = 0$. *Let*

$$x_h^* = -\partial\xi|_{V_h^- \setminus \{0_h^-\}} \ (h \in H), \qquad y_l^* = \partial\xi|_{V_l^+ \setminus \{0_l^+\}} \ (l \in L), \qquad p = \bar{p}|_{V_e^+ \setminus \{0_e^+\}},$$

where we regard x_h^*, y_l^* *and* p *as vectors on* K. *Then the following statements hold.*

(a) $x^\circ \in \arg\min \Psi[p]$.

(b) $x^\circ + \sum_{l \in L} y_l^* = \sum_{h \in H} x_h^*$.

(c) $x_h^* \in \arg\max U_h[-p] \qquad (h \in H)$.

(d) $y_l^* \in \arg\min C_l[-p] \qquad (l \in L)$.

Therefore, $((x_h^* \mid h \in H), (y_l^* \mid l \in L), p)$ *is an equilibrium if* $p \geq 0$. *Moreover, if there exists an equilibrium at all, then* $((x_h^* \mid h \in H), (y_l^* \mid l \in L))$ *is the consumptions and productions of some equilibrium.*

Proof. Statements (a), (b), (c) and (d) are straightforward consequences of Lemmas 4.1 and 4.2.

If there exists an equilibrium then $(-\partial_{\mathbf{R}} \Psi(x^\circ)) \cap \mathbf{R}_+^K \neq \emptyset$ holds. Statements (b), (c), (d) and statement (c) of Lemma 2.1 guarantee that $((x_h^* \mid h \in H), (y_l^* \mid l \in L))$ is the consumptions and productions of some equilibrium. ∎

Any algorithm for the MSFP will find a tuple $((x_h^* \mid h \in H), (y_l^* \mid l \in L), p)$, which gives an equilibrium for the initial total endowment x° if the optimal potential p happens to be nonnegative. We go on to show that the set of all nonnegative optimal potentials, or equivalently, the set of all equilibrium price vectors, can be expressed by a certain linear inequality system. Thus, the existence of an equilibrium price vector can be checked by solving a linear programming problem. We also show that if an equilibrium exists, then both the smallest and the largest equilibrium price vectors, whose existence is guaranteed in the sense of Theorem 3.10, can be found by solving linear programming problems. At the end of this section, we reduce these linear programming problems to shortest path problems which can be solved efficiently by combinatorial algorithms.

In order to give a linear inequality description of the set of equilibrium price vectors, we give a necessary and sufficient condition for (a) of Theorem 4.3. Lemma 2.1 and (1) yield the equivalence that

$$x^\circ \in \arg\min \Psi[q] \iff \begin{cases} x_h^* \in \arg\max U_h[-q] & (h \in H) \\ y_l^* \in \arg\min C_l[-q] & (l \in L). \end{cases} \tag{19}$$

By Theorem 3.2, we have

$$\begin{array}{l} y_l^* \in \arg\min C_l[-q] \\ \qquad \Updownarrow \\ \begin{cases} C_l(y_l^*) - C_l(y_l^* - \chi_j) \leq q(j) \leq C_l(y_l^* + \chi_j) - C_l(y_l^*) & (j \in K) \\ q(j) - q(i) \leq C_l(y_l^* - \chi_i + \chi_j) - C_l(y_l^*) & (i, j \in K, i \neq j) \end{cases} \end{array} \tag{20}$$

and (by a similar criterion for $\mathrm{M}^\natural$-concave functions)

$$
\begin{array}{l}
x_h^* \in \arg\max U_h[-q] \\
\qquad \Updownarrow \\
\left\{ \begin{array}{ll}
U_h(x_h^* + \chi_j) - U_h(x_h^*) \le q(j) \le U_h(x_h^*) - U_h(x_h^* - \chi_j) & (j \in K) \\
q(j) - q(i) \le U_h(x_h^*) - U_h(x_h^* + \chi_i - \chi_j) & (i, j \in K, i \ne j).
\end{array} \right.
\end{array} \tag{21}
$$

By (19), (20) and (21), we obtain

$$
x^\circ \in \arg\min \Psi[q] \iff \left\{ \begin{array}{ll}
l(j) \le q(j) \le u(j) & (j \in K) \\
q(j) - q(i) \le u(i,j) & (i, j \in K, i \ne j),
\end{array} \right. \tag{22}
$$

where

$$
\begin{aligned}
l(j) &= \max \left\{ \max_{h \in H} \{U_h(x_h^* + \chi_j) - U_h(x_h^*)\}, \right. \\
&\qquad \left. \max_{l \in L} \{C_l(y_l^*) - C_l(y_l^* - \chi_j)\} \right\}, \qquad (23) \\
u(j) &= \min \left\{ \min_{h \in H} \{U_h(x_h^*) - U_h(x_h^* - \chi_j)\}, \right. \\
&\qquad \left. \min_{l \in L} \{C_l(y_l^* + \chi_j) - C_l(y_l^*)\} \right\}, \qquad (24) \\
u(i,j) &= \min \left\{ \min_{h \in H} \{U_h(x_h^*) - U_h(x_h^* + \chi_i - \chi_j)\}, \right. \\
&\qquad \left. \min_{l \in L} \{C_l(y_l^* - \chi_i + \chi_j) - C_l(y_l^*)\} \right\}. \qquad (25)
\end{aligned}
$$

We note that $l(j) < +\infty$, $u(j) > -\infty$ and $u(i,j) > -\infty$ hold for any $i, j \in K$ $(i \ne j)$.

We recall that $P^*(x^\circ)$ denotes the set of all equilibrium price vectors for the initial total endowment x°. We have $P^*(x^\circ) = (-\partial_{\mathbf{R}} \Psi(x^\circ)) \cap \mathbf{R}_+^K$ by (b) of Lemma 2.1. By the above argument, $P^*(x^\circ)$ can be described as follows by using $((x_h^* \mid h \in H), (y_l^* \mid l \in L))$.

Theorem 4.4 *The set $P^*(x^\circ)$ of all equilibrium price vectors is a polyhedron described by*

$$
P^*(x^\circ) = \left\{ q \in \mathbf{R}^K \;\middle|\; \begin{array}{ll}
\max\{0, l(j)\} \le q(j) \le u(j) & (j \in K) \\
q(j) - q(i) \le u(i,j) & (i, j \in K, i \ne j)
\end{array} \right\}, \tag{26}
$$

where $l(j)$, $u(j)$ and $u(i,j)$ are defined in (23), (24) and (25) by $((x_h^* \mid h \in H), (y_l^* \mid l \in L))$.

Proof. The assertion follows from (1), (22) and $P^*(x^\circ) = (-\partial_{\mathbf{R}} \Psi(x^\circ)) \cap \mathbf{R}_+^K$ in (b) of Lemma 2.1. ∎

Theorem 4.4 guarantees that nonemptyness of $P^*(x^\circ)$ can be checked by linear programming. In particular, the largest equilibrium price vector, if any, can be found by solving a linear programming problem:

$$\begin{array}{lll} \text{Maximize} & \sum_{k\in K} q(k) & \\ \text{subject to} & \max\{0, l(j)\} \le q(j) \le u(j) & (j \in K) \\ & q(j) - q(i) \le u(i,j) & (i,j \in K, i \neq j), \end{array} \tag{27}$$

because the largest vector in $P^*(x^\circ)$ maximizes the sum of all components of a vector. Analogously, the smallest equilibrium price vector can be found by solving another linear programming problem:

$$\begin{array}{lll} \text{Minimize} & \sum_{k\in K} q(k) & \\ \text{subject to} & \max\{0, l(j)\} \le q(j) \le u(j) & (j \in K) \\ & q(j) - q(i) \le u(i,j) & (i,j \in K, i \neq j). \end{array} \tag{28}$$

Theorem 4.5 *There exists an equilibrium price vector if and only if problem (27) (as well as (28)) is feasible. In particular, both the largest and the smallest equilibrium price vectors, if any, can be found by solving problems (27) and (28), respectively.*

Next, we show that the existence of an equilibrium price vector can be characterized by graph-theoretical terms, and that problems (27) and (28) can be reduced to shortest path problems. Before doing this, we briefly explain the shortest path problem and related facts in a generic form without proofs (for details, see, e.g., Cook et al., 1998). Let $N = (V, A, c)$ be a directed network with vertex-set V, arc-set A and arc-length function $c : A \to \mathbf{R}$. A sequence $P = (v_0, v_1), (v_1, v_2), \ldots, (v_{t-1}, v_t)$ of arcs of A from vertex v_0 to vertex v_t is called a *path* from v_0 to v_t and its length $c(P)$ is defined as the sum of lengths of all arcs in P, i.e., $c(P) = \sum_{i=1}^{t} c(v_{i-1}, v_i)$. Path P is called a *cycle* if $v_0 = v_t$, and a cycle P is called a *negative cycle* if $c(P) < 0$. A *shortest path* from v_0 to v is a path from v_0 to v whose length is minimum among all the paths from v_0 to v. Here we consider a single-source shortest path problem, which is to find shortest paths from a specified starting vertex 0 in V to all other vertices, and call it simply the *shortest path problem*. A vector $q \in \mathbf{R}^V$ is called a *feasible potential* if $q(0) = 0$ and $q(v) - q(u) \le c(u, v)$ for all $(u, v) \in A$. The following relationships among negative cycles, feasible potentials and shortest paths, are well-known in combinatorial optimization. There exists a feasible potential if and only if there exists no negative cycle. A feasible potential gives lower bounds of the shortest lengths. That is, for any path $P = (v_0, v_1), (v_1, v_2), \ldots, (v_{t-1}, v_t)$ with $v_0 = 0$ and $v_t = v$,

we have

$$c(P) = \sum_{i=1}^{t} c(v_{i-1}, v_i) \geq \sum_{i=1}^{t} (q(v_i) - q(v_{i-1})) = q(v_t) - q(v_0) = q(v).$$

Moreover, it is known that the maximum vector among feasible potentials gives the lengths of shortest paths from the starting vertex 0 to all other vertices simultaneously. Thus, the shortest path problem is equivalent to the following problem

$$\begin{array}{ll} \text{Maximize} & \sum_{v \in V} q(v) \\ \text{subject to} & q(v) - q(u) \leq c(u,v) \quad ((u,v) \in A) \\ & q(0) = 0, \end{array} \tag{29}$$

which is called the *dual problem* of the shortest path problem. The shortest path problem is one of the most thoroughly studied combinatorial optimization problems and is well-known to be tractable, that is, it can be solved in polynomial time in the size of V. Hence, problem (29) and the problem of checking for the existence of a negative cycle can also be solved efficiently.

We finally give a graph theoretical characterization of the existence of an equilibrium price vector, and reduce problem (27) to the shortest path problem. Let $N(x^\circ) = (\{0\} \cup K, A, c)$ be a directed network with vertex-set $\{0\} \cup K$, where 0 is a starting vertex, arc-length function $c : (\{0\} \cup K) \times (\{0\} \cup K) \to \mathbf{R} \cup \{+\infty\}$ defined by

$$c(i,j) = \begin{cases} u(j) & \text{if } i = 0,\ j \in K \\ -\max\{0, l(j)\} & \text{if } i \in K,\ j = 0 \\ u(i,j) & \text{if } i, j \in K,\ i \neq j \end{cases} \tag{30}$$

and arc-set A defined by the set of all directed arcs of finite length. The existence of an equilibrium price vector can be characterized by using network $N(x^\circ)$.

Theorem 4.6 *There exists an equilibrium price vector if and only if there exists no negative cycle in network $N(x^\circ)$ with vertex-set $\{0\} \cup K$ and arc-length function c of (30).*

Proof. By (26) and (30), $P^*(x^\circ)$ is nonempty if and only if

$$\tilde{P}^*(x^\circ) = \left\{ \tilde{q} \in \mathbf{R}^{\{0\} \cup K} \;\middle|\; \begin{array}{l} \tilde{q}(j) - \tilde{q}(i) \leq c(i,j) \quad (i,j \in \{0\} \cup K, i \neq j) \\ \tilde{q}(0) = 0 \end{array} \right\}$$

is nonempty. Since $\tilde{P}^*(x^\circ)$ is the set of all feasible potentials of $N(x^\circ)$ with the starting vertex 0, the assertion holds. ∎

Problem (27) is obviously equivalent to problem (29) with vertex-set $V = \{0\} \cup K$ and arc-length function c defined by (30). This means that the linear programming problem (27) for the largest equilibrium price vector can be solved by finding the shortest paths from the vertex 0 to other vertices.

Theorem 4.7 *Both the smallest and the largest equilibrium price vectors, if any, can be found by solving the shortest path problem.*

Proof. The assertion for the largest equilibrium price vector is trivial. Since problem (28) is easily transformed to the shortest path problem with respect to the arc-length function c' defined by $c'(i,j) = c(j,i)$ for any (i,j), the assertion for the smallest equilibrium price vector also holds. ∎

Theorems 4.3, 4.6 and 4.7 are summarized in the following theorem as our conclusion.

Theorem 4.8 *The existence of an equilibrium in an instance of our economic model can be checked by the M-convex submodular flow problem and the shortest path problem. If an equilibrium exists, then the former problem provides the productions and consumptions of some equilibrium and the latter problem provides an equilibrium price vector (and furthermore, the smallest and the largest); otherwise, either problem is infeasible. Moreover, these conditions can be verified efficiently.*

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