CYCLE MAP FOR STRICTLY DECOMPOSABLE CYCLES

Andreas Rosenschon and Morihiko Saito

Abstract. We introduce a class of cycles, called nondegenerate, strictly decomposable cycles, and show that the image of each cycle in this class under the refined cycle map to an extension group in the derived category of arithmetic mixed Hodge structures does not vanish. This class contains certain cycles in the kernel of the Abel-Jacobi map. The construction may be viewed as a generalization of Nori's construction in the case of a self-product of a curve. As an application, we show that a higher cycle which is not annihilated by the reduced higher Abel-Jacobi map produces uncountablly many indecomposable higher cycles on the product with a variety having a nonzero global 1-form. This generalizes a result of Gordon and Lewis.

Introduction

Let X be a smooth projective variety over a field, and let $\mathrm{CH}^p(X)_{\mathbb{Q}}$ be the Chow group of codimension p cycles with rational coefficients modulo rational equivalence. The conjectures of Beilinson [3] and Bloch [5] predict that there is a decreasing filtration F on the Chow groups whose lower terms are described as follows (see also [20]): the first term F^1 consists of cycles homologically equivalent to zero; and for 0-cycles (i.e. if $p = \dim X$), the second term F^2 is the kernel of the albanese map. This filtration is closely related to a conjectural category of mixed motives. In fact, the cycle map of $\mathrm{CH}^p(X)_{\mathbb{Q}}$ to an analogue of Deligne cohomology for mixed motives should be bijective, and the above filtration would be induced by the filtration on the target which is associated with a Leray spectral sequence (see [3], [20]).

If the base field is $\mathbb{C}$, one approach toward these conjectures is based on the theory of mixed Hodge Modules ([24], [25], [26]). For a subfield k of $\mathbb{C}$, we have the category of arithmetic mixed Hodge structures (or more generally, of arithmetic realizations) $\mathcal{M}_{\langle k\rangle}$, see [28], [29] and also [1]. This is defined as the inductive limit of the categories of mixed Hodge Modules (or of systems of realizations) on $\operatorname{Spec} R$ with R running over the finitely generated smooth k-subalgebras of $\mathbb{C}$. Since any complex algebraic variety X has a model X_R over $\operatorname{Spec} R$ for some R as above, we can define the cohomology of X in this category. This construction was originally considered in order to define an analogue of "spreading out" of algebraic cycles [5] for mixed Hodge Modules (see [26], 1.9), and was greatly inspired by earlier work of M. Green [17], C. Voisin [31], [32] and others.

By the general theory of mixed sheaves [26], a generalization of Deligne cohomology $H^i_{\mathcal{D}}(X, \mathbb{Q}_{\langle k\rangle}(j))$ is defined as a certain extension group in the derived category of $\mathcal{M}_{\langle k\rangle}$,

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and is equal to the inductive limit of the extension group in the derived category of mixed Hodge Modules on X_R, see (1.2). We can naturally construct a refined cycle map

$$cl : \mathrm{CH}^p(X)_{\mathbb{Q}} \to H^{2p}_{\mathcal{D}}(X, \mathbb{Q}_{\langle k\rangle}(p)), \tag{0.1}$$

as the inductive limit of the cycle map on X_R. On the target, there is a decreasing filtration F_L associated with a Leray spectral sequence for the structure morphism $X_R \to \operatorname{Spec} R$. By (0.1) this induces a filtration F_L on $\mathrm{CH}^p(X)_{\mathbb{Q}}$ which is expected to coincide with the above conjectural filtration. (This filtration seems to coincide with a new filtration of M. Green explained in his talk at Azumino, July 2000.)

The problem here is the injectivity of the refined cycle map (0.1). For codimension 2 cycles, it is reduced to the injectivity of the Abel-Jacobi map for k-smooth projective models of X (see [28]), and this injectivity is conjectured in [2] when k is a number field. However, it is not easy to verify it for any single example (except for the case where the model does not have a nontrivial two form [5]). Therefore, we are interested to extend the class of cycles which are detected by the refined cycle map (0.1).

For simplicity, let X_1, X_2 be elliptic curves. Assume they are defined over a subfield k of $\mathbb{C}$, i.e. there exist smooth projective k-varieties $X_{i,k}$ such that $X_i = X_{i,k} \otimes_k \mathbb{C}$. Set $X = X_1 \times X_2, X_k = X_{1,k} \times_k X_{2,k}$. Assume the origin O_i of $X_{i,k}$ is also defined over k. For $P_i \in X_i(\mathbb{C})$, let $\zeta_i = [P_i] - [O_i] \in \mathrm{CH}^1(X_i)_{\mathbb{Q}}$. We assume that P_i is not a torsion point. Consider $\zeta := \zeta_1 \times \zeta_2 \in \mathrm{CH}^2(X)_{\mathbb{Q}}$. This belongs to the kernel of the Albanese map. It may be zero, for example, when X_1 and X_2 are the same elliptic curve, and $P_1 = P_2$, see (2.3.1). So we are interested in the following question: When is ζ not rationally equivalent to zero (up to torsion)? This seems nontrivial even in this simple case.

Let R be a finitely generated smooth k-subalgebra of $\mathbb{C}$ such that ζ_i is defined in $X_{i,k} \otimes_k R$. Set $S = \operatorname{Spec} R \otimes_k \mathbb{C}$. We assume that k is algebraically closed in R by replacing k if necessary. We denote by ξ_i^1 the Künneth component of the cohomology class of $\zeta_i \otimes_k \mathbb{C}$ in $H^1(X_i, \mathbb{Q}) \otimes H^1(S)$. Note that $\zeta_i \otimes_k \mathbb{C} \in \mathrm{CH}^1(X_i \times S)_{\mathbb{Q}}$ is constant (i.e. it is the pull-back of a cycle on X_i) if and only if $\xi_i^1 = 0$, see (2.2).

If $\xi_i^1 = 0$ for $i = 1, 2$ and k is a number field, then we may assume $R = k$, and it is conjectured by Beilinson [2] and Bloch that ζ would be rationally equivalent to zero up to torsion. If $\xi_1^1 \neq 0$ and $\xi_2^1 = 0$, then we can prove that $\zeta \neq 0$ in $\mathrm{CH}^2(X)_{\mathbb{Q}}$ by Theorem (0.2) below. This case is closely related to Nori's construction of cycles on a self-product of a curve of positive genus [30]. Assume now ξ_1^1 and ξ_2^1 are both nonzero. Then ζ can be rationally equivalent to zero again as in the case of a self-product explained above. However, if $R = R_1 \otimes_k R_2$ such that ζ_i is defined over R_i, then we can show the nonvanishing of ζ using the notion of strictly decomposable cycles as follows.

For smooth complex projective varieties X_1, X_2 in general, let X, $X_{i,k}$ and X_k be as above. Then a *strictly decomposable* cycle on X is represented by a product of cycles ζ_i on $X_{i,k} \otimes_k R_i\,(i = 1, 2)$, using the isomorphism

$$(X_{1,k} \otimes_k R_1) \times_k (X_{2,k} \otimes_k R_2) = X_k \otimes_k (R_1 \otimes_k R_2),$$

where the R_i are finitely generated smooth k-subalgebra of $\mathbb{C}$ such that k is algebraically closed in R_i and the canonical morphism $R_1 \otimes_k R_2 \to \mathbb{C}$ is injective (replacing k with a

finite extension k', and $X_{i,k}, X_k$ with their base changes by $k \to k'$). Let $p_i = \operatorname{codim} \zeta_i$. We denote by $\xi_i^0 \in H^{2p_i}(X_i, \mathbb{Q})(p_i)$ the usual cycle class of $\zeta_i \otimes_{R_i} \mathbb{C} \in \mathrm{CH}^{p_i}(X_i)_{\mathbb{Q}}$.

We have the notion of a *nondegenerate*, strictly decomposable cycle using the cycle map (0.1), see (2.4). For example, if $\xi_i^0 = 0$ for $i = 1, 2$ and $R_2 = k$, then ζ is nondegenerate if and only if the cohomology class ξ_1^1 of $\zeta_1 \otimes_k \mathbb{C}$ in $H^{2p_1-1}(X_1, \mathbb{Q}) \otimes H^1(S_1, \mathbb{Q})(p_1)$ is nonzero and the image of ζ_2 by the Abel-Jacobi map is nonzero. It is also nondegenerate when $\xi_i^0 = 0$ and $\xi_i^1 \neq 0$ for $i = 1, 2$. If $p_i = 1$ and k is a number field, it follows from the conjecture of Beilinson [2] and Bloch that a degenerate cycle would be rationally equivalent to zero. If ζ is nondegenerate, we can detect it by the refined cycle map (0.1) as follows.

0.2. Theorem. *Let ζ be a nondegenerate, strictly decomposable cycle of codimension p. Then $cl(\zeta) \neq 0$. More precisely, if we let $r = \#\{i : \xi_i^0 = 0\}$, then $\zeta \in F_L^r \mathrm{CH}^p(X)_{\mathbb{Q}}$ and $\mathrm{Gr}_{F_L}^r cl(\zeta) \neq 0$.*

The proof is reduced to a simple lemma on Hodge structures, see (2.7). This theorem is extended to the higher cycles case as in (3.3). Theorem(0.2) was considered in the case when both ξ_1^1 and ξ_2^1 are nonzero in order to show the nonvanishing of the composition of some extension classes (see [27]). In the study of Nori's construction [30], a special case of (0.2) was obtained for a self-product of an elliptic curve without complex multiplication [23]. Then these were generalized to (0.2). For higher cycles, we can prove

0.3. Theorem. *If $\Gamma(X_1, \Omega_{X_1}^1) \neq 0$ and the reduced higher Abel-Jacobi map (see (4.1.2)) for X_2 is not zero, then $\mathrm{CH}_{\mathrm{ind}}^{p+1}(X_1 \times X_2, 1)_{\mathbb{Q}}$ is uncountable.*

More precisely, let $\zeta_2 \in \mathrm{CH}^p(X_2, 1)$ whose image by (4.1.2) does not vanish, and let P denote the Picard variety of X_1. Assume X_i, ζ_2 and P are defined over k. Then for any $\zeta_1 \in P(\mathbb{C}) \setminus P(\overline{k})$, the image of $\zeta_1 \times \zeta_2$ in $\mathrm{CH}_{\mathrm{ind}}^{p+1}(X_1 \times X_2, 1)_{\mathbb{Q}}$ is nonzero. This gives a generalization of a result of Gordon and Lewis [16] in the case of a product of elliptic curves (see also [9]).

In Sect. 1, we recall the notion of arithmetic mixed Hodge structures and modules. In Sect. 2, we prove Theorem (0.3) by reducing it to Lemma (2.7). Sect. 3 treats the case of higher cycles, and the application to indecomposable higher cycles is given in Sect. 4.

In this paper, a variety means a separated scheme of finite type over a field. For a complex variety, cohomology with constant coefficients means that of the associated analytic space.

1. Arithmetic mixed Hodge structures

1.1. Let X be a complex algebraic variety, and k be a subfield of $\mathbb{C}$. Then there exists a finitely generated k-subalgebra R of $\mathbb{C}$ such that X is defined over R, i.e., there is an R-scheme X_R of finite type such that $X = X_R \otimes_R \mathbb{C}$. For a finitely generated smooth k-subalgebra R' of $\mathbb{C}$ containing R, let $X_{R'} = X_R \otimes_R R'$, $d_{R'} = \dim_k \operatorname{Spec} R'$, and let $k_{R'}$ be the algebraic closure of k in R'.

Let $\mathcal{M}(X_{R'}/k_{R'})$ be the category of mixed sheaves satisfying the axioms in [26]. For simplicity, we assume that $\mathcal{M}(X_{R'}/k_{R'})$ is either $\mathrm{MHM}(X_{R'}/k_{R'})$, the category of mixed Hodge Modules whose underlying filtered $\mathcal{D}$-Modules (and polarizations on graded pieces)

are defined over $k_{R'}$. (We can also consider $\mathcal{M}_{\mathrm{SR}}(X_{R'}/k_{R'})$, the category of systems of realizations, see [26], [28], [29]. This is a natural generalization of [11], [12], [19], and the l-adic part is given by an l-adic perverse sheaf [4].)

We define the category of arithmetic mixed sheaves to be the inductive limit of the categories of mixed sheaves on models of X, i.e.,

$$\mathcal{M}(X)_{\langle k\rangle} = \varinjlim \mathcal{M}(X_{R'}/k_{R'})[-d_{R'}], \tag{1.1.1}$$

where the inductive limit is taken over R' as above, and $\mathcal{M}(X_{R'}/k_{R'})[-d_{R'}]$ denotes $\mathcal{M}(X_{R'}/k_{R'})$ shifted by $-d_{R'}$ in the derived category. This shift is necessary, because a perverse sheaf is stable by smooth pull-back up to a shift of complexes. Since the inclusion $R' \to \mathbb{C}$ determines a geometric generic point of $\operatorname{Spec} R'$, we get a canonical functor

$$\mathcal{M}(X_{R'}/k_{R'})[-d_{R'}] \to \mathrm{MHM}(X) \tag{1.1.2}$$

by restricting to the geometric generic fiber X of $X_{R'}$. This gives a canonical functor

$$\iota : \mathcal{M}(X)_{\langle k\rangle} \to \mathrm{MHM}(X). \tag{1.1.3}$$

This is exact and faithful. We can construct canonically the standard functors

$$f_*, f_!, f^*, f^!, \mathbb{D}, \otimes, \mathcal{H}om, \text{etc.} \tag{1.1.4}$$

between the bounded derived categories $D^b\mathcal{M}(X)_{\langle k\rangle}$ in a compatible way with the functors (1.1.2-3), see loc. cit.

1.2. Cycle maps. For $X = \operatorname{Spec}\mathbb{C}$, we define the category of arithmetic mixed Hodge structures by

$$\mathcal{M}_{\langle k\rangle} = \mathcal{M}(\operatorname{Spec}\mathbb{C})_{\langle k\rangle}.$$

An object of this category is represented by an admissible variation of mixed Hodge structure on $\operatorname{Spec} R \otimes_{k_R} \mathbb{C}$ whose underlying $\mathcal{O}$-Module together with the connection, the Hodge and weight filtrations and the polarization on the graded pieces of the weight filtration is defined over $\operatorname{Spec} R$. There is a constant object $\mathbb{Q}_{\langle k\rangle}$ which is represented by a constant variation of Hodge structure of type $(0,0)$ on $\operatorname{Spec} R \otimes_{k_R} \mathbb{C}$. By (1.1.3) we have a canonical functor

$$\iota : \mathcal{M}_{\langle k\rangle} \to \mathrm{MHS},$$

where MHS denote the category of graded-polarizable mixed $\mathbb{Q}$-Hodge structures [10]. We define the cohomology $H^i(X, \mathbb{Q}_{\langle k\rangle})$ of a complex algebraic variety X in $\mathcal{M}_{\langle k\rangle}$ by taking a model of X over $\operatorname{Spec} R$, and applying (1.1.4) to the structure morphism $a_X : X \to \operatorname{Spec}\mathbb{C}$. An analogue of Deligne cohomology is defined as an extension group in the derived category $D^b\mathcal{M}_{\langle k\rangle}$:

$$H^i_{\mathcal{D}}(X, \mathbb{Q}_{\langle k\rangle}(j)) = \operatorname{Ext}^i(\mathbb{Q}_{\langle k\rangle}, (a_X)_* a_X^* \mathbb{Q}_{\langle k\rangle}(j)).$$

If X is a smooth proper complex algebraic variety, we have a refined cycle map

$$cl : \mathrm{CH}^p(X)_{\mathbb{Q}} \to H^{2p}_{\mathcal{D}}(X, \mathbb{Q}_{\langle k\rangle}(p)), \tag{1.2.1}$$

which is defined as the inductive limit of the cycle map ([28], [29])

$$cl : \mathrm{CH}^p(X_{R'})_{\mathbb{Q}} \to H^{2p}_{\mathcal{D}}(X_{R'}/k_{R'}, \mathbb{Q}_{k_{R'}}(p)). \tag{1.2.2}$$

Here the target is $\mathrm{Ext}^{2p}(\mathbb{Q}_{k_{R'}}, (a_{X_{R'}/k_{R'}})_* a^*_{X_{R'}/k_{R'}} \mathbb{Q}_{k_{R'}}(p))$ with $a_{X_{R'}/k_{R'}} : X_{R'} \to k_{R'}$ the structure morphism. The target of (1.2.1) has a filtration F_L associated with a Leray spectral sequence, which degenerates at E_2. This means

$$\mathrm{Gr}^i_{F_L} H^{2p}_{\mathcal{D}}(X, \mathbb{Q}_{\langle k\rangle}(p)) = \mathrm{Ext}^i_{M_{\langle k\rangle}}(\mathbb{Q}_{\langle k\rangle}, H^{2p-i}(X, \mathbb{Q}_{\langle k\rangle}(p)). \tag{1.2.3}$$

By (1.2.1), we get an induced filtration F_L on $\mathrm{CH}^p(X)_{\mathbb{Q}}$ such that

$$\mathrm{Gr}^i_{F_L} cl : \mathrm{Gr}^i_{F_L} \mathrm{CH}^p(X)_{\mathbb{Q}} \to \mathrm{Ext}^i_{M_{\langle k\rangle}}(\mathbb{Q}_{\langle k\rangle}, H^{2p-i}(X, \mathbb{Q}_{\langle k\rangle}(p)) \tag{1.2.4}$$

is injective.

1.3. Remarks. (i) Let K be a subfield of $\mathbb{C}$ containing k, and X a K-variety. Then we can define $\mathcal{M}(X/K)_{\langle k\rangle}$ together with a cycle map by taking the inductive limit over R' contained in K.

(ii) The refined cycle map (1.2.1) is compatible with the usual cycle map

$$\mathrm{CH}^p(X)_{\mathbb{Q}} \to H^{2p}_{\mathcal{D}}(X, \mathbb{Q}(p)), \tag{1.3.1}$$

by the functor (1.1.3), where the target is Deligne cohomology in the usual sense (see [2], [14], [15]). In particular, it induces Griffiths' Abel-Jacobi map [18] by using the Hodge-theoretic description of the intermediate Jacobian [8].

As a corollary of the compatibility of the cycle maps, $F^1_L \mathrm{CH}^p(X)_{\mathbb{Q}}$ coincides with the subgroup $\mathrm{CH}^p_{\mathrm{hom}}(X)_{\mathbb{Q}}$ of homologically equivalent to zero cycles, and $F^2_L \mathrm{CH}^p(X)_{\mathbb{Q}}$ is contained in the kernel of the Abel-Jacobi map $\mathrm{CH}^p_{\mathrm{AJ}}(X)_{\mathbb{Q}}$. Here we have the equality if $p = \dim X$. This follows from Murre's result on Albanese motives [22] together with the compatibility of the action of a correspondence (see e.g. [28], 3.6). Restricting to the subgroup $\mathrm{CH}^p_{\mathrm{alg}}(X)_{\mathbb{Q}}$ of algebraically equivalent to zero cycles, we have more generally the equality $F^2_L \mathrm{CH}^p_{\mathrm{alg}}(X)_{\mathbb{Q}} = \mathrm{CH}^p_{\mathrm{AJ}}(X)_{\mathbb{Q}} \cap \mathrm{CH}^p_{\mathrm{alg}}(X)_{\mathbb{Q}}$ for any p, see [29], 3.9.

We can show $\mathrm{Gr}^r_{F_L} \mathrm{CH}^p(X)_{\mathbb{Q}} = 0$ for $r > p$. It is expected that the filtration F_L coincides with a conjectural filtration of Beilinson [3] and Bloch [5].

2. Strictly decomposable cycles

2.1. Let X_1 and X_2 be smooth complex projective varieties defined over a subfield k of $\mathbb{C}$, i.e. there are smooth projective k-varieties $X_{i,k}$ such that $X_i = X_{i,k} \otimes_k \mathbb{C}$. Put $X_k = X_{1,k} \times_k X_{2,k}$, $X = X_k \otimes_k \mathbb{C}$ as in the introduction. We say that a cycle ζ on X is strictly decomposable if there exist subfields K_i of $\mathbb{C}$ finitely generated over k, together with cycles ζ_i on $X_{i,K_i} := X_i \otimes_k K_i$ for $i = 1, 2$ such that k is algebraically closed in K_i,

the canonical morphism $K_1 \otimes_k K_2 \to \mathbb{C}$ is injective, and ζ coincides with the base change of the cycle $\zeta_1 \times_k \zeta_2$ on

$$X_{1,K_1} \times_k X_{2,K_2} = X_k \otimes_k (K_1 \otimes_k K_2)$$

by $K_1 \otimes_k K_2 \to \mathbb{C}$ (replacing k with a finite extension if necessary). We say that a strictly decomposable cycle is of bicodimension (p_1, p_2) if $\operatorname{codim} \zeta_i = p_i$. Put $p = p_1 + p_2$.

Let R_i be a finitely generated smooth k-subalgebra of K_i such that the fraction field of R_i is K_i, and ζ_i is defined over R_i. Set $S_i = \operatorname{Spec} R_i \otimes_k \mathbb{C}$. We denote by $\zeta_i \otimes_k \mathbb{C}$ the base change of ζ_i by $k \to \mathbb{C}$. Note that $\zeta_i \otimes_k \mathbb{C} \in \mathrm{CH}^{p_i}(X_i \times S_i)$, and its restriction to $X_i \times \{\overline{\eta}_i\}$ coincides with $\zeta_i \otimes_{R_i} \mathbb{C}$, where $\overline{\eta}_i$ is defined by the inclusion $R_i \to \mathbb{C}$. Let

$$\xi_i^j \in \operatorname{Hom}_{\mathrm{MHS}}(\mathbb{Q}, H^{2p_i-j}(X_i, \mathbb{Q}) \otimes H^j(S_i, \mathbb{Q})(p_i))$$

denote the Künneth components of the cohomology class of $\zeta_i \otimes_k \mathbb{C}$ in $H^{2p_i}(X_i \times S_i, \mathbb{Q})(p_i)$. Define

$$M_i = H^{2p_i-1}(X_i, \mathbb{Q})(p_i),$$

and let M_{i,S_i} be its pull-back by $a_{S_i} : S_i \to \operatorname{Spec} \mathbb{C}$. If $\xi_i^0 = 0$, then ζ_i belongs to $F_L^1 \mathrm{CH}^{p_i}(X_{i,k} \otimes_k K_i)_{\mathbb{Q}}$, and $\mathrm{Gr}^1_{F_L} cl(\zeta_i)$ (see (1.2.2)) gives

$$\widetilde{\xi}_i^1 \in \operatorname{Ext}^1(\mathbb{Q}_{S_i}, M_{i,S_i}).$$

2.2. Proposition. *With the above notation, $\xi_i^1 = 0$ if and only if $\widetilde{\xi}_i^1$ comes from $\xi_i' \in \operatorname{Ext}^1(\mathbb{Q}, M_i)$ by the pull-back under a_S. In the case $p_i = 1$, these conditions are further equivalent to that ζ_i comes from $\mathrm{CH}^1(X_{i,k})_{\mathbb{Q}}$.*

Proof. By the Leray spectral sequence, we have an exact sequence

$$0 \to \operatorname{Ext}^1(\mathbb{Q}, M_i) \to \operatorname{Ext}^1(\mathbb{Q}_{S_i}, M_{i,S_i}) \to \operatorname{Hom}(\mathbb{Q}, H^1(S_i, \mathbb{Q}) \otimes M_i), \tag{2.2.1}$$

where the first morphism is induced by the pull-back functor for a_{S_i}. Since ξ_i^1 coincides with the image of $\widetilde{\xi}_i^1$ in the last term, we get the first equivalence. Then the second is clear because the cycle map to Deligne cohomology is an isomorphism in the divisor case and the base change by $k \to \mathbb{C}$ induces an injective morphism of Chow groups with rational coefficients.

2.5. Remarks. (i) By (2.2) the cohomology class ξ_i^1 measures how the cycle $\zeta_i \otimes_k \mathbb{C}$ varies along the parameter space S_i. If the first equivalent conditions of (2.2) are satisfied, we may assume $R_i = k$ as long as $\mathrm{Gr}^1_{F_L} cl(\zeta_i)$ is concerned, e.g. if $p_i = 1$. (Indeed, we can restrict to a k-valued point of $\operatorname{Spec} R_i$ by replacing k if necessary.)

(ii) Let E be an elliptic curve over $\mathbb{C}$ with the origin O. Then for any $P \in E(\mathbb{C})$

$$([P] - [O]) \times ([P] - [O]) = 0 \quad \text{in } \mathrm{CH}^2(E \times E). \tag{2.3.1}$$

This can be verified by taking Q such that $2Q = P$, and using the diagonal and antidiagonal curves passing (Q, Q) (which is viewed as a new origin).

2.4. Definition. We say that ζ_i is *degenerate* if $\xi_i^0 = \widetilde{\xi}_i^1 = 0$. A strictly decomposable cycle ζ is called *separately degenerate* if one of ζ_i is degenerate, and *cohomologically degenerate* if $\xi_i^0 = \xi_i^1 = 0$ for both i, and *degenerate* if it is separately degenerate or cohomologically degenerate. In other words, ζ is *nondegenerate* if one of ξ_i^0 and $\widetilde{\xi}_i^1$ does not vanish for each i and one of ξ_1^0, ξ_1^1, ξ_2^0, ξ_2^1 does not vanish.

2.5. Remark. Assume $p_i = 1$. Then ζ_i is degenerate if and only if it is zero in the Chow group with rational coefficients. If ζ is cohomologically degenerate, it comes from a cycle on X_k by the pull-back under the projection to X_k, see (2.2). If furthermore k is a number field, it is expected that ζ is rationally equivalent to zero by a conjecture of Beilinson [2] and Bloch. If ζ is nondegenerate, we can detect it by the refined cycle map due to Theorem (0.2).

2.6. Proof of Theorem (0.2). Let $\pi_i : Y_i := X_i \times S_i \to S_i$ and $\pi : Y := Y_1 \times Y_2 \to S := S_1 \times S_2$ denote the canonical projections. Then, by the cycle map, $\zeta_i \otimes_k \mathbb{C}$ and $\zeta \otimes_k \mathbb{C}$ induce morphisms in the derived category of mixed sheaves

$$\xi_i : \mathbb{Q}_{S_i} \to \widetilde{M}_i := (\pi_i)_* \mathbb{Q}_{Y_i}(p_i)[2p_i],$$
$$\xi : \mathbb{Q}_S \to \widetilde{M} := (\pi)_* \mathbb{Q}_Y(p)[2p],$$

and ξ is the external product of ξ_1 and ξ_2. If $\xi_i^0 = 0$, ξ_i factors through $\tau_{<0}\widetilde{M}_i$, where τ is associated with a t-structure which is shifted as in (1.1.1). So we get $\zeta \in F_L^r \mathrm{CH}^p(X)_{\mathbb{Q}}$ for r as above. Then the assertion is clear if one of ξ_i^0 does not vanish. Indeed, it is reduced to the case $R_i = k$ (replacing k with a finite extension if necessary), and follows from the nonvanishing of $\xi_{i'}^0$ or $\widetilde{\xi}_{i'}^1$ for $i' = 3 - i$. Thus we may assume $\xi_1^0 = \xi_2^0 = 0$.

Let $(M_1 \otimes M_2)_S$ be the pull-back of $M_1 \otimes M_2$ by $S \to \operatorname{Spec} \mathbb{C}$. Since it is isomorphic to the external product $\mathrm{pr}_1^* M_{1,S_1} \otimes \mathrm{pr}_2^* M_{2,S_2}$ of M_{1,S_1} and M_{2,S_2}, we see that

$$\widetilde{\xi}_1^1 \otimes \widetilde{\xi}_2^1 \in \mathrm{Ext}^2(\mathbb{Q}_S, (M_1 \otimes M_2)_S) \tag{2.6.1}$$

is induced by ξ. We have to show that the pull-back of $\widetilde{\xi}_1^1 \otimes \widetilde{\xi}_2^1$ by any dominant morphism $\operatorname{Spec} R' \to \operatorname{Spec} R$ does not vanish. Since the above construction is compatible with the base change by a finite extension of k, it is enough to show that the restriction of $\widetilde{\xi}_1^1 \otimes \widetilde{\xi}_2^1$ to any nonempty open subvariety U of S is nonzero by the same argument as in [28], 2.6. (Indeed, we can restrict to a subvariety of S' which is finite étale over an open subvariety of S, and then take the direct image.)

If both ξ_1^1 and ξ_2^1 are nonzero, the assertion is easy, and follows from the same argument as in [28], 4.4 (see also [27]). Indeed, if $H^2(S, \mathbb{Q})^\vee$ denotes the dual of $H^2(S, \mathbb{Q})$, then $\xi_1^1 \otimes \xi_2^1$ corresponds to a morphism of mixed Hodge structures

$$\xi^2 : H^2(S, \mathbb{Q})^\vee \to H^{2p-2}(X, \mathbb{Q})(p),$$

and its image has level 2 by hypothesis. (Here the level of a Hodge structure is the difference between the maximal and minimal numbers p such that $\mathrm{Gr}_F^p \neq 0$.) This level does not change by replacing S with any nonempty open subvariety U of S, see [10]. So we get the assertion in this case.

Now it remains to consider the case $\xi_1^1 \neq 0$ and $\xi_2^1 = 0$ (hence $\xi_2' \neq 0$). By the argument in (2.2), $\widetilde{\xi}_1^1 \otimes \widetilde{\xi}_2^1$ is the pull-back of $\widetilde{\xi}_1^1 \otimes \xi_2'$ by $S \to S_1$, and we may replace R_2 with k. So the assertion is reduced to the case $R_2 = k$, and it is sufficient to show the nonvanishing of

$$\xi_1^1 \otimes \xi_2' \in \mathrm{Ext}^1(\mathbb{Q}, H^1(S_1, \mathbb{Q}) \otimes M_1 \otimes M_2),$$

by replacing S_1 with any nonempty open subvariety U of S_1. Let

$$M_0 = H^1(S_1, \mathbb{Q}), \quad M_0' = \mathrm{Gr}_1^W M_0, \quad M_0'' = \mathrm{Gr}_2^W M_0.$$

Then $\xi_1^1 \in \mathrm{Hom}(\mathbb{Q}, M_0 \otimes M_1)$ comes from an element of $u \in \mathrm{Hom}(\mathbb{Q}, M_0' \otimes M_1)$, because M_1 is pure of weight -1. We denote by $e \in \mathrm{Ext}^1(M_0'', M_0')$ the extension class associated to the canonical short exact sequence.

Assume $\xi_1^1 \otimes \xi_2' = 0$. Using the long exact sequence associated with

$$0 \to M_0' \otimes M_1 \otimes M_2 \to M_0 \otimes M_1 \otimes M_2 \to M_0'' \otimes M_1 \otimes M_2 \to 0, \tag{2.6.2}$$

this means that $u \otimes \xi_2' \in \mathrm{Ext}^1(\mathbb{Q}, M_0' \otimes M_1 \otimes M_2)$ coincides with the composition of some $v \in \mathrm{Hom}(\mathbb{Q}, M_0'' \otimes M_1 \otimes M_2)$ with $e \otimes id$. We have to show that this implies $u = v = 0$. Since M_0'' is isomorphic to a direct sum of $\mathbb{Q}(-1)$, $e \in \mathrm{Ext}^1(M_0'', M_0')$ is identified with a direct sum of

$$e_j \in \mathrm{Ext}^1(\mathbb{Q}(-1), M_0'),$$

and $(e \otimes id) \circ v$ with $\sum_j e_j \otimes v_j$, where $v_j \in \mathrm{Hom}(\mathbb{Q}, M_1 \otimes M_2(-1))$. In particular, $(e \otimes id) \circ v$ factors through the inclusion $M_0' \otimes (\sum_j \mathrm{Im}\, v_j) \to M_0' \otimes M_1 \otimes M_2$. On the other hand, $u \otimes \xi_2'$ factors through $(\mathrm{Im}\, u) \otimes M_2 \to M_0' \otimes M_1 \otimes M_2$. Thus, by using a direct sum decomposition of $M_0' \otimes M_1 \otimes M_2$ compatible with $(\mathrm{Im}\, u) \otimes M_2$ and $M_0' \otimes (\sum_j \mathrm{Im}\, v_j)$, the assertion is reduced to the following:

2.7. Lemma. *Let H_i be a polarizable Hodge structure of weight 1 for $i = 0, 1, 2$. Let $u \in \mathrm{Hom}_{\mathrm{HS}}(\mathbb{Q}(-1), H_0 \otimes H_1)$, and $v_j \in \mathrm{Hom}_{\mathrm{HS}}(\mathbb{Q}(-1), H_1 \otimes H_2)$. Then*

$$(\mathrm{Im}\, u) \otimes H_2 \cap H_0 \otimes \Big(\sum_j \mathrm{Im}\, v_j\Big) = 0. \tag{2.7.1}$$

Proof. It is sufficient to show the assertion for the underlying $\mathbb{R}$-Hodge structure. Using the projections to the direct factors, we may assume that H_0, H_1, H_2 are simple (i.e. 2-dimensional), and furthermore H_0 is isomorphic to H_1, H_2 (because $u = 0$ or $v_j = 0$ otherwise). Let $\{e, \overline{e}\}$ be a basis of $H_0 = H_1 = H_2$ over $\mathbb{C}$ such that e generates F^1, and $\overline{e}$ is the complex conjugate of e. Then

$$u = a\, e \otimes \overline{e} + \overline{a}\, \overline{e} \otimes e \quad \text{and} \quad v_j = b_j e \otimes \overline{e} + \overline{b}_j \overline{e} \otimes e$$

for $a, b_j \in \mathbb{C}$. In particular, $H_0 \otimes (\sum_j \operatorname{Im} v_j)$ is contained in the subspace generated by $H_0 \otimes e \otimes \overline{e}$ and $H_0 \otimes \overline{e} \otimes e$. Thus we get (2.7.1). This completes the proof of Theorem (0.2).

3. The higher cycle case

3.1. For a smooth k-variety X and a positive integer m, let $\mathrm{CH}^p(X,m)_{\mathbb{Q}}$ denote the higher Chow group of X with rational coefficients [6]. By [28] we have the refined cycle map

$$cl : \mathrm{CH}^p(X,m)_{\mathbb{Q}} \to H^{2p-m}_{\mathcal{D}}(X, \mathbb{Q}_{\langle k\rangle}(p)). \tag{3.1.1}$$

This is expected to be compatible with the usual cycle map [7], [13]. (For $m = 1$, it is easy to verify this by reducing to the case $p = 1$.) If X is smooth proper, it induces

$$\mathrm{CH}^p(X,m)_{\mathbb{Q}} \to \mathrm{Ext}^1(\mathbb{Q}_{\langle k\rangle}, H^{2p-m-1}(X, \mathbb{Q}_{\langle k\rangle})(p)), \tag{3.1.2}$$

because $H^{2p-m}(X, \mathbb{Q}_{\langle k\rangle})$ is pure of weight $2p - m$.

Let X_i, $X_{i,k}$ $(i = 1, 2)$ and X, X_k be as in (2.1). In this paper we define the notion of a strictly decomposable higher cycle (in the strong sense) by allowing a higher cycle for ζ_2 in the notation of (2.1). More precisely, we say that a higher cycle $\zeta \in \mathrm{CH}^p(X,m)_{\mathbb{Q}}$ is *strictly decomposable* if there exist subfields K_1, K_2 of $\mathbb{C}$ and cycles $\zeta_1 \in \mathrm{CH}^{p_1}(X_{1,k} \otimes_k K_1)_{\mathbb{Q}}$, $\zeta_2 \in \mathrm{CH}^{p_2}(X_{2,k} \otimes_k K_2, m)_{\mathbb{Q}}$ such that the K_i satisfy the conditions in (2.1), $p_1 + p_2 = p$, and ζ is the base change of $\zeta_1 \times \zeta_2$ as before.

Let ξ_1^j, $\widetilde{\xi}_1^1$ and M_1 be as in (2.1). For $i = 2$, we have the Künneth component of the cohomology class

$$\xi_2^j \in \mathrm{Hom}(\mathbb{Q}, H^{2p_2-m-j}(X_2, \mathbb{Q}) \otimes H^j(S_2, \mathbb{Q})(p_2)).$$

Note that $\xi_2^0 = 0$ in this case. Let $M_2 = H^{2p_2-m-1}(X_2, \mathbb{Q})(p_2)$ with M_{2,S_2} its pull-back by $S_2 \to \operatorname{Spec} \mathbb{C}$ where $S_2 = \operatorname{Spec} R_2 \otimes_k \mathbb{C}$ as before. Then, applying (3.1.2) to ζ_2, we get

$$\widetilde{\xi}_2^1 \in \mathrm{Ext}^1(\mathbb{Q}_{S_2}, M_{2,S_2}).$$

By an exact sequence as in (2.2.1), it induces ξ_2^1 by taking the direct image under $S_2 \to \operatorname{Spec} \mathbb{C}$, and comes from $\xi_2' \in \mathrm{Ext}^1(\mathbb{Q}, M_2)$ if $\xi_2^1 = 0$. Thus we can define the notion of a nondegenerate, strictly decomposable higher cycle by the same way as in (2.4).

3.2. Remark. We have $m = 1$ if $\xi_2^1 \neq 0$, see the proof of (3.3) below.

3.3. Theorem. *The assertion of Theorem (0.2) holds also for nondegenerate, strictly decomposable higher cycles* $\zeta \in \mathrm{CH}^p(X,m)_{\mathbb{Q}}$.

Proof. The argument is similar to (2.6). We can identify ξ_2^1 with a morphism of mixed Hodge structures

$$H^1(S_2, \mathbb{Q})^\vee \to M_2 \,(= H^{2p_2-m-1}(X_2, \mathbb{Q})(p_2)), \tag{3.3.1}$$

where the source is the dual of $H^1(S_2,\mathbb{Q})$, and the target is pure of weight $-m-1$. Therefore the image of (3.3.1) is a direct sum of $\mathbb{Q}(1)$ if $m=1$, and vanishes otherwise. For a nonempty open subvariety S_2' of S_2, we have the injectivity of the restriction morphism

$$\mathrm{Gr}_2^W H^1(S_2,\mathbb{Q}) \to \mathrm{Gr}_2^W H^1(S_2',\mathbb{Q}), \tag{3.3.2}$$

because $H^1_Z(S_2,\mathbb{Q})=0$ with $Z=S_2\setminus S_2'$. Then, taking the tensor with $H^{2p_1}(X_1,\mathbb{Q})(p_1)\otimes M_2$, we get the assertion in the case $\xi_1^0\neq 0$ and $\xi_2^1\neq 0$ (where we may assume $R_1=k$). If $\xi_1^0\neq 0$ and $\xi_2^1=0$, we may assume $R_1=R_2=k$, and the assertion is clear. Thus the assertion is reduced to the case $\xi_1^0=0$.

The argument is similar if both ξ_1^1 and ξ_2^1 are nonzero (in particular, $m=1$). For any nonempty open subset S' of $S:=S_1\times S_2$, we have the injectivity of the restriction morphism

$$\mathrm{Gr}_3^W H^2(S,\mathbb{Q}) \to \mathrm{Gr}_3^W H^2(S',\mathbb{Q}),$$

because the local cohomology $H^2_Z(S,\mathbb{Q})$ is pure of type $(1,1)$, where $Z=S\setminus S'$. Thus we get the assertion in this case, and we may assume either ξ_1^1 or ξ_2^1 vanishes.

If $\xi_1^1\neq 0$ and $\xi_2^1=0$ (hence $\xi_2'\neq 0$), then the assertion follows immediately from the long exact sequence associated to (2.6.2), because $m>0$. If $\xi_1^1=0$ and $\xi_2^1\neq 0$ (hence $\xi_1'\neq 0$ and $m=1$), then we may assume $R_1=k$ and M_2 is a direct sum of $\mathbb{Q}(1)$ because the image of (3.3.1) is a direct factor of the target (and the similar assertion holds over k). Furthermore, it is sufficient to show the nonvanishing of the composition of $\xi_1'\otimes\xi_2^1$ with the morphism induced by the projection $H^1(S_2,\mathbb{Q})\to \mathrm{Gr}_2^W H^1(S_2,\mathbb{Q})$ after replacing S_2 with any nonempty open subvariety. Thus the assertion follows from the injectivity of (3.3.2). This finishes the proof of (3.3).

3.4. Remarks. (i) Theorems (0.2) and (3.3) hold also for a variety X over a subfield K of $\mathbb{C}$, where the K_i are assumed to be subfields of K, see Remark (i) of (1.3).

(ii) Let C be a curve of positive genus, and assume $X_1=X_2=C$, $K_1=k(X_{1,k})$, and $K_2=k$. Then (0.2) is quite similar to the construction of Nori explained in [30]. In this sense, Theorems (0.2) and (0.3) may be viewed as a generalization of Nori's construction.

4. Application to indecomposable higher cycles

4.1. Let X be a smooth complex projective variety. We have a canonical morphism

$$\mathrm{CH}^{p-1}(X)_{\mathbb{Q}}\otimes_{\mathbb{Z}}\mathbb{C}^*\to \mathrm{CH}^p(X,1)_{\mathbb{Q}}, \tag{4.1.1}$$

see [16], [21]. Its cokernel (resp. image) is denoted by $\mathrm{CH}^p_{\mathrm{ind}}(X,1)_{\mathbb{Q}}$ (resp. $\mathrm{CH}^p_{\mathrm{dec}}(X,1)_{\mathbb{Q}}$). An element of this group is called an indecomposable (resp. a decomposable) higher cycle. Let Hdg^{p-1} be the maximal subobject of $H^{2p-2}(X,\mathbb{Q})$ which is isomorphic to a direct sum of copies of $\mathbb{Q}(1-p)$ (i.e. the subgroup of Hodge cycles). It is a direct factor of $H^{2p-2}(X,\mathbb{Q})$ by semisimplicity. The usual cycle map induces the reduced higher Abel-Jacobi map

$$\mathrm{CH}^p_{\mathrm{ind}}(X,1)_{\mathbb{Q}}\to \mathrm{Ext}^1(\mathbb{Q},(H^{2p-2}(X,\mathbb{Q})/\mathrm{Hdg}^{p-1})(p)), \tag{4.1.2}$$

since the image of $\mathrm{CH}^{p-1}(X)_{\mathbb{Q}}\otimes_{\mathbb{Z}}\mathbb{C}^*$ is contained in $\mathrm{Ext}^1(\mathbb{Q},\mathrm{Hdg}^{p-1}(p))$, see loc. cit.

4.2. Theorem. *With the notation of (3.1), assume* $m = 1$, $\xi_1^0 = 0$, $\xi_1^1 \neq 0, R_2 = k$, *and the image of* ζ_2 *by (4.1.2) for* X_2 *is nonzero. Then* $\zeta \neq 0$ *in* $\mathrm{CH}^p_{\mathrm{ind}}(X,1)_{\mathbb{Q}}$.

Proof. Assume ζ is decomposable, i.e. ζ coincides with the image of $\sum_j \zeta'_j \otimes \alpha_j$ where $\zeta'_j \in \mathrm{CH}^{p-1}(X)_{\mathbb{Q}}$ and $\alpha_j \in \mathbb{C}^*$. Here we may assume that the α_j are linearly independent over $\mathbb{Q}$ in $\mathbb{C}^* \otimes_{\mathbb{Z}} \mathbb{Q}$, and hence $\zeta'_j \in \mathrm{CH}^{p-1}_{\mathrm{hom}}(X)_{\mathbb{Q}}$, because the image of ζ by the usual cycle map vanishes due to the assumption $\xi_1^0 = 0$, see (3.3). To simplify the notation, let $R = R_1$ and $S = \operatorname{Spec} R \otimes_k \mathbb{C}$. We may assume that the ζ'_j and α_j are defined over R replacing R if necessary. We denote by $\zeta \otimes_k \mathbb{C}$ the base change of ζ by $k \to \mathbb{C}$ as before, and similarly for $\zeta'_j \otimes_k \mathbb{C}$, etc. We will show that the images of $\zeta \otimes_k \mathbb{C}$ and $\sum_j (\zeta'_j \otimes \alpha_j) \otimes_k \mathbb{C}$ in the Deligne cohomology $H_{\mathcal{D}}^{2p-1}(S \times X, \mathbb{Q}(p))$ are different. Since $\mathrm{Hom}(\mathbb{Q}, H^{2p_2-1}(X_2, \mathbb{Q})(p_2)) = 0$, we see that the image of $\zeta \otimes_k \mathbb{C}$ in $\mathrm{Hom}(\mathbb{Q}, H^{2p-1}(S \times X, \mathbb{Q})(p))$ vanishes, and so does that of $\sum_j (\zeta'_j \otimes \alpha_j) \otimes_k \mathbb{C}$.

Let $M_1 = H^{2p_1-1}(X_1, \mathbb{Q})(p_1)$, $M_2 = H^{2p_2-2}(X_2, \mathbb{Q})(p_2)$ as before. We will show the difference of the direct factors of the images of $\zeta \otimes_k \mathbb{C}$ and $\sum_j (\zeta'_j \otimes \alpha_j) \otimes_k \mathbb{C}$ in

$$\mathrm{Ext}^1(\mathbb{Q}, H^1(S, \mathbb{Q}) \otimes M_1 \otimes M_2), \tag{4.2.1}$$

using the Künneth decomposition. We choose direct sum decompositions

$$(a_X)_* \mathbb{Q}_S \simeq \sum_i H^i(X, \mathbb{Q})[-i], \quad (a_S)_* \mathbb{Q}_S \simeq \sum_i H^i(S, \mathbb{Q})[-i], \tag{4.2.2}$$

in the bounded derived category $D^b\mathrm{MHS}$ of MHS. They exist because Ext^2 vanishes in MHS. Applying this to the image of $\zeta'_j \otimes_k \mathbb{C}$ by the cycle map to the Deligne cohomology of $S \times X$ and restricting to the relevant Künneth component, we get

$$\begin{aligned} &\xi'_{j,0} \in \mathrm{Hom}(\mathbb{Q}, H^1(S, \mathbb{Q}) \otimes M_1 \otimes M_2(-1)), \\ &\xi'_{j,1} \in \mathrm{Ext}^1(\mathbb{Q}, M_1 \otimes M_2(-1)), \end{aligned}$$

because $H^0(S, \mathbb{Q}) = \mathbb{Q}$. Here Ext^1 and Hom are taken in MHS. We see that the other Künneth components do not contribute to (4.2.1), because the $\alpha_j \otimes_k \mathbb{C}$, which are viewed as sections of $\mathcal{O}_S^*$, induce

$$\eta_{j,0} \in \mathrm{Hom}(\mathbb{Q}, H^1(S, \mathbb{Q})(1)), \quad \eta_{j,1} \in \mathrm{Ext}^1(\mathbb{Q}, \mathbb{Q}(1)),$$

by using (4.2.2). Thus the contribution of $\sum_j (\zeta'_j \otimes \alpha_j) \otimes_k \mathbb{C}$ to (4.2.1) is

$$\sum_j \eta_{j,0} \otimes \xi'_{j,1} + \sum_j \eta_{j,1} \otimes \xi'_{j,0}.$$

On the other hand, the contribution of $\zeta \otimes_k \mathbb{C}$ is $\xi_1^1 \otimes \zeta'_2$ with the notation of (3.2).

We have a direct sum decomposition

$$H^1(S, \mathbb{Q}) = H^1(S, \mathbb{Q})' \oplus H^1(S, \mathbb{Q})'', \tag{4.2.3}$$

such that $H^1(S,\mathbb{Q})''$ is a direct sum of copies of $\mathbb{Q}(-1)$, and

$$\mathrm{Hom}(\mathbb{Q}(-1),H^1(S,\mathbb{Q})) = \mathrm{Hom}(\mathbb{Q}(-1),H^1(S,\mathbb{Q})''),$$

because the composition $\mathbb{Q}(-1) \to H^1(S,\mathbb{Q}) \to \mathrm{Gr}_2^W H^1(S,\mathbb{Q})$ splits by semisimplicity. Clearly $\eta_{j,0} \in \mathrm{Hom}(\mathbb{Q},H^1(S,\mathbb{Q})''(1))$, and $\xi'_{j,0},\xi^1_1$ come respectively from

$$\begin{aligned}\xi''_{j,0} &\in \mathrm{Hom}(\mathbb{Q},\mathrm{Gr}_1^W H^1(S,\mathbb{Q})\otimes M_1\otimes M_2(-1)),\\ \xi''_1 &\in \mathrm{Hom}(\mathbb{Q},\mathrm{Gr}_1^W H^1(S,\mathbb{Q})\otimes M_1).\end{aligned}$$

So we get $\sum_j \eta_{j,0}\otimes\xi'_{j,1}=0$ due to the decomposition (4.2.3). Furthermore, we may replace $H^1(S,\mathbb{Q})$ with $\mathrm{Gr}_1^W H^1(S,\mathbb{Q})$, using the long exact sequence associated to (2.6.2) as in the proof of (3.3) (because $m>0$). Thus the assertion is reduced to the following lemma which we apply to

$$\begin{aligned}H_1 &= \mathrm{Gr}_1^W H^1(S,\mathbb{Q})\otimes H^{2p_1-1}(X_1,\mathbb{Q})(p_1),\\ H_2 &= (H^{2p_2-2}(X,\mathbb{Q})/\mathrm{Hdg}^{p_2-1})(p_2-1).\end{aligned}$$

4.3. Lemma. *Let H_i be a polarizable Hodge structure of weight 0 for $i=1,2$. Assume $\mathrm{Hom}(\mathbb{Q},H_2)=0$. Let $u\in\mathrm{Hom}(\mathbb{Q},H_1), e\in\mathrm{Ext}^1(\mathbb{Q},H_2(1))$, $u_j\in\mathrm{Hom}(\mathbb{Q},H_1\otimes H_2), e_j\in\mathrm{Ext}^1(\mathbb{Q},\mathbb{Q}(1))$ for $j=1,\dots,r$. Then $u\otimes e=0$ if $u\otimes e=\sum_j u_j\otimes e_j$ in $\mathrm{Ext}^1(\mathbb{Q},H_1\otimes H_2(1))$.*

Proof. Since $\mathrm{Im}\,u$ is a direct factor of H_1, we may assume $H_1=\mathbb{Q}$ by using the projection to the direct factor $H_1\to\mathbb{Q}$. Then $u_j=0$, and the assertion is clear.

4.4. Theorem. *Let X_1,X_2,X be as in (2.1). Assume $\Gamma(X_1,\Omega^1_{X_1})\neq 0$ and the reduced higher Abel-Jacobi map (4.1.2) for X_2 is not zero. Then $\mathrm{CH}^{p+1}_{\mathrm{ind}}(X,1)_{\mathbb{Q}}$ is uncountable. More precisely, let $\zeta_2\in\mathrm{CH}^p(X_2,1)$ whose image by (4.1.2) does not vanish. Let k be a subfield of $\mathbb{C}$ such that the X_i and ζ_2 are defined over k. Let P denote the Picard variety of X_1 which is defined over k. Then for any $\zeta_1\in P(\mathbb{C})\setminus P(\overline{k})$, the image of $\zeta_1\times\zeta_2$ in $\mathrm{CH}^{p+1}_{\mathrm{ind}}(X,1)_{\mathbb{Q}}$ is nonzero.*

Proof. The last assertion follows from (4.2) combined with (2.2). If $\mathrm{CH}^{p+1}_{\mathrm{ind}}(X,1)_{\mathbb{Q}}$ is countable, there are countablly many $\zeta_{1,j}\in P(\mathbb{C})$ $(j\in\mathbb{N})$ such that for any $\zeta_1\in P(\mathbb{C})$, there exists j such that the images of $\zeta_1\times\zeta_2$ and $\zeta_{1,j}\times\zeta_2$ in $\mathrm{CH}^{p+1}_{\mathrm{ind}}(X,1)_{\mathbb{Q}}$ coincide. We may assume $\zeta_{1,j}\in P(k)$ by replacing k with a countablly generated subfield of $\mathbb{C}$. Take any $\zeta_1\in P(\mathbb{C})\setminus P(\overline{k})$, and let $\zeta_{1,j}\in P(k)$ such that the image of $(\zeta_1-\zeta_{1,j})\times\zeta_2$ in $\mathrm{CH}^{p+1}_{\mathrm{ind}}(X,1)_{\mathbb{Q}}$ is zero. But this contradicts the above assertion, because $\zeta_1-\zeta_{1,j}\notin P(\overline{k})$.

Remark. A similar assertion was obtained by Gordon and Lewis [16] in the case of a product of elliptic curves, see also [9].

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Andreas Rosenschon
Department of Mathematics
Duke University
Durham, NC 27708
U.S.A
E-Mail: axr@math.duke.edu

Morihiko Saito
RIMS Kyoto University,
Kyoto 606–8502
Japan
E-Mail: msaito@kurims.kyoto-u.ac.jp

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