DELIGNE'S CONJECTURE ON 1-MOTIVES

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Abstract. We reformulate a conjecture of Deligne on 1-motives by using the integral weight filtration of Gillet and Soulé on cohomology, and prove it. This implies the original conjecture up to isogeny. If the degree of cohomology is at most two, we can prove the conjecture for the Hodge realization without isogeny, and even for 1-motives with torsion.

Let X be a complex algebraic variety. We denote by $H^j_{(1)}(X,\mathbb{Z})$ the maximal mixed Hodge substructure of $H^j(X,\mathbb{Z})$ which is of type $\{(0,0),\,(0,1),\,(1,0),\,(1,1)\}$. Let $H^j_{(1)}(X,\mathbb{Z})_{\rm fr}$ be the quotient of $H^j_{(1)}(X,\mathbb{Z})$ by the torsion subgroup. P. Deligne ([10], 10.4.1) conjectured that the 1-motive corresponding to $H^j_{(1)}(X,\mathbb{Z})_{\rm fr}$ admits a purely algebraic description, that is, there should exist a 1-motive $M_j(X)_{\rm fr}$ which is defined without using the associated analytic space, and whose image $r_{\mathcal{H}}(M_j(X)_{\rm fr})$ under the Hodge realization functor $r_{\mathcal{H}}$ (see loc. cit. and (1.5) below) is canonically isomorphic to $H^j_{(1)}(X,\mathbb{Z})_{\rm fr}(1)$ (and similarly for the l-adic and de Rham realizations).

This conjecture has been proved for curves [10], for the second cohomology of projective surfaces [9], and for the first cohomology of any varieties [2] (see also [25]). In general, a careful analysis of the weight spectral sequence in Hodge theory leads us to a candidate for $M_j(X)_{\rm fr}$ up to isogeny (see also [26]). However, since the torsion part cannot be handled by Hodge theory, it is a rather difficult problem to solve the conjecture without isogeny.

In this paper, we introduce the notion of an *effective* 1-motive which admits torsion. By modifying morphisms, we can get an abelian category of 1-motives which admit torsion, and prove that this is equivalent to the category of graded-polarizable mixed $\mathbb{Z}$-Hodge structures of the above type. However, our construction gives in general *nonreduced* effective 1-motives, that is, the discrete part has torsion and its image in the semiabelian variety is nontrivial.

Let Y be a closed subvariety of X. Using an appropriate 'resolution', we can define a canonical integral weight filtration W on the relative cohomology $H^j(X,Y;\mathbb{Z})$. This is due to Gillet and Soulé ([14], 3.1.2) if X is proper. See also (2.3) below. Let $H^j_{(1)}(X,Y;\mathbb{Z})$ be the maximal Hodge substructure of $H^j(X,Y;\mathbb{Z})$ of the considered type. It has the induced weight filtration W, and so do its torsion part $H^j_{(1)}(X,Y;\mathbb{Z})_{\rm tor}$ and its free part $H^j_{(1)}(X,Y;\mathbb{Z})_{\rm fr}$. Using the same resolution as above, we construct the desired effective

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1-motive $M_j(X,Y)$. In general, only its free part $M_j(X,Y)_{\rm fr}$ is independent of the choice of the resolution. By a similar idea, we can construct the derived relative Picard groups together with an exact sequence similar to Bloch's localization sequence for higher Chow groups [7], see (2.6). Our fist main result shows a close relation between the nonreduced structure of our 1-motive and the integral weight filtration:

0.1. Theorem. *There exists a canonical isomorphism of mixed Hodge structures*

$$\phi_{\rm fr} : r_{\mathcal{H}}(M_j(X,Y))_{\rm fr}(-1) \to W_2H^j_{(1)}(X,Y;\mathbb{Z})_{\rm fr},$$

such that the semiabelian part and the torus part of $M_j(X,Y)$ correspond respectively to $W_1H^j_{(1)}(X,Y;\mathbb{Z})_{\rm fr}$ and $W_0H^j_{(1)}(X,Y;\mathbb{Z})_{\rm fr}$. A quotient of its discrete part by some torsion subgroup is isomorphic to $\mathrm{Gr}^W_2H^j_{(1)}(X,Y;\mathbb{Z})$. Furthermore, similar assertions hold for the l-adic and de Rham realizations.

This implies Deligne's conjecture for the relative cohomology up to isogeny. As a corollary, the conjecture without isogeny is reduced to:

$$H^j_{(1)}(X,Y;\mathbb{Z})_{\rm fr} = W_2H^j_{(1)}(X,Y;\mathbb{Z})_{\rm fr}.$$

This is satisfied if the $\mathrm{Gr}^W_qH^j(X,Y;\mathbb{Z})$ are torsion-free for $q > 2$. The problem here is that we cannot rule out the possibility of the contribution of the torsion part of $\mathrm{Gr}^W_qH^j(X,Y;\mathbb{Z})$ to $H^j_{(1)}(X,Y;\mathbb{Z})_{\rm fr}$. By construction, $M_j(X,Y)$ does not have information on $W_1H^j_{(1)}(X,Y;\mathbb{Z})_{\rm tor}$, and the morphism $\phi_{\rm fr}$ in (0.1) is actually induced by a morphism of mixed Hodge structures

$$\phi : r_{\mathcal{H}}(M_j(X,Y))(-1) \to W_2H^j_{(1)}(X,Y;\mathbb{Z})/W_1H^j_{(1)}(X,Y;\mathbb{Z})_{\rm tor}.$$

0.2. Theorem. *The composition of ϕ and the natural inclusion*

$$r_{\mathcal{H}}(M_j(X,Y))(-1) \to H^j_{(1)}(X,Y;\mathbb{Z})/W_1H^j_{(1)}(X,Y;\mathbb{Z})_{\rm tor}$$

is an isomorphism if $j \le 2$ or if $j = 3$, X is proper, and has a resolution of singularities whose third cohomology with integral coefficients is torsion-free, and whose second cohomology is of type (1,1).

The proof of these Theorems makes use of a cofiltration on a complex of varieties, which approximates the weight filtration, and simplifies many arguments. The key point in the proof is the comparison of the extension classes associated with a 1-motives and a mixed Hodge structure, as indicated in Carlson's paper [9]. This is also the point which is not very clear in [26]. We solve this problem by using the theory of mixed Hodge complexes due to Deligne [10] and Beilinson [4]. For the comparison of algebraic structures on the Picard group, we use the theory of admissible normal functions [29]. This also shows the representability of the Picard type functor. However, for an algebraic construction of the semiabelian part of the 1-motive $M_j(X,Y)$, we have to verify the representability in a purely algebraic way [2] (see also [26]). The proof of (0.2) uses the weight spectral sequence [10] with integral coefficients, which is associated to the above resolution, see (4.4). It is then easy to show

0.3. Proposition. *Deligne's conjecture without isogeny is true if $E_\infty^{p,j-p}$ is torsion-free for $p \le j-3$. The morphism ϕ is injective if $E_2^{p,j-1-p}=0$ for $p \le j-4$ and $E_1^{j-3,2}$ is of type (1,1).*

The paper is organized as follows. In Sect. 1 we review the theory of 1-motives with torsion. In Sect. 2, the existence of a canonical integral filtration is deduced from [17] by using a complex of varieties. See also [14]. In Sect. 3, we construct the desired 1-motive by using a cofiltration on a complex of varieties, and show the compatibility for the l-adic and de Rham realizations. After reviewing mixed Hodge theory in Sect. 4, we prove the main theorems in Sect. 5.

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Notation. In this paper, a variety means a separated reduced scheme of finite type over a field.

1. 1-Motives

We review the theory of 1-motives with torsion by modifying slightly [10]. This would be known to some specialists.

1.1. Let k be a field of characteristic zero, and $\overline{k}$ an algebraic closure of k. (The argument in the positive characteristic case is more complicated due to the nonreduced part of finite commutative group schemes, see [22].)

An *effective* 1-motive $M=[\Gamma \xrightarrow{f} G]$ over k consists of a locally finite commutative group scheme Γ/k and a semiabelian variety G/k together with a morphism of k-group schemes $f:\Gamma \to G$ such that $\Gamma(\overline{k})$ is a finitely generated abelian group. Note that Γ is identified with $\Gamma(\overline{k})$ endowed with Galois action because $\operatorname{char} k=0$. Sometimes an effective 1-motive is simply called a 1-motive, since the category of 1-motives will be defined by modifying only morphisms. A locally finite commutative group scheme Γ/k and a semiabelian variety G/k are identified with 1-motives $[\Gamma \to 0]$ and $[0 \to G]$ respectively.

An *effective* morphism of 1-motives

$$u=(u_{\mathrm{lf}},u_{\mathrm{sa}}):M=[\Gamma \xrightarrow{f} G] \to M'=[\Gamma' \xrightarrow{f'} G']$$

consists of morphisms of k-group schemes $u_{\mathrm{lf}}:\Gamma \to \Gamma'$ and $u_{\mathrm{sa}}:G \to G'$ forming a commutative diagram (together with f,f'). We will denote by

$$\mathrm{Hom}_{\mathrm{eff}}(M,M')$$

the abelian group of effective morphisms of 1-motives.

An effective morphism $u = (u_{\mathrm{lf}}, u_{\mathrm{sa}})$ is called *strict*, if the kernel of u_{sa} is connected. We say that u is a *quasi-isomorphism* if u_{sa} is an isogeny and if we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E & \longrightarrow & \Gamma & \longrightarrow & \Gamma' & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & E & \longrightarrow & G & \longrightarrow & G' & \longrightarrow & 0 \end{array} \tag{1.1.1}$$

(i.e. if the right half of the diagram is cartesian).

We define morphisms of 1-motives by inverting quasi-isomorphisms from the right, i.e. a morphism is represented by $u \circ v^{-1}$ with v a quasi-isomorphism. More precisely, we define

$$\mathrm{Hom}(M, M') = \varinjlim \mathrm{Hom}_{\mathrm{eff}}(\widetilde{M}, M'), \tag{1.1.2}$$

where the inductive limit is taken over isogenies $\widetilde{G} \to G$, and $\widetilde{M} = [\widetilde{\Gamma} \to \widetilde{G}]$ with $\widetilde{\Gamma} = \Gamma \times_G \widetilde{G}$. (This is similar to the localization of a triangulated category in [33].) Here we may restrict to isogenies $n : G \to G$ for positive integers n, because they form a cofinal index subset. Note that the transition morphisms of the inductive system are injective by the surjectivity of isogenies together with the property of fiber product. By (1.2) below, 1-motives form a category which will be denoted by $\mathcal{M}_1(k)$.

Let Γ_{tor} denote the torsion part of Γ, and put $M_{\mathrm{tor}} = \Gamma_{\mathrm{tor}} \cap \mathrm{Ker}\, f$. This is identified with $[M_{\mathrm{tor}} \to 0]$, and is called the torsion part of M. We say that M is *reduced* if $f(\Gamma_{\mathrm{tor}}) = 0$, *torsion-free* if $M_{\mathrm{tor}} = 0$, *free* if $\Gamma_{\mathrm{tor}} = 0$, and *torsion* if Γ is torsion and $G = 0$ (i.e. if $M = M_{\mathrm{tor}}$). Note that M is free if and only if it is reduced and torsion-free. We say that M has *split torsion*, if $M_{\mathrm{tor}} \subset \Gamma_{\mathrm{tor}}$ is a direct factor of Γ_{tor}.

We define $M_{\mathrm{fr}} = [\Gamma/\Gamma_{\mathrm{tor}} \to G/f(\Gamma_{\mathrm{tor}})]$. This is free, and is called the free part of M. If M is torsion-free, M_{fr} is naturally quasi-isomorphic to M. This implies that $[\Gamma/M_{\mathrm{tor}} \to G]$ is quasi-isomorphic to M_{fr} in general, and (1.3) gives a short exact sequence

$$0 \to M_{\mathrm{tor}} \to M \to M_{\mathrm{fr}} \to 0.$$

Remark. If M is free, M is a 1-motive in the sense of Deligne [10]. We can show

$$\mathrm{Hom}_{\mathrm{eff}}(M, M') = \mathrm{Hom}(M, M') \tag{1.1.3}$$

for $M, M' \in \mathcal{M}_1(k)$ such that M' is free. This is verified by applying (1.1.1) to the isogenies $\widetilde{G} \to G$ in (1.1.2). In particular, the category of Deligne 1-motives, denoted by $\mathcal{M}_1(k)_{\mathrm{fr}}$, is a full subcategory of $\mathcal{M}_1(k)$. The functoriality of $M \mapsto M_{\mathrm{fr}}$ implies

$$\mathrm{Hom}(M_{\mathrm{fr}}, M') = \mathrm{Hom}(M, M') \tag{1.1.4}$$

for $M \in \mathcal{M}_1(k)$, $M' \in \mathcal{M}_1(k)_{\mathrm{fr}}$. In other words, the functor $M \mapsto M_{\mathrm{fr}}$ is left adjoint of the natural functor $\mathcal{M}_1(k)_{\mathrm{fr}} \to \mathcal{M}_1(k)$.

1.2. Lemma. *For any effective morphism $u : \widetilde{M} \to M'$ and any quasi-isomorphism $\widetilde{M}' \to M'$, there exists a quasi-isomorphism $\widehat{M} \to \widetilde{M}$ together with a morphism $v : \widehat{M} \to \widetilde{M}'$ forming a commutative diagram. Furthermore, v is uniquely determined by the other morphisms and the commutativity. In particular, we have a well-defined composition of morphisms of 1-motives (as in [33])*

$$\mathrm{Hom}(M, M') \times \mathrm{Hom}(M', M'') \to \mathrm{Hom}(M, M''). \tag{1.2.1}$$

Proof. For the existence of $\widehat{M}$, it is sufficient to consider the semiabelian part $\widehat{G}$ by the property of fiber product. Then it is clear, because the isogeny $n : G' \to G'$ factors through $\widetilde{G}' \to G'$ for some positive integer n, and it is enough to take $n : \widetilde{G} \to \widetilde{G}$. We have the uniqueness of v for $\widehat{G}$ since there is no nontrivial morphism of $\widehat{G}$ to the kernel of the isogeny $\widetilde{G}' \to \widetilde{G}$ which is a torsion group. The assertion for $\widehat{\Gamma}$ follows from the property of fiber product. Then the fist two assertions imply (1.2.1) using the injectivity of the transition morphisms.

1.3. Proposition. *Let $u : M \to M'$ be an effective morphism of 1-motives. Then there exists a quasi-isomorphism $\widetilde{M}' \to M'$ such that u is lifted to a strict morphism $u'\ M \to \widetilde{M}'$ (i.e. $\mathrm{Ker}\, u'_{\mathrm{sa}}$ is connected). In particular, $\mathcal{M}_1(k)$ is an abelian category.*

Proof. It is enough to show the following assertion for the semiabelian variety part: There exists an isogeny $\widetilde{G}' \to G'$ with a morphism $u'_{\mathrm{sa}} : G \to \widetilde{G}'$ lifting u_{sa} such that $\mathrm{Ker}\, u'_{\mathrm{sa}}$ is connected. (Indeed, the first assertion implies the existence of kernel and cokernel, and their independence of the representative of a morphism is easy.)

For the proof of the assertion, we may assume that $\mathrm{Ker}\, u'_{\mathrm{sa}}$ is torsion, dividing G by the identity component of $\mathrm{Ker}\, u_{\mathrm{sa}}$. Let n be a positive integer annihilating $E := \mathrm{Ker}\, u_{\mathrm{sa}}$. We have a commutative diagram

$$\begin{array}{ccccc} & & E & \xrightarrow{n} & E \\ & & \downarrow & & \downarrow \\ {}_nG & \longrightarrow & G & \xrightarrow{n} & G \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle u_{\mathrm{sa}}} \\ {}_nG' & \longrightarrow & G' & \xrightarrow{n} & G' \end{array} \tag{1.3.1}$$

Let $\widetilde{G}'$ be the quotient of G' by the image of ${}_nG$. Then it factorizes $n : G' \to G'$, and u_{sa} is lifted to a morphism u'_{sa} to $\widetilde{G}'$, because the image of ${}_nG$ in $\widetilde{G}'$ vanishes. Moreover, $\mathrm{Ker}\, u'_{\mathrm{sa}} = 0$, because it is the image of $E + {}_nG$ by the multiplication by n.

Remark. An isogeny of semiabelian varieties $G' \to G$ with kernel E corresponds to an *injective* morphism of 1-motives

$$[0 \to G'] \to [E \to G'] = [0 \to G].$$

1.4. Lemma. *Assume k is algebraically closed. Then, for a 1-motive M, there exists a quasi-isomorphism $M' \to M$ such that $M' = [\Gamma' \xrightarrow{f'} G']$ has split torsion.*

Proof. Let n be a positive integer such that $E := \Gamma_{\mathrm{tor}} \cap \mathrm{Ker}\, f$ is annihilated by n. Then G' is given by G with isogeny $G' \to G$ defined by the multiplication by n. Let $\Gamma' = \Gamma \times_G G'$. We have a diagram of the nine lemma

$$\begin{array}{ccccc} & & {}_nG & = & {}_nG \\ & & \downarrow & & \downarrow \\ E & \longrightarrow & \Gamma'_{\mathrm{tor}} & \longrightarrow & f'(\Gamma'_{\mathrm{tor}}) \\ \| & & \downarrow & & \downarrow \\ E & \longrightarrow & \Gamma_{\mathrm{tor}} & \longrightarrow & f(\Gamma_{\mathrm{tor}}) \end{array} \tag{1.4.1}$$

The l-primary torsion subgroup of G is identified with the quotient of $V_lG := T_lG \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ by $M := T_lG$. Let M' be the $\mathbb{Z}_l$-submodule of V_lG such that $M' \supset M$ and M'/M is isomorphic to the l-primary part of $f(\Gamma_{\mathrm{tor}})$. Then there exists a basis $\{e_i\}_{1 \le i \le r}$ of M' together with integers $c_i\,(1 \le i \le r)$ such that $\{l^{c_i} e_i\}_{1 \le i \le r}$ is a basis of M. So the assertion is reduced to the following (because the assumption on the second exact sequence $0 \to {}_nG \to f'(\Gamma'_{\mathrm{tor}}) \to f(\Gamma_{\mathrm{tor}}) \to 0$ is verified by the above argument):

Let $0 \to A_i \to B_i \to C \to 0$ be short exact sequences of finite abelian groups for $i = 1, 2$. Put $B = B_1 \times_C B_2$. Assume that the second exact sequence (i.e., for $i = 2$) is the direct sum of

$$0 \to \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/nb_j\mathbb{Z} \to \mathbb{Z}/b_j\mathbb{Z} \to 0,$$

such that A_1 is annihilated by n. Then the projection $B \to B_2$ splits.

For the proof of this, note that B corresponds to $(e_1, e_2) \in \mathrm{Ext}^1(C, A_1 \times A_2)$, where the $e_i \in \mathrm{Ext}^1(C, A_i)$ are defined by the exact sequences. Then it is enough to construct a morphism $u : A_2 \to A_1$ such that e_1 is the composition of e_2 and u, because this implies an automorphism of $A_1 \times A_2$ over A_2 which is defined by $(a_1, a_2) \mapsto (a_1 - u(a_2), a_2)$ so that (e_1, e_2) corresponds to $(0, e_2)$. (Indeed, it induces an automorphism of B over B_2 so that e_1 becomes 0.) But the existence of such u is clear by hypothesis.

The following is a generalization of Deligne's construction ([10], 10.1.3).

1.5. Proposition. *If $k = \mathbb{C}$, we have an equivalence of categories*

$$r_{\mathcal{H}} : \mathcal{M}_1(\mathbb{C}) \xrightarrow{\sim} \mathrm{MHS}_1, \tag{1.5.1}$$

where MHS_1 *is the category of mixed $\mathbb{Z}$-Hodge structures H of type $\{(0,0), (0,-1), (-1,0), (-1,-1)\}$ such that $\mathrm{Gr}^W_{-1} H_{\mathbb{Q}}$ is polarizable.*

Proof. The argument is essentially the same as in [10]. For a 1-motive $M = [\Gamma \xrightarrow{f} G]$, let $\mathrm{Lie}\, G \to G$ be the exponential map, and Γ_1 be its kernel. Then we have a commutative

diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma_1 & \longrightarrow & H_{\mathbb{Z}} & \longrightarrow & \Gamma & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Gamma_1 & \longrightarrow & \operatorname{Lie} G & \longrightarrow & G & \longrightarrow & 0 \end{array} \tag{1.5.2}$$

which defines $H_{\mathbb{Z}}$, and $F^0H_{\mathbb{C}}$ is given by the kernel of the projection $H_{\mathbb{C}} := H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C} \to \operatorname{Lie} G$. We get $W_{-1}H_{\mathbb{Q}}$ from Γ_1, and $W_{-2}H_{\mathbb{Q}}$ from the corresponding exact sequence for the torus part of G. (See also Remark below.)

We can verify that $H_{\mathbb{Z}}$ and F^0 are independent of the representative of M (i.e. a quasi-isomorphism induces isomorphisms of $H_{\mathbb{Z}}$ and F^0). Indeed, for an isogeny $M' \to M$, we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma_1' & \longrightarrow & \operatorname{Lie} G' & \longrightarrow & G' & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow & & \\ 0 & \longrightarrow & \Gamma_1 & \longrightarrow & \operatorname{Lie} G & \longrightarrow & G & \longrightarrow & 0 \end{array} \tag{1.5.3}$$

and the assertion follows by taking the base change by $\Gamma \to G$. So we get the canonical functor (1.5.1). We show that this is fully faithful and essentially surjective. (To construct a quasi-inverse, we have to choose a splitting of the torsion part of $H_{\mathbb{Z}}$ for any $H \in \mathrm{MHS}_1$.)

For the proof of the essential surjectivity, we may assume that H is either torsion-free or torsion. Note that we may assume the same for 1-motives by (1.4). But for these H we have a canonical quasi-inverse as in [10]. Indeed, if H is torsion-free, we lift the weight filtration W to $H_{\mathbb{Z}}$ so that the $\mathrm{Gr}_k^W H_{\mathbb{Z}}$ are torsion-free. Then we put

$$\Gamma = \mathrm{Gr}_0^W H_{\mathbb{Z}}, \quad G = J(W_{-1}H)\,(= \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}, W_{-1}H)),$$

(see [8]), and $f : \Gamma \to G$ is given by the boundary map

$$\mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Z}, \mathrm{Gr}_0^W H) \to \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}, W_{-1}H)$$

associated with $0 \to W_{-1}H \to H \to \mathrm{Gr}_0^W H \to 0$. It is easy to see that this is a quasi-inverse. The quasi-inverse for a torsion H is the obvious one.

As a corollary, we have the full faithfulness of $r_{\mathcal{H}}$ for free 1-motives using (1.1.3). So it remains to show that (1.5.1) induces

$$\mathrm{Hom}(M, M') = \mathrm{Hom}(r_{\mathcal{H}}(M), r_{\mathcal{H}}(M')) \tag{1.5.4}$$

when $M = [\Gamma \to G]$ is free and M' is torsion. Put $H = r_{\mathcal{H}}(M)$. We will identify both M' and $r_{\mathcal{H}}(M')$ with a finite abelian group Γ'.

Let $W_{-1}M = [0 \to G], \mathrm{Gr}_0^W M = [\Gamma \to 0]$. Then we have a short exact sequence

$$0 \to \mathrm{Hom}(\mathrm{Gr}_0^W M, M') \to \mathrm{Hom}(M, M') \to \mathrm{Hom}(W_{-1}M, M') \to 0,$$

because $\mathrm{Ext}^1(\mathrm{Gr}_0^W M, M') = \mathrm{Ext}^1(\Gamma, \Gamma') = 0$. Since we have the corresponding exact sequence for mixed Hodge structures and the assertion for $\mathrm{Gr}_0^W M$ is clear, we may assume $M = W_{-1}M$, i.e., $\Gamma = 0$.

Let $T(G)$ denote the Tate module of G. This is identified with the completion of $H_{\mathbb{Z}}$ using (1.5.3). Then

$$\mathrm{Hom}(M, M') = \mathrm{Hom}(T(G), \Gamma') = \mathrm{Hom}(H_{\mathbb{Z}}, \Gamma'),$$

and the assertion follows.

Remark. Let T be the torus part of G. Then we get in (1.5.2) the integral weight filtration W on $H := r_{\mathcal{H}}(M)$ by

$$W_{-1}H_{\mathbb{Z}} = \Gamma_1, \quad W_{-2}H_{\mathbb{Z}} = \Gamma_1 \cap \mathrm{Lie\ T}. \tag{1.5.5}$$

2. Geometric resolution

Using the notion of a complex of varieties together with some arguments from [17] (see also [14], [16]), we show the existence of a canonical integral weight filtration on cohomology.

2.1. Let $\mathcal{V}_k$ denote the *additive* category of k-varieties, where a morphism $X' \to X''$ is a (formal) finite $\mathbb{Z}$-linear combination $\sum_i [f_i]$ with f_i a morphism of connected component of X' to X''. It is identified with a cycle on $X' \times_k X''$ by taking the graph. (This is similar to a construction in [14].) We say that a morphism $\sum_i n_i[f_i]$ is proper, if each f_i is. The category of k-varieties in the usual sense is naturally viewed as a subcategory of the above category. For a k-variety X, we have similarly the additive category $\mathcal{V}_X$ consisting of *proper* k-varieties over X, where the morphisms are assumed to be defined over X in the above definition.

Since these are additive categories, we can define the categories of complexes $\mathcal{C}_k, \mathcal{C}_X$, and also the categories $\mathcal{K}_k, \mathcal{K}_X$ where morphisms are considered up to homotopy as in [33]. We will denote an object of $\mathcal{C}_X, \mathcal{K}_X$ (or $\mathcal{C}_k, \mathcal{K}_k$) by $(X_\bullet, d)$, where $d : X_j \to X_{j-1}$ is the differential, and will be often omitted to simplify the notation. The structure morphism is denoted by $\pi : X_\bullet \to X$. (This lower index of $X_\bullet$ is due to the fact that we consider only contravariant functors from this category.) For $i \in \mathbb{Z}$, we define the shift of complex by $(X_\bullet[i])_p = X_{p+i}$. We say that $Y_\bullet$ is a closed subcomplex of $X_\bullet$ if the Y_i are closed subvarieties of X_i, and are stable by the morphisms appearing in the differential of $X_\bullet$.

We will denote by $\mathcal{C}_X^b$ the full subcategory of $\mathcal{C}_X$ consisting of bounded complexes, and by $\mathcal{C}_{X^{\mathrm{nsqp}}}^b$ the full subcategory of $\mathcal{C}_X^b$ consisting of complexes of smooth quasi-projective varieties. (Here nsqp stands for nonsingular and quasiprojective.) Let D be a closed subvariety of X. We denote by $\mathcal{C}_{X\langle D\rangle^{\mathrm{nsqp}}}^b$ the full subcategory of $\mathcal{C}_{X^{\mathrm{nsqp}}}^b$ consisting of $X_\bullet$ such that $D_j := \pi^{-1}(D) \cap X_j$ is locally either a connected component or a divisor with simple normal crossings for any j. Here simple means that the irreducible components of D_j are smooth. For an integer j, let $\mathcal{C}_X^{b,\geq j}$ denote the full subcategory of $\mathcal{C}_X^b$ consisting of

complexes $X_\bullet$ such that $X_i = \emptyset$ for $i < j$, and similarly for $\mathcal{C}_X^{b,\geq j}{}_{\mathrm{nsqp}}$, $\mathcal{C}_{X\langle D\rangle}^{b,\geq j}{}^{\mathrm{nsqp}}$. Replacing $\mathcal{C}$ with $\mathcal{K}$, we define similarly $\mathcal{K}^b_{X^{\mathrm{nsqp}}}$, $\mathcal{K}^b_{X\langle D\rangle^{\mathrm{nsqp}}}$, etc.

We say that $X_\bullet \in \mathcal{K}_X^b$ is *strongly acyclic* if there exist $X'_\bullet \in \mathcal{K}_X^b$ isomorphic to $X_\bullet$ in $\mathcal{K}_X^b$ and a finite filtration G on $X'_\bullet$ such that the restriction of G to each component X'_j is given by direct factors, and for each integer i there exists a birational proper morphism of k-varieties $g : Y' \to Y$ together with a closed subvariety Z of Y satisfying the following condition: Letting $Z' = (Y' \times_Y Z)_{\mathrm{red}}$, the morphism $g : Y' \setminus Z' \to Y \setminus Z$ is an isomorphism and the graded piece $\mathrm{Gr}_i^G X'_\bullet$ is isomorphic in $\mathcal{K}_X^b$ to the single complex associated to

$$\begin{array}{ccc} Z' & \longrightarrow & Y' \\ \downarrow & & \downarrow \\ Z & \longrightarrow & Y \end{array} \tag{2.1.1}$$

up to a shift of complex. Clearly, this condition is stable by mapping cone. We say that a morphism $X'_\bullet \to X_\bullet$ in $\mathcal{C}_X^b$ or $\mathcal{K}_X^b$ is a *strong quasi-isomorphism* if its mapping cone is strongly acyclic in $\mathcal{K}_X^b$. This condition is stable by compositions, using the octahedral axiom of the triangulated category. Similarly, if vu and u or v are strongly acyclic, then so is the remaining. (It is not clear whether the strongly acyclic complexes form a thick subcategory in the sense of Verdier.)

We say that a proper morphism of k-varieties $X' \to X$ has the *lifting property* if it induces a surjective morphism $X'(K) \to X(K)$ for any field K (see [14]), or equivalently, if any irreducible subvariety of X can be lifted birationally to X'. We say that a morphism $u : X' \to X$ in $\mathcal{V}_k$ has the *lifting property*, if for any connected component X_i of X, there exists a connected component $X'_{i'}$ of X' such that the restriction of u to $X'_{i'}$ is given by a proper morphism $f_i : X'_{i'} \to X_i$ with coefficient ± 1 and f_i has the lifting property. We say that a morphism $u : X' \to X$ in $\mathcal{V}_k$ is *of birational type* if for any irreducible component X_i of X, there exists uniquely a connected component $X'_{i'}$ of X' such that the restriction of u to $X'_{i'}$ is defined by a birational proper morphism $f_i : X'_{i'} \to X_i$ with coefficient ± 1, and this gives a bijection between the irreducible components of X' and X.

For $X_\bullet \in \mathcal{C}_X^b$, we say that a morphism $u : X'_\bullet \to X_\bullet$ of $\mathcal{C}_X^b$ is a *quasi-projective resolution over* $X\langle D\rangle$, if $X'_\bullet \in \mathcal{C}^b_{X\langle D\rangle^{\mathrm{nsqp}}}$ and u is a strong quasi-isomorphism in $\mathcal{K}_X^b$. We say that u is a quasi-projective resolution *of degree* $\geq j$ over $X\langle D\rangle$, if furthermore $X'_\bullet, X_\bullet \in \mathcal{C}_X^{b,\geq j}$ and $u : X'_j \to X_j$ is of birational type. We denote by $\mathcal{K}^b_{X\langle D\rangle^{\mathrm{nsqp}}}(X_\bullet)$ the category of quasi-projective resolutions $u : X'_\bullet \to X_\bullet$ over $X\langle D\rangle$ (which are morphisms in $\mathcal{C}_X^b$). A morphism of u to v is a morphism w of the source of u to that of v in $\mathcal{K}_X^b$ such that $u = vw$ in $\mathcal{K}_X^b$. If $X_\bullet \in \mathcal{C}_X^{b,\geq j}$, we define similarly $\mathcal{K}^{b,\geq j}_{X\langle D\rangle^{\mathrm{nsqp}}}(X_\bullet)$ by assuming further the condition on degree $\geq j$.

For $X_\bullet \in \mathcal{C}_X^{b,\geq j}$ and a closed subcomplex $Y_\bullet$, we say that $u : (X'_\bullet, Y'_\bullet) \to (X_\bullet, Y_\bullet)$ is a *smooth quasi-projective modification of degree* $\geq j$, if $X'_\bullet \in \mathcal{C}^{b,\geq j}_{X\langle D\rangle^{\mathrm{nsqp}}}$, $Y'_\bullet = (X'_\bullet \times_{Y_\bullet} X_\bullet)_{\mathrm{red}}$, $u : X'_\bullet \to X_\bullet$ is a proper morphism inducing an isomorphism $X'_\bullet \setminus Y'_\bullet \to X_\bullet \setminus Y_\bullet$, and $u : X'_j \to X_j$ is of birational type.

Remarks. (i) A birational proper morphism $f : X' \to X$ has the lifting property. Indeed, according to Hironaka [18], there exists a variety X'' together with morphisms $g : X'' \to X$

and $h : X'' \to X'$, such that $fh = g$ and g is obtained by iterating blowing-ups with smooth centers. (Here we may assume that the centers are smooth using Hironaka's theory of resolution of singularities.) This implies that a proper morphism has the lifting property if the generic points of the irreducible components can be lifted.

(ii) For $X_\bullet \in \mathcal{C}_X^{b,\geq j}$ and a closed subcomplex $Y_\bullet$ such that $\dim Y_\bullet < \dim X_\bullet$, there exists a smooth quasi-projective modification $(X'_\bullet, Y'_\bullet) \to (X_\bullet, Y_\bullet)$ of degree $\geq j$ by replacing $Y_\bullet$ with a larger subcomplex of the same dimension if necessary. This follows from [17], I, 2.6, except the birationality of $X'_j \to X_j$, because there are connected components of $X'_\bullet$ which are not birational to irreducible components of $X_\bullet$. Indeed, if we denote by $Z_{i,a}$ the images of the irreducible components of $X_k\,(k \leq i)$ by morphisms to X_i which are obtained by composing morphisms appearing in the differential of $X_\bullet$, then the connected components of X'_i are 'sufficiently blown-up' resolutions of singularities of $Z_{i,a}$, and are defined by increasing induction on i, lifting the differential of $X_\bullet$ (see loc. cit). However, if $Z_{j,a}$ is a proper closed subvariety of some irreducible component $X_{j,b}$ of X_j, we may replace the resolution of $Z_{j,a}$ by its lifting to the resolution of $X_{j,b}$ using the lifting property, because the differential $X_j \to X_{j-1}$ is zero.

2.2. Proposition. *For any $X_\bullet \in \mathcal{C}_X^{b,\geq j}$, there exists a quasi-projective resolution $X'_\bullet \to X_\bullet$ of degree $\geq j$ over $X\langle D\rangle$, and the category $\mathcal{K}^{b,\geq j}_{X\langle D\rangle^{\mathrm{nsqp}}}(X_\bullet)$ is weakly directed in the following sense: For any $u_i \in \mathcal{K}^{b,\geq j}_{X\langle D\rangle^{\mathrm{nsqp}}}(X_\bullet)\,(i = 1, 2)$, there exists $u_3 \in \mathcal{K}^{b,\geq j}_{X\langle D\rangle^{\mathrm{nsqp}}}(X_\bullet)$ together with morphisms $u_3 \to u_i$ in $\mathcal{K}^{b,\geq j}_{X\langle D\rangle^{\mathrm{nsqp}}}(X_\bullet)$.*

Proof. We first show that $\mathcal{K}^{b,\geq j}_{X\langle D\rangle^{\mathrm{nsqp}}}(X_\bullet)$ is nonempty by induction on $n := \dim X_\bullet$. There exists a smooth quasi-projective modification $(X'_\bullet, Y'_\bullet) \to (X_\bullet, Y_\bullet)$ of degree $\geq j$ as in the above Remark (ii). Then we have a strong quasi-isomorphism

$$C(Y'_\bullet \to X'_\bullet \oplus Y_\bullet) \to X_\bullet$$

where the direct sum means the disjoint union. So it is enough to show by induction that $\widetilde{X}_\bullet := C(Y'_\bullet \to Y_\bullet)$ has a strong quasi-isomorphism

$$\widetilde{Z}_\bullet \to \widetilde{X}_\bullet \tag{2.2.1}$$

in $\mathcal{C}_X^b$ such that $\widetilde{Z}_\bullet \in \mathcal{C}^{b,\geq j}_{X\langle D\rangle^{\mathrm{nsqp}}}$ and for any irreducible component $\widetilde{Z}_{j,i}$ of $\widetilde{Z}_j$ the restriction of the differential to some irreducible component $\widetilde{Z}_{j+1,i}$ of $\widetilde{Z}_{j+1}$ is given by an isomorphism onto $\widetilde{Z}_{j,i}$ with coefficient ± 1, under the inductive hypothesis:

$$\widetilde{X}_{j+1} \to \widetilde{X}_j \text{ has the lifting property.} \tag{2.2.2}$$

Indeed, admitting this, $\widetilde{Z}_\bullet$ is then isomorphic to the mapping cone of

$$\widetilde{Z}'_\bullet \to \oplus_i C(\pm id : \widetilde{Z}_{j,i} \to \widetilde{Z}_{j,i})[-j]$$

with $\widetilde{Z}'_\bullet \in \mathcal{C}^{b,\geq j}_{X\langle D\rangle^{\mathrm{nsqp}}}$, and the mapping cone of $\pm id$ is isomorphic to zero in $\mathcal{K}_X^b$.

To show (2.2.1), we repeat the above argument with $X_\bullet$ replaced by $\widetilde{X}_\bullet$, and get a smooth quasi-projective modification $(\widetilde{X}'_\bullet, \widetilde{Y}'_\bullet) \to (\widetilde{X}_\bullet, \widetilde{Y}_\bullet)$. By the lifting property (2.2.2), we may assume that for any irreducible component $\widetilde{X}_{j,i}$ of $\widetilde{X}_j$, the corresponding irreducible component $\widetilde{X}'_{j,i}$ of $\widetilde{X}'_j$ has a morphism f_i to $\widetilde{X}_{j+1}$ such that the composition of f_i and $d : \widetilde{X}_{j+1} \to \widetilde{X}_j$ is the canonical morphism $\widetilde{X}'_{j,i} \to \widetilde{X}_{j,i}$ up to a sign. If $\dim \widetilde{X}'_{j,i} = \dim \widetilde{X}_\bullet$, then f_i induces a birational morphism to $\operatorname{Im} f_i$ and we may assume that there exists an irreducible component $\widetilde{X}'_{j+1,i}$ such that the restriction of d to $\widetilde{X}'_{j+1,i}$ is given by the isomorphism $\widetilde{X}'_{j+1,i} \to \widetilde{X}'_{j,i}$ by replacing $\widetilde{X}'_\bullet$ if necessary, because the differential d of $\widetilde{X}'_\bullet$ is defined by lifting d of $\widetilde{X}_\bullet$ (see loc. cit). Then we can modify the morphism $\widetilde{X}'_{j+1} \to \widetilde{X}_{j+1}$ by using f_i for $\dim \widetilde{X}_{j,i} < \dim \widetilde{X}_\bullet$, and replace $\widetilde{X}'_j$ with the union of the maximal dimensional components. So we may assume that $\widetilde{X}'_j$ is equidimensional, because the modified $\widetilde{X}'_\bullet \to \widetilde{X}_\bullet$ still induces an isomorphism over the complement of $\widetilde{Y}_\bullet$ by replacing $\widetilde{Y}_\bullet$ if necessary. Here we may assume also that $\widetilde{Y}_{j+1} \to \widetilde{Y}_j$ has the lifting property by taking $\widetilde{Y}_\bullet$ appropriately (due to (2.2.2) and the above Remark (i)). Then, considering the mapping cone of $\widetilde{Y}'_\bullet \to \widetilde{Y}_\bullet$, the first assertion follows by induction.

The proof of the second assertion is similar. Consider the shifted mapping cone (i.e. the first term has degree zero):

$$X'_\bullet = [X_{1,\bullet} \oplus X_{2,\bullet} \to X_\bullet], \tag{2.2.3}$$

where the morphism is given by $u_1 - u_2$. Then $X'_\bullet \to X_{a,\bullet}$ is a strong quasi-isomorphism. Note that the composition of the canonical morphism $X'_\bullet \to X_{a,\bullet}$ and u_a is independent of a up to homotopy.

By definition, for any irreducible component $X'_{j-1,i} = X_{j,i}$ of $X'_{j-1} = X_j$, there exist two connected components Z_i, Z'_i of X'_j such that the restrictions of d to Z_i, Z'_i are given by proper morphisms $Z_i \to X'_{j-1,i}, Z'_i \to X'_{j-1,i}$ which have the lifting property (with coefficient ± 1). Then by the same argument as above, we have a smooth quasi-projective modification $u' : (X''_\bullet, Y''_\bullet) \to (X'_\bullet, Y'_\bullet)$ of degree $\geq j-1$. Here we may assume that the connected component $X''_{j-1,i}$ of X''_{j-1} which is birational to $X'_{j-1,i}$ has morphisms to Z_i, Z'_i factorizing the morphisms to $X'_{j-1,i}$, and X''_j has two connected components such that the restriction of d to these components are given by isomorphisms onto $X''_{j-1,i}$ (with coefficients ± 1). We may also assume that $Y'_j \to Y'_{j-1}$ has the lifting property as before.

Then, applying the same argument to $C(Y''_\bullet \to Y'_\bullet)$, and using induction on dimension, we get a strong quasi-isomorphism

$$\widetilde{X}_\bullet \to X'_\bullet$$

such that $\widetilde{X}_\bullet \in \mathcal{C}^{b,\geq j-1}_{X\langle D\rangle^{\mathrm{nsqp}}}$, and for any irreducible component $\widetilde{X}_{j-1,i}$ of $\widetilde{X}_{j-1}$, $\widetilde{X}_j$ has two connected components such that the restrictions of d (resp. of the morphism to X'_j) to these components are given by isomorphisms onto $\widetilde{X}_{j-1,i}$ (resp. by birational proper morphisms to Z_i, Z'_i) with coefficients ± 1. Thus $\widetilde{X}_\bullet$ is isomorphic to the mapping cone of

$$\widetilde{X}'_\bullet \to \oplus_i C(\pm id : \widetilde{X}_{j-1,i} \to \widetilde{X}_{j-1,i})[-j+1] \tag{2.2.4}$$

where $\widetilde{X}'_\bullet \in \mathcal{C}^{b,\geq j-1}_{X\langle D\rangle^{\mathrm{nsqp}}}$. So the second assertion follows.

Remark. It is not clear if for any $u_i \in \mathcal{K}^{b,\geq j}_{X\langle D\rangle^{\mathrm{nsqp}}}(X_\bullet)(i=1,2)$ and $v_a : u_2 \to u_1$, there exists $u_3 \in \mathcal{K}^{b,\geq j}_{X\langle D\rangle^{\mathrm{nsqp}}}(X_\bullet)$ together with $w : u_3 \to u_2$ such that $v_1 w = v_2 w$. This condition is necessary to define an inductive limit over the category $\mathcal{K}^{b,\geq j}_{X\langle D\rangle^{\mathrm{nsqp}}}(X_\bullet)$. If we drop the condition on the degree $\geq j$, it can be proved for $\mathcal{K}^{b}_{X\langle D\rangle^{\mathrm{nsqp}}}(X_\bullet)$ by using the mapping cone (2.2.3). Indeed, let $K_{i,\bullet}$ be the source of u_i for $i=1,2$, and $K_{3,\bullet}$ the mapping cone of $(v_1 - v_2, 0) : C(K_{2,\bullet} \to 0) \to C(K_{1,\bullet} \to K_\bullet)[-2]$, choosing a homotopy h such that $dh + hd = u_1 v_1 - u_1 v_2$. Then $w : K_{3,\bullet} \to K_{2,\bullet}$ is given by the projection, and $v_1 w - v_2 w : K_{3,\bullet} \to K_{1,\bullet}$ factors through a morphism $(v_1 - v_2, 0) : K_{3,\bullet} \to C(K_{1,\bullet} \to K_\bullet)[-1]$, which is homotopic to zero.

2.3. Corollary. *For a complex algebraic variety X and a closed subvariety Y, there is a canonical integral weight filtration W on the relative cohomology $H^j(X,Y;\mathbb{Z})$. Furthermore, it is defined by a quasi-projective resolution $\overline{X}_\bullet \to C(\overline{Y} \to \overline{X})$ of degree ≥ 0 over $\overline{X}\langle D\rangle$, where $\overline{X}$ is a compactification of X, $\overline{Y}$ is the closure of Y in $\overline{X}$, and $D = \overline{X} \setminus X$.*

Remarks. (i) The first assertion is due to Gillet and Soulé ([14], 3.1.2) in the case X is proper (replacing $X_\bullet$ with a simplicial resolution). It is expected that their integral weight filtration coincides with ours.

(ii) If X is proper, we have

$$W_{i-1}H^i(X,Y;\mathbb{Z}) = \mathrm{Ker}(H^i(X,Y;\mathbb{Z}) \to H^i(X',\mathbb{Z})) \tag{2.3.1}$$

for any resolution of singularities $X' \to X$ (see also loc. cit). Note that $\pi^* : H^i(X',\mathbb{Z}) \to H^i(X'',\mathbb{Z})$ is injective for any birational proper morphism of smooth varieties $\pi : X'' \to X'$.

Proof of (2.3). The canonical mixed Hodge structure on the relative cohomology can also be defined by using any quasi-projective resolution $\overline{X}_\bullet \to C(\overline{Y} \to \overline{X})$ as in [10]. This gives an integral weight filtration together with an integral weight spectral sequence

$$E_1^{p,q} = \oplus_{k\geq 0} H^{q-2k}(\widetilde{D}^k_{p+k},\mathbb{Z})(-k) \Rightarrow H^{p+q}(X_\bullet,\mathbb{Z}) = H^{p+q}(X,Y;\mathbb{Z}), \tag{2.3.2}$$

where $\widetilde{D}^k_j$ the disjoint union of the intersections of k irreducible components of D_j, and the cohomology is defined by taking the canonical flasque resolution of Godement in the analytic or Zariski topology. By (2.2) we get a set of integral weight filtrations on $H^j(X,Y;\mathbb{Z})$ which is directed with respect to the natural ordering by the inclusion relation. Then this is stationary by the noetherian property. (It is constant if X is proper, see (2.5) below.) By the proof of (2.2) the limit is independent of the choice of the compactification $\overline{X}$. So the assertion follows.

2.4. Definition. For a complex of k-varieties $X_\bullet$ (see (2.1)), we define

$$\mathrm{Pic}(X_\bullet) = \mathbf{H}^1(X_\bullet, \mathcal{O}^*_{X_\bullet}) \text{ (see [2])}. \tag{2.4.1}$$

The right-hand side is defined by taking the canonical flasque resolution of Godement which is compatible with the pull-back by the differential of $X_\bullet$. For a k-variety X and closed subvariety Y, we define the *derived relative Picard groups* by

$$\mathbf{Pic}(X,Y;i) = \varinjlim \mathrm{Pic}(X_\bullet[i]), \tag{2.4.2}$$

where the inductive limit is taken over $X_\bullet \in \mathcal{K}^b_{X^{\mathrm{nsqp}}}(C(Y \to X))$. If Y is empty, $\mathbf{Pic}(X,Y;i)$ will be denoted by $\mathbf{Pic}(X,i)$, and i will be omitted if $i=0$.

Remark. We can define similarly the derived relative Chow cohomology group by

$$\mathbf{CH}^p(X,Y;i) = \varinjlim H^{p+i}(X_\bullet, \mathcal{K}_p), \tag{2.4.3}$$

where $\mathcal{K}_p$ is the Zariski sheafification of Quillen's higher K-group. (In the case X is smooth proper and Y is empty, this is related to Bloch's higher Chow group for $i=0,-1$.)

The following is a variant of a result of Gillet and Soulé [14], 3.1, and gives a positive answer to the question in [2], 4.4.4.

2.5. Proposition. *Assume X, Y proper. Then a strong quasi-isomorphism $u : X''_\bullet \to X'_\bullet$ in $\mathcal{C}^b_{X^{\mathrm{nsqp}}}$ induces (filtered) isomorphisms*

$$u^* : \mathrm{Pic}(X'_\bullet[i]) \to \mathrm{Pic}(X''_\bullet[i]), \quad u^* : (H^i(X'_\bullet,\mathbb{Z}),W) \to (H^i(X''_\bullet,\mathbb{Z}),W),$$

where we assume $k=\mathbb{C}$ for the second morphism. In particular, the inductive system in (2.4.2) is a constant system in this case.

Proof. It is sufficient to show that $\mathrm{Pic}(X_\bullet[i])=0$ and the E_1-complex ${}_{\mathbb{Z}}E_1^{\bullet,q}$ of the integral weight spectral sequence is acyclic, if $X_\bullet$ is strongly acyclic and the X_j are smooth. (Note that the E_1-complex for W is compatible with the mapping cone.) Considering the E_1-complex of the spectral sequence

$${}_PE_1^{p,q} = H^q(X_p, \mathcal{O}^*_{X_p}) \Rightarrow H^{p+q}(X_\bullet, \mathcal{O}^*_{X_\bullet}), \tag{2.5.1}$$

it is enough to show the acyclicity of the complexes ${}_PE_1^{\bullet,q}$ and ${}_{\mathbb{Z}}E_1^{\bullet,q}$ (where ${}_PE_1^{\bullet,q}=0$ for $q>1$, see (2.5.3) below).

By Gillet and Soulé ([14], 1.2) this is further reduced to the acyclicity of the Gersten complex of $X_\bullet \times V$ for any smooth proper variety V because it implies that the image of the complex $X_\bullet$ in the category of complexes of varieties whose differentials and morphisms are given by correspondences is homotopic to zero. Since the functor associating the Gersten complex preserves homotopy, it is sufficient to show that the Gersten complex of

$$Z' \to Z \oplus Y' \to Y$$

is acyclic in the notation of (2.1.1) (replacing it by the product with V). Consider the subcomplex of the Gersten complex given by the points of Z', Z, Y', Y contained in Z',

Z, Z', Z. It is clearly acyclic, and so is its quotient complex. This shows the desired assertion. (A similar argument works also for (2.4.3).)

Remark. Let X be a smooth irreducible k-variety, and $k(X)$ the function field of X. For closed subvariety D, let $k(X)^*_X$ and $\mathbb{Z}_D$ denote the constant sheaf in the Zariski topology on X and D with stalk $k(X)^*$ and $\mathbb{Z}$ respectively. Then we have a flasque resolution

$$0 \to \mathcal{O}^*_X \to k(X)^*_X \overset{\mathrm{div}}{\to} \oplus_D \mathbb{Z}_D \to 0, \tag{2.5.2}$$

where the direct sum is taken over irreducible divisors D on X. In particular, we get

$$H^i(X, \mathcal{O}^*_X) = 0 \quad \text{for } i > 1, \tag{2.5.3}$$

$$R^i j_* \mathcal{O}^*_X = 0 \quad \text{for } i > 0, \tag{2.5.4}$$

for an open immersion $j : U \to X$.

2.6. Proposition. *There is a canonical long exact sequence*

$$\to \mathbf{Pic}(X, Y; i) \to \mathbf{Pic}(X, i) \to \mathbf{Pic}(Y, i) \to \mathbf{Pic}(X, Y; i+1) \to . \tag{2.6.1}$$

Remark. This is an analogue of the localization sequence for higher Chow groups [7].

Proof of (2.6). The long exact sequence is induced by the distinguished triangle

$$\to Y \overset{i}{\to} X \to C(Y \to X) \to$$

in $\mathcal{K}^b_X$, because for any quasi-projective resolutions $u : X_\bullet \to X$ and $v : Y_\bullet \to Y$, there exists quasi-isomorphisms $u' : X'_\bullet \to X_\bullet$ and $v' : Y'_\bullet \to Y_\bullet$ together with $i' : Y'_\bullet \to X'_\bullet$ such that $u \circ u' \circ i' = i \circ v \circ v'$ in $\mathcal{K}^b_X$ by using the mapping cone (2.2.3).

2.7. Remark. Assume $k = \mathbb{C}$ and X is proper. Let $H^{i+2}_{\mathcal{D}}(X, Y; \mathbb{Z}(1))$ denote the relative Deligne cohomology. See [3] and also (5.2) below. Then we can show

$$\mathbf{Pic}(X, Y; i) = H^{i+2}_{\mathcal{D}}(X, Y; \mathbb{Z}(1)) \quad \text{for } i \leq 0, \tag{2.7.1}$$

This is analogous to the canonical isomorphisms for

$$\mathrm{CH}^1(X, i) = H^{\mathrm{AH}}_{2n-2+i}(X, \mathbb{Z}(n-1))$$

which holds for $i > 0$ and any variety X of dimension n [31].

Assume X is proper and normal. Let $\mathcal{H}^1_X \mathbb{Z}(1)$ be the Zariski sheaf associated with the presheaf $U \mapsto H^1(U, \mathbb{Z}(1))$. By the Leray spectral sequence we get a natural injective morphism $H^1_{\mathrm{Zar}}(X, \mathcal{H}^1_X \mathbb{Z}(1)) \to H^2(X, \mathbb{Z}(1))$. See [1], [6]. Define a subgroup $H^2_{\mathcal{D}}(X, \mathbb{Z}(1))_{\mathrm{alg}}$ of the Deligne cohomology $H^2_{\mathcal{D}}(X, \mathbb{Z}(1))$ (see (4.2) below) by the cartesian diagram

$$\begin{array}{ccc} H^2_{\mathcal{D}}(X, \mathbb{Z}(1))_{\mathrm{alg}} & \longrightarrow & F^1 \cap H^1_{\mathrm{Zar}}(X, \mathcal{H}^1_X \mathbb{Z}(1)) \\ \downarrow & & \downarrow \\ H^2_{\mathcal{D}}(X, \mathbb{Z}(1)) & \longrightarrow & F^1 \cap H^2(X, \mathbb{Z}(1)). \end{array} \tag{2.7.2}$$

Let $X_\bullet \to X$ be a quasi-projective resolution. Then

$$H^2_{\mathcal{D}}(X,\mathbb{Z}(1))_{\mathrm{alg}} = \mathrm{Im}(\mathrm{Pic}(X) \to \mathrm{Pic}(X_\bullet) = H^2_{\mathcal{D}}(X,\mathbb{Z}(1))). \tag{2.7.3}$$

Indeed, if we put $\mathrm{NS}(X_\bullet) := \mathrm{Im}(\mathrm{Pic}(X_\bullet) \to H^2(X,\mathbb{Z}(1)))$, and similarly for $\mathrm{NS}(X)$, this follows from a result of Biswas and Srinivas [6]:

$$\mathrm{NS}(X) = F^1 \cap H^1_{\mathrm{Zar}}(X, \mathcal{H}^1_X\mathbb{Z}(1)), \tag{2.7.4}$$

by using

$$\mathrm{Coker}(\mathrm{Pic}(X) \to \mathrm{Pic}(X_\bullet)) = \mathrm{NS}(X_\bullet)/\mathrm{NS}(X). \tag{2.7.5}$$

The last isomorphism follows from the morphism of long exact sequences

$$\begin{array}{ccccccc} H^1(X,\mathbb{Z}(1)) & \longrightarrow & H^1(X,\mathcal{O}_X) & \longrightarrow & \mathrm{Pic}(X) & \longrightarrow & H^2(X,\mathbb{Z}(1)) \\ \| & & \downarrow{\scriptstyle (*)} & & \downarrow & & \| \\ H^1(X_\bullet,\mathbb{Z}(1)) & \longrightarrow & H^1(X_\bullet,\mathcal{O}_{X_\bullet}) & \longrightarrow & \mathrm{Pic}(X_\bullet) & \longrightarrow & H^2(X_\bullet,\mathbb{Z}(1)) \end{array}$$

because $(*)$ is surjective by Hodge theory [10] (considering the morphism of $H^1(X,\mathbb{C})$).

3. Construction

We construct 1-motives associated with a complex of varieties, and show the compatibility for the l-adic and de Rham realizations. We assume k is an algebraically closed field of characteristic zero.

3.1. With the notation of (2.1), let $X_\bullet \in \mathcal{C}_k$ be a complex of smooth k-varieties (see (2.1)), and $\overline{X}_\bullet$ a smooth compactification of $X_\bullet$ such that $D_p := \overline{X}_p \setminus X_p$ is a divisor with simple normal crossings. We assume $X_\bullet$ is bounded below. The reader can also assume that $(\overline{X}_\bullet, D_\bullet)$ is the mapping cone $(\overline{Y}_\bullet, D'_\bullet) \to (\overline{X}_\bullet, D_\bullet)$ of simplicial resolutions of $f : (\overline{Y}, D') \to (\overline{X}, D)$ (see [10]), where $\overline{X}, \overline{Y}$ are proper k-varieties with closed subvarieties D, D' such that $D' = f^{-1}(D)$.

Let $j : X_\bullet \to \overline{X}_\bullet$ denote the inclusion. Put

$$\mathcal{K} = \Gamma(\overline{X}_\bullet, \mathcal{C}^\bullet(j_*\mathcal{O}^*_{X_\bullet})),$$

where $\mathcal{C}^\bullet$ is the canonical flasque resolution of Godement in the Zariski topology.

We define a *cofiltration*

$${}^oW'^pX_\bullet$$

to be the quotient complexes of $X_\bullet$ consisting of X_i for $i \geq p$ (and empty otherwise). This is similar to the filtration "bête" σ in [10]. It induces a decreasing filtration W' on $\mathcal{K}$ such that

$$W'^i\mathcal{K} = \Gamma({}^oW'^i\overline{X}_\bullet, \mathcal{C}^\bullet(j_*\mathcal{O}^*_{X_\bullet}))).$$

We define a cofiltration ${}^oW''^jX_\bullet$ for $j=-1,0,1$ by

$$ {}^oW''^{-1}X_\bullet = X_\bullet, \quad {}^oW''^0X_\bullet = \overline{X}_\bullet, \quad {}^oW''^1X_\bullet = \emptyset. $$

Since this depends on the compactification $\overline{X}_\bullet$, it is also denoted by ${}^oW''^j\overline{X}_\bullet\langle D_\bullet\rangle$ so that

$$ {}^oW''^{-1}\overline{X}_\bullet\langle D_\bullet\rangle = \overline{X}_\bullet\langle D_\bullet\rangle, \quad {}^oW''^0\overline{X}_\bullet\langle D_\bullet\rangle = \overline{X}_\bullet, \quad {}^oW''^1\overline{X}_\bullet\langle D_\bullet\rangle = \emptyset. $$

This corresponds to a decreasing filtration W'' on $\mathcal{K}$ such that

$$ W''^{-1}\mathcal{K} = \mathcal{K}, \quad W''^0\mathcal{K} = \Gamma(\overline{X}_\bullet, \mathcal{C}^\bullet(\mathcal{O}^*_{\overline{X}_\bullet})), \quad W''^1\mathcal{K} = 0. $$

Then $\mathrm{Gr}^{-1}_{W''}\mathcal{K}$ is quasi-isomorphic to $\Gamma(\widetilde{D}_\bullet, \mathbb{Z})$, and $\Gamma(\widetilde{D}_p, \mathbb{Z}) \simeq \mathbb{Z}^{\oplus r_p}$, where $\widetilde{D}_\bullet$ is the normalization of $D_\bullet$, and r_p is the number of irreducible components of D_p. (Indeed, a constant sheaf on an irreducible variety is flasque in the Zariski topology.)

Let W be the convolution of W' and W'' (i.e., $W^r = \sum_{i+j=r} W'^i \cap W''^j$) so that

$$ \mathrm{Gr}^r_W\mathcal{K} = \oplus_{i+j=r}\mathrm{Gr}^i_{W'}\mathrm{Gr}^j_{W''}\mathcal{K}, $$

(see [5], 3.1.2). This corresponds to the cofiltration ${}^oW^r$ such that

$$ {}^oW^rX_\bullet \ (\text{or } {}^oW^r\overline{X}_\bullet\langle D_\bullet\rangle) $$

consists of X_i (or $\overline{X}_\bullet\langle D_\bullet\rangle$) for $i > r$, and $\overline{X}_r$ for $i = r$. Then we have a natural quasi-isomorphism

$$ \Gamma(\overline{X}_r, \mathcal{C}^\bullet(\mathcal{O}^*_{\overline{X}_r}))[-r] \oplus \Gamma(\widetilde{D}_{r+1}, \mathbb{Z})[-r-1] \to \mathrm{Gr}^r_W\mathcal{K}. \tag{3.1.1} $$

Hence $H^j\mathrm{Gr}^r_W\mathcal{K}$ vanishes unless $j = r$ or $r+1$, and

$$ H^r\mathrm{Gr}^r_W\mathcal{K} = \Gamma(\overline{X}_r, \mathcal{O}^*_{\overline{X}_r}), \quad H^{r+1}\mathrm{Gr}^r_W\mathcal{K} = \mathrm{Pic}(\overline{X}_r) \oplus \Gamma(\widetilde{D}_{r+1}, \mathbb{Z}), \tag{3.1.2} $$

where $\mathrm{Pic}(\overline{X}_r) = H^1(\overline{X}_r, \mathcal{O}^*_{\overline{X}_r})$ is the Picard group of $\overline{X}_r$.

Let $\mathcal{K}' = \mathrm{Gr}^0_{W''}\mathcal{K}\,(= \Gamma(\overline{X}_\bullet, \mathcal{C}^\bullet(\mathcal{O}^*_{\overline{X}_\bullet})))$ with the induced filtration W. Then we have the spectral sequence

$$ E_1^{p,q} = \begin{cases} H^q(\overline{X}_p, \mathcal{O}^*_{\overline{X}_p}) & \text{for } p \geq r \\ 0 & \text{for } p < r \end{cases} \Rightarrow H^{p+q}(W^r\mathcal{K}'), \tag{3.1.3} $$

which degenerates at E_3. We define

$$ \begin{aligned} P_{\geq r}(\overline{X}_\bullet\langle D_\bullet\rangle) &:= H^{r+1}(W^r\mathcal{K}) = \mathrm{Pic}({}^oW^rX_\bullet[r]), \\ P_r(\overline{X}_\bullet\langle D_\bullet\rangle) &:= H^{r+1}(\mathrm{Gr}^r_W\mathcal{K}) = \mathrm{Pic}(\overline{X}_r) \oplus \Gamma(\widetilde{D}_{r+1}, \mathbb{Z}), \\ P_{\geq r}(\overline{X}_\bullet) &:= H^{r+1}(W^r\mathcal{K}') = \mathrm{Pic}({}^oW^r\overline{X}_\bullet[r]), \\ P_r(\overline{X}_\bullet) &:= H^{r+1}(\mathrm{Gr}^r_W\mathcal{K}') = \mathrm{Pic}(\overline{X}_r). \end{aligned} $$

Then we have an exact sequence

$$0 \to P_{\geq r}(\overline{X}_\bullet) \to P_{\geq r}(\overline{X}_\bullet\langle D_\bullet\rangle) \to \Gamma(\widetilde{D}_{r+1}, \mathbb{Z}).$$

By (3.2) below, $P_{\geq r}(\overline{X}_\bullet)$ has a structure of algebraic group $\mathcal{P}_{\geq r}(\overline{X}_\bullet)$ (locally of finite type) such that the identity component is a semiabelian variety. (This is well-known for $P_r(\overline{X}_\bullet)$.) Let $P_{\geq r}(\overline{X}_\bullet)^0$ denote the identity component of $P_{\geq r}(\overline{X}_\bullet)$. This is identified with $P_{\geq r}(\overline{X}_\bullet\langle D_\bullet\rangle)^0$ by the above exact sequence (and similarly for $P_r(\overline{X}^0_\bullet)$, $P_r(\overline{X}_\bullet\langle D_\bullet\rangle)^0$).

By the boundary map ∂ of the long exact sequence associated with

$$\to W^{r-1}\mathcal{K} \to W^{r-2}\mathcal{K} \to \mathrm{Gr}_W^{r-2}\mathcal{K} \to$$

we get a commutative diagram

$$\begin{array}{ccccc} P_{r-2}(\overline{X}_\bullet)^0 & \longrightarrow & P_{r-2}(\overline{X}_\bullet\langle D_\bullet\rangle) & \longrightarrow & P_{r-2}(\overline{X}_\bullet\langle D_\bullet\rangle)/P_{r-2}(\overline{X}_\bullet)^0 \\ \downarrow\partial & & \downarrow\partial & & \downarrow\partial \\ P_{\geq r-1}(\overline{X}_\bullet)^0 & \longrightarrow & P_{\geq r-1}(\overline{X}_\bullet\langle D_\bullet\rangle) & \longrightarrow & P_{\geq r-1}(\overline{X}_\bullet\langle D_\bullet\rangle)/P_{\geq r-1}(\overline{X}_\bullet)^0 \end{array} \tag{3.1.4}$$

Let $\Gamma'_r(\overline{X}_\bullet\langle D_\bullet\rangle)$ be the kernel of the right vertical morphism of (3.1.4). Put

$$\mathrm{NS}(\overline{X}_\bullet\langle D_\bullet\rangle)^j := P_j(\overline{X}_\bullet\langle D_\bullet\rangle)/P_j(\overline{X}_\bullet)^0 = \mathrm{NS}(\overline{X}_j) \oplus \Gamma(\widetilde{D}_{j+1}, \mathbb{Z}),$$

where

$$\mathrm{NS}(\overline{X}_j) := \mathrm{Pic}(\overline{X}_j)/\mathrm{Pic}(\overline{X}_j)^0 = \mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Z}, H^2(\overline{X}_j, \mathbb{Z})(1)).$$

Then $(\mathrm{NS}(\overline{X}_\bullet\langle D_\bullet\rangle)^\bullet, d^*)$ is the single complex associated with a double complex such that one of the differentials is the Gysin morphism

$$\Gamma(\widetilde{D}_{j+1}, \mathbb{Z}) \to \mathrm{NS}(\overline{X}_{j+1}).$$

Since $d^2 = 0$, d^* induces a morphism

$$d^* : \mathrm{NS}(\overline{X}_\bullet\langle D_\bullet\rangle)^{r-3} \to \Gamma'_r(\overline{X}_\bullet\langle D_\bullet\rangle). \tag{3.1.5}$$

We define

$$\begin{aligned} \Gamma_r(\overline{X}_\bullet\langle D_\bullet\rangle) &= \mathrm{Coker}(d^* : \mathrm{NS}(\overline{X}_\bullet\langle D_\bullet\rangle)^{r-3} \to \Gamma'_r(\overline{X}_\bullet\langle D_\bullet\rangle)), \\ G_r(\overline{X}_\bullet\langle D_\bullet\rangle) &= \mathrm{Coker}(\partial : \mathcal{P}_{r-2}(\overline{X}_\bullet)^0 \to \mathcal{P}_{\geq r-1}(\overline{X}_\bullet)^0), \end{aligned}$$

where $\Gamma_r(\overline{X}_\bullet\langle D_\bullet\rangle), \Gamma'_r(\overline{X}_\bullet\langle D_\bullet\rangle)$ are identified with locally finite commutative group schemes. Then (3.1.4) induces morphisms

$$\Gamma'_r(\overline{X}_\bullet\langle D_\bullet\rangle) \to G_r(\overline{X}_\bullet\langle D_\bullet\rangle), \quad \Gamma_r(\overline{X}_\bullet\langle D_\bullet\rangle) \to G_r(\overline{X}_\bullet\langle D_\bullet\rangle),$$

which define respectively

$$M'_r(\overline{X}_\bullet\langle D_\bullet\rangle), \quad M_r(\overline{X}_\bullet\langle D_\bullet\rangle).$$

(This construction is equivalent to the one in [26].)

Remark. By (3.1.3), $P_{\geq r}(\overline{X}_\bullet)$ is identified with the group of isomorphism classes of (L,γ) where L is a line bundle on $\overline{X}_r$ and

$$\gamma : \mathcal{O}_{\overline{X}_{r+1}} \xrightarrow{\sim} d^*L$$

is a trivialization such that

$$d^*\gamma : d^*\mathcal{O}_{\overline{X}_{r+1}} (= \mathcal{O}_{\overline{X}_{r+2}}) \xrightarrow{\sim} (d^2)^*L = \mathcal{O}_{\overline{X}_{r+2}}$$

is the identity morphism. (Note that d^*L is defined by using tensor of line bundles.) See also [2], [26].

For the construction of the group scheme $\mathcal{P}_{\geq r}(\overline{X}_\bullet)$, we need Grothendieck's theory of representable group functors (see [15], [23]) as follows:

3.2. Theorem. *There exists a k-group scheme locally of finite type $\mathcal{P}_{\geq r}(\overline{X}_\bullet)$ such that the group of its k-valued points is isomorphic to $P_{\geq r}(\overline{X}_\bullet)$. Moreover, $\mathcal{P}_{\geq r}(\overline{X}_\bullet)$ has the following universal property: For any k-variety S and any $(L,\gamma) \in P_{\geq r}(\overline{X}_\bullet \times_k S)$ as above, the set-theoretic map $S(k) \to P_{\geq r}(\overline{X}_\bullet)$ obtained by restricting (L,γ) to the fiber at $s \in S(k)$ comes from a morphism of k-schemes $S \to \mathcal{P}_{\geq r}(\overline{X}_\bullet)$.*

Proof. Essentially the same as in [2]. (This also follows from [24], see Remark after (3.3).)

Remark. We can easily construct a k-scheme locally of finite type S and $(L,\gamma) \in P_{\geq r}(\overline{X}_\bullet \times_k S)$ such that the associated map $f : S(k) \to P_{\geq r}(\overline{X}_\bullet)$ is surjective by using the theory of Hilbert scheme. Then $P_{\geq r}(\overline{X}_\bullet)$ has at most unique structure of k-algebraic group such that f is algebraic. But it is nontrivial that this really gives an algebraic structure on $P_{\geq r}(\overline{X}_\bullet)$, because it is even unclear if the inverse image of a closed point is a closed variety for example. The independence of the choice of (L,γ) and S is also nontrivial. So we have to use Grothendieck's general theory using sheafification in the *fppf* (faithfully flat and of finite presentation) topology.

If $k = \mathbb{C}$, we can prove (3.2) (for smooth varieties S) by using Hodge theory. See (5.3). In fact, this implies an isomorphism between the semiabelian parts of 1-motives (see also Remark after (5.3)). But this proof of (3.2) is not algebraic, and cannot be used to prove Deligne's conjecture.

3.3. Lemma. *The identity component $\mathcal{P}_{\geq r}(\overline{X}_\bullet)^0$ is a semiabelian variety.*

Proof. With the notation of (3.1), let

$$T_r(\overline{X}_\bullet) = H^r(\Gamma(\overline{X}_\bullet, \mathcal{O}^*_{\overline{X}_\bullet})), \quad \overline{P}_r(\overline{X}_\bullet) = \mathrm{Ker}(d^* : \mathrm{Pic}(\overline{X}_r) \to \mathrm{Pic}(\overline{X}_{r+1})).$$

Then (3.1.3) induces an exact sequence

$$0 \to T_{r+1}(\overline{X}_\bullet) \to P_{\geq r}(\overline{X}_\bullet) \to \overline{P}_r(\overline{X}_\bullet) \to T_{r+2}(\overline{X}_\bullet) \to . \tag{3.3.1}$$

Let $T_r(\overline{X}_\bullet)^0, \overline{P}_r(\overline{X}_\bullet)^0$ denote the identity components of $T_r(\overline{X}_\bullet), \overline{P}_r(\overline{X}_\bullet)$. Then (3.3.1) induces a short exact sequence

$$0 \to T_{r+1}(\overline{X}_\bullet)' \to P_{\geq r}(\overline{X}_\bullet)^0 \to \overline{P}_r(\overline{X}_\bullet)^0 \to 0, \tag{3.3.2}$$

where $T_{r+1}(\overline{X}_\bullet)'$ is a subgroup of $T_{r+1}(\overline{X}_\bullet)$ with finite index. This gives a structure of semiabelian variety

$$0 \to T_{r+1}(\overline{X}_\bullet)^0 \to P_{\geq r}(\overline{X}_\bullet)^0 \to \overline{P}_r(\overline{X}_\bullet)' \to 0, \tag{3.3.3}$$

with an isogeny of abelian varieties

$$0 \to T_{r+1}(\overline{X}_\bullet)'/T_{r+1}(\overline{X}_\bullet)^0 \to \overline{P}_r(\overline{X}_\bullet)' \to \overline{P}_r(\overline{X}_\bullet)^0 \to 0. \tag{3.3.4}$$

Remark. The representability of the Picard functor follows also from [24], Prop. 17.4, using (3.3.3). See also [26].

3.4. Theorem. *With the notation of (3.1), let $W_2H^r_{(1)}(X,Y;\mathbb{Z}_l)$ be the $\mathbb{Z}_l$-submodule of $W_2H^r_{\text{ét}}(X,Y;\mathbb{Z}_l)$ whose image in $\mathrm{Gr}^2_W H^r_{\text{ét}}(X,Y;\mathbb{Z}_l)$ is generated by the image of $\Gamma'_r(\overline{X}_\bullet\langle D_\bullet\rangle)$ under the cycle map. Then there is a canonical isomorphism*

$$r_l(M_r(X,Y))_{\text{fr}}(-1) = W_2H^r_{(1)}(X,Y;\mathbb{Z}_l)_{\text{fr}}$$

compatible with the weight filtration W.

Proof. Note first that W is defined by using a resolution $\overline{X}_\bullet$ in (2.3), and $X_\bullet := \overline{X}_\bullet \setminus D_\bullet$ will be denoted sometimes by $\overline{X}_\bullet\langle D_\bullet\rangle$ as in (3.1). Let

$$\mathcal{K}_{\text{ét}} = \Gamma(X_\bullet, C^\bullet_{\text{ét}}(\mathbb{G}_m)),$$

where $C^\bullet_{\text{ét}}$ denotes the canonical flasque resolution of Godement in the étale topology. Then $\mathcal{K}_{\text{ét}}$ has a filtration W in a generalized sense, which is induced by the cofiltration ${}^oW$ in (3.1) so that

$$W^j\mathcal{K}_{\text{ét}} = \Gamma(X^j_\bullet, C^\bullet_{\text{ét}}(\mathbb{G}_m)),$$

where $X^j_\bullet := {}^oW^j\overline{X}_\bullet\langle D_\bullet\rangle$. We define

$$\mathcal{K}_{\text{ét}}(r,n) = C(W^{r-1}\mathcal{K}_{\text{ét}} \xrightarrow{n} W^{r-2}\mathcal{K}_{\text{ét}}),$$

and similarly for $\mathcal{K}(r,n)$ in the Zariski topology.

By the Kummer sequence $0 \to \mu_n \to \mathbb{G}_m \xrightarrow{n} \mathbb{G}_m \to 0$, we have

$$H^i_{\text{ét}}(X^j_\bullet, \mu_n) = H^{i-1}(C(W^j\mathcal{K}_{\text{ét}} \xrightarrow{n} W^j\mathcal{K}_{\text{ét}})).$$

Note that the étale cohomology $H^i_{\text{ét}}(X^j_\bullet, \mu_n)$ has the weight filtration W, and

$$W_{q-2}H^i_{\text{ét}}(X^j_\bullet, \mu_n) = \mathrm{Im}(H^i_{\text{ét}}(X^{i-q}_\bullet, \mu_n) \to H^i_{\text{ét}}(X^j_\bullet, \mu_n)) \tag{3.4.1}$$

for $q \leq 2$ and $i - q \geq j$ as in (4.4.2). Here the shift of W by 2 comes from the Tate twist μ_n. We define

$$\begin{aligned} E'(r,n) &= \mathrm{Im}(H^{r-1}\mathcal{K}_{\text{ét}}(r,n) \to H^r_{\text{ét}}(X_\bullet^{r-2}, \mu_n)), \\ N(r,n) &= \mathrm{Im}(\partial : H^{r-1}\mathrm{Gr}_W^{r-2}\mathcal{K}_{\text{ét}} \to H^r_{\text{ét}}(X_\bullet^{r-2}/X_\bullet^{r-1}, \mu_n)) \\ &\quad (= \mathrm{Coker}(n : H^{r-1}\mathrm{Gr}_W^{r-2}\mathcal{K}_{\text{ét}} \to H^{r-1}\mathrm{Gr}_W^{r-2}\mathcal{K}_{\text{ét}})), \end{aligned}$$

where $\mathrm{Gr}_W^{r-2}\mathcal{K}_{\text{ét}}$ and $X_\bullet^{r-2}/X_\bullet^{r-1} := \mathrm{Gr}_{\circ W}^{r-2}\overline{X}_\bullet\langle D_\bullet\rangle$ are defined by using the mapping cones, and ∂ is induced by the boundary map of the Kummer sequence, and gives the cycle map. Note that

$$\begin{aligned} H^{r-1}\mathrm{Gr}_W^{r-2}\mathcal{K}_{\text{ét}} &= \mathrm{Pic}(\overline{X}_{r-2}) \oplus \Gamma(\widetilde{D}_{r-1}, \mathbb{Z}), \\ H^r_{\text{ét}}(X_\bullet^{r-2}/X_\bullet^{r-1}, \mu_n) &= H^2_{\text{ét}}(\overline{X}_{r-2}, \mu_n) \oplus H^0(\widetilde{D}_{r-1}, \mathbb{Z}/n), \end{aligned}$$

using Hilbert's theorem 90 for the first and the local cohomology for the second. Let

$$N'(r,n) = N(r,n) \cap \ker(H^r_{\text{ét}}(X_\bullet^{r-2}/X_\bullet^{r-1}, \mu_n) \to H^{r+1}_{\text{ét}}(X_\bullet^{r-1}, \mu_n)).$$

This coincides with the intersection of $N(r,n)$ with the image of $H^r_{\text{ét}}(X_\bullet^{r-2}, \mu_n)$ using the long exact sequence. So we get a short exact sequence

$$0 \to W_{-1}H^r_{\text{ét}}(X_\bullet^{r-2}, \mu_n) \to E'(r,n) \to N'(r,n) \to 0 \tag{3.4.2}$$

by considering the natural morphism between the distinguished triangles

$$\begin{array}{ccccccc} \longrightarrow & C(n : W^{r-1}\mathcal{K}_{\text{ét}}) & \longrightarrow & \mathcal{K}_{\text{ét}}(r,n) & \longrightarrow & \mathrm{Gr}_W^{r-2}\mathcal{K}_{\text{ét}} & \longrightarrow \\ & \| & & \downarrow & & \downarrow & \\ \longrightarrow & C(n : W^{r-1}\mathcal{K}_{\text{ét}}) & \longrightarrow & C(n : W^{r-2}\mathcal{K}_{\text{ét}}) & \longrightarrow & C(n : \mathrm{Gr}_W^{r-2}\mathcal{K}_{\text{ét}}) & \longrightarrow , \end{array}$$

where $C(n : A)$ is the abbreviation of $C(n : A \to A)$ for an abelian group A.

Let $E'(r, l^\infty)$ be the projective limit of $E'(r, l^m)$. By the Mittag-Leffler condition, it is identified with a $\mathbb{Z}_l$-submodule of $H^r_{\text{ét}}(X_\bullet^{r-2}, \mathbb{Z}_l)$ so that

$$W_{-1}H^r_{\text{ét}}(X_\bullet^{r-2}, \mathbb{Z}_l) \subset E'(r, l^\infty) \subset H^r_{\text{ét}}(X_\bullet^{r-2}, \mathbb{Z}_l)$$

and $E'(r, l^\infty)/W_{-1}H^r_{\text{ét}}(X_\bullet^{r-2}, \mathbb{Z}_l) \subset \mathrm{Gr}^0_W H^r_{\text{ét}}(X_\bullet^{r-2}, \mathbb{Z}_l)$ is generated by the images of $\Gamma'_r(\overline{X}_\bullet\langle D_\bullet\rangle)$ under the cycle map. Let

$$E(r, l^\infty) = \mathrm{Im}(E'(r, l^\infty) \to H^r_{\text{ét}}(X_\bullet^{r-3}, \mathbb{Z}_l)).$$

Since the integral weight spectral sequence degenerates at E_2 modulo torsion, we get

$$W_2H^r_{(1)}(X, Y; \mathbb{Z}_l)_{\text{fr}} = E(r, l^\infty)_{\text{fr}}. \tag{3.4.3}$$

We have to show that $E(r,l^\infty)_{\rm fr}$ is naturally isomorphic to the l-adic realization of $M_r(\overline{X}_\bullet\langle D_\bullet\rangle)_{\rm fr}$. Define a decreasing filtration G on $C(W^{r-2}\mathcal{K}_{\text{ét}} \overset{n}{\to} W^{r-2}\mathcal{K}_{\text{ét}})$ by

$$G^0 = C(W^{r-2}\mathcal{K}_{\text{ét}} \overset{n}{\to} W^{r-2}\mathcal{K}_{\text{ét}}), \quad G^1 = \mathcal{K}_{\text{ét}}(r,n), \quad G^2 = C(0 \to W^{r-1}\mathcal{K}_{\text{ét}}), \quad G^3 = 0.$$

Then $\mathrm{Gr}_G^0 = \mathrm{Gr}_W^{r-2}\mathcal{K}_{\text{ét}}[1], \mathrm{Gr}_G^1 = W^{r-1}\mathcal{K}_{\text{ét}}[1]\oplus\mathrm{Gr}_W^{r-2}\mathcal{K}_{\text{ét}}$. So there is a canonical morphism in the derived category $\beta : \mathrm{Gr}_W^{r-2}\mathcal{K}_{\text{ét}} \to \mathcal{K}_{\text{ét}}(r,n)$ whose mapping cone is isomorphic to $C(W^{r-2}\mathcal{K}_{\text{ét}} \overset{n}{\to} W^{r-2}\mathcal{K}_{\text{ét}})$. By the associated long exact sequence, we have

$$E'(r,n) = \mathrm{Coker}(\beta : H^{r-1}\mathrm{Gr}_W^r\mathcal{K}_{\text{ét}} \to H^{r-1}\mathcal{K}_{\text{ét}}(r,n)). \tag{3.4.4}$$

Here we can replace $\mathcal{K}_{\text{ét}}$ with $\mathcal{K}$, because the right-hand side does not change by doing it.

Using the induced filtration G on $\mathcal{K}(r,n)$, we have a long exact sequence

$$H^{r-1}(W^{r-1}\mathcal{K}) \oplus H^{r-2}(\mathrm{Gr}_W^{r-2}\mathcal{K}) \to H^{r-1}(W^{r-1}\mathcal{K})$$
$$\to H^{r-1}\mathcal{K}(r,n) \to H^r(W^{r-1}\mathcal{K}) \oplus H^{r-1}(\mathrm{Gr}_W^{r-2}\mathcal{K}) \to H^r(W^{r-1}\mathcal{K}),$$

where the first and the last morphisms are given by the sum of the multiplication by n and the boundary morphism ∂. In particular, the cokernel of the first morphism is a finite group, and is independent of $n = l^m$ for m sufficiently large, because

$$H^{r-1}(W^{r-1}\mathcal{K}) = \mathrm{Ker}(d^* : \Gamma(\overline{X}_{r-1}, \mathcal{O}^*_{\overline{X}_{r-1}}) \to \Gamma(\overline{X}_r, \mathcal{O}^*_{\overline{X}_r})).$$

Consider the cohomology $H(r,n)$ of

$$H^{r-1}(\mathrm{Gr}_W^{r-2}\mathcal{K}) \to H^r(W^{r-1}\mathcal{K}) \oplus H^{r-1}(\mathrm{Gr}_W^{r-2}\mathcal{K}) \to H^r(W^{r-1}\mathcal{K}),$$

where the first morphism is the composition of β with the third morphism of the above long exact sequence. Then, by the above argument, there is a surjective canonical morphism

$$E'(r,n) \to H(r,n), \tag{3.4.5}$$

whose kernel is a finite group, and is independent of $n = l^m$ for m sufficiently large. By definition, $H(r,n)$ is isomorphic to H^0 of

$$C(\partial : H^{r-1}(\mathrm{Gr}_W^{r-2}\mathcal{K}) \to H^r(W^{r-1}\mathcal{K})) \otimes C(n : \mathbb{Z} \to \mathbb{Z})[-1].$$

Since the Tate module of a finitely generated abelian group vanishes, we can replace the mapping cone of $\partial : H^{r-1}(\mathrm{Gr}_W^{r-2}\mathcal{K}) \to H^r(W^{r-1}\mathcal{K})$ with that of

$$\partial : \mathrm{Ker}(H^{r-1}(\mathrm{Gr}_W^{r-2}\mathcal{K}) \to P_{\geq r-1}(\overline{X}_\bullet)/P_{\geq r-1}(\overline{X}_\bullet)^0) \to P_{\geq r-1}(\overline{X}_\bullet)^0$$

in the notation of (3.1). Furthermore, H^0 of

$$C(\partial : P_{r-2}(\overline{X}_\bullet)^0 \to \partial(P_{r-2}(\overline{X}_\bullet)^0)) \otimes C(n : \mathbb{Z} \to \mathbb{Z})[-1]$$

is a finite group, and is independent of $n = l^m$ for m sufficiently large.

On the other hand, the l-adic realization $r_l(M'_r(\overline{X}_\bullet\langle D_\bullet\rangle))$ is the projective limit of H^0 of

$$C(\Gamma'_r(\overline{X}_\bullet\langle D_\bullet\rangle) \to G_r(\overline{X}_\bullet\langle D_\bullet\rangle)) \otimes C(n : \mathbb{Z} \to \mathbb{Z})[-1].$$

So we get a canonical surjective morphism

$$E'(r,l^\infty) \to r_l(M'_r(\overline{X}_\bullet\langle D_\bullet\rangle))$$

whose kernel is a finite group. This is clearly compatible with the weight filtration. Then the assertion follows by taking the image in $H^r_{\text{ét}}(X_\bullet, \mathbb{Z}_l)_{\rm fr}$, and using the E_2-degeneration of the weight spectral sequence modulo torsion.

3.5. Theorem. *With the notation of (3.1), let $H^r_{\mathrm{DR},(1)}(X,Y)$ be the k-submodule of $W_2H^r_{\mathrm{DR}}(X,Y)$ whose image in $\mathrm{Gr}^2_W H^r_{\mathrm{DR}}(X,Y)$ is generated by the image of $\Gamma'_r(\overline{X}_\bullet\langle D_\bullet\rangle)$ under the cycle map, where $H^r_{\mathrm{DR}}(X,Y)$ is defined by using $\overline{X}_\bullet\langle D_\bullet\rangle$ in (2.3). Then there is a canonical isomorphism*

$$r_{\mathrm{DR}}(M_r(X,Y))(-1) = H^r_{\mathrm{DR},(1)}(X,Y).$$

compatible with the Hodge filtration F and the weight filtration W.

Proof. By definition of de Rham realization [10], we have to construct first the universal $\mathbb{G}_a$-extension of $M_r(X,Y) = M_r(\overline{X}_\bullet\langle D_\bullet\rangle)$. Let $\widetilde{\mathcal{K}}$ be the shifted mapping cone $[\mathcal{K} \to \mathcal{K}^1]$ with

$$\mathcal{K} = \Gamma(\overline{X}_\bullet, \mathcal{C}^\bullet(j_*\mathcal{O}^*_{X_\bullet})), \quad \mathcal{K}^1 = \Gamma(\overline{X}_\bullet, \mathcal{C}^\bullet(\Omega^1_{\overline{X}_\bullet}(\log D_\bullet))),$$

where $\mathcal{K}$ has degree zero, and the morphism is induced by $g \mapsto g^{-1}dg$. Here $\mathcal{C}^\bullet$ is the canonical flasque resolution of Godement in the Zariski topology. Then $\widetilde{\mathcal{K}}$ has a filtration W in a generalized sense, which is induced by the cofiltration ${}^oW$ on $\overline{X}_\bullet\langle D_\bullet\rangle$.

Let G be the convolution of W with the Hodge filtration F defined by $\mathrm{Gr}^0_F = \mathcal{K}$ and $\mathrm{Gr}^1_F = \mathcal{K}^1$. Then $G^i\widetilde{\mathcal{K}} = [W^i\mathcal{K} \to W^{i-1}\mathcal{K}^1]$ with

$$W^i\mathcal{K} = \Gamma(\overline{X}^i_\bullet, \mathcal{C}^\bullet(j^i_*\mathcal{O}^*_{X^i_\bullet})), \quad W^{i-1}\mathcal{K}^1 = \Gamma(\overline{X}^{i-1}_\bullet, \mathcal{C}^\bullet(\Omega^1_{\overline{X}^{i-1}_\bullet}(\log D^{i-1}_\bullet))),$$

where $X^i_\bullet = {}^oW^i\overline{X}_\bullet\langle D_\bullet\rangle, D^i_\bullet = \overline{X}^i_\bullet \setminus X^i_\bullet$, and $\overline{X}^i_\bullet$ is the closure of $X^i_\bullet$ in $\overline{X}_\bullet$ with the inclusion $j^i : X^i_\bullet \to \overline{X}^i_\bullet$. This is compatible with W on $\mathcal{K}$ in (3.1). Note that $\Omega^1_{\overline{X}^i_j}(\log D^i_j) = \Omega^1_{\overline{X}_j}(\log D_j)$ for $j > i$, $\Omega^1_{\overline{X}_j}$ for $j = i$, and 0 for $j < i$. Let

$$\mathcal{K}' = \Gamma(\overline{X}_\bullet, \mathcal{C}(\mathcal{O}_{\overline{X}_\bullet})), \quad W^i\mathcal{K}' = \Gamma(\overline{X}^i_\bullet, \mathcal{C}^\bullet(\mathcal{O}_{\overline{X}^i_\bullet})),$$

and define W, F, G similarly on $\widetilde{\mathcal{K}}'$ so that $G^i\widetilde{\mathcal{K}}' = [W^i\mathcal{K}' \to W^{i-1}\mathcal{K}^1]$, where $\mathcal{K}^1$ and $W^{i-1}\mathcal{K}^1$ are as above, and the morphism is induced by d.

Then we have an exact sequence

$$0 \to H^{r-1}W^{r-2}\mathcal{K}^1 \to H^rG^{r-1}\widetilde{\mathcal{K}} \xrightarrow{\partial} H^rW^{r-1}\mathcal{K} \to, \tag{3.5.1}$$

and $\mathrm{Im}\,\partial$ contains $P_{\geq r-1}(\overline{X}_\bullet)^0$ with the notation of (3.1). Indeed, this is reduced to the case $k = \mathbb{C}$, and is verified by using the canonical morphism of the corresponding exact sequence

$$0 \to H^{r-1}W^{r-2}\mathcal{K}^1 \to H^rG^{r-1}\widetilde{\mathcal{K}}' \xrightarrow{\partial} H^rW^{r-1}\mathcal{K}' \to 0$$

to the above sequence, because $d : H^jW^i\mathcal{K}' \to H^jW^{i-1}\mathcal{K}^1$ vanishes by Hodge theory and the torsion of $H^{r-1}W^{r-1}\mathcal{K}$ comes from $H^{r-1}G^{r-1}\widetilde{\mathcal{K}}$.

Let $(H^rG^{r-1}\widetilde{\mathcal{K}})^0$ be the k-submodule of $H^rG^{r-1}\widetilde{\mathcal{K}}$ which contains $H^{r-1}W^{r-2}\mathcal{K}^1$, and whose image by ∂ is $P_{\geq r-1}(\overline{X}_\bullet)^0$. In the case $k = \mathbb{C}$, this is the image of $H^rG^{r-1}\widetilde{\mathcal{K}}'$. We

can verify that $(H^rG^{r-1}\widetilde{\mathcal{K}})^0$ has naturally a structure of a commutative k-group scheme. We consider the image of $(H^rG^{r-1}\widetilde{\mathcal{K}})^0$ by the canonical morphism

$$H^rG^{r-1}\widetilde{\mathcal{K}} \to H^{r-1}\mathrm{Gr}_W^{r-2}\mathcal{K}^1.$$

This coincides with the image of $H^{r-1}W^{r-2}\mathcal{K}^1$. Indeed, we can replace $H^rG^{r-1}\widetilde{\mathcal{K}}$ with $H^rG^{r-1}\widetilde{\mathcal{K}}'$ by reducing to the case $k=\mathbb{C}$.

Let $(H^{r-1}\mathrm{Gr}_W^{r-2}\mathcal{K}^1)_{\mathrm{alg}}$ be the k-submodule of

$$H^{r-1}\mathrm{Gr}_W^{r-2}\mathcal{K}^1 = H^1(\overline{X}_{r-2}, \Omega^1_{\overline{X}_{r-2}}) \oplus \Gamma(\widetilde{D}_{r-1}, \mathcal{O}_{\widetilde{D}_{r-1}})$$

generated by the divisor classes in $H^1(\overline{X}_{r-2}, \Omega^1_{\overline{X}_{r-2}})$ and by $\Gamma(\widetilde{D}_{r-1}, \mathcal{O}_{\widetilde{D}_{r-1}})$. Let $(H^rG^{r-1}\widetilde{\mathcal{K}})^0_{\mathrm{alg}}$ be the largest k-submodule of $(H^rG^{r-1}\widetilde{\mathcal{K}})^0$ whose image in $H^{r-1}\mathrm{Gr}_W^{r-2}\mathcal{K}^1$ is contained in $(H^{r-1}\mathrm{Gr}_W^{r-2}\mathcal{K}^1)_{\mathrm{alg}}$. We define similarly

$$(H^rG^{r-1}\widetilde{\mathcal{K}}')_{\mathrm{alg}} \subset H^rG^{r-1}\widetilde{\mathcal{K}}', \quad (H^{r-1}W^{r-2}\mathcal{K}^1)_{\mathrm{alg}} \subset H^{r-1}W^{r-2}\mathcal{K}^1.$$

These have the induced filtrations F and W so that

$$\mathrm{Gr}_F^0(H^rG^{r-1}\widetilde{\mathcal{K}})^0_{\mathrm{alg}} = P_{\geq r-1}(\overline{X}_\bullet)^0, \quad \mathrm{Gr}_F^1(H^rG^{r-1}\widetilde{\mathcal{K}})^0_{\mathrm{alg}} = (H^{r-1}W^{r-2}\mathcal{K}^1)_{\mathrm{alg}},$$
$$\mathrm{Gr}_F^0(H^rG^{r-1}\widetilde{\mathcal{K}}')_{\mathrm{alg}} = H^rW^{r-1}\mathcal{K}', \quad \mathrm{Gr}_F^1(H^rG^{r-1}\widetilde{\mathcal{K}}')_{\mathrm{alg}} = (H^{r-1}W^{r-2}\mathcal{K}^1)_{\mathrm{alg}},$$

and W on these spaces is calculated by using the weight spectral sequence which degenerates at E_2, see (4.4.2).

Consider the morphism

$$H^{r-1}\mathrm{Gr}_W^{r-2}\mathcal{K} \to H^rG^{r-1}\widetilde{\mathcal{K}}$$

induced by the distinguished triangle

$$\to G^{r-1}\widetilde{\mathcal{K}} \to [W^{r-2}\mathcal{K} \to W^{r-2}\mathcal{K}^1] \to \mathrm{Gr}_W^{r-2}\mathcal{K} \to .$$

Let $(H^{r-1}\mathrm{Gr}_W^{r-2}\mathcal{K})^{(0)}$ be the kernel of the morphism to $H^rW^{r-1}\mathcal{K}/P_{\geq r-1}(\overline{X}_\bullet)^0$. Then the above morphism induces

$$(H^{r-1}\mathrm{Gr}_W^{r-2}\mathcal{K})^{(0)} \to (H^rG^{r-1}\widetilde{\mathcal{K}})^0_{\mathrm{alg}}. \tag{3.5.2}$$

We divide the source by $\mathrm{Pic}(\overline{X}_{r-2})^0$ and the target by its image. (Note that the image of $\mathrm{Pic}(\overline{X}_{r-2})^0$ in $(H^rG^{r-1}\widetilde{\mathcal{K}})^0_{\mathrm{alg}}$ is isomorphic to that in $\mathrm{Gr}_F^0(H^rG^{r-1}\widetilde{\mathcal{K}})^0_{\mathrm{alg}}$ because there is no nontrivial morphism of an abelian variety to an affine space.) Since the source is then $\Gamma'_r(\overline{X}_\bullet\langle D_\bullet\rangle)$, we can divide these further by the images of $\mathrm{NS}(\overline{X}_\bullet\langle D_\bullet\rangle)^{r-3}$ and $(H^{r-2}\mathrm{Gr}_W^{r-3}\mathcal{K}^1)_{\mathrm{alg}}$ in the notation of (3.1). Let $\widetilde{G}_r(\overline{X}_\bullet\langle D_\bullet\rangle)$ be the commutative k-group scheme whose underlying group of k-valued points is the cokernel of the canonical morphism

$$\mathrm{Pic}(\overline{X}_{r-2})^0 \oplus (H^{r-2}\mathrm{Gr}_W^{r-3}\mathcal{K}^1)_{\mathrm{alg}} \to (H^rG^{r-1}\widetilde{\mathcal{K}})^0_{\mathrm{alg}},$$

which underlies naturally a morphism of groups schemes. Then we get

$$\widetilde{M}_r(\overline{X}_\bullet\langle D_\bullet\rangle) := [\Gamma_r(\overline{X}_\bullet\langle D_\bullet\rangle) \to \widetilde{G}_r(\overline{X}_\bullet\langle D_\bullet\rangle)].$$

The Lie algebra $\operatorname{Lie}\widetilde{G}_r(\overline{X}_\bullet\langle D_\bullet\rangle)$ is isomorphic to

$$\operatorname{Coker}(H^1(\overline{X}_{r-2},\mathcal{O}_{\overline{X}_{r-2}}) \oplus (H^{r-2}\operatorname{Gr}_W^{r-3}\mathcal{K}^1)_{\mathrm{alg}} \to (H^r G^{r-1}\widetilde{\mathcal{K}}')_{\mathrm{alg}})$$

by the standard argument using $\operatorname{Spec} k[\varepsilon]$ (cf. [21]). But this is isomorphic to the image of $(H^r G^{r-1}\widetilde{\mathcal{K}}')_{\mathrm{alg}}$ in $H^r_{\mathrm{DR}}(X_\bullet)/F^2$ by the E_2-degeneration of the weight spectral sequence together with the strictness of the Hodge filtration. Since it is isomorphic to $H^r_{\mathrm{DR},(1)}(X_\bullet)$, it is sufficient to show that $\widetilde{M}_r(\overline{X}_\bullet\langle D_\bullet\rangle)$ is the universal $\mathbb{G}_a$-extension of $M_r(\overline{X}_\bullet\langle D_\bullet\rangle)$. Then we may replace it with Gr^i_W, and the assertion is reduced to the well-known fact about the universal $\mathbb{G}_a$-extension of the Picard variety (see Remark (i) below). This completes the proof of (3.5).

Remarks. (i) Let $\widetilde{M} = [\Gamma \to \widetilde{G}]$ be the universal $\mathbb{G}_a$-extension of a 1-motive $M = [\Gamma \to G]$. Then the de Rham realization $r_{\mathrm{DR}}(M)$ is defined to be $\operatorname{Lie}\widetilde{G}$. It is known that the universal extension is given by a commutative diagram with exact rows

$$\begin{array}{ccccccccc} & & & & \Gamma & = & \Gamma & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \operatorname{Ext}^1(M,\mathbb{G}_a)^\vee & \longrightarrow & \widetilde{G} & \longrightarrow & G & \longrightarrow & 0 \end{array}$$

where $\operatorname{Ext}^1(M,\mathbb{G}_a)$ is a finite dimensional k-vector space, and its dual is identified with a group scheme. (See [10], [21] and also [2].) Indeed, if $0 \to V \to M' \to M \to 0$ is an extension by a k-vector space V which is identified with a k-group scheme, it gives a morphism $V^\vee (= \operatorname{Hom}(V,\mathbb{G}_a)) \to \operatorname{Ext}^1(M,\mathbb{G}_a)$ by composition, and its dual deduces the original extension from the universal extension. In particular, the functor $M \to \widetilde{M}$ is exact. If M is a torus, $\widetilde{M} = M$. If $M = \operatorname{Pic}(X)^0$ for a smooth proper variety, then $\operatorname{Lie}\widetilde{G} = H^1_{\mathrm{DR}}(X)$ and $\operatorname{Ext}^1(M,\mathbb{G}_a)^\vee = F^1H^1_{\mathrm{DR}}(X) = \Gamma(X,\Omega^1_X)$. If $M = [\Gamma \to 0]$, then $\widetilde{M} = [\Gamma \to \Gamma \otimes \mathbb{G}_a]$. See loc. cit.

(ii) With the above notation, assume $k = \mathbb{C}$. Then we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E & \longrightarrow & E' & \longrightarrow & \Gamma_r(\overline{X}_\bullet\langle D_\bullet\rangle) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & E & \longrightarrow & \operatorname{Lie}G_r(\overline{X}_\bullet\langle D_\bullet\rangle) & \xrightarrow{\exp} & G_r(\overline{X}_\bullet\langle D_\bullet\rangle) & \longrightarrow & 0 \end{array}$$

where exp is the exponential map. Note that the exponential map of a commutative Lie group depends on the analytic structure of the group, and it cannot be used for the proof of the coincidence of the natural analytic structure with the one coming from the

algebraic structure of the Picard variety, before we show that the generalized Abel-Jacobi map depends analytically on the parameter. See also Remark after (5.3).

We will later show that E' is identified with $H^r_{(1)}(X,Y;\mathbb{Z})$ modulo torsion. This is true if and only if we have the above commutative diagram with E' replaced by $H^r_{(1)}(X,Y;\mathbb{Z})$ (modulo torsion). But it is easy to see that the assertion is equivalent to the coincidence of the two extension classes associated with the 1-motive $M_r(X,Y)$ and the mixed Hodge structure $H^r_{(1)}(X,Y;\mathbb{Z})$. This will be proved in the proof of (5.4). This point is not clear in [26].

4. Mixed Hodge theory

We review the theory of mixed Hodge complexes ([10], [4]) and Deligne cohomology ([3], [4]). See also [11], [12], [13], [19], etc. We assume $k=\mathbb{C}$.

4.1. Mixed Hodge complexes. Let $\mathcal{C}_{\mathcal{H}}$ be the category of mixed Hodge complexes in the sense of Beilinson [4], 3.2. An object $K\in\mathcal{C}_{\mathcal{H}}$ consists of (filtered or bifiltered) complexes $K_{\mathbb{Z}},K'_{\mathbb{Q}},(K_{\mathbb{Q}},W),(K'_{\mathbb{C}},W),(K_{\mathbb{C}};F,W)$ over $\mathbb{Z},\mathbb{Q}$ or $\mathbb{C}$, together with (filtered) morphisms

$$\alpha_1:K_{\mathbb{Z}}\to K'_{\mathbb{Q}},\ \alpha_2:K_{\mathbb{Q}}\to K'_{\mathbb{Q}},\ \alpha_3:(K_{\mathbb{Q}},W)\to(K'_{\mathbb{C}},W),\ \alpha_4:(K_{\mathbb{C}},W)\to(K'_{\mathbb{C}},W)$$

which induce quasi-isomorphisms after scalar extensions. These complexes are bounded below, the $H^jK_{\mathbb{Z}}$ are finite $\mathbb{Z}$-modules and vanish for $j\gg 0$, the filtration F on $\mathrm{Gr}^W_iK_{\mathbb{C}}$ is strict, and $H^j\mathrm{Gr}^W_i(K_{\mathbb{Q}},(K_{\mathbb{C}},F))$ is a pure Hodge structure of weight i, using the isomorphism $H^j\mathrm{Gr}^W_iK_{\mathbb{Q}}\otimes_{\mathbb{Q}}\mathbb{C}=H^j\mathrm{Gr}^W_iK_{\mathbb{C}}$ given by α_3 and α_4. A morphism of $\mathcal{C}_{\mathcal{H}}$ is a family of morphisms of (filtered or bifiltered) complexes compatible with the α_i. A homotopy is defined similarly. We get $\mathcal{D}_{\mathcal{H}}$ by inverting bifiltered quasi-isomorphisms. See loc. cit. for details.

Similarly, we have categories $\mathcal{C}_{\mathcal{H}^p},\mathcal{D}_{\mathcal{H}^p}$ of mixed p-Hodge complexes. This is defined by modifying the above definition as follows:

Firstly, the weight of $H^j\mathrm{Gr}^W_i(K_{\mathbb{Q}},(K_{\mathbb{C}},F))$ is $i+j$ (as in [10]) instead of i, and it is assumed to be polarizable. A homotopy h preserves the Hodge filtration F. But it preserves the weight filtration W up to the shift -1, and $dh+hd$ preserves W. (This is necessary to show the acyclicity of the mapping cone of the identity.) The derived category $\mathcal{D}_{\mathcal{H}^p}$ is obtained by inverting quasi-isomorphisms (preserving F,W).

We have natural functors

$$\mathrm{Dec}:\mathcal{C}_{\mathcal{H}^p}\to\mathcal{C}_{\mathcal{H}},\quad \mathrm{Dec}:\mathcal{D}_{\mathcal{H}^p}\to\mathcal{D}_{\mathcal{H}},\tag{4.1.1}$$

by replacing the weight filtration W with $\mathrm{Dec}\,W$ (see [10]), which is defined by

$$(\mathrm{Dec}\,W)_jK^i_{\mathbb{Q}}=\mathrm{Ker}(d:W_{j-i}K^i_{\mathbb{Q}}\to\mathrm{Gr}^W_{j-i}K^{i+1}_{\mathbb{Q}}).$$

Note that (4.1.1) is well-defined, because $(F,\mathrm{Dec}\,W)$ is bistrict, and a quasi-isomorphism (preserving F,W) induces a bifiltered quasi-isomorphism for $(F,\mathrm{Dec}\,W)$. See [27], 1.3.8 or [30], A.2.

The Tate twist $K(m)$ of $K \in \mathcal{D}_{\mathcal{H}}$ (or $\mathcal{D}_{\mathcal{H}^p}$) for $n \in \mathbb{Z}$ is defined by twisting the complexes over $\mathbb{Z}$ or $\mathbb{Q}$ and shifting the Hodge filtration F and the weight filtration W as usual [10], e.g. $W_j(K(m)_{\mathbb{Q}}) = W_{j+2m}K_{\mathbb{Q}}(m), F^p(K(m)_{\mathbb{C}}) = F^{p+m}K_{\mathbb{C}}$.

For $K \in \mathcal{D}_{\mathcal{H}}$ we define $\Gamma_{\mathcal{D}}K$ and $\Gamma_{\mathcal{H}}K$ using the shifted mapping cones (i.e. the first terms have degree zero):

$$\Gamma_{\mathcal{D}}K = [K_{\mathbb{Z}} \oplus K_{\mathbb{Q}} \oplus F^0K_{\mathbb{C}} \to K'_{\mathbb{Q}} \oplus K'_{\mathbb{C}}],$$
$$\Gamma_{\mathcal{H}}K = [K_{\mathbb{Z}} \oplus W_0K_{\mathbb{Q}} \oplus F^0W_0K_{\mathbb{C}} \to K'_{\mathbb{Q}} \oplus W_0K'_{\mathbb{C}}],$$

where the morphisms of complexes are given by $(a,b,c) \mapsto (\alpha_1(a) - \alpha_1(b), \alpha_1(b) - \alpha_1(c))$. Note that we have a quasi-isomorphism

$$\Gamma_{\mathcal{D}}K \to [K_{\mathbb{Z}} \oplus F^0K_{\mathbb{C}} \to K'_{\mathbb{C}}], \tag{4.1.2}$$

where the morphism of complexes is given by $\alpha' \circ \alpha_1 - \alpha_4$, if there is a morphism $\alpha' : K'_{\mathbb{Q}} \to K'_{\mathbb{C}}$ such that $\alpha' \circ \alpha_2 = \alpha_3$. (Indeed, the quasi-isomorphism is given by $(a,b,c;b',c') \mapsto (a,c;b'+c')$.) We can also define a similar complex for polarizable mixed Hodge complexes. But it is not used in this paper. For $K \in \mathcal{D}_{\mathcal{H}^p}$, we define

$$\Gamma_{\mathcal{D}}K = \Gamma_{\mathcal{D}}(\mathrm{Dec}\, K), \quad \Gamma_{\mathcal{H}}K = \Gamma_{\mathcal{H}}(\mathrm{Dec}\, K),$$

using Dec in (4.1.1).

By Beilinson [4], 3.6, we have a canonical isomorphism

$$\mathrm{Hom}_{\mathcal{H}}(\mathbb{Z}, K) = H^0(\Gamma_{\mathcal{H}}K), \tag{4.1.3}$$

where $\mathrm{Hom}_{\mathcal{H}}$ means the group of morphisms in $\mathcal{D}_{\mathcal{H}}$. He also shows (loc. cit., 3.4) that the canonical functor induces an equivalence of categories

$$D^b\mathrm{MHS} \xrightarrow{\sim} \mathcal{D}_{\mathcal{H}}, \tag{4.1.4}$$

where the source is the bounded derived category of mixed $\mathbb{Z}$-Hodge structures.

4.2. Deligne cohomology. For a smooth complex algebraic variety X, let $\overline{X}$ be a smooth compactification such that $D := \overline{X} \setminus X$ is a divisor with simple normal crossings. Let $j : X \to \overline{X}$ denote the inclusion, and $\Omega^{\bullet}_{\overline{X}^{\mathrm{an}}}\langle D\rangle$ the complex of holomorphic logarithmic forms with the Hodge filtration F (defined by σ) and the weight filtration W. See [10]. Let $\mathcal{C}^{\bullet}$ denote the canonical flasque resolution of Godement. Then we define the mixed Hodge complex associated with $(\overline{X}, D)$:

$$K_{\mathcal{H}^p}(\overline{X}\langle D\rangle) \in \mathcal{C}_{\mathcal{H}^p}$$

as follows (see [4], [10]): Let

$$K_{\mathbb{Z}} = \Gamma(X^{\mathrm{an}}, \mathcal{C}^{\bullet}(\mathbb{Z}_{X^{\mathrm{an}}})), \;\; K_{\mathbb{Q}} = K'_{\mathbb{Q}} = \Gamma(\overline{X}^{\mathrm{an}}, j_*\mathcal{C}^{\bullet}(\mathbb{Q}_{X^{\mathrm{an}}})), \;\; K_{\mathbb{C}} = \Gamma(\overline{X}^{\mathrm{an}}, \Omega^{\bullet}_{\overline{X}^{\mathrm{an}}}\langle D\rangle).$$

where W on $K_{\mathbb{Q}}$ and $K_{\mathbb{C}}$ is induced by τ on $j_*\mathcal{C}^\bullet(\mathbb{Q}_{X^{\mathrm{an}}})$ and W on $\Omega^\bullet_{\overline{X}^{\mathrm{an}}}\langle D\rangle$ respectively, and the Hodge filtration F is induced by σ on $\Omega^\bullet_{\overline{X}^{\mathrm{an}}}\langle D\rangle$. We define $(K'_{\mathbb{C}}, W)$ by taking the global section functor of the mapping cone of

$$(\mathcal{C}^\bullet(\Omega^\bullet_{\overline{X}^{\mathrm{an}}}\langle D\rangle), \tau) \to (j_*\mathcal{C}^\bullet(\Omega^\bullet_{X^{\mathrm{an}}}), \tau) \oplus (\mathcal{C}^\bullet(\Omega^\bullet_{\overline{X}^{\mathrm{an}}}\langle D\rangle), W). \tag{4.2.1}$$

(The description in [30], 3.3 is not precise. We need a mapping cone as above.) Note that $K_{\mathcal{H}^p}(\overline{X}\langle D\rangle)$ has the weight filtration W defined over $\mathbb{Z}$.

We will denote the image of $K_{\mathcal{H}^p}(\overline{X}\langle D\rangle)$ in $\mathcal{D}_{\mathcal{H}^p}$ by

$$K_{\mathcal{H}^p}(X) \in \mathcal{D}_{\mathcal{H}^p},$$

because it is independent of the choice of the compactification $\overline{X}$ by definition of $\mathcal{D}_{\mathcal{H}^p}$. We define

$$K_{\mathcal{H}}(X) \in \mathcal{D}_{\mathcal{H}}$$

to be the image of $K_{\mathcal{H}^p}(X)$ by (4.1.1).

Let $X_\bullet, \overline{X}_\bullet$ be as in (3.1). Applying the above construction to each $\overline{X}_j, D_j$, we get

$$K_{\mathcal{H}^p}(\overline{X}_\bullet\langle D_\bullet\rangle) \in \mathcal{C}_{\mathcal{H}^p} \quad \text{and} \quad K_{\mathcal{H}^p}(X_\bullet) \in \mathcal{D}_{\mathcal{H}^p},$$

Here the filtration W for $K_{\mathcal{H}^p}(\overline{X}_j\langle D_j\rangle)$ is shifted by $-j$ when the complex is shifted. We define

$$K_{\mathcal{H}}(X_\bullet) \in \mathcal{D}_{\mathcal{H}}$$

to be the image of $K_{\mathcal{H}^p}(X_\bullet)$ by (4.1.1). Here W is not shifted depending on j, because we take Dec. Then we have a canonical isomorphism of mixed Hodge structures

$$H^i(X_\bullet) = H^i K_{\mathcal{H}}(X_\bullet). \tag{4.2.2}$$

We define

$$H^i_{\mathcal{D}}(X_\bullet, \mathbb{Z}(j)) = H^i\Gamma_{\mathcal{D}}(K_{\mathcal{H}}(X_\bullet)(j)),$$
$$H^i_{\mathrm{AH}}(X_\bullet, \mathbb{Z}(j)) = H^i\Gamma_{\mathcal{H}}(K_{\mathcal{H}}(X_\bullet)(j)).$$

For a closed subvariety Y of X, we apply the above construction to a resolution of $[\overline{Y} \to \overline{X}]$ as in the proof of (2.3), and get

$$K_{\mathcal{H}^p}(X, Y) \in \mathcal{D}_{\mathcal{H}^p}, \quad K_{\mathcal{H}}(X, Y) \in \mathcal{D}_{\mathcal{H}}.$$

These are independent of the choice of the resolution by definition of $\mathcal{D}_{\mathcal{H}^p}, \mathcal{D}_{\mathcal{H}}$. They will be denoted by $K_{\mathcal{H}^p}(X), K_{\mathcal{H}}(X)$ if Y is empty.

We define Deligne cohomology and absolute Hodge cohomology in the sense of Beilinson ([3], [4]) by

$$H^i_{\mathcal{D}}(X, Y; \mathbb{Z}(j)) = H^i\Gamma_{\mathcal{D}}(K_{\mathcal{H}}(X, Y)(j)),$$
$$H^i_{\mathrm{AH}}(X, Y; \mathbb{Z}(j)) = H^i\Gamma_{\mathcal{H}}(K_{\mathcal{H}}(X, Y)(j)),$$

See also [11], [12], [13], [19], etc. We will omit Y if it is empty.

By definition we have a natural morphism

$$H^i_{\mathrm{AH}}(X,Y;\mathbb{Z}(j)) \to H^i_{\mathcal{D}}(X,Y;\mathbb{Z}(j)). \tag{4.2.3}$$

4.3. Short exact sequences. Since higher extensions vanish in MHS, every complex is represented by a complex with zero differential in $D^b\mathrm{MHS}$. We see that $K \in \mathcal{D}_{\mathcal{H}}$ corresponds by (4.1.4) (noncanonically) to

$$\oplus_i (H^i K)[-i] \in D^b\mathrm{MHS}. \tag{4.3.1}$$

Then, using the t-structure on $\mathcal{D}_{\mathcal{H}}$, we have a canonical exact sequence

$$0 \to \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}, H^{i-1}K(j)) \to H^i\Gamma_{\mathcal{H}}(K(j)) \to \mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Z}, H^iK(j)) \to 0. \tag{4.3.2}$$

Similarly, we have

$$0 \to J(H^{i-1}K(j)) \to H^i\Gamma_{\mathcal{D}}(K(j)) \to H^iK_{\mathbb{Z}}(j) \cap F^jH^iK_{\mathbb{C}} \to 0, \tag{4.3.3}$$

where we put

$$J(H(j)) = H_{\mathbb{C}}/(H_{\mathbb{Z}}(j) + F^jH_{\mathbb{C}}),$$
$$H_{\mathbb{Z}}(j) \cap F^jH_{\mathbb{C}} = \mathrm{Ker}(H_{\mathbb{Z}}(j) \to H_{\mathbb{C}}/F^jH_{\mathbb{C}}),$$

for a mixed Hodge structure $H = (H_{\mathbb{Z}}, (H_{\mathbb{Q}}, W), (H_{\mathbb{C}}; F, W))$.

Comparing (4.3.2-3), we see that

$$H^i_{\mathcal{D}}(X_\bullet, \mathbb{Z}(j)) = H^i_{\mathrm{AH}}(X_\bullet, \mathbb{Z}(j)) \tag{4.3.4}$$

if $H^{i-1}(X_\bullet, \mathbb{Z})$ and $H^i(X_\bullet, \mathbb{Z})$ have weights $\leq 2j$ (using [8]).

4.4. Weight spectral sequence. Let W be the weight filtration of $K_{\mathcal{H}^p}(\overline{X}_\bullet\langle D_\bullet\rangle)$ in $\mathcal{C}_{\mathcal{H}^p}$, and $\widetilde{D}^k_j$ the disjoint union of the intersections of k irreducible components of D_j. See [10]. By definition we have

$$\mathrm{Gr}^W_{-p} K_{\mathcal{H}^p}(\overline{X}_\bullet\langle D_\bullet\rangle) = \oplus_{k\geq 0} K_{\mathcal{H}^p}(\widetilde{D}^k_{p+k})(-k)[-p-2k], \tag{4.4.1}$$

and this gives the integral weight spectral sequence (2.3.2). It depends on the choice of the compactification $\overline{X}_\bullet$ of $X_\bullet$. By loc. cit. this spectral sequence degenerates at E_2 modulo torsion.

Let ${}^oW_jX_\bullet$ be the cofiltration in (3.1). Then

$$\mathrm{Gr}^W_{-p} K_{\mathcal{H}^p}({}^oW^j\overline{X}_\bullet\langle D_\bullet\rangle) = \oplus_{k\geq 0} K_{\mathcal{H}^p}({}^oW^j\widetilde{D}^k_{p+k})(-k)[-p-2k],$$

where ${}^oW_j\widetilde{D}^k_{p+k} = \widetilde{D}^k_{p+k}$ if $p+k > j$ or $p-j = k = 0$, and ${}^oW_j\widetilde{D}^k_{p+k} = \emptyset$ otherwise. This implies

$$H^r(W^jK_{\mathcal{H}^p}(\overline{X}_\bullet\langle D_\bullet\rangle)) = H^r(X^j_\bullet)) \quad \text{for } r \leq j+2. \tag{4.4.2}$$

Indeed, we have $W^jK_{\mathcal{H}^p}({}^oW^j\overline{X}_\bullet\langle D_\bullet\rangle) = W^jK_{\mathcal{H}^p}(\overline{X}_\bullet\langle D_\bullet\rangle)$ and

$$H^r(K_{\mathcal{H}^p}({}^oW^j\overline{X}_\bullet\langle D_\bullet\rangle)/W^jK_{\mathcal{H}^p}({}^oW^j\overline{X}_\bullet\langle D_\bullet\rangle)) = 0 \quad \text{for } r \le j+2,$$

where $W^j = W_{-j}$. Note that the weight filtration on $H^r(X_\bullet,\mathbb{Z})$ is shifted by r as in [10] (i.e., it is induced by $\operatorname{Dec} W$).

4.5. Remark. Assume X is smooth proper. Then $H^i_{\mathcal{D}}(X,\mathbb{Z}(j))$ is the hypercohomology of the complex

$$\mathbb{Z}_{X^{\mathrm{an}}}(j) \to \mathcal{O}_{X^{\mathrm{an}}} \to \Omega^1_{X^{\mathrm{an}}} \to \cdots \to \Omega^{j-1}_{X^{\mathrm{an}}},$$

where the degree of $\mathbb{Z}_{X^{\mathrm{an}}}(j)$ is zero. In particular, using the exponential sequence, we have for $j=1$

$$H^i_{\mathcal{D}}(X,\mathbb{Z}(1)) = H^{i-1}(X^{\mathrm{an}},\mathcal{O}^*_{X^{\mathrm{an}}}). \tag{4.5.1}$$

By (4.1.3) and (4.3.2), we get in the smooth case

$$\begin{aligned} &H^i_{\mathcal{D}}(X,\mathbb{Z}(1)) = H^i_{\mathrm{AH}}(X,\mathbb{Z}(1)) \quad \text{for } i \le 2,\\ &H^i_{\mathrm{AH}}(X,\mathbb{Z}(1)) \text{ is torsion for } i > 3. \end{aligned} \tag{4.5.2}$$

Note that the algebraic cohomology $H^{i-1}(X,\mathcal{O}^*_X)$ coincides with $H^{i-1}(X^{\mathrm{an}},\mathcal{O}^*_{X^{\mathrm{an}}})$ for $i\le 2$ by GAGA, and vanishes otherwise by (2.5.3). In particular, $H^i_{\mathrm{AH}}(X,\mathbb{Z}(1))$ coincides with the algebraic cohomology $H^{i-1}(X,\mathcal{O}^*_X)$ up to torsion for $i\ne 2$ in the smooth case. (If we consider a polarizable version, we get isomorphisms up to torsion for every i.)

5. Comparison

In this section we prove Theorems (0.1–2).

5.1. Theorem. *Let $X_\bullet$ and $\overline{X}_\bullet$ be as in (3.1) with $k=\mathbb{C}$. Then there exist a decreasing filtration W'' in a generalized sense on $\Gamma_{\mathcal{D}}(K_{\mathcal{H}}(X_\bullet)(1))$ and a canonical morphism*

$$\Gamma(\overline{X}^{\mathrm{an}}_\bullet,\mathcal{C}^\bullet(j^m_*\mathcal{O}^*_{X^{\mathrm{an}}}[-1])) \to \Gamma_{\mathcal{D}}(K_{\mathcal{H}}(X_\bullet)(1)), \tag{5.1.1}$$

preserving the filtrations W' and W'', where $\mathcal{C}^\bullet$ denotes the canonical flasque resolution, $j^m_*\mathcal{O}^*_{X^{\mathrm{an}}}$ is the meromorphic extension, and W' on the target is induced by the degree of $X_\bullet$ as in (3.1). Furthermore (5.1.1) becomes a bifiltered quasi-isomorphism by replacing the target with W''^{-1}, and the mapping cone of $\mathrm{Gr}^p_{W'}$ of (5.1.1) is quasi-isomorphic to $(\tau_{>1}j_*\mathcal{C}^\bullet(\mathbb{Z}_{X_p^{\mathrm{an}}}(1)))[-p]$.

Remark. The filtration W'' is not given by subcomplexes, but by morphisms of complexes $0 = W''^1 \to W''^0 \to W''^{-1} \to W''^{-2} = \Gamma_{\mathcal{D}}(K_{\mathcal{H}}(X_\bullet)(1))$ compatible with W'. See [5] and [27], 1.3. For this we can define naturally the notion of bifiltered quasi-isomorphism.

Proof of (5.1). Let X be a smooth complex algebraic variety, and $\overline{X}$ a smooth compactification such that $D := \overline{X} \setminus X$ is a divisor with simple normal crossings. Then we have a short exact sequence

$$0 \to \mathcal{O}^*_{\overline{X}^{\mathrm{an}}} \to j^m_* \mathcal{O}^*_{X^{\mathrm{an}}} \to \oplus_i \mathbb{Z}_{D_i^{\mathrm{an}}} \to 0, \tag{5.1.2}$$

where the D_i are irreducible components of D. This implies a canonical isomorphism in the derived category

$$[\mathcal{O}_{\overline{X}^{\mathrm{an}}} \overset{\exp}{\to} j^m_* \mathcal{O}^*_{X^{\mathrm{an}}}] = \tau_{\leq 1} j_* \mathcal{C}^\bullet(\mathbb{Z}_{X^{\mathrm{an}}}(1)). \tag{5.1.3}$$

We can verify that $\Gamma_{\mathcal{D}}(K_{\mathcal{H}}(X)(1))$ is naturally quasi-isomorphic to the complex of global sections of the shifted mapping cone

$$[j_* \mathcal{C}^\bullet(\mathbb{Z}_{X^{\mathrm{an}}}(1)) \oplus \mathcal{C}^\bullet(\sigma_{\geq 1} \Omega^\bullet_{\overline{X}^{\mathrm{an}}}(\log D)) \to j_* \mathcal{C}^\bullet(\Omega^\bullet_{X^{\mathrm{an}}})],$$

where $\mathcal{C}^\bullet$ denotes the canonical flasque resolution of Godement. Since we have a natural morphism

$$[\mathcal{O}_{X^{\mathrm{an}}} \overset{\exp}{\to} \mathcal{O}^*_{X^{\mathrm{an}}}] \to \Omega^\bullet_{X^{\mathrm{an}}}$$

using $d\log$ (see [10]), the above shifted mapping cone is quasi-isomorphic to

$$\mathbb{Z}(1)_{\mathcal{D},X\langle D\rangle} := [j_* \mathcal{C}^\bullet([\mathcal{O}_{X^{\mathrm{an}}} \overset{\exp}{\to} \mathcal{O}^*_{X^{\mathrm{an}}}]) \oplus \mathcal{C}^\bullet(\sigma_{\geq 1} \Omega^\bullet_{\overline{X}^{\mathrm{an}}}(\log D)) \to j_* \mathcal{C}^\bullet(\Omega^\bullet_{X^{\mathrm{an}}})].$$

Consider then the shifted mapping cone

$$\mathbb{Z}(1)'_{\mathcal{D},X\langle D\rangle} := [\mathcal{C}^\bullet([\mathcal{O}_{\overline{X}^{\mathrm{an}}} \overset{\exp}{\to} j^m_* \mathcal{O}^*_{X^{\mathrm{an}}}]) \oplus \mathcal{C}^\bullet(\sigma_{\geq 1} \Omega^\bullet_{\overline{X}^{\mathrm{an}}}(\log D)) \to j_* \mathcal{C}^\bullet(\Omega^\bullet_{X^{\mathrm{an}}})].$$

It has a natural morphism to $\mathbb{Z}(1)_{\mathcal{D},X\langle D\rangle}$, and defines W''^{-1}. Here we can replace $j_* \mathcal{C}^\bullet(\Omega^\bullet_{X^{\mathrm{an}}})$ with $\mathcal{C}^\bullet(\Omega^\bullet_{\overline{X}^{\mathrm{an}}}(\log D))$ because the image of $d\log$ is a logarithmic form. So we see that $\mathbb{Z}(1)'_{\mathcal{D},X\langle D\rangle}$ is naturally quasi-isomorphic to $\mathcal{C}^\bullet(j^m_* \mathcal{O}^*_{X^{\mathrm{an}}})[-1]$. Furthermore, $\mathbb{Z}(1)'_{\mathcal{D},X\langle D\rangle}$ has a subcomplex

$$\mathbb{Z}(1)_{\mathcal{D},\overline{X}} := [\mathcal{C}^\bullet([\mathcal{O}_{\overline{X}^{\mathrm{an}}} \overset{\exp}{\to} \mathcal{O}^*_{\overline{X}^{\mathrm{an}}}]) \oplus \mathcal{C}^\bullet(\sigma_{\geq 1} \Omega^\bullet_{\overline{X}^{\mathrm{an}}}) \to \mathcal{C}^\bullet(\Omega^\bullet_{\overline{X}^{\mathrm{an}}})]$$

which is naturally quasi-isomorphic to $\Gamma_{\mathcal{D}}(K_{\mathcal{H}}(\overline{X})(1))$ and $\mathcal{C}^\bullet(\mathcal{O}^*_{\overline{X}^{\mathrm{an}}})[-1]$. This defines W''^0.

Thus we get a canonical filtered morphism

$$\Gamma(\overline{X}^{\mathrm{an}}, \mathcal{C}^\bullet(j_* \mathcal{O}^*_{X^{\mathrm{an}}}[-1])) \to \Gamma_{\mathcal{D}}(K_{\mathcal{H}}(X)(1)) \tag{5.1.4}$$

whose mapping cone is isomorphic to $\tau_{>1} j_* \mathcal{C}^\bullet(\mathbb{Z}_{X^{\mathrm{an}}}(1))$.

We apply this constriction to each component $\overline{X}_p$ of $\overline{X}_\bullet$. Then we get the morphism (5.1.1) preserving the filtrations W' and W''.

5.2. Corollary. *For $X_\bullet$ as in (5.1), we have a canonical morphism*

$$\mathrm{Pic}(X_\bullet) \to H^2_{\mathcal{D}}(X_\bullet, \mathbb{Z}(1)), \tag{5.2.1}$$

induced by (5.1.1). (See (2.4) for $\mathrm{Pic}(X_\bullet)$*.) This is injective if X_j is empty for $j < 0$, and is bijective if furthermore X_0 is proper. In particular, we have in the notation of (3.1) and (4.1)*

$$P_{\geq r}(\overline{X}_\bullet\langle D_\bullet\rangle) = H^{r+2}_{\mathcal{D}}({}^oW^rX_\bullet, \mathbb{Z}(1)). \tag{5.2.2}$$

Proof. This follows from (5.1) together with (2.5.3).

Remark. By (4.3.2) and (4.3.4) we get

$$H^{r+i}_{\mathrm{AH}}({}^oW^r\overline{X}_\bullet, \mathbb{Z}(1)) = H^{r+i}_{\mathcal{D}}({}^oW^rX_\bullet, \mathbb{Z}(1)) \quad \text{for } i \leq 2, \tag{5.2.3}$$

and a short exact sequence

$$\begin{aligned} 0 \to \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}, H^{r+1}({}^oW^rX_\bullet, \mathbb{Z})(1)) \to H^{r+2}_{\mathcal{D}}({}^oW^rX_\bullet, \mathbb{Z}(1)) \\ \to \mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Z}, H^{r+2}({}^oW^rX_\bullet, \mathbb{Z})(1)) \to 0, \end{aligned} \tag{5.2.4}$$

because $H^{r+i}({}^oW^rX_\bullet, \mathbb{Z})$ has weights $\leq i$ for $i \leq 2$ by (4.4.2). Furthermore, using the spectral sequence (2.3.2), we see

$$\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}, H^{r+1}({}^oW^r\overline{X}_\bullet, \mathbb{Z})(1)) = \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}, H^{r+1}({}^oW^rX_\bullet, \mathbb{Z})(1)), \tag{5.2.5}$$

and they have naturally a structure of semiabelian variety as is well-known. See [8], [10]. This coincides with the structure of semiabelian variety on $\mathcal{P}_{\geq r}(\overline{X}_\bullet)^0$ by the following:

5.3. Theorem. *Let S be a smooth complex algebraic variety, and $(L, \gamma) \in P_{\geq r}(\overline{X}_\bullet \times S)$ as in Remark after (3.1), where $D_\bullet$ is empty. Then the set-theoretic map $S(\mathbb{C}) \to H^{r+2}_{\mathcal{D}}({}^oW^r\overline{X}_\bullet, \mathbb{Z}(1))$ defined by restricting (L, γ) to the fiber at $s \in S(\mathbb{C})$ comes from a morphism of varieties.*

Proof. Let $\xi \in H^{r+2}_{\mathcal{D}}({}^oW^r\overline{X}_\bullet \times S, \mathbb{Z}(1))$ corresponding to (L, γ) by the injective morphism

$$P_{\geq r}(\overline{X}_\bullet \times S) \to H^{r+2}_{\mathcal{D}}({}^oW^r\overline{X}_\bullet \times S, \mathbb{Z}(1)) \tag{5.3.1}$$

given by (5.2.1) for $\overline{X}_\bullet \times S$. Since this morphism is compatible with the restriction to $X_\bullet \times \{s\}$, it is enough to show that the map

$$s \mapsto \xi_s \in H^{r+2}_{\mathcal{D}}({}^oW^r\overline{X}_\bullet, \mathbb{Z}(1)) = H^{r+2}_{\mathrm{AH}}({}^oW^r\overline{X}_\bullet, \mathbb{Z}(1))$$

is algebraic, where the last isomorphism follows from (5.2.3).

We first replace the Deligne cohomology of ${}^oW^r\overline{X}_\bullet \times S$ with the absolute Hodge cohomology. Since the restriction to the fiber is defined at the level of mixed Hodge complexes,

it is compatible with the canonical morphism induced by $\Gamma_{\mathcal{H}} \to \Gamma_{\mathcal{D}}$. So it is enough to show that ξ belongs to the image of the absolute Hodge cohomology. But this is verified by using ${}^oW^r(\overline{X}_\bullet \times S)$ which is defined by replacing the r-th component of ${}^oW^r\overline{X}_\bullet \times S$ with ${}^oW^r(\overline{X}_r \times \overline{S})$, where $\overline{S}$ is a smooth compactification of S. Indeed, the Deligne cohomology and the absolute Hodge cohomology coincide for this by (4.3.4), and the line bundle L can be extended to $\overline{X}_r \times \overline{S}$.

Now we reduce the assertion to the case

$$\xi_s \in \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}, H^{r+1}({}^oW^r\overline{X}_\bullet, \mathbb{Z})(1)).$$

Indeed, the image of ξ_s in $\mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Z}, H^{r+2}({}^oW^rX_\bullet, \mathbb{Z})(1))$ by (5.2.4) is constant, and we may assume it is zero by adding the pull-back of an element of $H^{r+2}_{\mathcal{D}}({}^oW^r\overline{X}_\bullet, \mathbb{Z}(1))$ by the projection $\overline{X}_\bullet \times S \to \overline{X}_\bullet$.

We then claim that $\{\xi_s\}_{s\in S(\mathbb{C})}$ is an admissible normal function in the sense of [29], i.e., it defines an extension between constant variations of mixed Hodge structures on S and the obtained extension is an admissible variation of mixed Hodge structure in the sense of Steenbrink-Zucker [32] and Kashiwara [20]. (Actually it is enough to show that $\{\xi_s\}$ is an analytic section for the proof of (5.4), because an analytic structure of a semiabelian variety is equivalent to an algebraic structure [10].)

Choosing a splitting of the exact sequence (4.3.2) for $K_{\mathcal{H}}({}^oW^r\overline{X}_\bullet \times S)$, we get a decomposition of ξ:

$$\xi' \in \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}, H^{r+1}({}^oW^r\overline{X}_\bullet \times S, \mathbb{Z})(1)),$$
$$\xi'' \in \mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Z}, H^{r+2}({}^oW^r\overline{X}_\bullet \times S, \mathbb{Z})(1)).$$

Then only the following Künneth components contribute to the restriction to the fiber at s:

$$\xi'^0 \in \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}, H^{r+1}({}^oW^r\overline{X}_\bullet, \mathbb{Z})(1) \otimes H^0(S, \mathbb{Z})),$$
$$\xi''^1 \in \mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Z}, H^{r+1}({}^oW^r\overline{X}_\bullet, \mathbb{Z})(1) \otimes H^1(S, \mathbb{Z})).$$

Clearly, ξ'^0 gives a constant section (where we may assume S connected), and the restriction of ξ''^1 is well-defined modulo constant section. We will show that the restrictions of ξ''^1 to the points of S form an admissible normal function.

The restriction is also defined by applying the functor $\Gamma_{\mathcal{H}}$ to the restriction morphism of mixed Hodge complexes

$$K_{\mathcal{H}}({}^oW^r\overline{X}_\bullet \times S)(1) \to K_{\mathcal{H}}({}^oW^r\overline{X}_\bullet)(1). \tag{5.3.2}$$

By (4.1.4) this corresponds to the tensor of $\oplus_i H^i({}^oW^r\overline{X}_\bullet, \mathbb{Z})[-i]$ with the restriction morphism under the inclusion $\{s\} \to S$:

$$\mathbf{R}\Gamma(S, \mathbb{Z})\,(\simeq \oplus_i H^i(S, \mathbb{Z})[-i]) \to \mathbb{Z} \quad \text{in } D^b\mathrm{MHS}. \tag{5.3.3}$$

Note that the restriction of the morphism to $H^i(S, \mathbb{Z})[-i]$ vanishes for $i > 1$, and choosing $s_0 \in S$, the restriction to $H^1(S, \mathbb{Z})[-1]$ for $s \neq s_0$ is expressed by the short exact sequence

$$0 \to \mathbb{Z} \to H^1(S, \{s_0, s\}; \mathbb{Z}) \to H^1(S, \{s_0\}; \mathbb{Z})\,(= H^1(S, \mathbb{Z})) \to 0,$$

using the corresponding distinguished triangle. Here $H^1(S,\mathbb{Z})[-1] \to \mathbb{Z}$ is zero for $s = s_0$.

The restriction of ξ'^0 and ξ''^1 is then obtained by tensoring (5.3.3) with $H := H^{r+1}({}^oW^r\overline{X}_\bullet,\mathbb{Z})(1)$, and applying the functor

$$\mathbf{R}\mathrm{Hom}_{D^b\mathrm{MHS}}(\mathbb{Z}[-1],*).$$

For ξ''^1, it is given by taking the pull-back of the above short exact sequence tensored with H by ξ''^1. Then we can construct the extended variation of mixed Hodge structure by using the diagonal of $S \times S$ as in [28], 3.8. This shows that $\{\xi_s\}$ determines an admissible normal function which will be denoted by ρ.

Let G be the semiabelian variety defined by $\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}, H^{r+1}({}^oW^r\overline{X}_\bullet,\mathbb{Z})(1))$ (see [8], [10]). Then ρ is a holomorphic section of $G^{\mathrm{an}} \times S^{\mathrm{an}} \to S^{\mathrm{an}}$. We have to show that this is algebraic using the property of admissible normal functions. Since G is semiabelian, there exist a torus T and an abelian variety A together with a short exact sequence

$$0 \to T \to G \to A \to 0.$$

As a variety, G may be viewed as a principal T-bundle. We choose an isomorphism

$$T = (\mathbb{G}_m)^n.$$

This gives compactifications $\overline{T} = (\mathbb{P}^1)^n$ of T, and also $\overline{G}$ of G.

Then, by GAGA, it is enough to show that the admissible normal function ρ is extended to a holomorphic section of $\overline{G}^{\mathrm{an}} \times \overline{S}^{\mathrm{an}} \to \overline{S}^{\mathrm{an}}$, where $\overline{S}$ is an appropriate smooth compactification of S such that $\overline{S} \setminus S$ is a divisor with normal crossings. Here we may assume $n = 1$ using the projections $\overline{T} \to \mathbb{P}^1$, because $\overline{G}$ is the fiber product of the $\mathbb{P}^1$-bundles over A.

So the assertion follows from the same argument as in [29], 4.4. Indeed, the group of connected components of the fiber of the Néron model of $G \times S$ at a generic point of $\overline{S} \setminus S$ is isomorphic to $\mathbb{Z}$ by an argument similar to (2.5.5) in loc. cit, and this corresponds to the order of zero or pole in an appropriate sense of a local section. By blowing up further, we may assume that these orders along any intersecting two of the irreducible components of the divisor have the same sign (including the case where one of them is zero). Then it can be extended to a section of $\overline{G}^{\mathrm{an}} \times \overline{S}^{\mathrm{an}} \to \overline{S}^{\mathrm{an}}$ as in the case of meromorphic functions (which corresponds to the case $A = 0$). This finishes the proof of (5.3).

Remark. It is easy to show that the map $S(\mathbb{C}) \to H^{r+2}_{\mathcal{D}}({}^oW^r\overline{X}_\bullet,\mathbb{Z}(1))$ is analytic using the long exact sequence associated with the direct image of the distinguished triangle $\to \mathbb{Z}(1) \to \mathcal{O} \to \mathcal{O}^* \to$ under the projection $\overline{X}_\bullet^{\mathrm{an}} \times S^{\mathrm{an}} \to S^{\mathrm{an}}$. This implies that the natural algebraic structure on $H^{r+2}_{\mathcal{D}}({}^oW^r\overline{X}_\bullet,\mathbb{Z}(1))$ is compatible with the one obtained by (3.2) and (5.2.2).

5.4. Theorem. *With the notation of (4.2), let W denote the decreasing weight filtration on the mixed Hodge complex $K := K_{\mathcal{H}^p}(\overline{X}_\bullet\langle D_\bullet\rangle)$ (i.e. $W^j = W_{-j}$). For a mixed Hodge structure H, let $H_{(1)}$ denote the maximal mixed Hodge substructure such that $\mathrm{Gr}_F^p = 0$ for $p \notin \{0,1\}$. Then we have natural isomorphisms of mixed Hodge structures*

$$r_{\mathcal{H}}(G_r(\overline{X}_\bullet\langle D_\bullet\rangle))(-1) = W_1H^r(W^{r-2}K)_{\mathrm{fr}}, \tag{5.4.1}$$

$$r_{\mathcal{H}}(\Gamma'_r(\overline{X}_\bullet\langle D_\bullet\rangle))(-1) = H^r(W^{r-2}K)_{(1)}/W_1H^r(W^{r-2}K), \tag{5.4.2}$$

$$r_{\mathcal{H}}(M'_r(\overline{X}_\bullet\langle D_\bullet\rangle))(-1) = H^r(W^{r-2}K)_{(1)}/W_1H^r(W^{r-2}K)_{\mathrm{tor}}, \tag{5.4.3}$$

and surjective morphisms of mixed Hodge structures

$$r_{\mathcal{H}}(\Gamma_r(\overline{X}_\bullet\langle D_\bullet\rangle))(-1) \to W_2H^r(W^{r-3}K)_{(1)}/W_1H^r(W^{r-2}K), \tag{5.4.4}$$

$$r_{\mathcal{H}}(M_r(\overline{X}_\bullet\langle D_\bullet\rangle))(-1) \to W_2H^r(W^{r-3}K)_{(1)}/W_1H^r(W^{r-3}K)_{\mathrm{tor}}, \tag{5.4.5}$$

whose kernels are torsion, and vanish if $H^2(\overline{X}_{r-3},\mathbb{Z})$ is of type (1,1).

Proof. By (4.4.2) and (5.3) we have

$$r_{\mathcal{H}}(\mathcal{P}_{\geq r-1}(\overline{X}_\bullet)^0)(-1) = H^r(W^{r-1}K)_{\mathrm{fr}}.$$

Since $r_{\mathcal{H}}(\mathcal{P}_{r-2}(\overline{X}_\bullet)^0)(-1) = H^1(\overline{X}_{r-2},\mathbb{Z})_{\mathrm{fr}} = H^{r-1}(\mathrm{Gr}_W^{r-2}K)_{\mathrm{fr}}$, we get

$$r_{\mathcal{H}}(G_r(\overline{X}_\bullet\langle D_\bullet\rangle))(-1) = \mathrm{Coker}(\partial : H^1(\overline{X}_{r-2},\mathbb{Z}) \to H^r(W^{r-1}K))_{\mathrm{fr}},$$

using the right exactness of $\mathrm{Ext}^1(\mathbb{Z},*)$. So (5.4.1) follows from the long exact sequence of mixed Hodge structures

$$\begin{aligned} &H^1(\overline{X}_{r-2},\mathbb{Z}) \to H^r(W^{r-1}K) \to H^r(W^{r-2}K) \\ &\to H^2(\overline{X}_{r-2},\mathbb{Z}) \oplus \Gamma(\widetilde{D}_{r-1},\mathbb{Z})(-1) \xrightarrow{\partial} H^{r+1}(W^{r-1}K). \end{aligned} \tag{5.4.6}$$

Here $H^{i+r}\mathrm{Gr}_W^{r-2}K = H^{i+2}(\overline{X}_{r-2},\mathbb{Z}) \oplus H^i(\widetilde{D}_{r-1},\mathbb{Z})(-1)$ for $i \leq 1$ by (4.4). Note that $W_iH^r(W^jK) = \mathrm{Im}(H^r(W^{r-i}K) \to H^r(W^jK)$ for $r-i \geq j$.

Similarly, we have by (5.2.2)

$$P_{\geq r-1}(\overline{X}_\bullet\langle D_\bullet\rangle)/P_{\geq r-1}(\overline{X}_\bullet)^0 = \mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Z}, H^{r+1}(W^{r-1}K)(1)).$$

Furthermore, the right vertical morphism of (3.1.4) is identified with the image by the functor $\mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Z}(-1),*)$ of the last morphism ∂ in (5.4.6). This is verified by using a canonical morphism of triangles in $\mathcal{D}_{\mathcal{H}}$

$$\begin{array}{ccccccccc} \longrightarrow & & W^{r-1}K & \longrightarrow & W^{r-2}K & \longrightarrow & \mathrm{Gr}_W^{r-2}K & \longrightarrow \\ & & \downarrow & & \downarrow & & \downarrow & \\ \longrightarrow & K_{\mathcal{H}}({}^oW^{r-1}X_\bullet) & \longrightarrow & K_{\mathcal{H}}({}^oW^{r-2}X_\bullet) & \longrightarrow & K_{\mathcal{H}}(\mathrm{Gr}_{{}^oW}^{r-2}X_\bullet) & \longrightarrow & \end{array}$$

where $\mathrm{Gr}_{{}^oW}^{r-2}X_\bullet$ is defined by the shifted mapping cone. Note that the left part of the diagram commutes without homotopy so that the morphism of the mapping cones is canonically defined, and the filtration $\mathrm{Dec}\,W$ on W^iK/W^jK coincides with the filtration induced by $\mathrm{Dec}\,W$ on K.

So (5.4.2) follows from (5.4.6), because the functor $\mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Z}(-1),*)$ is left exact, and its restriction to pure Hodge structures of weight 2 is identified with the functor $H \to H_{(1)}$. Note that a pure Hodge structure of type $(1,1)$ is identified with a $\mathbb{Z}$-module.

For (5.4.3), we consider the extension class in

$$\mathrm{Ext}^1_{\mathrm{MHS}}(\Gamma'_r(\overline{X}_\bullet\langle D_\bullet\rangle), W_1H^r(W^{r-2}K)_{\mathrm{fr}}(1)),$$

which is induced by (5.4.2) together with the natural exact sequence. Here $\Gamma'_r(\overline{X}_\bullet\langle D_\bullet\rangle)$ is identified with a mixed Hodge structure of type (0,0). In particular, the extension class is equivalent to the induced morphism

$$\mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Z}, \Gamma'_r(\overline{X}_\bullet\langle D_\bullet\rangle)) \to \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}, W_1H^r(W^{r-2}K)_{\mathrm{fr}}(1)). \tag{5.4.7}$$

So we have to show for the proof of (5.4.3) that this morphism is identified by (5.4.1) with

$$\Gamma'_r(\overline{X}_\bullet\langle D_\bullet\rangle) \to G_r(\overline{X}_\bullet\langle D_\bullet\rangle).$$

By (5.2) and (4.4.2), the commutative diagram (3.1.4) is identified with a morphism of the short exact sequences (4.3.2) induced by the morphism of absolute Hodge cohomologies

$$\partial : H^r\Gamma_{\mathcal{H}}(\mathrm{Gr}_W^{r-2}K(1)) \to H^{r+1}\Gamma_{\mathcal{H}}(W^{r-1}K(1)).$$

The last morphism ∂ is induced by the "boundary map" of the first distinguished triangle of the above diagram (which is defined by using the mapping cone). Then (5.4.3) follows from (4.1.3-4) by using (5.5) below, because $H^i\Gamma_{\mathcal{H}}$ is identified with $\mathrm{Ext}^i_{\mathrm{MHS}}(\mathbb{Z}, *)$ by the equivalence of categories (4.1.4) due to (4.1.3).

Finally, the surjectivity of (5.4.4) and (5.4.5) follows from the exact sequence

$$H^2(\overline{X}_{r-3}, \mathbb{Z}) \oplus H^0(\widetilde{D}_{r-2}, \mathbb{Z})(-1) \to H^r(W^{r-2}K) \xrightarrow{\partial} H^r(W^{r-3}K), \tag{5.4.8}$$

by comparing the (1,1) part of $\mathrm{Coker}\,\mathrm{Gr}_2^W\partial$ with the cokernel of the (1,1) part of $\mathrm{Gr}_2^W\partial$ (where $\mathrm{Gr}_2^W\mathrm{Coker}\,\partial = \mathrm{Coker}\,\mathrm{Gr}_2^W\partial$ because $H^r(W^{r-2}K) = W_2H^r(W^{r-2}K)$). The kernels of (5.4.4) and (5.4.5) come from the difference between these, and vanish if $H^2(\overline{X}_{r-3}, \mathbb{Z})$ is of type (1,1). Note that the intersection of $W_1H^r(W^{r-2}K)$ with

$$\mathrm{Im}(H^2(\overline{X}_{r-3}, \mathbb{Z}) \oplus H^0(\widetilde{D}_{r-2}, \mathbb{Z})(-1) \to H^r(W^{r-2}K))$$

is torsion by the strict compatibility of the weight filtration. This completes the proof of (5.4).

5.5. Remark. Let $\mathcal{A}$ be an abelian category such that $\mathrm{Ext}^i(A, B) = 0$ for any objects A, B and $i > 1$. Let $A_0 \in \mathcal{A}$, and $F(K) = \mathbf{R}\mathrm{Hom}(A_0, K)$ for $K \in D^b\mathcal{A}$ so that we have a canonical short exact sequence

$$0 \to \mathrm{Ext}^1(A_0, H^{-1}K) \to F(K) \to \mathrm{Hom}(A_0, H^0K) \to 0.$$

Let $\to K' \to K \to K'' \to$ be a distinguished triangle in $D^b\mathcal{A}$ with $\partial : K'' \to K'[1]$ the boundary map. Consider the morphism

$$\mathrm{Hom}(A_0, \mathrm{Ker}\,H^0\partial) \to \mathrm{Ext}^1(A_0, \mathrm{Coker}\,H^{-1}\partial), \tag{5.5.1}$$

obtained by the snake lemma together with the right exactness of $\mathrm{Ext}^1(A_0, *)$. (Here $H^i\partial$ is the abbreviation of $H^iK'' \to H^{i+1}K'$.) Then (5.5.1) coincides with the morphism induced by the short exact sequence

$$0 \to \mathrm{Coker}\, H^{-1}\partial \to H^0K \to \mathrm{Ker}\, H^0\partial \to 0. \tag{5.5.2}$$

Indeed, K', K'' are represented by complexes with *zero* differential, and the assertion is reduced to the case where $K'' = A, K' = B$ with $A, B \in \mathcal{A}$ (considering certain subquotients of K', K''). Then it follows from the well-known bijection between the extension group in the derived category and the set of extension classes in the usual sense.

5.6. Proof of (0.1–3). We take a resolution of $[Y \to X]$ so that the associated integral weight filtration is defined independently of the choice of the resolution as in the proof of (2.3) (e.g. we can take the simplicial resolution of Gillet and Soulé [14], 3.1.2 if X is proper). Then by (5.4), it is enough to show that the kernel of the canonical morphism

$$W_2H^r(W^{r-3}K) \to H^r(K)$$

is torsion, and it is contained in $W_1H^r(W^{r-3}K)_{\mathrm{tor}}$ if $E_2^{p,r-1-p} = 0$ for $p \le r-4$. But these can be verified by using a natural morphism between the weight spectral sequences (2.3.2) converging to $H^\bullet(W^{r-3}K)$ and $H^\bullet(K)$, because the spectral sequences degenerate at E_2 modulo torsion. Note that the weight filtration on the cohomology is shifted by the degree, and W_2 is induced by W^{r-2}. This completes the proof of (0.1–3).

References

[1] Barbieri-Viale, L., On algebraic 1-motives related to Hodge cycles, preprint.

[2] Barbieri-Viale, L. and Srinivas, V., Albanese and Picard 1-motives, preprint (math.AG/9906165), to appear in Mémoire Soc. Math. France.

[3] Beilinson, A., Higher regulators and values of L-functions, J. Soviet Math. 30 (1985), 2036–2070.

[4] ———, Notes on absolute Hodge cohomology, Contemporary Math. 55 (1986) 35–68.

[5] Beilinson, A., Bernstein, J. and Deligne, P., Faisceaux pervers, Astérisque, vol. 100, Soc. Math. France, Paris, 1982.

[6] Biswas, J. and Srinivas, V., A Lefschetz (1,1)-Theorem for normal projective varieties, Duke Math. J. 101 (2000), 427–458.

[7] Bloch, S., Algebraic cycles and higher K-theory, Advances in Math., 61 (1986), 267–304.

[8] Carlson, J., Extensions of mixed Hodge structures, in Journées de Géométrie Algébrique d'Angers 1979, Sijthoff-Noordhoff Alphen a/d Rijn, 1980, pp. 107–128.

[9] ———, The one-motif of an algebraic surface, Compos. Math. 56 (1985), 271–314.

[10] Deligne, P., Théorie de Hodge I, Actes Congrès Intern. Math., 1970, vol. 1, 425-430; II, Publ. Math. IHES, 40 (1971), 5–57; III ibid., 44 (1974), 5–77.

[11] El Zein, F. and Zucker, S., Extendability of normal functions associated to algebraic cycles, in Topics in transcendental algebraic geometry, Ann. Math. Stud., 106, Princeton Univ. Press, Princeton, N.J., 1984, pp. 269–288.

[12] Esnault, H. and Viehweg, E., Deligne-Beilinson cohomology, in Beilinson's conjectures on Special Values of L-functions, Academic Press, Boston, 1988, pp. 43–92.
[13] Gillet, H., Deligne homology and Abel-Jacobi maps, Bull. Amer. Math. Soc. 10 (1984), 285–288.
[14] Gillet, H. and Soulé, C., Descent, motives and K-theory, J. Reine Angew. Math. 478 (1996), 127–176.
[15] Grothendieck, A., Fondements de la Géométrie Algébrique, Benjamin, New York, 1966.
[16] Guillén, F. and Navarro Aznar, V., Un critère d'extension d'un foncteur défini sur les schémas lisses (preprint).
[17] Guillén, F., Navarro Aznar, V., Pascual-Gainza, P. and Puerta, F., Hyperrésolutions cubiques et descente cohomologique, Lect. Notes in Math., Springer, Berlin, vol. 1335, 1988.
[18] Hironaka, H., Flattening theorem in complex-analytic geometry, Amer. J. Math. 97 (1975), 503–547.
[19] Jannsen, U., Deligne homology, Hodge-D-conjecture, and motives, in Beilinson's conjectures on Special Values of L-functions, Academic Press, Boston, 1988, pp. 305–372.
[20] Kashiwara, M., A study of variation of mixed Hodge structure, Publ. RIMS, Kyoto Univ., 22 (1986), 991–1024.
[21] Mazur, B. and Messing W., Universal extensions and one dimensional crystalline cohomology, Lect. Notes in Math. vol. 370, Springer, Berlin, 1974.
[22] Mumford, D., Abelian variety, Oxford University Press, 1970
[23] Murre, J., On contravariant functors from the category of preschemes over a field into the category of abelian groups, Publ. Math. IHES 23 (1964), 5 – 43.
[24] Oort, F., Commutative group schemes, Lect. Notes in Math., vol. 15, Springer, Berlin, 1966.
[25] Ramachandran, N., Albanese and Picard motives of schemes, preprint (math.AG/9804042).
[26] ———, A conjecture of Deligne on one-motives, preprint (math.AG/9806117).
[27] Saito, M., Modules de Hodge polarisables, Publ. RIMS, Kyoto Univ., 24 (1988), 849–995.
[28] ———, Extension of mixed Hodge Modules, Compos. Math. 74 (1990), 209–234.
[29] ———, Admissible normal functions, J. Alg. Geom. 5 (1996), 235–276.
[30] ———, Mixed Hodge complexes on algebraic varieties, Math. Ann. 316 (2000), 283 –331.
[31] ———, Bloch's conjecture, Deligne cohomology and higher Chow groups, preprint (math.AG/9910113).
[32] Steenbrink, J. and Zucker, S., Variation of mixed Hodge structure I, Inv. Math., 80 (1985), 489–542.
[33] Verdier, J.-L., Catégories dérivées, in SGA 4 1/2, Lect. Notes in Math., Springer, Berlin, vol. 569, 1977, pp. 262–311.

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