ADDITIVITY PROPERTIES OF GENERALIZED BRIDGE NUMBERS OF KNOTS

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ABSTRACT. Suppose that K is the connected sum of K_1 and K_2 and that K admits a (g, b)-presentation. Then K_1 admits a $(g_1, 0)$-presentation and a $(g_1 - 1, 1)$-presentation and K_2 admits a (g_2, b)-presentation such that $g_1 + g_2 \leq 3g + 2b + 2$. In particular, if the 1-bridge genus of K_1 is $\bar{g}_1$ and the 1-bridge genus of K_2 is $\bar{g}_2$, then the 1-bridge genus of $K_1 \# K_2$ is at least $\frac{1}{3}(\bar{g}_1 + \bar{g}_2 - 4)$.

In "Über eine numerische Knoteninvariante" [11], Horst Schubert defined the bridge number of a knot and proved, among other things, that the bridge number of a knot is additive under connected sum of knots. Later, Helmut Doll generalized the the notion of bridge number of a knot. Schubert's notion of bridge number can be formulated in terms of a nice position of a knot with respect to a genus 0 Heegaard splitting of S^3, whereas Doll's more general notion of bridge number can be formulated in terms of a nice position of a knot with respect to a Heegaard splitting of S^3 of a given genus g (see [3]). Interestingly enough, Schubert's Theorems do not carry over to the more general setting. Specifically, Kanji Morimoto has constructed examples for which the 1-bridge genus of two knots degenerates under the operation of connected sum of knots (see [8] and [9]). Incidentally, Phoebe Hoidn has shown that this does not happen if the knots are small (see [5]).

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For standard definitions pertaining to knot theory see for instance [1] or [10]. For 3-manifolds see [4] or [6]. In the following, for L a submanifold of M, $\eta(L)$ will denote an open regular neighborhood of L in M and $N(L)$ will denote a closed regular neighborhood of L in M.

Definition 1. *A* <u>*compression body*</u> *is a* 3*-manifold* W *obtained from a connected closed orientable surface* S *by attaching* 2*-handles to* $S \times \{0\} \subset S \times I$ *and capping off any resulting* 2*-sphere boundary components. We denote* $S \times \{1\}$ *by* $\partial_+ W$ *and* $\partial W - \partial_+ W$ *by* $\partial_- W$*. Dually, a compression body is a connected orientable* 3*-manifold obtained from a (not necessarily connected) closed orientable surface* $\partial_- W \times I$ *or a* $3 - ball$ *by attaching* 1*-handles.*

In the case where $\partial_- W = \emptyset$ *(i.e., in the case where a* $3 - ball$ *was used in the dual construction of* W*), we also call* W *a* <u>*handlebody*</u>*. If* $W = \partial_- W \times I$*, we say that* W *is a* <u>*trivial compression body*</u>*.*

Define the <u>*index*</u> *of* W *by* $J(W) = \chi(\partial_- W) - \chi(\partial_+ W)$*. (Note that the index of a solid torus is* 0*.)*

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Definition 2. *A set of defining disks for a compression body W is a set of disks $\{D_1, \ldots, D_n\}$ properly imbedded in W with $\partial D_i \subset \partial_+ W$ for $i = 1, \ldots, n$ such that the result of cutting W along $D_1 \cup \cdots \cup D_n$ is homeomorphic to $\partial_- W \times I$ or a 3-ball in the case that W is a handlebody.*

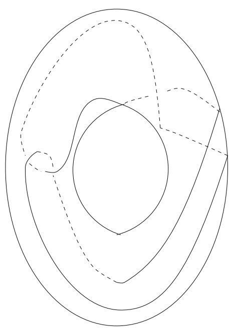
Definition 3. *A Heegaard splitting of a 3-manifold M is a decomposition $M = V \cup_S W$ in which V, W are compression bodies such that $V \cap W = \partial_+ V = \partial_+ W = S$ and $M = V \cup W$. We call S the splitting surface or Heegaard surface.*

Definition 4. *Suppose $M = V \cup_S W$ is a Heegaard splitting and a knot $K \subset M$ is transverse to S. Let γ be a component of $K \cap V$ (or $K \cap W$, respectively). If $\gamma = (point) \times I \subset \partial_- V \times I$ (or $\gamma = (point) \times I \subset \partial_- W \times I$, respectively), then we call γ a spanning arc. If there is a disk $D \subset V$, such that ∂D is the union of two subarcs, a, b with $a = \gamma$ and $b \subset \partial_+ V$ (or there is a disk $D \subset W$, such that ∂D is the union of two subarcs, a, b with $a = \gamma$ and $b \subset \partial_+ W$, respectively), then we call γ a bridge and D a parallelizing disk for γ.*

We say that $K \subset M$ admits a (g, b)-presentation if there is a Heegaard splitting $M = V \cup_S W$ with $g(S) = g$ such that each component of $K \cap V$ and $K \cap W$ is either a spanning arc or a bridge and such that the number of bridges in V is b. (It follows that the number of bridges in W is also b.) We call g the genus of the presentation and b the number or bridges.

We further say that $K \subset M$ admits a $(g, 0)$-presentation if there is a Heegaard splitting $M = V \cup_S W$ with $g(S) = g$ such that K is parallel into S.

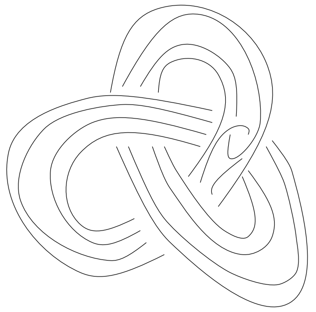
We say that the (g, b)-presentation of K is minimal, if g is the smallest genus for which K admits a (g, b) presentation, and b is the smallest number bridges for which K admits a (g, b) presentation.



(1,2)-presentation for the figure 8 knot

Definition 5. *Let $K = K_1 \# K_2$ and let R be the sphere defining the connected sum. Then $\partial N(K \cup R)$ has two components T_1, T_2, each called a swallow-follow torus. More specifically, let T_i be the component of $\partial(N(K \cup R))$ in $C(K_i)$, then T_1 is the swallow-follow torus that swallows K_2 and follows K_1 and T_2 is the swallow-follow torus that swallows K_1 and follows K_2.*

The notion of strong irreducibility of a Heegaard splitting was introduced by A. Casson and C. McA. Gordon in [2] and has proven extremely useful.



A swallow-follow torus

Definition 6. *A Heegaard splitting $M = V \cup_S W$ is strongly irreducible if for any pair of essential disks $D \subset V$ and $E \subset W$, $\partial D \cap \partial E \neq \emptyset$.*

Though all compact 3-manifolds admit Heegaard splittings, many do not admit strongly irreducible Heegaard splitting. This fact prompted M. Scharlemann and A. Thompson to introduce the following notion of generalized Heegaard splittings.

Definition 7. *A generalized Heegaard splitting of a compact orientable 3-manifold M is a decomposition $M = (V_1 \cup_{S_1} W_1) \cup_{F_1} (V_2 \cup_{S_2} W_2) \cup_{F_2} \cdots \cup_{F_{m-1}} (V_m \cup_{S_m} W_m)$ such that each of the V_i and W_i is a union of compression bodies with $\partial_+ V_i = S_i = \partial_+ W_i$ (i.e., $V_i \cup_{S_i} W_i$ is a union of Heegaard splittings of a union of submanifolds of M) and $\partial_- W_i = F_i = \partial_- V_{i+1}$.*

We say that a generalized Heegaard splitting is strongly irreducible if each Heegaard splitting $V_i \cup_{S_i} W_i$ is strongly irreducible and each F_i is incompressible in M. We will denote $\cup_i F_i$ by $\mathcal{F}$ and $\cup_i S_i$ by $\mathcal{S}$. The surfaces in $\mathcal{F}$ are called the thin levels and the surfaces in $\mathcal{S}$ the thick levels.

Let $M = V \cup_S W$ be an irreducible Heegaard splitting. We may think of M as being obtained from $\partial_- V \times I$ by attaching all 1-handles in V (dual definition of compression body) followed by all 2-handles in W (standard definition of compression body), followed, perhaps, by 3-handles. A weak reduction of $M = V \cup_S W$ is a rearrangement of the order in which the 1-handles of V and the 2-handles of W are attached such that the resulting generalized Heegaard splitting is strongly irreducible. Note that $\partial_- V_1 \subset \partial_- V$ and this inclusion may be a strict inclusion. For convenience, we will occasionally denote $\partial_- V_1$ by F_0.

The Main Theorem in [14] together with the calculation [15, Lemma 2] implies the following:

Theorem 1. *Suppose M is an irreducible closed 3-manifold. Then M possesses a genus g Heegaard splitting*
$$M = V \cup_S W$$
if and only if M possesses a strongly irreducible generalized Heegaard splitting
$$M = (V_1 \cup_{S_1} W_1) \cup_{F_1} (V_2 \cup_{S_2} W_2) \cup_{F_2} \cdots \cup_{F_{m-1}} (V_m \cup_{S_m} W_m)$$
such that
$$\sum_{i=1}^{m} J(V_i) = \sum_{i=1}^{m} J(W_i) = 2g(S) - 2 + \chi(\partial_- V).$$

1. Orbifold Version of Weak Reduction

The procedure of weak reduction of a Heegaard splitting, motivated by the idea of thin position for 3-manifolds in [14] due to Scharlemann and Thompson, is described in detail in [11]. (There the term "untelescoping" is used, but more recently, the term "weak reduction" has been used by M. Scharlemann and others.) We here sketch an orbifold version of this procedure. (The "orbifold" terminology is motivated by the use of this word in a similar context by T. Kobayashi in [7].)

Definition 8. *Let $K \subset M$ be a knot and let $M = (V_1 \cup_{S_1} W_1) \cup_{F_1} \cdots \cup_{F_{n-1}} (V_n \cup_{S_n} W_n)$ be a generalized Heegaard splitting. If each component of $K \cap V_i$ and of $K \cap W_i$ is either a spanning arc or a bridge, for $i = 1, \dots, n$, then we say that $M = (V_1 \cup_{S_1} W_1) \cup_{F_1} \cdots \cup_{F_{n-1}} (V_n \cup_{S_n} W_n)$ is a generalized bridge presentation for K.*

Given a generalized bridge presentation for K, we call $D \subset V_i - K$ (or $D \subset W_i - K$, respectively) an essential disk, if D is not boundary parallel in $V_i - K$ (or $W_i - K$, respectively). If any pair of essential disks, (D, E), with $D \subset V_i$ and $E \subset W_i$ must have $\partial D \cap \partial E \neq \emptyset$, then we say that the generalized bridge presentation for K is strongly irreducible. We also say that $V_i \cup_{s_i} W_i$ is K-strongly irreducible. If the total number of bridges in $\cup_i V_i$ is b and $\sum_i J(V_i) = 2g$, then we call $M = (V_1 \cup_{S_1} W_1) \cup_{F_1} \cdots \cup_{F_{n-1}} (V_n \cup_{S_n} W_n)$ a generalized (g, b)-presentation.

Lemma 1. *K admits a (g, b)-presentation if and only if K admits a strongly irreducible generalized (g, b)-presentation.*

Proof. The proof is identical to that of the analogous theorem for untelescoping of Heegaard splittings. We include a sketch here. Suppose that $K \subset M$, $M = V \cup_S W$ with $genus(S) = g$ and that there is a pair (D, E) of essential disks with $D \subset V$ and $E \subset W$ with $\partial D \cap \partial E = \emptyset$. Choose collections of essential disks $\mathcal{D} \subset V$, $\mathcal{E} \subset W$, such that $D \epsilon \mathcal{D}$, $E \epsilon \mathcal{E}$ and such that $\mathcal{D}$ and $\mathcal{E}$ are maximal collections of nonparallel disjoint essential disks with the property that $\partial \mathcal{D} \cap \partial \mathcal{E} = \emptyset$. Denote by S_1 the result of performing ambient 2-surgery on S along $\partial \mathcal{D}$, by F_1 the result of performing ambient 2-surgery on S along $\partial(\mathcal{D} \cup \mathcal{E})$, and by S_2 the result of performing ambient 2-surgery on S along $\partial \mathcal{E}$.

A pushoff of S_1 into V bounds a (perhaps disconnected) handlebody V_1 (a copy of V minus some 1-handles) and a pushoff of S_2 into W also bounds a (perhaps disconnected) handlebody W_2 (a copy of W minus some 2-handles). Furthermore, S_1 and F_1 cobound a compression body W_1 ($F_1 \times I \cup (1 - handles)$, with each 1-handle dual to one of the 2-handles defined by a component of $\mathcal{E}$) and F_1 and S_2 cobound a compression body V_2 ($F_1 \times I \cup (1 - handles)$, with each 1-handle dual to one of the 2-handles defined by a component of $\mathcal{D}$). Here $\partial_+ W_1 = S_1$, $\partial_- W_1 = F_1$, $\partial_- V_2 = F_1$, $\partial_+ V_2 = S_2$ and $\sum_i J(V_i) = 2g$.

Repeating this procedure as necessary provides the strongly irreducible generalized (g, b)-presentation.

Let $\mathcal{D}'$ be a collection of parallelizing disks for $K \cap V$. We may assume, by rechoosing $\mathcal{D}'$, if necessary, that $\mathcal{D}' \cap \mathcal{D} = \emptyset$. Then shrunk versions of $\mathcal{D}'$ provide a set of parellelizing disks for $K \cap V_i$ in V_i.

Conversely, suppose that $S^3 = (V_1 \cup_{S_1} W_1) \cup_{F_1} \cdots \cup_{F_{n-1}} (V_n \cup_{S_n} W_n)$ is a generalized (g, b)-presentation for K. Then the underlying Heegaard splitting may be amalgamated to $S^3 = V \cup_S W$. Furthermore, let D' be a parallelizing disk for $\gamma \subset V_1$. In the process of amalgamation, the portion of W_i homeomorphic to $\partial_- W_i \times I$ along with the portion of V_{i+1} homeomorphic to $\partial V_{i+1} \times I$ is identified to a copy of F_i lying in S (i.e., squashed down). Thus, since the map that identifies $\partial_- W_i \times I\{0,1\}$ and the map that identifies $\partial V_{i+1} \times \{0,1\}$ in this process can be chosen so that parallelizing disks avoid attaching disks of 1-handles, the parallelizing disks for K survive the process of amalgamation. □

Lemma 2. *Suppose that M is a boundary irreducible 3-manifold and that $M = (V_1 \cup_{S_1} W_1) \cup_{F_1} \cdots \cup_{F_{n-1}} (V_n \cup_{S_n} W_n)$ is a strongly irreducible generalized (g,b)-presentation for K and that $|\mathcal{F}|$ is minimal over all strongly irreducible generalized (g,b)-presentations. Then $\mathcal{F}$ is incompressible in $M - K$.*

Proof. Suppose that D is a compressing disk for $\mathcal{F}$ in $S^3 - K$. By choosing an innermost subdisk of D cut out by $D \cap \mathcal{F}$ if necessary, we may assume that $D \cap \mathcal{F} = \partial D$. Say, $D \subset V_i \cup W_i$. By Haken's Theorem, D may be isotoped to intersect S_i in a single circle. We may assume (by symmetry) that $D \cap V_i$ is a spanning annulus in $\partial_- V_i \times I$ and that $D \cap W_i$ is an essential disk. We must show that after this isotopy we may still assume that $D \cap K = \emptyset$.

Note that the endpoints of spanning arcs in $K \cap V_i$ and $K \cap W_i$ can be considered to lie in a subdisk $\bar{D}$ of S_i. Then clearly D may be made disjoint from $\bar{D} \times I \subset \partial_- V_i \times I$ and $\bar{D} \times I \subset \partial_- W_i \times I$ without affecting the fact that $|D \cap S_i| = 1$. So we may assume that D is disjoint from the spanning arcs among $K \cap V_i$ and $K \cap W_i$.

Now let $\tilde{D}$ be a parallelizing disk for a bridge among $K \cap V_i$ or $K \cap W_i$. Since compression bodies are irreducible, closed components of $D \cap \tilde{D}$ may also be eliminated without affecting the fact that $|D \cap S_i| = 1$. Finally, a subdisk of $\tilde{D}$ cut off by an outermost arc of $D \cap \tilde{D}$ defines an isotopy of D, reducing the number of components of $D \cap \tilde{D}$, that does not create any new components of intersection with K. This isotopy thus reduces the number of such components by 1 without affecting the fact that $|D \cap S_i| = 1$.

Now $D \cap V_i$ is a spanning annulus in $\partial_- V_i \times I$, $D \cap W_i$ is an essential disk and $D \cap K = \emptyset$. Since $D \cap V_i \subset \partial_- V_i \times I \subset V_i$, there is a set of defining disks for V_i disjoint from D. A nonempty set of defining disks would contradict the K-strong irreducibility of $V_i \cup_{S_i} W_i$. Hence V_i must be a trivial compression body. But then F_{i-1} is superfluous, i.e., $W_{i-1} \cup V_i \cup W_i$ is a compression body, contradicting the minimality of $|\mathcal{F}|$. □

Definition 9. *For W a compression body and Γ a collection of spanning arcs and bridges in W, a spine of (W, Γ) is the union of $\partial_- W$, Γ, and a properly imbedded 1-complex C such that $\partial_- W \cup \Gamma \cup C$ is connected and such that W collapses to $\partial_- W \cup \Gamma \cup C$.*

Lemma 3. *Suppose that $K = K_1 \# K_2$ and that T is a swallow-follow torus. Further suppose that $S^3 = (V_1 \cup_{S_1} W_1) \cup_{F_1} \cdots \cup_{F_{n-1}} (V_n \cup_{S_n} W_n)$ is a strongly irreducible (g,b)-presentation for K such that $|\mathcal{F}|$ is minimal over all (g,b)-presentations for K. Then T may be isotoped so that $T \cap (\mathcal{F} \cup \mathcal{S})$ is essential in both T and $\mathcal{F} \cup \mathcal{S}$.*

Proof. Except for the slight difference in the definition of a spine of a compression body in this orbifold setting, the proof of this lemma is identical to the proof of [13, Lemma 6]. We provide a sketch of the argument. Since both T and $\mathcal{F}$ are incompressible in $S^3 - K$, they may be isotoped so that each component of $T \cap \mathcal{F}$ is essential in both T and $\mathcal{F}$. After such an isotopy, we isotope $\mathcal{S}$ fixing $\mathcal{F}$. Consider $T \cap (V_i \cup_{S_i} W_i)$. Here V_i may be shrunk to be a small regular neighborhood of a spine of $(V_i, K \cap V_i)$. Then $T \cap V_i$ consists of a collar of $T \cap F_{i-1}$ in T (i.e.,

$T \cap \partial_- V_i \times I$ and disks. On the other hand, V_i may be expanded, and W_i shrunk to a small regular neighborhood of a spine of $(W_i, K \cap W_i)$. Then $T \cap W_i$ consists of a collar of $T \cap F_i$ in T (i.e., $T \cap \partial_- W_i \times I$ and disks. If at one of these two extremes there are no disks, then we are done. If there are disks at both extremes, note that during the isotopy from one extreme to the other, the subinterval of I for which there are disks in $T \cap V_i$ is closed, as is the subinterval of I for which there are disks in $T \cap W_i$. Since $S^3 = (V_1 \cup_{S_1} W_1) \cup_{F_1} \cdots \cup_{F_{n-1}} (V_n \cup_{S_n} W_n)$ is a strongly irreducible generalized bridge presentation for K, the intervals can't overlap, hence there is a point in I the complement of their union, i.e., and isotopy for which there are no disks in $T \cap V_i$ nor in $T \cap W_i$. $\square$

2. Generalized Bridge Position for Knot Summands

We use an extension of the following:

Proposition 2. *([12, Proposition 2]) Suppose that $M = (V_1 \cup_{S_1} W_1) \cup_{F_1} \cdots \cup_{F_{n-1}} (V_n \cup_{S_n} W_n)$ is a generalized Heegaard splitting and Q a compact orientable surface in M. Let M^j be the closures of the components of $M - Q$ and let n_j be the minimal number of parallelism classes of essential annuli in M^j representatives of which can be simultaneously imbedded. Further suppose that each component of $Q \cap (\mathcal{F} \cup \mathcal{S})$ is essential in both Q and $\mathcal{F} \cup \mathcal{S}$, such that the number of such components is minimal subject to this condition, and such that no component of Q is parallel to a component of $\mathcal{F}$. Then there are generalized Heegaard splittings $M^j = (A_1^j \cup_{G_1^j} B_1^j) \cup_{P_1^j} \cdots \cup_{P_{n-1}^j} (A_n^j \cup_{G_n}^j B_n^j)$ for $M^j, j = 1, \dots k$ that satisfy the following inequality*

$$\sum_i J(V_i) \geq \frac{1}{3}\Big(\sum_j \Big(\sum_i J(A_i^j) - 2n_j\Big) + 2\chi(Q)\Big).$$

Proposition 2 is laboriously proven in [12]. The construction there still applies in the orbifold context except that the following orbifold version of the definition of a "dipping annulus" and of Lemma 4 must replace the more specialized definition and Lemma 8 there.

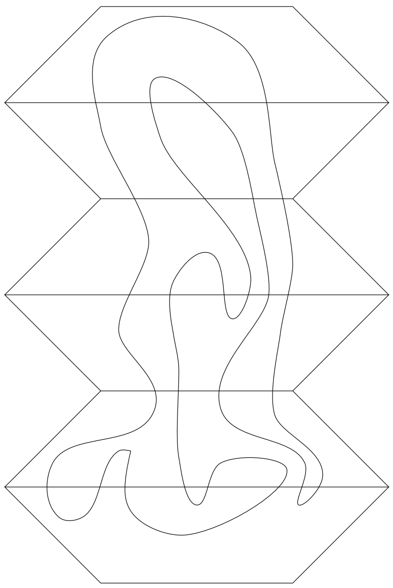
Definition 10. *Let W be a compression body and $K \subset W$ a 1-manifold consisting of spanning arcs and bridges. An essential annulus A in $W - K$ with $\partial A \subset \partial_+ W$ is called a <u>dipping annulus in (W, K)</u>.*

Lemma 4. *Let W be a compression body and $K \subset W$ a 1-manifold consisting of spanning arcs and bridges. Further suppose that the number of bridges is b. If $\mathcal{A}$ is a collection of nonparallel dipping annuli in (W, K) and $\partial_- W \neq \emptyset$, then $|\mathcal{A}| \leq J(W) + 2b$. If $\mathcal{A}$ is a collection of nonparallel dipping annuli in (W, K) and $\partial_- W = \emptyset$, then $|\mathcal{A}| \leq J(W) + 2b - 2$.*

Proof. The proof is by induction on $(J(W), b)$. If $(J(W), b) = (0, 0)$, then $\mathcal{A} = \emptyset$. The same is true if $\partial_- W = \emptyset$ and $(J(W), b) = (0, 1), (2, 0)$.

Otherwise, let $\mathcal{D}$ be a set of defining disks for W and let $\mathcal{D}'$ be a collection of parallelizing disks for the bridges of K in W. We may assume that $\mathcal{A}$ has been isotoped so that $|\mathcal{A} \cap (\mathcal{D} \cup \mathcal{D}')|$ is minimal.

First consider a component D' of $\mathcal{D}'$. If $D' \cap \mathcal{A} \neq \emptyset$, then let a be an outermost arc of $D' \cap \mathcal{A}$ in D' and let A be the annulus that gives rise to a. Cutting A along a and attaching copies of the outermost disk cut off of D' by a yields a disk $\bar{D}$. If



Schematic rendition of swallow-follow
in generalized Heegaard splitting

$\bar{D}$ were inessential, then A would have to be inessential, hence $\bar{D}$ must be essential. Furthermore, $\bar{D}$ is disjoint from $\mathcal{A}$.

Case 1: Cutting W along $\bar{D}$ yields a compression body W' homeomorphic to W and a 3-ball B.

Since $\bar{D}$ is essential, there must be bridges in B. Let b' be the number of bridges in W' and b'' the number of bridges in B. Then

$$(J(W'), b') = (J(W), b') \leq (J(W), b)$$

$$(J(B), b'') = (0, b'') \leq (J(W), b)$$

and

$$(J(W'), b') + (J(B), b'') = (J(W), b).$$

Here $\mathcal{A} \subset W' \cup B$, but some components of $\mathcal{A}$ may now be boundary parallel. An annulus that was essential in W and is boundary parallel in W' or B must cut off a solid torus from W' or W'' that contains at least one remnant of $\bar{D}$. Thus there can be only two such annuli. Let $\tilde{A}'$ be the collection of annuli in $W' \cup B$ obtained by deleting those two annuli, then the inductive hypothesis applies to $\mathcal{A}' \subset W'$ and $\mathcal{A}' \subset B$. Whence $|\mathcal{A}| \leq 2 + J(W) + 2b' + 2b'' - 2 = J(W) + 2b$.

Case 2: Cutting W along $\bar{D}$ yields one compression body W' with $J(W') = J(W) - 2$.

Again, $\mathcal{A} \subset W'$, but two components of $\mathcal{A}$ may now be boundary parallel. The result follows by induction, $|\mathcal{A}| \leq 2 + J(W') + 2b = 2 + J(W) - 2 + 2b = J(W) + 2b$.

Case 3: Cutting W along $\bar{D}$ yields two compression bodies W' and W'' such that $J(W') + J(W'') = J(W) - 2$.

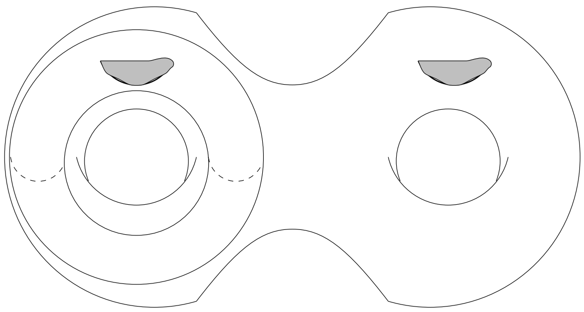
Again, $\mathcal{A} \subset W' \cup W''$, but two components of $\mathcal{A}$ may now be boundary parallel. The result follows by induction, $|\mathcal{A}| \leq 2 + J(W') + J(W'') + 2b = 2 + J(W) - 2 + 2b = J(W) + 2b$.

Case 4: Cutting W along $\bar{D}$ yields two compression bodies W' and W'' such that $J(W') + J(W'') = J(W)$.

This occurs if either W' or $J(W'')$ or both are handlebodies. So suppose W'' is a handlebody. As above, the result follows by induction, $|\mathcal{A}| \leq 2 + J(W') + J(W'') - 2 + 2b = 2 + J(W) - 2 + 2b = J(W) + 2b$.

If $\mathcal{D}' \neq \emptyset$ but $\mathcal{A} \cap \mathcal{D} = \emptyset$, then let $\bar{D}$ be the boundary of a collar of a component of $\mathcal{D}'$ in W. The Lemma then follows by induction as above.

If $\mathcal{D}' = \emptyset$, repeat the above argument with $\mathcal{D}$ instead of $\mathcal{D}'$. $\square$



An essential annulus in (W, K)

Theorem 3. *Suppose that $K = K_1 \# K_2$ and that K admits a (g, b)-presentation. Then K_1 admits a $(g_1, 0)$-presentation and a $(g_1 - 1, 1)$-presentation and K_2 admits a (g_2, b)-presentation such that $g_1 + g_2 \leq 3g + 2b + 2$.*

Proof. Given the (g, b)-presentation of $K = K_1 \# K_2$, let $S^3 = (V_1 \cup_{S_1} W_1) \cup_{F_1} \cdots \cup_{F_{n-1}} (V_n \cup_{S_n} W_n)$ be the strongly irreducible generalized bridge presentation for K furnished by Lemma 1. Let T be a swallow-follow torus for $K = K_1 \# K_2$. By Lemma 3, $\mathcal{F} \cup \mathcal{S}$ may be isotoped so that each component of $T \cap (\mathcal{F} \cup \mathcal{S})$ is essential in both T and $\mathcal{F} \cup \mathcal{S}$ and such that the number of such components is minimal subject to this condition.

Apply Proposition 2 to the underlying generalized Heegaard splitting of S^3 in the strongly irreducible generalized bridge presentation. Though it is not evident from the statement of Proposition 2, it is in fact the case that the arduous construction in [12] yielding this Proposition can be performed in the complement of K and that parallelizing disks for K survive this procedure (as in the proof of Lemma 1).

Replacing Lemma 8 of [12] with Lemma 4 here changes the numbers in the inequality to incorporate b. Specifically, if we assume that T is the swallow-follow torus that "swallows" K_2 and "follows" K_1, then this yields a generalized Heegaard splitting $C(K_1) = (A_1^1 \cup_{G_1^1} B_1^1) \cup_{P_1^1} \cdots \cup_{P_{n-1}^1} (A_n^1 \cup_{G_n}^1 B_n^1)$ and $C(S^3 - (solid\ \ torus), K_2) = (A_1^2 \cup_{G_1^2} B_1^2) \cup_{P_1^2} \cdots \cup_{P_{n-1}^2} (A_n^2 \cup_{G_n}^2 B_n^2)$ that satisfy the following inequality

$$\sum_i J(V_i) \geq \frac{1}{3}(\sum_i J(A_i^1) + \sum_i J(A_i^2) + 2\chi(T) - 4b) =$$

$$\frac{1}{3}(\sum_i J(A_i^1) + \sum_i J(A_i^2) - 4b),$$

or

$$\sum_i J(A_i^1) + \sum_i J(A_i^2) \leq 3\sum_i J(V_i) + 4b.$$

(Recall that knot complements are anannular.) Furthermore, the parallelizing disks for K all become parallelizing disks for K_2.

(From the point of view of the destabilizations performed in [12] it would be tempting to try to remove the $4b$ summand in the above. But doing so in this setting would interfere with the existence of appropriate parallelizing disks.)

Amalgamating the generalized Heegaard splittings of $C(K_1)$ and $C(S^3 - (solid\ \ torus), K_2)$ yields Heegaard splittings

$$C(K_1) = V_1 \cup_{S_1} W_1$$

and

$$C(S^3 - (solid\ \ torus), K_2) = V_2 \cup_{S_2} W_2$$

such that $g(S_j) = \frac{1}{2}(\sum_i J(A_i^j)) + 1$. Thus adding a solid torus with K_1 as a core to V_1 yields a $(g(S_1), 0)$-bridge presentation for K_1.

Here the underlying Heegaard splitting $S^3 = (V_1 \cup (solid\ \ torus)) \cup_{S_1} W_1$ must be trivial, hence there is a disk E in W_1 that intersects a cocore D of $(solid\ \ torus)$ exactly once. Since K_1 is a core of $(solid\ \ torus)$, we may assume that meets K_1 meets D exactly once. Replace $V_1 \cup (solid\ \ torus)$ by $(V_1 \cup (solid\ \ torus)) - \eta(D)$, W_1 by $W_1 \cup N(D)$. Then E can be extended into V_1 along $\partial E \cap \partial N(D)$ to yield a parallelizing disk for $K \cap N(D)$.

Abstractly adding a solid torus V to V_2 in such a way that adding V to $(S^3 - (solid\ \ torus), K_2)$ yields (S^3, K_2) provides a $(g(S_2), b)$-presentation for K_2. Setting $g_1 = g(S_1)$ and $g_2 = g(S_2)$ we obtain

$$g_1 + g_2 = \frac{1}{2}\sum_i (J(A_i^1) + J(A_i^2)) + 2) \leq$$

$$\frac{1}{2}(3\sum_i J(V_i) + 4b) + 2 = \frac{1}{2}(3(2g) + 4b) + 2 = 3g + 2b + 2.$$

□

One of the more interesting cases here is that of 1-bridge genus.

Corollary 1. *If the* 1*-bridge genus of* K_1 *is* $\bar{g}_1$ *and the* 1*-bridge genus of* K_2 *is* $\bar{g}_2$*, then the* 1*-bridge genus of* $K_1 \# K_2$ *is at least* $\frac{1}{3}(\bar{g}_1 + \bar{g}_2 - 4)$.

Proof. By the above, K_1 admits a $(g_1 - 1, 1)$-bridge presentation and K_2 admits a $(g_2, 1)$-bridge presentation with $g_1 + g_2 \leq 3g + 4$, for g the 1-bridge genus of $K_1 \# K_2$. Thus

$$g \geq \frac{1}{3}(g_1 + g_2 - 4) \geq \frac{1}{3}(\bar{g}_1 + \bar{g}_1 - 4).$$

□

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