HEEGAARD GENUS FORMULA FOR HAKEN MANIFOLDS

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Abstract. Suppose M is a compact orientable 3-manifold and $Q \subset M$ a compact orientable incompressible surface. Further suppose that $M^1, \ldots, M^k$ are the closures of components of $M - Q$ and that n_j is the maximal number of parallelism classes of essential annuli for which representatives can be simultaneously imbedded in M^j. If $M = (V_1 \cup_{S_1} W_1) \cup_{F_1} \cdots \cup_{F_{n-1}} (V_n \cup_{S_n} W_n)$ is a generalized Heegaard splitting of M with $\partial Q \subset \cup_i \partial_- V_i$, then there are generalized Heegaard splittings $M^j = (A_1^j \cup_{G_1^j} B_1^j) \cup_{P_1^j} \cdots \cup_{P_{n-1}^j} (A_n^j \cup_{G_n}^j B_n^j)$ for $j = 1, \ldots, k$ that satisfy the following inequality

$$\sum_i J(V_i) \geq \frac{1}{3}\left(\sum_j \sum_i J(A_i^j) + 2\chi(Q) - 2\sum_j n_j\right).$$

Heegaard splittings have long been used in the study of 3-manifolds. One reason for their continued importance in this study is that the Heegaard genus of a compact 3-manifold has proven to capture the topology of the 3-manifold more accurately than many other invariants. In particular, it provides an upper bound for the rank of the fundamental group of the 3-manifold, and this upper bound can be a strict upper bound, as seen in the examples provided by M. Boileau and H. Zieschang in [1].

The formula derived in this paper provides a topological analogue to the algebraic formula provided by R. Weidmann for the rank of the fundamental group. The correspondence between the two results makes the formula derived here particularly interesting, since it seems to indicate that at least in some respect, rank and genus of a 3-manifold behave in the same way. The construction and techniques used here are an immediate generalization of joint work with M. Scharlemann in [12], [13] and [14].

This formula provides a variant of Johannson's formula $g(M - Q) \leq 5g(M) + 2g(Q)$ that is valid in the case that M is closed. (See [7, Proposition 23.40].) In his formula, the coefficient of $g(M)$ is larger than in the one proven in this paper, whereas the coefficient of $g(Q)$ is smaller. One is tempted to conjecture that the smaller coefficients simultaneously suffice. But upon closer inspection, this does not appear to be the case. The discrepancy may indeed not be due to an oversight in either case, but rather a necessary consequence of the techniques used. In a nutshell, Johannson's technique allows one compression body to "capture all of $\chi(Q)$" (where it does not contribute to the count) at the expense of having more intersections of the splitting surface with Q. The technique here, on the other hand, puts greater restraints on the number of intersections of the splitting surfaces of the generalized Heegaard splitting with Q, but at the cost of $\chi(Q)$ playing a greater role in the construction; because it might be entirely "captured in" the "wrong"

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compression bodies. A better conjecture then, is that the roles of $g(M)$ and $\chi(Q)$ are not independent in the ideal formula. (Of course, for all practical purposes one would work with the inequality that is stronger for a given combination of $g(M)$ and $\chi(Q)$.)

Section 2 of this paper shows how the generalized Heegaard splitting of M induces generalized Heegaard splittings of M_1 and M_2. This construction provides the generalized Heegaard splittings, but gives little control over their complexity. Section 3 provides more specifices on the construction in Section 2 that provide a series of destabilizations. Section 4 locates additional ways in which the complexity of the resulting generalized Heegaard splittings may be reduced. Section 5 proves the Main Theorem. This proof consists entirely of adding up and subtracting the appropriate numbers from Sections 2, 3, and 4. Section 6 provides a sketch of how this construction may be used to obtain a bound on the genus of a Heegaard splitting that is a common stabilization of two strongly irreducible Heegaard splittings of a closed 3-manifold. The bound obtained is an improvement over that obtained in [9].

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1. Preliminaries

For standard definitions concerning 3-manifolds, see [5] or [6].

Definition 1. *For L a properly imbedded submanifold of M, we denote an open regular neighborhood of L in M by $\eta(L)$ and a closed regular neighborhood of L by $N(L)$.*

Definition 2. *A compression body is a 3-manifold W obtained from a connected closed orientable surface S by attaching 2-handles to $S \times \{0\} \subset S \times I$ and capping off any resulting 2-sphere boundary components. We denote $S \times \{1\}$ by $\partial_+ W$ and $\partial W - \partial_+ W$ by $\partial_- W$. Dually, a compression body is a connected orientable 3-manifold obtained from a (not necessarily connected) closed orientable surface $\partial_- W \times I$ or a $3-$ball by attaching 1-handles.*

In the case where $\partial_- W = \emptyset$ (i.e., in the case where a $3-$ball was used in the dual construction of W), we also call W a handlebody.

Define the index of W by $J(W) = \chi(\partial_- W) - \chi(\partial_+ W)$. (Note that the index of a solid torus is 0.)

Definition 3. *A set of defining disks for a compression body W is a set of disks $\{D_1, \ldots, D_n\}$ properly embedded in W with $\partial D_i \subset \partial_+ W$ for $i = 1, \ldots, n$ such that the result of cutting W along $D_1 \cup \cdots \cup D_n$ is homeomorphic to $\partial_- W \times I$ or a 3-ball in the case that W is a handlebody.*

Definition 4. *A Heegaard splitting of a 3-manifold M is a decomposition $M = V \cup_S W$ in which V, W are compression bodies such that $V \cap W = \partial_+ V = \partial_+ W = S$ and $M = V \cup W$. We call S the splitting surface or Heegaard surface.*

Definition 5. *A Heegaard splitting is reducible (resp. weakly reducible) if there are essential disks D_1 and D_2, such that $\partial D_1 = \partial D_2$ (resp. $\partial D_1 \cap \partial D_2 = \emptyset$). A Heegaard splitting which is not (weakly) reducible is (strongly) irreducible.*

The notion of strong irreducibility, due to Casson and Gordon in [3], prompted the following definition due to Scharlemann and Thompson.

Definition 6. *A generalized Heegaard splitting of a compact orientable 3-manifold M is a decomposition $M = (V_1 \cup_{S_1} W_1) \cup_{F_1} (V_2 \cup_{S_2} W_2) \cup_{F_2} \cdots \cup_{F_{n-1}} (V_n \cup_{S_n} W_n)$. Each of the V_i and W_i is a union of compression bodies, $\partial_+ V_i = S_i = \partial_+ W_i$, (i.e., $V_i \cup_{S_i} W_i$ is a union of Heegaard splittings of a submanifold of M) and $\partial_- W_i = F_i = \partial_- V_{i+1}$. We say that a generalized Heegaard splitting is strongly irreducible if each Heegaard splitting $V_i \cup_{S_i} W_i$ is strongly irreducible and each F_i is incompressible in M. We will denote $\cup_i F_i$ by $\mathcal{F}$ and $\cup_i S_i$ by $\mathcal{S}$. The surfaces in $\mathcal{F}$ are called the thin levels and the surfaces in $\mathcal{S}$ the thick levels.*

Let $M = V \cup_S W$ be an irreducible Heegaard splitting. We may think of M as being obtained from $\partial_- V \times I$ by attaching all 1-handles in V followed by all 2-handles in W, followed, perhaps, by 3-handles. A weak reduction of $M = V \cup_S W$ is a rearrangement of the order in which the 1-handles of V and the 2-handles of W are attached. This rearrangement yields a generalized Heegaard splitting. For convenience, we will occasionally denote $\partial_- V = \partial_- V_1$ by F_0.

The Main Theorem in [11] together with the calculation [13, Lemma 2] implies the following:

Theorem 1. *Suppose M is an irreducible compact 3-manifold. Then M possesses a Heegaard splitting*

$$M = V \cup_S W$$

if and only if M possesses a strongly irreducible generalized Heegaard splitting

$$M = (V_1 \cup_{S_1} W_1) \cup_{F_1} (V_2 \cup_{S_2} W_2) \cup_{F_2} \cdots \cup_{F_{n-1}} (V_n \cup_{S_n} W_n)$$

such that

$$\sum_{i=1}^{n} J(V_i) = J(V).$$

The details can be found in [10]. Roughly speaking, one implication comes from taking a weak reduction of a given Heegaard splitting of genus g, the other from thinking of a given generalized Heegaard splitting as a weak reduction of some Heegaard splitting. The latter process is called the *amalgamation* (for details see [15]) of the generalized Heegaard splitting.

A strongly irreducible Heegaard splitting can be isotoped so that its splitting surface, S intersects an incompressible surface, Q, only in curves essential in both S and Q. This is a deep fact and is proven, for instance, in [16, Lemma 6]. This fact, together with the fact that incompressible surfaces can be isotoped to meet only in essential curves, establishes the following:

Lemma 1. *Let Q be a properly imbedded incompressible surface in an irreducible 3-manifold M and let $M = (V_1 \cup_{S_1} W_1) \cup_{F_1} \cdots \cup_{F_{n-1}} (V_n \cup_{S_n} W_n)$ be a strongly irreducible generalized Heegaard splitting of M. Then $\mathcal{F} \cup \mathcal{S}$ can be isotoped to intersect Q only in curves that are essential in both Q and $\mathcal{F} \cup \mathcal{S}$.*

Definition 7. *Let $M = V \cup_S W$ be a Heegaard splitting. A Heegaard splitting is stabilized if there are disks $D \subset V, E \subset W$ such that $|\partial D \cap \partial E| = 1$. A destabilization of $M = V \cup_S W$ is a Heegaard splitting obtained from $M = V \cup_S W$ by cutting along the cocore of a 1-handle. (E.g., if $D \subset V, E \subset W$ is a stabilizing pair of disks, then D is the cocore of a 1-handle of V and the existence of E guarantees* *that the result of cutting along D results in a Heegaard splitting.) We say that a Heegaard splitting $M = X \cup_T Y$ is a* <u>*stabilization*</u> *of $M = V \cup_S W$, if there is a sequence of Heegaard splittings $M = X^1 \cup_{T^1} Y^1, \dots, M = X^l \cup_{T^l} Y^l$ with $X^1 = X, Y^1 = Y, X^l = V, Y^l = W$ and $X^r \cup_{S^r} Y^r$ is obtained from $X^{r-1} \cup_{S^{r-1}} Y^{r-1}$ by a destabilization.*

2. The Construction

In this section, we suppose that M is a compact orientable 3-manifold with generalized Heegaard splitting $M = (V_1 \cup_{S_1} W_1) \cup_{F_1} \cdots \cup_{F_{n-1}} (V_n \cup_{S_n} W_n)$ and that Q is a compact orientable surface in M. Assuming that Q can be isotoped so that all components of $Q \cap (\mathcal{S} \cup \mathcal{F})$ are essential in both Q and $\mathcal{S} \cup \mathcal{F}$ (which can be guaranteed, if, for instance, Q is assumed, in addition, to be incompressible), we describe a construction for a generalized Heegaard splitting of the closure of a component of $M - Q$.

A more concrete approach to this sort of construction would be to consider a Heegaard splitting $M = V \cup_S W$, make V very thin, so it intersects Q in a collar of ∂Q and disks, cut along Q and add a collar of the copies of components of Q in the boundary of the component C of $M - Q$ to $V \cap C$. This would indeed yield a Heegaard splitting of C and is the basis for the construction in [7]. However, the approach there, though simpler at the outset, requires far more work to gain (slightly less) control over the number of disks in $V \cap Q$. Thus, although the construction in this section appears to be more complicated than required, it will become evident in the following sections, that it allows for more satisfactory control over the sum of indices of the resulting generalized Heegaard splittings.

Denote the closures of the components of $M - Q$ by $M^1, \dots, M^k$.

Definition 8. *For a submanifold $N \subset M$, we will denote $N \cap M^j$ by N^j. E.g., $S_3^1 = S_3 \cap M^1$, $W_5^2 = W_5 \cap M^2$, $Q^j = Q \cap M^j$. For P a compact subsurface of Q, we call $P^j \subset \partial M^j$ the* <u>*remnants of P*</u> *for $j = 1, \dots, k$.*

Note that S_i^j and F_i^j will typically not be closed surfaces. Also, V_i^j and W_i^j will typically not be compression bodies. Also, V_i^j and W_i^j will not necessarily be connected. We can only say the following:

Lemma 2. *Let X be a compression body and let P be an essential surface in X. Let C be the result of cutting $\partial_- X$ along $P \cap \partial_- X$. Then the result of cutting X along P is $(C \times I) \cup (1 - handles)$.*

Proof. Let $\mathcal{D}$ be a set of defining disks for X, chosen so that the number of components of $P \cap \mathcal{D}$ is minimal. In particular, since P is incompressible, this intersection will contain no closed components. Cutting X along $\mathcal{D}$ yields $\partial_- W \times I$ (or a $3 - ball$ in the case that X is a handlebody). The components of $P \cap (\partial_- W \times I)$ (or $P \cap (3 - ball)$, respectively) will either be spanning annuli or boundary parallel incompressible surfaces. Furthermore, an incompressible surface that is boundary parallel in $\partial_- W \times I$ cannot be parallel into $\partial_- W$. Thus further cutting along $P \cap (\partial_- W \times I)$ (or $P \cap (3 - ball)$, respectively) yields $C \times I$ together with handlebodies (including 3-balls). Now the result of cutting X along P may be reconstructed from the components of $(C \times I) \cup (handlebodies)$ by glueing along the disk components of $\mathcal{D} - P \cap \mathcal{D}$. But this corresponds to attaching $1 - handles$. The lemma follows. □

By Lemma 2, $V_i^j = (F_{i-1}^j \times I) \cup (1-handles)$ and $W_i^j = (F_i^j \times I) \cup (1-handles)$. The following construction appears to be a fairly natural way of "capping off" the components of $\partial F_{i-1}^j \times I$ (thought of in V_i^j) and of $\partial F_i^j \times I$ (thought of in W_i^j) with appropriate $((punctured\ \ surface) \times I)$'s. This is the first step in constructing a generalized Heegaard splittings on M^j for $j = 1, \dots, k$. The difficulty lies in "capping off" $\partial F_{i-1}^j \times I$ (thought of in V_i^j) in a way that is consistent with the way in which $\partial F_{i-1}^j \times I$ (thought of in W_{i-1}^j) and $\partial F_i^j \times I$ (thought of in W_i^j) are "capped off".

Lemma 3. *(The Key Lemma) Suppose M is a compact orientable 3-manifold with generalized Heegaard splitting*

$$M = (V_1 \cup_{S_1} W_1) \cup_{F_1} \cdots \cup_{F_{n-1}} (V_n \cup_{S_n} W_n)$$

and suppose Q is a compact orientable surface in M that cuts M into 3-manifolds $M^1, \dots, M^k$. Further suppose that

$$\partial Q \subset \cup_i \partial_- V_i$$

and that $\mathcal{S} \cup \mathcal{F}$ has been isotoped so that all components of $Q \cap (\mathcal{S} \cup \mathcal{F})$ are essential in both Q and $\mathcal{S} \cup \mathcal{F}$ and so that the number of such components is minimal. Then

$$M = (V_1 \cup_{S_1} W_1) \cup_{F_1} \cdots \cup_{F_{n-1}} (V_n \cup_{S_n} W_n)$$

induces generalized Heegaard splittings on $M^1, \dots, M^k$.

fig. 1

Proof. Two cases must be considered.
Case 1: Q is parallel to a subset $\bar{F}$ of $\mathcal{F}$, for some l.

In this case, isotope Q to coincide with $\bar{F}$. Then let M^j be the closure of a component of $M - Q$ and let $V_{i_1}, \dots, V_{i_l}$, $W_{i_1}, \dots, W_{i_l}$ be the compression bodies among $V_1, \dots, V_n, W_1, \dots, W_n$ that meet M^j. Set $A_i^j = V_i$, if $i \epsilon \{i_1, \dots, i_l\}$ and $A_i^j = \emptyset$ otherwise, $B_i^j = W_i$, if $i \epsilon \{i_1, \dots, i_l\}$ and $B_i^j = \emptyset$ otherwise, $G_i^j = S_i$ if $i \epsilon \{i_1, \dots, i_l\}$ and $G_i^j = \emptyset$ otherwise, and $P_i^j = F_i$ if $i \epsilon \{i_1, \dots, i_l\}$ and $P_i^j = \emptyset$ otherwise. Note that in this case the Main Theorem follows immediately.
Case 2: Q is not parallel to F_l, for any l.

Cut M along Q, as above, to obtain copies of $M^1, \dots, M^k$. Let $h : M \to [0,1]$ be a Morse function corresponding to $M = (V_1 \cup_{S_1} W_1) \cup_{F_1} \cdots \cup_{F_{n-1}} (V_n \cup_{S_n} W_n)$ with $S_i = h^{-1}(s_i)$ and $F_i = h^{-1}(f_i)$ for appropriate regular values $s_1, \dots, s_n$ and $f_1, \dots, f_{n-1}$. Set $q_t = Q \cap h^{-1}(t)$ and $Q_{[t_1,t_2]} = Q \cap h^{-1}([t_1,t_2])$. Here q_t will consist of a collection of circles (at least for regular values of h_Q) and $Q_{[t_1,t_2]}$ will be a subsurface of Q with $\partial Q_{[t_1,t_2]} = q_{t_1} \cup q_{t_2}$.

Consider $Q \times I$ and extend h_Q across $Q \times I$ in a way that is consistent with the product structure. There is a proper imbedding of $Q_{[s_1,1]} \sqcup Q_{[f_1,1]} \sqcup \cdots \sqcup Q_{[f_{n-1},1]} \sqcup Q_{[s_n,1]}$ into $Q \times I$, with $\partial Q_{[s_1,1]} \sqcup \partial Q_{[f_1,1]} \sqcup \cdots \sqcup \partial Q_{[f_{n-1},1]} \sqcup \partial Q_{[s_n,1]} \subset Q \times \{0\}$, that is level preserving. (This imbedding is unique up to isotopy.) Each of the imbedded subsurfaces of Q is in fact boundary parallel in $Q \times I$, but the fact used latter on is that the portion a_i cut out of $Q \times I$ by $Q_{[f_{i-1},1]}$ and $Q_{[s_i,1]}$ (or just $Q_{[s_1,1]}$ for $i = 1$) is homeomorphic to $Q_{[f_{i-1},1]} \times I/ \sim$, where $(x,t) \sim (y,s)$ if and only $x = y$ and $h(x) = h(y) = f_{i-1}$ (or simply $Q \times I$ for $i = 1$), and the portion b_i cut out of $Q \times I$

by $Q_{[s_i,1]}$ and $Q_{[f_i,1]}$ (or just $Q_{[s_n,1]}$ for $i = n$) is homeomorphic to $Q_{[s_i,1]} \times I/\sim$, where similarly, $(x,t) \sim (y,s)$ if and only if $x = y$ and $h(x) = h(y) = s_i$.

Denote the portion of Q remnants of which lie in ∂M^j by Q^j. Here Q^j will consist of a union of components of Q. Note that a component of Q^j either lies in $Q^{j'}$ for some $j' \neq j$ and appears exactly once as a remnant in ∂M^j and $\partial M^{j'}$, or it lies exclusively in Q^j and appears exactly twice as a remnant in ∂M^j. Identify M^j and $Q^j \times I$ along the remnant or remnants of Q^j in ∂M^j and $Q^j \times \{0\} \subset Q \times I$ in a level preserving way, to obtain a manifold $M^j \cup_{Q^j} (Q^j \times I)$ homeomorphic to M^j. Denote the components of a_i that meet $Q^j \times \{0\}$ by a_i^j and the components of b_i that meet $Q^j \times \{0\}$ by b_i^j. Similarly, denote the components of $Q_{[s_i,1]}$ and $Q_{[f_i,1]}$,respectively, that meet $Q^j \times \{0\}$ by $Q^j_{[s_i,1]}$ and $Q^j_{[f_i,1]}$, respectively.

Set $\tilde{S}_i^j = S_i^j \cup Q^j_{[s_i,1]}$ and $\tilde{F}_i^j = F_i^j \cup Q^j_{[f_i,1]}$. Then $\tilde{S}_i^j$ and $\tilde{F}_i^j$ are closed surfaces. Further, set $\tilde{V}_i^j = V_i^j \cup a_i^j$ and $\tilde{W}_i^j = W_i^j \cup b_i^j$. Here $\tilde{V}_i^j$ is obtained by attaching V_i^j to a_i^j along a subsurface of $Q^j \times \{0\}$ homeomorphic to $Q^j_{[f_{i-1},s_i]}$ and $\tilde{W}_i^j$ is obtained by attaching W_i^j to b_i^j along a subsurface of $Q^j \times \{0\}$ homeomorphic to $Q^j_{[s_i,f_i]}$. Thus $\tilde{V}_i^j$ will be a cobordism between $\tilde{F}^j_{i-1}$ and $\tilde{S}_i^j$ and $\tilde{W}_i^j$ will be a cobordism between $\tilde{S}_i^j$ and $\tilde{F}_i^j$, but at this stage, neither $\tilde{V}_i^j$ nor $\tilde{W}_i^j$ need be a compression body.

Now let α_i^j be a union of properly imbedded arcs in $Q^j_{[f_{i-1},s_i]}$ disjoint from $q_{f_{i-1}}$ that cut $Q^j_{[f_{i-1},s_i]}$ into disks and into annuli with exactly one boundary component in $q_{f_{i-1}}$. I.e., $Q^j_{[f_{i-1},s_i]} - \alpha_i^j = (q^j_{f_{i-1}} \times I) \cup (disks)$. Analogously, choose β_i^j in $Q^j_{[s_i,f_i]}$. Then set $\dot{V}_i^j = (\tilde{V}_i^j - \eta(\alpha_i^j)) \cup \eta(\beta_i^j)$ and $\dot{W}_i^J = (\tilde{W}_i^j - \eta(\beta_i^j)) \cup \eta(\alpha_i^j)$.

Claim: $\dot{V}_i^j$ and $\dot{W}_i^j$ are compression bodies for $i = 1, \dots, n$ and $j = 1, \dots, k$.

Consider the construction of $\dot{V}_i^j$. By Lemma 2, $V_i^j = (F_{i-1} \times I) \cup (1 - handles)$. $\dot{V}_i^j$ is obtained from $V_i^j = (F_i^j \times I) \cup (1 - handles)$ and a_i^j by glueing along $Q_{[f_{i-1},s_i]} - \eta(\alpha_i^j)$ and then adding the $1 - handles$ $N(\beta_i^j)$. Since $Q^j_{[f_{i-1},s_i]} - \eta(\alpha_i^j) = (q^j_{f_{i-1}} \times I) \cup (disks)$, we see that $\dot{V}_i^j = ((F_i^j \cup_{q_{f-1}} Q_{[f_{i-1},1]}) \times I) \cup (1 - handles)$. Similarly for $\dot{W}_i^j$. This proves the Claim.

Set $\dot{S}_i^j = \partial_+ \dot{V}_i^j = \partial_+ \dot{W}_i^j$ for $i = 1, \dots, n$ and $j = 1, \dots, k$ and $\dot{F}_i^j = \partial_- \dot{W}_i^j = \partial_- \dot{V}^j_{i+1}$ for $i = 1, \dots n-1$ and $j = 1, \dots, k$. The generalized Heegaard splitting induced on M_j is $M^j = (\dot{V}_1^j \cup_{\dot{S}_1^j} \dot{W}_1^j) \cup_{\dot{F}_1^j} \cdots \cup_{\dot{F}^j_{n-1}} (\dot{V}_n^j \cup^j_{\dot{S}_n} \dot{W}_n^j)$. $\square$

For α_i^j as above, denote $\cup_j \alpha_i^j$ by α_i, similarly for β_i^j.

Lemma 4. *(Preliminary Count) For the construction above,*

$$\sum_j \sum_i J(\dot{V}_i^j) = \sum_i (J(V_i) + 2\chi(Q_{[f_{i-1},s_i]}) + 4(|\alpha_i| + |\beta_i|)).$$

Proof. Consider the Euler characteristics of the surfaces in the construction above. Since the Euler characteristic of a circle is 0, $\sum_j \chi(S_i^j) = \chi(S_i)$, for $i = 1, \dots, n$ and $\sum_j \chi(F_i^j) = \chi(F_i)$, for $i = 0, \dots, n-1$. Furthermore,

$$\sum_j \sum_i \chi(\dot{S}_i^j) = \sum_i \sum_j (\chi(S_i^j) + 2\chi(Q^j_{[s_i,1]}) - 4|\alpha_i^j| - 4|\beta_i^j|) =$$

$$\sum_i(\sum_j \chi(S_i^j) + 2\chi(Q_{[s_i,1]}) - 4|\alpha_i| - 4|\beta_i|)$$

, and $\sum_j \chi(\dot{F}_i^j) = \sum_j \chi(F_i^j) + 2\chi(Q_{[f_i,1]})$. Thus

$$\sum_i \sum_j J(\dot{V}_i^j) =$$

$$\sum_i \sum_j (\chi(\dot{F}_{i-1}^j) - \chi(\dot{S}_i^j)) =$$

$$\sum_i \sum_j (\chi(F_{i-1}^j) - \chi(S_i^j)) + 2(\chi(Q_{[f_{i-1},1]}) - \chi(Q_{[s_i,1]})) + 4(|\alpha_i| + |\beta_i|)) =$$

$$\sum_i (\chi(F_{i-1} - \chi(S_i) + 2\chi(Q_{[f_{i-1},s_i]}) + 4(|\alpha_i| + |\beta_i|)) =$$

$$\sum_i (J(V_i) + 2\chi(Q_{[f_{i-1},s_i]}) + 4(|\alpha_i| + |\beta_i|)).$$

□

3. Destabilizations

The idea in this section is to systematically locate stabilizing pairs of disks in $M^j = (\dot{V}_1^j \cup_{\dot{S}_1^j} \dot{W}_1^j) \cup_{\dot{F}_1^j} \cdots \cup_{\dot{F}_{n-1}^j} (\dot{V}_n^j \cup^j_{\dot{S}_n} \dot{W}_n^j)$ and to destabilize along the pairs. To this end, let $\mathcal{D}_i$ be a defining set of disks for V_i and let $\mathcal{E}_i$ be a defining set of disks for W_i. We may assume that $\mathcal{D}_i$ and $\mathcal{E}_i$ have been chosen to minimize $\mathcal{D}_i \cap Q_{[f_{i-1},s_i]}$ and $\mathcal{E}_i \cap Q_{[s_i,f_i]}$. In particular, since $Q_{[f_{i-1},s_i]}$ and $Q_{[s_i,f_i]}$ are incompressible in V_i and W_i, respectively, these intersections will contain no closed components. In what follows, we will assume that the collection of arcs α_i used in the construction of Lemma 3 is $\mathcal{D}_i \cap Q^j_{[f_{i-1},s_i]}$ and the collection of arcs β_i is $\mathcal{E}_i \cap Q^j_{[s_i,f_i]}$. This will allow us to locate subdisks of $\mathcal{D}_i$, $\mathcal{E}_i$ and of Q, that meet an arc component of α_i or β_i exactly once. This will yield a stabilizing pair of disks. See fig. 2.

fig. 2

Though the result here is a natural extension of the result in [14], the strategy here is somewhat different. The reason for this difference is the fact that [14, Lemma 4.8] fails when Q no longer consists of annuli, which necessitates a slightly more careful procedure. The philosophy remains the same.

Definition 9. *Let T_{V_i} be the tree obtained as follows: assign a vertex to each component of $\mathcal{D}_i - \alpha_i$; connect two vertices by an edge, if and only if the corresponding components of $\mathcal{D}_i - \alpha_i$ abut along an arc component in α_i. See fig. 3.*

fig. 3

Note that distinct components of $\mathcal{D}_i - \alpha_i$ lying in V_i^j do not meet.

Definition 10. *We assign weights to the vertices of T_{V_i} inductively. Select an outermost vertex of T_{V_i} that is not isolated and assign it a weight of* 1. *Then delete that vertex from $T_{V_i} = T^1$ to obtain T^2. Inductively, select an outermost vertex of T^l that is not isolated and assign it a weight of l, then delete the vertex to obtain T^{l+1}.*

Let e be an edge of T_{V_i}. It corresponds to an arc a in α_i. There are two copies of a in $M^1, \dots, M^k$. We refer to them by a^+ and a^-. Here we consider a^+ and a^- to be pushoffs of a into the two components of $\mathcal{D}_i - \alpha_i$ adjacent to a and take a^+ to be the pushoff into the region corresponding to the vertex of T_{V_i} of higher weight. Correspondingly, we refer to the two components of $\mathcal{D}_i - \alpha_i$ by D_{a^+} and D_{a^-}. Note that here a^+ is a component of α_i^j for some j and a^- is a component of α_i^l for some l, where, possibly, $j = l$.

Now $N(a^+)$ lies in $\dot{W}_i^j$ for some j, $N(A^-)$ lies in $\dot{W}_i^l$ for some l and shrunk versions of D_{a^+} and D_{a^-} lie in $\dot{V}_i^j$ and $\dot{V}_i^l$. Thus a cocore of $N(a^+)$ together with the shrunk version of D_{a^+} forms a stabilizing pair of disks for $\dot{V}_i^j \cup_{\dot{S}_i^j} \dot{W}_i^j$ and a cocore of $N(a^-)$ together with the shrunk version of D_{a^-} forms a stabilizing pair of disks for $\dot{V}_i^l \cup_{\dot{S}_i^l} \dot{W}_i^l$ for some l. Note however, that it may not be possible to perform the two corresponding destabilizations in sequence. We wish to keep track of the number of destabilizations of this type that can be performed in sequence.

Definition 11. *For convenience, we assign weights to the edges of T_{V_i} as well, by setting $w(e) = w(v)$, where v, v' are the vertices of e, and $w(v) < w(v')$.*

Procedure I:

Step 1: Perform the destabilization corresponding to the cocore of $N(a^-)$ together with the shrunk version of D_{a^-} arising as above, for a the component of $\mathcal{D}_i - \alpha_i$ corresponding to the edge e of T_{V_i} such that $w(e) = 1$. I.e., cut along the cocore of $N(A^-)$. See fig. 4.

fig. 4

Step k: Proceed as in Step 1, but using the edge of weight k. Our choices guarantee that the shrunk version of D_{a^-}, for a the component of $\mathcal{D}_i - \alpha_i$ corresponding to the edge e such that $w(e) = k$, is properly imbedded in V_i^j, for some j. fig. 5.

fig. 5

Strictly speaking, we could perform one more destabilization per component of $\mathcal{D}_i$, corresponding to the case in which a component of T^k has exactly one edge, since in this case both the shrunk version of D_{a^-} and the shrunk version of D_{a^+}, for a the component of $\mathcal{D}_i - \alpha_i$ corresponding to the unique edge meeting the vertex v of weight k, are properly imbedded in V_i^j, for some j. See fig. 6. But this causes unnecessary technicalities in Procedure II, where any gain here would be lost.

fig. 6

Lemma 5. *Procedure I locates a combined total of*

$$\sum_i |\alpha_i| + \sum_i |\beta_i|$$

destabilizations for

$$M^j = (\dot{V}_1^j \cup_{\dot{S}_1^j} \dot{W}_1^j) \cup_{\dot{F}_1^j} \cdots \cup_{\dot{F}_{n-1}^j} (\dot{V}_n^j \cup_{\dot{S}_n^j} \dot{W}_n^j) \;\; for \;\; j = 1, \dots, k$$

that can be performed in sequence.

Proof. The total number of destabilizations performed during Procedure I is exactly $|\alpha_i|$ (one per edge of T_{V_i}). Applying the same procedure to $\mathcal{E}_i$ yields $|\beta_i|$ destabilizations. The lemma follows by summing over i. $\square$

Remark 1. *For ease of notation, we will continue to denote the result of performing the destabilizations arising in Procedure I on*

$$M^j = (\dot{V}_1^j \cup_{\dot{S}_1^j} \dot{W}_1^j) \cup_{\dot{F}_1^j} \cdots \cup_{\dot{F}_{n-1}^j} (\dot{V}_n^j \cup_{\dot{S}_n^j} \dot{W}_n^j)$$

by

$$M^j = (\dot{V}_1^j \cup_{\dot{S}_1^j} \dot{W}_1^j) \cup_{\dot{F}_1^j} \cdots \cup_{\dot{F}_{n-1}^j} (\dot{V}_n^j \cup_{\dot{S}_n^j} \dot{W}_n^j).$$

For each arc component a of α_i, ambient 2-surgery is performed on the boundary of the cocore of $N(a^-)$ in Procedure I. (The destabilizations occur on the side of Q opposite the one that would be termed the "root side" in [14].)

Definition 12. *Let a be an arc component of α_i. We sometimes refer to a^+ as the surviving arc for a. We also define a function from the components of α_i to $1, \ldots, k$ by setting $c(a) = j$ if a^+ lies in M^j.*

Remark 2. *With one exception per component of $\mathcal{D}_i$, it is the case that for each component D of $\mathcal{D}_i - \alpha_i$, there is an arc a in α_i, such that $D = D_{a^-}$.*

The collections α_i and β_i above may be a rather inefficient choice of arcs if considered in terms of their number of components, but they are an efficient choice for keeping track of the number of destabilizations of the type just discussed. See fig. 7.

fig. 7

Now we will locate stabilizing pairs of disks that counterbalance the redundancy in the choice of α_i and β_i. To do so, consider a disk component D of $Q_{[f_{i-1},s_i]} - \alpha_i$. The boundary of D is partitioned into subarcs lying alternately in S_i and α_i. Before Procedure I, there were two copies of D, in V_i^j for $j = 1, \ldots, k$, whose boundary was partitioned into subarcs lying alternately in S_i^j and arc components of α_i^j (the boundary of a shrunk version of D^j would lie entirely in $\partial_+ \dot{V}_i^j$). After Procedure I, however, the subarcs of the boundary of D^j lying on copies of the arc components of α_i along which destabilizations occurred, lie in the interior of $\dot{V}_i^j$.

Let a be an arc component of α_i contained in the boundary of D. One copy of D in $M^1, \ldots, M^k$ contains a^+, another contains a^-. If M^j, say, contains the copy D^j of D containing a^-, then a destabilization occurred along a^-. We may augment D^j by adding D_{a^-} along a^-. See fig. 8 and fig. 9.

fig. 8

fig. 9

Definition 13. *Define $\hat{D}^j$ to be the result of performing all appropriate augmentations of this form on D^j. (Note that more than one arc component of α_i^j in ∂D^j may have been affected.)*

It follows that for each disk component D of $Q_{[f_{i-1},s_i]} - \alpha_i$, even after Procedure I, there will still be two disks of the form $\hat{D}^j$ (possibly both in the same M^j) such that $\partial \hat{D}^j$ decomposes into subarcs lying alternately in S_i^j and α_i^j and such that a shrunk version of $\hat{D}^j$ lies in $\dot{V}_i^j$. However, if D meets some arc component a of α_i exactly once, we may not necessarily have located a stabilizing pair of disks. This is due to the fact that the augmentations after Procedure I may produce disks $\hat{D}^j$ such that $\partial \hat{D}^j$, say, contains more than one copy of a^+.

The following remarks and definitions enable a more careful choice of D and a that does locate stabilizing pairs of disks.

Remark 3. *Each arc component a of α_i meets two components of $\mathcal{D}_i - \alpha_i$ and two components of $Q_{[f_{i-1},s_i]} - \alpha_i$. Let D be a component of $Q_{[f_{i-1},s_i]} - \alpha_i$ that meets a once along its boundary. Let $a_1, \dots, a_k$ be the other components of α_i in the boundary of D. Suppose $c(a) = j$ and consider $\hat{D}^j$. In V_i^j, a copy of exactly one component, D_{a^+}, of $\mathcal{D}_i - \alpha_i$ meets a^j. If $D_{a^+} = D_{b^-}$ (see Remark 2), for $b = a_l$ for $l \epsilon \{1, \dots, k\}$, then, and only then, will more than one copy of a^+ lie in the boundary of $\hat{D}^j$. See fig. 10.*

fig. 10

Definition 14. *Define a function f from the components of α_i to α_i by letting $f(a)$ be the subset of α_i lying in ∂D_{a^+}.*

See fig. 11.

fig. 11

Definition 15. *Let $T_{Q_{[f_{i-1},s_i]}}$ be the graph obtained as follows: assign a vertex to each component of $Q_{[f_{i-1},s_i]} - \alpha_i$; connect two vertices by an edge if and only if the corresponding components of $Q_{[f_{i-1},s_i]} - \alpha_i$ abut along an edge of α_i. Then delete isolated vertices corresponding to disk components of $Q_{[f_{i-1},s_i]} - \alpha_i$ and edges with both ends on one vertex corresponding to a disk component of $Q_{[f_{i-1},s_i]} - \alpha_i$ to obtain T^1.*

Procedure II:

Step 1: Let v be a vertex of T^1 corresponding to a disk component D of $Q_{[f_{i-1},s_i]} - \alpha_i$. Let e be the edge emanating from v such that $w(e)$ is maximal. Let a be the arc corresponding to e. Let v' be the other endpoint of e and let E be the component of $Q_{[f_{i-1},s_i]} - \alpha_i$ corresponding to v'. Since $w(b) > w(a)$, for all b in $f(a)$, $f(a) \cap D = \emptyset$. Thus a shrunk version of $\partial\hat{D}^j$, for $j = c(a)$ is properly imbedded in $\dot{V}_i^j$ and meets the boundary of the cocore of $N(a^+)$ exactly once. I.e., the two disks define a stabilizing pair. Delete a from $\alpha_i = \alpha_i^1$ to obtain a new set of arcs α_i^2 and construct a new graph T^2 accordingly.

Step k: Proceed as in Step 1, working with a vertex in T^k that corresponds to a disk component of $Q_{f_{i-1},s_i]} - \alpha_i^k$. Then construct T^{k+1} analogously.

Definition 16. *Let qv_i be the number of components of $Q_{[f_{i-1},s_i]}$ that do not meet $q_{f_{i-1}}$ and let qw_i be the number of components of $Q_{[s_i,f_i]}$ that do not meet q_{f_i}.*

Lemma 6. *Procedure II locates a combined total of at least*

$$\sum_i (|\alpha_i| + |\beta_i| - qv_i - qw_i) + \chi(Q)$$

destabilizations for

$$M^j = (\dot{V}_1^1 \cup_{\dot{S}_1^1} \dot{W}_1^1) \cup_{\dot{F}_1^1} \cdots \cup_{\dot{F}_{n-1}^1} (\dot{V}_n^1 \cup_{\dot{S}_n^1} \dot{W}_n^1) \ \ for \ \ j = 1, \dots, k$$

that can be performed in sequence.

Proof. Procedure II terminates when no vertices of T^l correspond to disk components of $Q_{[f_{i-1},s_i]} - \alpha_i^l$. Recall that $Q_{[f_{i-1},s_i]} - \alpha_i$ consists of disks and of annuli with exactly one boundary component in $q_{f_{i-1}}$. The deletion of arc components of α_i in Procedure II occurs only as long as one component of $Q_{[f_{i-1},s_i]} - \alpha_i^k$ meeting the arc is a disk. Thus $Q_{[f_{i-1},s_i]} - \alpha_i^l$ still consists of disks and of annuli with exactly one boundary component in $q_{f_{i-1}}$. More specifically, Procedure II terminates, when there is one component (either a disk or an annulus with exactly one boundary component in $q_{f_{i-1}}$) of $Q_{[f_{i-1},s_i]} - \alpha_i^l$ left per component of $Q_{[f_{i-1},s_i]}$. I.e., after

$$|\alpha_i| + \chi(Q_{[f_{i-1},s_i]}) - qv_i$$

steps.

At each step, the cocore of $N(a^+)$ together with a shrunk version of $\hat{D}^j$ yield a stabilizing pair of disks. Destabilize by performing ambient 2-surgery along the boundary of a cocore of $N(a^j)$. Preceding steps in no way interfere with subsequent steps.

Similarly, for $\beta_i - Q_{[s_i,f_i]}$. Now sum over i. $\square$

Remark 4. *For ease of notation, we will continue to denote the result of performing the destabilizations arising in Procedure II on*

$$M^j = (\dot{V}_1^j \cup_{\dot{S}_1^j} \dot{W}_1^j) \cup_{\dot{F}_1^j} \cdots \cup_{\dot{F}_{n-1}^j} (\dot{V}_n^j \cup_{\dot{S}_n^j} \dot{W}_n^j)$$

by

$$M^j = (\dot{V}_1^j \cup_{\dot{S}_1^j} \dot{W}_1^j) \cup_{\dot{F}_1^j} \cdots \cup_{\dot{F}_{n-1}^j} (\dot{V}_n^j \cup_{\dot{S}_n^j} \dot{W}_n^j).$$

Procedure III:

This procedure exploits the crucial difference between V_i and W_i in the construction of Lemma 3. If a component Q_c of $Q_{[s_i,f_i]}$ does not meet $\partial_- W_i$, then $Q_c \subset Q_{[s_i,1]}$ with $\partial Q_c \subset q_{s_i}$, and the copy of Q_c in $Q \times I$ is boundary parallel. More specifically, the portion of $Q \times I$ cut off by Q_c is $Q_c \times I/\sim$, where $(x,s) \sim (y,t)$ if and only if $x = y$ and $h(x) = h(y) = s_i$, i.e., $(x,s) \sim (y,t)$ for $x \epsilon \partial Q_c$. This portion is disjoint from any other components of $Q_{[s_1,1]} \sqcup Q_{[f_1,1]} \sqcup \cdots \sqcup Q_{[f_{n-1},1]} \sqcup Q_{[s_n,1]}$ in $Q \times I$. This means that if $Q_c \subset Q^j$, then $Q_c \times I/\sim$ is contained in b_i^j (as in the proof of Lemma 3).

Let $b_1, \ldots, b_{-\chi(Q_c)+1}$ be the subarcs of β_i left in Q_c after Procedure II. For two values of j, a copy of $Q_c \times I/\sim$ is attached to W_i^j in the process of creating $\dot{W}_i^j$. Thus for each of $b_1, \ldots, b_{-\chi(Q_c)+1}$, there will be one surviving arc. Since each of the b_m yields one disk $D_m = (b_m \times I/\sim) \subset (Q_c \times I/\sim)$, the surviving arc b_m^j will have the property that the boundary of the cocore of $N(b_m^j)$ intersects the boundary of a properly imbedded shrunk version of D_m in $\dot{W}_i^j$ exactly once along its boundary.

Lemma 7. *Procedure III locates a combined total of at least*

$$\sum_i qw_i$$

destabilizations that can be performed in sequence.

Proof. Destabilize by performing ambient 2-surgery along the boundary of the cocore of $N(b_m^-)$ for $m = 1, \ldots, -\chi(Q_c) + 1$ and for all such components Q_c. $\square$

(Intuitively speaking, the difference between V_i and W_i in this construction is that in W_i, Q_c is cut off of Q by a collection of curves in q_{s_i} and there is nothing "above" Q_c. Whereas a component Q_d of $Q_{[f_{i-1},s_i]}$ that does not meet $\partial_- V_i$ is cut off by a collection of curves in q_{s_i} and there will be something "above" Q_d.) See fig. 1.

Remark 5. *We will denote the result of performing the destabilizations arising in Procedure III on*

$$M^j = (\dot{V}_1^j \cup_{\dot{S}_1^j} \dot{W}_1^j) \cup_{\dot{F}_1^j} \cdots \cup_{\dot{F}_{n-1}^j} (\dot{V}_n^j \cup^j_{\dot{S}_n} \dot{W}_n^j)$$

by

$$M^j = (\hat{V}_1^j \cup_{\hat{S}_1^j} \hat{W}_1^j) \cup_{\hat{F}_1^j} \cdots \cup_{\hat{F}_{n-1}^j} (\hat{V}_n^j \cup^j_{\hat{S}_n} \hat{W}_n^j).$$

4. Genus Reductions

The procedures above show that the number of components of $Q_{[f_{i-1},s_i]}$ that do not meet $q_{f_{i-1}}$ plays a role in the complexities of the Heegaard splittings constructed. This section is devoted to controlling the contribution arising from annular components of this type.

Definition 17. *An essential annulus A in a compression body W with $\partial A \subset \partial_+ W$ is called a dipping annulus.*

Lemma 8. *If $\mathcal{A}$ is a collection of nonparallel dipping annuli in the compression body W, then $|\mathcal{A}| \leq J(W)$.*

Proof. The proof is by induction on $J(W)$. If $J(W) = 0$, then $\mathcal{A} = \emptyset$. Otherwise, let $\mathcal{D}$ be a set of defining disks for W. Let a be an outermost arc of $\mathcal{A} \cap \mathcal{D}$ in $\mathcal{D}$, A the component of $\mathcal{A}$ and D the component of $\mathcal{D}$ that give rise to a and D' the outermost disk cut off by a in D. Let $\bar{D}$ be the result of replacing a collar of a in A by two parallel copies of D' and performing a small isotopy so that $\bar{D}$ is disjoint from A. Since A is an essential annulus, $\bar{D}$ is an essential disk. Furthermore, $\bar{D}$ is disjoint from $\mathcal{A}$.

Cut W along $\bar{D}$ to obtain either one compression body, W', with $J(W') = J(W)-2$, or two compression bodies, W' and W'', with $J(W')+J(W'') = J(W)-2$. Here $\mathcal{A} \subset \mathcal{W}' \cup \mathcal{W}''$, but some components of $\mathcal{A}$ may now be boundary parallel. An annulus that was essential in W and is boundary parallel in W' or W'' must cut off a solid torus from W' or W'' that contains at least one remnant of $\bar{D}$. Thus as the components of $\mathcal{A}$ are nonparallel, at most two components of $\mathcal{A}$ are boundary parallel in W' and W''. Hence $|\mathcal{A}|-2 \leq J(W')+J(W'')-2$ by inductive hypothesis, whence $|\mathcal{A}| \leq J(W)$. □

In the construction of the preceding sections, however, there is no reason to suppose that there are no parallel dipping annuli. The following lemma shows that with the exception of one parallel pair of dipping annuli per essential annulus in M (up to isotopy), parallel pairs of dipping annuli in $Q \cap V_i$ allow for a reduction in genus of one of the $\dot{V}_i^j \cup_{\dot{S}_i^j} \dot{W}_i^j$. See fig. 12.

fig. 12

Definition 18. *Let p_i be the number of (unordered) pairs (A_1, A_2) of parallel dipping annuli in $Q_{[f_{i-1},s_i]} \cap V_i$ such that there is no other such annulus in the product region between A_1 and A_2. Let n_j be the maximal number of parallelism classes of annuli for which there exist representatives that can be simultaneously imbedded in M^j.*

Lemma 9. *There are generalized Heegaard splittings*

$$M^j = (A_1^j \cup_{G_1^j} B_1^j) \cup_{P_1^j} \cdots \cup_{P_{n-1}^j} (A_n^j \cup_{G_n^j} B_n^j),$$

for $j = 1, \ldots, k$, such that

$$\sum_j \sum_i J(A_i^j) \leq (\sum_j (\sum_i J(\hat{V}_i^j)) - 2p_i) + 2n_j).$$

Proof. Let A_1, A_2 be parallel dipping annuli in $Q \cap V_i$. Let A_3, A_4 be the two annuli in S_i cobounded by $\partial(A_1 \cup A_2)$ in S_i. Then copies of $A_1 \cup A_2 \cup A_3 \cup A_4$ cobound a solid torus in $\hat{V}_i^j$ for some j, say in $\hat{V}_i^1$. At the outset, there was at least one spanning arc of A_1 and A_2 among the copies of α_i in $\hat{V}_i^1$ and there was a correspondance between such arcs in A_1 and in A_2. (This is because as parallel annuli, they intersected each component of $\mathcal{D}_{V_i}$ in parallel arcs.) After Procedure I, pairs of arcs which were parallel in $\mathcal{D}_{V_i}$ lost one representative. Depending on the distribution of weights in T_{V_i}, arcs may have remained in both annuli or only in one.

In Procedure II, arcs would have been deleted if they cut A_1 or A_2 into more than one disk, i.e., if there was more than one such arc in A_1 or A_2. Hence at this stage, there is at least one spanning arc, s, left in either A_1 or A_2 (and possibly in both, though that cannot be guaranteed).

Here $A_3 \subset S_i^1$ and $A_4 \subset S_i^1$ can be augmented by annuli among $q_{s_i} \times I \subset Q \times I$ providing properly imbedded annuli $\tilde{A}_3$ and $\tilde{A}_4$ in M^1. If $\tilde{A}_3$ and $\tilde{A}_4$ extend beyond $\dot{V}_i^1 \cup_{\dot{S}_i^1} \dot{W}_i^1$, then $\tilde{A}_3 - A_3$ and $\tilde{A}_4 - A_4$ lie in a union of product regions of the form $\partial_- V_l^1 \times I$, $\partial_- W_l^1 \times I$. Hence for the purposes of the construction below, we may assume that $\tilde{A}_3, \tilde{A}_4 \subset \dot{V}_i^1 \cup_{\dot{S}_i^1} \dot{W}_i^1$. (Extend across product regions as appropriate.)

Case I: $\tilde{A}_3$, hence also $\tilde{A}_4$, is boundary parallel. See fig. 13.

fig. 13

We may assume that $\tilde{A}_3$ is outermost among $\tilde{A}_3$, $\tilde{A}_4$. Denote the solid torus cut off of M^1 by $\tilde{A}_4$ by T. The result of cutting $\hat{W}_i^1$ along the annulus A_4, and deleting $\hat{W}_i^1 \cap T$ is a compression body. The result of cutting $\hat{V}_i^1$ along the two spanning annuli $\tilde{A}_4 \cap \partial_- \partial V_i \times I$, and replacing $\hat{V}_i^1 \cap T$ with $\tilde{A}_4 \times I$, to obtain a manifold homeomorphic to M^1, is also a compression body. The genus of the resulting Heegaard splitting is at least one less than that of $\hat{V}_i^1 \cup_{\hat{S}_i^1} \hat{W}_i^1$.

fig. 14

In this case, by choosing the pair A_1, A_2 so that $\tilde{A}_3$ is an outermost boundary parallel annulus arising in this way, we may assure that the reduction in genus may be performed for all (unordered) pairs of parallel dipping annuli giving rise to boundary parallel annuli $\tilde{A}_3, \tilde{A}_4$ in M^1.

Case II: $\tilde{A}_3$ and $\tilde{A}_4$ are not boundary parallel.

Let $\mathcal{N}_1$ be a maximal collection of simultaneously imbedded (in particular, disjoint) essential annuli in M^1 containing $\tilde{A}_3$ and containing at most one component per parallelism class of essential annuli in M^1.

Choose $N \subset \mathcal{N}_j$ such that $\tilde{A}_3$ and $\tilde{A}_4$ are parallel to B. Consider all pairs $(\tilde{A}^1_3, \tilde{A}^1_4), \ldots, (\tilde{A}^l_3, \tilde{A}^l_4)$ arising in this way that are parallel to N. See fig. 15.

fig. 15

After renaming, if necessary, $\tilde{A}^1_4, \tilde{A}^2_3, \tilde{A}^2_4, \ldots, \tilde{A}^{l-1}_4, \tilde{A}^l_3$ lie in the product region between $\tilde{A}^1_3$ and $\tilde{A}^l_4$. Cut M^j along $\tilde{A}^1_3 \cup \tilde{A}^l_3$ and glue the result along the remnants of $\tilde{A}^1_3 \cup \tilde{A}^l_3$. The result is a Heegaard splitting whose genus is reduced by at least $l-1$. See fig. 16. $\square$

5. Putting it all together

By performing the construction, counting indices, subtracting amounts corresponding to performing destabilizations and genus reductions, we arrive at two propositions that imply the Main Theorem.

Proposition 2. *Suppose that $M = (V_1 \cup_{S_1} W_1) \cup_{F_1} \cdots \cup_{F_{n-1}} (V_n \cup_{S_n} W_n)$ is a generalized Heegaard splitting and Q a surface in M parallel to a subsurface in $\mathcal{F}$. Then for $M^1, \ldots, M^j$ the closures of components of $M - Q$, there are generalized Heegaard splittings $M^j = (A^j_1 \cup_{G^j_1} B^j_1) \cup_{P^j_1} \cdots \cup_{P^j_{n-1}} (A^j_n \cup^j_{G_n} B^j_n)$ for $j = 1, \ldots, k$ such that*

$$\sum_i J(V_i) = \sum_j \sum_i J(A^j_i).$$

Proof. This follows immediately from Case 1 in Lemma 3. $\square$

Proposition 3. *Suppose M is a compact orientable 3-manifold and $Q \subset M$ a compact orientable surface. Further suppose that $M^1, \ldots, M^k$ are the closures of components of $M - Q$ and that n_j is the maximal number of parallelism classes of essential annuli for which representatives can be simultaneously imbedded in M^j. If $M = (V_1 \cup_{S_1} W_1) \cup_{F_1} \cdots \cup_{F_{n-1}} (V_n \cup_{S_n} W_n)$ is a generalized Heegaard splitting such that each component of $Q \cap (\mathcal{F} \cup \mathcal{S})$ is essential in both Q and $\mathcal{F} \cup \mathcal{S}$, such that the number of such components is minimal subject to this condition, and such that no component of Q is parallel to a component of $\mathcal{F}$, then there are generalized Heegaard splittings $M^j = (A^j_1 \cup_{G^j_1} B^j_1) \cup_{P^j_1} \cdots \cup_{P^j_{n-1}} (A^j_n \cup^j_{G_n} B^j_n)$ of M^j for $j = 1, \ldots, k$ that satisfy the following inequality*

$$\sum_i J(V_i) \geq \frac{1}{3}\Big(\sum_j \Big(\sum_i J(A^j_i) - 2n_j\Big) + 2\chi(Q)\Big).$$

Proof. By Lemma 4, the construction in Lemma 3 gives generalized Heegaard splittings $M^j = (\dot{V}^j_1 \cup_{\dot{S}^j_1} \dot{W}^j_1) \cup_{\dot{F}^j_1} \cdots \cup_{\dot{F}^j_{n-1}} (\dot{V}^j_n \cup^j_{\dot{S}_n} \dot{W}^j_n)$ for which

$$\sum_j \sum_i J(\dot{V}^j_i) = \sum_i (J(V_i) + 2\chi(Q_{[f_{i-1}, s_i]}) + 4(|\alpha_i| + |\beta_i|)).$$

Procedure I leads to a total of

$$\sum_i (|\alpha_i| + |\beta_i|$$

destabilizations; Procedure II leads to an additional total of at least

$$\sum_i ((|\alpha_i| + |\beta_i|) - (qv_i + qw_i)) + \chi(Q)$$

destabilizations; Procedure III leads to a combined total of at least

$$\sum_i qw_i$$

destabilizations. The total number of destabilizations in Procedures I, II, and III, is thus at least

$$\sum_i (2(|\alpha_i| + |\beta_i|) - qv_i) + \chi(Q).$$

Hence the generalized Heegaard splittings $M^j = (\hat{V}_1^j \cup_{\hat{S}_1^j} \hat{W}_1^j) \cup_{\hat{F}_1^j} \cdots \cup_{\hat{F}_{n-1}^j} (\hat{V}_n^j \cup_{\hat{S}_n^j} \hat{W}_n^j)$ for $j = 1, \dots, k$, obtained by performing Procedures I, II, and III on $M^j = (\dot{V}_1^j \cup_{\dot{S}_1^j} \dot{W}_1^j) \cup_{\dot{F}_1^j} \cdots \cup_{\dot{F}_{n-1}^j} (\dot{V}_n^j \cup_{\dot{S}_n^j} \dot{W}_n^j)$ satisfy

$$\sum_j \sum_i J(\hat{V}_i^j) \leq \sum_j \sum_i J(\dot{V}_i^j) - 2(\sum_i (2(|\alpha_i| + |\beta_i|) - qv_i) + \chi(Q)) =$$

$$\sum_i (J(V_i) + 2\chi(Q_{[f_{i-1}, s_i]}) + 4(|\alpha_i| + |\beta_i|) - 4(|\alpha_i| + |\beta_i|) + 2qv_i) - 2\chi(Q)) \leq$$

$$\sum_i (J(V_i) + 2qv_i - 2\chi(Q_{[s_i, f_i]})).$$

Let da_i be the number of dipping annuli in $Q_{[f_{i-1}, s_i]} \cap V_i$, then

$$qv_i \leq -\chi(Q_{[f_{i-1}, s_i]}) + da_i$$

and

$$\sum_i da_i \leq \sum_i (J(V_i) + p_i).$$

Whence

$$\sum_i \sum_j J(\hat{V}_i^j) \leq \sum_i (J(V_i) - 2\chi(Q_{[s_i, f_i]}) - 2\chi(Q_{[f_{i-1}, s_i]}) + 2J(V_i) + 2p_i) \leq$$

$$\sum_i (3J(V_i) + 2p_i) - 2\chi(Q).$$

There will be at least

$$\sum_i p_i - \sum_j n_j$$

genus reductions in Lemma 9.

Let $M^j = (A_1^j \cup_{G_1^j} B_1^j) \cup_{P_1^J} \cdots \cup_{P_{n-1}^j} (A_n^j \cup_{G_n^j} B_n^j)$ be the result of performing all genus reductions as in Lemma 9 on $M^j = (\hat{V}_1^j \cup_{\hat{S}_1^j} \hat{W}_1^j) \cup_{\hat{F}_1^j} \cdots \cup_{\hat{F}_{n-1}^j} (\hat{V}_n^j \cup_{\hat{S}_n^j} \hat{W}_n^j)$ for $j = 1, \dots, k$. Then

$$\sum_j \sum_i J(A_i^j) \leq \sum_i \sum_j J(\hat{V}_i^j) - 2\sum_i p_i + 2\sum_j n_j \leq$$

$$\sum_i 3J(V_i) - 2\chi(Q) + 2\sum_j n_j.$$

Whence

$$\sum_i J(V_i) \geq \frac{1}{3}(\sum_j(\sum_i J(A_i^j) - 2n_j) + 2\chi(Q)).$$

□

Theorem 4. *(The Main Theorem) Suppose that $M = (V_1 \cup_{S_1} W_1) \cup_{F_1} \cdots \cup_{F_{n-1}} (V_n \cup_{S_n} W_n)$ is a strongly irreducible generalized Heegaard splitting and Q a compact orientable incompressible surface in M. Let $M^1, \dots, M^k$ be the closures of the components of $M - Q$ and let n_j be the maximal number of parallelism classes of essential annuli in M^j for which there are representatives that can be simultaneously imbedded in M^j. Then there are generalized Heegaard splittings $M^j = (A_1^j \cup_{G_1^j} B_1^j) \cup_{P_1^j} \cdots \cup_{P_{n-1}^j} (A_n^j \cup_{G_n}^j B_n^j)$ for $M^1, \dots, M^k$ that satisfy the following inequality*

$$\sum_i (J(V_i)) \geq \frac{1}{3}(\sum_j(\sum_i J(A_i^j) - n_j) + 2\chi(Q)).$$

Proof. By Lemma 1, Q may be isotoped so that each component of $Q \cap (\mathcal{F} \cup \mathcal{S})$ is essential in both Q and $\mathcal{F} \cup \mathcal{S}$. We may assume that the number of components of $Q \cap (\mathcal{F} \cup \mathcal{S})$ is minimal subject to this condition. We then first decompose M along all components of Q parallel to components of $\mathcal{F}$ and apply Proposition 2. We then further decompose the resulting manifolds along all remaining components of Q and apply Proposition 3. The result follows. □

Theorem 5. *Let M be a compact orientable 3-manifold and Q a compact orientable incompressible surface in M and suppose that the Heegaard genus of M is realized by a Heegaard splitting $M = V \cup_S W$ with $\partial Q \subset \partial_- V$. Then for $M^1, \dots M^k$ the submanifolds into which Q decomposes M, and n_j the number of parallelism classes of essential annuli for which representatives can be simultaneously imbedded in M^j,*

$$g(M) \geq \frac{1}{3}(\sum_j(g(M^j) - n_j) + 3 - k - \chi(\partial_- V) + 2\chi(Q)).$$

Proof. The weak reduction of $M = V \cup_S W$ is a strongly irreducible generalized Heegaard splitting $M = (V_1 \cup_{S_1} W_1) \cup_{F_1} \cdots \cup_{F_{n-1}} (V_n \cup_{S_n} W_n)$ as required in the Main Theorem, which yields generalized Heegaard splittings $M^j = (A_1^j \cup_{G_1^j} B_1^j) \cup_{P_1^j} \cdots \cup_{P_{n-1}^j} (A_n^j \cup_{G_n^j} B_n^j)$ for which

$$\sum_i J(V_i) \geq \frac{1}{3}(\sum_j(\sum_i J(A_i^j) - 2n_j) + 2\chi(Q)).$$

Amalgamating $M^j = (A_1^j \cup_{G_1^j} B_1^j) \cup_{P_1^j} \cdots \cup_{P_{n-1}^j} (A_n^j \cup_{G_n^j} B_n^j)$ yields a Heegaard splitting $M^j = V^j \cup_{S^j} W^j$ with

$$\sum_i J(A_i^j) = 2g(S^j) - 2 + \chi(\partial_- V^j).$$

Hence

$$2g(S) - 2 + \chi(\partial_- V) = \sum_i J(V_i) \geq$$

$$\frac{1}{3}(\sum_j(\sum_i J(A_i^j) - 2n_j) + 2\chi(Q)) =$$

$$\frac{1}{3}(\sum_j(2g(S^j) - 2 + \chi(\partial_- V^j) - 2n_j) + 2\chi(Q))$$

Here

$$\sum_j \chi(\partial_- V^j) = \chi(\partial_- V) + 2\chi(Q).$$

Therefore

$$2g(S) - 2 + \chi(\partial_- V) \geq \frac{1}{3}(\sum_j(2g(S^j) - 2 - 2n_j) + \chi(\partial_- V) + 4\chi(Q)).$$

Whence

$$g(S) \geq \frac{1}{3}(\sum_j(g(S^j) - n_j) + 3 - k - \chi(\partial_- V) + 2\chi(Q))$$

□

Finally, we consider two interesting cases encompassed by this construction. In [4], M. Eudave-Muñoz constructs a family of tunnel number 1 knots whose complements contain incompressible surfaces of arbitrarily high genus. For elementary definitons pertaining to knot theory, see for instance [2] or [8].

Remark 6. *Let K be a tunnel number one knot whose complement $C(K) = S^3 - \eta(K)$ contains a closed incompressible surface F of genus g. Let M^1, M^2 be the manifolds obtained by cutting $C(K)$ along F. (Here M^1, say, is the complement of an open regular neighborhood of the toroidal graph in the construction of Eudave-Muñoz and M^2 is the component containing $\partial C(K)$.) In the examples of Eudave-Muñoz the genus is $g(F) + 1$ for both M^1 and M^2, i.e.,*

$$g(M^1) + g(M^2) = 2g(F) + 2.$$

This number is significantly lower than the upper bound derived here. This leads one to believe that the bound suggested is too high. However, the constructions in this paper show that the upper bound in the inequality would be obtained if the splitting surface of a minimal genus Heegaard splitting had to intersect the incompressible surface many times. The examples of Eudave-Muñoz are very well behaved in this respect, as the splitting surface of the genus 2 Heegaard splitting may be isotoped to intersect the incompressible surface in only one curve inessential in F or two curves essential in F.

Remark 7. *Consider $Q \times I$ for Q a closed surface. The genus of $Q \times I$ is $g(Q)$. On the other hand, the genus of the manifold M obtained by identifying $Q \times \{0\}$ to $Q \times \{1\}$ is 2 for a suitably chosen glueing homeomorphism. Here the right hand side in Theorem 5 is negative unless Q is a torus. This is disconcerting, but merely means that the result is automatically true. Again, the genus 2 Heegaard splitting in this example is well behaved with respect to its intersection with Q; it too may be isotoped so that it intersects Q either in one curve inessential in Q or two curves essential in Q.*

Recall that in the appendix to [14], A. Casson provided a class of 3-manifolds each having a strongly irreducible generalized Heegaard splitting and each containing an essential annulus such that the number of dipping annuli can be arbitrarily large. These examples provide one reason to believe the upper bound provided here is not larger than necessary.

6. Application to Stabilizations of Heegaard Splittings

A variant on the construction above provides an improvement over the inequality obtained by H. Rubinstein and M. Scharlemann in [9] concerning stabilizations of Heegaard splittings for closed non Haken 3-manifolds.

Lemma 10. *Let M be a closed 3-manifold. If $M = V \cup_S W$ and $M = X \cup_T Y$ are strongly irreducible Heegaard splittings of M, then S may be isotoped so that each component of $S \cap T$ is essential in both S and T.*

Proof. See [9, Corollary 2]. □

Theorem 6. *Let M be a closed 3-manifold. If $M = V \cup_S W$ and $M = X \cup_T Y$ are strongly irreducible Heegaard splittings of M, then they admit a common stabilization of genus at most*

$$3genus(S) + 5genus(T) - 4.$$

Remark 8. *By contrast, the upper bound in [9] is*

$$8genus(S) + 5genus(T) - 9.$$

Proof. By Lemma 10, S may be isotoped so that each component of $S \cap T$ is essential in both S and T and so that the number of such components is minimal subject to this condition. If S and T are parallel surfaces in M, then the conclusion is immediate. Otherwise, apply Proposition 3 to the Heegaard splitting $M = V \cup_S W$ and the surface T. This yields Heegaard splittings $X = V^X \cup_{S^X} W^X$ and $Y = V^Y \cup_{S^Y} W^Y$ such that

$$J(V^X) + J(V^Y) \leq 3J(V) - 2\chi(T) + 2(n_x + n_y) \leq 3J(V) - 4\chi(T).$$

The construction in Lemma 3 can be performed in reverse order. I.e., in this setting, the tubes attached to S^X and S^Y can be attached to S in M. We attach only the tubes remaining after Procedures I, II, and III. The defining set of disks remaining after the destabilizations of Procedure I, II, and III demonstrate that attaching these tubes corresponds to stabilizing $M = V \cup_S W$. Denote the resulting Heegaard splitting by $M = A \cup_R B$. This line of reasoning exhibits $M = A \cup_R B$ as the amalgamation of $X = V^X \cup_{S^X} W^X$ and $Y = V^Y \cup_{S^Y} W^Y$ along T. In particular, $M = A \cup_R B$ is a stabilization of $M = X \cup_T Y$.

Here A is obtained by deleting the collar of T in, say, V^X and attaching the resulting "loose ends" of 1-handles of V^X to W^Y. This means that

$$J(A) = J(V^X) + J(W^Y).$$

Note that since M is closed, $\partial_- V^X = \partial_- V^Y = T$ and W^X, W^Y are handlebodies. Thus

$$g(S^X) = g(T) + \frac{1}{2}(J(V^X))$$

and

$$g(S^Y) = g(T) + \frac{1}{2}(J(V^Y)).$$

In particular,

$$J(W^Y) = 2g(T) + J(V^Y).$$

Whence

$$J(A) = J(V^X) + 2g(T) + J(V^Y) \leq$$
$$3J(V) - 2\chi(T) - 4\chi(T) + 2g(T) = 3J(V) - 12 + 14g(T).$$

Therefore, since M is closed, i.e., since A and V are handlebodies,

$$g(R) \leq 3g(S) + 7g(T) - 14.$$

□

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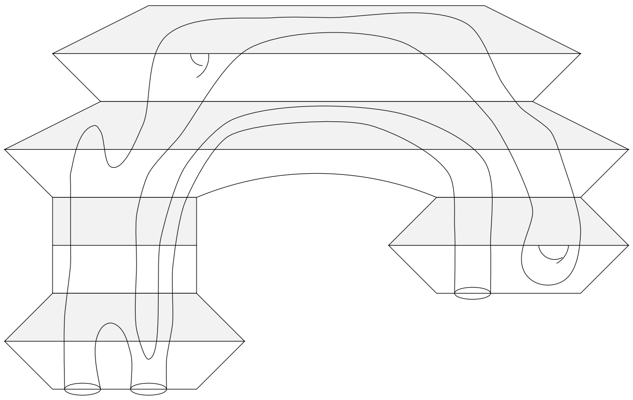


fig. 1

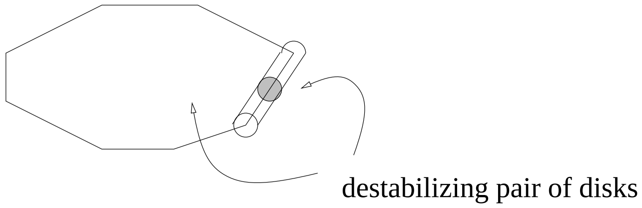


fig. 2

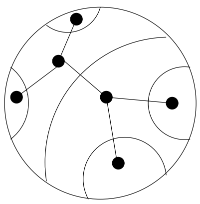


fig. 3

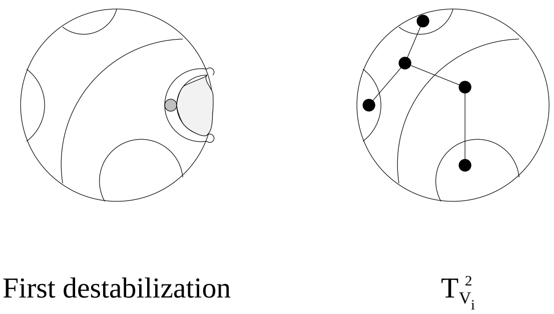


fig. 4

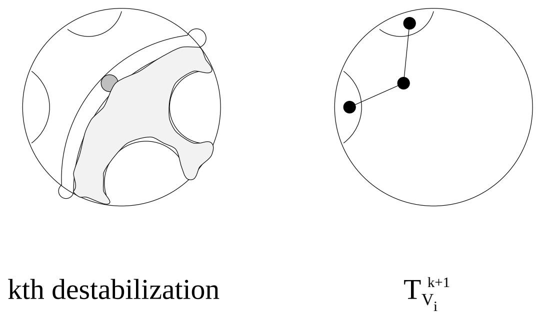


fig. 5

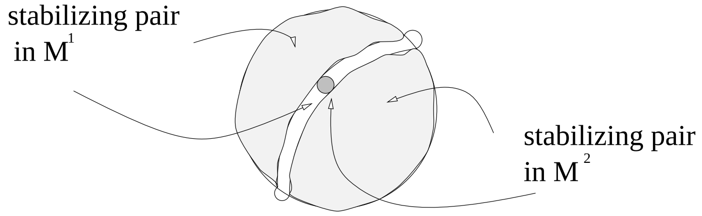


fig. 6

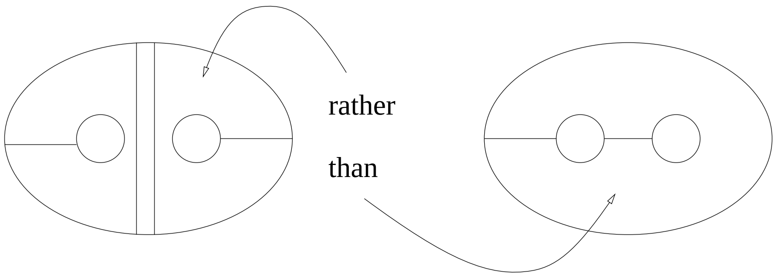


fig. 7

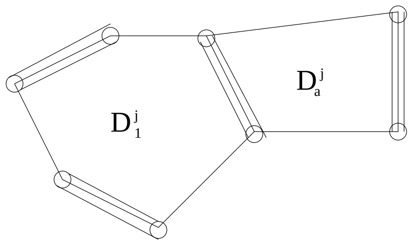


fig. 8

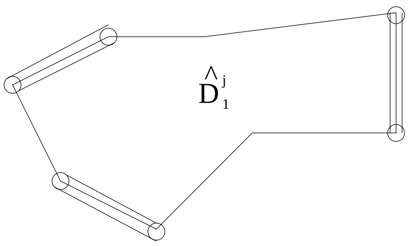


fig. 9

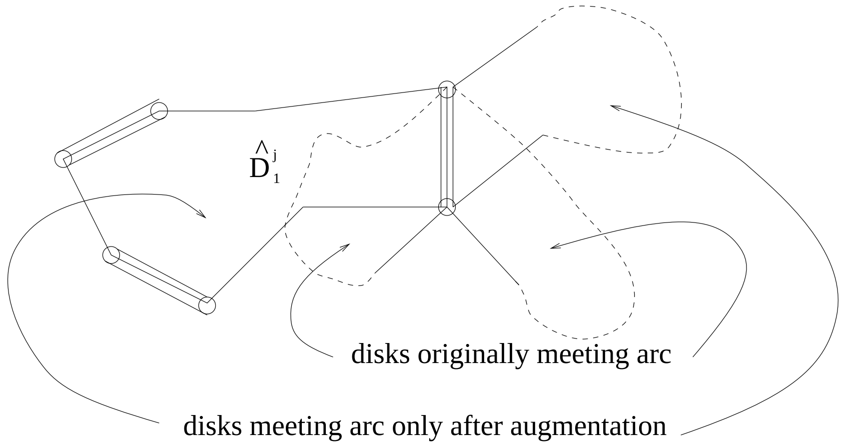
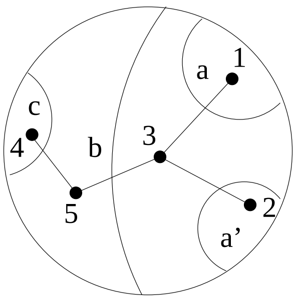
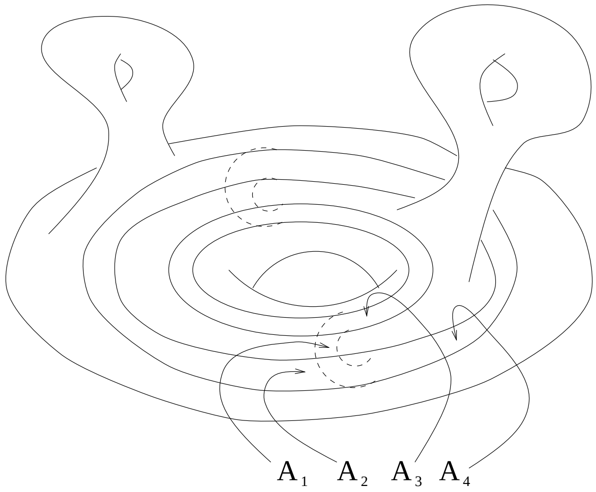


fig. 10



f(a) = f(a') = b, f(c) = b, f(b) = 0

fig. 11



Parallel Dipping Annuli

fig. 12

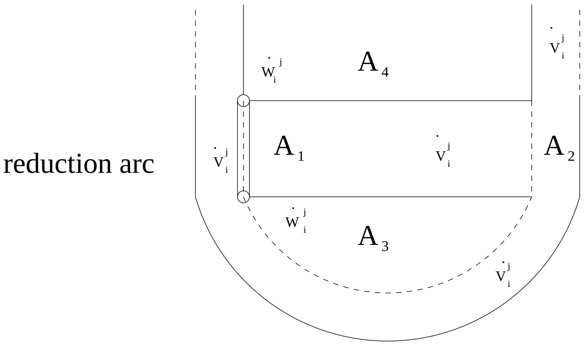
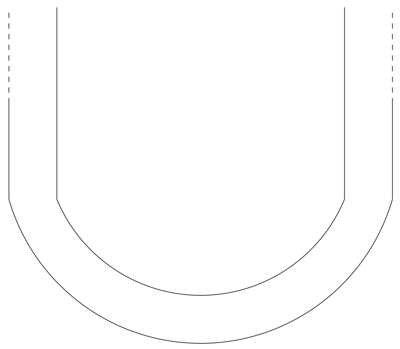


fig. 13



Local picture of alternative generalized Heegaard splitting

fig. 14

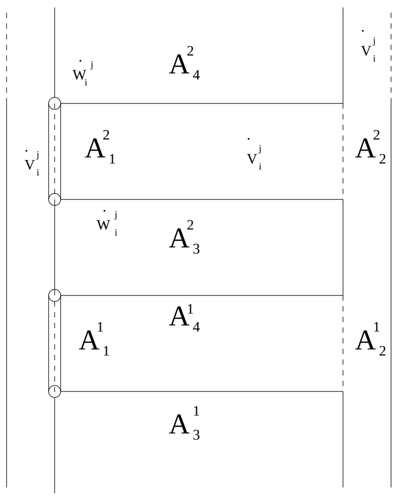
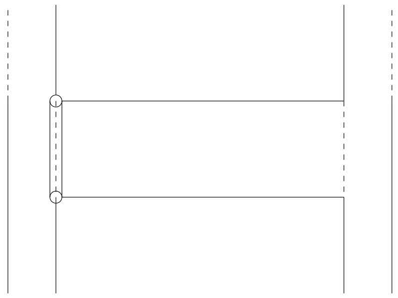


fig. 15



Local picture of alternative generalized Heegaard splitting

fig. 16