On solutions of the q-hypergeometric equation with $q^N = 1$

Yoshihiro Takeyama[◇]

Research Institute for Mathematical Sciences, Kyoto University, Kyoto 6068502, Japan

Abstract. We consider the q-hypergeometric equation with $q^N = 1$ and $\alpha, \beta, \gamma \in \mathbb{Z}$. We solve this equation on the space of functions given by a power series multiplied by a power of the logarithmic function. We prove that the subspace of solutions is two-dimentional over the field of quasi-constants. We get a basis for this space explicitly. In terms of this basis, we represent the q-hypergeometric function of the Barnes type constructed by Nishizawa and Ueno. Then we see that this function has logarithmic singularity at the origin. This is a difference between the q-hypergeometric functions with $0 < |q| < 1$ and at $|q| = 1$.

1 Introduction

Consider the q-hypergeometric equation

$$\left\{(1 - D_q)(1 - q^{\gamma-1} D_q) - t(1 - q^{\alpha} D_q)(1 - q^{\beta} D_q)\right\} \varphi(t) = 0, \tag{1.1}$$

where D_q is the q-difference operator defined by $(D_q \varphi)(t) := \varphi(qt)$. In this paper, we solve (1.1) with $q^N = 1$ and $\alpha, \beta, \gamma \in \mathbb{Z}$ explicitly on a special space of functions and represent the q-hypergeometric function of the Barnes type at $|q| = 1$ in terms of our solutions.

Let us recall some results about solutions to (1.1). In the case of $0 < |q| < 1$, one of the solutions to (1.1) is the basic hypergeometric function $\varphi(\alpha, \beta, \gamma; t)$ [GR] defined by

$$\varphi(\alpha, \beta, \gamma; t) := \sum_{k=0}^{\infty} \frac{(q^{\alpha})_k (q^{\beta})_k}{(q)_k (q^{\gamma})_k} t^k, \tag{1.2}$$

where $(x)_n := \prod_{j=0}^{n-1} (1 - q^j x)$. We can get this solution by setting

$$\varphi(t) = \sum_{k=0}^{\infty} c_k t^k, \quad c_k \in \mathbb{C} \tag{1.3}$$

◇ Research Fellow of the Japan Society for the Promotion of Science.
e-mail:ninihuni@kurims.kyoto-u.ac.jp

and solving a recursion relation for $\{c_k\}$. In the case of $|q| = 1$, we can not get solutions in this manner because the coefficient in (1.2) does not converge. However, some solutions are constructed in terms of a contour integral by Nishizawa and Ueno [NU]. In this paper, we compare the solutions of these two types under the condition that $q^N = 1$ and $\alpha, \beta, \gamma \in \mathbb{Z}$.

First we try to find solutions in a similar way to the case of $0 < |q| < 1$. Then we consider solutions of more general form than (1.3) because of the following reason. In the limit as $q \to 1$, the equation (1.1) goes to the hypergeometric differential equation:

$$\left\{ t(1-t)\frac{d^2}{dt^2} + (\gamma - (\alpha + \beta + 1)t)\frac{d}{dt} - \alpha\beta \right\} F(t) = 0. \tag{1.4}$$

It is known that if $\gamma \notin \mathbb{Z}$ there exist two independent solutions at $t = 0$ of the form

$$F(t) = \sum_{k=0}^{\infty} a_k t^{k+\rho}. \tag{1.5}$$

However, if $\gamma \in \mathbb{Z}$ one of the two independent solutions is represented as

$$F_1(t) \log t + F_2(t) \tag{1.6}$$

where $F_1(t)$ and $F_2(t)$ are functions of the from (1.5) (see [AAR], for example).

Now let us return to the equation (1.1) with $0 < |q| < 1$. When we consider solutions of the form (1.5), we get two independent solutions if $\gamma \notin \mathbb{Z}$. One of them is the basic hypergeometric function (1.2) and the other is given by

$$t^{1-\gamma}\varphi(\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma; t) = \sum_{k=0}^{\infty} \frac{(q^{\alpha-\gamma+1})_k (q^{\beta\gamma+1})_k}{(q)_k (q^{2-\gamma})_k} t^{k+1-\gamma}. \tag{1.7}$$

Here we note that if $\gamma \in \mathbb{Z}$ one of these solutions does not make sense. Then we can construct another solution in the form (1.6) (see [E] for details).

From the consideration above, we formulate our problem as follows. We set $t = q^x$ and rewrite (1.1) as a difference equation for a function of x, see (2.1). We try to find solutions of the following form:

$$\sum_{j=0}^{n} \sum_{k=0}^{\infty} c_{jk} x^j q^{kx}, \quad c_{jk} \in \mathbb{C}. \tag{1.8}$$

Here the part of $j > 0$ corresponds to the term in (1.6) with logarithmic singularity. Now we note that the function q^{Nx} is invariant under the shift $x \mapsto x + 1$. Hence we can rewrite (1.8) as

$$\sum_{j=0}^{n} \sum_{k=0}^{N-1} f_{jk}(x) x^j q^{kx}, \tag{1.9}$$

where $f_{jk}(x)$ is a periodic function with a period 1, that is a quasi-constant for the difference equation (2.1). We solve the q-hypergeometric equation with $q^N = 1$ and $\alpha, \beta, \gamma \in \mathbb{Z}$ on the space of functions of the form (1.9).

The result is as follows (Theorem 2.2). The space of solutions is two-dimentional over the field of quasi-constants, and all the solutions are represented as (1.6), that is $n = 0$ or 1 in (1.9).

Next we consider solutions to (1.1) with $|q| = 1$. One of the solutions is the q-hypergeometric function of the Barnes type at $|q| = 1$, see (3.1). We can deal with the integral representation of this function in the framework of the q-twisted cohomology at $|q| = 1$ [T]. It is shown that, if q is not a root of unity and the parameters α, β and γ are generic, we can construct two independent solutions to (1.1) in terms of the integral of the Barnes type by taking two independent homologies. However, in the case that $q^N = 1$ and $\alpha, \beta, \gamma \in \mathbb{Z}$, the function (3.1) is a unique solution of this form. In Theorem 3.1, we write down an explicit formula for this function in terms of the basis constructed in Theorem 2.2. Then we see that if the parameters α, β and γ satisfy some condition, the q-hypergeometric function at $|q| = 1$ has logarithmic singularity. On the other hand, the q-hypergeometric function with $0 < |q| < 1$ defined by (1.2) has no logarithmic singularity. This is a difference between the cases of $0 < |q| < 1$ and $|q| = 1$.

Acknoledgement

The author thanks Masaki Kashiwara and Tetsuji Miwa for valuable remarks.

2 A basis for the space of solutions

Let N be a integer with $N \geqslant 2$. The q-hypergeometric difference equation is defined by

$$L\Psi(x) = 0, \quad L := (1-D)(1-q^{\gamma-1}D) - q^x(1-q^\alpha D)(1-q^\beta D), \tag{2.1}$$

where D is the difference operator defined by $D\Psi(x) := \Psi(x+1)$. In this paper, we consider the equation (2.1) with

$$q := e^{\frac{2\pi i}{N}}, \quad \alpha, \beta, \gamma \in \{1, \cdots, N\}, \quad \beta \leqslant \alpha. \tag{2.2}$$

Note that the equation (2.1) is symmetric with respect to α and β, and hence we can assume $\beta \leqslant \alpha$ without loss of generality.

We denote by $\mathcal{C}$ the field of periodic meromorphic functions of x with a period 1. This field is a space of quasi-constants for (2.1) in the sense that if Ψ is a solution to (2.1) then $f\Psi$ is also a solution for any $f \in \mathcal{C}$.

Let us find a solution Ψ of the following form:

$$\Psi(x) = \sum_{j=0}^{n} \sum_{k=0}^{N-1} f_{jk} x^j q^{kx}, \quad f_{jk} \in \mathcal{C}. \tag{2.3}$$

It is easy to see the following.

Proposition 2.1 *The expression (2.3) is unique, that is*

$$\sum_{j=0}^{n} \sum_{k=0}^{N-1} f_{jk} x^j q^{kx} = 0 \quad \Longrightarrow \quad \forall f_{jk} = 0, \tag{2.4}$$

where $f_{jk} \in \mathcal{C}$.

Let $\mathcal{S}$ be the space of functions of the form (2.3):

$$\mathcal{S} := \left\{ \sum_{j=0}^{n} \sum_{k=0}^{N-1} f_{jk} x^j q^{kx} \mid n \geqslant 0, f_{jk} \in \mathcal{C} \right\}. \tag{2.5}$$

Set

$$\mathcal{P} := \left\{ \sum_{k=0}^{N-1} z_k q^{kx} \mid z_k \in \mathcal{C} \right\}. \tag{2.6}$$

Note that $L\mathcal{P} \subset \mathcal{P}$ and $L\mathcal{S} \subset \mathcal{S}$ because $q^{Nx} \in \mathcal{C}$.

The following result holds.

Theorem 2.2 *The subspace of solutions to (2.1) in* $\mathcal{S}$ *is two-dimensional over* $\mathcal{C}$. *A basis* $\{\Psi_1, \Psi_2\}$ *of this space is given as follows:*

1. $\gamma \leqslant \beta \leqslant \alpha$ *case.*

$$\begin{aligned}
\Psi_1(x) &= \sum_{k=0}^{N-\alpha} \frac{b_0 \cdots b_{k-1}}{a_1 \cdots a_k} q^{kx}, \\
\Psi_2(x) &= x\Psi_1(x) + \sum_{k=1}^{N-\alpha} \frac{b_0 \cdots b_{k-1}}{a_1 \cdots a_k} \sum_{j=1}^{k} \left(\frac{1-q^{\gamma+2j-1}}{a_j} - \frac{1-q^{\alpha+\beta+2(j-1)}}{b_{j-1}} \right) q^{kx} \\
&\quad -(1-q^{\beta-\alpha}) \frac{b_0 \cdots b_{N-\alpha-1}}{a_1 \cdots a_{N-\alpha+1}} \sum_{k=N-\alpha+1}^{N-\beta} \frac{b_{N-\alpha+1} \cdots b_{k-1}}{a_{N-\alpha+2} \cdots a_k} q^{kx} \\
&\quad -(1-q^{\gamma-1}) \sum_{k=N-\gamma+1}^{N-1} \frac{a_{k+1} \cdots a_{N-1}}{b_k \cdots b_{N-1} q^{Nx}} q^{kx}.
\end{aligned} \tag{2.7}$$

2. $\beta < \gamma \leqslant \alpha$ *case.*

$$\Psi_1(x) = \sum_{k=0}^{N-\alpha} \frac{b_0 \cdots b_{k-1}}{a_1 \cdots a_k} q^{kx}, \quad \Psi_2(x) = \sum_{k=N-\gamma+1}^{N-\beta} \frac{b_{N-\gamma+1} \cdots b_{k-1}}{a_{N-\gamma+2} \cdots a_k} q^{kx}. \tag{2.8}$$

3. $\beta \leqslant \alpha < \gamma$ *case.*

$$\begin{aligned}
\Psi_1(x) &= \sum_{k=N-\gamma+1}^{N-\alpha} \frac{b_{N-\gamma+1} \cdots b_{k-1}}{a_{N-\gamma+2} \cdots a_k} q^{kx}, \\
\Psi_2(x) &= x\Psi_1(x) - (1-q^{N-\gamma+1}) \sum_{k=0}^{N-\gamma} \frac{a_{k+1} \cdots a_{N-\gamma}}{b_k \cdots b_{N-\gamma}} q^{kx} \\
&\quad + \sum_{k=N-\gamma+2}^{N-\alpha} \frac{b_{N-\gamma+1} \cdots b_{k-1}}{a_{N-\gamma+2} \cdots a_k} \sum_{j=N-\gamma+2}^{k} \left(\frac{1-q^{\gamma+2j-1}}{a_j} - \frac{1-q^{\alpha+\beta+2(j-1)}}{b_{j-1}} \right) q^{kx} \\
&\quad -(1-q^{\beta-\alpha}) \frac{b_{N-\gamma-1} \cdots b_{N-\alpha-1}}{a_{N-\gamma+2} \cdots a_{N-\alpha+1}} \sum_{k=N-\alpha+1}^{N-\beta} \frac{b_{N-\alpha+1} \cdots b_{k-1}}{a_{N-\alpha+2} \cdots a_k} q^{kx}.
\end{aligned} \tag{2.9}$$

Here we set

$$a_k := (1-q^k)(1-q^{\gamma-1+k}), \quad b_k := (1-q^{\alpha+k})(1-q^{\beta+k}). \tag{2.10}$$

Proof. Here we prove the theorem in the first case. The proof for the other case is similar.

Set

$$\Psi = \sum_{j=0}^{n} x^j P_j, \quad P_j := \sum_{k=0}^{N-1} f_{jk} q^{kx} \in \mathcal{P}. \tag{2.11}$$

From Proposition 2.1, we see that $L\Psi = 0$ is equivalent to the following:

$$LP_j + \sum_{t=j+1}^{n} \binom{t}{t-j} L_{t-j} P_t = 0, \quad (j = n, n-1, \cdots, 0), \tag{2.12}$$

where

$$L_k := 2^k (1-q^x) D^2 - ((1+q^{\gamma-1}) - q^x(q^\alpha + q^\beta)) D. \tag{2.13}$$

It is easy to solve (2.12) for $j = n$ and $n-1$. The solution is given by

$$P_n = f_n \Psi_1, \quad P_{n-1} = f_{n-1}\Psi_1 + n f_n (\Psi_2 - x\Psi_1), \tag{2.14}$$

where $f_n, f_{n-1} \in \mathcal{C}$.

Now we prove that $n < 2$. If $n \geqslant 2$, there is a solution P_{n-2} to

$$LP_{n-2} + (n-1)(L_1 P_{n-1} + \frac{n}{2} L_2 P_n) = 0, \tag{2.15}$$

where P_{n-1} and P_n are given in (2.14). Especially, we have

$$Q_n := (n-1)(L_1 P_{n-1} + \frac{n}{2} L_2 P_n) \in L\mathcal{P}. \tag{2.16}$$

On the other hand, we can see that

$$L\mathcal{P} \ni \sum_{k=0}^{N-1} z_k q^{kx}, \quad (z_k \in \mathcal{C}) \quad \Longleftrightarrow \quad \begin{cases} \displaystyle\sum_{j=N-\beta+1}^{N-\gamma} \frac{b_j \cdots b_{N-\gamma}}{a_j \cdots a_{N-\gamma}} z_j + z_{N-\gamma+1} = 0, & (\gamma \neq 1), \\ \displaystyle\sum_{j=N-\beta+1}^{N-1} \frac{b_j \cdots b_{N-1}}{a_j \cdots a_{N-1}} z_j + q^{-Nx} z_0 = 0, & (\gamma = 1). \end{cases} \tag{2.17}$$

However, it can be checked that Q_n does not satisfy (2.17) if $n > 2$ and $f_n \neq 0$. This contradicts (2.16).

Hence $n = 0$ or 1, and $\{\Psi_1, \Psi_2\}$ is a basis. $\square$

3 The q-hypergeometric function at $q^N = 1$

We recall the definition of the q-hypergeometric function at $|q| = 1$.

Set $q = e^{2\pi i\omega}, \omega > 0$. The q-hypergeometric function of the Barnes type is given as follows [NU, T]:

$$\Psi(\alpha, \beta, \gamma; x) := \frac{\langle\alpha\rangle\langle\beta\rangle}{\langle 1\rangle\langle\gamma\rangle} \int_C q^{xz} \frac{\langle z+1+\frac{1}{\omega}\rangle\langle z+\gamma\rangle}{\langle z+\alpha\rangle\langle z+\beta\rangle} dz. \tag{3.1}$$

Here the function $\langle z \rangle$ is defined by

$$\langle z \rangle := \exp\left(\frac{\pi i}{2}\left((1+\omega)z - \omega z^2\right)\right) S_2(z|1, \frac{1}{\omega}), \tag{3.2}$$

where $S_2(z)$ is the double sine function. We refer the reader to [JM] for the double sine function. The contour C is the imaginary axis $(-i\infty, i\infty)$ except that the poles at

$$-\alpha + \mathbb{Z}_{\leqslant 0} + \frac{1}{\omega}\mathbb{Z}_{\leqslant 0}, \quad -\beta + \mathbb{Z}_{\leqslant 0} + \frac{1}{\omega}\mathbb{Z}_{\leqslant 0} \tag{3.3}$$

are on the left of C and the poles at

$$\mathbb{Z}_{\geqslant 0} + \frac{1}{\omega}\mathbb{Z}_{\geqslant 0}, \quad -\gamma + \mathbb{Z}_{\geqslant 1} + \frac{1}{\omega}\mathbb{Z}_{\geqslant 1} \tag{3.4}$$

are on the right of C. The integral (3.1) is absolutely convergent if

$$0 < \mathrm{Re}x < 1 + \frac{1}{\omega} + \mathrm{Re}\gamma - \mathrm{Re}\alpha - \mathrm{Re}\beta. \tag{3.5}$$

Then the function $\Psi(x)$ satisfies the equation (2.1) with $q = e^{2\pi i\omega}$.

Now we consider $\Psi(x)$ under the condition (2.2). We also assume that

$$\alpha + \beta \leqslant N - \gamma. \tag{3.6}$$

Then the integral (3.1) converges if $0 < \mathrm{Re}x < 1$ because $\omega = 1/N$ and (3.5).

Theorem 3.1 *Under the conditions (2.2) and (3.6), the q-hypergeometric function Ψ satisfies $\Psi \in \mathcal{S}$ and is represented explicitly as follows:*

1. $\gamma \leqslant \beta \leqslant \alpha$ case.

$$\Psi = \frac{1}{1 - q^{Nx}}\Psi_1. \tag{3.7}$$

2. $\beta < \gamma \leqslant \alpha$ case.

$$\Psi = \frac{1}{1 - q^{Nx}}\left\{\Psi_1 + \frac{(q)_{\gamma-1}}{(q)_{\alpha-1}(q)_{\beta-1}}\frac{(q)_{\alpha-\gamma}(q)_{N-\gamma+\beta}}{(q)_{N-\gamma+1}}\Psi_2\right\}. \tag{3.8}$$

3. $\beta \leqslant \alpha < \gamma$ case.

$$\Psi = \frac{1}{1 - q^{Nx}}\frac{(q)_{\gamma-1}}{(q)_{\alpha-1}(q)_{\beta-1}}\frac{1}{N}\frac{(q)_{N-\gamma+\alpha}(q)_{N-\gamma+\beta}}{(q)_{N-\gamma+1}}\left(C^{\gamma}_{\alpha,\beta}\Psi_1 - \Psi_2\right), \tag{3.9}$$

$$C^{\gamma}_{\alpha,\beta} := 1 - \left(\sum_{j=1}^{N-\gamma+1} + \sum_{j=N-\gamma+\alpha+1}^{N-1} + \sum_{j=N-\gamma+\beta+1}^{N-1}\right)\frac{q^j}{1-q^j}. \tag{3.10}$$

Here $\{\Psi_1, \Psi_2\}$ in (3.7), (3.8) and (3.9) is the basis of the space of solutions given in (2.7), (2.8) and (2.9), respectively.

Proof. Let us calculate the integral in (3.1). We denote by $\Phi(z)$ the integrand of (3.1):

$$\Phi(z) := q^{xz}\frac{\langle z+1+\frac{1}{\omega}\rangle\langle z+\gamma\rangle}{\langle z+\alpha\rangle\langle z+\beta\rangle}. \tag{3.11}$$

By using

$$\frac{\langle z+N\rangle}{\langle z\rangle} = \frac{1}{1-e^{2\pi iz}}, \tag{3.12}$$

we see the following under the condition (2.2):

$$\Phi(z+N) = q^{Nx}\Phi(z). \tag{3.13}$$

Hence, we have

$$(1-q^{Nx})\int_C \Phi(z)dz = \left(\int_C - \int_{C+N}\right)\Phi(z)dz \tag{3.14}$$

From (2.2), we can take the line $-\frac{1}{2}+i\mathbb{R}$ as the contour C. Then the right hand side of (3.14) is given by the sum of residues at $z=0,\cdots,N-1$. Therefore, we get

$$\Psi(x) = \frac{-2\pi i}{1-q^{Nx}}\frac{\langle\alpha\rangle\langle\beta\rangle}{\langle 1\rangle\langle\gamma\rangle}\sum_{k=0}^{N-1}\mathrm{res}_{z=k}\Phi(z). \tag{3.15}$$

By using

$$\frac{\langle z+1\rangle}{\langle z\rangle} = \frac{1}{1-q^z}, \tag{3.16}$$

we can represent the function $\Phi(z)$ in terms of q^z and calculate residues of this function explicitly.

It is easy to see that all the poles at $z=0,\cdots N-1$ are simple if $\beta\leqslant\alpha$ and $\gamma\leqslant\alpha$. Then we find the formulae (3.7) and (3.8) easily by using

$$(1-q)(1-q^2)\cdots(1-q^{N-1}) = N. \tag{3.17}$$

Let us consider the case of $\beta\leqslant\alpha<\gamma$. The poles at $z=0,\cdots,N-\gamma$ and $z=N-\alpha+1,\cdots,\beta$ are simple, and it is easy to calculate residues at these poles. The result is as follows:

$$\mathrm{res}_{z=k} = -\frac{1}{N}\frac{(q)_{N-\gamma+\alpha}(q)_{N-\gamma+\beta}}{(q)_{N-\gamma+1}}(1-q^{N-\gamma+1})\frac{a_{k+1}\cdots a_{N-\gamma}}{b_k\cdots b_{N-\gamma}},\quad (k=0,\cdots,N-\gamma) \tag{3.18}$$

and

$$\begin{aligned}\mathrm{res}_{z=k} &= -\frac{1}{N}\frac{(q)_{N-\gamma+\alpha}(q)_{N-\gamma+\beta}}{(q)_{N-\gamma+1}}\\ &\times (1-q^{\beta-\alpha})\frac{b_{N-\gamma-1}\cdots b_{N-\alpha-1}}{a_{N-\gamma+2}\cdots a_{N-\alpha+1}}\frac{b_{N-\alpha+1}\cdots b_{k-1}}{a_{N-\alpha+2}\cdots a_k},\quad (k=N-\alpha+1,\cdots,\beta)\end{aligned} \tag{3.19}$$

Here we used (3.17).

Next we calculate residues at $z = N-\gamma+1, \cdots, N-\alpha$. Note that these points are double poles. For $k = N-\gamma+1, \cdots, N-\alpha$, we have

$$\begin{aligned}&\mathrm{res}_{z=k}\Phi(z) = \\ &= q^{kx}\mathrm{res}_{z=0}\left(\frac{q^{zx}}{(1-q^z)^2}\prod_{j=1}^{k}\frac{1}{1-q^{z+j}}\prod_{j=\alpha+k}^{N-1}\frac{1}{1-q^{z+j}}\prod_{j=1}^{k+\gamma-1-N}\frac{1}{1-q^{z+j}}\prod_{j=\beta+k}^{N-1}\frac{1}{1-q^{z+j}}\right) \quad (3.20)\end{aligned}$$

By substituting

$$\begin{aligned}&q^{zx} = 1+\frac{2\pi i}{N}xz+o(z), \\ &\frac{1}{1-q^{z+j}} = \begin{cases} -\dfrac{N}{2\pi i}z^{-1}+\dfrac{1}{2}+o(1), & (j\equiv 0 \bmod N), \\ \dfrac{1}{1-q^j}+\dfrac{2\pi i}{N}\dfrac{q^j}{(1-q^j)^2}z+o(z), & (j\not\equiv 0 \bmod N), \end{cases} \quad \text{as} \quad z\to 0, \quad (3.21)\end{aligned}$$

we find

$$\begin{aligned}(3.20) &= q^{kx}\mathrm{res}_{z=0}\left((\frac{N}{2\pi i})^2 D_k z^{-2}+\frac{N}{2\pi i}D_k(x-E_k)z^{-1}+o(1)\right) \\ &= \frac{N}{2\pi i}D_k(x-E_k)q^{kx}, \quad (3.22)\end{aligned}$$

where

$$\begin{aligned}D_k &= \prod_{j=1}^{k}\frac{1}{1-q^j}\prod_{j=\alpha+k}^{N-1}\frac{1}{1-q^j}\prod_{j=1}^{k+\gamma-1-N}\frac{1}{1-q^j}\prod_{j=\beta+k}^{N-1}\frac{1}{1-q^j}, \\ E_k &= 1-\left(\sum_{j=1}^{k}+\sum_{j=\alpha+k}^{N-1}+\sum_{j=1}^{k+\gamma-1-N}+\sum_{j=\beta+k}^{N-1}\right)\frac{q^j}{1-q^j}. \quad (3.23)\end{aligned}$$

By using (3.17), we see

$$D_k = \frac{1}{N^2}\frac{(q)_{N-\gamma+\alpha}(q)_{N-\gamma+\beta}}{(q)_{N-\gamma+1}}\frac{b_{N-\gamma+1}\cdots b_{k-1}}{a_{N-\gamma+2}\cdots a_k}. \quad (3.24)$$

Moreover, we have

$$E_{N-\gamma+1} = C^{\gamma}_{\alpha,\beta}, \quad (3.25)$$

$$E_k - E_{k-1} = \frac{1-q^{\gamma+2k-1}}{a_k}-\frac{1-q^{\alpha+\beta+2(k-1)}}{b_{k-1}}, \quad (3.26)$$

where $C^{\gamma}_{\alpha,\beta}$ is given by (3.10). Hence we find

$$E_k = C^{\gamma}_{\alpha,\beta}+\sum_{j=N-\gamma+2}^{k}\left(\frac{1-q^{\gamma+2j-1}}{a_j}-\frac{1-q^{\alpha+\beta+2(j-1)}}{b_{j-1}}\right). \quad (3.27)$$

From this calculation, we get the relation (3.9). $\square$

Theorem 3.1 implies that the q-hypergeometric function of the Barnes type has logarithmic singularity in the case that $q^N = 1$ and $\beta \leqslant \alpha < \gamma$.

References

[AAR] Andrews, G. E., Askey, R. and Roy, R., *Special functions*, Encyclopedia of Mathematics and its Applications 71. Cambridge Univ. Press., 1999.

[E] Exton, H., *q-hypergeometric functions and applications*, Ellis Horwood Limited., 1983.

[GR] Gasper, G. and Rahman, M., *Basic hypergeometric series*, Encyclopedia of Mathematics and its Applications 35. Cambridge Univ. Press., 1990.

[JM] Jimbo, M. and Miwa, T., *Quantum KZ equation with* $|q| = 1$ *and correlation functions of the XXZ model in the gapless regime*, J. Phys. A: Math. Gen. **29** (1996), 2923-2958.

[NU] Nishizawa, M. and Ueno, K., *Integral solutions of q-difference equations of the hypergeometric type with* $|q| = 1$, Proceedings of the workshop "Infinite Analysis - Integral Systems and Representation Theory", IIAS Report No. 1997-001, 247-255. (`q-alg/9612014`)

[T] Takeyama, Y., *The q-twisted cohomology and the q-hypergeometric function at* $|q| = 1$, Publ. RIMS, Kyoto Univ., **37** (2001), 71-89.