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Quadratic M-convex and L-convex Functions*

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Abstract

The concepts of L-convexity and M-convexity are introduced by Murota (1996) for functions defined over the integer lattice, and recently extended to polyhedral convex functions by Murota–Shioura (2000). M-convex and L-convex functions are deeply connected with well-solvability in combinatorial optimization problems with convex objective functions. In this paper, we consider these concepts to quadratic functions, and investigate the combinatorial properties of quadratic L-convex and M-convex functions and the structure of the coefficient matrices of such quadratic functions. It is shown that quadratic L-convex functions and their coefficient matrices are deeply connected with well-known concepts such as Dirichlet forms and diagonally-dominant symmetric M-matrices. The conjugacy relationship between quadratic L-convex and M-convex functions is also discussed.

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1 Introduction

A quadratic function $f : \mathbf{R}^n \to \mathbf{R}$ is defined by a symmetric matrix $A \in \mathbf{R}^{n\times n}$ as

$$f(x) = \frac{1}{2}x^{\mathrm{T}}Ax \qquad (x \in \mathbf{R}^n). \tag{1.1}$$

It is well-known that f is convex if and only if A is positive semidefinite. In this paper, we consider quadratic convex functions equipped with "combinatorial structures." This is tantamount to investigating symmetric matrices with some combinatorial properties in addition to positive semidefiniteness.

Many different classes of matrices with combinatorial properties have been investigated in the area of matrix theory. A typical example is the class of M-matrices that appear in various areas such as control engineering [26], numerical analysis [1, 28], mathematical economics [21], etc. A real matrix $A \in \mathbf{R}^{n\times n}$ is called an M-matrix if it is given by $A = sI - B$ with a nonnegative matrix $B \in \mathbf{R}^{n\times n}$ and a real number $s \geq \rho(B)$, where $\rho(B)$ denotes the maximum modulus of an eigenvalue of B. In particular, a nonsingular M-matrix is a matrix such that each off-diagonal component is nonpositive and its inverse is a nonnegative matrix. We here consider diagonally-dominant symmetric M-matrices, where it is noted that any symmetric M-matrix can be transformed into a diagonally-dominant symmetric M-matrix by a symmetric diagonal scaling. A diagonally-dominant symmetric M-matrix $A \in \mathbf{R}^{n\times n}$ can be characterized by the following property of a combinatorial nature:

$$a_{ij} \leq 0 \quad (\forall i, j \in N\ i \neq j), \qquad \sum_{j=1}^{n} a_{ij} \geq 0 \quad (\forall i \in N), \tag{1.2}$$

where $N = \{1, 2, \cdots, n\}$. Note that a symmetric matrix A is called a compartmental matrix [13] if $-A$ satisfies the property (1.2). See [2] for mathematical properties of M-matrices.

The concept of Dirichlet forms is deeply connected with M-matrices. Indeed, a quadratic function $(1/2)x^{\mathrm{T}}Ax$ is a Dirichlet form (of finite dimension) if and only if A is a diagonally-dominant symmetric M-matrix. The Dirichlet form is an important tool in the theory of Markov processes and the potential theory [7]. It is known that a Dirichlet form $f(x) = (1/2)x^{\mathrm{T}}Ax$ can be characterized by the following combinatorial property:

$$f(x) \geq f((\mathbf{0} \vee x) \wedge \mathbf{1}) \qquad (\forall x \in \mathbf{R}^n),$$

where $\mathbf{0}$ and $\mathbf{1}$ denote the vectors with all components being equal to zero and one, respectively, and for any $x, y \in \mathbf{R}^n$ the vectors $x \wedge y, x \vee y \in \mathbf{R}^n$ are given by

$$(x \wedge y)_i = \min\{x_i, y_i\}, \qquad (x \vee y)_i = \max\{x_i, y_i\} \qquad (i = 1, \cdots, n).$$

Such a combinatorial property of Dirichlet forms can be seen as a reflection of the combinatorial structure of diagonally-dominant symmetric M-matrices.

In the area of combinatorial optimization, on the other hand, "combinatorial convexity," i.e., convexity equipped with nice combinatorial properties has also been investigated [3, 4, 5, 9, 12, 14, 23].

Combinatorial optimization problems with nonlinear objective functions have been dealt with more often than before due to theoretical interest and needs of practical applications. The nonlinear resource allocation problem (see [8, 11]) and the convex cost submodular flow problem (see [5, 10]) are well-studied examples. Both of the problems have nice combinatorial structures, which lead to efficient combinatorial algorithms. Extensive studies have been done for revealing the essence of the well-solvability in nonlinear combinatorial optimization problems [14, 3].

By extracting nice combinatorial structure in combinatorial optimization problems with convex objective functions, Murota [15, 16, 17] introduced the concept of L-convexity and M-convexity for functions defined over the integer lattice. Later, the concepts of L$^\natural$-convexity and M$^\natural$-convexity are introduced by Fujishige–Murota [6] and by Murota–Shioura [19], respectively, as variants of L-convexity and M-convexity.

The concepts of L-convexity and M-convexity are extended to polyhedral convex functions by Murota–Shioura [20]. A polyhedral convex function $g : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ is said to be L-convex if $\mathrm{dom}\, g \neq \emptyset$ and g satisfies (LF1) and (LF2):

(LF1) $\quad g(p) + g(q) \geq g(p \wedge q) + g(p \vee q) \qquad (\forall p, q \in \mathrm{dom}\, g)$,

(LF2) $\quad \exists r \in \mathbf{R}$ such that $g(p + \lambda\mathbf{1}) = g(p) + \lambda r \quad (\forall p \in \mathrm{dom}\, g,\ \lambda \in \mathbf{R})$,

where $\mathrm{dom}\, g = \{p \in \mathbf{R}^n \mid g(p) < +\infty\}$. A polyhedral convex function $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ is said to be M-convex if $\mathrm{dom}\, f \neq \emptyset$ and f satisfies (M-EXC):

(M-EXC) $\forall x, y \in \mathrm{dom}\, f$, $\forall i \in \mathrm{supp}^+(x-y)$, $\exists j \in \mathrm{supp}^-(x-y)$, $\exists \alpha_0 > 0$:

$$f(x) + f(y) \geq f(x - \alpha(\chi_i - \chi_j)) + f(y + \alpha(\chi_i - \chi_j)) \qquad (0 < \forall \alpha \leq \alpha_0),$$

where

$$\mathrm{supp}^+(x-y) = \{i \in N \mid x_i > y_i\}, \qquad \mathrm{supp}^-(x-y) = \{i \in N \mid x_i < y_i\}.$$

Variants of polyhedral L-convex and M-convex functions are also introduced in [20]. Put $\widehat{N} = \{0\} \cup N$. A polyhedral convex function $g : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ is said to be L$^\natural$-convex if the function $\widehat{g} : \mathbf{R}^{\widehat{N}} \to \mathbf{R} \cup \{+\infty\}$ defined by

$$\widehat{g}(p_0, p) = g(p - p_0\mathbf{1}) \qquad ((p_0, p) \in \mathbf{R}^{\widehat{N}})$$

is L-convex. A polyhedral convex function $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ is said to be M$^\natural$-convex if the function $\widehat{f} : \mathbf{R}^{\widehat{N}} \to \mathbf{R} \cup \{+\infty\}$ defined by

$$\widehat{f}(x_0, x) = \begin{cases} f(x) & (x_0 = -x(N)), \\ +\infty & (x_0 \neq -x(N)), \end{cases} \qquad ((x_0, x) \in \mathbf{R}^{\widehat{N}})$$

is M-convex.

Although the concepts of L-/L$^\natural$-convexity and M-/M$^\natural$-convexity are defined for polyhedral convex functions, we can naturally extend these concepts to quadratic functions on the basis of (LF1) and (LF2), and (M-EXC), respectively. The main aim of this paper is to investigate the combinatorial properties of quadratic L-/L$^\natural$-convex and M-/M$^\natural$-convex functions and the structure of their coefficient matrices.

We show that a quadratic function $g : \mathbf{R}^n \to \mathbf{R}$ given by $g(p) = (1/2)p^{\mathrm{T}}Bp$ ($p \in \mathbf{R}^n$) is L$^\natural$-convex if and only if its coefficient matrix B satisfies the property (1.2), i.e., B is a diagonally-dominant symmetric M-matrix. This means, in particular, that a quadratic L$^\natural$-convex function coincides with a Dirichlet form of finite dimension. A similar characterization of a quadratic L-convex function is given in terms of the sign pattern of the coefficient matrix.

As for M-/M$^\natural$-convexity, a quadratic function $f : \mathbf{R}^n \to \mathbf{R}$ given by (1.1) with a nonsingular coefficient matrix is M$^\natural$-convex if and only if its coefficient matrix A satisfies the following combinatorial property:

$$x^{\mathrm{T}}a_i > \min\left\{0, \min_{j \neq i} x^{\mathrm{T}}a_j\right\} \qquad (\forall x \in \mathbf{R}^n,\ \forall i \in \mathrm{supp}^+(x)). \tag{1.3}$$

It is mentioned here that the property (1.3) immediately yields the positive definiteness of matrix A, to be shown in Theorem 2.13. Moreover, the combinatorial property (1.3) turns out to be equivalent to the property that the inverse of matrix A is a diagonally-dominant symmetric M-matrix. This reveals a nice conjugacy relationship between quadratic L$^\natural$-convex functions and M$^\natural$-convex functions expressed in terms of the coefficient matrices. The conjugacy here means the conjugacy under the Legendre–Fenchel transformation. Given a function f : $\mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$, the conjugate $f^\bullet : \mathbf{R}^n \to \mathbf{R} \cup \{\pm\infty\}$ of f is defined by

$$f^\bullet(p) = \sup\{p^{\mathrm{T}}x - f(x) \mid x \in \mathbf{R}^n\} \qquad (p \in \mathbf{R}^n).$$

Functions $f, g : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ are said to be conjugate to each other if $f^\bullet = g$ and $g^\bullet = f$.

Finally, we consider L-/L$^\natural$-convex and M-/M$^\natural$-convex functions defined over $\mathbf{Z}^n$ (the integer lattice), and compare them to those over $\mathbf{R}^n$. This is to indicate a subtlety between the discrete case and the continuous case, especially with M-convexity. It is shown that the classes of coefficient matrices of L-/L$^\natural$-convex functions over $\mathbf{Z}^n$ coincide with those of quadratic L-/L$^\natural$-convex functions over $\mathbf{R}^n$, whereas the coefficient matrices of quadratic M-/M$^\natural$-convex functions over $\mathbf{Z}^n$ constitute more restrictive classes than those of quadratic M-/M$^\natural$-convex functions over $\mathbf{R}^n$.

The organization of this paper is as follows. We investigate the structures of quadratic L-/L$^\natural$-convex and M-/M$^\natural$-convex functions in Section 2, while the proofs are given in Section 5. The conjugacy relationship between quadratic L-/L$^\natural$-convex and M-/M$^\natural$-convex functions are discussed in Section 3. In Section 4, we provide characterizations of quadratic L-/L$^\natural$-convex and M-/M$^\natural$-convex functions over $\mathbf{Z}^n$.

2 Results

We describe the main theorems in this paper. The proofs are given in Section 5.

2.1 Definitions and Notation

We denote by $\mathbf{R}$ the set of reals. Throughout this paper, we assume that n is a positive integer, and denote $N = \{1, 2, \cdots, n\}$. For any finite set S, its cardinality is denoted by $|S|$. A family $\{N_1, N_2, \cdots, N_m\}$ of subsets of N is called a *subpartition* of N if $N_i \cap N_j = \emptyset$ for any distinct $i, j \in \{1, \cdots, m\}$, and a *partition* if it is a subpartition and $\bigcup_{i=1}^{m} N_i = N$.

The characteristic vector of a subset $S \subseteq N$ is denoted by χ_S ($\in \{0, 1\}^n$), i.e.,

$$(\chi_S)_i = \begin{cases} 1 & (i \in S), \\ 0 & (i \in N \setminus S). \end{cases}$$

In particular, we use the notation $\mathbf{0} = \chi_\emptyset$, $\mathbf{1} = \chi_N$. We sometimes denote $\chi_0 = \mathbf{0}$.

For $x \in \mathbf{R}^n$, we define

$$\mathrm{supp}^+(x) = \{v \in N \mid x_i > 0\}, \quad \mathrm{supp}^-(x) = \{v \in N \mid x_i < 0\},$$
$$x(S) = \textstyle\sum_{i \in S} x_i \ (S \subseteq N).$$

For a subset $S \subseteq N$, we denote by $x|_S \in \mathbf{R}^S$ the restriction of x to S.

For any $p, q \in \mathbf{R}^n$, $p \wedge q$ and $p \vee q$ denote the vectors in $\mathbf{R}^n$ such that

$$(p \wedge q)_i = \min\{p_i, q_i\}, \quad (p \vee q)_i = \max\{p_i, q_i\} \quad (i \in N).$$

A set $H \subseteq \mathbf{R}^n$ is called a *linear subspace* of $\mathbf{R}^n$ if

$$\alpha x + \beta y \in H \qquad (\forall x, y \in H, \ \alpha, \beta \in \mathbf{R}).$$

Given a set of vectors $x_1, x_2, \cdots, x_m \in \mathbf{R}^n$, we define

$$\mathrm{span}\{x_i \mid i = 1, 2, \cdots, m\} = \Big\{ \sum_{i=1}^{m} \alpha_i x_i \mid \alpha_i \in \mathbf{R} \ (i = 1, \cdots, m) \Big\}.$$

We denote by $\mathcal{S}_n$ the set of $n \times n$ real symmetric matrices. For a symmetric matrix $A \in \mathcal{S}_n$ and subsets I and J of N, we denote by $A[I, J]$ the submatrix of A induced by the row set I and the column set J. We also use the notation O for the zero matrix (of appropriate size).

For a function $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$, we denote

$$\mathrm{dom}\, f = \{x \in \mathbf{R}^n \mid f(x) < +\infty\},$$
$$\arg\min f = \{x \in \mathbf{R}^n \mid f(x) \leq f(y) \ (\forall y \in \mathbf{R}^n)\}.$$

2.2 Quadratic L-convex/L$^\natural$-convex Functions

For a symmetric matrix $B \in \mathcal{S}_n$ and a linear subspace $K \subseteq \mathbf{R}^n$, we define a quadratic function $g : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ over K by

$$g(p) = \begin{cases} \dfrac{1}{2} p^{\mathrm{T}} B p & (p \in K), \\ +\infty & (p \notin K). \end{cases} \tag{2.4}$$

We call such a quadratic function g *L-convex* if it satisfies the properties (LF1) and (LF2):

> **(LF1)** g is submodular, i.e., $g(p) + g(q) \geq g(p \wedge q) + g(p \vee q) \qquad (\forall p, q \in \mathrm{dom}\, g)$,
> **(LF2)** $g(p + \lambda \mathbf{1}) = g(p) \qquad (\forall p \in \mathrm{dom}\, g,\ \forall \lambda \in \mathbf{R})$.

It is easy to see that the effective domain $\mathrm{dom}\, g = K$ satisfies the following properties:

> **(LS1)** $p,\ q \in K \Longrightarrow p \wedge q,\ p \vee q \in K$,
> **(LS2)** $p \in K \Longrightarrow p + \lambda \mathbf{1} \in K\ (\forall \lambda \in \mathbf{R})$.

We call a linear subspace *L-convex* if it satisfies the properties (LS1) and (LS2).

We call a quadratic function $g : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ given by (2.4) *L$^\natural$-convex* if the function $\widehat{g} : \mathbf{R}^{\widehat{N}} \to \mathbf{R} \cup \{+\infty\}$, where $\widehat{N} = \{0\} \cup N$, defined by

$$\widehat{g}(p_0, p) = g(p - p_0 \mathbf{1}) \qquad ((p_0, p) \in \mathbf{R}^{\widehat{N}}) \tag{2.5}$$

is L-convex. Then, the linear subspace $\widehat{K} \subseteq \mathbf{R}^{\widehat{N}}$ defined by

$$\widehat{K} = \{(p_0, p) \in \mathbf{R}^{\widehat{N}} \mid p - p_0 \mathbf{1} \in K\} \tag{2.6}$$

is L-convex. We call a linear subspace $K \subseteq \mathbf{R}^n$ *L$^\natural$-convex* if the linear subspace $\widehat{K}$ defined by (2.6) is L-convex.

Proposition 2.1. *Let $K \subseteq \mathbf{R}^n$ be a nonempty set.*
(i) *K is an L-convex linear subspace if and only if there exists a partition $\{N_k \mid k = 1, 2, \cdots, m\}$ of N such that $K = \mathrm{span}\{\chi_{N_k} \mid k = 1, \cdots, m\}$.*
(ii) *K is an L$^\natural$-convex linear subspace if and only if there exists a subpartition $\{N_k \mid k = 1, 2, \cdots, m\}$ of N such that $K = \mathrm{span}\{\chi_{N_k} \mid k = 1, \cdots, m\}$.*

2.2.1 L-convex Functions

We first consider a special case when the function g is defined over the entire space $\mathbf{R}^n$:

$$g(p) = \frac{1}{2} p^{\mathrm{T}} B p \qquad (p \in \mathbf{R}^n). \tag{2.7}$$

We show that the coefficent matrix of such a quadratic L-convex function has a nice sign pattern (Theorem 2.2) and block-diagonal structure (Theorem 2.4). Based on these results, we then investigate the structure of general quadratic L-convex function (Theorem 2.5).

We first provide a characterization of a quadratic L-convex function of the form (2.7) in terms of the coefficient matrix.

Theorem 2.2. *Let $B \in \mathcal{S}_n$ be a symmetric matrix. Then, a quadratic function $g : \mathbf{R}^n \to \mathbf{R}$ given by (2.7) is L-convex if and only if it satisfies*

$$b_{ij} \leq 0 \quad (i, j \in N,\ i \neq j), \qquad \sum_{j=1}^{n} b_{ij} = 0 \quad (i \in N). \tag{2.8}$$

Noting that the matrix B satisfying (2.8) is singular, i.e., $\operatorname{rank} B \leq n-1$, we denote

$$\begin{aligned} \mathcal{L} &= \bigcup_{n=1}^{\infty} \{B \in \mathcal{S}_n \mid \text{the function (2.7) over } \mathbf{R}^n \text{ is L-convex},\ \operatorname{rank} B = n-1\}, \\ \overline{\mathcal{L}} &= \bigcup_{n=1}^{\infty} \{B \in \mathcal{S}_n \mid \text{the function (2.7) over } \mathbf{R}^n \text{ is L-convex}\}. \end{aligned}$$

As is well known, any matrix in $\overline{\mathcal{L}}$ is positive semidefinite [1, 28].

Proposition 2.3. *A symmetric matrix $B \in \mathcal{S}_n$ with the property (2.8) is positive semidefinite.*

Therefore, quadratic L-convex functions of the form (2.7) are convex functions in the ordinary sense.

We next reveal the block-diagonal structure of matrices in $\overline{\mathcal{L}}$.

Theorem 2.4. *A symmetric matrix $B \in \mathcal{S}_n$ belongs to $\overline{\mathcal{L}}$ if and only if there exists a partition $\{N_k \subseteq N \mid k = 1, 2, \cdots, m\}$ of N such that*

$$\left.\begin{array}{ll} B[N_k, N_k] \in \mathcal{L} & (k \in \{1, \cdots, m\}), \\ B[N_k, N_l] = O & (k, l \in \{1, \cdots, m\},\ k \neq l), \end{array}\right\} \tag{2.9}$$

where $m = n - \operatorname{rank} B$.

We then consider an L-convex function of the general form (2.4) with a symmetric matrix $B \in \mathcal{S}_n$ and a linear subspace $K \subseteq \mathbf{R}^n$.

Theorem 2.5. *Let $B \in \mathcal{S}_n$, and $K \subseteq \mathbf{R}^n$ be a linear subspace. Then, a quadratic function $g : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ over K, given by (2.4), is L-convex if and only if there exists a partition $\{N_k \mid k = 1, 2, \cdots, d\}$ of N such that K is given as*

$$K = \operatorname{span}\{\chi_{N_k} \mid k = 1, 2, \cdots, d\} \tag{2.10}$$

and the "aggregated" matrix $\widetilde{B} \in \mathbf{R}^{d\times d}$ *defined by*

$$\widetilde{b}_{kl} = \sum_{i\in N_k}\sum_{j\in N_l} b_{ij} \qquad (k,l \in \{1,2,\cdots,d\}) \tag{2.11}$$

belongs to $\overline{\mathcal{L}}$*, where* $d = \dim K$*.*

By Theorems 2.4 and 2.5, we see that the function g defined over K is L-convex if and only if the "aggregated" matrix $\widetilde{B}$ of the coefficient matrix B defined by (2.11) is a block-diagonal matrix satisfying the property (2.9).

2.2.2 L$^\natural$-convex Functions

Theorems 2.2, 2.4, and 2.5 above for L-convex functions are rephrased for L$^\natural$-convex functions as follows. We denote

$$\begin{aligned}
\mathcal{L}^\natural &= \bigcup_{n=1}^{\infty}\{B \in \mathcal{S}_n \mid \text{the function (2.7) over } \mathbf{R}^n \text{ is L}^\natural\text{-convex},\ \operatorname{rank} B = n\},\\
\overline{\mathcal{L}^\natural} &= \bigcup_{n=1}^{\infty}\{B \in \mathcal{S}_n \mid \text{the function (2.7) over } \mathbf{R}^n \text{ is L}^\natural\text{-convex}\}.
\end{aligned}$$

The following is easy to see from the definitions.

Proposition 2.6.
(i) $B \in \mathcal{S}_n$ *belongs to* $\mathcal{L}$ *(resp.* $\overline{\mathcal{L}}$*) if and only if the submatrix* $B[N\setminus\{n\}, N\setminus\{n\}] \in \mathcal{S}_{n-1}$ *belongs to* $\mathcal{L}^\natural$ *(resp.* $\overline{\mathcal{L}^\natural}$*).*
(ii) $B \in \mathcal{S}_n$ *belongs to* $\mathcal{L}^\natural$ *(resp.* $\overline{\mathcal{L}^\natural}$*) if and only if the symmetric matrix* $\widehat{B} \in \mathbf{R}^{\widehat{N}\times\widehat{N}}$ *defined by*

$$\widehat{b}_{ij} = \begin{cases} b_{ij} & (i,j \in N),\\ -\sum_{k=1}^{n} b_{ik} & (i \in N,\ j = 0),\\ -\sum_{k=1}^{n} b_{kj} & (i = 0,\ j \in N),\\ \sum_{k=1}^{n}\sum_{l=1}^{n} b_{kl} & (i = j = 0) \end{cases} \tag{2.12}$$

belongs to $\mathcal{L}$ *(resp.* $\overline{\mathcal{L}}$*), where* $\widehat{N} = \{0\} \cup N$*.*

Theorem 2.7. *Let* $B \in \mathcal{S}_n$ *be a symmetric matrix. Then, a quadratic function* $g : \mathbf{R}^n \to \mathbf{R}$ *given by (2.7) is L*$^\natural$*-convex if and only if* B *satisfies*

$$b_{ij} \le 0 \quad (\forall i,j \in N,\ i \ne j), \qquad \sum_{j=1}^{n} b_{ij} \ge 0 \quad (\forall i \in N). \tag{2.13}$$

Proposition 2.8. *A symmetric matrix* $B \in \mathcal{S}_n$ *with the property (2.13) is positive semidefinite.*

Therefore, quadratic L$^\natural$-convex functions of the form (2.7) are convex functions in the ordinary sense.

The following properties follow immediately from Theorems 2.2 and 2.7.

Proposition 2.9.
(i) $\overline{\mathcal{L}}$ *is properly contained in* $\overline{\mathcal{L}}^\natural$.
(ii) *For a matrix* $B \in \overline{\mathcal{L}}^\natural$, *any principal submatrix of* B *belongs to* $\overline{\mathcal{L}}^\natural$.

We next state the block-diagonal structure of matrices in $\overline{\mathcal{L}}^\natural$.

Theorem 2.10. *A symmetric matrix* $B \in \mathcal{S}_n$ *belongs to* $\overline{\mathcal{L}}^\natural$ *if and only if there exists a subpartition* $\{N_k \subseteq N \mid k = 1, 2, \cdots, m\}$ *of* N *such that*

$$\begin{aligned}
&\operatorname{rank} B[N_k, N_k] \in \mathcal{L} \qquad (k \in \{1, \cdots, m\}),\\
&\operatorname{rank} B[N_{m+1}, N_{m+1}] \in \mathcal{L}^\natural,\\
&B[N_k, N_l] = O \qquad (k, l \in \{1, 2, \cdots, m, m+1\},\ k \neq l),
\end{aligned}$$

where $m = n - \operatorname{rank} B$ *and* $N_{m+1} = N \setminus \bigcup_{l=1}^{m} N_l$.

We then consider an L$^\natural$-convex function of the general form (2.4) with a symmetric matrix $B \in \mathcal{S}_n$ and a linear subspace $K \subseteq \mathbf{R}^n$.

Theorem 2.11. *Let* $B \in \mathcal{S}_n$, *and* $K \subseteq \mathbf{R}^n$ *be a linear subspace. Then, the quadratic function* $g : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ *over* K, *given by (2.4), is L$^\natural$-convex if and only if there exists a subpartition* $\{N_k \mid k = 1, 2, \cdots, d\}$ *of* N *such that* K *is given as (2.10) and the "aggregated" matrix* $\widetilde{B} \in \mathbf{R}^{d \times d}$ *defined by*

$$\widetilde{b}_{kl} = \sum_{i \in N_k} \sum_{j \in N_l} b_{ij} \quad (k, l \in \{1, 2, \cdots, d\})$$

belongs to $\overline{\mathcal{L}}^\natural$, *where* $d = \dim K$.

2.3 Quadratic M-convex/M$^\natural$-convex Functions

For a symmetric matrix $A \in \mathcal{S}_n$ and a linear subspace $H \subseteq \mathbf{R}^n$, we define a quadratic function $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ over H by

$$f(x) = \begin{cases} \dfrac{1}{2} x^{\mathrm{T}} A x & (x \in H), \\ +\infty & (x \notin H). \end{cases} \tag{2.14}$$

We call such a quadratic function f *M-convex* if it satisfies the property (M-EXC):

(M-EXC) $\forall x, y \in \operatorname{dom} f$, $\forall i \in \operatorname{supp}^+(x - y)$, $\exists j \in \operatorname{supp}^-(x - y)$, $\exists \alpha_0 > 0$:

$$f(x) + f(y) \geq f(x - \alpha(\chi_i - \chi_j)) + f(y + \alpha(\chi_i - \chi_j)) \qquad (0 < \forall \alpha \leq \alpha_0).$$

It is easy to see that the effective domain $\mathrm{dom}\, f = H$ satisfies the following property:

(B-EXC) $\forall x, y \in H$, $\forall i \in \mathrm{supp}^+(x-y)$, $\exists j \in \mathrm{supp}^-(x-y)$, $\exists \alpha_0 > 0$:

$$x - \alpha(\chi_i - \chi_j) \in H, \quad y + \alpha(\chi_i - \chi_j) \in H \qquad (0 < \forall \alpha \leq \alpha_0).$$

We call a linear space *M-convex* if it satisfies the property (B-EXC). An M-convex linear space is contained in a hyperplane of the form $\{x \in \mathbf{R}^n \mid x(N) = 0\}$ (see, e.g., [5]).

A quadratic function $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ given by (2.14) is called M$^\natural$-*convex* if the function $\widehat{f} : \mathbf{R}^{\widehat{N}} \to \mathbf{R} \cup \{+\infty\}$, where $\widehat{N} = \{0\} \cup N$, defined by

$$\widehat{f}(x_0, x) = \begin{cases} f(x) & (x_0 = -x(N)), \\ +\infty & (x_0 \neq -x(N)), \end{cases} \qquad ((x_0, x) \in \mathbf{R}^{\widehat{N}}) \tag{2.15}$$

is M-convex. Then, the linear subspace $\widehat{H} \subseteq \mathbf{R}^{\widehat{N}}$ defined by

$$\widehat{H} = \{(x_0, x) \in \mathbf{R}^{\widehat{N}} \mid x \in H,\ x_0 = -x(N)\} \tag{2.16}$$

is M-convex. We call a linear subspace $H \subseteq \mathbf{R}^n$ *M$^\natural$-convex* if the linear subspace $\widehat{H}$ defined by (2.16) is M-convex. As shown below, the class of quadratic M-convex functions is (properly) contained in that of quadratic M$^\natural$-convex functions, and we mainly deal with M$^\natural$-convex functions in this paper.

For discussion, we need a characterization of linear subspaces with M-/M$^\natural$-convexity.

Proposition 2.12. *Let $H \subseteq \mathbf{R}^n$ be a nonempty set.*
(i) *H is an M-convex linear subspace if and only if there exists a partition $\{N_k \mid k = 1, 2, \cdots, m\}$ of N such that*

$$\begin{aligned} H &= \{x \in \mathbf{R}^n \mid x(N_k) = 0\ (k = 1, \cdots, m)\} \\ &= \mathrm{span}\{\chi_i - \chi_j \mid i, j \in N_k,\ k = 1, \cdots, m\}. \end{aligned} \tag{2.17}$$

(ii) *H is an M$^\natural$-convex linear subspace if and only if there exists a subpartition $\{N_k \mid k = 1, 2, \cdots, m\}$ of N such that*

$$\begin{aligned} H &= \{x \in \mathbf{R}^n \mid x(N_k) = 0\ (k = 1, \cdots, m)\} \\ &= \mathrm{span}\{\chi_i - \chi_j \mid i, j \in N_k,\ k = 1, \cdots, m\} \cup \{\chi_i \mid i \in N \setminus \bigcup_{k=1}^{m} N_k\}). \end{aligned}$$

2.3.1 M$^\natural$-convex Functions

We first consider a special case when the function f is defined over the entire space $\mathbf{R}^n$:

$$f(x) = \frac{1}{2} x^{\mathrm{T}} A x \qquad (x \in \mathbf{R}^n). \tag{2.18}$$

In particular, we first assume that A is nonsingular, and provide various characterizations of such quadratic M$^\natural$-convex functions by combinatorial properties (Theorem 2.14). Then, we consider the case when B is possibly singular, and show that such a coefficient matrix has a nice block structure (Theorem 2.17). Based on the results on quadratic M$^\natural$-convex functions of the form (2.18), we finally investigate the structure of general quadratic M$^\natural$-convex functions (Theorem 2.19).

We denote

$$\begin{aligned}\mathcal{M}^\natural &= \bigcup_{n=1}^{\infty}\{A \in \mathcal{S}_n \mid \text{the function (2.18) over } \mathbf{R}^n \text{ is M}^\natural\text{-convex},\ \operatorname{rank} A = n\},\\ \overline{\mathcal{M}}^\natural &= \bigcup_{n=1}^{\infty}\{A \in \mathcal{S}_n \mid \text{the function (2.18) over } \mathbf{R}^n \text{ is M}^\natural\text{-convex}\}.\end{aligned}$$

Let $f : \mathbf{R}^n \to \mathbf{R}$ be a quadratic function of the form (2.18) with a nonsingular $A \in \mathcal{S}_n$. We will show in Theorem 2.14 that M$^\natural$-convexity of such f is characterized by the following properties.

(M$^\natural$-EXC$^+$) $\forall x, y \in \operatorname{dom} f$, $\forall i \in \operatorname{supp}^+(x-y)$, $\exists j \in \operatorname{supp}^-(x-y) \cup \{0\}$, $\exists \alpha_0 > 0$:

$$f(x) + f(y) > f(x - \alpha(\chi_i - \chi_j)) + f(y + \alpha(\chi_i - \chi_j)) \qquad (0 < \forall \alpha \le \alpha_0),$$

(M$^\natural$-EXC$^+_\mathrm{d}$) $\quad x^\mathrm{T} a_i > \min\{0, \min_{j \in \operatorname{supp}^-(x)} x^\mathrm{T} a_j\} \qquad (\forall x \in \mathbf{R}^n,\ \forall i \in \operatorname{supp}^+(x))$,

(M$^\natural$-EXC$^+_\mathrm{d}$-RL) $\quad x^\mathrm{T} a_i > \min\{0, \min_{j \in N\setminus\{i\}} x^\mathrm{T} a_j\} \qquad (\forall x \in \mathbf{R}^n,\ \forall i \in \operatorname{supp}^+(x))$,

(L$^\natural$-INV) A is nonsingular and $A^{-1} \in \mathcal{L}^\natural$,

where a_i $(i \in N)$ denotes the i-th column of A.

We note first that the property (M$^\natural$-EXC$^+_\mathrm{d}$-RL) implies the positive definiteness of a matrix.

Proposition 2.13. *A symmetric matrix $A \in \mathcal{S}_n$ that satisfies the property* (M$^\natural$-EXC$^+_\mathrm{d}$-RL) *is positive definite.*

Proof. Let $x \in \mathbf{R}^n$ be an eigenvector of A and $\lambda \in \mathbf{R}$ be the corresponding eigenvalue. We may also assume that $\operatorname{supp}^+(x) \neq \emptyset$, and let $i_* \in \operatorname{supp}^+(x)$ minimize the value $x^\mathrm{T} a_{i_*}$. From (M$^\natural$-EXC$^+_\mathrm{d}$-RL) follows

$$\lambda x_{i_*} = x^\mathrm{T} a_{i_*} > \min\{0, \min_{j \in N\setminus\{i_*\}} x^\mathrm{T} a_j\} = \min\{0, \min_{j \in N\setminus\{i_*\}} \lambda x_j\}. \tag{2.19}$$

Moreover, we have

$$\begin{aligned}\min\{0, \min_{j \in N\setminus\{i_*\}} \lambda x_j\} &= \min\{0, \min_{j \in \operatorname{supp}^+(x)\setminus\{i_*\}} \lambda x_j, \min_{j \in N\setminus \operatorname{supp}^+(x)} \lambda x_j\}\\ &\ge \min\{0, \lambda x_{i_*}, \min_{j \in N\setminus \operatorname{supp}^+(x)} \lambda x_j\}\\ &= \min\{0, \min_{j \in N\setminus \operatorname{supp}^+(x)} \lambda x_j\},\end{aligned} \tag{2.20}$$

where the last equality is by the inequality (2.19). From (2.19) and (2.20), we have $\lambda x_{i_*} > 0$ or $\lambda x_{i_*} > \lambda x_j$ for some $j \in N \setminus \mathrm{supp}^+(x)$. Since $x_{i_*} > 0$ and $x_j \leq 0$, we have $\lambda > 0$. $\square$

Theorem 2.14. *Let $A \in \mathcal{S}_n$ be a symmetric matrix, and define a quadratic function $f : \mathbf{R}^n \to \mathbf{R}$ over $\mathbf{R}^n$ by (2.18). Then,*

$$A \in \mathcal{M}^\natural \iff (\mathrm{M}^\natural\text{-EXC}^+) \iff (\mathrm{M}^\natural\text{-EXC}^+_\mathrm{d}) \iff (\mathrm{M}^\natural\text{-EXC}^+_\mathrm{d}\text{-RL}) \iff (\mathrm{L}^\natural\text{-INV}).$$

As a corollary of Theorem 2.14, we have the following relationship between $\mathcal{L}^\natural$ and $\mathcal{M}^\natural$.

Theorem 2.15. *For $B \in \mathcal{L}^\natural$ and $A \in \mathcal{M}^\natural$, we have $B^{-1} \in \mathcal{M}^\natural$ and $A^{-1} \in \mathcal{L}^\natural$. Hence, the mappings $B \mapsto B^{-1}$ ($B \in \mathcal{L}^\natural$) and $A \mapsto A^{-1}$ ($A \in \mathcal{M}^\natural$) provide a one-to-one correspondence between $\mathcal{L}^\natural$ and $\mathcal{M}^\natural$.*

We next reveal the structure of the (possibly singular) coefficient matrix $A \in \mathcal{S}_n$ that defines a quadratic M$^\natural$-convex function over $\mathbf{R}^n$.

Proposition 2.16. *A symmetric matrix $A \in \overline{\mathcal{M}}^\natural$ is positive semidefinite.*

Theorem 2.17. *A symmetric matrix $A \in \mathcal{S}_n$ belongs to $\overline{\mathcal{M}}^\natural$ if and only if there exist a subpartition $\{N_k \mid k = 1, 2, \cdots, m\}$ of N and an $m \times m$ matrix $\widetilde{A} \in \mathcal{M}^\natural$ such that*

$$\begin{array}{ll} a_{ij} = \widetilde{a}_{kl} & (i \in N_k,\ j \in N_l,\ k, l \in \{1, \cdots, m\}), \\ a_{ij} = 0 & (i \in N \setminus \bigcup_{k=1}^m N_k \text{ or } j \in N \setminus \bigcup_{k=1}^m N_k), \end{array}$$

where $m = \mathrm{rank}\, A$.

M$^\natural$-convexity of quadratic functions over $\mathbf{R}^n$ can also be characterized by the following properties:

(M$^\natural$-EXC) $\forall x, y \in \mathrm{dom}\, f$, $\forall i \in \mathrm{supp}^+(x-y)$, $\exists j \in \mathrm{supp}^-(x-y) \cup \{0\}$, $\exists \alpha_0 > 0$:

$$f(x) + f(y) \geq f(x - \alpha(\chi_i - \chi_j)) + f(y + \alpha(\chi_i - \chi_j)) \qquad (0 < \forall \alpha \leq \alpha_0),$$

(M$^\natural$-EXC$_\mathbf{d}$) $\quad x^\mathrm{T} a_i \geq \min\{0, \min_{j \in \mathrm{supp}^-(x)} x^\mathrm{T} a_j\} \qquad (\forall x \in \mathbf{R}^n,\ \forall i \in \mathrm{supp}^+(x)).$

Theorem 2.18. *Let $A \in \mathcal{S}_n$ be a symmetric matrix, and define a quadratic function $f : \mathbf{R}^n \to \mathbf{R}$ by (2.18). Then, $A \in \overline{\mathcal{M}}^\natural \iff (\mathrm{M}^\natural\text{-EXC}) \iff (\mathrm{M}^\natural\text{-EXC}_\mathrm{d})$.*

We finally clarify the structure of the coefficient matrix of a quadratic M$^\natural$-convex function over a linear subspace.

Theorem 2.19. *Let $A \in \mathcal{S}_n$, and $H \subseteq \mathbf{R}^n$ be a linear subspace. Then, a quadratic function $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ over H, given by (2.14), is M$^\natural$-convex if and only if there exists a subpartition $\{N_k \subseteq N \mid k = 1, 2, \cdots, m\}$ of N such that*

$$H = \{x \in \mathbf{R}^n \mid x(N_k) = 0 \ (k = 1, 2, \cdots, m)\} \tag{2.21}$$

and

$$A[N_k, N_k] \in \overline{\mathcal{M}} \qquad (k = 1, 2, \cdots, m), \qquad A[N_{m+1}, N_{m+1}] \in \overline{\mathcal{M}}^\natural,$$
$$a_{ij} - a_{i'j} = a_{ij'} - a_{i'j'} \qquad (i, i' \in N_k,\ j, j' \in N_l,\ k, l \in \{1, 2, \cdots, m\},\ k \neq l), \tag{2.22}$$
$$a_{ij} = a_{ij'} \qquad (i \in N_{m+1},\ j, j' \in N_k,\ k \in \{1, 2, \cdots, m\}), \tag{2.23}$$

where $N_{m+1} = N \setminus \bigcup_{k=1}^{m} N_k$ and $m = n - \dim H$.

Remark 2.20. If the linear subspace H is given by (2.21) and the symmetric matrix A satisfies the properties (2.22) and (2.23), then the quadratic function $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ over H, given by (2.14), can be represented as

$$f(x) = \sum_{k=1}^{m+1} (x|_{N_k})^{\mathrm{T}} A[N_k, N_k] \, (x|_{N_k}) \qquad (x \in H), \tag{2.24}$$

where $x|_{N_k} \in \mathbf{R}^{N_k}$ $(k = 1, \cdots, m+1)$ denotes the restriction of $x \in \mathbf{R}^n$ to N_k (see Section 5.2.6). This fact shows that the coefficient matrix A is effectively a block-diagonal matrix. □

2.3.2 M-convex Functions

Theorems 2.18 and 2.19 above for M$^\natural$-convex functions are rephrased for M-convex functions as follows.

For our study, it is convenient to consider a special case of the following form:

$$f(x) \ = \ \begin{cases} \dfrac{1}{2} x^{\mathrm{T}} A x & (x \in \mathbf{R}^n,\ x(N) = 0), \\ +\infty & (x \in \mathbf{R}^n,\ x(N) \neq 0). \end{cases} \tag{2.25}$$

We denote

$$\overline{\mathcal{M}} \ = \ \bigcup_{n=1}^{\infty} \{A \in \mathcal{S}_n \mid \text{the function (2.25) over } \{x \in \mathbf{R}^n \mid x(N) = 0\} \text{ is M-convex}\}.$$

The following properties are easy to see from the definitions.

Proposition 2.21.
(i) *$A \in \mathcal{S}_n$ belongs to $\overline{\mathcal{M}}$ if and only if $A' \in \mathcal{S}_{n-1}$ given by*

$$a'_{ij} = a_{ij} - a_{in} - a_{nj} + a_{nn} \qquad (i, j = 1, \cdots, n-1) \tag{2.26}$$

belongs to $\overline{\mathcal{M}}^\natural$*.*
(ii) $A \in \mathcal{S}_n$ *belongs to* $\overline{\mathcal{M}}^\natural$ *if and only if the symmetric matrix* $\widehat{A} \in \mathbf{R}^{\widehat{N} \times \widehat{N}}$ *given by*

$$\widehat{a}_{ij} = \begin{cases} a_{ij} & (i, j \in N), \\ 0 & (i = 0 \text{ or } j = 0) \end{cases} \tag{2.27}$$

belongs to $\overline{\mathcal{M}}$*, where* $\widehat{N} = \{0\} \cup N$*.*

Proof. Let $A \in \mathcal{S}_n$. For any $x \in \mathbf{R}^n$ with $x(N) = 0$, we have

$$x^{\mathrm{T}} A x = \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} (a_{ij} - a_{in} - a_{nj} + a_{nn}) x_i x_j,$$

which, together with the definitions of M-/M$^\natural$-convexity, yields (i). The claim (ii) follows immediately from (i). □

Proposition 2.22.
(i) *For a matrix* $A \in \overline{\mathcal{M}}$*, any principal submatrix of* A *also belongs to* $\overline{\mathcal{M}}$*.*
(ii) *For a matrix* $A \in \overline{\mathcal{M}}^\natural$*, any principal submatrix of* A *also belongs to* $\overline{\mathcal{M}}^\natural$*.*

Proof. We prove (i) only since (ii) can be shown in the similar way.

For $I \subseteq N$, we have $A[I, I] \in \overline{\mathcal{M}}$ if and only if the following property holds:

> $\forall x, y \in \mathbf{R}^n$ with $x(N) = y(N) = 0$ and $x_i = y_i = 0$ $(i \in I)$, $\forall i \in \mathrm{supp}^+(x - y)$, $\exists j \in \mathrm{supp}^-(x - y)$, $\exists \alpha_0 > 0$:
>
> $$f(x) + f(y) \geq f(x - \alpha(\chi_i - \chi_j)) + f(y + \alpha(\chi_i - \chi_j)) \qquad (0 < \forall \alpha \leq \alpha_0),$$

which is implied by (M-EXC) for f. □

We first give a characterization of quadratic M-convex functions $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ over the hyperplane $\{x \in \mathbf{R}^n \mid x(N) = 0\}$ given by (2.25). We show that M-convexity for such f can be characterized by the following property:

$$\textbf{(M-EXC}_\mathbf{d}\textbf{)} \quad x^{\mathrm{T}} a_i \geq \min_{j \in \mathrm{supp}^-(x)} x^{\mathrm{T}} a_j \qquad (\forall x \in \mathbf{R}^n \text{ with } x(N) = 0,\ \forall i \in \mathrm{supp}^+(x)).$$

Theorem 2.23. *Let* $A \in \mathcal{S}_n$ *be a symmetric matrix, and define a quadratic function* $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ *by (2.25). Then,* $A \in \overline{\mathcal{M}} \iff$ (M-EXC$_\mathrm{d}$).

We next state the structure of a quadratic M-convex function over a linear subspace.

Theorem 2.24. *Let $A \in \mathcal{S}_n$, and $H \in \mathbf{R}^n$ be a linear subspace. Then, a quadratic function $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ over H, given by (2.14), is M-convex if and only if there exists a partition $\{N_k \mid k = 1, 2, \cdots, m\}$ of N such that H is represented as (2.21) and*

$$\begin{aligned} & A[N_k, N_k] \in \overline{\mathcal{M}} \qquad (k = 1, 2, \cdots, m), \\ & a_{ij} - a_{i'j} = a_{ij'} - a_{i'j'} \qquad (i, i' \in N_k,\ j, j' \in N_l,\ k, l \in \{1, 2, \cdots, m\},\ k \neq l), \end{aligned} \tag{2.28}$$

where $m = n - \dim H$.

Remark 2.25. If the linear subspace H is given by (2.21) and the symmetric matrix A satisfies the property (2.28), then the quadratic function $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ over H, given by (2.14), can be represented as (2.24) (see Section 5.2.6). □

Remark 2.26. By definitions, we have $\overline{\mathcal{M}^\natural} \subseteq \overline{\mathcal{M}}$. On the other hand, the reverse inclusion does not hold. For example, the matrix $A \in \mathcal{S}_3$ given by

$$A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 0 & -3 \\ 0 & -3 & 0 \end{bmatrix}$$

belongs to $\overline{\mathcal{M}}$ since

$$x^{\mathrm{T}} A x = 2x_1^2 + 3x_2^2 + 3x_3^2 \quad \text{if} \quad x_1 + x_2 + x_3 = 0$$

while $A \notin \overline{\mathcal{M}^\natural}$ by Proposition 2.16. □

3 Conjugacy Between Quadratic L-convex and M-convex Functions

In this section, we investigate the conjugacy relationship between quadratic L-/L$^\natural$-convex and M-/M$^\natural$-convex functions.

3.1 Review of the Fundamental Results on Conjugacy

For any function $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$, we define a function $f^\bullet : \mathbf{R}^n \to \mathbf{R} \cup \{\pm\infty\}$ by

$$f^\bullet(p) = \sup_{x \in \mathbf{R}^n} \{p^{\mathrm{T}} x - f(x)\} \qquad (p \in \mathbf{R}^n).$$

We call the transformation from f to $f^\bullet$ the *Legendre–Fenchel transformation*, and the function $f^\bullet$ is called the *conjugate* of f. We have $(f^\bullet)^\bullet = f$ for a closed convex function. See [22, 27] for details on the conjugacy of convex functions.

Suppose that f is a quadratic convex function given by

$$f(x) = \begin{cases} \frac{1}{2} x^{\mathrm{T}} A x & (x \in H), \\ +\infty & (x \notin H) \end{cases}$$

with a symmetric positive semidefinite matrix $A \in \mathcal{S}_n$ and a linear subspace $H \subseteq \mathbf{R}^n$. Then, the conjugate $f^{\bullet}$ is also a quadratic convex function given by

$$f^{\bullet}(p) = \begin{cases} \frac{1}{2} p^{\mathrm{T}} B p & (p \in K), \\ +\infty & (p \notin K) \end{cases}$$

with a symmetric positive semidefinite matrix $B \in \mathcal{S}_n$ and a linear subspace $K \subseteq \mathbf{R}^n$ such that

$$K = (H \cap \ker A)^{\perp}, \qquad H = (K \cap \ker B)^{\perp}, \tag{3.29}$$

where

$$\begin{aligned} X^{\perp} &= \{p \in \mathbf{R}^n \mid p^{\mathrm{T}} x = 0 \ (\forall x \in X)\} \qquad (X \subseteq \mathbf{R}^n), \\ \ker A &= \{x \in \mathbf{R}^n \mid Ax = 0\} \qquad (A \in \mathbf{R}^{n \times n}). \end{aligned}$$

We may write this also as

$$\mathrm{dom}\, f^{\bullet} = (\arg\min f)^{\perp}, \qquad \mathrm{dom}\, f = (\arg\min f^{\bullet})^{\perp}. \tag{3.30}$$

In particular, if $H = \mathbf{R}^n$ and A is positive definite, then the conjugate $f^{\bullet}$ is explicitly written as

$$f^{\bullet}(p) = \frac{1}{2} p^{\mathrm{T}} A^{-1} p \qquad (p \in \mathbf{R}^n). \tag{3.31}$$

3.2 Conjugacy Between L-convexity and M-convexity

We show that quadratic L-/L$^{\natural}$-convex and M-/M$^{\natural}$-convex functions are conjugate to each other under the Legendre–Fenchel transformation.

Theorem 3.1.
(i) *Let $B \in \mathcal{S}_n$, and $K \subseteq \mathbf{R}^n$ be a linear subspace. Suppose that a quadratic function $g : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ over K, given by (2.4), is L-convex (resp. L$^{\natural}$-convex). Then, the conjugate $g^{\bullet} : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ is a quadratic M-convex (resp. M$^{\natural}$-convex) function defined over an M-convex (resp. M$^{\natural}$-convex) linear subspace.*
(ii) *Let $A \in \mathcal{S}_n$, and $H \subseteq \mathbf{R}^n$ be a linear subspace. Suppose that a quadratic function $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ over H, given by (2.14), is M-convex (resp. M$^{\natural}$-convex). Then, the conjugate $f^{\bullet} : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ is a quadratic L-convex (resp. L$^{\natural}$-convex) function defined over an L-convex (resp. L$^{\natural}$-convex) linear subspace.*

Note that we have already seen the essence of the above conjugacy in Theorem 2.15 and (3.31). The former reveals the inverse-matrix relationship between the coefficient matrices of quadratic L$^\natural$-convex and M$^\natural$-convex functions, and the latter shows a fundamental fact that the coefficient matrices of a conjugate pair of quadratic convex functions are the inverse of each other. The proof of Theorem 3.1, to be given below, is a rather straightforward argument to build up the conjugacy in this basic case by means of the structure theorems such as Theorems 2.10 and 2.19 for the general case. We also use the following (easy) fact, indicating the conjugacy between L-/L$^\natural$-convex and M-/M$^\natural$-convex linear subspaces.

Proposition 3.2. *For a pair of linear subspaces*

$$K = \mathrm{span}\{\chi_{N_k} \mid k = 1, 2, \cdots, m\}, \qquad H = \{x \in \mathbf{R}^n \mid x(N_k) = 0 \ (k = 1, 2, \cdots, m)\}$$

defined by a (sub)partition $\{N_k \mid k = 1, 2, \cdots, m\}$ *of* N*, we have* $K = H^\perp$ *and* $H = K^\perp$*.*

3.2.1 Conjugate of Quadratic L-convex Functions

Suppose that a quadratic function $g : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ over K given by (2.4) with $B \in \mathcal{S}_n$ is L-convex. We show that the conjugate $g^\bullet$ is a quadratic M-convex function. To do this, we consider the following three cases:

Case 1: $K = \mathbf{R}^n$ and $\mathrm{rank} B = n - 1$, i.e., $B \in \mathcal{L}$.
Case 2: $K = \mathbf{R}^n$ and $\mathrm{rank} B$ is not necessarily $n - 1$, i.e., $B \in \overline{\mathcal{L}}$.
Case 3: K is a general L-convex linear subspace.

[Case 1: $K = \mathbf{R}^n$, $B \in \mathcal{L}$] The submatrix $B' = B[N \setminus \{n\}, N \setminus \{n\}] \in \mathcal{S}_{n-1}$ belongs to $\mathcal{L}^\natural$ by Proposition 2.6, and $\mathrm{rank}\, B' = n - 1$ in particular. We show that the conjugate $f = g^\bullet : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ of g is given by (2.14) with a symmetric matrix $A \in \mathcal{S}_n$ and a linear subspace $H \subseteq \mathbf{R}^n$ such that

$$\begin{aligned} A &= \left[\begin{array}{ccc|c} & & & 0 \\ & (B')^{-1} & & \vdots \\ & & & 0 \\ \hline 0 & \cdots & 0 & 0 \end{array}\right], \qquad (3.32) \\ H &= \{x \in \mathbf{R}^n \mid x(N) = 0\}. \end{aligned}$$

We have $\arg\min g = \{\lambda \mathbf{1} \mid \lambda \in \mathbf{R}\}$. Hence, $\mathrm{dom}\, g^\bullet = H$ follows from (3.30) and Proposition 3.2. For $x \in \mathrm{dom}\, g^\bullet$, it holds that

$$\begin{aligned} g^\bullet(x) &= \sup\{p^{\mathrm{T}} x - \frac{1}{2} p^{\mathrm{T}} B p \mid p \in \mathbf{R}^n\} \\ &= \sup\{p^{\mathrm{T}} x - \frac{1}{2} p^{\mathrm{T}} B p \mid p \in \mathbf{R}^n, \ p_n = 0\} \\ &= \sup\{(p')^{\mathrm{T}} x' - \frac{1}{2} (p')^{\mathrm{T}} B' p' \mid p' \in \mathbf{R}^{n-1}\} \ = \ \frac{1}{2} (x')^{\mathrm{T}} (B')^{-1} x', \end{aligned}$$

where $x' = x|_{N\setminus\{n\}} \in \mathbf{R}^{n-1}$ is the restriction of the vector x to $N \setminus \{n\}$. Hence, we have the claim.

By Proposition 2.21 (ii) and Theorem 2.15, the matrix $A \in \mathcal{S}_n$ given by (3.32) belongs to $\overline{\mathcal{M}}$, which yields the M-convexity of $g^\bullet$. We denote by ψ the mapping from $B \in \mathcal{L}$ to $A \in \overline{\mathcal{M}}$ given by (3.32).

[**Case 2:** $K = \mathbf{R}^n$, $B \in \overline{\mathcal{L}}$] By Theorem 2.4, there exists a partition $\{N_k \mid k = 1, 2, \cdots, m\}$ of N such that

$$\begin{aligned} &B_k = B[N_k, N_k] \in \mathcal{L} \quad (k \in \{1, \cdots, m\}),\\ &B[N_k, N_l] = O \quad (k, l \in \{1, \cdots, m\},\ k \neq l), \end{aligned}$$

where $m = n - \operatorname{rank} B$. Therefore, g is a convex function such that

$$\begin{aligned} g(p) &= \sum_{k=1}^{m} g_k(p|_{N_k}) \qquad (p \in \mathbf{R}^n),\\ g_k(p|_{N_k}) &= \frac{1}{2}(p|_{N_k})^{\mathrm{T}} B_k (p|_{N_k}), \end{aligned}$$

where $p|_{N_k} \in \mathbf{R}^{N_k}$ is the restriction of p to N_k. Since $g^\bullet = \sum_{k=1}^{m} g_k^\bullet$, the argument in Case 1 implies that $f = g^\bullet$ is given by (2.14) with a symmetric matrix $A \in \mathcal{S}_n$ and a linear subspace $H \subseteq \mathbf{R}^n$ such that

$$\begin{aligned} A &= \begin{bmatrix} \psi(B_1) & O & \cdots & O \\ O & \psi(B_2) & & O \\ \vdots & & \ddots & \vdots \\ O & O & \cdots & \psi(B_m) \end{bmatrix},\\ H &= \{x \in \mathbf{R}^n \mid x(N_k) = 0 \ (k = 1, \cdots, m)\}. \end{aligned}$$

Since $\psi(B_k) \in \overline{\mathcal{M}}$ for all $k = 1, \cdots, m$, Theorem 2.24 implies that the conjugate $g^\bullet$ is a quadratic M-convex function.

[**Case 3: general** K] By Theorem 2.5, there exists a partition $\{N_k \mid k = 1, 2, \cdots, d\}$ of N such that

$$K = \operatorname{span}\{\chi_{N_k} \mid k = 1, 2, \cdots, d\} = \Big\{ \sum_{k=1}^{d} q_k \chi_{N_k} \mid q \in \mathbf{R}^d \Big\}$$

and the "aggregated" matrix $\widetilde{B} \in \mathcal{S}_d$ defined by

$$\widetilde{b}_{kl} = \sum_{i \in N_k} \sum_{j \in N_{k'}} b_{ij} \qquad (k, k' \in \{1, 2, \cdots, d\})$$

belongs to $\overline{\mathcal{L}}$, where $d = \dim K$. By Theorem 2.4, there exists a partition $\{I_l \mid l = 1, 2, \cdots, m\}$ of the set $\{1, \cdots, d\}$ such that

$$\begin{array}{l} \widetilde{B}_l = \widetilde{B}[I_l, I_l] \in \mathcal{L} \quad (l \in \{1, \cdots, m\}), \\ \widetilde{B}[I_l, I_{l'}] = O \quad (l, l' \in \{1, \cdots, m\},\ l \neq l'). \end{array}$$

Define $\widetilde{g} : \mathbf{R}^d \to \mathbf{R}$ by $\widetilde{g}(q) = (1/2)q^{\mathrm{T}}\widetilde{B}q$ ($q \in \mathbf{R}^d$). For $x \in \mathbf{R}^n$, it holds that

$$\begin{array}{rcl} g^{\bullet}(x) & = & \sup\{p^{\mathrm{T}}x - g(p) \mid p \in K\} \\ & = & \sup\{(\sum_{k=1}^{d} q_k \chi_{N_k})^{\mathrm{T}} x - \frac{1}{2}(\sum_{k=1}^{d} q_k \chi_{N_k})^{\mathrm{T}} B (\sum_{k=1}^{d} q_k \chi_{N_k}) \mid q \in \mathbf{R}^d\} \\ & = & \sup\{\sum_{k=1}^{d} q_k x(N_k) - \frac{1}{2} q^{\mathrm{T}} \widetilde{B} q \mid q \in \mathbf{R}^d\} \\ & = & \sup\{q^{\mathrm{T}}y - \widetilde{g}(q) \mid q \in \mathbf{R}^d\} \ = \ \widetilde{g}^{\bullet}(y), \end{array}$$

where $y \in \mathbf{R}^d$ is the vector with $y_k = x(N_k)$ ($k = 1, \cdots, d$) and $\widetilde{g}^{\bullet} = (\widetilde{g})^{\bullet}$. From the argument in Case 2, the conjugate $\widetilde{g}^{\bullet}$ is written as

$$\begin{array}{l} \mathrm{dom}\, \widetilde{g}^{\bullet} = \{y \in \mathbf{R}^d \mid y(I_l) = 0 \ (l = 1, \cdots, m)\}, \\ \widetilde{g}^{\bullet}(y) = (1/2) y^{\mathrm{T}} \widetilde{A} y \qquad (y \in \mathrm{dom}\, \widetilde{g}^{\bullet}) \end{array}$$

with $\widetilde{A} \in \mathcal{S}_d$ given by

$$\widetilde{A} = \begin{bmatrix} \psi(\widetilde{B}_1) & O & \cdots & O \\ O & \psi(\widetilde{B}_2) & & O \\ \vdots & & \ddots & \vdots \\ O & O & \cdots & \psi(\widetilde{B}_m) \end{bmatrix}.$$

Note that $\psi(\widetilde{B}_l) \in \overline{\mathcal{M}}$ for all $l = 1, \cdots, d$. Hence, $f = g^{\bullet}$ is given by (2.14) with a symmetric matrix $A \in \mathcal{S}_n$ and a linear subspace $H \subseteq \mathbf{R}^n$ such that

for any $k, k' \in \{1, \cdots, d\}$, each element in $A[N_k, N_{k'}]$ is equal to $\widetilde{a}_{kk'}$,
$H = \{x \in \mathbf{R}^n \mid x(\bigcup_{k \in I_l} N_k) = 0 \ (l = 1, \cdots, m)\}$.

From this result, it is easy to check the M-convexity of $g^{\bullet}$.

3.2.2 Conjugate of Quadratic M-convex Functions

Suppose that a quadratic function $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ over H given by (2.14) with $A \in \mathcal{S}_n$ is M-convex. We show that the conjugate $f^{\bullet}$ is a quadratic L-convex function. To do this, we consider the following two cases:

Case 1: $H = \{x \in \mathbf{R}^n \mid x(N) = 0\}$.
Case 2: H is a general M-convex linear subspace.

[Case 1: $H = \{x \in \mathbf{R}^n \mid x(N) = 0\}$] By Proposition 2.21 (i) (and its proof), we may assume that $A \in \overline{\mathcal{M}}$ is of the following form:

$$A = \left[\begin{array}{ccc|c} & & & 0 \\ & A' & & \vdots \\ & & & 0 \\ \hline 0 & \cdots & 0 & 0 \end{array}\right], \qquad A' \in \overline{\mathcal{M}}^{\natural}. \tag{3.33}$$

In the following, we first consider the case when A' is nonsingular, and then the general case when A' is not necessarily nonsingular.

[Case 1-1: $\operatorname{rank} A' = n-1$] Since $\arg\min f = \{\mathbf{0}\}$, we have $\operatorname{dom} f^{\bullet} = \mathbf{R}^n$ by (3.30). For $p \in \mathbf{R}^n$, it holds that

$$\begin{aligned} f^{\bullet}(p) &= \sup\{p^{\mathrm{T}}x - \frac{1}{2}x^{\mathrm{T}}Ax \mid x \in \mathbf{R}^n,\ x(N) = 0\} \\ &= \sup\{(p' - p_n\mathbf{1}')^{\mathrm{T}}x' - \frac{1}{2}(x')^{\mathrm{T}}A'x' \mid x' \in \mathbf{R}^{n-1}\} \\ &= \frac{1}{2}(p' - p_n\mathbf{1}')^{\mathrm{T}}(A')^{-1}(p' - p_n\mathbf{1}'), \end{aligned}$$

where $p' \in \mathbf{R}^{n-1}$ is the restriction of p to $\{1, \cdots, n-1\}$ and $\mathbf{1}' \in \mathbf{R}^{n-1}$ is the vector with all components equal to one. Hence $g = f^{\bullet} : \mathbf{R}^n \to \mathbf{R}$ is written as (2.7) with a symmetric matrix $B \in \mathcal{S}_n$ given by

$$B = \left[\begin{array}{c|c} (A')^{-1} & (A')^{-1}\mathbf{1}' \\ \hline (\mathbf{1}')^{\mathrm{T}}(A')^{-1} & (\mathbf{1}')^{\mathrm{T}}(A')^{-1}\mathbf{1}' \end{array}\right]. \tag{3.34}$$

We have $(A')^{-1} \in \mathcal{L}^{\natural}$ by Theorem 2.15. Therefore, $B \in \mathcal{L}$ follows from its definition and Proposition 2.6 (ii).

We denote by φ the mapping from a matrix $A \in \overline{\mathcal{M}}$ of the form (3.33) with $\operatorname{rank} A' = n-1$ to a matrix $B \in \mathcal{L}$ given by (3.34).

[Case 1-2: general A'] By Theorem 2.17 (and its proof), there exist a partition $\{N_k \mid k = 1, 2, \cdots, m\}$ of N and an $m \times m$ symmetric matrix $\widetilde{A} \in \overline{\mathcal{M}}$ such that

$$\begin{aligned} &\arg\min f = \{x \in \mathbf{R}^n \mid x(N_k) = 0\ (k = 1, \cdots, m)\}, \\ &\widetilde{A}[\{1, \cdots, m-1\}, \{1, \cdots, m-1\}] \in \mathcal{M}^{\natural}, \\ &\widetilde{a}_{km} = \widetilde{a}_{mk} = 0 \qquad (k = 1, \cdots, m), \\ &a_{ij} = \widetilde{a}_{kl} \qquad (i \in N_k,\ j \in N_l,\ k, l \in \{1, \cdots, m\}). \end{aligned}$$

Hence, (3.30) and Proposition 3.2 yield dom $f^\bullet = \mathrm{span}\{\chi_{N_k} \mid k = 1, \cdots, m\}$.

Define $\widetilde{f} : \mathbf{R}^m \to \mathbf{R} \cup \{+\infty\}$ by

$$\widetilde{f}(y) = \begin{cases} (1/2) y^{\mathrm{T}} \widetilde{A} y & (y \in \mathbf{R}^m,\ \sum_{k=1}^m y_k = 0), \\ +\infty & (\text{otherwise}). \end{cases}$$

For $p = \sum_{k=1}^m q_k \chi_{N_k} \in \mathrm{dom}\, f^\bullet$ with $q \in \mathbf{R}^d$, it holds that

$$\begin{aligned} f^\bullet(p) &= \sup\{p^{\mathrm{T}} x - \frac{1}{2} x^{\mathrm{T}} A x \mid x \in \mathbf{R}^n,\ x(N) = 0\} \\ &= \sup\{\sum_{k=1}^m q_k x(N_k) - \frac{1}{2} \sum_{k=1}^m \sum_{l=1}^m \widetilde{a}_{kl}\, x(N_k)\, x(N_l) \mid x \in \mathbf{R}^n,\ x(N) = 0\} \\ &= \sup\{q^{\mathrm{T}} y - \widetilde{f}(y) \mid y \in \mathbf{R}^m\} \ = \ \widetilde{f}^\bullet(q), \end{aligned}$$

where $y \in \mathbf{R}^m$ is the vector with $y_k = x(N_k)$ $(k = 1, \cdots, m)$ and $\widetilde{f}^\bullet = (\widetilde{f})^\bullet$. From the argument in Case 1-1, the conjugate $\widetilde{f}^\bullet$ is given by

$$\widetilde{f}^\bullet(q) = \frac{1}{2} q^{\mathrm{T}} \widetilde{B} q \qquad (q \in \mathbf{R}^m),$$

where $\widetilde{B} \in \mathcal{L}$ is given by $\widetilde{B} = \varphi(\widetilde{A})$. Hence, $g = f^\bullet$ is written as (2.4) with a symmetric matrix $B \in \mathcal{S}_n$ and a linear subspace $K \subseteq \mathbf{R}^n$ such that

for $k, l \in \{1, \cdots, m\}$, the sum $\displaystyle\sum_{i \in N_k} \sum_{j \in N_l} b_{ij}$ is equal to the (k,l)-component of $\varphi(\widetilde{A})$,

$K = \mathrm{span}\{\chi_{N_k} \mid k = 1, \cdots, m\}$.

From this fact and Theorem 2.5 follows the L-convexity of $f^\bullet$.

[**Case 2: general** H] By Theorem 2.24, there exists a partition $\{N_k \mid k = 1, 2, \cdots, m\}$ of N such that

$$\begin{aligned} & H = \{x \in \mathbf{R}^n \mid x(N_k) = 0\ (k = 1, 2, \cdots, m)\}, \\ & A_k = A[N_k, N_k] \in \overline{\mathcal{M}} \qquad (k = 1, 2, \cdots, m), \\ & a_{ij} - a_{i'j} = a_{ij'} - a_{i'j'} \qquad (i, i' \in N_k,\ j, j' \in N_l,\ k, l \in \{1, 2, \cdots, m\},\ k \neq l). \end{aligned}$$

Therefore, the quadratic M-convex function $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ is given as

$$\begin{aligned} f(x) &= \sum_{k=1}^m f_k(x|_{N_k}) \qquad (x \in \mathbf{R}^n), \\ f_k(y) &= \begin{cases} (1/2) y^{\mathrm{T}} A_k y & (y \in \mathbf{R}^{N_k},\ y(N_k) = 0), \\ +\infty & (\text{otherwise}) \end{cases} \end{aligned}$$

(see Remark 2.25). Therefore, we can reduce this case to Case 1-2. For $k = 1, \cdots, m$, the conjugate $(f_k)^\bullet : \mathbf{R}^{N_k} \to \mathbf{R} \cup \{+\infty\}$ of f_k is a quadratic L-convex function given as

$$(f_k)^\bullet(q) = \begin{cases} (1/2)q^{\mathrm{T}} B_k q & (q = \sum_{l \in I_k} \lambda_l \chi_{N_{kl}} \in \mathbf{R}^{N_k}), \\ +\infty & (\text{otherwise}), \end{cases}$$

with a symmetric matrix $B_k \in \mathcal{S}_{|N_k|}$ and a partition $\{N_{kl} \mid l \in I_k\}$ of N_k. Hence, the conjugate $f^\bullet$ is also a quadratic L-convex function given as the sum of the conjugate functions $(f_k)^\bullet$:

$$f^\bullet(p) = \sum_{k=1}^{m} (f_k)^\bullet(p|_{N_k}) \qquad (p \in \mathbf{R}^n).$$

3.2.3 Conjugate of Quadratic L$^\natural$-convex and M$^\natural$-convex Functions

Conjugacy between quadratic L$^\natural$-convex and M$^\natural$-convex functions can be derived from the arguments in Sections 3.2.1 and 3.2.2.

Suppose that a quadratic function $g : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ over K given by (2.4) with $B \in \mathcal{S}_n$ is L$^\natural$-convex. Then, the function $\widehat{g} : \mathbf{R}^{\widehat{N}} \to \mathbf{R} \cup \{+\infty\}$, where $\widehat{N} = \{0\} \cup N$, defined by (2.5) is L-convex. From the argument in Section 3.2.1, we see that the conjugate $(\widehat{g})^\bullet$ is a quadratic M-convex function. For $x \in \mathbf{R}^n$, we have

$$\begin{aligned} g^\bullet(x) &= \sup\{q^{\mathrm{T}} x - g(q) \mid q \in \mathbf{R}^n\} \\ &= \sup\{(p - p_0\mathbf{1})^{\mathrm{T}} x - g(p - p_0\mathbf{1}) \mid (p_0, p) \in \mathbf{R}^{\widehat{N}}\} \\ &= \sup\{(p_0, p)^{\mathrm{T}}(-x(N), x) - \widehat{g}(p_0, p) \mid (p_0, p) \in \mathbf{R}^{\widehat{N}}\} \\ &= (\widehat{g})^\bullet(-x(N), x), \end{aligned}$$

which, together with the definition of M$^\natural$-convexity, implies that $g^\bullet$ is a quadratic M$^\natural$-convex.

We next suppose that a quadratic function $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ over H given by (2.14) with $A \in \mathcal{S}_n$ is M$^\natural$-convex. Then, the function $\widehat{f} : \mathbf{R}^{\widehat{N}} \to \mathbf{R} \cup \{+\infty\}$ defined by (2.15) is M-convex. From the argument in Section 3.2.2, we see that the conjugate $(\widehat{f})^\bullet$ is a quadratic L-convex function. For $p \in \mathbf{R}^n$, we have

$$\begin{aligned} f^\bullet(p) &= \sup\{p^{\mathrm{T}} x - f(x) \mid x \in \mathbf{R}^n\} \\ &= \sup\{(0, p)^{\mathrm{T}}(x_0, x) - \widehat{f}(x_0, x) \mid (x_0, x) \in \mathbf{R}^{\widehat{N}}\} \\ &= (\widehat{f})^\bullet(0, p_0), \end{aligned}$$

which, together with the definition of L$^\natural$-convexity, implies that $f^\bullet$ is a quadratic L$^\natural$-convex.

4 Quadratic L-convex and M-convex Functions over the Integer Lattice

In this section, we reveal the structure of quadratic L-convex and M-convex functions defined over the integer lattice $\mathbf{Z}^n$. This is to indicate a subtlety between the discrete case and the continuous case.

4.1 L-convex/L$^\natural$-convex Functions

We denote by $\mathbf{Z}$ the set of integers. For a symmetric matrix $B \in \mathcal{S}_n$, we define a quadratic function $g : \mathbf{Z}^n \to \mathbf{R}$ by

$$g(p) = \frac{1}{2} p^{\mathrm{T}} B p \qquad (p \in \mathbf{Z}^n). \tag{4.35}$$

We call such a function *L-convex* if it satisfies the properties (LF1[$\mathbf{Z}$]) and (LF2[$\mathbf{Z}$]):

> **(LF1[Z])** g is submodular, i.e., $g(p)+g(q) \geq g(p \wedge q)+g(p \vee q) \qquad (\forall p, q \in \mathrm{dom}_{\mathbf{Z}}\, g)$,
> **(LF2[Z])** $g(p+\lambda\mathbf{1}) = g(p) \qquad (\forall p \in \mathrm{dom}_{\mathbf{Z}}\, g,\ \forall \lambda \in \mathbf{Z})$,

where $\mathrm{dom}_{\mathbf{Z}}\, g = \{p \in \mathbf{Z}^n \mid g(p) < +\infty\}$.

A quadratic function $g : \mathbf{Z}^n \to \mathbf{R}$ given by (4.35) is called *L$^\natural$-convex* if the function $\widehat{g} : \mathbf{Z}^{\widehat{N}} \to \mathbf{R}$, where $\widehat{N} = \{0\} \cup N$, defined by

$$\widehat{g}(p_0, p) = g(p - p_0\mathbf{1}) \qquad ((p_0, p) \in \mathbf{Z}^{\widehat{N}})$$

is L-convex.

We denote

$$\begin{aligned} \overline{\mathcal{L}}[\mathbf{Z}] &= \bigcup_{n=1}^{\infty} \{B \in \mathcal{S}_n \mid \text{the function (4.35) over } \mathbf{Z}^n \text{ is L-convex}\}, \\ \overline{\mathcal{L}}^\natural[\mathbf{Z}] &= \bigcup_{n=1}^{\infty} \{B \in \mathcal{S}_n \mid \text{the function (4.35) over } \mathbf{Z}^n \text{ is L}^\natural\text{-convex}\}. \end{aligned}$$

L-/L$^\natural$-convexity of quadratic functions over $\mathbf{Z}^n$ can be characterized by the same properties as those over $\mathbf{R}^n$. Proofs are quite similar and therefore omitted.

Theorem 4.1. *A symmetric matrix $B \in \mathcal{S}_n$ belongs to $\overline{\mathcal{L}}[\mathbf{Z}]$ if and only if it satisfies the property (2.8).*

Theorem 4.2. *A symmetric matrix $B \in \mathcal{S}_n$ belongs to $\overline{\mathcal{L}}^\natural[\mathbf{Z}]$ if and only if it satisfies the property (2.13).*

Corollary 4.3. $\overline{\mathcal{L}}[\mathbf{Z}] = \overline{\mathcal{L}}$ *and* $\overline{\mathcal{L}}^{\natural}[\mathbf{Z}] = \overline{\mathcal{L}}^{\natural}$.

Proposition 4.4.
(i) $\overline{\mathcal{L}}[\mathbf{Z}]$ *is properly contained in* $\overline{\mathcal{L}}^{\natural}[\mathbf{Z}]$.
(ii) *For a matrix* $B \in \overline{\mathcal{L}}^{\natural}[\mathbf{Z}]$, *any principal submatrix of* B *belongs to* $\overline{\mathcal{L}}^{\natural}[\mathbf{Z}]$.

4.2 M-convex/M$^{\natural}$-convex Functions

For a symmetric matrix $A \in \mathcal{S}_n$, we define a quadratic function $f : \mathbf{Z}^n \to \mathbf{R} \cup \{+\infty\}$ by

$$f(x) = \begin{cases} \dfrac{1}{2}x^{\mathrm{T}}Ax & (x \in \mathbf{Z}^n,\ x(N) = 0), \\ +\infty & (x \in \mathbf{Z}^n,\ x(N) \neq 0). \end{cases} \tag{4.36}$$

We call such a quadratic function f *M-convex* if it satisfies the property (M-EXC[$\mathbf{Z}$]):

(M-EXC[Z]) $\forall x, y \in \mathrm{dom}_{\mathbf{Z}} f$, $\forall i \in \mathrm{supp}^+(x-y)$, $\exists j \in \mathrm{supp}^-(x-y)$:

$$f(x) + f(y) \geq f(x - \chi_i + \chi_j) + f(y + \chi_i - \chi_j).$$

A quadratic function $f : \mathbf{Z}^n \to \mathbf{R}$

$$f(x) = \frac{1}{2}x^{\mathrm{T}}Ax \qquad (x \in \mathbf{Z}^n) \tag{4.37}$$

is called M$^{\natural}$-*convex* if the function $\widehat{f} : \mathbf{Z}^{\widehat{N}} \to \mathbf{R} \cup \{+\infty\}$, where $\widehat{N} = \{0\} \cup N$, defined by

$$\widehat{f}(x_0, x) = \begin{cases} f(x) & (x_0 = -x(N)), \\ +\infty & (x_0 \neq -x(N)), \end{cases} \qquad ((x_0, x) \in \mathbf{Z}^{\widehat{N}})$$

is M-convex. M$^{\natural}$-convex functions over $\mathbf{Z}^n$ can also be characterized by an exchange property:

(M$^{\natural}$-EXC[Z]) $\forall x, y \in \mathrm{dom}_{\mathbf{Z}} f$, $\forall i \in \mathrm{supp}^+(x-y)$, $\exists j \in \mathrm{supp}^-(x-y) \cup \{0\}$:

$$f(x) + f(y) \geq f(x - \chi_i + \chi_j) + f(y + \chi_i - \chi_j).$$

Proposition 4.5 ([19, Theorem 4.2]). *A quadratic function* $f : \mathbf{Z}^n \to \mathbf{R}$ *given by (4.37) is M$^{\natural}$-convex if and only if it satisfies* (M$^{\natural}$-EXC[$\mathbf{Z}$]).

We denote

$$\begin{aligned} \overline{\mathcal{M}}[\mathbf{Z}] &= \bigcup_{n=1}^{\infty} \{A \in \mathcal{S}_n \mid \text{the function (4.36) over } \{x \in \mathbf{Z}^n \mid x(N) = 0\} \text{ is M-convex}\}, \\ \overline{\mathcal{M}}^{\natural}[\mathbf{Z}] &= \bigcup_{n=1}^{\infty} \{A \in \mathcal{S}_n \mid \text{the function (4.37) over } \mathbf{Z}^n \text{ is M}^{\natural}\text{-convex}\}. \end{aligned}$$

We first provide a characterization of the coefficient matrices of quadratic M-convex functions over the set $\{x \in \mathbf{Z}^n \mid x(N) = 0\}$.

Theorem 4.6. *A symmetric matrix $A \in \mathcal{S}_n$ belongs to $\overline{\mathcal{M}}[\mathbf{Z}]$ if and only if it satisfies*

$$a_{ij} + a_{kl} \geq \min\{a_{ik} + a_{jl}, a_{il} + a_{jk}\} \qquad (\forall i, j, k, l \in N \text{ with } \{i, j\} \cap \{k, l\} = \emptyset). \qquad (4.38)$$

To prove this theorem, we use the characterization of M-convex functions by a localized version of the property (M-EXC[$\mathbf{Z}$]). For $x \in \mathbf{Z}^N$, we define $||x||_1 = \sum_{i=1}^n |x_i|$.

(M-EXC[Z]-loc) $\forall x, y \in \mathrm{dom}_{\mathbf{Z}} f$ with $||x - y||_1 = 4$, $\exists i \in \mathrm{supp}^+(x - y)$, $\exists j \in \mathrm{supp}^-(x - y)$:

$$f(x) + f(y) \geq f(x - \chi_i + \chi_j) + f(y + \chi_i - \chi_j).$$

Lemma 4.7 (cf. [16, Th. 3.1], [25, Th. 2]). *For a quadratic function $f : \mathbf{Z}^n \to \mathbf{R} \cup \{+\infty\}$ given by (4.36),* (M-EXC[$\mathbf{Z}$]) $\iff$ (M-EXC[$\mathbf{Z}$]-loc).

Proof of Theorem 4.6. By Lemma 4.7, we have only to show the equivalence between (M-EXC[$\mathbf{Z}$]-loc) and (4.38).

Let $x, y \in \mathrm{dom}_{\mathbf{Z}} f$ be vectors with $||x - y||_1 = 4$, and suppose that $y = x - \chi_i - \chi_j + \chi_k + \chi_l$ for some $i, j, k, l \in N$ with $\{i, j\} \cap \{k, l\} = \emptyset$. Then, (M-EXC[$\mathbf{Z}$]-loc) for x and y is equivalent to

$$f(x) + f(y) \geq \min\{f(x - \chi_i + \chi_k) + f(y + \chi_i - \chi_k), f(x - \chi_i + \chi_l) + f(y + \chi_i - \chi_l)\}. \quad (4.39)$$

We have

$$\begin{aligned} & \{f(x - \chi_i + \chi_k) - f(x)\} + \{f(y + \chi_i - \chi_k) - f(y)\} \\ = \;& (y - x)^{\mathrm{T}} A(\chi_i - \chi_k) + (\chi_i - \chi_k)^{\mathrm{T}} A(\chi_i - \chi_k) \\ = \;& (-\chi_i - \chi_j + \chi_k + \chi_l)^{\mathrm{T}} A(\chi_i - \chi_k) + (\chi_i - \chi_k)^{\mathrm{T}} A(\chi_i - \chi_k) \\ = \;& (-\chi_j + \chi_l)^{\mathrm{T}} A(\chi_i - \chi_k) \; = \; a_{il} + a_{jk} - a_{ij} - a_{kl}. \end{aligned}$$

Similarly, we have

$$\{f(x - \chi_i + \chi_l) - f(x)\} + \{f(y + \chi_i - \chi_l) - f(y)\} = a_{ik} + a_{jl} - a_{ij} - a_{kl}.$$

Hence, the inequality (4.39) is rewritten as (4.38). □

From Theorem 4.6 immediately follows a characterization of the coefficient matrices of quadratic M$^\natural$-convex functions over $\mathbf{Z}^n$.

Theorem 4.8. *A quadratic function $f : \mathbf{Z}^n \to \mathbf{R}$ given by (4.37) is M$^\natural$-convex if and only if the symmetric matrix $A \in \mathcal{S}_n$ satisfies*

$$\begin{aligned} & a_{ij} \geq 0 \qquad (\forall i, j \in N), \\ & a_{ij} \geq \min\{a_{ik}, a_{jk}\} \qquad (\forall i, j, k \in N \text{ with } \{i, j\} \cap \{k\} = \emptyset), \\ & a_{ij} + a_{kl} \geq \min\{a_{ik} + a_{jl}, a_{il} + a_{jk}\} \qquad (\forall i, j, k, l \in N \text{ with } \{i, j\} \cap \{k, l\} = \emptyset). \end{aligned}$$

Proof. Put $\widehat{N} = \{0\} \cup N$, and let $\widehat{A} \in \mathbf{R}^{\widehat{N} \times \widehat{N}}$ be a symmetric matrix given by (2.27). By the definition of M$^\natural$-convexity, we have $A \in \overline{\mathcal{M}}^\natural[\mathbf{Z}]$ if and only if $\widehat{A} \in \overline{\mathcal{M}}[\mathbf{Z}]$. Therefore, the claim follows from Theorem 4.6. $\square$

From Theorems 4.6 and 4.8 follows $\overline{\mathcal{M}}^\natural[\mathbf{Z}] \subseteq \overline{\mathcal{M}}[\mathbf{Z}]$. Moreover, we have the following properties of the classes $\overline{\mathcal{M}}[\mathbf{Z}]$ and $\overline{\mathcal{M}}^\natural[\mathbf{Z}]$.

Proposition 4.9.
(i) *For a matrix $A \in \overline{\mathcal{M}}[\mathbf{Z}]$, any principal submatrix of A also belongs to $\overline{\mathcal{M}}[\mathbf{Z}]$.*
(ii) *For a matrix $A \in \overline{\mathcal{M}}^\natural[\mathbf{Z}]$, any principal submatrix of A also belongs to $\overline{\mathcal{M}}^\natural[\mathbf{Z}]$.*
(iii) $\overline{\mathcal{M}}[\mathbf{Z}] \subseteq \overline{\mathcal{M}}$ *and* $\overline{\mathcal{M}}^\natural[\mathbf{Z}] \subseteq \overline{\mathcal{M}^\natural}$.

Proof. The statements (i) and (ii) are immediate from Theorems 4.6 and 4.8

[$\overline{\mathcal{M}}[\mathbf{Z}] \subseteq \overline{\mathcal{M}}$] Let $A \in \overline{\mathcal{M}}[\mathbf{Z}]$, and define $f : \mathbf{Z}^n \to \mathbf{R} \cup \{+\infty\}$ by (4.36). The property (M-EXC[$\mathbf{Z}$]) for f can be rewritten as

$$\min_{j \in \mathrm{supp}^-(x)} \{x^{\mathrm{T}} a_j - x^{\mathrm{T}} a_i + a_{ii} + a_{jj} - 2a_{ij}\} \leq 0 \qquad (\forall x \in \mathbf{Z}^n \text{ with } x(N) = 0,\ \forall i \in \mathrm{supp}^+(x)).$$

From Theorem 4.6 follows

$$\min_{j \in \mathrm{supp}^-(x)} \{x^{\mathrm{T}} a_j - x^{\mathrm{T}} a_i\} \leq 0 \qquad (\forall x \in \mathbf{Z}^n \text{ with } x(N) = 0,\ \forall i \in \mathrm{supp}^+(x)).$$

Since x can be any integral vector with $x(N) = 0$, the property (M-EXC$_\mathrm{d}$) holds for A, i.e., $A \in \overline{\mathcal{M}}$ by Theorem 2.23.

[$\overline{\mathcal{M}}^\natural[\mathbf{Z}] \subseteq \overline{\mathcal{M}^\natural}$] Let $A \in \mathcal{M}^\natural[\mathbf{Z}]$. By Proposition 4.5, we have

$$\min_{j \in \mathrm{supp}^-(x) \cup \{0\}} \{x^{\mathrm{T}} a_j - x^{\mathrm{T}} a_i + a_{ii} + a_{jj} - 2a_{ij}\} \leq 0 \qquad (\forall x \in \mathbf{Z}^n,\ \forall i \in \mathrm{supp}^+(x)),$$

where $a_0 = \mathbf{0}$ and $a_{i0} = a_{0i} = 0$ for $i \in N \cup \{0\}$. From Theorem 4.8 follows

$$\min_{j \in \mathrm{supp}^-(x) \cup \{0\}} \{x^{\mathrm{T}} a_j - x^{\mathrm{T}} a_i\} \leq 0 \qquad (\forall x \in \mathbf{Z}^n,\ \forall i \in \mathrm{supp}^+(x)).$$

Since x can be any integral vector, the property (M$^\natural$-EXC$_\mathrm{d}$) holds for A, i.e., $A \in \overline{\mathcal{M}}^\natural$ by Theorem 2.18. $\square$

Remark 4.10. We have $\overline{\mathcal{M}}^\natural[\mathbf{Z}] \neq \overline{\mathcal{M}}[\mathbf{Z}]$. See Remark 2.26. $\square$

Remark 4.11. We have $\overline{\mathcal{M}}[\mathbf{Z}] \neq \overline{\mathcal{M}}$ and $\overline{\mathcal{M}}^\natural[\mathbf{Z}] \neq \overline{\mathcal{M}}^\natural$. For example, the symmetric matrix $A \in \mathcal{S}_3$ given by

$$A = \begin{bmatrix} 5 & 2 & 4 \\ 2 & 5 & 3 \\ 4 & 3 & 6 \end{bmatrix}$$

belongs to $\overline{\mathcal{M}}^\natural$ and not to $\overline{\mathcal{M}}^\natural[\mathbf{Z}]$. $\square$

5 Proofs

5.1 Proofs for Quadratic L-convex/L$^\natural$-convex Functions

5.1.1 Proof of Proposition 2.1

We show the "only if" part of (i) only. The "if" part of (i) is obvious, and (ii) can be immediate from (i) and the definition of L$^\natural$-convex linear subspace.

Since K is a polyhedron satisfying (LS1) and (LS2), there exists a function $\gamma : N \times N \to \mathbf{R} \cup \{+\infty\}$ satisfying

$$\gamma(i,i) = 0 \ (i \in N), \qquad \gamma(i,j) + \gamma(j,k) \geq \gamma(i,k) \ (i,j,k \in N)$$

such that K is given as

$$K = \{p \in \mathbf{R}^n \mid p_j - p_i \leq \gamma(i,j) \ (i,j \in N)\}$$

(see [20, Theorem 3.23]). Since K is a linear subspace, γ satisfies $\gamma(i,j) = \gamma(j,i) \in \{0, +\infty\}$ $(i,j \in N)$. Hence, we have the claim.

5.1.2 Proof of Theorem 2.2

The property (LF1) for g is equivalent to the local submodularity: for any $p \in \mathbf{R}^n$, distinct $i,j \in N$, and $\lambda, \mu \geq 0$,

$$g(p + \lambda\chi_i) + g(p + \mu\chi_j) \geq g(p) + g(p + \lambda\chi_i + \mu\chi_j),$$

which is in turn equivalent to the former condition in (2.8). We see that (LF2) for g is equivalent to the latter condition in (2.8).

5.1.3 Proofs of Propositions 2.3 and 2.8

Since the property (2.8) is a special case of (2.13), we have only to prove Proposition 2.8. In fact, the positive semidefiniteness of a matrix B with the property (2.13) is well-known in numerical linear algebra [1, 28]. We afford a proof for completeness.

The proof of the claim is by induction on the size n of the matrix B. We use the fact that B is positive semidefinite if and only if the determinant of every principal submatrix of B is nonnegative. Since any principal submatrix of B satisfies (2.13), the inductive hypothesis implies that for every principal submatrix of B of size at most $n-1$, the determinant is nonnegative. Hence, it suffices to show that the determinant of B is nonnegative. Partition the matrix B as

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

with a submatrix $B_{11} \in \mathcal{S}_{n-1}$ and $B_{12}^{\mathrm{T}} = B_{21} = (b_{n1}, b_{n2}, \cdots, b_{n,n-1})$, $B_{22} = b_{nn}$. The inductive assumption implies $\det B_{11} \geq 0$. If $b_{nn} = 0$, then (2.13) yields $b_{in} = b_{ni} = 0$ for $i = 1, \cdots, n-1$, and therefore $\det B = 0$. Hence, we assume $b_{nn} > 0$. Putting $\widehat{B} = B_{11} - (1/b_{nn})B_{12}B_{21}$, we have $\det B = b_{nn} \det \widehat{B}$. For any $i, j \in \{1, \cdots, n-1\}$ with $i \neq j$, we have $\widehat{b}_{ij} = b_{ij} - b_{in}b_{nj}/b_{nn} \leq 0$ since $b_{ij}, b_{in}, b_{nj} \leq 0$ and $b_{nn} > 0$. For any $i \in \{1, \cdots, n-1\}$, we have

$$\sum_{j=1}^{n-1} \widehat{b}_{ij} = \sum_{j=1}^{n-1} b_{ij} - (b_{in}/b_{nn}) \sum_{j=1}^{n-1} b_{nj} \geq -b_{in} - (b_{in}/b_{nn}) \sum_{j=1}^{n-1} b_{nj} \geq 0.$$

Hence, $\widehat{B}$ satisfies (2.13), and the inductive hypothesis yields $\det B = b_{nn} \det \widehat{B} \geq 0$.

5.1.4 Proof of Theorem 2.4

We use the following fundamental properties of L-convex functions.

Proposition 5.1 (cf. [20, Theorem 4.34]). *Let $B \in \mathcal{S}_n$, and $K \subseteq \mathbf{R}^n$ be a linear subspace, and define a quadratic function $g : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ over K by (2.4). If g is L-convex, then the set of minimizers* $\arg\min g$ *is an L-convex linear subspace.*

Proof of Theorem 2.4. The "if" part is immediate from Theorem 2.2. In the following, we prove the "only if" part.

Suppose that $B \in \overline{\mathcal{L}}$. Then, B satisfies the condition (2.8) by Theorem 2.2. Since the set $\arg\min g$ is an L-convex linear subspace, there exists a partition $\{N_k \mid k = 1, 2, \cdots, m\}$ of N such that

$$\arg\min g = \mathrm{span}\{\chi_{N_k} \mid k = 1, \cdots, m\}$$

by Proposition 2.1 (i). Since $\ker B = \arg\min g$, we have

$$b_i^{\mathrm{T}} \chi_{N_k} = \sum_{j \in N_k} b_{ij} = 0 \qquad (i \in N,\ k = 1, \cdots, m),$$

where b_i is the i-th column of B. This property, together with (2.8), implies that the matrix B is a block-triangular matrix satisfying (2.9). In particular, the property $\mathrm{rank}\, B[N_k, N_k] = |N_k| - 1$ follows from the fact that $\ker B[N_k, N_k] = \chi_{N_k}$. $\square$

5.1.5 Proof of Theorem 2.5

We first prove the "only if" part. Suppose that g is L-convex. Then, $\mathrm{dom}\, g = K$ is an L-convex linear subspace, and therefore can be represented as

$$K = \mathrm{span}\{\chi_{N_k} \mid k = 1, 2, \cdots, d\}$$

by some partition $\{N_k \mid k = 1, 2, \cdots, d\}$ of N. Substituting $p = \sum_{k=1}^{d} q_k \chi_{N_k}$ $(q \in \mathbf{R}^d)$ in (2.4), we have

$$
\begin{aligned}
g(p) &= \frac{1}{2}\Big(\sum_{k=1}^{d} q_k\, \chi_{N_k}\Big)^{\mathrm{T}} B \Big(\sum_{l=1}^{d} q_l\, \chi_{N_l}\Big) \\
&= \frac{1}{2}\sum_{k=1}^{d}\sum_{l=1}^{d} q_k\, q_l\, \chi_{N_k}^{\mathrm{T}} B \chi_{N_l} \;=\; \frac{1}{2}\sum_{k=1}^{d}\sum_{l=1}^{d} q_k\, q_l\, \widetilde{b}_{kl}, \qquad (5.40)
\end{aligned}
$$

where $\widetilde{b}_{kl}$ is given by (2.11). We define $\widetilde{g} : \mathbf{R}^d \to \mathbf{R}$ by $\widetilde{g}(q) = (1/2)q^{\mathrm{T}}\widetilde{B}q$ $(q \in \mathbf{R}^d)$. Then, from the equation (5.40) follows that $\widetilde{g}$ is a quadratic L-convex function with $\mathrm{dom}\,\widetilde{g} = \mathbf{R}^d$. Hence, Theorem 2.4 implies $\widetilde{B} \in \overline{\mathcal{L}}$.

The "if" part is obvious from the discussion above and Proposition 2.1 (i).

5.1.6 Proofs of Theorems 2.7, 2.10, and 2.11

Suppose that a quadratic function $g : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ is given by (2.4) with a symmetric matrix $B \in \mathcal{S}_n$ and a linear subspace $K \subseteq \mathbf{R}^n$. Put $\widehat{N} = \{0\} \cup N$, and define an $(n+1) \times (n+1)$ symmetric matrix $\widehat{B} = (\widehat{b}_{ij} \mid i, j \in \widehat{N})$ by (2.12) and a linear subspace $\widehat{K} \subseteq \mathbf{R}^{\widehat{N}}$ by

$$\widehat{K} = \{(p_0, p) \in \mathbf{R}^{\widehat{N}} \mid p - p_0\mathbf{1} \in K\}.$$

By Proposition 2.1 (i), $\widehat{K}$ is an L-convex linear subspace if and only if there exists some subpartition $\{N_k \mid k = 1, 2, \cdots, m\}$ of N such that $K = \mathrm{span}\{\chi_{N_k} \mid k = 1, 2, \cdots, m\}$. We also define a function $\widehat{g} : \mathbf{R}^{\widehat{N}} \to \mathbf{R} \cup \{+\infty\}$ by

$$\widehat{g}(\widehat{p}) = \begin{cases} \dfrac{1}{2}\widehat{p}^{\mathrm{T}}\, \widehat{B}\, \widehat{p} & (\widehat{p} = (p_0, p) \in \widehat{K}), \\ +\infty & (\widehat{p} = (p_0, p) \notin \widehat{K}). \end{cases}$$

Then, we have $\widehat{g}(\widehat{p}) = g(p - p_0\mathbf{1})$ for any $\widehat{p} = (p_0, p) \in \mathbf{R}^{\widehat{N}}$. Therefore, L$^\natural$-convexity of g is equivalent to L-convexity of $\widehat{g}$. Hence, Theorems 2.7, 2.10, and 2.11 are immediate from Theorems 2.2, 2.4, and 2.5.

5.2 Proofs for Quadratic M-convex/M$^\natural$-convex Functions

5.2.1 Proof of Proposition 2.12

We prove (i) only, since (ii) follows immediately from (i) and the definition of M$^\natural$-convex linear subspaces.

It is easy to see that H defined by (2.17) satisfies (B-EXC). Hence, we prove the "only if" part only.

The property (B-EXC) for H implies that there exists a submodular set function $\rho : 2^N \to \mathbf{R} \cup \{+\infty\}$ such that

$$H = \{x \in \mathbf{R}^n \mid x(S) \le \rho(S) \ (S \subseteq N), \ x(N) = \rho(N)\}.$$

Since H is a linear subspace, the function ρ satisfies $\rho(S) \in \{0, +\infty\}$ $(S \subseteq N)$ and $\rho(\emptyset) = \rho(N) = 0$. Hence, $\mathcal{F}_0 = \{S \subseteq N \mid \rho(S) = 0\}$ satisfies the following properties:

$$\begin{aligned}
&\emptyset, N \in \mathcal{F}_0, \\
&S, T \in \mathcal{F}_0 \Longrightarrow S \cap T, S \cup T \in \mathcal{F}_0, \\
&S \in \mathcal{F}_0 \iff N \setminus S \in \mathcal{F}_0.
\end{aligned}$$

Also, H can be represented as $H = \{x \in \mathbf{R}^n \mid x(S) = 0 \ (S \in \mathcal{F}_0)\}$. These properties imply that $\mathcal{F} \subseteq 2^N$ given by

$$\mathcal{F} = \{S \in \mathcal{F}_0 \mid S \ne \emptyset, \ \text{minimal in } \mathcal{F}_0\}$$

is a partition of N, and the linear subspace can be represented as (2.17) with $\mathcal{F} = \{N_1, \cdots, N_m\}$.

5.2.2 Proof of Proposition 2.16

By the definition of M$^\natural$-convexity, the function $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ given by (2.14) is M$^\natural$-convex if and only if $\widehat{f}$ defined by (2.15) satisfies (M-EXC). A straightforward translation of this condition yields the following condition (M$^\natural$-PRJ) for f:

(M$^\natural$-PRJ)
(a) $\forall x, y \in \mathrm{dom}\, f$ with $x(N) > y(N)$, $\forall i \in \mathrm{supp}^+(x - y)$, $\exists j \in \mathrm{supp}^-(x - y) \cup \{0\}$, $\exists \alpha_0 > 0$:

$$f(x) + f(y) \ge f(x - \alpha(\chi_i - \chi_j)) + f(y + \alpha(\chi_i - \chi_j)) \qquad (0 < \forall \alpha \le \alpha_0), \quad (5.41)$$

(b) $\forall x, y \in \mathrm{dom}\, f$ with $x(N) \le y(N)$, $\forall i \in \mathrm{supp}^+(x - y)$, $\exists j \in \mathrm{supp}^-(x - y)$, $\exists \alpha_0 > 0$ satisfying (5.41),
(c) $\forall x, y \in \mathrm{dom}\, f$ with $x(N) < y(N)$, $\exists j \in \mathrm{supp}^-(x - y)$, $\exists \alpha_0 > 0$ satisfying (5.41) with $i = 0$,

where $\chi_0 = \mathbf{0}$. By using this property, we prove Proposition 2.16.

We show that any eigenvalue of A is nonnegative. Let $\lambda \in \mathbf{R}$ be an eigenvalue of A and $x \in \mathbf{R}^n$ be a corresponding eigenvector. We may assume that $\mathrm{supp}^+(x) \ne \emptyset$. We denote $a_0 = \mathbf{0}$ and $x_0 = 0$.

By applying (M$^\natural$-PRJ) to x and $\mathbf{0}$, we have some $i \in \mathrm{supp}^+(x)$, $j \in \mathrm{supp}^-(x) \cup \{0\}$, and $\alpha_0 > 0$ such that

$$\begin{aligned} & f(x - \alpha(\chi_i - \chi_j)) + f(\alpha(\chi_i - \chi_j)) - f(x) - f(\mathbf{0}) \\ = \;& \alpha x^{\mathrm{T}}(a_j - a_i) + \alpha^2(\chi_i - \chi_j)^{\mathrm{T}} A(\chi_i - \chi_j) \le 0 \qquad (0 < \forall \alpha \le \alpha_0). \end{aligned}$$

This inequality implies $x^{\mathrm{T}}(a_j - a_i) = \lambda(x_j - x_i) \le 0$ since α can be chosen to be sufficiently small. We also have $x_i > 0$ and $x_j \le 0$, from which follows $\lambda \ge 0$.

5.2.3 Proof of Theorem 2.14

To prove Theorem 2.14, we use three other properties:

(M$^\natural$-PRJ$^+$)
(a) $\forall x, y \in \mathrm{dom}\, f$ with $x(N) > y(N)$, $\forall i \in \mathrm{supp}^+(x - y)$, $\exists j \in \mathrm{supp}^-(x - y) \cup \{0\}$, $\exists \alpha_0 > 0$ such that

$$f(x) + f(y) > f(x - \alpha(\chi_i - \chi_j)) + f(y + \alpha(\chi_i - \chi_j)) \qquad (0 < \forall \alpha \le \alpha_0), \quad (5.42)$$

(b) $\forall x, y \in \mathrm{dom}\, f$ with $x(N) \le y(N)$, $\forall i \in \mathrm{supp}^+(x - y)$, $\exists j \in \mathrm{supp}^-(x - y)$, $\exists \alpha_0 > 0$ satisfying (5.42),
(c) $\forall x, y \in \mathrm{dom}\, f$ with $x(N) < y(N)$, $\exists j \in \mathrm{supp}^-(x - y)$, $\exists \alpha_0 > 0$ satisfying (5.42) with $i = 0$,

(M$^\natural$-PRJ$^+_{\mathrm{d}}$)

$$\begin{array}{lll} \text{(a)} & x^{\mathrm{T}} a_i > \min\{0, \min\limits_{j \in \mathrm{supp}^-(x)} x^{\mathrm{T}} a_j\} & (\forall x \in \mathbf{R}^n \text{ with } x(N) > 0,\ \forall i \in \mathrm{supp}^+(x)), \\ \text{(b)} & x^{\mathrm{T}} a_i > \min\limits_{j \in \mathrm{supp}^-(x)} x^{\mathrm{T}} a_j & (\forall x \in \mathbf{R}^n \text{ with } x(N) \le 0,\ \forall i \in \mathrm{supp}^+(x)), \\ \text{(c)} & 0 > \min\limits_{j \in \mathrm{supp}^-(x)} x^{\mathrm{T}} a_j & (\forall x \in \mathbf{R}^n \text{ with } x(N) < 0), \end{array}$$

(M$^\natural$-PRJ$^+_{\mathrm{d}}$-RL)

$$\begin{array}{lll} \text{(a)} & x^{\mathrm{T}} a_i > \min\{0, \min\limits_{j \in N \setminus \{i\}} x^{\mathrm{T}} a_j\} & (\forall x \in \mathbf{R}^n \text{ with } x(N) > 0,\ \forall i \in \mathrm{supp}^+(x)), \\ \text{(b)} & x^{\mathrm{T}} a_i > \min\limits_{j \in N \setminus \{i\}} x^{\mathrm{T}} a_j & (\forall x \in \mathbf{R}^n \text{ with } x(N) \le 0,\ \forall i \in \mathrm{supp}^+(x)), \\ \text{(c)} & 0 > \min\limits_{j \in N} x^{\mathrm{T}} a_j & (\forall x \in \mathbf{R}^n \text{ with } x(N) < 0). \end{array}$$

In the following, we prove the following implications in turn:

$$\begin{array}{ccccccc} A \in \mathcal{M}^\natural & & & & & & \\ \Updownarrow & & & & & & \\ (\mathrm{M}^\natural\text{-PRJ}^+) & \Longleftrightarrow & (\mathrm{M}^\natural\text{-PRJ}^+_{\mathrm{d}}) & \Longleftarrow & (\mathrm{M}^\natural\text{-PRJ}^+_{\mathrm{d}}\text{-RL}) & \Longleftrightarrow & (\mathrm{L}^\natural\text{-INV}) \\ \Downarrow & & & & \Updownarrow & \swarrow\!\!\nearrow & \\ (\mathrm{M}^\natural\text{-EXC}^+) & \Longleftrightarrow & (\mathrm{M}^\natural\text{-EXC}^+_{\mathrm{d}}) & \Longrightarrow & (\mathrm{M}^\natural\text{-EXC}^+_{\mathrm{d}}\text{-RL}). & & \end{array}$$

Lemma 5.2. $A \in \mathcal{M}^\natural \iff$ (M$^\natural$-PRJ$^+$).

Proof. We have $A \in \mathcal{M}^\natural \iff$ (M$^\natural$-PRJ). Hence, it suffices to show that if $A \in \mathcal{S}_n$ is nonsingular, then (M$^\natural$-PRJ) $\Longrightarrow$ (M$^\natural$-PRJ$^+$).

Assume that A is nonsingular and f satisfies (M$^\natural$-PRJ). Then, f is a strictly convex function by Proposition 2.16. For any $x, y \in \mathbf{R}^n$ and $i, j \in N \cup \{0\}$ we define a function $\varphi : \mathbf{R} \to \mathbf{R}$ by

$$\varphi(\alpha) = f(x - \alpha(\chi_i - \chi_j)) + f(y + \alpha(\chi_i - \chi_j)) - f(x) - f(y) \qquad (\alpha \in \mathbf{R}).$$

Then, φ is also a strictly convex function. This fact, together with (M$^\natural$-PRJ), implies (M$^\natural$-PRJ$^+$). $\square$

Let $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ be a convex function. For $x \in \mathrm{dom}\, f$ and $d \in \mathbf{R}^n$, the *directional derivative* $f'(x; d)$ of f at x in the direction d is defined by

$$f'(x; d) = \lim_{\alpha \downarrow 0} \frac{f(x + \alpha d) - f(x)}{\alpha},$$

where the existence of the limit is assured by the convexity of f. In particular, if f is a quadratic function given by (2.14), then we have $f'(x; d) = x^{\mathrm{T}} A d$. For a sufficiently small $\alpha > 0$, we have

$$f(x + \alpha d) = f(x) + \alpha f'(x; d) + \mathrm{O}(\alpha^2).$$

Therefore, the inequality (5.42) is equivalent to

$$\begin{aligned} f'(x; -\chi_i + \chi_j) + f'(y; \chi_i - \chi_j) &= (x - y)^{\mathrm{T}} A(-\chi_i + \chi_j) \\ &= (x - y)^{\mathrm{T}}(-a_i + a_j) \ < \ 0. \end{aligned} \tag{5.43}$$

Since the vectors x, y can be chosen arbitrarily, we may assume that $y = \mathbf{0}$ in (5.43). Hence, we have "(M$^\natural$-PRJ$^+$) $\iff$ (M$^\natural$-PRJ$^+_{\mathrm{d}}$)" and "(M$^\natural$-EXC$^+$) $\iff$ (M$^\natural$-EXC$^+_{\mathrm{d}}$)."

The implications "(M$^\natural$-PRJ$^+$) $\Longrightarrow$ (M$^\natural$-EXC$^+$)" and "(M$^\natural$-EXC$^+_{\mathrm{d}}$) $\Longrightarrow$ (M$^\natural$-EXC$^+_{\mathrm{d}}$-RL)" are obvious.

We then prove the equivalence of three properties (M$^\natural$-PRJ$^+_{\mathrm{d}}$-RL), (M$^\natural$-EXC$^+_{\mathrm{d}}$-RL), and (L$^\natural$-INV). We require the following three lemmas.

Lemma 5.3 (Farkas' lemma). *For any matrix $A \in \mathbf{R}^{m \times n}$ and any vector $b \in \mathbf{R}^m$, the following conditions* (a) *and* (b) *are equivalent:*

(a) *there exists some nonnegative vector $x \in \mathbf{R}^n$ such that $Ax = b$,*
(b) *$y^{\mathrm{T}} b \geq 0$ for every $y \in \mathbf{R}^m$ with $y^{\mathrm{T}} A \geq \mathbf{0}^{\mathrm{T}}$.*

Lemma 5.4. *If $A \in \mathcal{S}_n$ satisfies* (L$^\natural$-INV)*, then any principal submatrix of A satisfies* (L$^\natural$-INV)*.*

Proof. Put $B = A^{-1}$. Then, B satisfies (2.13). Partition A and B as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \qquad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

with submatrices

$$\begin{aligned} &A_{11}, B_{11} \in \mathcal{S}_{n-1}, \\ &A_{12}^{\mathrm{T}} = A_{21} = (a_{n1}, a_{n2}, \cdots, a_{n,n-1}), \\ &B_{12}^{\mathrm{T}} = B_{21} = (b_{n1}, b_{n2}, \cdots, b_{n,n-1}), \\ &A_{22} = a_{nn}, \quad B_{22} = b_{nn}. \end{aligned}$$

We will show that A_{11} is nonsingular and $(A_{11})^{-1}$ satisfies (2.13).

Since B is a nonsingular matrix satisfying the property (2.13), we have $b_{nn} > 0$. Therefore, the inverse of A_{11} is given by $\widehat{B} = B_{11} - (1/b_{nn})B_{12}B_{21}$, which implies that A_{11} is nonsingular. For any $i, j \in \{1, \cdots, n-1\}$ with $i \neq j$, we have $\widehat{b}_{ij} = b_{ij} - b_{in}b_{nj}/b_{nn} \leq 0$ since $b_{ij}, b_{in}, b_{nj} \leq 0$ and $b_{nn} > 0$. For any $i \in \{1, \cdots, n-1\}$, we have

$$\sum_{j=1}^{n-1} \widehat{b}_{ij} = \sum_{j=1}^{n-1} b_{ij} - (b_{in}/b_{nn}) \sum_{j=1}^{n-1} b_{nj} \geq -b_{in} - (b_{in}/b_{nn}) \sum_{j=1}^{n-1} b_{nj} \geq 0.$$

Hence, $\widehat{B} = (A_{11})^{-1}$ satisfies (2.13). □

Lemma 5.5. *Let* $A \in \mathcal{S}_n$ *be a symmetric matrix with the property* ($\mathrm{L}^\natural$-INV). *Then, we have* $a_{ii} \geq a_{ij} \geq 0$ *for any* $i, j \in N$.

Proof. Let $i, j \in N$ and assume $i \neq j$. By Lemma 5.4, the submatrix $A[\{i,j\},\{i,j\}]$ also satisfies ($\mathrm{L}^\natural$-INV). The inverse of $A[\{i,j\},\{i,j\}]$ is given as

$$A[\{i,j\},\{i,j\}]^{-1} = \frac{1}{a_{ii}a_{jj} - a_{ij}^2} \begin{bmatrix} a_{jj} & -a_{ij} \\ -a_{ij} & a_{ii} \end{bmatrix}.$$

Since $A[\{i,j\},\{i,j\}]^{-1}$ is positive definite by Proposition 2.8, the submatrix $A[\{i,j\},\{i,j\}]$ is also positive definite. Hence $a_{ii}a_{jj} - a_{ij}^2 > 0$ follows, which, together with the property (2.13) for $A[\{i,j\},\{i,j\}]^{-1}$, implies $a_{ii} - a_{ij} \geq 0$ and $-a_{ij} \leq 0$. □

Lemma 5.6. *For* $A \in \mathcal{S}_n$, ($\mathrm{L}^\natural$-INV) $\iff$ ($\mathrm{M}^\natural$-$\mathrm{EXC}_{\mathrm{d}}^{+}$-RL) $\iff$ ($\mathrm{M}^\natural$-$\mathrm{PRJ}_{\mathrm{d}}^{+}$-RL).

Proof. The implication "($\mathrm{M}^\natural$-$\mathrm{PRJ}_{\mathrm{d}}^{+}$-RL) $\Longrightarrow$ ($\mathrm{M}^\natural$-$\mathrm{EXC}_{\mathrm{d}}^{+}$-RL)" is obvious. In the following, we prove "($\mathrm{M}^\natural$-$\mathrm{EXC}_{\mathrm{d}}^{+}$-RL) $\Longrightarrow$ ($\mathrm{L}^\natural$-INV)" and "($\mathrm{L}^\natural$-INV) $\Longrightarrow$ ($\mathrm{M}^\natural$-$\mathrm{EXC}_{\mathrm{d}}^{+}$-RL)."

[($\mathrm{M}^\natural$-$\mathrm{EXC}_{\mathrm{d}}^{+}$-RL) $\Longrightarrow$ ($\mathrm{L}^\natural$-INV)] The matrix A is nonsingular by Proposition 2.13. Put $B = A^{-1}$. It suffices to show the property (2.13) for B. Since $AB = I$, we have $\sum_{j=1}^{n} b_{ji}a_j = \chi_i$ for all $i \in N$. This equation can be rewritten as

$$\Bigl(\sum_{j=1}^{n} b_{ji}\Bigr) a_i + \sum_{j \neq i} (-b_{ji})(a_i - a_j) = \chi_i.$$

Note that the vector

$$x = (-b_{1i}, \cdots, -b_{i-1,i}, \sum_{j=1}^{n} b_{ji}, -b_{i+1,i}, \cdots, -b_{ni}) \in \mathbf{R}^n$$

is the unique solution of the equation $x_i a_i + \sum_{j \neq i} x_j (a_i - a_j) = \chi_i$. The property (M$^\natural$-EXC$^+_{\rm d}$-RL) says, on the other hand, that for any $y \in \mathbf{R}^n$, if $y^{\rm T}\chi_i > 0$ then either $y^{\rm T} a_i > 0$ or $y^{\rm T}(a_i - a_j) > 0$ for some $j \neq i$. From Lemma 5.3 and the uniqueness of x follows $x \geq \mathbf{0}$, i.e., the property (2.13) holds for B.

[(L$^\natural$-INV) $\Longrightarrow$ (M$^\natural$-PRJ$^+_{\rm d}$-RL)] We first prove the following inequalities:

Claim:

$$x^{\rm T} a_i > \min\left\{0, \min_{j \in N \setminus \{i\}} x^{\rm T} a_j\right\} \qquad (x \in \mathbf{R}^n,\ i \in \mathrm{supp}^+(x)), \tag{5.44}$$

$$0 > \min_{j \in N} x^{\rm T} a_j \qquad (x \in \mathbf{R}^n \text{ with } x(N) < 0). \tag{5.45}$$

[Proof of Claim] Put $B = A^{-1}$. Then, B satisfies the property (2.13).

We first prove (5.44). Since $AB = I$, we have $\sum_{j=1}^{n} b_{ji} a_j = \chi_i$ for all $i \in N$. This equation can be rewritten as

$$\Big(\sum_{j=1}^{n} b_{ji}\Big) a_i + \sum_{j \neq i}^{n} (-b_{ji})(a_i - a_j) = \chi_i.$$

By (2.13), we have $(\sum_{j=1}^{n} b_{ji}) \geq 0$ and $-b_{ji} \geq 0$ for all $j \neq i$. From Lemma 5.3 follows that if $x^{\rm T}\chi_i = x_i > 0$ then either $x^{\rm T} a_i > 0$ or $x^{\rm T}(a_i - a_j) > 0$ for some $j \neq i$.

We next prove (5.45). Since $AB = I$, we have $\sum_{j=1}^{n} \sum_{i=1}^{n} b_{ji} a_j = \mathbf{1}$. By (2.13), we have $\sum_{i=1}^{n} b_{ji} \geq 0$. From Lemma 5.3 follows that if $x^{\rm T}\mathbf{1} = x(N) < 0$ then $x^{\rm T} a_j < 0$ for some $j \in N$.

[End of Claim]

We now prove the property (M$^\natural$-PRJ$^+_{\rm d}$-RL). Since (a) and (c) are immediate from Claim above, we show (b) only.

Let $x \in \mathbf{R}^n$ be a vector with $x(N) \leq 0$.

[Case 1: $x(N) < 0$] If $x^{\rm T} a_i \leq 0$, then the inequality (5.44) implies

$$x^{\rm T} a_i > \min_{j \in N \setminus \{i\}} x^{\rm T} a_j.$$

Otherwise, we have $x^{\rm T} a_i > 0$, and therefore the property (5.45) implies

$$x^{\rm T} a_i > 0 > \min_{j \in N} x^{\rm T} a_j = \min_{j \in N \setminus \{i\}} x^{\rm T} a_j.$$

[Case 2: $x(N) = 0$] Let $i \in \mathrm{supp}^+(x)$, and ε be a positive number with $\varepsilon < x_i$. Putting $x' = x - \varepsilon\chi_i$, we have $x'(N) < 0$ and $\mathrm{supp}^+(x') = \mathrm{supp}^+(x)$. Therefore, the argument in Case 1 implies that

$$x^{\rm T} a_i - \varepsilon a_{ii} = (x')^{\rm T} a_i > \min_{j \in N \setminus \{i\}} (x')^{\rm T} a_j = \min_{j \in N \setminus \{i\}} (x^{\rm T} a_j - \varepsilon a_{ij}).$$

Since $a_{ii} \geq a_{ij}$ by Lemma 5.5, we have $x^{\mathrm{T}} a_i > \min_{j \in N \setminus \{i\}} x^{\mathrm{T}} a_j$. $\square$

To complete the proof of Theorem 2.14, we finally prove the implication "(M$^\natural$-PRJ$_{\mathrm{d}}^{+}$-RL) $\Longrightarrow$ (M$^\natural$-PRJ$_{\mathrm{d}}^{+}$)."

Lemma 5.7. *For $A \in \mathcal{S}_n$, (M$^\natural$-PRJ$_{\mathrm{d}}^{+}$-RL) $\Longrightarrow$ (M$^\natural$-PRJ$_{\mathrm{d}}^{+}$).*

Proof. Assume (M$^\natural$-PRJ$_{\mathrm{d}}^{+}$-RL) for A. We prove the properties (a), (b), and (c) of (M$^\natural$-PRJ$_{\mathrm{d}}^{+}$) in turn.

(a): Let $x \in \mathbf{R}^n$ with $x(N) > 0$. It suffices to show that

$$\min_{i \in \mathrm{supp}^+(x)} x^{\mathrm{T}} a_i > \min \left\{ 0, \min_{j \in \mathrm{supp}^-(x)} x^{\mathrm{T}} a_j \right\}.$$

Let $i_* \in \mathrm{supp}^+(x)$ minimize the value $x^{\mathrm{T}} a_{i_*}$, and put $S = \{ j \in N \mid x_i \neq 0 \}$. Let $\overline{x} = x|_S \in \mathbf{R}^S$ be the restriction of the vector x to S. Put $\overline{A} = A[S, S]$, and denote the columns of $\overline{A}$ by $\overline{a}_j$ ($j \in S$). Note that

$$\begin{array}{c} i \in \mathrm{supp}^+(\overline{x}) = \mathrm{supp}^+(x), \qquad \mathrm{supp}^-(\overline{x}) = \mathrm{supp}^-(x), \\ \overline{x}_j \neq 0, \quad x^{\mathrm{T}} a_j = \overline{x}^{\mathrm{T}} \overline{a}_j \qquad (\forall j \in S). \end{array}$$

By Lemmas 5.4 and 5.6, the matrix $\overline{A}$ satisfies (M$^\natural$-PRJ$_{\mathrm{d}}^{+}$-RL). Hence follows

$$\overline{x}^{\mathrm{T}} \overline{a}_{i_*} > \min \left\{ 0, \min_{j \in S \setminus \{i_*\}} \overline{x}^{\mathrm{T}} \overline{a}_j \right\}. \tag{5.46}$$

This inequality, together with the choice of i_*, implies

$$\begin{aligned} \min \left\{ 0, \min_{j \in S \setminus \{i_*\}} \overline{x}^{\mathrm{T}} \overline{a}_j \right\} &= \min \left\{ 0, \min_{j \in \mathrm{supp}^+(\overline{x}) \setminus \{i_*\}} \overline{x}^{\mathrm{T}} \overline{a}_j, \min_{j \in \mathrm{supp}^-(\overline{x})} \overline{x}^{\mathrm{T}} \overline{a}_j \right\} \\ &= \min \left\{ 0, \min_{j \in \mathrm{supp}^-(\overline{x})} \overline{x}^{\mathrm{T}} \overline{a}_j \right\}. \end{aligned} \tag{5.47}$$

Combining (5.46) and (5.47), we have

$$\begin{aligned} x^{\mathrm{T}} a_{i_*} &= \overline{x}^{\mathrm{T}} \overline{a}_{i_*} \\ &> \min \left\{ 0, \min_{j \in \mathrm{supp}^-(\overline{x})} \overline{x}^{\mathrm{T}} \overline{a}_j \right\} = \min \left\{ 0, \min_{j \in \mathrm{supp}^-(x)} x^{\mathrm{T}} a_j \right\}. \end{aligned}$$

(b): The proof is quite similar to that for (a), and therefore omitted.

(c): Let $x \in \mathbf{R}^n$ with $x(N) < 0$. The property (M$^\natural$-PRJ$_{\mathrm{d}}^{+}$-RL) (c) yields $\min_{j \in N} x^{\mathrm{T}} a_j < 0$, and the property (M$^\natural$-PRJ$_{\mathrm{d}}^{+}$) (b) shown above yields

$$\min_{i \in \mathrm{supp}^+(x)} x^{\mathrm{T}} a_i > \min_{j \in \mathrm{supp}^-(x)} x^{\mathrm{T}} a_j.$$

Hence, $\min_{j \in \mathrm{supp}^-(x)} x^{\mathrm{T}} a_j < 0$. $\square$

5.2.4 Proof of Theorem 2.17

["only if" part] Let $A \in \overline{\mathcal{M}}^\natural$. Then, the quadratic function $f : \mathbf{R}^n \to \mathbf{R}$ given by (2.18) is M$^\natural$-convex. Since the set $\arg\min f$ is an M$^\natural$-convex linear subspace, there exists a subpartition $\{N_k \mid k = 1, 2, \cdots, m\}$ of N such that

$$\arg\min f = \mathrm{span}(\bigcup_{k=1}^{m}\{\chi_j - \chi_i \mid i, j \in N_k\} \cup \{\chi_i \mid i \in N \setminus \bigcup_{k=1}^{m} N_k\})$$

by Proposition 2.12 (ii). Since $\ker A = \arg\min f$, we have

$$a_h^{\mathrm{T}}(\chi_j - \chi_i) = a_{jh} - a_{ih} = 0 \quad (h \in N,\ i, j \in N_k,\ k \in \{1, \cdots, m\}), \tag{5.48}$$

$$a_h^{\mathrm{T}}\chi_i = a_{ih} = 0 \quad (h \in N,\ i \in N \setminus \bigcup_{k=1}^{m} N_k). \tag{5.49}$$

We define a matrix $\widetilde{A} \in \mathcal{S}_m$ by

$$\widetilde{a}_{kl} = a_{ij} \qquad (i \in N_k,\ j \in N_l,\ k, l \in \{1, \cdots, m\}),$$

which is well-defined by (5.48) and (5.49).

Claim: The matrix $\widetilde{A}$ is nonsingular.

[Proof of Claim 1] Assume, to the contrary, that there exists some vector $\widetilde{x} \in \mathbf{R}^m$ with $\widetilde{A}\widetilde{x} = 0$. For each $k = 1, 2, \cdots, m$, we choose $i_k \in N_k$ arbitrarily, and put $x = \sum_{k=1}^{m} \widetilde{x}_k \chi_{i_k}$. Then, we have $Ax = 0$, i.e., $x \in \arg\min f$. This, however, is a contradiction since such x cannot be contained in $\arg\min f$. [End of Claim]

We have $\widetilde{A} \in \mathcal{M}^\natural$ by the claim above and Proposition 2.22 (ii). This completes the proof of "only if" part.

["if" part] Define a quadratic function $\widetilde{f} : \mathbf{R}^m \to \mathbf{R}$ by $\widetilde{f}(\widetilde{x}) = (1/2)\widetilde{x}^{\mathrm{T}}\widetilde{A}\widetilde{x}$ ($\widetilde{x} \in \mathbf{R}^m$). Then, $\widetilde{f}$ satisfies the property (M$^\natural$-PRJ$^+$) by Lemma 5.2. By using this fact, we prove that the function f satisfies the property (M$^\natural$-PRJ).

Let $x, y \in \mathbf{R}^n$. We define two vectors $\widetilde{x}, \widetilde{y} \in \mathbf{R}^m$ by

$$\widetilde{x}_k = \sum_{i \in N_k} x_i, \qquad \widetilde{y}_k = \sum_{i \in N_k} y_i \qquad (k = 1, \cdots, m).$$

It is easy to see that

$$\sum_{k=1}^{m} \widetilde{x}_k = x(N), \quad \sum_{k=1}^{m} \widetilde{y}_k = y(N),$$
$$\widetilde{f}(\widetilde{x}) = f(x), \quad \widetilde{f}(\widetilde{y}) = f(y).$$

To prove (M$^\natural$-PRJ) (a) for f, we suppose $x(N) > y(N)$. Let $i \in \mathrm{supp}^+(x - y)$ and assume $i \in N_k$.

Suppose $\widetilde{x}_k > \widetilde{y}_k$. Then, (M$^\natural$-PRJ$^+$) (a) for $\widetilde{f}$ implies that there exist some $l \in \mathrm{supp}^-(\widetilde{x} - \widetilde{y}) \cup \{0\}$ and $\alpha_0 > 0$ such that

$$\widetilde{f}(\widetilde{x}) + \widetilde{f}(\widetilde{y}) > \widetilde{f}(\widetilde{x} - \alpha(\widetilde{\chi}_k - \widetilde{\chi}_l)) + \widetilde{f}(\widetilde{y} + \alpha(\widetilde{\chi}_k - \widetilde{\chi}_l)) \qquad (0 < \forall\alpha \leq \alpha_0),$$

where $\widetilde{\chi}_k, \widetilde{\chi}_l \in \{0,1\}^m$ are the characteristic vectors of k and l, respectively. If $l \neq 0$, then put $j = 0$, and if $l \neq 0$, then we take some $j \in \mathrm{supp}^-(x - y) \cap N_l$, where we note that $\mathrm{supp}^-(x - y) \cap N_l \neq \emptyset$. In either case, we have

$$\begin{aligned} f(x) + f(y) &= \widetilde{f}(\widetilde{x}) + \widetilde{f}(\widetilde{y}) \\ &> \widetilde{f}(\widetilde{x} - \alpha(\widetilde{\chi}_k - \widetilde{\chi}_l)) + \widetilde{f}(\widetilde{y} + \alpha(\widetilde{\chi}_k - \widetilde{\chi}_l)) \\ &= f(x - \alpha(\chi_i - \chi_j)) + f(y + \alpha(\chi_i - \chi_j)) \quad (0 < \forall\alpha \leq \alpha_0). \end{aligned}$$

Suppose that $\widetilde{x}_k \leq \widetilde{y}_k$. Then, $\mathrm{supp}^-(x - y) \cap N_k$ is nonempty, and for any $j \in \mathrm{supp}^-(x - y) \cap N_k$ we have

$$\begin{aligned} f(x) + f(y) &= \widetilde{f}(\widetilde{x}) + \widetilde{f}(\widetilde{y}) \\ &= f(x - \alpha(\chi_i - \chi_j)) + f(y + \alpha(\chi_i - \chi_j)) \quad (\forall\alpha \geq 0). \end{aligned}$$

The proof for (M$^\natural$-PRJ) (b) is similar to that for (a) and therefore omitted.

We finally prove (M$^\natural$-PRJ) (c) for f. Suppose $x(N) < y(N)$. Then, (M$^\natural$-PRJ$^+$) (c) for $\widetilde{f}$ implies that there exist some $l \in \mathrm{supp}^-(\widetilde{x} - \widetilde{y})$ and $\alpha_0 > 0$ such that

$$\widetilde{f}(\widetilde{x}) + \widetilde{f}(\widetilde{y}) > \widetilde{f}(\widetilde{x} + \alpha\widetilde{\chi}_l) + \widetilde{f}(\widetilde{y} - \alpha\widetilde{\chi}_l) \qquad (0 < \forall\alpha \leq \alpha_0).$$

We note that $\mathrm{supp}^-(x - y) \cap N_l \neq \emptyset$, and for any $j \in \mathrm{supp}^-(x - y) \cap N_l$ we have

$$\begin{aligned} f(x) + f(y) &= \widetilde{f}(\widetilde{x}) + \widetilde{f}(\widetilde{y}) \\ &> \widetilde{f}(\widetilde{x} + \alpha\widetilde{\chi}_l) + \widetilde{f}(\widetilde{y} - \alpha\widetilde{\chi}_l) \\ &= f(x + \alpha\chi_j) + f(y - \alpha\chi_j) \quad (0 < \forall\alpha \leq \alpha_0). \end{aligned}$$

5.2.5 Proofs of Theorems 2.18 and 2.23

To prove Theorem 2.18, we use another property:

(M$^\natural$-PRJ$_\mathbf{d}$)

(a) $x^{\mathrm{T}}a_i \geq \min\{0, \min_{j \in \mathrm{supp}^-(x)} x^{\mathrm{T}}a_j\}$ $\quad (\forall x \in \mathbf{R}^n$ with $x(N) > 0,\ \forall i \in \mathrm{supp}^+(x))$,

(b) $x^{\mathrm{T}}a_i \geq \min_{j \in \mathrm{supp}^-(x)} x^{\mathrm{T}}a_j$ $\quad (\forall x \in \mathbf{R}^n$ with $x(N) \leq 0,\ \forall i \in \mathrm{supp}^+(x))$,

(c) $0 \geq \min_{j \in \mathrm{supp}^-(x)} x^{\mathrm{T}}a_j$ $\quad (\forall x \in \mathbf{R}^n$ with $x(N) < 0)$.

In the following, we prove the following implications:

$$\begin{array}{ccc}
(\mathrm{M}^\natural\text{-PRJ}) & \Longleftarrow & (\mathrm{M}^\natural\text{-PRJ}_\mathrm{d}) \\
\Downarrow & & \Uparrow \\
(\mathrm{M}^\natural\text{-EXC}) & \Longrightarrow & (\mathrm{M}^\natural\text{-EXC}_\mathrm{d}).
\end{array}$$

The implication "($\mathrm{M}^\natural$-PRJ) $\Longrightarrow$ ($\mathrm{M}^\natural$-EXC)" is obvious, and "($\mathrm{M}^\natural$-EXC) $\Longrightarrow$ ($\mathrm{M}^\natural$-EXC$_\mathrm{d}$)" follows from the equation:

$$\begin{aligned}
&f(x - \alpha(\chi_i - \chi_j)) + f(y + \alpha(\chi_i - \chi_j)) - f(x) - f(y) \\
&\qquad = \ \alpha(x - y)^\mathrm{T}(a_j - a_i) + \alpha^2(a_{ii} + a_{jj} - 2a_{ij}) \qquad (x, y \in \mathbf{R}^n,\ i, j \in N \cup \{0\},\ \alpha \in \mathbf{R}).
\end{aligned} \tag{5.50}$$

Hence, it suffices to show "($\mathrm{M}^\natural$-EXC$_\mathrm{d}$) $\Longrightarrow$ ($\mathrm{M}^\natural$-PRJ$_\mathrm{d}$)" and "($\mathrm{M}^\natural$-PRJ$_\mathrm{d}$) $\Longrightarrow$ ($\mathrm{M}^\natural$-PRJ)."

Proposition 5.8. *A symmetric matrix $A \in \mathcal{S}_n$ satisfying* ($\mathrm{M}^\natural$-EXC$_\mathrm{d}$) *is positive semidefinite.*

Proof. The proof is quite similar to that of Proposition 2.13, and therefore omitted. □

Lemma 5.9. ($\mathrm{M}^\natural$-EXC$_\mathrm{d}$) $\Longrightarrow$ ($\mathrm{M}^\natural$-PRJ$_\mathrm{d}$).

Proof. Assume ($\mathrm{M}^\natural$-EXC$_\mathrm{d}$) for A. For $\beta > 0$, we define a symmetric matrix $A_\beta \in \mathcal{S}_n$ by $A_\beta = A + \beta I$. Then, A_β is positive definite by Proposition 5.8 and satisfies the property ($\mathrm{M}^\natural$-EXC$^+_\mathrm{d}$) since

$$x^\mathrm{T} a_i + \beta x_i > x^\mathrm{T} a_i \geq \min\{0, \min_{j \in \mathrm{supp}^-(x)} x^\mathrm{T} a_j\} \geq \min\{0, \min_{j \in \mathrm{supp}^-(x)} (x^\mathrm{T} a_j + \beta x_j)\}$$

for $x \in \mathbf{R}^n$ and $i \in \mathrm{supp}^+(x)$. From Theorem 2.14 and Lemma 5.2 follows the property ($\mathrm{M}^\natural$-PRJ$^+_\mathrm{d}$) for A_β ($\forall \beta > 0$).

We now prove the properties (a), (b), and (c) in ($\mathrm{M}^\natural$-PRJ$_\mathrm{d}$) for A. Assume, to the contrary, that

$$x^\mathrm{T} a_i < \min\{0, \min_{j \in \mathrm{supp}^-(x)} x^\mathrm{T} a_j\}$$

for some $x \in \mathbf{R}^n$ and $i \in \mathrm{supp}^+(x)$, i.e., ($\mathrm{M}^\natural$-PRJ$_\mathrm{d}$) (a) does not hold for A. Take any $\beta > 0$ with

$$\beta\{x_i - \min_{j \in \mathrm{supp}^-(x)} x_j\} \leq \min\{0, \min_{j \in \mathrm{supp}^-(x)} x^\mathrm{T} a_j\} - x^\mathrm{T} a_i.$$

Then, we have

$$x^\mathrm{T}(a_i + \beta\chi_i) \leq \min\{0, \min_{j \in \mathrm{supp}^-(x)} x^\mathrm{T}(a_j + \beta\chi_j)\},$$

a contradiction to ($\mathrm{M}^\natural$-PRJ$^+_\mathrm{d}$) (a) for A_β. We can prove ($\mathrm{M}^\natural$-PRJ$_\mathrm{d}$) (b) and (c) in a similar way. □

Lemma 5.10. (M$^\natural$-PRJ$_\mathrm{d}$) $\Longrightarrow$ (M$^\natural$-PRJ).

Proof. From the equation (5.50) we have the equivalence between the property (M$^\natural$-PRJ) and the following properties (a$'$), (b$'$), and (c$'$) in terms of matrices, where ε is a sufficiently small positive number.

(a$'$) $\forall x \in \mathbf{R}^n$ with $x(N) > 0$, $\forall i \in \mathrm{supp}^+(x)$, $\exists j \in \mathrm{supp}^-(x) \cup \{0\}$ such that

$$x^\mathrm{T}(a_j - a_i) + \varepsilon(a_{ii} + a_{jj} - 2a_{ij}) \leq 0, \tag{5.51}$$

(b$'$) $\forall x \in \mathbf{R}^n$ with $x(N) \leq 0$, $\forall i \in \mathrm{supp}^+(x)$, $\exists j \in \mathrm{supp}^-(x)$ such that

$$x^\mathrm{T}(a_j - a_i) + \varepsilon(a_{ii} + a_{jj} - 2a_{ij}) \leq 0,$$

(c$'$) $\forall x \in \mathbf{R}^n$ with $x(N) < 0$, $\exists j \in \mathrm{supp}^-(x)$ such that

$$x^\mathrm{T} a_j + \varepsilon a_{jj} \leq 0,$$

where $a_0 = \mathbf{0}$ and $a_{00} = a_{i0} = 0$. For example, the inequality (5.51) reads as follows:

$$x^\mathrm{T}(a_j - a_i) < 0, \qquad \text{or} \qquad x^\mathrm{T}(a_j - a_i) = 0 \text{ and } a_{ii} + a_{jj} - 2a_{ij} \leq 0.$$

In the following, we show that the property (M$^\natural$-PRJ$_\mathrm{d}$) implies (a$'$), (b$'$), and (c$'$).

[(M$^\natural$-PRJ$_\mathrm{d}$) $\Longrightarrow$ (a$'$)] Let $x \in \mathbf{R}^n$ be a vector with $x(N) > 0$, and $i \in \mathrm{supp}^+(x)$. Define

$$\tilde{x} = x - 2(k+1)\varepsilon\chi_i + 2\varepsilon\chi_S$$

with $S = \mathrm{supp}^-(x)$ and $k = |S|$. Then, we have $i \in \mathrm{supp}^+(\tilde{x}) = \mathrm{supp}^+(x)$, $\mathrm{supp}^-(\tilde{x}) = \mathrm{supp}^-(x)$, and $\tilde{x}(N) > 0$. By the property (M$^\natural$-PRJ$_\mathrm{d}$) (a) applied to $\tilde{x}$ and i, there exists $j \in \mathrm{supp}^-(\tilde{x}) \cup \{0\}$ such that

$$\tilde{x}^\mathrm{T}(a_j - a_i) = x^\mathrm{T}(a_j - a_i) + \{2\varepsilon\chi_S - 2(k+1)\varepsilon\chi_i\}^\mathrm{T}(a_j - a_i) \leq 0. \tag{5.52}$$

Put $x' = 2\chi_S - \chi_j - (2k+1)\chi_i$. Since $x'(N) < 0$ and $\mathrm{supp}^-(x') = \{i\}$, if $j \in \mathrm{supp}^-(\tilde{x}) = \mathrm{supp}^+(x')$ then the property (M$^\natural$-PRJ$_\mathrm{d}$) (b) yields

$$\{2\chi_S - \chi_j - (2k+1)\chi_i\}^\mathrm{T}(a_i - a_j) = \{-2\chi_S + \chi_j + (2k+1)\chi_i\}^\mathrm{T}(a_j - a_i) \leq 0, \tag{5.53}$$

and if $j = 0$ then the property (M$^\natural$-PRJ$_\mathrm{d}$) (c) yields the same inequality (5.53). Combining (5.52) and (5.53), we have

$$x^\mathrm{T}(a_j - a_i) + \varepsilon(\chi_j - \chi_i)^\mathrm{T}(a_j - a_i) = x^\mathrm{T}(a_j - a_i) + \varepsilon(a_{ii} + a_{jj} - 2a_{ij}) \leq 0.$$

[(M$^\natural$-PRJ$_\mathrm{d}$) $\Longrightarrow$ (b′)] Let $x \in \mathbf{R}^n$ be a vector with $x(N) \leq 0$, and $i \in \mathrm{supp}^+(x)$. Define

$$\tilde{x} = x - 2k\varepsilon\chi_i + 2\varepsilon\chi_S$$

with $S = \mathrm{supp}^-(x)$ and $k = |S|$. Then, we have $i \in \mathrm{supp}^+(\tilde{x}) = \mathrm{supp}^+(x)$, $\mathrm{supp}^-(\tilde{x}) = \mathrm{supp}^-(x)$, and $\tilde{x}(N) = x(N) \leq 0$. By the property (M$^\natural$-PRJ$_\mathrm{d}$) (b) applied to $\tilde{x}$ and i, there exists $j \in \mathrm{supp}^-(\tilde{x})$ such that

$$\tilde{x}^\mathrm{T}(a_j - a_i) = x^\mathrm{T}(a_j - a_i) + (2\varepsilon\chi_S - 2k\varepsilon\chi_i)^\mathrm{T}(a_j - a_i) \leq 0. \tag{5.54}$$

Put $x' = 2\chi_S - \chi_j - (2k-1)\chi_i$. Since $\mathrm{supp}^-(x') = \{i\}$, $j \in \mathrm{supp}^-(\tilde{x}) = \mathrm{supp}^+(x')$, and $x'(N) = 0$, the property (M$^\natural$-PRJ$_\mathrm{d}$) (b) yields

$$\{2\chi_S - \chi_j - (2k-1)\chi_i\}^\mathrm{T}(a_i - a_j) = \{-2\chi_S + \chi_j + (2k-1)\chi_i\}^\mathrm{T}(a_j - a_i) \leq 0. \tag{5.55}$$

Combining (5.54) and (5.55), we have

$$x^\mathrm{T}(a_j - a_i) + \varepsilon(\chi_j - \chi_i)^\mathrm{T}(a_j - a_i) = x^\mathrm{T}(a_j - a_i) + \varepsilon(a_{ii} + a_{jj} - 2a_{ij}) \leq 0.$$

[(M$^\natural$-PRJ$_\mathrm{d}$) $\Longrightarrow$ (c′)] Let $x \in \mathbf{R}^n$ be a vector with $x(N) < 0$. Define

$$\tilde{x} = x + 2\varepsilon\chi_S$$

with $S = \mathrm{supp}^-(x)$ and $k = |S|$. Then, we have $\mathrm{supp}^-(\tilde{x}) = \mathrm{supp}^-(x)$ and $\tilde{x}(N) < 0$. By the property (M$^\natural$-PRJ$_\mathrm{d}$) (c) applied to $\tilde{x}$, there exists $j \in \mathrm{supp}^-(\tilde{x})$ such that

$$\tilde{x}^\mathrm{T}a_j = x^\mathrm{T}a_j + 2\varepsilon\chi_S^\mathrm{T}a_j \leq 0. \tag{5.56}$$

Put $x' = 2\chi_S - \chi_j$. Since $j \in \mathrm{supp}^-(\tilde{x}) = \mathrm{supp}^+(x')$, $\mathrm{supp}^-(x') = \emptyset$, and $x'(N) > 0$, the property (M$^\natural$-PRJ$_\mathrm{d}$) (a) yields

$$(2\chi_S - \chi_j)^\mathrm{T}a_j \geq 0. \tag{5.57}$$

Combining (5.56) and (5.57), we have

$$x^\mathrm{T}a_j + \varepsilon\chi_j^\mathrm{T}a_j = x^\mathrm{T}a_j + \varepsilon a_{jj} \leq 0.$$

□

We then prove Theorem 2.23. Given a symmetric matrix $A \in \mathcal{S}_n$, we define an $(n-1)\times(n-1)$ symmetric matrix $A' \in \mathcal{S}_{n-1}$ by (2.26). Then, we have

$$\begin{aligned} A \in \overline{\mathcal{M}} \quad &\Longleftrightarrow \quad A' \in \overline{\mathcal{M}}^\natural, \\ A \text{ satisfies (M-EXC}_\mathrm{d}) \quad &\Longleftrightarrow \quad A' \text{ satisfies (M}^\natural\text{-PRJ}_\mathrm{d}). \end{aligned}$$

Hence, the claim follows from the equivalence "(M$^\natural$-PRJ) $\Longleftrightarrow$ (M$^\natural$-PRJ$_\mathrm{d}$)" shown above.

5.2.6 Proofs of Theorems 2.19 and 2.24

We first prove Theorem 2.24, and then 2.19.

Proof of 2.24. ["only if" part] Suppose that f is a quadratic M-convex function. Then, $\mathrm{dom}\, f = H$ is an M-convex linear subspace. By Proposition 2.12 (i), there exists a partition $\{N_k \mid k = 1, 2, \cdots, m\}$ of N such that H is represented as

$$H = \{x \in \mathbf{R}^n \mid x(N_k) = 0 \ (k = 1, 2, \cdots, m)\}. \tag{5.58}$$

We then prove $A[N_k, N_k] \in \overline{\mathcal{M}}$ $(k = 1, \cdots, m)$. Let $x, y \in H$ be any vectors with $\{i \in N \mid x_i \neq 0 \text{ or } y_i \neq 0\} \subseteq N_k$, and $i \in \mathrm{supp}^+(x - y) \cap N_k$. By the M-convexity of f, there exist some $j \in \mathrm{supp}^-(x - y)$ and $\alpha_0 > 0$ satisfying

$$f(x) + f(y) \geq f(x - \alpha(\chi_i - \chi_j)) + f(y + \alpha(\chi_i - \chi_j)) \qquad (0 < \forall\alpha \leq \alpha_0).$$

In particular, we have $x - \alpha(\chi_i - \chi_j) \in H$ and $y + \alpha(\chi_i - \chi_j) \in H$ for any α with $0 < \alpha \leq \alpha_0$, implying $j \in N_k$. This fact shows that the quadratic function $f_k : \mathbf{R}^{N_k} \to \mathbf{R}$ defined by

$$f_k(x') = \begin{cases} (1/2)(x')^{\mathrm{T}} A[N_k, N_k] x' & (x' \in \mathbf{R}^{N_k},\ x'(N_k) = 0), \\ +\infty & (x' \in \mathbf{R}^{N_k},\ x'(N_k) \neq 0), \end{cases}$$

is M-convex, implying $A[N_k, N_k] \in \overline{\mathcal{M}}$.

We next show $a_{ij} - a_{i'j} = a_{ij'} - a_{i'j'}$ for $i, i' \in N_k$, $j, j' \in N_l$ and distinct $k, l \in \{1, 2, \cdots, m\}$. Put

$$x = (\chi_i - \chi_{i'}) + \beta(\chi_j - \chi_{j'}), \qquad y = \mathbf{0},$$

where β is any real number. Since $\mathrm{dom}\, f = H$ is represented as (5.58), the property (M-EXC) applied to $x, y \in H$ and $i \in \mathrm{supp}^+(x - y)$ implies that there exists some $\alpha > 0$ such that

$$\begin{aligned} 0 &\geq f(x - \alpha(\chi_i - \chi_{i'})) + f(y + \alpha(\chi_i - \chi_{i'})) - f(x) - f(y) \\ &= \alpha x^{\mathrm{T}}(a_{i'} - a_i) + \alpha^2(a_{ii} + a_{i'i'} - 2a_{ii'}) \\ &= \alpha(2a_{ii'} - a_{ii} - a_{i'i'}) + \alpha\beta(a_{i'j} - a_{i'j'} - a_{ij} + a_{ij'}) + \alpha^2(a_{ii} + a_{i'i'} - 2a_{ii'}). \end{aligned}$$

Since β can be chosen arbitrarily, we have $a_{i'j} - a_{i'j'} - a_{ij} + a_{ij'} = 0$.

["if" part] It suffices to show that the function f is rewritten as

$$f(x) = \sum_{k=1}^{m} (x|_{N_k})^{\mathrm{T}} A[N_k, N_k]\, (x|_{N_k}) \qquad (x \in H). \tag{5.59}$$

Let $k, l \in \{1, 2, \cdots, m\}$ be any distinct integers, and $x' \in \mathbf{R}^{N_k}$ and $y' \in \mathbf{R}^{N_l}$ are such that $x'(N_k) = y'(N_l) = 0$. Also, let $i_* \in N_k$ and $j_* \in N_l$ be any elements. Then, we have

$$\sum_{i \in N_k} \sum_{j \in N_l} a_{ij}\, x'_i\, y'_j \ = \sum_{i \in N_k \setminus \{i_*\}} \ \sum_{j \in N_l \setminus \{j_*\}} (a_{ij} - a_{ij_*} - a_{i_*j} + a_{i_*j_*})\, x'_i\, y'_j \ = \ 0,$$

where the last equality is by the property (2.28). From this fact follows (5.59). □

We now prove Theorem 2.19. Put $\widehat{N} = \{0\} \cup N$. For $A \in \mathcal{S}_n$ and $H \subseteq \mathbf{R}^n$, we define a symmetric matrix $\widehat{A} \in \mathbf{R}^{\widehat{N} \times \widehat{N}}$ by (2.27) and a linear subspace $\widehat{H} \subseteq \mathbf{R}^{\widehat{N}}$ by

$$\widehat{H} = \{(x_0, x) \in \mathbf{R}^{\widehat{N}} \mid x \in H,\ x_0 = -x(N)\}.$$

By the definition of M$^\natural$-convexity, a quadratic function f over H, given by (2.14) is M$^\natural$-convex if and only if the quadratic function $\widehat{f} : \mathbf{R}^{\widehat{N}} \to \mathbf{R} \cup \{+\infty\}$ over $\widehat{H}$ given by

$$\widehat{f}(\widehat{x}) = \begin{cases} \frac{1}{2}\widehat{x}^{\mathrm{T}}\widehat{A}\widehat{x} & (\widehat{x} \in \widehat{H}), \\ +\infty & (\widehat{x} \notin \widehat{H}) \end{cases}$$

is M-convex. Hence, Theorem 2.19 is just a translation of Theorem 2.24 for $\widehat{A}$ and $\widehat{H}$ in terms of A and H.

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