# The Elliptic Representation of the General Painlevé 6 Equation

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## 1 Introduction

In this paper we study the elliptic representation of the sixth Painlevé equation

$$\frac{d^2y}{dx^2}=\frac{1}{2}\left[\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-x}\right]\left(\frac{dy}{dx}\right)^2-\left[\frac{1}{x}+\frac{1}{x-1}+\frac{1}{y-x}\right]\frac{dy}{dx}$$
$$+\frac{y(y-1)(y-x)}{x^2(x-1)^2}\left[\alpha+\beta\frac{x}{y^2}+\gamma\frac{x-1}{(y-1)^2}+\delta\frac{x(x-1)}{(y-x)^2}\right], \qquad \text{(PVI)}.$$

For all values of $\alpha,\beta,\gamma,\delta$ we study the analytic properties of the solutions in elliptic representation and we derive their critical behavior close to the critical points $x=0,1,\infty$. Moreover, we solve the connection problem for generic values of $\alpha,\beta,\gamma,\delta$ and for the special (non-generic) case $\beta=\gamma=1-2\delta=0$, which is important in 2-D topological field theory.

The six classical Painlevé equations were discovered by Painlevé [29] and Gambier [9], who classified all the second order ordinary differential equations of the type

$$\frac{d^2y}{dx^2}=\mathcal{R}\left(x,y,\frac{dy}{dx}\right)$$

where $\mathcal{R}$ is rational in $\frac{dy}{dx}$, $x$ and $y$. The Painlevé equations satisfy the *Painlevé property* of absence of movable critical singularities. The only movable singularities, namely the singularities whose position depends on the initial data, are poles. The general solution of the $6^{\text{th}}$ Painlevé equation can be analytically continued to a meromorphic function on the universal covering of $\mathbf{P}^1\backslash\{0,1,\infty\}$. Essential singularities or branch points of a solution may only be at $x=0,1,\infty$, which are called *critical points*. The behavior of a solution $y(x)$ at such points is called *critical behavior*. For generic values of the integration constants and of the parameters in the equation, the solution can not be expressed via elementary or classical transcendental functions. For this reason, the solution is called a *Painlevé transcendent*.

The Painlevé equations turned out to be important in physical models, since the works [24] [25] [26]. The reader may find a review of the physical applications in [13]. More recently, the Painlevé 6 equation found applications in topological field theory [4], [5].

The main analytical problem with Painlevé equations is to determine the critical behavior of the transcendents at the critical points. Such a behavior must depend on two parameters, which are integration constants. The second problem, called *connection problem*, is to find the relation between the couples of parameters at different critical points.

The method of isomonodromic deformations developed in [16] [17] was applied to the Painlevé 6 equation in [15], to solve such problems for generic values of the parameters. The non-generic case $\beta=\gamma=1-2\delta=0$ is studied in [7] [11] for its applications to topological field theory. Studies on the critical behavior can be also found in [35].

In the present paper we show that the elliptic representation is a valuable tool to study the critical behavior of the Painlevé 6 transcendents. We obtain results which include and extend the results of [15] [7] [11]. On the other hand, we make use of the isomonodromic deformation theory to solve the connection problem, to be formulated below, for the elliptic representation.

The elliptic representation was introduced by R. Fuchs in [8]. We consider a differential operator

$$\mathcal{L}:=x(1-x)\frac{d^2}{dx^2}+(1-2x)\frac{d}{dx}-\frac{1}{4}$$

Let $\wp(z;\omega_1,\omega_2)$ denote the Weierstrass elliptic function of the independent variable $z\in\mathbf{P}^1$, with *half-periods* $\omega_1,\omega_2$. A transcendent has the representation

$$y(x)=\wp\left(\frac{u(x)}{2};\omega_1(x),\omega_2(x)\right)+\frac{1+x}{3}$$

where $\omega_1(x)$, $\omega_2(x)$ are two independent solutions of the *hyper-geometric equation* $\mathcal{L}\omega=0$, given in section 2, and $u(x)$ solves the following non-linear differential equation which is equivalent to Painlevé 6::

$$\mathcal{L}(u)=\frac{1}{2x(1-x)}\frac{\partial}{\partial u}\left\{2\alpha\left[\wp\left(\frac{u}{2};\omega_1,\omega_2\right)+\frac{1+x}{3}\right]-2\beta\frac{x}{\wp\left(\frac{u}{2};\omega_1,\omega_2\right)+\frac{1+x}{3}}+\right.$$
$$\left.+2\gamma\frac{1-x}{\wp\left(\frac{u}{2};\omega_1,\omega_2\right)+\frac{x-2}{3}}+(1-2\delta)\frac{x(1-x)}{\wp\left(\frac{u}{2};\omega_1,\omega_2\right)+\frac{1-2x}{3}}\right\} \quad (1)$$

The algebraic-geometrical properties of the elliptic representations where studied in [22]. Nevertheless, the analytic properties of the function $u(x)$ have not been studied so far, except for some special cases. The most simple case is $\alpha=\beta=\gamma=1-2\delta=0$. The function $u(x)$ is a linear combination of $\omega_1$ and $\omega_2$. This case was well known to Picard [30], and the critical behavior was studied in [23].

A more general case was studied in [11], for $\beta=\gamma=1-2\delta=0$ and $\alpha$ any complex number. The motivation of [11] was that this case is equivalent to the WDVV equations of associativity in 2-D topological field theory introduced by Witten [36], Dijkgraaf, Verlinde E., Verelinde H. [2]. Such an equivalence is discussed in [4] and it is a consequence of the theory of Frobenius manifolds [1].

In this paper, we study the analytic properties of $u(x)$ for *any* value of $\alpha,\beta,\gamma,\delta$.

In **Theorem 1** of section 3 we construct a class of two-parameter solutions of the equation (1) in a neighborhood of $x=0$, for any value of $\alpha,\beta,\gamma,\delta$. Such solutions can be written as follows:

$$\frac{u(x)}{2}=\nu_1\omega_1(x)+\nu_2\omega_2(x)+v(x) \quad (2)$$

$\nu_1,\nu_2$ are complex parameters with the only constraint that

$$\nu_2\not\in(-\infty,0]\cup\{1\}\cup[2,+\infty)$$

[note that the periodicity of the $\wp$-function implies that the contraint is equivalent to the exclusion of *only* two points: namely $\nu_2\neq 0,1$].

If $\nu_2$ is real, we prove that for sufficiently small $r<1$ there exists $v(x)$ holomorphic in the domain $\mathcal{D}_0(r):=\{x\in\widetilde{\mathbf{C}_0}$ such that $|x|<r\}$. Here $\widetilde{\mathbf{C}_0}$ denotes the universal covering of $\mathbf{C}_0:=\mathbf{C}\backslash\{0\}$. On the domain, $v$ has a convergent expansion

$$\sum_{n\geq 1}a_nx^n+\sum_{n\geq 0,m\geq 1}b_{nm}x^n\left[e^{-i\pi\nu_1}\left(\frac{x}{16}\right)^{1-\nu_2}\right]^m+\sum_{n\geq 0,m\geq 1}c_{nm}x^n\left[e^{i\pi\nu_1}\left(\frac{x}{16}\right)^{\nu_2}\right]^m \quad (3)$$

if $0 < \nu_2 < 1$. Otherwise, it has expansion

$$\sum_{n\geq 1} a_n x^n + \sum_{n\geq 0, m\geq 1} b_{nm} x^n \left[ e^{-i\pi\nu_1} \left(\frac{x}{16}\right)^{2-\nu_2}\right]^m + \sum_{n\geq 0, m\geq 1} c_{nm} x^n \left[ e^{i\pi\nu_1} \left(\frac{x}{16}\right)^{\nu_2-1}\right]^m$$

if $1 < \nu_2 < 2$. The coefficients $a_n$, $b_{nm}$, $c_{nm}$ are certain rational functions of $\nu_2$, which can be determined from PVI. Moreover, $v(x) = O(x)$ for $x \to 0$.

If $\Im \nu_2 \neq 0$, we prove that for sufficiently small $r < 1$ there exists $v(x)$ holomorphic in the domain

$$\mathcal{D}(r;\nu_1,\nu_2) := \left\{ x \in \widetilde{\mathbf{C}_0} \text{ such that } |x| < r, \left| \frac{e^{-i\pi\nu_1}}{16^{1-\nu_2}} x^{1-\nu_2} \right| < r, \left| \frac{e^{i\pi\nu_1}}{16^{\nu_2}} x^{\nu_2} \right| < r \right\}$$

In this domain $v$ has convergent expansion (3). Moreover, there exists a positive constant $M(\nu_2)$ such that

$$|v(x)| \leq M(\nu_2) \left( |x| + \left| e^{-i\pi\nu_1} \left(\frac{x}{16}\right)^{1-\nu_2} \right| + \left| e^{i\pi\nu_1} \left(\frac{x}{16}\right)^{\nu_2} \right| \right)$$

The above result is important because it makes it possible to compute the critical behavior at $x = 0$ of the corresponding transcendents $y(x) = \wp\Big(\nu_1\omega_1(x) + \nu_2\omega_2(x) + v(x);\ \omega_1(x), \omega_2(x)\Big) + \frac{1+x}{3}$, for $x \in \mathcal{D}(r;\nu_1,\nu_2)$ or $\mathcal{D}_0(r)$. In **Theorem 2** of section 5 we derive the critical behavior for $x \to 0$. The transcendent defined in $\mathcal{D}(r;\nu_1,\nu_2)$ for non-real $\nu_2$ and the transcendent defined in $\mathcal{D}_0(r)$ for $0 < \nu_2 < 1$ have behavior

$$y(x) = -\frac{1}{4} \left[ \frac{e^{i\pi\nu_1}}{16^{\nu_2-1}} \right] x^{\nu_2} \left(1 + O(|x^{\nu_2}| + |x^{1-\nu_2}|)\right)$$

In the case $\Im \nu_2 \neq 0$, there are two exceptions to the above behavior: if $x \to 0$ along a direction asymptotic to the special path

$$\arg x = \arg x_0 + \frac{\Re \nu_2}{\Im \nu_2} \ln \frac{|x|}{|x_0|}, \qquad x_0 \in \mathcal{D}_1(r;\nu_1,\nu_2),$$

the behavior is

$$y(x) = \big(1 + O(x)\big) \left[ \frac{x}{2} + \sin^{-2}\left( -i\frac{\nu_2}{2} \ln \frac{x}{16} + \frac{\pi\nu_1}{2} + \sum_{m\geq 1} c_{0m} \left[ e^{i\pi\nu_1} \left(\frac{x}{16}\right)^{\nu_2}\right]^m \right) \right]$$

Along the special path

$$\arg x = \arg x_0 + \frac{\Re \nu_2 - 1}{\Im \nu_2} \ln \frac{|x|}{|x_0|}, \quad x_0 \in \mathcal{D}_1(r;\nu_1,\nu_2),$$

the behavior is

$$y(x) = (1 + O(x))\ x\ \sin^2\left( i\frac{1-\nu_2}{2} \ln \frac{x}{16} + \frac{\pi\nu_1}{2} + \sum_{m\geq 1} b_{0m} \left[ e^{-i\pi\nu_1} \left(\frac{x}{16}\right)^{1-\nu_2}\right]^m \right)$$

Finally, the transcendent defined in $\mathcal{D}_0(r)$ for $1 < \nu_2 < 2$, has behavior

$$y(x) = -\frac{1}{4} \left[ \frac{e^{i\pi\nu_1}}{16^{\nu_2-1}} \right]^{-1} x^{2-\nu_2} \left(1 + O(|x^{2-\nu_2}| + |x^{\nu_2-1}|)\right)$$

Let us define $\omega_1^{(0)} := \omega_1$, $\omega_2^{(0)} := \omega_2$; $\omega_1^{(1)} := \omega_2$, $\omega_2^{(1)} := \omega_1$ and $\omega_1^{(\infty)} := \omega_1 + \omega_2$, $\omega_2^{(\infty)} := \omega_2$. In **Theorem 3** of section 7 we construct solutions

$$\frac{u(x)}{2} = \nu_1^{(1)}\omega_1^{(1)}(x) + \nu_2^{(1)}\omega_2^{(1)}(x) + v^{(1)}(x)$$

in a neighborhood of $x = 1$, and solutions

$$\frac{u(x)}{2} = \nu_1^{(\infty)}\omega_1^{(\infty)}(x) + \nu_2^{(\infty)}\omega_2^{(\infty)}(x) + v^{(\infty)}(x)$$

in a neighborhood of $x = \infty$. The functions $v^{(j)}$, $j = 1, \infty$ are holomorphic in suitable domains and we construct their convergent expansion, repeating the considerations of Theorem 1. Again, the constraint

$$\nu_2^{(j)} \notin (-\infty, 0] \cup \{1\} \cup [2, +\infty)$$

must be imposed. In section 7 we also compute the critical behaviors for $x \to 1$ and $x \to \infty$.

The elliptic representation allows us to obtained detailed information about the critical behavior of the Painlevé transcendents. On the other hand, the local analysis does not solve the *connection problem.* This is the problem of determining the critical behavior of a given transcendent at both $x = 0$, $x = 1$ and $x = \infty$. In our framework, we ask if a transcendent may have, at the same time, three representations

$$\begin{aligned} y(x) &= \wp(\nu_1^{(0)}\omega_1^{(0)} + \nu_2^{(0)}\omega_2^{(0)} + v^{(0)}) + \frac{1+x}{3} \\ &= \wp(\nu_1^{(1)}\omega_1^{(1)} + \nu_2^{(1)}\omega_2^{(1)} + v^{(1)}) + \frac{1+x}{3} \\ &= \wp(\nu_1^{(\infty)}\omega_1^{(\infty)} + \nu_2^{(\infty)}\omega_2^{(\infty)} + v^{(\infty)}) + \frac{1+x}{3} \end{aligned}$$

in the domains of Theorems 1 and 3. Moreover, we look for formulae which connect the three couples of parameters $(\nu_1^{(0)}, \nu_2^{(0)})$, $(\nu_1^{(1)}, \nu_2^{(1)})$, $(\nu_1^{(\infty)}, \nu_2^{(\infty)})$.

The connection problem may be solved using the method of isomonodromic deformations. Works on this problem are [15], [7] and [11]. The PVI is the isomonodromy deformation equation of a Fuchsian system of differential equations

$$\frac{dY}{dz} = \left[\frac{A_0(x)}{z} + \frac{A_x(x)}{z-x} + \frac{A_1(x)}{z-1}\right] Y$$

The $2 \times 2$ matrices $A_i(x)$ ($i = 0, x, 1$ are labels) depend on $x$ in such a way that the monodromy of a fundamental solution $Y(z, x)$ does not change for small deformations of $x$. The eigenvalues of $A_i(x)$, $i = 0, 1, x$ depend on $\alpha, \beta, \gamma, \delta$ – see section 9. Let $M_0$, $M_1$, $M_x$ be the monodromy matrices at $z = 0, 1, x$, for a given basis in the fundamental group of $\mathbf{P}^1 \backslash \{0, 1, x, \infty\}$. Such basis is chosen as in figure 2.

In section 9 we solve the connection problem for the elliptic representation for generic values of $\alpha$, $\beta$, $\gamma$, $\delta$ (the meaning of "generic" will be explained) and in the special case $\beta = \gamma = 1 - 2\delta = 0$, and $\alpha$ any complex number.

The special case is relevant for its applications to Frobenius manifolds [5], quantum cohomology [19] [12] and topological filed theory. In this case we can give a representation of $u(x)$ in a domain which is wider that the general case. Namely,

$$\mathcal{D}(r; \nu_1, \nu_2) := \left\{ x \in \tilde{\mathbf{C}}_0 \mid |x| < r, \ \left| e^{-i\pi\nu_1} \left(\frac{x}{16}\right)^{2-\nu_2} \right| < r, \ \left| e^{i\pi\nu_1} \left(\frac{x}{16}\right)^{\nu_2} \right| < r \right\}$$

In this domain $v(x)$ is holomorphic with convergent expansion

$$v(x) = \sum_{n \geq 1} a_n x^n + \sum_{n \geq 0, m \geq 1} b_{nm} x^n \left[ e^{-i\pi\nu_1} \left(\frac{x}{16}\right)^{2-\nu_2} \right]^m + \sum_{n \geq 0, m \geq 1} c_{nm} x^n \left[ e^{i\pi\nu_1} \left(\frac{x}{16}\right)^{\nu_2} \right]^m$$

If $\nu_2$ is real, the value $\nu_2 = 1$ is now allowed, namely, the constraint is $\nu_2 \notin (-\infty, 0] \cup [2, +\infty)$. Therefore, by periodicity of the $\wp$-function we can assume

$$0 \leq \Re\nu_2 < 2, \qquad \nu_2 \neq 0$$

A similar result holds at $x = 1$ and $x = \infty$. We call the functions $v$ with the notations $v^{(0)}(x; \nu_1^{(0)}, \nu_2^{(0)})$, $v^{(1)}(x; \nu_1^{(1)}, \nu_2^{(1)})$, $v^{(\infty)}(x; \nu_1^{(\infty)}, \nu_2^{(\infty)})$ for neighborhoods of $x = 0, 1$, and $\infty$ respectively. Their series and domains of definition are explicitly presented in sub-section 9.2 in terms of the parameters $\nu_1^{(i)}$, $\nu_2^{(i)}$, $i = 0, 1, \infty$.

According to [7], we define $2 - x_0^2 := \text{tr } M_0 M_x$, $2 - x_1^2 := \text{tr } M_1 M_x$, $2 - x_\infty^2 := \text{tr } M_0 M_1$. There is a one to one correspondence between triples $(x_0, x_1, x_\infty)$ (defined up to the change of two signs) and Painlevé transcendents, provided that at most one $x_i$ is zero and not all the $x_i$ are $\pm 2$ at the same time. We show that there exists a unique transcendent associated to a triple. It is:

$$y(x; x_0, x_1, x_\infty) = \wp\big(\nu_1^{(0)}\omega_1^{(0)}(x) + \nu_2^{(0)}\omega_2^{(0)}(x) + v^{(0)}(x; \nu_1^{(0)}, \nu_2^{(0)})\big) + \frac{1+x}{3}$$

$$= \wp\big(\nu_1^{(1)}\omega_1^{(1)}(x) + \nu_2^{(1)}\omega_2^{(1)}(x) + v^{(1)}(x; \nu_1^{(1)}, \nu_2^{(1)})\big) + \frac{1+x}{3} \tag{4}$$

$$= \wp\big(\nu_1^{(\infty)}\omega_1^{(\infty)}(x) + \nu_2^{(\infty)}\omega_2^{(\infty)}(x) + v^{(\infty)}(x; \nu_1^{(\infty)}, \nu_2^{(\infty)})\big) + \frac{1+x}{3} \tag{5}$$

The parameters $\nu_2^{(i)}$ are obtained from

$$\cos \pi \nu_2^{(i)} = \frac{x_i^2}{2} - 1, \quad 0 \le \Re \nu_2^{(i)} \le 1, \quad \nu_2^{(i)} \neq 0, \quad i = 0, 1, \infty$$

The further constraint that $x_i \neq \pm 2$, $i = 0, 1, \infty$, has to be imposed, because $\nu_2^{(i)} \neq 0$. The parameter $\nu_1^{(0)}$ is obtained by the formula

$$e^{i\pi\nu_1^{(0)}} = -\frac{i\Gamma^4\left(1 - \frac{\nu_2^{(0)}}{2}\right)}{2\sin(\pi\nu_2^{(0)})\Gamma^2\left(\frac{3}{2} - \mu - \frac{\nu_2^{(0)}}{2}\right)\Gamma^2\left(\frac{1}{2} + \mu - \frac{\nu_2^{(0)}}{2}\right)} \Big[2(1 - e^{i\pi\nu_2^{(0)}}) -$$

$$- f(x_0, x_1, x_\infty)(x_\infty^2 - e^{i\pi\nu_2^{(0)}} x_1^2)\Big] f(x_0, x_1, x_\infty)$$

where

$$f(x_0, x_1, x_\infty) := \frac{4 - x_0^2}{x_1^2 + x_\infty^2 - x_0 x_1 x_\infty}, \quad \alpha = \frac{(2\mu - 1)^2}{2}$$

Moreover, $\exp\{-i\pi\nu_1^{(1)}\}$, $\exp\{i\pi\nu_1^{(\infty)}\}$ are given by an analogous formula with the substitutions $(x_0, x_1, x_\infty) \mapsto (x_1, x_0, x_0 x_1 - x_\infty)$, $\nu_2^{(0)} \mapsto \nu_2^{(1)}$ and $(x_0, x_1, x_\infty) \mapsto (x_\infty, -x_1, x_0 - x_1 x_\infty)$, $\nu_2^{(0)} \mapsto \nu_2^{(\infty)}$ respectively.

We recall that the most general choice of $\nu_2$ is $0 \le \Re \nu_2 < 2$. This corresponds to the fact that the transcendent $y(x; x_0, x_1, x_\infty)$ has three representations

$$y(x; x_0, x_1, x_\infty) = \wp\big(\tilde{\nu}_1^{(0)}\omega_1^{(0)}(x) + \tilde{\nu}_2^{(0)}\omega_2^{(0)}(x) + v^{(0)}(x; \tilde{\nu}_1^{(0)}, \tilde{\nu}_2^{(0)})\big) + \frac{1+x}{3}$$

$$= \wp\big(\tilde{\nu}_1^{(1)}\omega_1^{(1)}(x) + \tilde{\nu}_2^{(1)}\omega_2^{(1)}(x) + v^{(1)}(x; \tilde{\nu}_1^{(1)}, \tilde{\nu}_2^{(1)})\big) + \frac{1+x}{3}$$

$$= \wp\big(\tilde{\nu}_1^{(\infty)}\omega_1^{(\infty)}(x) + \tilde{\nu}_2^{(\infty)}\omega_2^{(\infty)}(x) + v^{(\infty)}(x; \tilde{\nu}_1^{(\infty)}, \tilde{\nu}_2^{(\infty)})\big) + \frac{1+x}{3}$$

where

$$\cos \pi \tilde{\nu}_2^{(i)} = \frac{x_i^2}{2} - 1, \quad 1 \le \Re \nu_2^{(i)} < 2, \quad i = 0, 1, \infty$$

The parameter $\tilde{\nu}_1^{(0)}$ is obtained by the formula

$$e^{-i\pi\tilde{\nu}_1^{(0)}} = \frac{i\Gamma^4\left(\frac{\tilde{\nu}_2^{(0)}}{2}\right)}{2\sin(\pi\tilde{\nu}_2^{(0)})\Gamma^2\left(\frac{1}{2} - \mu + \frac{\tilde{\nu}_2^{(0)}}{2}\right)\Gamma^2\left(-\frac{1}{2} + \mu + \frac{\nu_2}{2}\right)} \Big[2(1 - e^{-i\pi\tilde{\nu}_2^{(0)}}) -$$

$$- f(x_0, x_1, x_\infty)(x_\infty^2 - e^{-i\pi\tilde{\nu}_2^{(0)}} x_1^2)\Big] f(x_0, x_1, x_\infty).$$

$\exp\{i\pi\tilde{\nu}_1^{(1)}\}$, $\exp\{-i\pi\tilde{\nu}_1^{(\infty)}\}$ are given by an analogous formula with the substitutions $(x_0, x_1, x_\infty) \mapsto (x_1, x_0, x_0x_1 - x_\infty)$, $\tilde{\nu}_2^{(0)} \mapsto \tilde{\nu}_2^{(1)}$ and $(x_0, x_1, x_\infty) \mapsto (x_\infty, -x_1, x_0 - x_1x_\infty)$, $\tilde{\nu}_2^{(0)} \mapsto \tilde{\nu}_2^{(\infty)}$ respectively.

There are some exceptions to the formulae above for $\sigma = 0, \pm 2\mu + 2m$, $m$ integer. They are explained in subsection 9.2.

Conversely, a transcendent

$$y(x) = \wp\big(\nu_1\omega_1^{(0)}(x) + \nu_2\omega_2^{(0)}(x) + v^{(0)}(x;\nu_1,\nu_2)\big) + \frac{1+x}{3}, \qquad \text{at } x = 0 \tag{6}$$

coincides with $y(x;\ x_0, x_1, x_\infty)$, with the following monodromy data.

If $0 \le \Re\nu_2 \le 1$:

$$x_0 = 2\cos\left(\frac{\pi}{2}\nu_2\right)$$

$$x_1 = \left[\frac{4^{-\nu_2}\ 2\ e^{i\frac{\pi}{2}\nu_1}}{f(\nu_2,\mu)G(\nu_2,\mu)} + \frac{G(\nu_2,\mu)}{4^{-\nu_2}\ 2\ e^{i\frac{\pi}{2}\nu_1}}\right]$$

$$x_\infty = \left[\frac{4^{-\nu_2}\ 2\ e^{i\frac{\pi}{2}(\nu_1-\nu_2)}}{f(\nu_2,\mu)G(\nu_2,\mu)} + \frac{G(\nu_2,\mu)}{4^{-\nu_2}\ 2\ e^{i\frac{\pi}{2}(\nu_1-\nu_2)}}\right]$$

where

$$f(\nu_2,\mu) = -\frac{2\sin^2\left(\frac{\pi}{2}\nu_2\right)}{\cos(\pi\nu_2) + \cos(2\pi\mu)}, \qquad G(\nu_2,\mu) = 4^{-\nu_2}\ 2\ \frac{\Gamma\left(1 - \frac{\nu_2}{2}\right)^2}{\Gamma\left(\frac{3}{2} - \mu - \frac{\nu_2}{2}\right)\Gamma\left(\frac{1}{2} + \mu - \frac{\nu_2}{2}\right)}$$

If $1 \le \Re\nu_2 < 2$:

$$x_0 = 2\cos\left(\frac{\pi}{2}\nu_2\right)$$

$$x_1 = \left[\frac{e^{-i\frac{\pi}{2}\nu_1}}{4^{1-\nu_2}\ 2\ f(\nu_2,\mu)G_1(\nu_2,\mu)} + \frac{4^{1-\nu_2}\ 2\ G_1(\nu_2,\mu)}{e^{-i\frac{\pi}{2}\nu_1}}\right]$$

$$x_\infty = \left[\frac{e^{i\frac{\pi}{2}(\nu_2-\nu_1)}}{4^{1-\nu_2}\ 2\ f(\nu_2,\mu)G_1(\nu_2,\mu)} + \frac{4^{1-\nu_2}\ 2\ G_1(\nu_2,\mu)}{e^{i\frac{\pi}{2}(\nu_2-\nu_1)}}\right]$$

where

$$G_1(\nu_2,\mu) = \frac{1}{4^{1-\nu_2}\ 2}\frac{\Gamma\left(\frac{\nu_2}{2}\right)^2}{\Gamma\left(\frac{1}{2} - \mu + \frac{\nu_2}{2}\right)\Gamma\left(-\frac{1}{2} + \mu + \frac{\nu_2}{2}\right)}$$

After computing the monodromy data, we can write the elliptic representations of $y(x;x_0,x_1,x_\infty)$ at $x = 1$ and $x = \infty$, namely (4), (5). Since they are the elliptic representations at $x = 1$, $x = \infty$ of (6), we have solved the connection problem for (6).

For generic values of $\alpha$, $\beta$, $\gamma$, $\delta$ the connection problem is solved in sub-section 9.3, using (a generalization of) the results of [15]. The meaning of "generic" is explained in sub-section 9.3 (in condition (121)).

We expect that in all non-generic cases we can solve the connection problem and express the parameters $\nu_1$, $\nu_2$ in terms of monodromy data. From the conceptual point of view nothing should change with respect to [11] and [15], but the technical details may require a long time for computations.

Finally, we remark that there is still an open problem to be solved, namely we are not yet able to give the critical behavior (and the elliptic representation) of transcendents corresponding to monodromy data such that $\mathrm{tr}(M_0M_x) = \pm 2$ for $x = 0$ (this corresponds to $\nu_2^{(0)} = 0$); $\mathrm{tr}(M_1M_x) = \pm 2$ for $x = 1$ ( this corresponds to $\nu_2^{(1)} = 0$) ; $\mathrm{tr}(M_0M_1) = \pm 2$ for $x = \infty$ ( this corresponds to $\nu_2^{(\infty)} = 0$). [The critical behavior for $\nu_2^{(i)} = 1$ can be found in [15] in the generic case, or in [11] – and in the present paper – for the special case $\beta = \gamma = 1 - 2\delta = 0$.]

## 2 The Elliptic Representation

We derive the elliptic form for the general Painlevé 6 equation, denoted PVI in the following. We follow [8]. Let

$$u=\int_{\infty}^{y}\frac{d\lambda}{\sqrt{\lambda(\lambda-1)(\lambda-x)}} \tag{7}$$

We observe that

$$\frac{du}{dx}=\frac{\partial u}{\partial y}\frac{dy}{dx}+\frac{\partial u}{\partial x}=\frac{1}{\sqrt{y(y-1)(y-x)}}\frac{dy}{dx}+\frac{\partial u}{\partial x}$$

from which we compute

$$\frac{d^2u}{dx^2}+\frac{2x-1}{x(x-1)}\frac{du}{dx}+\frac{u}{4x(x-1)}=$$

$$=\frac{1}{\sqrt{y(y-1)(y-x)}}\left[\frac{d^2y}{dx^2}+\left(\frac{1}{x}+\frac{1}{x-1}+\frac{1}{y-x}\right)\frac{dy}{dx}-\frac{1}{2}\left(\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-x}\right)\left(\frac{dy}{dx}\right)^2\right]$$

$$+\frac{\partial^2u}{\partial x^2}+\frac{2x-1}{x(x-1)}\frac{\partial u}{\partial x}+\frac{u}{4x(x-1)}$$

By direct calculation we have:

$$\frac{\partial^2u}{\partial x^2}+\frac{2x-1}{x(x-1)}\frac{\partial u}{\partial x}+\frac{u}{4x(x-1)}=-\frac{1}{2}\frac{\sqrt{y(y-1)(y-x)}}{x(x-1)}\frac{1}{(y-x)^2}$$

Therefore, $y(x)$ satisfies PVI if and only if

$$\frac{d^2u}{dx^2}+\frac{2x-1}{x(x-1)}\frac{du}{dx}+\frac{u}{4x(x-1)}=\frac{\sqrt{y(y-1)(y-x)}}{2x^2(1-x)^2}\left[2\alpha+2\beta\frac{x}{y^2}+\gamma\frac{x-1}{(y-1)^2}+\left(\delta-\frac{1}{2}\right)\frac{x(x-1)}{(y-x)^2}\right] \tag{8}$$

We invert the function $u=u(y)$ by observing that we are dealing with an elliptic integral. Therefore, we write

$$y=f(u,x)$$

where $f(u,x)$ is an elliptic function of $u$. This implies that

$$\frac{\partial y}{\partial u}=\sqrt{y(y-1)(y-x)}$$

The above equality allows us to rewrite (8) in the following way:

$$x(1-x)\frac{d^2u}{dx^2}+(1-2x)\frac{du}{dx}-\frac{1}{4}u=\frac{1}{2x(1-x)}\frac{\partial}{\partial u}\psi(u,x), \tag{9}$$

where

$$\psi(u,x):=2\alpha f(u,x)-2\beta\frac{x}{f(u,x)}+2\gamma\frac{1-x}{f(u,x)-1}+(1-2\delta)\frac{x(x-1)}{f(u,x)-x}$$

The last step concerns the form of $f(u,x)$. We observe that $4\lambda(\lambda-1)(\lambda-x)$ is not in Weierstrass canonical form. We change variable:

$$\lambda=t+\frac{1+x}{3},$$

and we get the Weierstrass form:

$$4\lambda(\lambda-1)(\lambda-x)=4t^3-g_2t-g_3,\quad g_2=\frac{4}{3}(1-x+x^2),\quad g_3:=\frac{4}{27}(x-2)(2x-1)(1+x)$$

Thus

$$\frac{u}{2}=\int_{\infty}^{y-\frac{1+x}{3}}\frac{dt}{\sqrt{4t^3-g_2t-g_3}}$$

which implies

$$f(u,x) = \mathcal{P}\left(\frac{u}{2};\omega_1,\omega_2\right) + \frac{1+x}{3}$$

We still need to explain what are the *half periods* $\omega_1$, $\omega_2$. In order to do that, we first observe that the Weierstrass form is

$$4t^3 - g_2 t - g_3 = 4(t-e_1)(t-e_2)(t-e_3)$$

where

$$e_1 = \frac{2-x}{3}, \quad e_2 = \frac{2x-1}{3}, \quad e_3 = -\frac{1+x}{3}.$$

Therefore

$$g := \sqrt{e_1 - e_2} = 1, \quad \kappa^2 := \frac{e_2 - e_3}{e_1 - e_3} = x, \quad {\kappa'}^2 := 1 - \kappa^2 = 1 - x$$

We identify $e_1 = \wp(\omega_1)$, $e_2 = \wp(\omega_1 + \omega_2)$, $e_3 = \wp(\omega_2)$. Therefore, the half-periods are

$$\omega_1 = \frac{1}{g}\int_0^1 \frac{d\xi}{\sqrt{(1-\xi^2)(1-\kappa^2\xi^2)}} = \int_0^1 \frac{d\xi}{\sqrt{(1-\xi^2)(1-x\xi^2)}} = \mathbf{K}(x)$$

$$\omega_2 = \frac{i}{g}\int_0^1 \frac{d\xi}{\sqrt{(1-\xi^2)(1-{\kappa'}^2\xi^2)}} = i\int_0^1 \frac{d\xi}{\sqrt{(1-\xi^2)(1-(1-x)\xi^2)}} = i\mathbf{K}(1-x)$$

The elliptic integral $\mathbf{K}(x)$ coincides, up to a factor, with the hyper-geometric function $F\left(\frac{1}{2},\frac{1}{2},1;x\right)$, whose convergent series for $|x|<1$ is:

$$F\left(\frac{1}{2},\frac{1}{2},1;x\right) = \sum_{n=0}^{\infty} \frac{\left[\left(\frac{1}{2}\right)_n\right]^2}{(n!)^2} x^n.$$

Namely:

$$\mathbf{K}(x) = \frac{\pi}{2}\, F\left(\frac{1}{2},\frac{1}{2},1;x\right)$$

Moreover, $\mathbf{K}(x)$ and $\mathbf{K}(1-x)$ are two linearly independent solutions of the hyper-geometric equation

$$x(1-x)\omega'' + (1-2x)\omega' - \frac{1}{4}\omega = 0.$$

Observe that for $|\arg(x)| < \pi$:

$$-\pi F\left(\frac{1}{2},\frac{1}{2},1;1-x\right) = F\left(\frac{1}{2},\frac{1}{2},1;x\right)\ln(x) + F_1(x)$$

where

$$F_1(x) := \sum_{n=0}^{\infty} \frac{\left[\left(\frac{1}{2}\right)_n\right]^2}{(n!)^2} 2\left[\psi(n+\frac{1}{2}) - \psi(n+1)\right] x^n,$$

$$\psi(z) = \frac{d}{dz}\ln\Gamma(z), \quad \psi\left(\frac{1}{2}\right) = -\gamma - 2\ln 2, \quad \psi(1) = -\gamma, \quad \psi(a+n) = \psi(a) + \sum_{l=0}^{n-1}\frac{1}{a+l}.$$

Therefore

$$\omega_2(x) = -\frac{i}{2}[F\left(\frac{1}{2},\frac{1}{2},1;x\right)\ln(x) + F_1(x)]$$

In the following we use sometimes the abbreviation $F(x)$ for $F\left(\frac{1}{2},\frac{1}{2},1;x\right)$. The series of $F(x)$ and $F_1(x)$ converge for $|x|<1$.

Let

$$\mathcal{L}(u) := x(1-x)\frac{d^2u}{dx^2} + (1-2x)\frac{du}{dx} - \frac{1}{4}u$$

The PVI equation becomes

$$\mathcal{L}(u) = \frac{1}{2x(1-x)} \frac{\partial}{\partial u} \left\{ 2\alpha \left[ \wp\left(\frac{u}{2};\omega_1,\omega_2\right) - e_3 \right] - 2\beta \frac{x}{\wp\left(\frac{u}{2};\omega_1,\omega_2\right) - e_3} + \right.$$

$$\left. +2\gamma \frac{1-x}{\wp\left(\frac{u}{2};\omega_1,\omega_2\right) - e_1} + (1-2\delta) \frac{x(1-x)}{\wp\left(\frac{u}{2};\omega_1,\omega_2\right) - e_2} \right\} \tag{10}$$

We recall that

$$\frac{1}{\wp\left(\frac{u}{2}\right) - e_3} = \frac{1}{(e_1-e_3)(e_2-e_3)} \left[ \wp\left(\frac{u}{2} + \omega_2\right) - e_3 \right]$$

$$\frac{1}{\wp\left(\frac{u}{2}\right) - e_1} = \frac{1}{(e_1-e_2)(e_1-e_3)} \left[ \wp\left(\frac{u}{2} + \omega_1\right) - e_1 \right]$$

$$\frac{1}{\wp\left(\frac{u}{2}\right) - e_2} = \frac{1}{(e_1-e_2)(e_3-e_2)} \left[ \wp\left(\frac{u}{2} + \omega_1 + \omega_2\right) - e_2 \right].$$

We observe also that $e_1 - e_3 = 1$, $e_2 - e_3 = x$, $e_1 - e_2 = 1 - x$. Therefore, (10) becomes:

$$\mathcal{L}(u) = \frac{1}{2x(1-x)} \left[ 2\alpha \frac{\partial}{\partial u} \wp\left(\frac{u}{2}\right) - 2\beta \frac{\partial}{\partial u} \wp\left(\frac{u}{2} + \omega_2\right) + \right.$$

$$\left. +2\gamma \frac{\partial}{\partial u} \wp\left(\frac{u}{2} + \omega_1\right) + (1-2\delta) \frac{\partial}{\partial u} \wp\left(\frac{u}{2} + \omega_1 + \omega_2\right) \right] \tag{11}$$

# 3 Local Representation - Theorem 1

The equation $\mathcal{L}(u) = 0$ has a general solution

$$u_0 = 2\nu_1\omega_1 + 2\nu_2\omega_2, \qquad \nu_1,\ \nu_2 \in \mathbf{C}$$

We look for a solution of (11) of the form $u(x) = 2\nu_1\omega_1 + 2\nu_2\omega_2(x) + 2v(x)$, where $v(x)$ is a perturbation of $u_0$.

We state the main result. Let $\mathbf{C}_0 := \mathbf{C}\backslash\{0\}$, $\widetilde{\mathbf{C}_0}$ the universal covering and let $0 < r < 1$. We define the domains

$$\mathcal{D}(r;\nu_1,\nu_2) := \left\{ x \in \widetilde{\mathbf{C}_0} \ \text{ such that } |x| < r, \left| \frac{e^{-i\pi\nu_1}}{16^{1-\nu_2}} x^{1-\nu_2} \right| < r, \left| \frac{e^{i\pi\nu_1}}{16^{\nu_2}} x^{\nu_2} \right| < r \right\} \tag{12}$$

$$\mathcal{D}_0(r) := \left\{ x \in \widetilde{\mathbf{C}_0} \ \text{ such that } |x| < r \right\} \tag{13}$$

Consider

$$y(x) = \wp\left(\nu_1\omega_1(x) + \nu_2\omega_2(x) + v(x); \omega_1(x), \omega_2(x)\right) + \frac{1+x}{3}$$

We observe that the translations $\nu_i \mapsto \nu_i + 2N_i$, $i = 1,2$, $N_i \in \mathbf{Z}$ do not change $y(x)$, due to the periodicity of the $\wp$-function. Therefore, one can take $0 \leq \Re\nu_i < 2$, $i = 1,2$. Nevertheless, we don't need to suppose such a range explicitly. Only in the case $\Im\nu_2 = 0$ we will suppose that

$$0 \leq \nu_2 < 2$$

We introduce the following expansion:

$$v(x;\nu_1,\nu_2) := \sum_{n\geq 1} a_n x^n + \sum_{n\geq 0, m\geq 1} b_{nm} x^n \left[ e^{-i\pi\nu_1} \left(\frac{x}{16}\right)^{1-\nu_2} \right]^m + \sum_{n\geq 0, m\geq 1} c_{nm} x^n \left[ e^{i\pi\nu_1} \left(\frac{x}{16}\right)^{\nu_2} \right]^m \tag{14}$$

**Theorem 1:** *Let $\nu_1$, $\nu_2$ be two complex numbers.*

**I)** *For any complex $\nu_1$, $\nu_2$ such that*

$$\Im\nu_2 \neq 0$$

*there exist a positive number $r<1$ and a transcendent*

$$y(x)=\wp\Big(\nu_1\omega_1(x)+\nu_2\omega_2(x)+v(x;\nu_1,\nu_2);~\omega_1(x),\omega_2(x)\Big)+\frac{1+x}{3}$$

*such that $v(x;\nu_1,\nu_2)$ is holomorphic in the domain $\mathcal{D}(r;\nu_1,\nu_2)$ and it is given by the expansion (14) which is convergent in $\mathcal{D}(r;\nu_1,\nu_2)$. The coefficients $a_n$, $b_{nm}$, $c_{nm}$, $i=1,2$, are certain rational functions of $\nu_2$. Moreover, there exists a positive constant $M(\nu_2)$ such that*

$$|v(x;\nu_1,\nu_2)|\le M(\nu_2)\left(|x|+\left|e^{-i\pi\nu_1}\left(\frac{x}{16}\right)^{1-\nu_2}\right|+\left|e^{i\pi\nu_1}\left(\frac{x}{16}\right)^{\nu_2}\right|\right)\qquad \textit{in } \mathcal{D}(r;\nu_1,\nu_2) \tag{15}$$

**II)** *For any complex $\nu_1$ and real $\nu_2$, with the constraint $0<\nu_2<1$ or $1<\nu_2<2$, there exists a positive $r<1$ and a transcendent*

$$y(x)=\wp\Big(\nu_1\omega_1(x)+\nu_2\omega_2(x)+v(x;\nu_1,\nu_2);~\omega_1(x),\omega_2(x)\Big)+\frac{1+x}{3},\quad \textit{if } 0<\nu_2<1$$

*or*

$$y(x)=\wp\Big(\nu_1\omega_1(x)+\nu_2\omega_2(x)+v(x;-\nu_1,2-\nu_2);~\omega_1(x),\omega_2(x)\Big)+\frac{1+x}{3},\quad \textit{if } 1<\nu_2<2$$

*such that $v(x;\nu_1,\nu_2)$ and $v(x;-\nu_1,2-\nu_2)$ are holomorphic in $\mathcal{D}_0(r)$, with convergent expansion (14) and bound (15) (for $1<\nu_2<2$ substitute $\nu_1\mapsto-\nu_1$, $\nu_2\mapsto 2-\nu_2$).*

Note that in the theorem

$$\nu_2\neq 0,1$$

We stress that in case **II)**, if $\nu_2$ is greater that 2 or less then 0, we can always make a translation $\nu_2\mapsto\nu_2+2N$ to obtain $0<\nu_2<2$ (on the other hand, if $-2N<\nu_2<2-2N$, the formulae of case **II)** hold with the substitution $\nu_2\mapsto\nu_2+2N$).

**Observation 1:** As a consequence of the theorem, for any $N\in\mathbf{Z}$ and for any complex $\nu_1,\nu_2$ such that $\Im\nu_2\neq0$ there exists $r_N<1$ and a transcendent $y(x)=\wp\Big(\nu_1\omega_1(x)+[\nu_2+2N]\omega_2(x)+v(x;\nu_1,\nu_2+2N);~\omega_1(x),\omega_2(x)\Big)+\frac{1+x}{3}$ in $\mathcal{D}(r;\nu_1,\nu_2+2N)$. By periodicity of the $\wp$-function we re-write the transcendent as follows:

$$y(x)=\wp\Big(\nu_1\omega_1(x)+\nu_2\omega_2(x)+v(x;\nu_1,\nu_2+2N);~\omega_1(x),\omega_2(x)\Big)+\frac{1+x}{3}\quad\text{in } \mathcal{D}(r;\nu_1,\nu_2+2N).$$

We will show in section 9 that if a transcendent has elliptic representation

$$y(x)=\wp\Big(\nu_1\omega_1(x)+\nu_2\omega_2(x)+v(x;\nu_1,\nu_2);~\omega_1(x),\omega_2(x)\Big)+\frac{1+x}{3}$$

in $\mathcal{D}(r,\nu_1,\nu_2)$ for some $\nu_1,\nu_2$, $\Im\nu_2\neq0$, then it has representation

$$y(x)=\wp\Big(\nu_1'\omega_1(x)+\nu_2\omega_2(x)+v(x;\nu_1,\nu_2+2N);~\omega_1(x),\omega_2(x)\Big)+\frac{1+x}{3}$$

in $\mathcal{D}(r,\nu_1',\nu_2+2N)$ for a suitable $\nu_1'$ (depending on $\nu_1$, $\nu_2$ and $N$).

**Observation 2:** Another consequence of the theorem is that for any complex $\nu_1,\nu_2$ such that $\Im\nu_2\neq0$ there exists $y(x)=\wp\Big(-\nu_1\omega_1(x)+[2-\nu_2]~\omega_2(x)+v(x;-\nu_1,2-\nu_2);~\omega_1(x),\omega_2(x)\Big)+\frac{1+x}{3}$. Again we use the fact that the $\wp$-function is periodic w.r.t. $2\omega_2$ and it is an even function. Therefore the transcendent becomes

$$y(x)=\wp\Big(\nu_1\omega_1(x)+\nu_2\omega_2(x)-v(x;-\nu_1,2-\nu_2);~\omega_1(x),\omega_2(x)\Big)+\frac{1+x}{3},\quad\text{in } \mathcal{D}(r;-\nu_1,2-\nu_2)$$

Note that the series $-v(x;-\nu_1,2-\nu_2)$ is of the form

$$\sum_{n\ge1}a_nx^n+\sum_{n\ge0,m\ge1}b_{nm}x^n\left[e^{-i\pi\nu_1}\left(\frac{x}{16}\right)^{2-\nu_2}\right]^m+\sum_{n\ge0,m\ge1}c_{nm}x^n\left[e^{i\pi\nu_1}\left(\frac{x}{16}\right)^{\nu_2-1}\right]^m$$

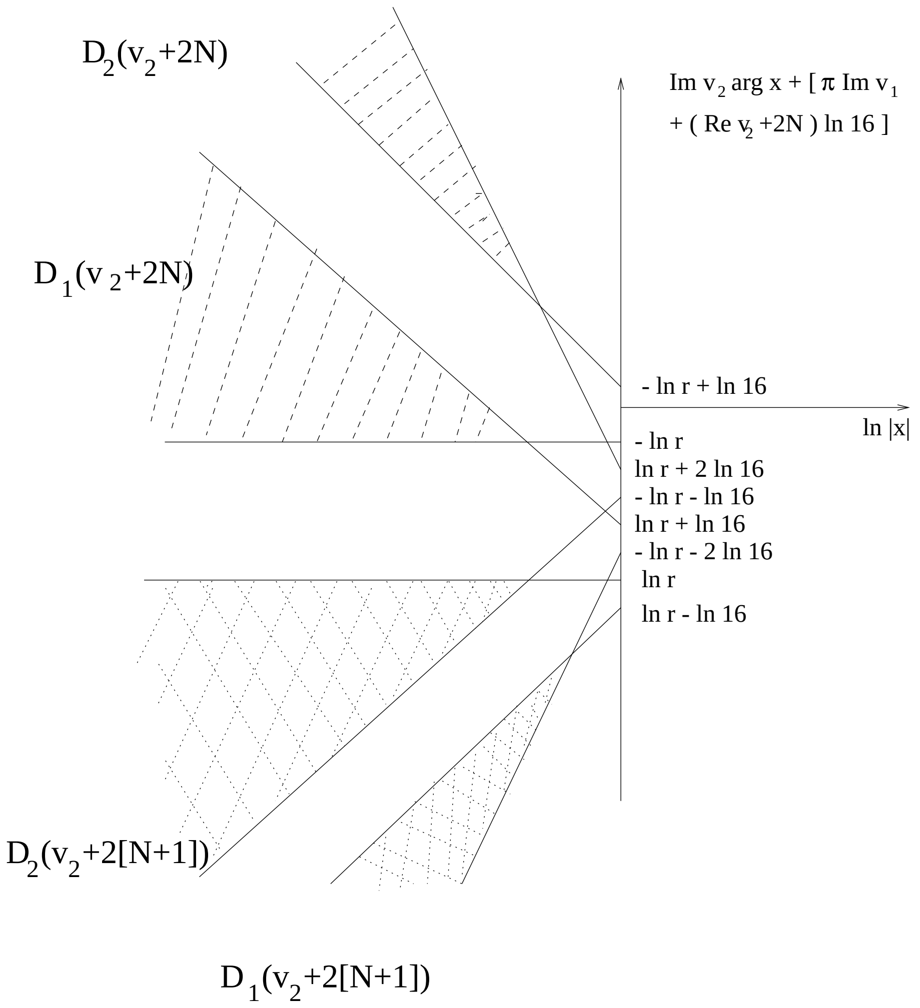


Figure 1: The domains $\mathcal{D}_1(r;\nu_1,\nu_2{+}2N):=\mathcal{D}(r;\nu_1,\nu_2{+}2N)$, $\mathcal{D}_2(r;\nu_1,\nu_2{+}2N):=\mathcal{D}(r;-\nu_1,2{-}\nu_2{-}2N)$ and $\mathcal{D}_1(r;\nu_1,\nu_2+2[N+1])$, $\mathcal{D}_2(r;\nu_1,\nu_2+2[N+1])$ for arbitrarily fixed values of $\nu_1$, $\nu_2$, $N$. They are represented in the plane $(\ln|x|,\ \Im\nu_2\arg x+[\pi\Im\nu_1+(\Re\nu_2+2N)\ln 16]$.

where we have re-named the constants $a_n, b_{nm}, c_{nm}$.

**Remark:** The domain $\mathcal{D}(r_N;\nu_1,\nu_2+2N)$ can be written as follows:

$$(\Re\nu_2+2N)\ln\frac{|x|}{16}-\pi\Im\nu_1-\ln r_N<\Im\nu_2\arg x<$$

$$<(\Re\nu_2-1+2N)\ln\frac{|x|}{16}-\pi\Im\nu_1+\ln r_N,\qquad |x|<r_N$$

Therefore the domain $\mathcal{D}(r_N,-\nu_1,2-\nu_2-2N)$ is

$$(\Re\nu_2-1+2N)\ln\frac{|x|}{16}-\pi\Im\nu_1-\ln r_N<\Im\nu_2\arg x<$$

$$<(\Re\nu_2-2+2N)\ln\frac{|x|}{16}-\pi\Im\nu_1+\ln r_N,\qquad |x|<r_N$$

We can draw their picture in the $(\ln|x|,\Im\nu_2\arg x)$-plane. See figure 1.

## 4 Proof of Theorem 1

For technical reason it is convenient to introduce the following definitions:

$$\mathcal{D}_1(r;\nu_1,\nu_2):=\left\{x\in\widetilde{\mathbf{C}_0}\ \text{ such that } |x|<r, \left|\frac{e^{-i\pi\nu_1}}{16^{1-\nu_2}}x^{1-\nu_2}\right|<r, \left|\frac{e^{i\pi\nu_1}}{16^{\nu_2}}x^{\nu_2}\right|<r\right\} \tag{16}$$

$$\mathcal{D}_2(r;\nu_1,\nu_2):=\left\{x\in\widetilde{\mathbf{C}_0}\ \text{ such that } |x|<r, \left|\frac{e^{-i\pi\nu_1}}{16^{2-\nu_2}}x^{2-\nu_2}\right|<r, \left|\frac{e^{i\pi\nu_1}}{16^{\nu_2-1}}x^{\nu_2-1}\right|<r\right\} \tag{17}$$

Note that $\mathcal{D}_1(r;\nu_1,\nu_2)\equiv\mathcal{D}(r;\nu_1,\nu_2)$ in (12) and $\mathcal{D}_2(r;\nu_1,\nu_2)\equiv\mathcal{D}(r;-\nu_1,2-\nu_2)$. Let

$$\tau(x):=\frac{\omega_2(x)}{\omega_1(x)}=\frac{1}{\pi}(\arg x-i\ln\frac{|x|}{16})+O(x)$$

be the modular parameter of the elliptic function, obtained expanding $\omega_1(x)$, $\omega_2(x)$ at $x=0$. We expand the r.h.s of (11) in Fourier series. We are dealing with Weierstrass $\wp$-functions of the form $\wp\left(\frac{u}{2}+\varepsilon_1\omega_1+\varepsilon_2\omega_2\right)$, where $\varepsilon_i\in\{0,1\}$, $i=1,2$. Due to periodicity of $\wp$ we are going to expand

$$\wp\left(\frac{u}{2}+[\varepsilon_1+2N_1]\,\omega_1+[\varepsilon_2+2N_2]\,\omega_2\right),\qquad \varepsilon_i\in\{0,1\},\quad N_i\in\mathbf{Z},\quad i=1,2$$

As it is well known [33], the Fourier expansion of the $\wp$-function is

$$\wp(z;\omega_1,\omega_2)=\left(\frac{\pi}{2\omega_1}\right)^2\left[-\frac{1}{3}+\frac{1}{\sin^2\left(\frac{\pi z}{2\omega_1}\right)}+8\sum_{n\geq 1}\frac{ne^{2i\pi n\tau}}{1-e^{2i\pi n\tau}}\ \left[1-\cos\left(\frac{n\pi z}{\omega_1}\right)\right]\right],\quad \text{for } \Im\tau>\left|\Im\left(\frac{z}{2\omega_1}\right)\right|$$

Therefore, the expansion is possible provided that

$$\left|\Im\left(\frac{u(x)}{4\omega_1(x)}+\frac{\varepsilon_1+2N_1}{2}+\frac{\varepsilon_2+2N_2}{2}\ \tau(x)\right)\right|<\Im\tau(x) \tag{18}$$

We observe that for $x\to 0$

$$\Im\tau(x)=-\frac{1}{\pi}\ln\frac{|x|}{16}+O(x)\to+\infty$$

The condition (18) becomes

$$-\Im\tau<\frac{1}{2}\Im\nu_1+\frac{1}{2}[\varepsilon_2+2N_2+\Re\nu_2]\ \Im\tau+\frac{1}{2}\Im\nu_2\ \Re\tau+\Im\left(\frac{v}{2\omega_1}\right)<\Im\tau \tag{19}$$

If $\nu_2$ is real, we divide by $\Im\tau$, we let $x\to 0$ and we obtain

$$-2-2N_2-\varepsilon_2<\nu_2<2-2N_2-\epsilon_2$$

provided that $v(x)$ is bounded. In general, (19) becomes

$$(\Re\nu_2+2+\varepsilon_2+2N_2)\ln\frac{|x|}{16}-\left[\pi\Im\nu_1+\Im\left(\frac{\pi v}{\omega_1}\right)\right]+O(x)$$

$$\leq\Im\nu_2\arg x\leq$$

$$(\Re\nu_2-2+\varepsilon_2+2N_2)\ln\frac{|x|}{16}-\left[\pi\Im\nu_1+\Im\left(\frac{\pi v}{\omega_1}\right)\right]+O(x) \tag{20}$$

In the domain defined by the above condition, and $|x|<1$, we do the Fourier expansion as follows

$$\frac{\partial}{\partial u}\wp\left(\frac{u}{2}+[\varepsilon_1+2N_1]\omega_1+[\varepsilon_2+2N_2]\omega_2\right)=$$

$$=\frac{\pi^3}{8\omega_1^3}\left\{8\sum_{n\geq 1}\frac{n^2e^{2i\pi n\tau}}{1-e^{2i\pi n\tau}}\sin\left[\pi n\left(\frac{u}{2\omega_1}+\epsilon_1+(\epsilon_2+2N_2)\tau\right)\right]-\frac{\cos\left[\frac{\pi}{2}\left(\frac{u}{2\omega_1}+\epsilon_1+(\epsilon_2+2N_2)\tau\right)\right]}{\sin^3\left[\frac{\pi}{2}\left(\frac{u}{2\omega_1}+\epsilon_1+(\epsilon_2+2N_2)\tau\right)\right]}\right\}$$

$$= \frac{i\pi^3}{2\omega_1^3}\left\{ \frac{e^{i\frac{f(x)}{2}} + e^{-i\frac{f(x)}{2}}}{\left[e^{i\frac{f(x)}{2}} - e^{-i\frac{f(x)}{2}}\right]^3} - \sum_{n\geq 1} \frac{n^2 e^{2i\pi n\tau}}{1-e^{2i\pi n\tau}}\left[e^{inf(x)} - e^{-inf(x)}\right]\right\} \tag{21}$$

where

$$e^{if(x)} = \exp\left\{ i\pi\left[(\nu_1+\varepsilon_1) + (\nu_2+\varepsilon_2+2N_2)\tau(x) + \frac{v(x)}{\omega_1(x)}\right]\right\}$$

Note that $N_1$ does not appear in the expansion. So, we can take $N_1 = 0$ in the following.

Now we make the *further assumption* that

$$\left|e^{if(x)}\right| < 1 \tag{22}$$

and we rewrite (21) as follows:

$$\frac{i\pi^3}{2\omega_1^3}\left\{ \frac{e^{2if(x)} + e^{if(x)}}{\left[e^{if(x)} - 1\right]^3} - \sum_{n\geq 1} \frac{n^2 e^{i\pi n\left[-(\nu_1+\varepsilon_1)+(2-\nu_2-\varepsilon_2-2N_2)\tau - \frac{v}{\omega_1}\right]}}{1-e^{2\pi n\tau}}\ \left[e^{2inf(x)} - 1\right]\right\}$$

Due to (22), the denominator in the first term does not vanish, so that the expansion has no poles. The condition (22) is more restrictive than (20) and yields:

$$(\Re\nu_2 + \varepsilon_2 + 2N_2)\ln\frac{|x|}{16} - \left[\pi\Im\nu_1 + \Im\left(\frac{\pi v}{\omega_1}\right)\right] + O(x)$$

$$\leq \Im\nu_2 \arg x \leq$$

$$(\Re\nu_2 - 2 + \varepsilon_2 + 2N_2)\ln\frac{|x|}{16} - \left[\pi\Im\nu_1 + \Im\left(\frac{\pi v}{\omega_1}\right)\right] + O(x) \tag{23}$$

We observe that, for any complex number $C$,

$$e^{i\pi C\tau(x)} = h(x)^C\ \left[\frac{x}{16}\right]^C,$$

where

$$h(x) := e^{C\left(\frac{F_1(x)}{F(x)} + 4\ln 2\right)} = 1 + O(x), \quad x \to 0$$

Therefore,

$$\frac{\partial}{\partial u}\wp\left(\frac{u}{2} + (\varepsilon_1 + 2N_1)\omega_1 + (\varepsilon_2+2N_2)\omega_2\right) = \frac{i\pi^3}{2\omega_1^3}\left\{\frac{A^2+A}{(A-1)^3} - \sum_{n\geq 1}\frac{n^2B^n}{1-h(x)^{2n}\left(\frac{x}{16}\right)^{2n}}\left(A^{2n}-1\right)\right\}$$

where

$$A = A\left(x, e^{i\pi\nu_1}\left(\frac{x}{16}\right)^{\nu_2+\epsilon_2+2N_2} e^{i\pi\frac{v}{\omega_1}}\right) := h(x)^{\nu_2+\varepsilon_2+2N_2} e^{i\pi(\varepsilon_1+\nu_1)}\left(\frac{x}{16}\right)^{\nu_2+\varepsilon_2+2N_2} e^{i\frac{\pi v}{\omega_1}} \tag{24}$$

$$B = B\left(x, e^{-i\pi\nu_1}\left(\frac{x}{16}\right)^{2-\nu_2-\varepsilon_2-2N_2} e^{-i\pi\frac{v}{\omega_1}}\right) := h(x)^{2-\nu_2-\varepsilon_2-2N_2} e^{-i\pi(\nu_1+\varepsilon_1)}\left(\frac{x}{16}\right)^{2-\nu_2-\varepsilon_2-2N_2} e^{-i\pi\frac{v}{\omega_1}} \tag{25}$$

In the r.h.s. of equation (11), we have four $\wp$-functions with different arguments. We recall that we can take $N_1 = 0$. As for $N_2$, it can be different in each $\wp$-function. There are only two possibilities in order for the four resulting domains (23) to intersect: the $\wp$-functions must be

$$\wp\left(\frac{u}{2} + 2N_2\omega_2\right), \qquad \wp\left(\frac{u}{2} + \omega_1 + 2N_2\omega_2\right), \quad \wp\left(\frac{u}{2} + \omega_2 + 2N_2'\omega_2\right), \qquad \wp\left(\frac{u}{2} + \omega_1 + \omega_2 + 2N_2'\omega_2\right),$$

and

$$(a): \qquad N_2' = N_2$$

or

$$(b): \qquad N_2' = N_2 - 1$$

Thus, the r.h.s. of (11) is expanded in Fourier series if $|x|<1$, in the following domains.

Case (a):

$$(\Re\nu_2 + 2N)\ln\frac{|x|}{16} - \left[\pi\Im\nu_1 + \Im\left(\frac{\pi v}{\omega_1}\right)\right] + O(x) < \Im\nu_2 \arg x <$$

$$< (\Re\nu_2 - 1 + 2N)\ln\frac{|x|}{16} - \left[\pi\Im\nu_1 + \Im\left(\frac{\pi v}{\omega_1}\right)\right] + O(x), \qquad \text{for } N_2' = N_2 =: N \tag{26}$$

Case (b):

$$(\Re\nu_2 - 1 + 2N)\ln\frac{|x|}{16} - \left[\pi\Im\nu_1 + \Im\left(\frac{\pi v}{\omega_1}\right)\right] + O(x) < \Im\nu_2 \arg x <$$

$$< (\Re\nu_2 - 2 + 2N)\ln\frac{|x|}{16} - \left[\pi\Im\nu_1 + \Im\left(\frac{\pi v}{\omega_1}\right)\right] + O(x), \qquad \text{for } N_2' + 1 = N_2 =: N \tag{27}$$

Note that if $\Im\nu_2 = 0$ any value of $\arg(x)$ is allowed, but dividing the previous expressions by $\ln|x|$ and letting $x \to 0$ we have

$$-2N < \nu_2 < 1 - 2N \text{ for (a)}, \qquad 1 - 2N < \nu_2 < 2 - 2N \text{ for (b)}$$

provided that $v(x)$ is bounded fort $x \to 0$. Taking (24) (25) into account in the two cases, we conclude that:

In case (a), for $N_2' = N_2 = N$, the following powers appear in the r.h.s. of (11):

$$x^{2-\nu_2-2N}, \quad x^{\nu_2+2N}, \quad x^{1-\nu_2-2N}, \quad x^{1+\nu_2+2N}$$

The dominant powers are then

$$x^{\nu_2+2N}, \quad x^{1-\nu_2-2N}$$

and we can write (11) as;

$$\mathcal{L}(u) = \frac{1}{2x(1-x)}\mathcal{F}\left(x, e^{-i\pi\nu_1}\left(\frac{x}{16}\right)^{1-\nu_2-2N} e^{-i\pi\frac{v}{\omega_1}}, e^{i\pi\nu_1}\left(\frac{x}{16}\right)^{\nu_2+2N} e^{i\pi\frac{v}{\omega_1}}\right)$$

on the domain defined by $|x|<1$ and (26), assuming that $v(x)$ is bounded.

In case (b), for $N_2' + 1 = N_2 = N$, the following powers appear in the r.h.s. of (11):

$$x^{2-\nu_2-2N}, \quad x^{\nu_2+2N}, \quad x^{3-\nu_2-2N}, \quad x^{\nu_2-1+2N}$$

The dominant powers are then

$$x^{2-\nu_2-2N}, \quad x^{\nu_2-1+2N}$$

and we can write (11) as;

$$\mathcal{L}(u) = \frac{1}{2x(1-x)}\mathcal{F}\left(x, e^{-i\pi\nu_1}\left(\frac{x}{16}\right)^{2-\nu_2-2N} e^{-i\pi\frac{v}{\omega_1}}, e^{i\pi\nu_1}\left(\frac{x}{16}\right)^{\nu_2-1+2N} e^{i\pi\frac{v}{\omega_1}}\right)$$

on the domain defined by $|x|<1$ and (27), assuming that $v(x)$ is bounded.

Let $\epsilon < 1$ be sufficiently small: in both cases, $\mathcal{F}(x,y,z)$ is holomorphic for $|x|,|y|,|z| < \epsilon$ and $\mathcal{F}(0,0,0) = 0$. In (a) we decompose

$$\mathcal{F}\left(x, e^{-i\pi\nu_1}\left(\frac{x}{16}\right)^{1-\nu_2-2N} e^{-i\pi\frac{v}{\omega_1}}, e^{i\pi\nu_1}\left(\frac{x}{16}\right)^{\nu_2+2N} e^{i\pi\frac{v}{\omega_1}}\right) =$$

$$= \mathcal{F}\left(x, e^{-i\pi\nu_1}\left(\frac{x}{16}\right)^{1-\nu_2-2N}, e^{i\pi\nu_1}\left(\frac{x}{16}\right)^{\nu_2+2N}\right) + \mathcal{G}\left(x, e^{-i\pi\nu_1}\left(\frac{x}{16}\right)^{1-\nu_2-2N}, e^{i\pi\nu_1}\left(\frac{x}{16}\right)^{\nu_2+2N}, v\right)$$

where

$$\mathcal{G}\left(x, e^{-i\pi\nu_1}\left(\frac{x}{16}\right)^{1-\nu_2-2N}, e^{i\pi\nu_1}\left(\frac{x}{16}\right)^{\nu_2+2N}, v\right) :=$$

$$\mathcal{F}\left(x, e^{-i\pi\nu_1}\left(\frac{x}{16}\right)^{1-\nu_2-2N} e^{-i\pi\frac{v}{\omega_1}}, e^{i\pi\nu_1}\left(\frac{x}{16}\right)^{\nu_2+2N} e^{i\pi\frac{v}{\omega_1}}\right) -$$

$$-\mathcal{F}\left(x, e^{-i\pi\nu_1}\left(\frac{x}{16}\right)^{1-\nu_2-2N}, e^{i\pi\nu_1}\left(\frac{x}{16}\right)^{\nu_2+2N}\right)$$

We note that $\mathcal{L}(u) = \mathcal{L}(u_0 + 2v)$=$\mathcal{L}(u_0) + 2\mathcal{L}(v) \equiv 2\mathcal{L}(v)$. Let us put

$$w(x) := x\frac{d}{dx}v(x)$$

Equation (11) becomes the system

$$\begin{cases} x\frac{dv}{dx} = w \\ x\frac{dw}{dx} = \Phi + \Psi \end{cases}$$

where

$$\Phi = \Phi\left(x, e^{-i\pi\nu_1}\left(\frac{x}{16}\right)^{1-\nu_2-2N}, e^{i\pi\nu_1}\left(\frac{x}{16}\right)^{\nu_2+2N}\right) :=$$

$$= \frac{1}{4(1-x)^2}\mathcal{F}\left(x, e^{-i\pi\nu_1}\left(\frac{x}{16}\right)^{1-\nu_2-2N}, e^{i\pi\nu_1}\left(\frac{x}{16}\right)^{\nu_2+2N}\right)$$

and

$$\Psi = \Psi\left(x, e^{-i\pi\nu_1}\left(\frac{x}{16}\right)^{1-\nu_2-2N}, e^{i\pi\nu_1}\left(\frac{x}{16}\right)^{\nu_2+2N}, v, w\right) :=$$

$$= \frac{x(w+v/4)}{1-x} + \frac{1}{4(1-x)^2}\mathcal{G}\left(x, e^{-i\pi\nu_1}\left(\frac{x}{16}\right)^{1-\nu_2-2N}, e^{i\pi\nu_1}\left(\frac{x}{16}\right)^{\nu_2+2N}, v\right)$$

In the same way, in case (b) we obtain a similar system, where

$$\Phi = \Phi\left(x, e^{-i\pi\nu_1}\left(\frac{x}{16}\right)^{2-\nu_2-2N}, e^{i\pi\nu_1}\left(\frac{x}{16}\right)^{\nu_2-1+2N}\right)$$

$$\Psi = \Psi\left(x, e^{-i\pi\nu_1}\left(\frac{x}{16}\right)^{2-\nu_2-2N}, e^{i\pi\nu_1}\left(\frac{x}{16}\right)^{\nu_2-1+2N}, v, w\right)$$

By construction, in both cases $\Phi(x, y, z)$ and $\Psi(x, y, z, v, w)$ are holomorphic of their arguments for $|x|, |y|, |z|, |v|, |w| < \epsilon$ for sufficiently small $\epsilon < 1$ (assuming $v(x)$ bounded in a neighborhood of $x = 0$). Moreover

$$\Phi(0,0,0) = \Psi(0,0,0,v,w) = \Psi(x,y,z,0,0) = 0$$

We complete the proof of the theorem in case (a). Case (b) is analogous. We reduce the system of differential equations to a system of integral equations

$$w(x) = \int_{L(x)} \frac{1}{s}\left\{\Phi(s, e^{-i\pi\nu_1}\left(\frac{s}{16}\right)^{1-\nu_2-2N}, e^{i\pi\nu_1}\left(\frac{s}{16}\right)^{\nu_2+2N})+\right.$$

$$\left.+\Psi(s, e^{-i\pi\nu_1}\left(\frac{s}{16}\right)^{1-\nu_2-2N}, e^{i\pi\nu_1}\left(\frac{s}{16}\right)^{\nu_2+2N}, v(s), w(s))\right\}\ ds$$

$$v(x) = \int_{L(x)} \frac{1}{s}w(s)\ ds$$

The point $x$ and the path of integration are chosen to belong to the domain where $|x|$, $|\frac{e^{-i\pi\nu_1}}{16^{1-\nu_2-2N}}x^{1-\nu_2-2N}|$, $|\frac{e^{i\pi\nu_1}}{16^{\nu_2+2N}}x^{\nu_2+2N}|$, $|v(x)|$, $|w(x)|$ are less than $\epsilon$, in such a way that $\Phi$ and $\Psi$ are holomorphic. That such a domain is not empty will be shown below. In particular, we'll show that if we require that $|x| < r$,

$|\frac{e^{-i\pi\nu_1}}{16^{1-\nu_2}}x^{1-\nu_2}|<r$, $|\frac{e^{i\pi\nu_1}}{16^{\nu_2}}x^{\nu_2}|<r$, where $r<\epsilon$ is small enough, also the solutions of the integral equations $|v(x)|$ and $|w(x)|$ are less than $\epsilon$. Such a domain is contained in (26).

We choose the path of integration $L(x)$ connecting 0 to $x$, defined by

$$\arg(s)=\arg(x)+\frac{\Re\nu_2+2N-\nu^*}{\Im\nu_2}\log\frac{|s|}{|x|},\quad 0<|s|\le|x|,\quad \nu^*\in\mathbf{C}$$

If $x$ belongs to the domain (26) in case (a) and to (27) in case (b), than the path does not leave the domain when $s\to 0$, provided that $0\le\nu^*\le 1$ in case (a), or $1\le\nu^*\le 2$ in case (b). If $\Im\nu_2=0$ we take the radial path $\arg s=\arg x$, $0<|s|\le|x|$, namely $\nu^*=\nu_2$. The parameterization of the path is

$$s=\rho\ e^{i\left\{\arg x+\frac{\Re\nu_2+2N-\nu^*}{\Im\nu_2}\log\frac{\rho}{|x|}\right\}},\quad 0<\rho\le|x|$$

therefore

$$|ds|=P(\nu_2+2N,\nu^*)\ d\rho,\qquad P(\nu_2+2N,\nu^*):=\sqrt{1+\left(\frac{\Re\nu_2+2N-\nu^*}{\Im\nu_2}\right)^2}$$

For any complex numbers $A$, $B$ we have

$$\int_{L(x)}\frac{1}{|s|}\left(|s|+|As^{1-\nu_2-2N}|+|Bs^{\nu_2+2N}|\right)^n\ |ds|\le\frac{P(\nu_2+2N,\nu^*)}{n\min(\nu^*,1-\nu^*)}\left(|x|+|Ax^{1-\nu_2-2N}|+|Bx^{\nu_2+2N}|\right)^n \tag{28}$$

The quantity $\min(\nu^*,1-\nu^*)$ is substituted by $\min(\nu^*-1,2-\nu^*)$ in case (b). To prove (28) we observe that on $L(x)$ we have

$$|s^{\nu_2+2N+\alpha}|=|x^{\nu_2+2N+\alpha}|\frac{|s|^{\nu^*+\alpha}}{|x|^{\nu^*+\alpha}},\qquad\forall\alpha\in\mathbf{C}.$$

Therefore

$$\int_{L(x)}\frac{1}{|s|}|s|^i|As^{1-\nu_2-2N}|^j|Bs^{\nu_2+2N}|^k\ |ds|=$$

$$\frac{|Ax^{1-\nu_2-2N}|^j\ |Bx^{\nu_2+2N}|^k}{|x|^{(1-\nu^*)j}\ |x|^{\nu^*k}}\ P(\nu_2+2N,\nu^*)\int_0^{|x|}d\rho\ \rho^{i-1+(1-\nu^*)j+\nu^*k}$$

$$=\frac{P(\nu_2+2N,\nu^*)}{i+j(1-\nu^*)+k\nu^*}|x|^i\ |Ax^{1-\nu_2-2N}|^j\ |Bx^{\nu_2+2N}|^k$$

$$\le\frac{P(\nu_2+2N,\nu^*)}{(i+j+k)\min(\nu^*,1-\nu^*)}|x|^i\ |Ax^{1-\nu_2-2N}|^j\ |Bx^{\nu_2+2N}|^k$$

from which (28) follows, provided that $0<\nu^*<1$. For $\Im\nu_2=0$, this yields again $0<\nu_2<1$.

We observe that a solution of the integral equations is also a solution of the differential equations, by virtue of the following lemma:

**Lemma 1:** *Let $f(x)$ be a holomorphic function in the domain $|x|<\epsilon$, $|Ax^{1-\nu_2-2N}|<\epsilon$, $|Bx^{\nu_2+2N}|<\epsilon$, such that $f(x)=O(|x|+|Ax^{1-\nu_2-2N}|+|Bx^{\nu_2+2N}|)$, $A,B\in\mathbf{C}$. Let $L(x)$ be the path of integration defined above for $0<\nu^*<1$ and*

$$F(x):=\int_{L(x)}\frac{1}{s}f(s)\ ds$$

*Then, $F(x)$ is holomorphic on the domain and $\frac{dF(x)}{dx}=\frac{1}{x}f(x)$*

*The same holds for a function $f(x)=O(|x|+|Ax^{2-\nu_2-2N}|+|Bx^{\nu_2-1+2N}|)$ in the domain $|x|<\epsilon$, $|Ax^{2-\nu_2-2N}|<\epsilon$, $|Bx^{\nu_2-1+2N}|<\epsilon$ ( $L(x)$ being defined for $1<\nu^*<2$).*

*Proof:* We give the proof in case (a). We choose a point $x+\Delta x$ close to $x$ and we prove that $\int_{L(x)}-\int_{L(x+\Delta x)}=\int_x^{x+\Delta x}$, where the last integral is on a segment from $x$ to $x+\Delta x$. We consider a small disk $U_R$ centered at $x=0$ of small radius $R<|x|$ and the points $x_R:=L(x)\cap U_R$, $x_R':=L(x+\Delta x)\cap U_R$. Since the integral of $f/s$ on a finite close curve (not containing 0) is zero we have:

$$\left(\int_{L(x)}-\int_{L(x+\Delta x)}-\int_x^{x+\Delta x}\right)ds\ \frac{f(s)}{s}=\left(\int_{L(x_R)}-\int_{L(x_R')}-\int_{\gamma(x_R,x_R')}\right)ds\ \frac{f(s)}{s} \tag{29}$$

The last integral is on the arc $\gamma(x_R,x'_R)$ from $x_R$ to $x'_R$ on the circle $|s|=R$. We have also kept into account the obvious fact that $L(x_R)$ is contained in $L(x)$ and $L(x'_R)$ is contained in $L(x+\Delta x)$. We then take $R\to 0$ and we prove that the r.h.s. of (29) vanishes.

Taking into account that $f(x)=O(|x|+|Ax^{1-\nu_2-2N}|+|Bx^{\nu_2+2N}|)$ and (28) we have

$$\left|\int_{L(x_R)}\frac{1}{s}f(s)\ ds\right|\leq\int_{L(x_R)}\frac{1}{|s|}O(|s|+|As^{1-\nu_2-2N}|+|Bs^{\nu_2+2N}|)\ |ds|$$

$$\leq\frac{P(\nu_2+2N,\nu^*)}{\min(\nu^*,1-\nu^*)}O(|x_R|+|Ax_R^{1-\nu_2-2N}|+|Bx_R^{\nu_2+2N}|)=$$

$$=\frac{P(\nu_2+2N,\nu^*)}{\min(\nu^*,2-\nu^*)}\ O(R^{\min\{\nu^*,1-\nu^*\}})$$

The last step follows from $|x_R^{\nu_2+2N}|=\frac{|x^{\nu_2+2N}|}{|x|^{\nu^*}}R^{\nu^*}$. So the integral vanishes for $R\to 0$. The same is proved for $\int_{L(x+\Delta x)}$. As for the integral on the arc we have

$$|\arg x_R-\arg x'_R|=\left|\arg x-\arg(x+\Delta x)+\frac{\Re\nu_2+2N-\nu^*}{\Im\nu_2}\log\left|1+\frac{\Delta x}{x}\right|\right|$$

or $|\arg x_R-\arg x'_R|=|\arg x-\arg(x+\Delta x)|$ if $\Im\nu_2=0$. This is independent of $R$, therefore the length of the arc is $O(R)$ and

$$\left|\int_{\gamma(x_R,x'_R)}\frac{1}{|s|}|f(s)|\ |ds|\right|=O(R^{\min\{\nu^*,1-\nu^*\}})\to 0 \text{ for } R\to 0$$

$\square$

Now we prove a fundamental lemma.

**Lemma 2:** *For any $N\in\mathbf{Z}$ and for any complex $\nu_1$, $\nu_2$ such that*

$$\nu_2\notin(-\infty,-2N]\cup\{1-2N\}\cup[2-2N,+\infty)$$

*there exists a sufficiently small $r_N<1$ such that the system of integral equations has a solution $v_1(x;\nu_1,\nu_2+2N)$ holomorphic in the domain $\mathcal{D}_1(r_N;\nu_1,\nu_2+2N)$ defined in (16). Moreover, there exists a positive constant $M_1(\nu_2+2N)$ depending on $\nu_2+2N$ such that*

$$|v_1(x;\nu_1,\nu_2+2N)|\leq M_1(\nu_2+2N)\left(|x|+\left|\frac{e^{-i\pi\nu_1}}{16^{1-\nu_2-2N}}x^{1-\nu_2-2N}\right|+\left|\frac{e^{i\pi\nu_1}}{16^{\nu_2+2N}}x^{\nu_2+2N}\right|\right)$$

*in $\mathcal{D}_1(r;\nu_1,\nu_2)$.*

*The system of integral equations has* another *solution $v_2(x;\nu_1,\nu_2+2N)$ holomorphic in $\mathcal{D}_2(r;\nu_1,\nu_2)$ defined in (17). There exists a positive constant $M_2(\nu_2+2N)$ such that*

$$|v_2(x;\nu_1,\nu_2+2N)|\leq M_2(\nu_2+2N)\left(|x|+\left|\frac{e^{-i\pi\nu_1}}{16^{2-\nu_2-2N}}x^{2-\nu_2-2N}\right|+\left|\frac{e^{i\pi\nu_1}}{16^{\nu_2-1+2N}}x^{\nu_2-1+2N}\right|\right)$$

*in $\mathcal{D}_2(r;\nu_1,\nu_2)$.*

Note that $\mathcal{D}_i(r_N;\nu_1,\nu_2+2N)=\mathcal{D}_i(r_N;\nu_1+2N_1,\nu_2+2N)$, $i=1,2$, for any $N_1\in\mathbf{Z}$. To prove Lemma 2 we need some sub-lemmas

**Sub-Lemma 1:** *Let $\Phi(x,y,z)$ and $\Psi(x,y,z,v,w)$ be two holomorphic functions of their arguments for $|x|,|y|,|z|,|v|,|w|<\epsilon$, satisfying*

$$\Phi(0,0,0)=0,\qquad \Psi(0,0,0,v,w)=\Psi(x,y,z,0,0)=0$$

*Then, there exists a constant $c>0$ such that:*

$$|\Phi(x,y,z)|\leq c\ \ (|x|+|y|+|z|) \tag{30}$$

$$|\Psi(x,y,z,v,w)| \leq c\ \ (|x|+|y|+|z|) \tag{31}$$

$$|\Psi(x,y,z,v_2,w_2)-\Psi(x,y,z,v_1,w_1)| \leq c\,(|x|+|y|+|z|)\ \ (|v_2-v_1|+|w_2-w_1|) \tag{32}$$

*for* $|x|,|y|,|z|,|v|,|w|<\epsilon$.

*Proof:* Let's proof (31).

$$\Psi(x,y,z,v,w)=\int_0^1 \frac{d}{d\lambda}\Psi(\lambda x,\lambda y,\lambda z,v,w)\ d\lambda$$

$$x\int_0^1 \frac{\partial\Psi}{\partial x}(\lambda x,\lambda y,\lambda z,v,w)\ d\lambda\ +\ y\int_0^1 \frac{\partial\Psi}{\partial y}(\lambda x,\lambda y,\lambda z,v,w)\ d\lambda\ +\ z\int_0^1 \frac{\partial\Psi}{\partial z}(\lambda x,\lambda y,\lambda z,v,w)\ d\lambda$$

Moreover, for $\delta$ small:

$$\frac{\partial\Psi}{\partial x}(\lambda x,\lambda y,\lambda z,v,w)=\int_{|\zeta-\lambda x|=\delta}\frac{\Psi(\zeta,\lambda y,\lambda z,v,w)}{(\zeta-\lambda x)^2}\ \frac{d\zeta}{2\pi i}$$

which implies that $\frac{\partial\Psi}{\partial x}$ is holomorphic and bounded when its arguments are less than $\epsilon$. The same holds true for $\frac{\partial\Psi}{\partial y}$ and $\frac{\partial\Psi}{\partial z}$. This proves (31), $c$ being a constant which bounds $\left|\frac{\partial\Psi}{\partial x}\right|$, $\left|\frac{\partial\Psi}{\partial y}\right|$ $\left|\frac{\partial\Psi}{\partial z}\right|$. The inequality (30) is proved in the same way. We turn to (32). First we prove that for $|x|,|y|,|z|,|v_1|,|w_1|,|v_2|,|w_2|<\epsilon$ there exist two holomorphic and bounded functions $\psi_1(x,y,z,v_1,w_1,v_2,w_2)$, $\psi_2(x,y,z,v_1,w_1,v_2,w_2)$ such that

$$\Psi(x,y,z,v_2,w_2)-\Psi(x,y,z,v_1,w_1)$$
$$=(v_2-v_1)\ \psi_1(x,y,z,v_1,w_1,v_2,w_2)+(w_1-w_2)\ \psi_2(x,y,z,v_1,w_1,v_2,w_2) \tag{33}$$

In order to prove this, we write

$$\Psi(x,y,z,v_2,w_2)-\Psi(x,y,z,v_1,w_1)=$$

$$=\int_0^1 \frac{d}{d\lambda}\Psi(x,y,z,\lambda v_2+(1-\lambda)v_1,\lambda w_2+(1-\lambda)w_1)\ d\lambda$$

$$=(v_2-v_1)\int_0^1\frac{\partial\Psi}{\partial v}(x,y,z,\lambda v_2+(1-\lambda)v_1,\lambda w_2+(1-\lambda)w_1)\ d\lambda\ +$$

$$+\ (w_2-w_1)\int_0^1\frac{\partial\Psi}{\partial w}(x,y,z,\lambda v_2+(1-\lambda)v_1,\lambda w_2+(1-\lambda)w_1)\ d\lambda$$

$$=:(v_2-v_1)\ \psi_1(x,y,z,v_1,w_1,v_2,w_2)+(w_2-w_1)\ \psi_2(x,y,z,v_1,w_1,v_2,w_2)$$

Moreover, for small $\delta$,

$$\frac{\partial\Psi}{\partial v}(x,y,z,v,w)=\int_{|\zeta-v|=\delta}\frac{\Psi(x,y,z,\zeta,w)}{(\zeta-v)^2}\ \frac{dz}{2\pi i}$$

which implies that $\psi_1$ is holomorphic and bounded for its arguments less than $\epsilon$. We also obtain $\frac{\partial\Psi}{\partial v}(0,0,0,v,w)=0$, then $\psi_1(0,0,0,v_1,w_1,v_2,w_2)=0$. The proof for $\psi_2$ is analogous. We use (33) to complete the proof of (32). Actually, we observe that

$$\psi_i(x,y,z,v_1,w_1,v_2,w_2)=\int_0^1\frac{d}{d\lambda}\psi_i(\lambda x,\lambda y,\lambda z,v_1,w_1,v_2,w_2)\ d\lambda$$

$$=x\int_0^1\frac{\partial\psi_i}{\partial x}d\lambda+y\int_0^1\frac{\partial\psi_i}{\partial y}d\lambda+z\int_0^1\frac{\partial\psi_i}{\partial z}d\lambda$$

and we conclude as in the proof of (31).

□

We solve the system of integral equations by successive approximations. We can choose any path $L(x)$ such that $0<\nu^*<1$ in case (a), or $1<\nu^*<2$ in case (b). Here we complete the proof only for the case (a). Therefore, the solution $v(x)$ we are going to find is $v_1(x;\nu_1,\nu_2+2N)$. Case (b) is analogous and we don't need to repeat the proof for it. It yields the function $v_2(x;\nu_1,\nu_2+2N)$.

We choose $\nu^* = \frac{1}{2}$, therefore $\min\{\nu^*, 1-\nu^*\} = \frac{1}{2}$. For convenience, we put

$$A := \frac{e^{-i\pi\nu_1}}{16^{1-\nu_2-2N}}, \qquad B := \frac{e^{i\pi\nu_1}}{16^{\nu_2+2N}}$$

Therefore, for any $n \geq 1$ the successive approximations are:

$$v_0 = w_0 = 0$$

$$w_n(x) = \int_{L(x)} \frac{1}{t} \left\{\Phi(s, As^{1-\nu_2-2N}, Bs^{\nu_2+2N}) + \Psi(s, As^{1-\nu_2-2N}, Bs^{\nu_2+2N}, v_{n-1}(s), w_{n-1}(s))\right\}\ ds \quad (34)$$

$$v_n(x) = \int_{L(x)} \frac{1}{s}\ w_n(s)\ ds \quad (35)$$

**Sub-Lemma 2:** *There exists a sufficiently small $\epsilon' < \epsilon$ such that for any $n \geq 0$ the functions $v_n(x)$ and $w_n(x)$ are holomorphic in the domain*

$$\mathcal{D}_1(\epsilon'; \nu_1, \nu_2 + 2N) := \left\{x \in \tilde{\mathbf{C}}_0 \ \ \textit{such that}\ |x| < \epsilon', \left|Ax^{1-\nu_2-2N}\right| < \epsilon', \left|Bx^{\nu_2+2N}\right| < \epsilon'\right\}$$

*They are also correctly bounded, namely $|v_n(x)| < \epsilon$, $|w_n(x)| < \epsilon$ for any $n$. They satisfy*

$$|v_n - v_{n-1}| \leq \frac{(2c)^n (2P(\nu_2+2N))^{2n}}{n!} \left(|x| + |Ax^{1-\nu_2-2N}| + |Bx^{\nu_2+2N}|\right)^n \quad (36)$$

$$|w_n - w_{n-1}| \leq \frac{(2c)^n (2P(\nu_2+2N))^{2n}}{n!} \left(|x| + |Ax^{1-\nu_2-2N}| + |Bx^{\nu_2+2N}|\right)^n \quad (37)$$

*where $P(\nu_2 + 2N) := P(\nu_2 + 2N, \nu^* = 1/2)$ and $c$ is the constant appearing in Sub-Lemma 1. Moreover*

$$x \frac{dv_n}{dx} = w_n$$

**Proof:** We proceed by induction.

$$w_1 = \int_{L(x)} \frac{1}{s} \Phi(s, As^{1-\nu_2-2N}, Bs^{\nu_2+2N})\ ds, \qquad v_1 = \int_{L(x)} \frac{1}{s} w_1(s) ds$$

It follows from Lemma 1 and (30) that $w_1(x)$ is holomorphic for $|x|, |Ax^{1-\nu_2-2N}|, |Bx^{\nu_2+2N}| < \epsilon$. From (28) and (30) we have

$$|w_1(x)| \leq \int \frac{1}{|s|} |\Phi(s, As^{1-\nu_2-2N}, Bs^{\nu_2+2N})|\ |ds|$$

$$\leq 2cP(\nu_2+2N)(|x| + |Ax^{1-\nu_2-2N}| + |Bx^{\nu_2+2N}|) \leq 6cP(\nu_2+2N)\epsilon' < \epsilon$$

on $\mathcal{D}_1(\epsilon'; \nu_1, \nu_2 + 2N)$, provided that $\epsilon'$ is small enough. By Lemma 1, also $v_1(x)$ is holomorphic for $|x|, |Ax^{1-\nu_2-2N}|, |Bx^{\nu_2+2N}| < \epsilon$ and

$$x \frac{dv_1}{dx} = w_1$$

By (28) we also have

$$|v_1(x)| \leq c(2P(\nu_2+2N))^2 (|x| + |Ax^{1-\nu_2-2N}| + |Bx^{\nu_2+2N}|) \leq 12cP(\nu_2)^2 \epsilon' < \epsilon$$

on $\mathcal{D}_1(\epsilon'; \nu_1, \nu_2 + 2N)$, if $\epsilon'$ is small enough. Note that $P(\nu_2 + 2N) \geq 1$, so (37) (36) are true for $n = 1$. Now we suppose that the statement of the sub-lemma is true for $n$ and we prove it for $n + 1$. Consider:

$$|w_{n+1}(x) - w_n(x)| = \left|\int_{L(x)} \frac{1}{s}\ \left[\Psi(s, As^{1-\nu_2-2N}, Bs^{\nu_2+2N}, v_n, w_n) - \right.\right.$$

$$-\Psi(s, As^{1-\nu_2-2N}, Bs^{\nu_2+2N}, v_{n-1}, w_{n-1})\big]\ \ ds|$$

By (32) the above is

$$\leq c\ \int_{L(x)} \frac{1}{|s|}(|s| + |As^{1-\nu_2-2N}| + |Bs^{\nu_2+2N}|)\ (|v_n - v_{n-1}| + |w_n - w_{n-1}|)\ |ds|$$

By induction this is

$$\leq 2c\ \frac{(2c)^n(2P(\nu_2+2N))^{2n}}{n!}\ \int_{L(x)} \frac{1}{|s|}(|s| + |As^{1-\nu_2-2N}| + |Bs^{\nu_2+2N}|)^{n+1}\ |ds|$$

$$\leq 2c\ \frac{(2c)^n(2P(\nu_2+2N))^{2n}}{n!}\ \frac{2P(\nu_2)}{n+1}\ (|x| + |Ax^{1-\nu_2-2N}| + |Bx^{\nu_2+2N}|)^{n+1}$$

$$\leq \frac{(2c)^{n+1}(2P(\nu_2+2N))^{2(n+1)}}{(n+1)!}(|x| + |Ax^{1-\nu_2-2N}| + |Bx^{\nu_2+2N}|)^{n+1}$$

This proves (37). Now we estimate

$$|v_{n+1}(x) - v_n(x)| \leq \int_{L(x)} |w_{n+1}(s) - w_n(s)|\ |ds|$$

$$\leq \frac{(2c)^{n+1}(2P(\nu_2+2N))^{2n+1}}{(n+1)!}\ \int_{L(x)} \frac{1}{|s|}(|s| + |As^{1-\nu_2-2N}| + |Bs^{\nu_2+2N}|)^{n+1}\ |ds|$$

$$\leq \frac{(2c)^{n+1}(2P(\nu_2+2N))^{2(n+1)}}{(n+1)\ (n+1)!}\ (|x| + |Ax^{1-\nu_2-2N}| + |Bx^{\nu_2+2N}|)^{n+1}$$

$$\leq \frac{(2c)^{n+1}(2P(\nu_2+2N))^{2(n+1)}}{(n+1)!}\ (|x| + |Ax^{1-\nu_2-2N}| + |Bx^{\nu_2+2N}|)^{n+1}$$

This proves (36). From Lemma 1 we also conclude that $w_n$ and $v_n$ are holomorphic in $\mathcal{D}(\epsilon', \nu_1, \nu_2)$ and

$$x\frac{dv_n}{dx} = w_n$$

Finally we see that

$$|v_n(x)| \leq \sum_{k=1}^{n} |v_k(x) - v_{k-1}(x)| \leq \exp\{2c(2P(\nu_2+2N))^2(|x| + |Ax^{1-\nu_2-2N}| + |Bx^{\nu_2+2N}|)\} - 1 \leq$$

$$\leq \exp\{24cP^2(\nu_2+2N)\epsilon'\} - 1$$

and the same for $|w_n(x)|$. Therefore, if $\epsilon'$ is small enough we have $|v_n(x)| < \epsilon$, $|w_n(x)| < \epsilon$ on $\mathcal{D}_1(\epsilon', \nu_1, \nu_2)$.

Note that $P(\nu_2+2N)$ grows as $N^2$, so $\epsilon'$ decreases as $N^{-2}$.

□

Let's define

$$v(x) := \lim_{n\to\infty} v_n(x), \qquad w(x) := \lim_{n\to\infty} w_n(x)$$

if they exist. We can also rewrite

$$v(x) = \lim_{n\to\infty} v_n(x) = \sum_{n=1}^{\infty}(v_n(x) - v_{n-1}(x)).$$

We see that the series converges uniformly in $\mathcal{D}(\epsilon', \nu_1, \nu_2+2N)$ because

$$|\sum_{n=1}^{\infty}(v_n(x) - v_{n-1}(x))|$$

$$\leq \sum_{n=1}^{\infty} \frac{(2c)^n (2P(\nu_2+2N))^{2n}}{n!} \ (|x| + |Ax^{1-\nu_2-2N}| + |Bx^{\nu_2+2N}|)^n$$

$$= \exp\{8cP^2(\nu_2)(|x| + |Ax^{1-\nu_2-2N}| + |Bx^{\nu_2+2N}|)\} - 1$$

The same holds for $w_n(x)$. Therefore, $v(x)$ and $w(x)$ define holomorphic functions in $\mathcal{D}_1(\epsilon', \nu_1, \nu_2 + 2N)$. From Sub-Lemma 2 we also have

$$x \frac{dv(x)}{dx} = w(x)$$

in $\mathcal{D}_1(\epsilon', \nu_1, \nu_2 + 2N)$.

We show that $v(x), w(x)$ solve the initial integral equations. The l.h.s. of (34) converges to $w(x)$ for $n \to \infty$. Let's prove that the r.h.s. also converges to

$$\int_{L(x)} \frac{1}{s} \left\{ \Phi(s, As^{1-\nu_2-2N}, Bs^{\nu_2+2N}) + \Psi(s, As^{1-\nu_2-2N}, Bs^{\nu_2+2N}, v(s), w(s)) \right\} \ ds.$$

We have to evaluate the following difference:

$$\left| \int_{L(x)} \frac{1}{s} \Psi(s, As^{2-\nu_2-2N}, Bs^{\nu_2+2N}, v(s), w(s)) \ ds \ - \ \int_{L(x)} \frac{1}{s} \Psi(s, As^{2-\nu_2-2N}, Bs^{\nu_2+2N}, v_n(s), w_n(s)) \ ds \right|$$

By (32) the above is

$$\leq \ c \int_{L(x)} \frac{1}{|s|} (|s| + |As^{1-\nu_2-2N}| + |Bs^{\nu_2+2N}|) \ (|v - v_n| + |w - w_n|) \ |ds| \qquad (38)$$

Now we observe that

$$|v(x) - v_n(x)| \leq \sum_{k=n+1}^{\infty} |v_k - v_{k-1}|$$

$$= \sum_{k=n+1}^{\infty} \frac{(2c)^k (2P(\nu_2+2N))^{2k}}{k!} (|x| + |Ax^{1-\nu_2-2N}| + |Bx^{\nu_2+2N}|)^k$$

$$\leq (|x|+|Ax^{2-\nu_2-2N}|+|Bx^{\nu_2+2N}|)^{n+1} \sum_{k=0}^{\infty} \frac{(2c)^{k+n+1} P(\nu_2+2N)^{2(k+n+1)}}{(k+n+1)!} (|x|+|Ax^{2-\nu_2-2N}|+|Bx^{\nu_2+2N}|)^k$$

The series converges. Its sum is less than some constant $S(\nu_2 + 2N)$ independent of $n$. We obtain

$$|v(x) - v_n(x)| \leq S(\nu_2 + 2N)(|x| + |Ax^{1-\nu_2-2N}| + |Bx^{\nu_2+2N}|)^{n+1}.$$

The same holds for $|w - w_n|$. Thus, (38) is

$$\leq 2c \ S(\nu_2 + 2N) \ \int_{L(x)} \frac{1}{|s|} (|s| + |As^{1-\nu_2-2N}| + |Bs^{\nu_2+2N}|)^{n+2} \ |ds|$$

$$\leq \frac{2cS(\nu_2+2N) \ 2P(\nu_2+2N)}{n+2} \ (|x| + |Ax^{1-\nu_2-2N}| + |Bx^{\nu_2+2N}|)^{n+2}$$

Namely:

$$\left| \int_{L(x)} \frac{1}{s} \Psi(s, As^{1-\nu_2-2N}, Bs^{\nu_2+2N}, v(s), w(s)) \ ds \ - \ \int_{L(x)} \frac{1}{s} \Psi(s, As^{1-\nu_2-2N}, Bs^{\nu_2+2N}, v_n(s), w_n(s)) \ ds \right|$$

$$\leq \frac{2cS(\nu_2) \ 2P(\nu_2+2N)}{n+2} \ (3\epsilon')^{n+2}$$

In a similar way, the r.h.s. of (35) is

$$\left| \int \frac{1}{s} (w(s) - w_n(s)) \ ds \right| \leq \frac{S(\nu_2+2N) \ 2P(\nu_2)}{n+1} (3\epsilon')^{n+1}$$

Therefore, the r.h. sides of (34) (35) converge on the domain $\mathcal{D}(r,\nu_1,\nu_2)$ for $r<\min\{\epsilon',1/3\}$. We note that $r=r_N$, namely it depends on $N$ in the same way as $\epsilon'$ does. Thus, it decreases as $N^{-2}$ as $N$ increases.

We finally observe that $|v(x)|$ and $|w(x)|$ are bounded on $\mathcal{D}(r)$. For example

$$|v(x)|\leq(|x|+|Ax^{1-\nu_2-2N}|+|Bx^{\nu_2+2N}|)\sum_{k=0}^{\infty}\frac{(2c)^{k+1}(2P(\nu_2))^{2(k+1)}}{(k+1)!}(|x|+|Ax^{1-\nu_2-2N}|+|Bx^{\nu_2+2N}|)^k$$

$$=:M(\nu_2+2N)(|x|+|Ax^{1-\nu_2-2N}|+|Bx^{\nu_2+2N}|)$$

where the sum of the series is less than a constant $M(\nu_2+2N)$. We have proved Lemma 2 for the case (a). Case (b) is analogous.

□

**Remark:** The domain $\mathcal{D}_1(r_N,\nu_1,\nu_2)$ is

$$(\Re\nu_2+2N)\ln\frac{|x|}{16}-\pi\Im\nu_1-\ln r_N<\Im\nu_2\arg x<$$

$$<(\Re\nu_2-1+2N)\ln\frac{|x|}{16}-\pi\Im\nu_1+\ln r_N,\qquad |x|<r_N$$

while the domain $\mathcal{D}_2(r_N,\nu_1,\nu_2)$ is

$$(\Re\nu_2-1+2N)\ln\frac{|x|}{16}-\pi\Im\nu_1-\ln r_N<\Im\nu_2\arg x<$$

$$<(\Re\nu_2-2+2N)\ln\frac{|x|}{16}-\pi\Im\nu_1+\ln r_N,\qquad |x|<r_N$$

We invite the reader to observe the $\ln r_N$ terms.

Also, note that for $\nu_2\in\mathbf{R}$ the domain is specified by $\max\left\{|x|,\left|\frac{e^{-i\pi\nu_1}}{16^{1-\nu_2}}x^{1-\nu_2}\right|,\left|\frac{e^{i\pi\nu_1}}{16^{\nu_2}}x^{\nu_2}\right|\right\}<r$. We can make the notation easer by simply re-writing $|x|<r$ for sufficiently small $r$. So, we obtain the domain $\mathcal{D}_0(r)$.

We have proved that the structure of the integral equations implies that $v(x)$ is bounded (namely $|v(x)|=O(r_N)$). The proof of Lemma 2 only makes use of the properties of $\Phi$ and $\Psi$, regardless of how these functions have been constructed.

In our case, $\Phi$ and $\Psi$ have been constructed from the Fourier expansion of elliptic functions. We see that the domain (26) contains $\mathcal{D}_1(r_N;\nu_1,\nu_2+2N)$ and (27) contains $\mathcal{D}_2(r_N;\nu_1,\nu_2+2N)$ because the term $\Im\frac{\pi v}{2\omega_1}$ in (26) and (27) is $O(r_N)$, while in $\mathcal{D}_i(r,\nu_1,\nu_2+2N)$, $i=1,2$, the term $\ln r_N$ appear (see the Remark).

To conclude the proof of Theorem 1, we have to work out the series of $v(x)$. Let's do that in case (a), namely for $v_1(x;\nu_1,\nu_2+2N)$. Case (b) is analogous. We observe that $w_1$ and $v_1$ are series of the type

$$\sum_{p,q,r\geq 0}c_{pqr}(\nu_2+2N)\ x^p\ (Ax^{1-\nu_2-2N})^q\ (Bx^{\nu_2+2N})^r \tag{39}$$

where $c_{pqr}(\nu_2+2N)$ is rational in $\nu_2$. This follows from

$$w_1(x)=\int_{L(x)}\Phi(s,As^{1-\nu_2-2N},Bs^{\nu_2+2N})\ ds$$

and from the fact that $\Phi(x,Ax^{1-\nu_2-2N},Bx^{\nu_2+2N})$ itself is a series (39) by construction, with coefficients which are rational functions of $\nu_2+2N$. The same holds true for $\Psi$. We conclude that $w_n(x)$ and $v_n(x)$ have the form (39) for any $n$. This implies that the limit $v(x)$ is also a series of type (39). We can reorder such a series. Consider the term

$$c_{pqr}(\nu_2)\ x^p\ (Ax^{1-\nu_2-2N})^q\ (Bx^{\nu_2+2N})^r,$$

and recall that by definition $B=\frac{1}{16^2\ A}$. We absorb $16^{-2r}$ into $c_{pqr}(\nu_2)$ and we study the factor

$$A^{q-r}x^{p+(1-\nu_2-2N)q+(\nu_2+2N)r}=A^{q-r}x^{p+q+(r-q)(\nu_2+2N)}$$

We have three cases:

1) $r=q$, then we have $x^{p+q}=: x^n$, $n=p+q$.

2) $r>q$, then we have $x^{p+q}\ \left[\frac{1}{A}x^{\nu_2+2N}\right]^{r-q}=: x^n\ \left[\frac{1}{A}x^{\nu_2+2N}\right]^m$, $n=p+q$, $m=q-r$.

3) $r<q$, then we have $A^{q-r}\ x^{p+r}\ \left[Ax^{1-\nu_2-2N}\right]^{q-r}=: x^n\ \left[Ax^{1-\nu_2-2N}\right]^m$, $n=p+r$, $m=q-r$.

Therefore, the series of the type (39) can be re-written in the form of $v(x)$ of Theorem 1. The series of $v(x)$ is uniquely determined by the Painlevé equation, because it is constructed from $\Phi$ and $\Psi$ by successive approximations of the solutions of the integral equations.

This concludes the proof. We only remark that if $\nu_2$ is real, we make a translation $\nu_2\mapsto\nu_2+2N$ which yields $0<\nu_2<1$ or $1<\nu_2<2$ and all the formulae can be read for $N=0$.

The statement of Theorem 1 as it follows from our proof is

*Let $\nu_1$, $\nu_2$ be two complex numbers.*

**I)** *For any $N\in\mathbf{Z}$ and for any complex $\nu_1$, $\nu_2$ such that*

$$\Im\nu_2\neq 0$$

*there exist a positive number $r_N<1$ and a transcendent*

$$y(x)=\wp\Big(\nu_1\omega_1(x)+\nu_2\omega_2(x)+v_1(x;\nu_1,\nu_2+2N);\ \omega_1(x),\omega_2(x)\Big)+\frac{1+x}{3}$$

*such that $v_1(x;\nu_1,\nu_2+2N)$ is holomorphic in the domain $\mathcal{D}_1(r_N;\nu_1,\nu_2+2N)$, where it has convergent expansion*

$$v_1(x;\nu_1,\nu_2+2N)=\sum_{n\geq 1}a_n^{(1)}x^n+\sum_{n\geq 0,m\geq 1}b_{nm}^{(1)}x^n\left[e^{-i\pi\nu_1}\left(\frac{x}{16}\right)^{1-\nu_2-2N}\right]^m+$$

$$+\sum_{n\geq 0,m\geq 1}c_{nm}^{(1)}x^n\left[e^{i\pi\nu_1}\left(\frac{x}{16}\right)^{\nu_2+2N}\right]^m$$

*There also exists a transcendent*

$$y(x)=\wp\Big(\nu_1\omega_1(x)+\nu_2\omega_2(x)+v_2(x;\nu_1,\nu_2+2N);\ \omega_1(x),\omega_2(x)\Big)+\frac{1+x}{3}$$

*such that $v_2(x;\nu_1,\nu_2+2N)$ is holomorphic in the domain $\mathcal{D}_2(r_N;\nu_1,\nu_2+2N)$, where it has convergent expansion*

$$v_2(x;\nu_1,\nu_2+2N)=\sum_{n\geq 1}a_n^{(2)}x^n+\sum_{n\geq 0,m\geq 1}b_{nm}^{(2)}x^n\left[e^{-i\pi\nu_1}\left(\frac{x}{16}\right)^{2-\nu_2-2N}\right]^m+$$

$$+\sum_{n\geq 0,m\geq 1}c_{nm}^{(2)}x^n\left[e^{i\pi\nu_1}\left(\frac{x}{16}\right)^{\nu_2-1+2N}\right]^m$$

*For both transcendents, the coefficients $a_n^{(i)}$, $b_{nm}^{(i)}$, $c_{nm}^{(i)}$, $i=1,2$, are rational functions of $\nu_2+2N$. On $\mathcal{D}_i(r_N;\nu_1,\nu_2+2N)$ there exists a positive constant $M_i(\nu_2+2N)$ such that*

$$|v_1(x;\nu_1,\nu_2+2N)|\leq M_1(\nu_2+2N)\left(|x|+\left|e^{-i\pi\nu_1}\left(\frac{x}{16}\right)^{1-\nu_2-2N}\right|+\left|e^{i\pi\nu_1}\left(\frac{x}{16}\right)^{\nu_2+2N}\right|\right)\qquad(40)$$

$$|v_2(x;\nu_1,\nu_2+2N)|\leq M_2(\nu_2+2N)\left(|x|+\left|e^{-i\pi\nu_1}\left(\frac{x}{16}\right)^{2-\nu_2-2N}\right|+\left|e^{i\pi\nu_1}\left(\frac{x}{16}\right)^{\nu_2-1+2N}\right|\right)\qquad(41)$$

**II)** *For any complex $\nu_1$ and for any real $\nu_2$ such that*

$$0<\nu_2<1\qquad or\qquad 1<\nu_2<2$$

*there exists a positive $r<1$ and a transcendent*

$$y(x)=\wp\Big(\nu_1\omega_1(x)+\nu_2\omega_2(x)+v(x;\nu_1,\nu_2);\ \omega_1(x),\omega_2(x)\Big)+\frac{1+x}{3}$$

*such that $v(x;\nu_1,\nu_2)$ is holomorphic in $\mathcal{D}_0(r;\nu_1,\nu_2)$, where it has the convergent expansion*

$$v(x;\nu_1,\nu_2)=\sum_{n\geq 1}a_nx^n+\sum_{n\geq 0,m\geq 1}b_{nm}x^n\left[e^{-i\pi\nu_1}\left(\frac{x}{16}\right)^{1-\nu_2}\right]^m+$$

$$+\sum_{n\geq 0,m\geq 1}c_{nm}x^n\left[e^{i\pi\nu_1}\left(\frac{x}{16}\right)^{\nu_2}\right]^m$$

*if $0<\nu_2<1$; and*

$$v(x)=\sum_{n\geq 1}a_nx^n+\sum_{n\geq 0,m\geq 1}b_{nm}x^n\left[e^{-i\pi\nu_1}\left(\frac{x}{16}\right)^{2-\nu_2}\right]^m+$$

$$+\sum_{n\geq 0,m\geq 1}c_{nm}x^n\left[e^{i\pi\nu_1}\left(\frac{x}{16}\right)^{\nu_2-1}\right]^m$$

*if $1<\nu_2<2$. The coefficients are rational functions of $\nu_2$ and $|v(x)|$ is bounded as in (40), with $N=0$, for $0<\nu_2<1$ and as in (41), with $N=0$, for $1<\nu_2<2$.*

The above statement is completely equivalent to the statement of Theorem 1 in section 3 , due to the two observations we made there. In particular, $v_1(x;\nu_1,\nu_2)=v(x;\nu_1,\nu_2)$ of Theorem 1 and $v_2(x;\nu_1,\nu_2)=-v(x;-\nu_1,2-\nu_2)$

A technique similar to that used in the proof of Theorem 1 was first introduced by S. Shimomura in [35] for a class of functional equations.

# 5 Critical Behavior – Theorem 2

Let $\Im\nu_2\neq 0$ and $\mathcal{V}\in\mathbf{C}$. We consider the following family of paths joining a point $x_0$ to $x=0$

$$\arg x=\arg x_0+\frac{\Re\nu_2-\mathcal{V}}{\Im\nu_2}\ln\frac{|x|}{|x_0|}\tag{42}$$

It is contained in $\mathcal{D}(r;\nu_1,\nu_2)$ if $0\leq\mathcal{V}\leq 1$ and $x_0$ belongs to $\mathcal{D}(r;\nu_1,\nu_2)$. If $\Im\nu_2=0$, and $x_0\in\mathcal{D}_0(r)$, any path joining it to $x=0$ is contained in $\mathcal{D}_0(r)$.

**Theorem 2:** *Let $\nu_1$, $\nu_2$ be given.*

*If $\Im\nu_2\neq 0$, the critical behavior of the transcendent $y(x)=\wp(\nu_1\omega_1+\nu_2\omega_2+v(x;\nu_1,\nu_2);\omega_1,\omega_2)+(1+x)/3$ when $x\to 0$ along the path (42) is:*

*For $0<\mathcal{V}<1$:*

$$y(x)=-\frac{1}{4}\left[\frac{e^{i\pi\nu_1}}{16^{\nu_2-1}}\right]\ x^{\nu_2}\ \left(1+O(|x^{\nu_2}|+|x^{1-\nu_2}|)\right).\tag{43}$$

*For $\mathcal{V}=0$:*

$$y(x)=\big(1+O(x)\big)\ \left[\frac{x}{2}+\sin^{-2}\left(-i\frac{\nu_2}{2}\ln\frac{x}{16}+\frac{\pi\nu_1}{2}+\sum_{m\geq 1}c_{0m}\left[e^{i\pi\nu_1}\left(\frac{x}{16}\right)^{\nu_2}\right]^m\right)\right].\tag{44}$$

*For $\mathcal{V}=1$:*

$$y(x)=(1+O(x))\ x\ \sin^2\left(i\frac{1-\nu_2}{2}\ln\frac{x}{16}+\frac{\pi\nu_1}{2}+\sum_{m\geq 1}b_{0m}\left[e^{-i\pi\nu_1}\left(\frac{x}{16}\right)^{1-\nu_2}\right]^m\right).\tag{45}$$

*For $\nu_2$ real we have two cases. For $0<\nu_2<1$, the transcendent $y(x)=\wp(\nu_1\omega_1+\nu_2\omega_2+v(x;\nu_1,\nu_2);\omega_1,\omega_2)+(1+x)/3$ defined in $\mathcal{D}_0(r)$ has behavior*

$$y(x)=-\frac{1}{4}\left[\frac{e^{i\pi\nu_1}}{16^{\nu_2-1}}\right]\ x^{\nu_2}\ \left(1+O(|x^{\nu_2}|+|x^{1-\nu_2}|)\right),\quad 0<\nu_2<1 \tag{46}$$

*For $1<\nu_2<2$, the transcendent $y(x)=\wp(\nu_1\omega_1+\nu_2\omega_2+v(x;-\nu_1,2-\nu_2);\omega_1,\omega_2)+(1+x)/3$ defined in $\mathcal{D}_0(r)$ has behavior*

$$y(x)=-\frac{1}{4}\left[\frac{e^{i\pi\nu_1}}{16^{\nu_2-1}}\right]^{-1}\ x^{2-\nu_2}\ \left(1+O(|x^{2-\nu_2}|+|x^{\nu_2-1}|)\right),\quad 1<\nu_2<2 \tag{47}$$

*Proof:* Let $x_0\in\mathcal{D}(r;\nu_1,\nu_2)$. We expand $y(x)$ in Fourier series. This is possible on $\mathcal{D}(r;\nu_1,\nu_2)$ as it is clear from the proof of Theorem 1. Let $f:=\nu_1+\nu_2\tau+\frac{v}{\omega_1}$.

$$y(x)=\wp(\nu_1\omega_1+\nu_2\omega_2+v;\omega_1,\omega_2)+\frac{1+x}{3}=$$

$$\left(\frac{\pi}{2\omega_1}\right)^2\left\{-\frac{1}{3}+\sin^{-2}\left(\frac{\pi}{2}f\right)+8\sum_{n\geq1}\frac{ne^{2i\pi n\tau}}{1-e^{2i\pi n\tau}}\left[1-\cos(\pi nf)\right]\right\}+\frac{1+x}{3}=$$

$$=\left(\frac{\pi}{2\omega_1}\right)^2\left\{-\frac{1}{3}-4\left[e^{i\frac{\pi}{2}f}-e^{-i\frac{\pi}{2}f}\right]^{-2}+4\sum_{n\geq1}\frac{ne^{2i\pi n\tau}}{1-e^{2i\pi n\tau}}\left[2-e^{i\pi nf}-e^{-i\pi nf}\right]\right\}+\frac{1+x}{3}$$

Observe that for $x\to0$ $\frac{\pi^2}{4\omega_1^2}=\frac{1}{F(x)^2}=1-\frac{1}{2}x+O(x^2)$. We recall that $e^{i\pi C\tau(x)}=h(x)^C(x/16)^C$, where $h(x)=1+O(x)$. We define

$$\mathcal{X}(x):=h(x)^{2-\nu_2}e^{-i\pi\nu_1}\left(\frac{x}{16}\right)^{2-\nu_2}e^{-i\pi\frac{v(x)}{\omega_1(x)}}$$

$$\mathcal{Y}(x):=h(x)^{\nu_2}e^{i\pi\nu_1}\left(\frac{x}{16}\right)^{\nu_2}e^{i\pi\frac{v(x)}{\omega_1(x)}}$$

Thus

$$y(x)=\frac{x}{2}+O(x^2)+4\left(1-\frac{x}{2}+O(x^2)\right)\left\{\sum_{n\geq1}\frac{n\mathcal{X}^n(x)}{1-h(x)^{2n}(x/16)^{2n}}\left[2\mathcal{Y}^n(x)-\mathcal{Y}^{2n}(x)-1\right]-\frac{\mathcal{Y}(x)}{\left[1-\mathcal{Y}(x)\right]^2}\right\}$$

Note that the denominator $[1-\mathcal{Y}(x)]^2$ in the last term does not vanish on $\mathcal{D}$.

The above formula immediately yields (43) and (46), (47). It is enough to recall that in $\mathcal{D}(r;\nu_1,\nu_2)$ the dominant powers are $x^{\nu_2}$, $x^{1-\nu_2}$. We also need to observe that

$$|x^{\nu_2+\alpha}|=|x|^{\mathcal{V}+\alpha}\ \frac{|x_0^{\nu_2+\alpha}|}{|x_0|^{\mathcal{V}+\alpha}},\qquad\forall\alpha\in\mathbf{C}.$$

and that $v(x)\to0$ for $x\to0$, according to (15). Therefore, (43) and (46), (47) follow simply taking the leading terms in the expansion.

As for (45), we observe that if $\mathcal{V}=1$, $v(x)$ does not vanish, namely:

$$\phi(x):=\sum_{m\geq1}b_{0m}\left[e^{-i\pi\nu_1}\left(\frac{x}{16}\right)^{1-\nu_2}\right]^m\not\to0\text{ as }x\to0$$

and $e^{i\pi\frac{v}{\omega_1}}=e^{2i\phi_1}(1+O(x))$. This implies that the dominant terms in the Fourier expansion are:

$$y(x)=\left[\frac{x}{2}-4e^{i\pi\nu_1}\left(\frac{x}{16}\right)^{\nu_2}e^{2i\phi(x)}-4e^{-i\pi\nu_1}\left(\frac{x}{16}\right)^{2-\nu_2}e^{-2i\phi(x)}\right]\ (1+O(x))$$

from which (45) follows. Note that $x$, $x^{\nu_2}$ and $x^{2-\nu_2}$ are of the same order.

As for (44), $v(x)$ does not vanish when $\mathcal{V}=0$, because $x^{\nu_2} \not\to 0$ . Namely

$$\psi(x) := \sum_{m\geq 1} c_{0m} \left[ e^{i\pi\nu_1} \left(\frac{x}{16}\right)^{\nu_2} \right]^m \not\to 0 \text{ as } x\to 0$$

In this case, it is convenient to keep the term $\sin^{-2}(\pi f/2)$. It is

$$\sin^{-2}\left[-i\frac{\nu_2}{2}\ln\frac{x}{16}+\frac{\pi\nu_1}{2}+\psi(x)+O(x)\right]=$$

$$=\sin^{-2}\left[-i\frac{\nu_2}{2}\ln\frac{x}{16}+\frac{\pi\nu_1}{2}+\psi(x)\right](1+O(x))$$

The last step is possible because $-i\frac{\nu_2}{2}\ln\frac{x}{16}+\frac{\pi\nu_1}{2}+\psi(x)\neq 0$ on $\mathcal{D}(r;\nu_1,\nu_2)$. So, (44) follows. Note that $\sin\left[-i\frac{\nu_2}{2}\ln\frac{x}{16}+\frac{\pi\nu_1}{2}+\psi(x)\right]\neq 0$ on the domain, namely $y(x)$ in (44) does not have poles in $\mathcal{D}(r;\nu_1,\nu_2)$.

□

**Remark:** In (44) the $\sin^{-2}(...)$ is never infinite on the domain. Actually, it is clear from the proof of Theorem 1 that the domains $\mathcal{D}_i(r_N;\nu_1,\nu_2+2N)$, $i=1,2$, where exactly chosen in such a way that $\sin(...)\neq 0$. Therefore, the *movable poles* of the transcendent must lie outside $\mathcal{D}(r;\nu_1,\nu_2)$.

# 6 Special Cases

There are some special cases when the domain $\mathcal{D}(r;\nu_1,\nu_2)$ can be enlarged.

**FIRST CASE:** $\beta=1-2\delta=0$.

Theorem 1 can be stated as follows:

**Theorem 1 – special case** $\beta=1-2\delta=0$ **:** *For any complex $\nu_1$, $\nu_2$ with the constraint*

$$0<\nu_2<2 \qquad \textit{if } \nu_2 \textit{ is real}$$

*there exist a positive number $r<1$ and a transcendent*

$$y(x)=\wp\Big(\nu_1\omega_1(x)+\nu_2\omega_2(x)+v(x;\nu_1,\nu_2);\ \omega_1(x),\omega_2(x)\Big)+\frac{1+x}{3}$$

*such that $v(x;\nu_1,\nu_2)$ is holomorphic in the domain*

$$\mathcal{D}(r;\nu_1,\nu_2)=\left\{x\in\tilde{\mathbf{C}}_0 \mid |x|<r,\ \left|e^{-i\pi\nu_1}\left(\frac{x}{16}\right)^{2-\nu_2}\right|<r,\ \left|e^{i\pi\nu_1}\left(\frac{x}{16}\right)^{\nu_2}\right|<r\right\} \quad \textit{if } \Im\nu_2\neq 0$$

*or in the domain $\mathcal{D}_0(r)$ if $0<\nu_2<2$. On the domain it has convergent expansion :*

$$v(x;\nu_1,\nu_2)=\sum_{n\geq 1}a_nx^n+\sum_{n\geq 0,m\geq 1}b_{nm}x^n\left[e^{-i\pi\nu_1}\left(\frac{x}{16}\right)^{2-\nu_2}\right]^m+\sum_{n\geq 0,m\geq 1}c_{nm}x^n\left[e^{i\pi\nu_1}\left(\frac{x}{16}\right)^{\nu_2}\right]^m$$

Note that the domain is larger than the general case and that $\nu_2=1$ is now allowed.

*Proof:* If we go back to the proof of Theorem 1 we see that in the equation (11) we only have the functions $\wp(u/2+\varepsilon_1\omega_1+2N_2\omega_2)$. This means that we do not have $N_2'$. Let us put $N_2=N$. Then, we do the proof of Theorem 1 in the domain (23) with $N_2=N$ and $\varepsilon_2=0$. A remarkable fact happens. While in proof for the general case we have to distinguish (for given $N$, $\nu_1$, $\nu_2$) between $\mathcal{D}_1(r_N;\nu_1,\nu_2+2N)$ and $\mathcal{D}_2(r_N;\nu_1,\nu_2+2N)$, in the present special case we prove the existence of $v(x)$ on the larger domain

$$\mathcal{D}(r_N;\nu_1,\nu_2+2N):=\left\{x\in\tilde{\mathbf{C}}_0 \mid |x|<r_N,\ \left|e^{-i\pi\nu_1}\left(\frac{x}{16}\right)^{2-\nu_2-2N}\right|<r,\ \left|e^{i\pi\nu_1}\left(\frac{x}{16}\right)^{\nu_2+2N}\right|<r\right\} \quad (48)$$

where $v(x;\nu_1,\nu_2+2N)$ is represented as:

$$\sum_{n\geq 1} a_n x^n + \sum_{n\geq 0, m\geq 1} b_{nm} x^n \left[e^{-i\pi\nu_1}\left(\frac{x}{16}\right)^{2-\nu_2-2N}\right]^m + \sum_{n\geq 0, m\geq 1} c_{nm} x^n \left[e^{i\pi\nu_1}\left(\frac{x}{16}\right)^{\nu_2+2N}\right]^m$$

We note that, if $\Im\nu_2 = 0$, the constraint $0 < \nu_2 < 2$ holds. Namely, now $\nu_2 = 1$ is allowed. When $\nu_2$ is real, $v(x)$ is equal to (48) with $N = 0$ and the domain is simply given by the condition $\max\{|x|, \left|e^{-i\pi\nu_1}\left(\frac{x}{16}\right)^{2-\nu_2}\right|, |e^{i\pi\nu_1}\left(\frac{x}{16}\right)^{\nu_2}|\} < r$, namely $|x| < r$ for $r$ small enough. This is the domain $\mathcal{D}_0(r)$.

□

The critical behavior for $x \to 0$ can be studied along

$$\arg x = \arg x_0 + \frac{\Re\nu_2 - \mathcal{V}}{\Im\nu_2} \ln\frac{|x|}{|x_0|}$$

contained in $\mathcal{D}(r;\nu_1,\nu_2)$ for $x_0 \in \mathcal{D}(r;\nu_1,\nu_2)$ and $0 \leq \mathcal{V} \leq 2$. If $\Im\nu_2 = 0$ any path going to $x = 0$ is contained in $\mathcal{D}_0(r)$. The critical behavior is obtained through the Fourier expansion. We give the result.

If $\Im\nu_2 \neq 0$, the transcendent $y(x) = \wp\big(\nu_1\omega_1(x) + \nu_2\omega_2(x) + v(x;\nu_1,\nu_2)\big) + (1+x)/3$ has the following behaviors on $\mathcal{D}(r;\nu_1,\nu_2)$:

For $0 < \mathcal{V} < 1$:

$$y(x) = -\frac{1}{4}\left[\frac{e^{i\pi\nu_1}}{16^{\nu_2-1}}\right] x^{\nu_2} \ (1 + O(|x^{\nu_2}|)). \tag{49}$$

For $1 < \mathcal{V} < 2$:

$$y(x) = -\frac{1}{4}\left[\frac{e^{i\pi\nu_1}}{16^{\nu_2-1}}\right]^{-1} x^{2-\nu_2} \ \left(1 + O(|x^{2-\nu_2}|)\right). \tag{50}$$

For $\mathcal{V} = 1$:

$$y(x) = (1 + O(x))\ x\ \sin^2\left(i\frac{1-\nu_2}{2}\ln\frac{|x|}{16} + \frac{\pi\nu_1}{2}\right). \tag{51}$$

For $\mathcal{V} = 0$:

$$y(x) = \big(1 + O(x)\big)\ \left[\frac{x}{2} + \sin^{-2}\left(-i\frac{\nu_2}{2}\ln\frac{x}{16} + \frac{\pi\nu_1}{2} + \sum_{m\geq 1} c_{0m}\left[e^{i\pi\nu_1}\left(\frac{x}{16}\right)^{\nu_2}\right]^m\right)\right]. \tag{52}$$

For $\mathcal{V} = 2$:

$$y(x) = \big(1 + O(x)\big)\ \left[\frac{x}{2} + \sin^{-2}\left(i\frac{2-\nu_2}{2}\ln\frac{x}{16} + \frac{\pi\nu_1}{2} + \sum_{m\geq 1} b_{0m}\left[e^{-i\pi\nu_1}\left(\frac{x}{16}\right)^{2-\nu_2}\right]^m\right)\right]. \tag{53}$$

For $\nu_2$ real, satisfying the constraints $0 < \nu_2 < 2$, the transcendent $y(x) = \wp\big(\nu_1\omega_1(x) + \nu_2\omega_2(x) + v(x;\nu_1,\nu_2)\big) + (1+x)/3$ has the following behaviors on $\mathcal{D}_0(r)$:

$$y(x) = -\frac{1}{4}\left[\frac{e^{i\pi\nu_1}}{16^{\nu_2-1}}\right] x^{\nu_2} \ (1 + O(|x^{\nu_2}|)), \quad 0 < \nu_2 < 1 \tag{54}$$

$$y(x) = -\frac{1}{4}\left[\frac{e^{i\pi\nu_1}}{16^{\nu_2-1}}\right]^{-1} x^{2-\nu_2} \ \left(1 + O(|x^{2-\nu_2}|)\right), \quad 1 < \nu_2 < 2 \tag{55}$$

$$y(x) = (1 + O(x))\ x\ \sin^2\left(\frac{\pi\nu_1}{2}\right), \quad \nu_2 = 1, \quad (\nu_1 \neq 0) \tag{56}$$

**SECOND CASE:** $\alpha = \gamma = 0$.

Theorem 1 is restated as follows:

**Theorem 1 – special case $\alpha=\gamma=0$:** *For any complex $\nu_1$, $\nu_2$ with the constraint*

$$-1<\nu_2<1 \qquad \text{if } \nu_2 \text{ is real}$$

*there exist a positive number $r<1$ and a transcendent*

$$y(x)=\wp\Big(\nu_1\omega_1(x)+\nu_2\omega_2(x)+v(x;\nu_1,\nu_2);\ \omega_1(x),\omega_2(x)\Big)+\frac{1+x}{3}$$

*such that $v(x;\nu_1,\nu_2)$ is holomorphic in the domain*

$$\mathcal{D}(r;\nu_1,\nu_2)=\left\{x\in\tilde{\mathbf{C}}_0 \mid |x|<r,\ \left|e^{-i\pi\nu_1}\left(\frac{x}{16}\right)^{1-\nu_2}\right|<r,\ \left|e^{i\pi\nu_1}\left(\frac{x}{16}\right)^{\nu_2+1}\right|<r\right\}$$

*or in the domain $\mathcal{D}_0(r)$ if $-1<\nu_2<1$, where it has convergent expansion :*

$$v(x;\nu_1,\nu_2)=\sum_{n\geq1}a_nx^n+\sum_{n\geq0,m\geq1}b_{nm}x^n\left[e^{-i\pi\nu_1}\left(\frac{x}{16}\right)^{1-\nu_2}\right]^m+\sum_{n\geq0,m\geq1}c_{nm}x^n\left[e^{i\pi\nu_1}\left(\frac{x}{16}\right)^{\nu_2+1}\right]^m$$

Note that now $\nu_2=0$ is allowed.

*Proof:* If we go back to the proof of Theorem 1, we see that in the equation (11) we only have the functions $\wp(u/2+\varepsilon_1\omega_1+(2N_2'+1)\omega_2)$. This means that we do not have $N_2$. This time, let us put $N_2'=N$. Therefore we do the proof of Theorem 1 for the domain (23) with $N_2'=N$ and $\varepsilon_2=1$. Therefore, on the new (bigger) domain

$$\mathcal{D}(r_N;\nu_1,\nu_2+2N)=\left\{x\in\tilde{\mathbf{C}}_0 \mid |x|<r_N,\ \left|e^{-i\pi\nu_1}\left(\frac{x}{16}\right)^{1-\nu_2-2N}\right|<r,\ \left|e^{i\pi\nu_1}\left(\frac{x}{16}\right)^{\nu_2+1+2N}\right|<r\right\} \quad (57)$$

there exist one $v(x)$, holomorphic with convergent expansion:

$$v(x;\nu_1,\nu_2+2N)=\sum_{n\geq1}a_nx^n+\sum_{n\geq0,m\geq1}b_{nm}x^n\left[e^{-i\pi\nu_1}\left(\frac{x}{16}\right)^{1-\nu_2-2N}\right]^m+$$

$$+\sum_{n\geq0,m\geq1}c_{nm}x^n\left[e^{i\pi\nu_1}\left(\frac{x}{16}\right)^{\nu_2+1+2N}\right]^m$$

In a sense, in the case $\beta=1-2\delta=0$ the disjoint union $\mathcal{D}_1(r_N;\nu_1,\nu_2+2N)\cup\mathcal{D}_2(r_N;\nu_1,\nu_2+2N)$ was replaced by the bigger domain (48). In the present case the disjoint union $\mathcal{D}_1(r_N;\nu_1,\nu_2+2N)\cup\mathcal{D}_2(r_{N+1};\nu_1,\nu_2+2[N+1])$ is replaced by the bigger domain (57).

If $\nu_2$ is real, this time $\nu_2=0$ is allowed, but not $\nu_2=1$. Then, it is convenient to choose the constraint

$$-1<\nu_2<1$$

by doing $\nu_2\mapsto\nu_2-2$ if $1<\nu_2<2$. The domain is simply specified by $\max\left\{|x|,\ \left|e^{-i\pi\nu_1}\left(\frac{x}{16}\right)^{1-\nu_2}\right|,\ \left|e^{i\pi\nu_1}\left(\frac{x}{16}\right)^{\nu_2+1}\right|\right\}<r$, namely $|x|<r$ for $r$ small enough. This is $\mathcal{D}_0(r)$.

$\square$

We compute the critical behaviors for $x\to0$ along

$$\arg x=\arg x_0+\frac{\Re\nu_2-\mathcal{V}}{\Im\nu_2}\ln\frac{|x|}{|x_0|},$$

which is contained in $\mathcal{D}(r;\nu_1,\nu_2)$ if and only if $x_0\in\mathcal{D}(r;\nu_1,\nu_2)$ and $-1\leq\mathcal{V}\leq1$. If $\Im\nu_2=0$ any path converging to $x=0$ is contained in $\mathcal{D}_0(r)$. We use Fourier expansion; the result is the following.

For $0<\mathcal{V}<1$:

$$y(x)=-\frac{1}{4}\left[\frac{e^{i\pi\nu_1}}{16^{\nu_2-1}}\right]\ x^{\nu_2}\ \left(1+O(|x^{1-\nu_2}|)\right). \quad (58)$$

For $-1 < \mathcal{V} < 0$:

$$y(x) = -\frac{1}{4}\left[\frac{e^{i\pi\nu_1}}{16^{\nu_2+1}}\right]^{-1} x^{-\nu_2} \left(1 + O(|x^{\nu_2+1}|)\right). \tag{59}$$

For $\mathcal{V} = 1$:

$$y(x) = (1 + O(x))\; x\; \sin^2\left(i\frac{1-\nu_2}{2}\ln\frac{|x|}{16} + \frac{\pi\nu_1}{2} + \sum_{m\geq 1} b_{0m}\left[e^{-i\pi\nu_1}\left(\frac{x}{16}\right)^{1-\nu_2}\right]^m\right). \tag{60}$$

For $\mathcal{V} = -1$:

$$y(x) = (1 + O(x))\; x\; \sin^2\left(-i\frac{\nu_2+1}{2}\ln\frac{|x|}{16} + \frac{\pi\nu_1}{2} + \sum_{m\geq 1} c_{0m}\left[e^{i\pi\nu_1}\left(\frac{x}{16}\right)^{\nu_2+1}\right]^m\right). \tag{61}$$

For $\mathcal{V} = 0$:

$$y(x) = \left(1 + O(x)\right)\; \left[\frac{x}{2} + \sin^{-2}\left(-i\frac{\nu_2}{2}\ln\frac{x}{16} + \frac{\pi\nu_1}{2}\right)\right]. \tag{62}$$

For $\nu_2$ real, the behaviors on $\mathcal{D}_0(r)$ are:

$$y(x) = -\frac{1}{4}\left[\frac{e^{i\pi\nu_1}}{16^{\nu_2-1}}\right] x^{\nu_2} \left(1 + O(|x^{1-\nu_2}|)\right), \quad \text{if } 0 < \nu_2 < 1. \tag{63}$$

$$y(x) = -\frac{1}{4}\left[\frac{e^{i\pi\nu_1}}{16^{\nu_2+1}}\right]^{-1} x^{-\nu_2} \left(1 + O(|x^{\nu_2+1}|)\right), \quad \text{if } -1 < \nu_2 < 0. \tag{64}$$

$$y(x) = (1 + O(x))\; \left[\frac{1}{\sin^2\left(\frac{\pi\nu_1}{2}\right)} + \frac{x}{2}\right], \quad \text{if } \nu_2 = 0, \quad (\nu_1 \neq 0). \tag{65}$$

# 7 Representations at $x = 1$, $x = \infty$ – Theorem 3

Two independent solutions of the hyper-geometric equation $\mathcal{L}(u) = 0$ are the hyper-geometric function $F(x; 1/2, 1/2, 1)$ and

$$g\left(\frac{1}{2},\frac{1}{2},1;x\right) := F_1(x) + F\left(\frac{1}{2},\frac{1}{2},1;x\right)\ln x$$

The following connection formulae hold [27]:

i) Connection $0 - 1$

$$F\left(\frac{1}{2},\frac{1}{2},1;x\right) = -\frac{1}{\pi}g\left(\frac{1}{2},\frac{1}{2},1;1-x\right), \quad |\arg(1-x)| < \pi$$

$$g\left(\frac{1}{2},\frac{1}{2},1;x\right) = -\pi F\left(\frac{1}{2},\frac{1}{2},1;1-x\right), \quad |\arg x| < \pi$$

ii) Connection $0 - \infty$

$$F\left(\frac{1}{2},\frac{1}{2},1;x\right) = \frac{x^{-\frac{1}{2}}}{\pi}\left[ig\left(\frac{1}{2},\frac{1}{2},1;\frac{1}{x}\right) + \pi F\left(\frac{1}{2},\frac{1}{2},1;\frac{1}{x}\right)\right], \quad -2\pi < \arg x < 0$$

$$g\left(\frac{1}{2},\frac{1}{2},1;x\right) = x^{-\frac{1}{2}}g\left(\frac{1}{2},\frac{1}{2},1;\frac{1}{x}\right), \quad |\arg x| < \pi$$

We recall that $\omega_1(x)$ and $\omega_2(x)$ are $\pi/2\; F(1/2, 1/2, 1; x)$ and $-i/2\; g(1/2, 1/2, 1; x)$ respectively. This representation is convenient in a neighborhood of $x = 0$. In this section we will use the notation $\omega_1^{(0)}$ and $\omega_2^{(0)}$ instead of $\omega_1$ and $\omega_2$. Namely

$$\omega_1^{(0)}(x) = \frac{\pi}{2}F\left(\frac{1}{2},\frac{1}{2},1;x\right), \quad \omega_2^{(0)}(x) = -\frac{i}{2}g\left(\frac{1}{2},\frac{1}{2},1;x\right)$$

Taking into account the above connection formulae, we define in a neighborhood of $x=1$:

$$\omega_1^{(1)}(x) := \omega_2^{(0)}(x) = \frac{i\pi}{2} F\left(\frac{1}{2},\frac{1}{2},1;1-x\right), \quad |\arg x| < \pi$$

$$\omega_2^{(1)}(x) := \omega_1^{(0)}(x) = -\frac{1}{2} g\left(\frac{1}{2},\frac{1}{2},1;1-x\right), \quad |\arg(1-x)| < \pi$$

We also define, in a neighborhood of $x=\infty$:

$$\omega_1^{(\infty)}(x) := \omega_1^{(0)}(x) + \omega_2^{(0)}(x) = \frac{\pi}{2} x^{-\frac{1}{2}} F\left(\frac{1}{2},\frac{1}{2},1;\frac{1}{x}\right), \quad -\pi < \arg x < 0$$

$$\omega_2^{(\infty)}(x) := \omega_2^{(0)}(x) = -\frac{i}{2} x^{-\frac{1}{2}} g\left(\frac{1}{2},\frac{1}{2},1;\frac{1}{x}\right), \quad |\arg x| < \pi$$

The above functions have branch cuts specified by the constrains on $\arg x$. Once they are so defined, they are continued on the universal covering of $\mathbf{P}^1\backslash\{0,1,\infty\}$.

In the following, we use superscripts 0, 1, $\infty$ for solutions $u(x)$ of (11) represented around $x=0,1,\infty$ respectively. At $x=0$ we looked for solutions

$$\frac{u^{(0)}}{2} = \nu_1^{(0)}\omega_1^{(0)} + \nu_2^{(0)}\omega_2^{(0)} + v^{(0)}$$

and we found the representations of $v^{(0)}(x)$ in Theorem 1. Now we look for a solution in a neighborhood of $x=1$ of the type

$$\frac{u^{(1)}}{2} = \nu_1^{(1)}\omega_1^{(1)} + \nu_2^{(1)}\omega_2^{(1)} + v^{(1)}$$

and for a solution in a neighborhood of $x=\infty$ of the type

$$\frac{u^{(\infty)}}{2} = \nu_1^{(\infty)}\omega_1^{(\infty)} + \nu_2^{(\infty)}\omega_2^{(\infty)} + v^{(\infty)}$$

They yield Painlevé transcendents $y(x) = \wp(u^{(1)}/2;\omega_1^{(0)},\omega_2^{(0)}) + (1+x)/3 \equiv \wp(u^{(1)}/2;\omega_1^{(1)},\omega_2^{(1)}) + (1+x)/3$ and $y(x) = \wp(u^{(\infty)}/2;\omega_1^{(0)},\omega_2^{(0)}) + (1+x)/3 \equiv \wp(u^{(\infty)}/2;\omega_1^{(\infty)},\omega_2^{(\infty)}) + (1+x)/3$.

The definitions of $\omega_j^{(i)}$ are convenient because the proof of Theorem 1 can be repeated with no changes in a neighborhood of $x=1$ and $\infty$. So, we can just give the result in a short form. Let $\mathbf{P}_i^1 := \mathbf{P}^1\backslash\{i\}$, $i=1,\infty$, and let $\widetilde{\mathbf{P}_i^1}$ be the universal covering.

**Theorem 3:**

**In a Neighborhood of $x=1$.**

*For any complex $\nu_1^{(1)}$ $\nu_2^{(1)}$ such that $\Im\nu_2^{(1)} \neq 0$ there exists a transcendent $y(x) = \wp(\nu_1^{(1)}\omega_1^{(1)} + \nu_2^{(1)}\omega_2^{(1)} + v^{(1)}(x;\nu_1^{(1)},\nu_2^{(1)})) + \frac{1+x}{3}$ such that $v^{(1)}(x)$ is holomorphic and has the following convergent expansion:*

$$v^{(1)}(x;\nu_1^{(1)},\nu_2^{(1)}) = \sum_{n\geq 1} a_n(1-x)^n + \sum_{n\geq 0, m\geq 1} b_{nm}(1-x)^n \left[e^{i\pi\nu_1^{(1)}}\left(\frac{1-x}{16}\right)^{1-\nu_2^{(1)}}\right]^m +$$
$$+ \sum_{n\geq 0, m\geq 1} c_{nm}(1-x)^n \left[e^{-i\pi\nu_1^{(1)}}\left(\frac{1-x}{16}\right)^{\nu_2^{(1)}}\right]^m \tag{66}$$

*in*

$$\mathcal{D}(r;\nu_1^{(1)},\nu_2^{(1)}) := \left\{ x \in \widetilde{\mathbf{P}_1^1} \text{ such that } |1-x| < r, \left|e^{i\pi\nu_1^{(1)}}\left(\frac{1-x}{16}\right)^{1-\nu_2^{(1)}}\right| < r, \right.$$
$$\left. \left|e^{-i\pi\nu_1^{(1)}}\left(\frac{1-x}{16}\right)^{\nu_2^{(1)}}\right| < r \right\}$$

*When $\nu_2^{(1)}$ is real, the constraint*

$$0<\nu_2^{(1)}<1 \quad \text{or} \quad 1<\nu_2^{(1)}<2$$

*holds. If $0<\nu_2^{(1)}<1$ there exists a sufficiently small $r$ and a transcendent $y(x)=\wp(\nu_1^{(1)}\omega_1^{(1)}+\nu_2^{(1)}\omega_2^{(1)}+v^{(1)}(x;\nu_1^{(1)},\nu_2^{(1)}))+\frac{1+x}{3}$ such that $v^{(1)}(x;\nu_1^{(1)},\nu_2^{(1)})$ is holomorphic in*

$$\mathcal{D}_0(r):=\Big\{x\in\widetilde{\mathbf{P}_1^1} \quad \text{such that} \quad |1-x|<r\Big\}$$

*where it has convergent expansion (66). If $1<\nu_2^{(1)}<2$ we repeat the statement for $y(x)=\wp(\nu_1^{(1)}\omega_1^{(1)}+\nu_2^{(1)}\omega_2^{(1)}+v^{(1)}(x;-\nu_1^{(1)},2-\nu_2^{(1)}))+\frac{1+x}{3}$.*

Note that $\nu_1^{(1)}$ comes with sign changed w.r.t. $\nu_1^{(0)}$; this is due to the definition of $\omega_1^{(1)}$.

**In a Neighborhood of $x=\infty$.**

*For any complex $\nu_1^{(\infty)}$ $\nu_2^{(\infty)}$ such that $\Im\nu_2^{(\infty)}\neq 0$ there exists a transcendent $y(x)=\wp(\nu_1^{(\infty)}\omega_1^{(\infty)}+\nu_2^{(\infty)}\omega_2^{(\infty)}+v^{(\infty)}(x;\nu_1^{(\infty)},\nu_2^{(\infty)}))+\frac{1+x}{3}$ such that $v^{(\infty)}(x;\nu_1^{(\infty)},\nu_2^{(\infty)})$ is holomorphic and has the convergent expansion:*

$$x^{\frac{1}{2}}v^{(\infty)}(x;\nu_1^{(\infty)},\nu_2^{(\infty)})=\sum_{n\geq 1}a_n\left(\frac{1}{x}\right)^n+\sum_{n\geq 0,m\geq 1}b_{nm}\left(\frac{1}{x}\right)^n\left[e^{-i\pi\nu_1^{(\infty)}}\left(\frac{16}{x}\right)^{1-\nu_2^{(\infty)}}\right]^m+$$

$$+\sum_{n\geq 0,m\geq 1}c_{nm}\left(\frac{1}{x}\right)^n\left[e^{i\pi\nu_1^{(\infty)}}\left(\frac{16}{x}\right)^{\nu_2^{(\infty)}}\right]^m \tag{67}$$

*in*

$$\mathcal{D}(r;\nu_1^{(\infty)},\nu_2^{(\infty)}):=\Bigg\{x\in\widetilde{\mathbf{P}_\infty^1} \quad \text{such that} \quad |x^{-1}|<r, \quad \left|e^{-i\pi\nu_1^{(\infty)}}\left(\frac{16}{x}\right)^{1-\nu_2^{(\infty)}}\right|<r,$$

$$\left|e^{i\pi\nu_1^{(\infty)}}\left(\frac{16}{x}\right)^{\nu_2^{(\infty)}}\right|<r\Bigg\}$$

*When $\nu_2^{(\infty)}$ is real, the constraint*

$$0<\nu_2^{(\infty)}<1 \quad \text{or} \quad 1<\nu_2^{(\infty)}<2$$

*holds. If $0<\nu_2^{(\infty)}<1$ there exists a sufficiently small $r$ and a transcendent $y(x)=\wp(\nu_1^{(\infty)}\omega_1^{(\infty)}+\nu_2^{(\infty)}\omega_2^{(\infty)}+v^{(\infty)})+\frac{1+x}{3}$ such that $v^{(\infty)}(x)=v^{(\infty)}(x;\nu_1^{(\infty)},\nu_2^{(\infty)})$ is holomorphic in*

$$\mathcal{D}_0(r):=\Big\{x\in\widetilde{\mathbf{P}_\infty} \quad \text{such that} \quad |x^{-1}|<r\Big\}$$

*where it has convergent expansion (67). If $1<\nu_2^{(\infty)}<2$ the statement holds with $v^{(\infty)}(x;-\nu_1^{(\infty)},2-\nu_2^{(\infty)})$.*

**Note:** We have used the notations $\mathcal{D}(r;\nu_1,\nu_2)$ for the domains both at $x=0$, $x=1$ and $x=\infty$. We believe this will not confuse the reader, because it is always clear which is the critical point we are considering.

## 7.1 Critical Behaviors at $x=1,\infty$

From Theorem 3 and the Fourier expansion of the Weierstrass function we obtain the critical behaviors of the transcendents of Theorem 3.

**In a neighborhood of $x=1$:**

If $\Im\nu_2^{(1)} = 0$, $x \to 1$ along any path. Otherwise, we consider the path

$$\arg(1-x) = \arg(1-x_0) + \frac{\Re\nu_2^{(1)} - \mathcal{V}}{\Im\nu_2^{(1)}} \ln\frac{|1-x|}{|1-x_0|}, \quad 0 \le \mathcal{V} \le 1$$

joining $x_0$ to $x = 1$. The path lies in $\mathcal{D}(r;\nu_1^{(1)},\nu_2^{(1)})$ if $x_0 \in \mathcal{D}(r;\nu_1^{(1)},\nu_2^{(1)})$.

If $\Im\nu_2^{(1)} \neq 0$, the transcendent defined in $\mathcal{D}(r;\nu_1^{(1)},\nu_2^{(1)})$ has behavior:

For $0 < \mathcal{V} < 1$:

$$y(x) = 1 + \frac{1}{4}\left[\frac{e^{-i\pi\nu_1^{(1)}}}{16^{\nu_2^{(1)}-1}}\right](1-x)^{\nu_2^{(1)}}\ \left(1 + O(|1-x|^{\nu_2^{(1)}} + |1-x|^{1-\nu_2^{(1)}})\right). \tag{68}$$

For $\mathcal{V} = 0$:

$$y(x) = 1 - \left\{\frac{1-x}{2} + \sin^{-2}\left(i\frac{\nu_2^{(1)}}{2}\ln\frac{1-x}{16} + \frac{\pi\nu_1^{(1)}}{2} - i\sum_{m\ge1} c_{0m}\left[e^{-i\pi\nu_1^{(1)}}\left(\frac{1-x}{16}\right)^{\nu_2^{(1)}}\right]^m\right)\right\}.$$

For $\mathcal{V} = 1$:

$$y(x) = 1 - (1-x)\ [\ 1 + O(|1-x|)\ ]\ \sin^2\left(i\frac{\nu_2^{(1)}-1}{2}\ln\frac{1-x}{16} + \frac{\pi\nu_1^{(1)}}{2} - \right.$$
$$\left. -i\sum_{m\ge1} b_{0m}\left[e^{i\pi\nu_1^{(1)}}\left(\frac{1-x}{16}\right)^{1-\nu_2^{(1)}}\right]^m\right).$$

For $\Im\nu_2 = 0$, the transcendent defined on $\mathcal{D}_0(r)$ has behavior (68) if $0 < \nu_2^{(1)} < 1$, or behavior

$$y(x) = 1 + \frac{1}{4}\left[\frac{e^{-i\pi\nu_1^{(1)}}}{16^{\nu_2^{(1)}-1}}\right]^{-1}(1-x)^{2-\nu_2^{(1)}}\ \left(1 + O(|1-x|^{2-\nu_2^{(1)}} + |1-x|^{\nu_2^{(1)}-1}|)\right). \tag{69}$$

if $1 < \nu_2^{(1)} < 2$.

**In a neighborhood of $x = \infty$:**

If $\Im\nu_2^{(\infty)} = 0$, $x \to \infty$ along any path. Otherwise, we consider the path

$$\arg x = \arg x_0 + \frac{\Re\nu_2^{(\infty)} - \mathcal{V}}{\Im\nu_2^{(\infty)}}\ln\frac{|x|}{|x_0|}, \quad 0 \le \mathcal{V} \le 1$$

If $\Im\nu_2^{(\infty)} \neq 0$ the transcendent defined in $D(r;\nu_1^{(\infty)},\nu_2^{(\infty)})$ has behavior:

For $0 < \mathcal{V} < 1$:

$$y(x) = -\frac{1}{4}\left[\frac{e^{i\pi\nu_1^{(\infty)}}}{16^{\nu_2^{(\infty)}-1}}\right]x^{1-\nu_2^{(\infty)}}\ \left(1 + O(|x|^{-\nu_2^{(\infty)}} + |x|^{\nu_2^{(\infty)}-1})\right). \tag{70}$$

For $\mathcal{V} = 0$:

$$y(x) = \left(1 + O(x^{-1})\right)\left\{\frac{1}{2} + x\sin^{-2}\left(-i\frac{\nu_2^{(\infty)}}{2}\ln\frac{16}{x} + \frac{\pi\nu_1^{(\infty)}}{2} + \right.\right.$$
$$\left.\left.\sum_{m\ge1} c_{0m}\left[e^{i\pi\nu_1^{(\infty)}}\left(\frac{16}{x}\right)^{\nu_2^{(\infty)}}\right]^m\right)\right\}.$$

For $\mathcal{V}=1$:

$$y(x)=\left(1+O(x^{-1})\right)\sin^2\left(i\frac{1-\nu_2^{(\infty)}}{2}\ln\frac{16}{x}+\frac{\pi\nu_1^{(\infty)}}{2}+\sum_{m\geq 1}b_{0m}\left[e^{-i\pi\nu_1^{(\infty)}}\left(\frac{16}{x}\right)^{1-\nu_2^{(\infty)}}\right]^m\right).$$

For $\Im\nu_2=0$, the transcendent defined on $\mathcal{D}_0(r)$ has behavior (70) if $0<\nu_2^{(\infty)}<1$, or it has behavior

$$y(x)=-\frac{1}{4}\left[\frac{e^{i\pi\nu_1^{(\infty)}}}{16^{\nu_2^{(\infty)}-1}}\right]^{-1}x^{\nu_2^{(\infty)}-1}\left(1+O(|x|^{\nu_2^{(\infty)}-2}+|x|^{1-\nu_2^{(\infty)}})\right). \tag{71}$$

if $1<\nu_2^{(\infty)}<2$.

# 8 Some Considerations on Analytic Continuation

We can easily study the effect of a small loop around a critical point on the transcendents of Theorem 1 and 3.

Consider a transcendent of Theorem 1

$$y(x)=\wp\left(\nu_1\omega_1^{(0)}+\nu_2\omega_2^{(0)}+v(x;\nu_1,\nu_2);\ \omega_1^{(0)},\omega_2^{(0)}\right)+\frac{1+x}{3}$$

defined on $\mathcal{D}(r;\nu_1,\nu_2)$. If $\nu_2$ is real, it is defined on $\mathcal{D}_0(r)$.

We do the loop $x\mapsto xe^{2\pi i}$, where $x\in\mathcal{D}(r;\nu_1,\nu_2)$ (or $\mathcal{D}_0(r)$), therefore $|x|<1$. From the monodromy properties of $F(1/2,1/2,1;x)$ and $g(1/2,1/2,1;x)$ we have

$$\omega_1^{(0)}(x)\mapsto\omega_1^{(0)}(xe^{2\pi i})=\omega_1^{(0)}(x),$$

$$\omega_2^{(0)}(x)\mapsto\omega_2^{(0)}(xe^{2\pi i})=\omega_2^{(0)}(x)+2\omega_1^{(0)}(x)$$

We also have

$$v(x;\nu_1,\nu_2)\mapsto v(xe^{2\pi i};\nu_1,\nu_2)\equiv v(x;\nu_1+2\nu_2,\nu_2)$$

The last step of the above equalities follows from the explicit expansion of $v(xe^{2\pi i};\nu_1,\nu_2)$ given in Theorem 1, *in the hypothesis* that also $xe^{2\pi i}\in\mathcal{D}(r;\nu_1,\nu_2)$ – or, which is the same thing, $x\in\mathcal{D}(r;\nu_1+2\nu_2,\nu_2)$ [this always happens if $\nu_2$ is real for $\mathcal{D}_0(r)$]. Therefore

$$y(x)\mapsto y(xe^{2\pi i})=:y'(x)=$$

$$=\wp\left(\nu_1\omega_1^{(0)}(x)+\nu_2[\omega_2^{(0)}(x)+2\omega_1^{(0)}(x)]+v(x;\nu_1+2\nu_2,\nu_2);\ \omega_1^{(0)}(x),\omega_2^{(0)}(x)+2\omega_1^{(0)}(x)\right)+\frac{1+x}{3}$$

$$\equiv\wp\left((\nu_1+2\nu_2)\omega_1^{(0)}(x)+\nu_2\omega_2^{(0)}(x)+v(x;\nu_1+2\nu_2,\nu_2);\ \omega_1^{(0)}(x),\omega_2^{(0)}(x)\right)+\frac{1+x}{3}$$

So, the effect of the loop is simply the transformation

$$(\nu_1,\nu_2)\mapsto(\nu_1+2\nu_2,\nu_2)$$

From the above, we also obtain the critical behavior according to Theorem 2.

We remark that the above considerations hold only if both $x$ and $xe^{2\pi i}$ belongs to $\mathcal{D}(\nu_1,\nu_2)$ – namely $x\in\mathcal{D}(\nu_1+2\nu_2,\nu_2)$. Otherwise, we can not represent $v(xe^{2\pi i};\nu_1,\nu_2)$ as $v(x;\nu_1+2\nu_2,\nu_2)$. This last case may actually occur when $\Im\nu_2\neq 0$: the point $xe^{2\pi i}$ lies outside the domain and it may be a pole of $y(x)$.

The same procedure is applied to

$$y(x)=\wp\left(\nu_1^{(1)}\omega_1^{(1)}+\nu_2^{(1)}\omega_2^{(1)}+v(x;\nu_1^{(1)},\nu_2^{(1)});\ \omega_1^{(1)},\omega_2^{(1)}\right)+\frac{1+x}{3}$$

Now we denote by $\mathcal{D}(r;\nu_1^{(1)},\nu_2^{(1)})$ the domain around $x=1$ [see the Note at the end of section 7]. The effect of the loop $(1-x)\mapsto(1-x)e^{2\pi i}$, $x\in\mathcal{D}(r;\nu_1^{(1)},\nu_2^{(1)})$ is

$$(\nu_1^{(1)},\nu_2^{(1)})\mapsto(\nu_1^{(1)}-2\nu_2^{(1)},\nu_2^{(1)})$$

if also $x \in \mathcal{D}(r; \nu_1^{(1)} - 2\nu_1^{(2)}, \nu_2^{(1)})$.

The procedure is applied to

$$y(x) = \wp\Big(\nu_1^{(\infty)}\omega_1^{(\infty)} + \nu_2^{(\infty)}\omega_2^{(\infty)} + v(x; \nu_1^{(\infty)}, \nu_2^{(\infty)});\ \omega_1^{(\infty)}, \omega_2^{(\infty)}\Big) + \frac{1+x}{3}$$

Now $\mathcal{D}(r, \nu_1^{(\infty)}, \nu_2^{(\infty)})$ is the domain around $x = \infty$. The effect of the loop $x \mapsto xe^{-2\pi i}$, $x \in \mathcal{D}(r; \nu_1^{(\infty)}, \nu_2^{(\infty)})$ is

$$(\nu_1^{(\infty)}, \nu_2^{(\infty)}) \mapsto (\nu_1^{(\infty)} + 2\nu_2^{(\infty)}, \nu_2^{(\infty)})$$

provided that also $x \in \mathcal{D}(r; \nu_1^{(\infty)} + 2\nu_1^{(\infty)}, \nu_2^{(\infty)})$.

# 9 Connection Problem

The analysis so far developed is only local. Two natural questions arise.

**Question 1).** This question makes sense in $\nu_2$ is not real. Suppose that $y(x)$ has the representation of Theorem 1 close to $x = 0$: $y(x) = \wp\big(\nu_1\omega_1 + \nu_2\omega_2 + v(x; \nu_1, \nu_2)\big) + (1+x)/3$ in $\mathcal{D}(r; \nu_1, \nu_2)$. We ask if $y(x)$ may have a representation $y(x) = \wp\big(\nu_1'\omega_1 + \nu_2\omega_2 + v(x; \nu_1, \nu_2 + 2N)\big) + (1+x)/3$ in $\mathcal{D}(r; \nu_1', \nu_2 + 2N)$ for some integer $N$. If this happens, we should be able to express $\nu_1'$ as a function of $\nu_1, \nu_2, N$.

**Question 2).** [Connection Problem]. May a given transcendent have three representations of Theorem 1 and Theorem 3 at $x = 0, 1, \infty$ at the same time? If this happens, which is the relation between the three sets $(\nu_1^{(0)}, \nu_2^{(0)})$, $(\nu_1^{(1)}, \nu_2^{(1)})$, $(\nu_1^{(\infty)}, \nu_2^{(\infty)})$?

The answer to this questions is positive, but it can not be achieved with the method so far used, which only allows to obtain local results. For example, suppose we want to solve question 2) for $y(x)$ represented as in Theorem 1. We can write

$$\begin{aligned} y(x) &= \wp(\nu_1^{(0)}\omega_1^{(0)}(x) + \nu_2^{(0)}\omega_2^{(0)}(x) + v^{(0)}(x)) + \frac{1+x}{3} \\ &\equiv \wp(\nu_2^{(0)}\omega_1^{(1)}(x) + \nu_1^{(0)}\omega_2^{(1)}(x) + v^{(0)}(x)) + \frac{1+x}{3} \end{aligned} \tag{72}$$

There also exists a transcendent

$$y(x) = \wp(\nu_1^{(1)}\omega_1^{(1)}(x) + \nu_2^{(1)}\omega_2^{(1)}(x) + v^{(1)}(x)) + \frac{1+x}{3} \tag{73}$$

where $v^{(1)}(x)$ is represented by Theorem 3. In particular, $v^{(1)}(x)$ is bounded as $x \to 1$. On the other hand, the function $v^{(0)}$ is only known for small $x$, and it may diverge as $x \to 1$. In the best hypothesis, we may suppose that after the rescaling

$$v^{(0)}(x) \mapsto v^{(0)}(x) - \delta\nu_1\ \omega_1^{(1)}(x) - \delta\nu_2\ \omega_2^{(1)}(x), \qquad \delta\nu_1, \delta\nu_2 \in \mathbf{C}$$

the new function $v^{(0)} - \delta\nu_1\omega_1^{(1)} - \delta\nu_2\omega_2^{(1)}$ vanishes as $x \to 1$ (this may happen, since the divergence of $\omega_2^{(1)}$, which is $\ln(1-x)$ as $x \to 1$, may cancel the divergence of $v^{(0)}(x)$ as $x \to 1$). Therefore, (72) is a transcendent (73) where

$$\nu_1^{(1)} = \nu_2^{(0)} + \delta\nu_1, \qquad \nu_2^{(1)} = \nu_1^{(0)} + \delta\nu_2$$

and

$$v^{(1)}(x) = v^{(0)} - \delta\nu_1\ \omega_1^{(1)} - \delta\nu_2\ \omega_2^{(1)}$$

Unfortunately, we are not able to say if the above rescaling is possible. And if it is, we do not know $\delta\nu_1$, $\delta\nu_2$.

The answer to questions 1) and 2) can be obtained by the method of isomonodromic deformations (which, on the other hand, has some limitations in providing the local behavior).

A simultaneous use of the elliptic representation and the isomonodromic deformation method can be found in our paper [11] for the special case $\beta = \gamma = 1 - 2\delta = 0$.

## 9.1 Picard Solutions

Before answering Questions 1 and 2 in general, we recall that there is one case when the local analysis in elliptic representation becomes global, namely when $v(x)=0$. This happens when $\alpha=\beta=\gamma=1-2\delta=0$. This case was already well known to Picard [30] and it was studied in [23]. The equation (11) reduces to the hyper-geometric equation $\mathcal{L}(u)=0$, and it has general solution $u(x)/2=\nu_1\omega_1^{(0)}(x)+\nu_2\omega_2^{(0)}(x)$. Therefore,

$$y(x)=\wp\left(\nu_1\omega_1^{(0)}(x)+\nu_2\omega_2^{(0)}(x);\omega_1^{(0},\omega_2^{(0)}\right)+\frac{1+x}{3}$$

Question 1) can be answered immediately. There is no need to prove Theorem 1 because we have no $v(x)$! The function $u(x)/2=\nu_1\omega_1^{(0)}(x)+\nu_2\omega_2^{(0)}(x)$ in the argument of $\wp$ is defined for any $x\neq 0,1,\infty$.

We obtain the critical behavior by Fourier-expanding

$$\wp\left(\nu_1\omega_1^{(0)}(x)+\nu_2\omega_2^{(0)}(x);\omega_1^{(0},\omega_2^{(0)}\right)=\wp\left(\nu_1\omega_1^{(0)}(x)+[\nu_2+2N]\omega_2^{(0)}(x);\omega_1^{(0},\omega_2^{(0)}\right)$$

For any fixed $N$, the expansion is performed if $|x|<1$ and

$$\left|\Im\left[\frac{\nu_1}{2}+\left(\frac{\nu_2}{2}+N\right)\tau(x)\right]\right|<\Im\tau(x)$$

Namely:

$$(\Re\nu_2+2+2N)\ln\frac{|x|}{16}+O(x)<\Im\nu_2\ \arg x+\pi\ \Im\nu_1<(\Re\nu_2-2+2N)\ln\frac{|x|}{16}+O(x) \qquad (74)$$

Note that this defines a domain $\mathcal{D}(r;\nu_1,\nu_2+2N)$ which contains the union of $\mathcal{D}_1(r<1;\nu_1,\nu_2+2N)$, $\mathcal{D}_2(r<1;\nu_1,\nu_2+2N)$, $\mathcal{D}_1(r<1;\nu_1,\nu_2+2[N+1])$, $\mathcal{D}_2(r<1;\nu_1,\nu_2+2[N+1])$ introduced in the proof of Theorem 1.

The critical behavior along

$$\arg x=\arg x_0+\frac{\Re\nu_2+2N-\mathcal{V}}{\Im\nu_2}\ln|x|,\qquad -2\leq\mathcal{V}\leq 2,\quad \Im\nu_2\neq 0$$

is:

For $0<\mathcal{V}<1$

$$y(x)=-\frac{1}{4}\left[\frac{e^{i\pi\nu_1}}{16^{\nu_2+2N-1}}\right]x^{\nu_2+2N}\ (1+O(x^{\nu_2+2N}))$$

For $1<\mathcal{V}<2$

$$y(x)=-\frac{1}{4}\left[\frac{e^{i\pi\nu_1}}{16^{\nu_2+2N-1}}\right]^{-1}x^{2-\nu_2-2N}\ (1+O(x^{2-\nu_2-2N}))$$

For $\mathcal{V}=1$

$$y(x)=(1+O(x))\ x\ \sin^2\left(i\frac{1-\nu_2-2N}{2}\ln\frac{x}{16}+\frac{\pi\nu_1}{2}\right)$$

For $\mathcal{V}=0$

$$y(x)=\left[\frac{x}{2}+\sin^{-2}\left(-i\frac{\nu_2+2N}{2}\ln\frac{x}{16}+\frac{\pi\nu_1}{2}-i\frac{\nu_2+2N}{2}\frac{F_1(x)}{F(x)}\right)\right]$$

For $\mathcal{V}=2$

$$y(x)=\left[\frac{x}{2}+\sin^{-2}\left(i\frac{2-\nu_2-2N}{2}\ln\frac{x}{16}+\frac{\pi\nu_1}{2}+i\frac{2-\nu_2-2N}{2}\frac{F_1(x)}{F(x)}\right)\right]$$

For $-1<\mathcal{V}<0$: it is like $1<\mathcal{V}<2$ with $N\mapsto N+1$.

For $-2<\mathcal{V}<-1$: it is like $0<\mathcal{V}<1$ with $N\mapsto N+1$.

For $\mathcal{V}=-1$: it is like $\mathcal{V}=1$ with $N\mapsto N+1$.

For $\mathcal{V}=-2$: it is like $\mathcal{V}=0$ with $N\mapsto N+1$.

We observe that the choice of $N$ is arbitrary, therefore the *same* transcendent has different critical behaviors on different domains (74) specified by different values of $N$. This answers Question 1.

**Remark:** Note that in the cases $\mathcal{V} = -2, 0, 2$, the $\sin^2(...)$ may vanish, therefore there may be (movable) poles. Actually, this happens because the domain now is bigger than that of the generic case since we did not impose that $\sin^2(...) \neq 0$, as we did in the proof of Theorem 1. This is also the reason why we have to keep $\frac{F_1(x)}{F(x)}$ in the argument of $\sin^2(...)$ in the Fourier expansion.

If $\Im\nu_2 = 0$, we choose the convention $0 \leq \nu_i < 2$. The critical behavior for $0 < \nu_2 < 1$ is the same of the case $\Im\nu_2 \neq 0$ with $N = 0$ and $0 < \mathcal{V} < 1$; for $1 < \nu_2 < 2$ it is the same of the case $\Im\nu_2 \neq 0$ with $N = 0$ and $1 < \mathcal{V} < 2$. Finally,

$$y(x) = (1 + O(x))\ x\ \sin^2\left(\frac{\pi\nu_1}{2}\right), \quad \text{if } \nu_2 = 1$$

$$y(x) = \left[\frac{x}{2} + \sin^{-2}\left(\frac{\pi\nu_1}{2}\right)\right](1 + O(x)), \quad \text{if } \nu_2 = 0, \quad \nu_1 \neq 0$$

Question 2) can be answered. Actually

$$\begin{aligned} y(x) &= \wp\left(\nu_1\omega_1^{(0)}(x) + \nu_2\omega_2^{(0)}(x); \omega_1^{(0}, \omega_2^{(0)}\right) + \frac{1+x}{3} \\ &= \wp\left(\nu_2\omega_1^{(1)}(x) + \nu_1\omega_2^{(1)}(x); \omega_1^{(0}, \omega_2^{(0)}\right) + \frac{1+x}{3} \\ &= \wp\left(\nu_1\omega_1^{(\infty)}(x) + (\nu_2 - \nu_1)\omega_2^{(\infty)}(x); \omega_1^{(0}, \omega_2^{(0)}\right) + \frac{1+x}{3} \end{aligned}$$

Namely, the couples $(\nu_1^{(i)}, \nu_2^{(i)})$, $i = 0, 1, \infty$, are:

$$(\nu_1, \nu_2), \quad (\nu_2, \nu_1), \quad (\nu_1, \nu_2 - \nu_1)$$

## 9.2 Solution of the Connection Problem in a Non-Generic Case

We first give the solutions of Questions 1) and 2) in the special case $\beta = \gamma = 1 - 2\delta = 0$ studied in [7] and in [11], because of its connection to 2-D topological field theory and to Frobenius Manifolds [4] [5]. In the next sub-section we consider the generic case. The Painlevé VI equation is the isomonodromic deformation equation of the fuchsian system [17]

$$\frac{dY}{dz} = \left(\frac{A_0(x)}{z} + \frac{A_1(x)}{z-1} + \frac{A_x(x)}{z-x}\right)\ Y \equiv A(z,x)\ Y \tag{75}$$

where $A_i(x)$ $(i = 0, 1, x)$ are $2 \times 2$ matrices. They depend on $x$ in such a way that the monodromy matrices at $z = 0, 1, x$, do not change for small deformations of $x$. Moreover,

$$A_0(x) + A_1(x) + A_x(x) = -\frac{1}{2}\begin{pmatrix} \theta_\infty & 0 \\ 0 & -\theta_\infty \end{pmatrix}$$

$$\text{the eigenvalues of } A_i(x) = \pm\frac{1}{2}\theta_i, \quad i = 0, 1, x$$

The constants $\theta_\nu$, $\nu = 0, 1, x, \infty$ are linked to the parameters of PVI:

$$\alpha = \frac{1}{2}(\theta_\infty - 1)^2, \quad \beta = -\frac{1}{2}\theta_0^2, \quad \gamma = \frac{1}{2}\theta_1^2, \quad \delta = \frac{1}{2}(1 - \theta_x^2)$$

A Painlevé transcendent is the solution $z = z(x)$ of $A(z,x)_{12} = 0$: actually $A_{12}(z,x) = \kappa(x)(z - y(x))/z(z-1)(z-x)$, where the function $\kappa(x)$ does not interests us now (see [17]). The monodromy matrices give a representation of the fundamental group of $\mathbf{P}^1 \backslash \{0, 1, x, \infty\}$. They are usually denoted by $M_\nu$, $\nu = 0, 1, x, \infty$ and they correspond to counterclockwise loops around $z = 0, 1, x, \infty$ respectively. We order the loops as in figure 2 (an arbitrary base point is chosen). The monodromy matrices are not independent, because

$$M_1 M_x M_0 = M_\infty$$

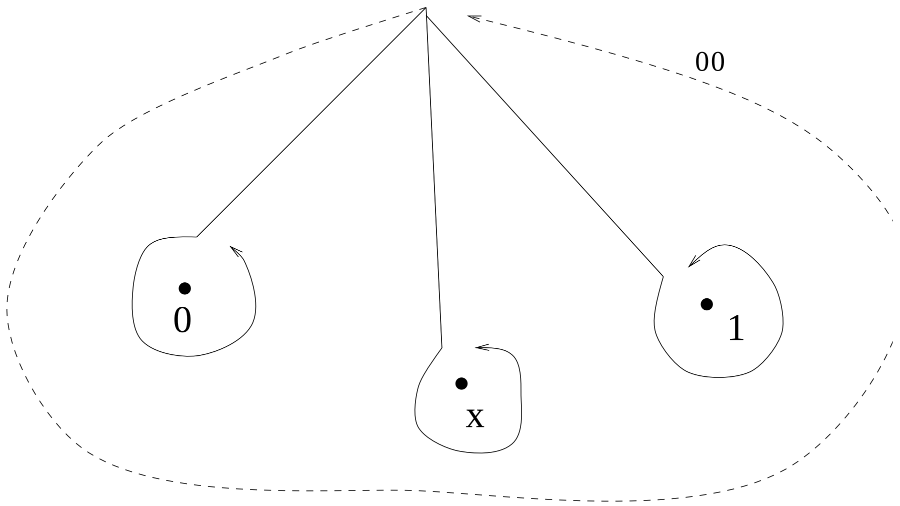


Figure 2: The order of the basis of loops of the Fuchsian system.

Moreover

$$M_\infty \text{ is similar to } e^{2\pi i\begin{pmatrix}-\frac{\theta_\infty}{2} & 0\\ 0 & \frac{\theta_\infty}{2}\end{pmatrix}} e^{2\pi i R}, \qquad R=\begin{cases} 0 & \text{if } \theta_\infty\notin \mathbf{Z}\\ \begin{pmatrix}0 & b\\ 0 & 0\end{pmatrix} & \text{if } 0<\theta_\infty\in\mathbf{Z}\\ \begin{pmatrix}0 & 0\\ b & 0\end{pmatrix} & \text{if } 0>\theta_\infty\in\mathbf{Z}\end{cases}$$

where $b\in\mathbf{C}$. Let us introduce the notation

$$2-x_0^2=\operatorname{tr}\, M_0M_x,\quad 2-x_1^2=\operatorname{tr}\, M_1M_x,\quad 2-x_\infty^2=\operatorname{tr}\, M_0M_1$$

Let us put $\alpha =: \frac{(2\mu-1)^2}{2}$ (namely $\theta_\infty=2\mu$, $\theta_0=\theta_x=\theta_1=0$). For any fixed $\mu$, it is proved in [7] that there is a one to one correspondence between Painlevé transcendents and triples of monodromy data $(x_0,x_1,x_\infty)$, defined up to the change of two signs, satisfying $x_i\neq\pm2$, $i=0,1,\infty$ and at most one $x_i=0$.

A transcendent in one to one correspondence to $(x_0,x_1,x_\infty)$ will be denoted $y=y(x;x_0,x_1,x_\infty)$. Let $\sigma$ be a complex number such that $2\cos(\pi\sigma)=\operatorname{tr}(M_0M_x)$, namely:

$$\cos\pi\sigma=1-\frac{x_0^2}{2},\qquad 0\le\Re\sigma\le1,\qquad \sigma\neq1$$

The critical behavior of $y=y(x;x_0,x_1,x_\infty)$ was obtained in [11] in the domains

$$\Re(\pm\sigma+2n)\ln|x|+\vartheta_2\ \Im(\pm\sigma)\le\Im(\pm\sigma)\arg x<[\Re(\pm\sigma+2n)-\tilde\sigma]\ \ln|x|+\vartheta_1\ \Im(\pm\sigma),\qquad |x|<\epsilon_n$$

where $n\in\mathbf{Z}$, $\epsilon_n<1$ is small (and decreases as $n$ increases), $0<\tilde\sigma<1$ is arbitrarily close to 1 and $\vartheta_1$, $\vartheta_2$ are constants, $\Im(\pm\sigma)\ \vartheta_1$ decreasing as $-\ln|n|$. We call the domain $D(\epsilon_n;\pm\sigma+2n)$. More simply, note that

$$D(\epsilon;\sigma)=\{x\in\widetilde{\mathbf{C}_0}\text{ s.t. } |x|<\epsilon,\ \ e^{-\vartheta_1\Im\sigma}|x|^{\tilde\sigma}\le|x^\sigma|\le e^{-\vartheta_2\Im\sigma},\quad 0<\tilde\sigma<1\}.$$

where $\widetilde{\mathbf{C}_0}$ is the universal covering of $\mathbf{C}\backslash\{0\}$. The critical behavior in the domains $D(\epsilon_n;\pm\sigma+2n)$ is derived in [11]:

$$y(x;x_0,x_1,x_\infty)=a(\pm\sigma+2n;x_0,x_1,x_\infty)\ x^{1-[\pm\sigma+2n]}(1+O(x^{\pm\sigma+2n}+x^{1-[\pm\sigma+2n]}))\tag{76}$$

where

$$a(\sigma;x_0,x_1,x_\infty)=\frac{i16^\sigma\Gamma\left(\frac{\sigma+1}{2}\right)^4}{8\sin(\pi\sigma)\Gamma\left(1-\mu+\frac{\sigma}{2}\right)^2\Gamma\left(\mu+\frac{\sigma}{2}\right)^2}\left[2(1+e^{-i\pi\sigma})-\right.$$

$$-f(x_0,x_1,x_\infty)(x_\infty^2+e^{-i\pi\sigma}x_1^2)\big]\, f(x_0,x_1,x_\infty) \tag{77}$$

and

$$f(x_0,x_1,x_\infty):=\frac{4-x_0^2}{x_1^2+x_\infty^2-x_0x_1x_\infty}$$

Conversely, a transcendent with behavior $ax^{1-\sigma}(1+$ higher orders$)$, for given $\sigma$ and $a$, is *unique* and coincides with $y(x;x_0,x_1,x_\infty)$ for

$$x_0=2\sin\left(\frac{\pi}{2}\sigma\right)$$

$$x_1=i\left(\frac{\sqrt{a}}{f(\sigma,\mu)G(\sigma,\mu)}-\frac{G(\sigma,\mu)}{\sqrt{a}}\right)$$

$$x_\infty=\frac{e^{\frac{i\pi\sigma}{2}}\sqrt{a}}{f(\sigma,\mu)G(\sigma,\mu)}+\frac{G(\sigma,\mu)}{e^{\frac{i\pi\sigma}{2}}\sqrt{a}}$$

where

$$f(\sigma,\mu):=\frac{2\cos^2\left(\frac{\pi}{2}\sigma\right)}{\cos(\pi\sigma)-\cos(2\pi\mu)}\equiv f(x_0,x_1,x_\infty),\qquad G(\sigma,\mu):=\frac{4^\sigma\Gamma\left(\frac{\sigma+1}{2}\right)^2}{2\Gamma\left(1-\mu+\frac{\sigma}{2}\right)\Gamma\left(\mu+\frac{\sigma}{2}\right)}$$

We refer to [11] for details.

We observe that in the special case $\beta=\gamma=1-2\delta=0$ there is only $\frac{\partial}{\partial u}\wp(u/2)$ in the r.h.s. of (11). Therefore, $\varepsilon_1=\varepsilon_2=0$ in the proof of Theorems 1 and 3. As a consequence, not only at $x=0$, but also at $x=1,\infty$ the domains $\mathcal{D}_1$, $\mathcal{D}_2$ are substituted by one domain $\mathcal{D}$, where only one function $v(x)$ is defined. More precisely, at $x=0$ the result of section 6 applies: *For any $N\in\mathbf{Z}$, and for any complex $\nu_1^{(0)},\nu_2^{(0)}$, such that $0<\nu_2^{(0)}<2$ if $\nu_2^{(0)}$ is real, there exists a transcendent $y=\wp\big(\nu_1^{(0)}\omega_1^{(0)}+\nu_2^{(0)}\omega_2^{(0)}+v^{(0)}\big)+\frac{1+x}{3}$ such that $v^{(0)}(x)$ is holomorphic in*

$$\mathcal{D}(r_N;\nu_1^{(0)},\nu_2^{(0)}+2N):=\left\{x\in\tilde{\mathbf{C}}_0\ \mid\ |x|<r_N,\ \ \left|e^{-i\pi\nu_1^{(0)}}\left(\frac{x}{16}\right)^{2-\nu_2^{(0)}-2N}\right|<r,\ \ \left|e^{i\pi\nu_1^{(0)}}\left(\frac{x}{16}\right)^{\nu_2^{(0)}+N}\right|<r\right\}$$

*where it has the convergent expansion*

$$v(x;\nu_1^{(0)},\nu_2^{(0)}+2N)=\sum_{n\geq1}a_nx^n+\sum_{n\geq0,m\geq1}b_{nm}x^n\left[e^{-i\pi\nu_1^{(0)}}\left(\frac{x}{16}\right)^{2-\nu_2^{(0)}-2N}\right]^m+$$

$$+\sum_{n\geq0,m\geq1}c_{nm}x^n\left[e^{i\pi\nu_1^{(0)}}\left(\frac{x}{16}\right)^{\nu_2^{(0)}+2N}\right]^m$$

At $x=1$. *For any $N\in\mathbf{Z}$, and for any complex $\nu_1^{(1)},\nu_2^{(1)}$, such that $0<\nu_2^{(1)}<2$ if $\nu_2^{(1)}$ is real, there exists a transcendent $y=\wp\big(\nu_1^{(1)}\omega_1^{(1)}+\nu_2^{(1)}\omega_2^{(1)}+v^{(1)}\big)+\frac{1+x}{3}$ such that $v^{(1)}(x)$ is holomorphic in*

$$\mathcal{D}(r_N;\nu_1^{(1)},\nu_2^{(1)}+2N):=\left\{x\in\widetilde{\mathbf{P}_1^1}\ \ \text{such that}\ \ |1-x|<r_N,\ \ \left|e^{i\pi\nu_1^{(1)}}\left(\frac{1-x}{16}\right)^{2-\nu_2^{(1)}-2N}\right|<r_N,\right.$$

$$\left.\left|e^{-i\pi\nu_1^{(1)}}\left(\frac{1-x}{16}\right)^{\nu_2^{(1)}+2N}\right|<r_N\right\}$$

*where it has a convergent expansion*

$$v^{(1)}(x;\nu_1^{(1)},\nu_2^{(1)}+2N)=\sum_{n\geq1}a_n(1-x)^n+\sum_{n\geq0,m\geq1}b_{nm}(1-x)^n\left[e^{i\pi\nu_1^{(1)}}\left(\frac{1-x}{16}\right)^{2-\nu_2^{(1)}-2N}\right]^m+$$

$$+\sum_{n\geq 0, m\geq 1} c_{nm}(1-x)^n \left[e^{-i\pi\nu_1^{(1)}}\left(\frac{1-x}{16}\right)^{\nu_2^{(1)}+2N}\right]^m$$

At $x=\infty$. *For any $N\in\mathbf{Z}$, and for any complex $\nu_1^{(\infty)},\nu_2^{(\infty)}$, such that $0<\nu_2^{(\infty)}<2$ if $\nu_2^{(\infty)}$ is real, there exists a transcendent $y=\wp(\nu_1^{(\infty)}\omega_1^{(\infty)}+\nu_2^{(\infty)}\omega_2^{(\infty)}+v^{(\infty)})+\frac{1+x}{3}$ such that $v^{(\infty)}(x)$ is holomorphic in*

$$\mathcal{D}(r_N;\nu_1^{(\infty)},\nu_2^{(\infty)}+2N):=\left\{x\in\widetilde{\mathbf{P}_\infty}\ \ \textit{such that}\ \ |x^{-1}|<r_N,\ \left|e^{-i\pi\nu_1^{(\infty)}}\left(\frac{16}{x}\right)^{2-\nu_2^{(\infty)}-2N}\right|<r_N,\right.$$

$$\left.\left|e^{i\pi\nu_1^{(\infty)}}\left(\frac{16}{x}\right)^{\nu_2^{(\infty)}+2N}\right|<r_N\right\}$$

*where it has a convergent expansion*

$$x^{\frac{1}{2}}v^{(\infty)}(x;\nu_1^{(\infty)},\nu_2^{(\infty)}+2N)=\sum_{n\geq 1}a^{(1)}\left(\frac{1}{x}\right)^n+\sum_{n\geq 0,m\geq 1}b_{nm}\left(\frac{1}{x}\right)^n\left[e^{-i\pi\nu_1^{(\infty)}}\left(\frac{16}{x}\right)^{2-\nu_2^{(\infty)}-2N}\right]^m+$$

$$+\sum_{n\geq 0,m\geq 1}c^{(1)}\left(\frac{1}{x}\right)^n\left[e^{i\pi\nu_1^{(\infty)}}\left(\frac{16}{x}\right)^{\nu_2^{(\infty)}+2N}\right]^m$$

Note that we have used the same notations for the coefficients $a_n$, $b_{nm}$, $c_{nm}$ both at $x=0,1$ and $x=\infty$. This is for simplicity of notations, but they are different!

We recall that $y(x)=\wp(\nu_1^{(0)}\omega_1^{(0)}+\nu_2^{(0)}\omega_2^{(0)}+v^{(0)}(x;\nu_1^{(0)},\nu_2^{(0)}+2N))+\frac{1+x}{3}$ has behavior

$$y(x)=-\frac{1}{4}\left[\frac{e^{i\pi\nu_1}}{16^{\nu_2+2N-1}}\right]x^{\nu_2+2N}(1+\text{ higher orders}),\quad 0<\mathcal{V}<1$$

$$=-\frac{1}{4}\left[\frac{e^{i\pi\nu_1}}{16^{\nu_2+2N-1}}\right]^{-1}x^{2-\nu_2-2N}(1+\text{ higher orders}),\quad 1<\mathcal{V}<2$$

On the other hand, a transcendent $y(x;x_0,x_1,x_\infty)$ has behavior (76):

$$y(x;x_0,x_1,x_\infty)=ax^{1-[\pm\sigma+2n]}(1+\text{ higher orders})$$

Due to uniqueness of this behavior, the transcendent in the elliptic representation, for given $\nu_1,\nu_2,N$, must coincide with some $y(x;x_0,x_1,x_\infty)$.

Let us choose the convention

$$0\leq\Re\sigma\leq 1,\quad 0\leq\Re\nu_2<2$$

We take $N=n=0$. A necessary condition for the transcendents to coincide is that

$$D(\epsilon_0;\sigma)\cap\mathcal{D}(r_0;\nu_1,\nu_2)\neq\emptyset$$

This occurs for:

$$(1):\quad \nu_2=1-\sigma,\quad 0\leq\Re\nu_2\leq 1$$

$$(2):\quad \nu_2=1+\sigma,\quad 1\leq\Re\nu_2<2$$

(recall that $\sigma\neq 1$ and $\nu_2\neq 0,2$). The transcendents coincide if and only if

$$a=-\frac{1}{4}\left[\frac{e^{i\pi\nu_1}}{16^{\nu_2-1}}\right]\ \text{ in case (1)}$$

$$a=-\frac{1}{4}\left[\frac{e^{i\pi\nu_1}}{16^{\nu_2-1}}\right]^{-1}\ \text{ in case (2)}$$

The transformation $\sigma \mapsto -\sigma$ is $\sigma = 1-\nu_2 \mapsto \nu_2 - 1$ in case (1), and $\sigma = \nu_2 - 1 \mapsto 1-\nu_2$ in case (2). Namely, in case (1) the behavior of $y(x;x_0,x_1,x_\infty)$ is

$$y(x;x_0,x_1,x_\infty) = a(\sigma;x_0,x_1,x_\infty)x^{1-\sigma}(1+ \text{ higher orders}), \quad \text{in } D(\epsilon_0;\sigma)$$

and

$$y(x;x_0,x_1,x_\infty) = a(-\sigma;x_0,x_1,x_\infty)x^{1-[-\sigma]}(1+ \text{ higher orders}), \quad \text{in } D(\epsilon_0;-\sigma)$$

This corresponds to the behaviors obtained in elliptic representation:

$$y(x) = -\frac{1}{4}\left[\frac{e^{i\pi\nu_1}}{16^{\nu_2-1}}\right] x^{\nu_2}(1+ \text{ higher orders}), \quad 0<\mathcal{V}<1$$

and

$$y(x) = -\frac{1}{4}\left[\frac{e^{i\pi\nu_1}}{16^{\nu_2-1}}\right]^{-1} x^{2-\nu_2}(1+ \text{ higher orders}), \quad 1<\mathcal{V}<2$$

respectively, with

$$e^{i\pi\nu_1} = -\frac{i\Gamma^4\left(1-\frac{\nu_2}{2}\right)}{2\sin(\pi\nu_2)\Gamma^2\left(\frac{3}{2}-\mu-\frac{\nu_2}{2}\right)\Gamma^2\left(\frac{1}{2}+\mu-\frac{\nu_2}{2}\right)}\left[2(1-e^{i\pi\nu_2})- \right.$$
$$\left. -f(x_0,x_1,x_\infty)(x_\infty^2-e^{i\pi\nu_2}x_1^2)\right] f(x_0,x_1,x_\infty) \tag{78}$$

In case (2), the behavior of $y(x;x_0,x_1,x_\infty)$ is still

$$y(x;x_0,x_1,x_\infty) = a(\sigma;x_0,x_1,x_\infty)x^{1-\sigma}(1+ \text{ higher orders}), \quad \text{in } D(\epsilon_0;\sigma)$$

and

$$y(x;x_0,x_1,x_\infty) = a(-\sigma;x_0,x_1,x_\infty)x^{1-[-\sigma]}(1+ \text{ higher orders}), \quad \text{in } D(\epsilon_0;-\sigma)$$

This corresponds to

$$y(x) = -\frac{1}{4}\left[\frac{e^{i\pi\nu_1}}{16^{\nu_2-1}}\right]^{-1} x^{2-\nu_2}(1+ \text{ higher orders}), \quad 1<\mathcal{V}<2$$

and

$$y(x) = -\frac{1}{4}\left[\frac{e^{i\pi\nu_1}}{16^{\nu_2-1}}\right] x^{\nu_2}(1+ \text{ higher orders}), \quad 0<\mathcal{V}<1$$

respectively, with

$$e^{-i\pi\nu_1} = \frac{i\Gamma^4\left(\frac{\nu_2}{2}\right)}{2\sin(\pi\nu_2)\Gamma^2\left(\frac{1}{2}-\mu+\frac{\nu_2}{2}\right)\Gamma^2\left(-\frac{1}{2}+\mu+\frac{\nu_2}{2}\right)}\left[2(1-e^{-i\pi\nu_2})- \right.$$
$$\left. -f(x_0,x_1,x_\infty)(x_\infty^2-e^{-i\pi\nu_2}x_1^2)\right] f(x_0,x_1,x_\infty \tag{79}$$

In particular, in both cases we have $a(\sigma;x_0,x_1,x_\infty)^{-1} = 16\ a(-\sigma;x_0,x_1,x_\infty)$.

Let us now study the effect of the shift $\sigma \mapsto \sigma+2n$ on the elliptic representation of $y(x;x_0,x_1,x_\infty)$. $y(x;x_0,x_1,x_\infty)$ has critical behaviors on the domains $D(\epsilon_n;\sigma+2n)$, with exponents $\sigma+2n$ and coefficients $a = a(\sigma+2n;x_0,x_1,x_\infty)$. The shift corresponds to $\nu_2 \mapsto \nu_2 - 2n$ in case (1), and $\nu_2 \mapsto \nu_2+2n$ in case (2). Using the properties of the $\Gamma$-function and (78),(79) we obtain, both for case (1) and case (2):

$$e^{i\pi\nu_1}\Big|_{\nu_2+2n} = e^{i\pi\nu_1}\Big|_{\nu_2} K(\nu_2,n)$$

where

$$K(\nu_2,n) := \begin{cases} 1, & n=0 \\ \Pi_{k=1}^{n}\frac{[\nu_2-1+2\mu+2(k-1)]^2[\nu_2-1-2\mu+2k]^2}{(\nu_2+2k-1)^4}, & n>0 \\ \Pi_{k=1}^{|n|}\frac{(\nu_2-2k)^4}{[\nu_2-1+2\mu-2k]^2[\nu_2-1-2\mu-2(k-1)]^2}, & n<0 \end{cases}$$

Thus

$$\nu_1\Big|_{\nu_2+2n} = \nu_1\Big|_{\nu_2} - \frac{i}{\pi}\ln K(\nu_2,n)$$

In this way we have answered to Question 1): the representations

$$y(x) = \wp\left(\left[\nu_1 - \frac{i}{\pi}\ln K(\nu_2,N)\right]\,\omega_1 + \nu_2\,\omega_2 + v\big(x;\nu_1 - \frac{i}{\pi}\ln K(\nu_2,N),\nu_2+2N\big)\right) + \frac{1+x}{3}$$

define the same transcendent on different domains $\mathcal{D}\left(r_N;\nu_1 - \frac{i}{\pi}\ln K(\nu_2,N),\nu_2+2N\right)$.

**Remark:** We now know the elliptic representation and the critical behavior of the transcendent $y(x;x_0,x_1,x_\infty)$ on the union of the domains $\mathcal{D}(r_N;\nu_1-\frac{i}{\pi}\ln K(\nu_2,N),\nu_2+2N)$, $n\in\mathbf{Z}$ (for $\Im\nu_2\neq 0$). However, some regions in the $(\ln|x|,\Im\nu_2\arg x)$-plane, for $x$ close to 0, are not included in the union. In these regions the transcendent may have movable poles.

Question 2) has positive answer. In [7],[11] the connection problem was solved by showing that a given transcendent $y(x;x_0,x_1,x_\infty)$ has critical behaviors:

$$y(x;x_0,x_1,x_\infty) = a^{(0)}x^{1-\sigma^{(0)}}(1+\text{ higher orders in } x),\quad x\to 0 \tag{80}$$

$$= 1 - a^{(1)}(1-x)^{1-\sigma^{(1)}}(1+\text{ higher orders in } (1-x)),\quad x\to 1 \tag{81}$$

$$= a^{(\infty)}x^{\sigma^{(\infty)}}(1+\text{ higher orders in } x^{-1}),\quad x\to\infty \tag{82}$$

where

$$\cos(\pi\sigma^{(i)}) = 1-\frac{x_i^2}{2},\quad 0\le\Re\sigma^{(i)}\le 1,\quad \sigma^{(i)}\neq 1,\quad i=0,1,\infty$$

The parameters $a^{(0)}$, $\sigma^{(0)}$ stand for $a,\sigma$ used before. $a^{(1)},a^{(\infty)}$ are obtained by the formula (77) with the substitutions $(x_0,x_1,x_\infty)\mapsto(x_1,x_0,x_0x_1-x_\infty)$, $\sigma^{(0)}\mapsto\sigma^{(1)}$ and $(x_0,x_1,x_\infty)\mapsto(x_\infty,-x_1,x_0-x_1x_\infty)$, $\sigma^{(0)}\mapsto\sigma^{(\infty)}$ respectively. The behavior (82) holds in a domain

$$D(M;\sigma^{(\infty)}) := \{x\in\widetilde{\mathbf{P}^1\backslash\{\infty\}}\text{ s.t. } |x|>M,\ e^{-\vartheta_1\Im\sigma^{(\infty)}}|x|^{-\tilde{\sigma}}\le|x^{-\sigma^{(\infty)}}|\le e^{-\vartheta_2\Im\sigma^{(\infty)}}\}$$

where $M>0$ is sufficiently big; the behavior (81) holds in

$$D(\epsilon;\sigma^{(1)}) := \{x\in\widetilde{\mathbf{C}\backslash\{1\}}\text{ s.t. } |1-x|<\epsilon,\ \ e^{-\vartheta_1\Im\sigma}|1-x|^{\tilde{\sigma}}\le|(1-x)^{\sigma^{(1)}}|\le e^{-\vartheta_2\Im\sigma}\}$$

At $x=1$, $x=\infty$ we repeat the analysis we did above for $x=0$, identifying $y(x;x_0,x_1,x_\infty)$ with a transcendent in elliptic representation, because they have the same critical behaviors. Thus, we conclude that the transcendent $y(x;x_0,x_1,x_\infty)$ has a representation

$$y(x) = \wp(\nu_1^{(0)}\omega_1^{(0)}+\nu_2^{(0)}\omega_2^{(0)}+v^{(0)})+\frac{1+x}{3}$$

at $x=0$; it has a representation

$$y(x) = \wp(\nu_1^{(1)}\omega_1^{(1)}+\nu_2^{(1)}\omega_2^{(1)}+v^{(1)})+\frac{1+x}{3}$$

at $x=1$ and it has a representation

$$y(x) = \wp(\nu_1^{(\infty)}\omega_1^{(\infty)}+\nu_2^{(\infty)}\omega_2^{(\infty)}+v^{(\infty)})+\frac{1+x}{3}$$

with parameters $\nu_2^{(i)}$ obtained from

$$\cos\pi\nu_2^{(i)} = \frac{x_i^2}{2}-1,\quad 0\le\Re\nu_2^{(i)}\le 1,\ (\nu_2^{(i)}\neq 0),\quad\text{or}\ \ 1\le\Re\nu_2^{(i)}<2\qquad i=0,1,\infty$$

The parameter $\nu_1^{(0)}$ is obtained from (78) for the choice $0\le\Re\nu_2^{(0)}\le 1$ and from (79) for the choice $1\le\Re\nu_2^{(0)}<2$. Moreover, $\exp\{\mp i\pi\nu_1^{(1)}\}$, $\exp\{\pm i\pi\nu_1^{(\infty)}\}$ are given by formulae analogous to (78), (79) with the substitutions $(x_0,x_1,x_\infty)\mapsto(x_1,x_0,x_0x_1-x_\infty)$, $\nu_2^{(0)}\mapsto\nu_2^{(1)}$ and $(x_0,x_1,x_\infty)\mapsto(x_\infty,-x_1,x_0-x_1x_\infty)$, $\nu_2^{(0)}\mapsto\nu_2^{(\infty)}$ respectively.

Conversely, given

$$y(x)=\wp\big(\nu_1\omega_1^{(0)}(x)+\nu_2\omega_2^{(0)}(x)+v^{(0)}(x;\nu_1,\nu_2)\big)+\frac{1+x}{3},\qquad \text{at } x=0 \tag{83}$$

it coincides with $y(x;\ x_0,x_1,x_\infty)$, with the following monodromy data.
If $0\leq\Re\nu_2\leq 1$:

$$x_0=2\cos\left(\frac{\pi}{2}\nu_2\right)$$

$$x_1=\left[\frac{4^{-\nu_2}\ 2\ e^{i\frac{\pi}{2}\nu_1}}{f(\nu_2,\mu)G(\nu_2,\mu)}+\frac{G(\nu_2,\mu)}{4^{-\nu_2}\ 2\ e^{i\frac{\pi}{2}\nu_1}}\right]$$

$$x_\infty=\left[\frac{4^{-\nu_2}\ 2\ e^{i\frac{\pi}{2}(\nu_1-\nu_2)}}{f(\nu_2,\mu)G(\nu_2,\mu)}+\frac{G(\nu_2,\mu)}{4^{-\nu_2}\ 2\ e^{i\frac{\pi}{2}(\nu_1-\nu_2)}}\right]$$

where

$$f(\nu_2,\mu)=-\frac{2\sin^2\left(\frac{\pi}{2}\nu_2\right)}{\cos(\pi\nu_2)+\cos(2\pi\mu)},\quad G(\nu_2,\mu)=4^{-\nu_2}\ 2\ \frac{\Gamma\left(1-\frac{\nu_2}{2}\right)^2}{\Gamma\left(\frac{3}{2}-\mu-\frac{\nu_2}{2}\right)\Gamma\left(\frac{1}{2}+\mu-\frac{\nu_2}{2}\right)}$$

If $1\leq\Re\nu_2<2$:

$$x_0=2\cos\left(\frac{\pi}{2}\nu_2\right)$$

$$x_1=\left[\frac{e^{-i\frac{\pi}{2}\nu_1}}{4^{1-\nu_2}\ 2\ f(\nu_2,\mu)G_1(\nu_2,\mu)}+\frac{4^{1-\nu_2}\ 2\ G_1(\nu_2,\mu)}{e^{-i\frac{\pi}{2}\nu_1}}\right]$$

$$x_\infty=\left[\frac{e^{i\frac{\pi}{2}(\nu_2-\nu_1)}}{4^{1-\nu_2}\ 2\ f(\nu_2,\mu)G_1(\nu_2,\mu)}+\frac{4^{1-\nu_2}\ 2\ G_1(\nu_2,\mu)}{e^{i\frac{\pi}{2}(\nu_2-\nu_1)}}\right]$$

where

$$G_1(\nu_2,\mu)=\frac{1}{4^{1-\nu_2}\ 2}\frac{\Gamma\left(\frac{\nu_2}{2}\right)^2}{\Gamma\left(\frac{1}{2}-\mu+\frac{\nu_2}{2}\right)\Gamma\left(-\frac{1}{2}+\mu+\frac{\nu_2}{2}\right)}$$

After computing the monodromy data, we can write the elliptic representations of $y(x;x_0,x_1,x_\infty)$ at $x=1$ and $x=\infty$. These are the elliptic representations at $x=1$, $x=\infty$ of (83). Thus, we have solved the connection problem for (83).

**Remark:** There are some special cases, studied in [11], where the form of the coefficient $a(\sigma;x_0,x_1,x_\infty)$ is slightly different from (77). Namely

I) $x_0=0$.

$$\sigma=0,\qquad a=\frac{x_\infty^2}{x_1^2+x_\infty^2}.$$

provided that $x_1\neq 0$ and $x_\infty\neq 0$.

II) $x_0^2=4\sin^2(\pi\mu)$. Then $x_\infty^2=-x_1^2\ \exp(\pm 2\pi i\mu)$ . There are four sub-cases:

II1) $x_\infty^2=-x_1^2e^{-2\pi i\mu}$

$$\sigma=2\mu+2m,\qquad a=-\frac{1}{4x_1^2}\frac{16^{2\mu+2m}\Gamma(\mu+m+\frac{1}{2})^4}{\Gamma(m+1)^2\Gamma(2\mu+m)^2},\quad m=0,1,2,...$$

II2) $x_\infty^2=-x_1^2e^{2\pi i\mu}$

$$\sigma=2\mu+2m,\qquad m=-1,-2,-3,...$$

$$a=-\frac{\cos^4(\pi\mu)}{4\pi^4}16^{2\mu+2m}\Gamma(\mu+m+\frac{1}{2})^4\Gamma(-2\mu-m+1)^2\Gamma(-m)^2\ x_1^2$$

II3) $x_\infty^2=-x_1^2e^{2\pi i\mu}$

$$\sigma=-2\mu+2m,\qquad a=-\frac{1}{4x_1^2}\frac{16^{-2\mu+2m}\Gamma(-\mu+m+\frac{1}{2})^4}{\Gamma(m-2\mu+1)^2\Gamma(m)^2},,\qquad m=1,2,3,...$$

II4) $x_\infty^2 = -x_1^2 e^{-2\pi i\mu}$

$$\sigma = -2\mu + 2m, \qquad m = 0, -1, -2, -3, ...$$

$$a = -\frac{\cos^4(\pi\mu)}{4\pi^4} 16^{-2\mu+2m} \Gamma(-\mu + m + \frac{1}{2})^4 \Gamma(2\mu - m)^2 \Gamma(1-m)^2 \; x_1^2$$

In any case, $y(x) = \wp(\nu_1\omega_1 + \nu_2\omega_2 + v) + \frac{1+x}{3}$ coincides with $y(x; x_0, x_1, x_\infty)$, with parameters

$$\nu_2 = 1 - \sigma, \quad e^{i\pi\nu_1} = -4a(\sigma)\; 16^{-\sigma} \quad \text{if } 0 \le \Re\nu_2 \le 1$$

$$\nu_2 = 1 + \sigma, \quad e^{-i\pi\nu_1} = -4a(\sigma)\; 16^{-\sigma} \quad \text{if } 1 \le \Re\nu_2 \le 2$$

**Remark:** We observed that there is a one to one correspondence between Painlevé transcendents and triples of monodromy data $(x_0, x_1, x_\infty)$, defined up to the change of two signs, satisfying $x_i \neq \pm 2$, $i = 0, 1, \infty$ (i.e. $\sigma^{(i)} \neq 1$) and at most one $x_i = 0$. For other monodromy matrices the reader can refer to [23]. Finally, if $x_i = \pm 2$ (namely the trace is $-2$) the problem of finding the critical behavior at the corresponding critical point $x = i$ is still open (except when *all the three* $x_i$ are $\pm 2$: in this case there is a one-parameter class of solutions called *Chazy solutions* in [23]).

## 9.3 Solution of the Connection Problem in the Generic Case

We solve the connection problem for the elliptic representation in the *generic case*:

$$\sigma,\ \theta_0,\ \theta_x,\ \theta_1,\ \theta_\infty \ \notin \mathbf{Z}; \qquad \frac{\pm\sigma \pm \theta_1 \pm \theta_\infty}{2},\ \frac{\pm\sigma \pm \theta_0 \pm \theta_x}{2} \ \notin \mathbf{Z}$$

In the Appendix it is shown that for *any* value of $\alpha$, $\beta$, $\gamma$, $\delta$, for any $a \neq 0$ and for any complex $\sigma \notin (-\infty, 0) \cup [1, +\infty)$ there exists a transcendent

$$y(x; \sigma, a) = a x^{1-\sigma}(1 + \text{ higher orders}), \quad x \to 0$$

for $x \in D(\epsilon; \sigma)$.

In the generic case we show in the Appendix, following [15], that there is a one to one correspondence between transcendents and the monodromy data $\theta_0, \theta_x, \theta_1, \theta_\infty, \mathrm{tr}(M_0 M_x), \mathrm{tr}(M_0 M_1), \mathrm{tr}(M_1 M_x)$ of the fuchsian system (75). For this reason we write

$$y(x) = y\big(x;\ \theta_0, \theta_x, \theta_1, \theta_\infty, \mathrm{tr}(M_0 M_x), \mathrm{tr}(M_0 M_1), \mathrm{tr}(M_1 M_x)\ \big) \tag{84}$$

Let us fix

$$0 \le \Re\sigma \le 1$$

In the Appendix we show that the transcendent (84) coincides with $y(x; \pm\sigma + 2n, a)$ in the domains $D(\epsilon_n; \pm\sigma + 2n)$ (note that only $0 < \sigma < 1$ is allowed if $\sigma$ is real, and in this case no translation $\pm\sigma + 2n$ is possible [$\sigma = 0$ is discussed in [15]]). The relation between $\sigma$, and the monodrmoy data is given by

$$2\cos(\pi\sigma) = \mathrm{tr}(M_0 M_x)$$

The coefficient $a$ is a more complicated function

$$a = a\big(\sigma; \theta_0, \theta_x, \theta_1, \theta_\infty, \mathrm{tr}(M_0 M_x), \mathrm{tr}(M_0 M_1), \mathrm{tr}(M_1 M_x)\big)$$

which can be computed explicitly in terms of trigonometric and $\Gamma$ functions of the arguments. We refer to the Appendix for more details. We note that $a$ depends on the monodromy data and on $\sigma$. Namely, it changes if we do $\sigma \mapsto \pm\sigma + 2n$ (being the monodromy data fixed). In the following, for brevity we write $a = a(\sigma)$, understanding the dependence on the monodromy data.

Conversely, for given $\theta_0$, $\theta_x$, $\theta_1$, $\theta_\infty$ and $y(x; \sigma, a)$ we can compute $\mathrm{tr}(M_0 M_x)$, $\mathrm{tr}(M_1 M_x)$, $\mathrm{tr}(M_0 M_1)$ as functions of $a, \sigma, \theta_\nu$ ($\nu = 0, x, 1, \infty$) by the formulae of the Appendix (in particular $\mathrm{tr}(M_0 M x) = 2\cos(\pi\sigma)$). Therefore,

$$y(x; \sigma, a) = y\big(x; \theta_0, \theta_x, \theta_1, \theta_\infty, \mathrm{tr}(M_0 M_x), \mathrm{tr}(M_0 M_1), \mathrm{tr}(M_1 M_x)\big)$$

We claim that

**Proposition A1:** *Let $\sigma \not\in (-\infty,0]\cup[1,+\infty)$ and $a\neq 0$. Let $y(x)$ be a solution of $PVI$ in the generic case such that $y(x)=ax^{1-\sigma}$ (1+higher order terms) as $x\to 0$ in an open domain contained in $D(\epsilon;\sigma)$. Then, $y(x)$ coincides with $y(x;\sigma,a)$.*

The proposition is proved in the Appendix and it allows us to identify the transcendents $y(x;\sigma,a)$ above with the transcendents in elliptic representation. We fix the range of $\nu_2$ and $\sigma$ by

$$0\leq \Re\nu_2<2, \quad 0\leq \Re\sigma\leq 1$$

If $\Im\nu_2\neq 0$, we consider a transcendent

$$y(x)=\wp\Big(\nu_1\omega_1(x)+\nu_2\omega_2(x)+v(x;\nu_1,\nu_2);\ \omega_1(x),\omega_2(x)\Big)+\frac{1+x}{3}$$

on the domain $\mathcal{D}(r;\nu_1,\nu_2)$. For $x\to 0$ it has behavior

$$y(x)=-\frac{1}{4}\left[\frac{e^{i\pi\nu_1}}{16^{\nu_2-1}}\right]\ x^{\nu_2}\ (1+\text{ higher orders }).$$

Thus it coincides with

$$y(x;\sigma,a(\sigma))=a(\sigma)x^{1-\sigma}\ \ (1+\text{ higher orders })\quad \text{ if } 0\leq\Re\nu_2\leq 1 \tag{85}$$

$$y(x;-\sigma,a(-\sigma))=a(-\sigma)x^{1+\sigma}\ \ (1+\text{ higher orders })\quad \text{ if } 1\leq\Re\nu_2\leq 2 \tag{86}$$

where

$$\nu_2=1-\sigma,\quad e^{i\pi\nu_1}=-4a(\sigma)16^{-\sigma}\equiv -4a(1-\nu_2)\ 16^{\nu_2-1}\quad \text{ in case (85)}$$
$$\nu_2=1+\sigma,\quad e^{i\pi\nu_1}=-4a(-\sigma)16^{\sigma}\equiv -4a(1-\nu_2)\ 16^{\nu_2-1}\quad \text{ in case (86)}$$

We note that the corresponding domains $\mathcal{D}$ and $D$ have non-empty intersection, therefore Proposition A1 holds.

If $\nu_2$ is real, we consider a transcendent

$$y(x)=\wp\Big(\nu_1\omega_1(x)+\nu_2\omega_2(x)+v(x;\nu_1,\nu_2);\ \omega_1(x),\omega_2(x)\Big)+\frac{1+x}{3}$$
$$=-\frac{1}{4}\left[\frac{e^{i\pi\nu_1}}{16^{\nu_2-1}}\right]x^{\nu_2}\ \ (1+\text{ higher orders }),\quad \text{ if } 0<\nu_2<1 \tag{87}$$

or

$$y(x)=\wp\Big(\nu_1\omega_1(x)+\nu_2\omega_2(x)+v(x;-\nu_1,2-\nu_2);\ \omega_1(x),\omega_2(x)\Big)+\frac{1+x}{3}$$
$$=-\frac{1}{4}\left[\frac{e^{i\pi\nu_1}}{16^{\nu_2-1}}\right]^{-1}\ x^{2-\nu_2}\ \ (1+\text{ higher orders }),\quad \text{ if } 1<\nu_2<2 \tag{88}$$

In both cases, they coincide with

$$y(x;\sigma,a(\sigma))=a(\sigma)x^{1-\sigma}\ \ (1+\text{ higher orders }),\quad 0<\sigma<1$$

where

$$\nu_2=1-\sigma,\quad e^{i\pi\nu_1}=-4a(\sigma)16^{-\sigma}\equiv -4a(1-\nu_2)16^{\nu_2-1}\quad \text{ in case (87)}$$
$$\nu_2=1+\sigma,\quad e^{-i\pi\nu_1}=-4a(\sigma)16^{-\sigma}\equiv -4a(\nu_2-1)16^{1-\nu_2}\quad \text{ in case (88)}$$

The above identifications give $\nu_1$ and $\nu_2$ in terms of the monodromy data associated to $a,\ \sigma$.

From another point of view, we can start from $y(x;\sigma,a)$ ($0\leq\Re\sigma\leq 1$, $\sigma\neq 0,1$) and find its elliptic representation. This will be $y(x)=\wp\Big(\nu_1\omega_1+\nu_2\omega_2+v(x;\nu_1,\nu_2)\Big)+\frac{1+x}{3}$, $\quad 0\leq\Re\nu_2\leq 1\quad (\nu_2\neq 0,1)$ on the domain $\mathcal{D}(r;\nu_1,\nu_2)$ if $\Im\sigma\neq 0$, or $\mathcal{D}_0(r)$ if $\Im\sigma=0$, with

$$\nu_2=1-\sigma,\quad e^{i\pi\nu_1}=-4a(\sigma)\ 16^{-\sigma}$$

Equivalently - according to the Observation 2 in section 3 - $y(x;\sigma,a)$ coincides with $y(x)=\wp\Big(\tilde{\nu}_1\omega_1+\tilde{\nu}_2\omega_2(x)+v(x;-\tilde{\nu}_1,2-\tilde{\nu}_2)\Big)+\frac{1+x}{3}$, $\quad 1\leq \Re\tilde{\nu}_2\leq 2\quad (\tilde{\nu}_2\neq 1,2)$ represented in the domain $\mathcal{D}(r;-\tilde{\nu}_1,2-\tilde{\nu}_2)$, with

$$\tilde{\nu}_2=1+\sigma,\qquad e^{-i\pi\tilde{\nu}_1}=-4a(\sigma)\ 16^{-\sigma}$$

We are ready to solve the connection problem. In the Appendix we show, following [15], that

$$y\big(x;\ \theta_0,\theta_x,\theta_1,\theta_\infty,\mathrm{tr}(M_0M_x),\mathrm{tr}(M_0M_1),\mathrm{tr}(M_1M_x)\ \big)=$$

$$=a^{(0)}x^{1-\sigma^{(0)}}\ (1+\text{ higher orders}),\quad x\to 0$$

$$=1-a^{(1)}(1-x)^{1-\sigma^{(1)}}\ (1+\text{ higher orders}),\quad x\to 1 \tag{89}$$

$$=a^{(\infty)}x^{\sigma^{(\infty)}}\ (1+\text{ higher orders}),\quad x\to \infty \tag{90}$$

in suitable domains. The parameters are computed in terms of the monodromy data by

$$2\cos(\pi\sigma^{(0)})=\mathrm{tr}M_0M_x,\quad 2\cos(\pi\sigma^{(1)})=\mathrm{tr}M_1M_x,\quad 2\cos(\pi\sigma^{(\infty)})=\mathrm{tr}M_0M_1$$

$$0\leq\Re\sigma^{(i)}\leq 1,\quad \sigma^{(i)}\neq 0,1,\quad (i=0,1,\infty)$$

The coefficients $a^{(0)}$, $a^{(1)}$, $a^{(\infty)}$ are given in terms of the monodromy data by explicit formulae in the Appendix. Note that here we denote $\sigma$, $a$ introduced before with $\sigma^{(0)}$, $a^{(0)}$.

By uniqueness of critical behavior, the elliptic representation at $x=1$ of (89) is $y(x)=\wp(\nu_1^{(1)}\omega_1^{(1)}+\nu_2^{(1)}\omega_2^{(1)}+v^{(1)}(x;\nu_1^{(1)},\nu_2^{(1)}))+\frac{1+x}{3}$, with

$$\nu_2^{(1)}=1-\sigma^{(1)},\quad e^{-i\pi\nu_1^{(1)}}=-4a^{(1)}(1-\nu_2^{(1)})\ 16^{\nu_2^{(1)}-1}$$

At $x=\infty$ the representation of (90) is $y(x)=\wp(\nu_1^{(\infty)}\omega_1^{(\infty)}+\nu_2^{(\infty)}\omega_2^{(\infty)}+v^{(\infty)}(x;\nu_1^{(\infty)},\nu_2^{(\infty)}))+\frac{1+x}{3}$, with

$$\nu_2^{(\infty)}=1-\sigma^{(\infty)},\quad e^{i\pi\nu_1^{(\infty)}}=-4a^{(\infty)}(1-\nu_2^{(\infty)})\ 16^{\nu_2^{(\infty)}-1}$$

Therefore, the solution of the connection problem is a follows. We start from an elliptic representation in $\mathcal{D}(r;\nu_1^{(0)},\nu_2^{(0)})$ close to $x=0$:

$$y(x)=\wp(\nu_1^{(0)}\omega_1^{(0)}+\nu_2^{(0)}\omega_2^{(0)}+v^{(0)}(x;\nu_1^{(0)},\nu_2^{(0)}))+\frac{1+x}{3} \tag{91}$$

which is defined for $\Im\nu_2^{(0)}\neq 0$. We identify it with $y(x;\sigma^{(0)},a^{(0)}(\sigma^{(0)}))$ or $y(x;-\sigma^{(0)},a^{(0)}(-\sigma^{(0)}))$, according to the value of $\Re\nu_2^{(0)}$ as in (85) and (86). When $\Im\nu_2^{(0)}=0$ we start from

$$y(x)=\wp(\nu_1^{(0)}\omega_1^{(0)}+\nu_2^{(0)}\omega_2^{(0)}+v^{(0)}(x;\nu_1^{(0)},\nu_2^{(0)}))+\frac{1+x}{3}\qquad 0<\nu_2^{(0)}<1 \tag{92}$$

or

$$y(x)=\wp(\nu_1^{(0)}\omega_1^{(0)}+\nu_2^{(0)}\omega_2^{(0)}+v^{(0)}(x;-\nu_1^{(0)},2-\nu_2^{(0)}))+\frac{1+x}{3}\qquad 1<\nu_2^{(0)}<2 \tag{93}$$

which we identify with $y(x;\sigma,a)$ as in cases (87) and (88) respectively.

We compute monodromy data corresponding to $\sigma^{(0)}$, $a^{(0)}$, and therefore we compute $\sigma^{(1)}$, $a^{(1)}$ and $\sigma^{(\infty)}$, $a^{(\infty)}$ from the monodromy data. Finally, we write the elliptic representation of (89), (90). These are the elliptic representations of (91) at $x=1$ and $x=\infty$. This answers Question 2.

As for Question 1, we recall that $y\big(x;\ \theta_0,\theta_x,\theta_1,\theta_\infty,\mathrm{tr}(M_0M_x),\mathrm{tr}(M_0M_1)\mathrm{tr}(M_1M_x)\ \big)$ has representations $y(x;\pm\sigma+2n,a(\pm\sigma+2n))$, $n\in\mathbf{Z}$, on different domains. Let us consider the family of transcendents in elliptic representation (at $x=0$)

$$y(x)=\wp\big(\nu_1(N)\omega_1+\nu_2\omega_2+v(x;\nu_1(N),\nu_2+2N)\big)+\frac{1+x}{3}$$

$$\equiv\wp\big(\nu_1(N)\omega_1+[\nu_2+2N]\ \omega_2+v(x;\nu_1(N),\nu_2+2N)\big)+\frac{1+x}{3}$$

where $\nu_1 = \nu_1(N)$ means that $\nu_1$ changes with $N$. They are the elliptic representations of the same transcendent in different domains if and only if they coincide with

$$y(x; \sigma - 2N, a(\sigma - 2N)) \qquad \text{if } 0 \leq \Re\nu_2 \leq 1,$$

$$y(x; -\sigma - 2N, a(-\sigma - 2N)) \qquad \text{if } 1 \leq \Re\nu_2 \leq 2,$$

where $y(x; \sigma - 2N, a(\sigma - 2N))$, or $y(x; -\sigma - 2N, a(-\sigma - 2N))$, correspond to the same monodromy data (i.e. they are representations of $y\big(x;\ \theta_0, \theta_x, \theta_1, \theta_\infty, \mathrm{tr}(M_0M_x), \mathrm{tr}(M_0M_1)\mathrm{tr}(M_1M_x)\ \big)$). We can explicitly compute ( through the formulae of the Appendix) $\nu_1(N) = \text{function}(N, \nu_2, \nu_1(N = 0))$ from

$$e^{i\pi\nu_1(N)} = -4a(1 - \nu_2 - 2N)16^{\nu_2 + 2N - 1}$$

as we did in (85) and (86). In section 9.2 we gave the explicit expression for a non-generic case.

**Remark:** We have found the elliptic representations and the critical behaviors of

$$y\big(x;\ \theta_0, \theta_x, \theta_1, \theta_\infty, \mathrm{tr}(M_0M_x), \mathrm{tr}(M_0M_1)\mathrm{tr}(M_1M_x)\ \big)$$

in the union of the domains $\mathcal{D}(r_N; \nu_1(N), \nu_2+2N)$ ($\Im\nu_2 \neq 0$). However, some regions in the $(\ln|x|, \Im\nu_2 \arg x)$-plane, for $x$ close to 0, are not included in the union. In these regions there may be movable poles (see figure 1).

**Remark:** In the generic case, we prove in the Appendix the one-to-one correspondence between transcendents and monodromy data. However, the condition $\sigma^{(i)} \neq 1$, namely $\nu_2^{(i)} = 0, 2$, implies that we can not give the critical behaviors (and the elliptic representation) of

$$y\big(x;\ \theta_0, \theta_x, \theta_1, \theta_\infty, \mathrm{tr}(M_0M_x), \mathrm{tr}(M_0M_1)\mathrm{tr}(M_1M_x)\ \big)$$

at $x = 0$ for $\mathrm{tr}(M_0M_x) = \pm 2$, at $x = 1$ for $\mathrm{tr}(M_1M_x) = \pm 2$, at $x = \infty$ for $\mathrm{tr}(M_0M_1) = \pm 2$. We recall that in the case $\sigma^{(i)} = 0$ the connection problem is solved in [15].

# 10 APPENDIX

We give a brief account of the solution of the connection problem for the generic PVI, following [15], with a generalization of the domain of the independent variable $x$ where the critical behavior is computed. This generalization is necessary to identify the transcendent with the elliptic representation. Such a generalization is proved exactly as in [11], to which we refer for a detailed analysis of the non-generic case $\beta = \gamma = 1 - 2\delta = 0$.

The Painlevé VI equation is the isomonodromic deformation equation of the fuchsian system [17]

$$\frac{dY}{dz} = \left(\frac{A_0(x)}{z} + \frac{A_1(x)}{z-1} + \frac{A_x(x)}{z-x}\right)\ Y \equiv A(z,x)\ Y$$

where $A_i(x)$ $(i = 0, 1, x)$ are $2\times 2$ matrices with the following properties:

$$A_0(x) + A_1(x) + A_x(x) = -\frac{1}{2}\begin{pmatrix} \theta_\infty & 0 \\ 0 & -\theta_\infty \end{pmatrix}, \qquad \text{eigenvalues of } A_i(x) = \pm\frac{1}{2}\theta_i, \quad i = 0, 1, x \tag{94}$$

$$\alpha = \frac{1}{2}(\theta_\infty - 1)^2, \quad \beta = -\frac{1}{2}\theta_0^2, \quad \gamma = \frac{1}{2}\theta_1^2, \quad \delta = \frac{1}{2}(1 - \theta_x^2)$$

The dependence on $x$ is isomonodromic. A Painlevé transcendent is obtained from the $(1,2)$-component of $A(z,x)$: namely $A_{12}(z,x) = \kappa(x)(z - y(x))/z(z-1)(z-x)$, where $\kappa(x)$ is a function of a quadrature of $y(x)$ which is not important now. $y(x)$ is the solution $z = z(x)$ of $A_{12}(z,x) = 0$.

The Fuchsian system is obtained from the following Riemann-Hilbert problem. We fix $\theta_0, \theta_x, \theta_1, \theta_\infty$ and monodromy matrices $M_0$, $M_x$, $M_1$ corresponding to the loops in the basis of figure 2, with eigenvalues $\exp\{\pm i\pi\theta_i\}$, $i = 0, x, 1$ respectively. At infinity, the monodromy is $M_1M_xM_0$, and we require that this has eigenvalues $\exp\{\pm i\pi\theta_\infty\}$. Having assigned the monodromy matrices and their eigenvalues, there exist $2\times 2$ invertible matrices $C_0$, $C_x$, $C_1$, $C_\infty$ such that

$$M_i = C_i^{-1}\ \exp\left\{2\pi i \begin{pmatrix} \frac{\theta_i}{2} & 0 \\ 0 & -\frac{\theta_i}{2}\end{pmatrix}\right\}\ C_i, \quad i=0,x,1 \tag{95}$$

$$M_1 M_x M_0 = C_\infty^{-1}\ \exp\left\{2\pi i \begin{pmatrix} -\frac{\theta_\infty}{2} & 0 \\ 0 & \frac{\theta_\infty}{2}\end{pmatrix}\right\} C_\infty$$

In order to construct the fuchsian system, we have to find a $2\times 2$ invertible matrix $Y(z;x)$ holomorphic in $\mathbf{P}^1\backslash\{0,x,1,\infty\}$ with the assigned monodromy at $z=0,1,x,\infty$. We also fix the *indices* of the Riemann-Hilbert problem, namely we require that $Y(z;x)$ be normalized as follows:

$$Y(z;x)=\begin{cases} \left(I+O\left(\left(\frac{1}{z}\right)\right)\right)\ z^{\begin{pmatrix} -\frac{\theta_\infty}{2} & 0 \\ 0 & \frac{\theta_\infty}{2}\end{pmatrix}}\ C_\infty, \quad z\to\infty \\ G_i(I+O(z-i))\ (z-i)^{\begin{pmatrix} \frac{\theta_i}{2} & 0 \\ 0 & -\frac{\theta_i}{2}\end{pmatrix}}\ C_i, \quad z\to i=0,x,1 \end{cases} \tag{96}$$

Here $G_i$ are invertible matrices. The fuchsian system is obtained from

$$A(z;x):=\frac{dY(z;x)}{dz}\ Y(z;x)^{-1} \quad\Longrightarrow\quad A_i = G_i \begin{pmatrix} \frac{\theta_i}{2} & 0 \\ 0 & -\frac{\theta_i}{2}\end{pmatrix} G_i^{-1} \tag{97}$$

The fuchsian system satisfies (94). We remark that we have to assume

$$\theta_0,\ \theta_x,\ \theta_1,\ \theta_\infty\ \notin \mathbf{Z}$$

otherwise the general form of $Y(z;x)$ would be different from (96).

A $2\times 2$ Riemann-Hilbert always has solution [1]. In our case, the solution is unique, up to diagonal conjugation $A_i\mapsto DA_iD^{-1}$, where $D$ is any diagonal matrix. To prove this fact, let $C_\nu$, $\tilde{C}_\nu$, $\nu = 0,x,1,\infty$, be such that

$$C_\nu^{-1}\ \exp\left\{2\pi i \begin{pmatrix} \frac{\theta_\nu}{2} & 0 \\ 0 & -\frac{\theta_\nu}{2}\end{pmatrix}\right\}\ C_\nu = \tilde{C}_\nu^{-1}\ \exp\left\{2\pi i \begin{pmatrix} \frac{\theta_\nu}{2} & 0 \\ 0 & -\frac{\theta_\nu}{2}\end{pmatrix}\right\}\ \tilde{C}_\nu$$

If follows that $C_\nu\tilde{C}_\nu^{-1}$ is any diagonal matrix. We denote $Y$ and $\tilde{Y}$ the solutions (96) with $C_\nu$ and $\tilde{C}_\nu$ respectively. Since $C_\nu\tilde{C}_\nu^{-1}$ is diagonal we conclude that $Y\ \tilde{Y}^{-1}$ is holomorphic at $z=0,1,x,\infty$. Therefore, being holomorphic on $\mathbf{P}^1$, it is a constant diagonal matrix $D:=C_\infty\tilde{C}_\infty^{-1}$. This proves uniqueness up to diagonal conjugation. We remark that if some $\theta_\nu$ ($\nu=0,x,1,\infty$) is integer, the uniqueness may fail. One such case is discussed in [23], [7].

Once the Riemann-Hilbert problem is solved, we compute $y(x)$ from $A(z;x)_{12}=0$. It is clear that $y(x)$ depends on the monodromy data $M_0$, $M_x$, $M_1$, $\theta_\nu$ ($\nu=0,1,x,\infty$). Note that for any invertible matrix $C$ we obtain the fuchsian system (97) from any $Y(z;u)C$. The solution $Y(z;u)C$ corresponds to the monodromy matrices $C^{-1}M_iC$. This implies that $y(x)$ depends on the invariants of $M_i$ ($i=0,x,1$) with respect to the conjugation $M_i\mapsto C^{-1}M_iC$. They are traces and determinants of the products on the $M_i$'s. The determinants are 1, the traces of $M_i$ are specified by the eigenvalues and the traces of the products $M_{i_1}M_{i_2}...M_{i_n}$ for $n>2$ are functions of the traces of $M_{i_1}M_{i_2}$. Hence:

$$y(x)=y\big(x;\ \theta_0,\theta_x,\theta_1,\theta_\infty,\mathrm{tr}(M_0M_x),\mathrm{tr}(M_0M_1),\mathrm{tr}(M_1M_x)\ \big)$$

PVI is equivalent to the Schlesinger equations which determine the isomonodromicity of the fuchsian system [16]:

$$\begin{aligned} \frac{dA_0}{dx} &= \frac{[A_x,A_0]}{x} \\ \frac{dA_1}{dx} &= \frac{[A_1,A_x]}{1-x} \\ \frac{dA_x}{dx} &= \frac{[A_x,A_0]}{x}+\frac{[A_1,A_x]}{1-x} \end{aligned} \tag{98}$$

The system (98) is a particular case of

$$\begin{array}{c} \frac{dA_\mu}{dx}=\sum_{\nu=1}^{n_2}[A_\mu,B_\nu]\ f_{\mu\nu}(x) \\ \frac{dB_\nu}{dx}=-\frac{1}{x}\sum_{\nu'=1}^{n_2}[B_\nu,B_{\nu'}]+\sum_{\mu=1}^{n_1}[B_\nu,A_\mu]\ g_{\mu\nu}(x)+\sum_{\nu'=1}^{n_2}[B_\nu,B_{\nu'}]\ h_{\nu\nu'}(x) \end{array} \tag{99}$$

where the functions $f_{\mu\nu}$, $g_{\mu\nu}$, $h_{\mu\nu}$ are meromorphic with poles at $x=1,\infty$ and

$$\sum_\nu B_\nu+\sum_\mu A_\mu=-A_\infty,\qquad A_\infty:=\begin{pmatrix}\frac{\theta_\infty}{2} & 0\\ 0 & -\frac{\theta_\infty}{2}\end{pmatrix}$$

System (98) is obtained for $f_{\mu\nu}=g_{\mu\nu}=b_\nu/(a_\mu-xb_\nu)$, $h_{\mu\nu}=0$, $n_1=1$, $n_2=2$, $a_1=b_2=1$, $b_1=0$ and $B_1=A_0$, $B_2=A_x$, $A_1=A_1$.

Let $\sigma$ be a complex number such that

$$\sigma\notin(-\infty,0)\cup[1,+\infty)$$

We introduce the following domain for real $\sigma$

$$D(\sigma;\epsilon):=\{x\in\widetilde{\mathbf{C}_0}\text{ s.t. }|x|<\epsilon\},\qquad 0\leq\sigma<1 \tag{100}$$

where $\widetilde{\mathbf{C}_0}$ denotes the universal covering of $\mathbf{C}\backslash\{0\}$. For non-real $\sigma$, we need to introduce some additional parameters $\vartheta_1$, $\vartheta_2\in\mathbf{R}$, $0<\tilde{\sigma}<1$. We define the domain

$$D(\epsilon;\sigma;\vartheta_1,\vartheta_2,\tilde{\sigma}):=\{x\in\widetilde{\mathbf{C}_0}\text{ s.t. }|x|<\epsilon,\ \ e^{-\vartheta_1\Im\sigma}|x|^{\tilde{\sigma}}\leq|x^\sigma|\leq e^{-\vartheta_2\Im\sigma}|x|^0,\quad 0<\tilde{\sigma}<1\}.$$

The domain can be written as

$$|x|<\epsilon,\qquad \Re\sigma\log|x|+\vartheta_2\Im\sigma\leq\Im\sigma\arg(x)\leq(\Re\sigma-\tilde{\sigma})\log|x|+\vartheta_1\Im\sigma$$

The definition of $D(\epsilon;\sigma;\vartheta_1,\vartheta_2,\tilde{\sigma})$ makes sense for any $\sigma\notin(\infty,0)\cup[1,+\infty)$ and includes (100), though it is superfluous to specify $\vartheta_1,\vartheta_2,\tilde{\sigma}\equiv\sigma$ for (100).

In [11] the reader can find the proof of the following lemma, which is a generalization in $D(\epsilon;\sigma;\vartheta_1,\vartheta_2,\tilde{\sigma})$ of [31], page 262

**Lemma A1:** *Consider matrices $B_\nu^0$ ($\nu=1,..,n_2$), $A_\mu^0$ ($\mu=1,..,n_1$) and $\Lambda$, independent of $x$ and such that*

$$\sum_\nu B_\nu^0+\sum_\mu A_\mu^0=-A_\infty$$

$$\sum_\nu B_\nu^0=\Lambda,\qquad \textit{eigenvalues}(\Lambda)=\frac{\sigma}{2},\ -\frac{\sigma}{2},\quad \sigma\notin(-\infty,0)\cup[1,+\infty).$$

*Suppose that $f_{\mu\nu}$, $g_{\mu\nu}$, $h_{\mu\nu}$ are holomorphic if $|x|<\epsilon'$, for some small $\epsilon'<1$.*

*For any $0<\tilde{\sigma}<1$ and $\vartheta_1,\vartheta_2$ real there exists a sufficiently small $0<\epsilon<\epsilon'$ such that the system (99) has holomorphic solutions $A_\mu(x)$, $B_\nu(x)$ in $D(\epsilon;\sigma;\vartheta_1,\vartheta_2,\tilde{\sigma})$ satisfying:*

$$||A_\mu(x)-A_\mu^0||\leq C\ |x|^{1-\sigma_1}$$

$$||x^{-\Lambda}B_\nu(x)\ x^\Lambda-B_\nu^0||\leq C\ |x|^{1-\sigma_1}$$

*Here $C$ is a positive constant and $\tilde{\sigma}<\sigma_1<1$*

**Lemma A2:** *Let $\theta_\infty\neq0$. Let $r$, $s$ be two complex numbers not equal to zero. Let $\hat{G}_0$ be such that*

$$\hat{G}_0^{-1}\Lambda\hat{G}_0=\text{diag}\left(\frac{\sigma}{2},-\frac{\sigma}{2}\right),\qquad\sigma\neq0$$

*The general solution of*

$$A_0^0+A_x^0+A_1^0=\text{diag}\left(-\frac{\theta_\infty}{2},\frac{\theta_\infty}{2}\right),\quad A_0^0+A_x^0=\Lambda,\quad \textit{eigenvalues }A_i=\pm\frac{\theta_i}{2}\quad(i=0,1,x)$$

*is*

$$A_0^0=\hat{G}_0\begin{pmatrix}\frac{\alpha(\beta+1-\gamma)}{\alpha-\beta} & s\\ \frac{\alpha\beta(\beta+1-\gamma)(\gamma-1-\alpha)}{(\alpha-\beta)^2}\ \frac{1}{s} & \frac{\beta(\gamma-\alpha-1)}{\alpha-\beta}\end{pmatrix}\hat{G}_0^{-1}+\frac{\theta_0}{2},$$

$$A_x^0=\hat{G}_0\begin{pmatrix}\frac{\alpha(\gamma-\alpha-1)}{\alpha-\beta} & -s\\ -(A_0^0)_{21} & \frac{\beta(\beta+1-\gamma)}{\alpha-\beta}\end{pmatrix}\hat{G}_0^{-1}+\frac{\theta_x}{2}$$

*where*

$$\alpha:=\frac{-\sigma+\theta_0+\theta_x}{2},\quad \beta:=\frac{\sigma+\theta_0+\theta_x}{2},\quad \gamma:=1+\theta_0$$

*and*

$$\Lambda=\begin{pmatrix}\frac{\alpha(\beta+1-\gamma)}{\alpha-\beta} & r\\ \frac{\alpha\beta(\beta+1-\gamma)(\gamma-1-\alpha)}{(\alpha-\beta)^2}\ \frac{1}{r} & \frac{\beta(\gamma-\alpha-1)}{\alpha-\beta}\end{pmatrix}+\frac{\sigma}{2},\quad A_1^0=\begin{pmatrix}\frac{\alpha(\gamma-\alpha-1)}{\alpha-\beta} & -r\\ -(\Lambda)_{21} & \frac{\beta(\beta+1-\gamma)}{\alpha-\beta}\end{pmatrix}+\frac{\theta_1}{2}$$

$$\hat{G}_0=\begin{pmatrix}1 & 1\\ \frac{\alpha(1+\beta-\gamma)}{\beta-\alpha}\frac{1}{r} & \left\{\frac{\alpha(1+\beta-\gamma)}{\beta-\alpha}+1-\gamma\right\}\frac{1}{r}\end{pmatrix}$$

*where*

$$\alpha:=\frac{\theta_\infty+\theta_1+\sigma}{2},\quad \beta:=\frac{-\theta_\infty+\theta_1+\sigma}{2},\quad \gamma:=1+\sigma$$

(note that $\alpha,\beta,\gamma$ are not the coefficients of PVI. We apologize for using the same symbols.)

Setting $A(z;x)_{12}=0$ we obtain from Lemma A1 and Lemma A2:

$$y(x)=-x\ \frac{(A_0)_{12}}{(A_1)_{12}}\ (1+O(|x|^{1-\tilde{\sigma}}))$$

$$=\frac{1}{16\sigma^3}\left[\theta_0+\theta_x+\sigma\right]\left[-\theta_0+\theta_x+\sigma\right]\left[\theta_0+\theta_x-\sigma\right]\left[\theta_0-\theta_x+\sigma\right]\ \frac{1}{s}\ x^{1-\sigma}(1+O(|x|^{\Delta}))\tag{101}$$

where $\Delta$ is a small positive number and $x\to 0$ in $D(\epsilon;\sigma;\vartheta_1,\vartheta_2,\tilde{\sigma}))$. Namely, we have the following theorem (for a detailed proof see [11]):

**Theorem A1:** *Let $\theta_\infty\neq 0$, $\sigma\neq 0$. For any complex $\sigma\notin(-\infty,0)\cup[1,+\infty)$, for any complex $a\neq 0$, for any $\vartheta_1,\vartheta_2\in\mathbf{R}$ and for any $0<\tilde{\sigma}<1$, there exists a sufficiently small positive $\epsilon$ and a transcendent $y(x;\sigma,a)$ with behavior*

$$y(x;\sigma,a)=ax^{1-\sigma}\left(1+O(|x|^{\Delta})\right),\qquad 0<\Delta<1,\tag{102}$$

*as $x\to 0$ in $D(\epsilon;\sigma;\vartheta_1,\vartheta_2,\tilde{\sigma})$.*

*The above local behavior is valid with an exception which occurs if $\Im\sigma\neq 0$ and $x\to 0$ along the special paths $\Im\sigma\arg(x)=\Re\sigma\log|x|+b$, where $b$ is a constant such that the path is contained in $D(\epsilon;\sigma;\vartheta_1,\vartheta_2)$: the behavior becomes*

$$y(x;\sigma,a)=\sin^2\left(\frac{i\sigma}{2}\ln x-\frac{i}{2}\ln(4a)-\frac{\pi}{2}\right)\ x\ (1+O(|x|^{\Delta}))\tag{103}$$

For brevity, let us denote the domain $D(\epsilon;\sigma)$. We assume in the following that

$$\theta_\nu\notin\mathbf{Z},\qquad \nu=0,x,1,\infty$$

We choose a solution of the fuchsian system

$$\frac{dY}{dz}=\left[\frac{A_0(x)}{z}+\frac{A_x(x)}{z-x}+\frac{A_1(x)}{z-1}\right]\ Y\tag{104}$$

normalized as follows

$$Y(z,x)=\left(I+O\left(\frac{1}{z}\right)\right)\ z^{-A_\infty},\quad z\to\infty,\quad A_\infty=\mathrm{diag}\left(\frac{\theta_\infty}{2},-\frac{\theta_\infty}{2}\right).\tag{105}$$

As a corollary of Lemma A1, the limits

$$\hat{Y}(z) := \lim_{x\to 0} Y(z,x), \qquad \tilde{Y}(z) := \lim_{x\to 0}\ x^{-\Lambda} Y(xz,x)$$

exist when $x \to 0$ in $D(\epsilon;\sigma)$. They satisfy

$$\frac{d\hat{Y}}{dz} = \left[\frac{A_1^0}{z-1} + \frac{\Lambda}{z}\right]\ \hat{Y} \tag{106}$$

$$\frac{d\tilde{Y}}{dz} = \left[\frac{A_0^0}{z} + \frac{A_x^0}{z-1}\right]\ \tilde{Y} \tag{107}$$

For the system (106) we choose a fundamental matrix solution normalized as follows

$$\hat{Y}_N(z) = \left(I + O\left(\frac{1}{z}\right)\right)\ z^{-A_\infty}, \qquad z\to\infty \tag{108}$$

$$= (I+O(z))\ z^{\Lambda}\ \hat{C}_0, \qquad z \to 0$$

$$= \hat{G}_1(I+O(z-1))\ (z-1)^{\mathrm{diag}\left(\frac{\theta_1}{2},-\frac{\theta_1}{2}\right)}\ \hat{C}_1, \qquad z\to 1$$

Where $\hat{G}_1^{-1}A_1^0\hat{G}_1 = \mathrm{diag}\left(\frac{\theta_1}{2},-\frac{\theta_1}{2}\right)$. $\hat{C}_0$, $\hat{C}_1$ are connection matrices. For (107) we choose a fundamental matrix solution normalized as follows

$$\tilde{Y}_N(z) = \left(I + O\left(\frac{1}{z}\right)\right)\ z^{\Lambda}, \qquad z\to\infty \tag{109}$$

$$= \tilde{G}_0(I+O(z))\ z^{\mathrm{diag}\left(\frac{\theta_0}{2},-\frac{\theta_0}{2}\right)}\ \tilde{C}_0, \qquad z \to 0$$

$$= \tilde{G}_1(I+O(z-1))\ (z-1)^{\mathrm{diag}\left(\frac{\theta_x}{2},-\frac{\theta_x}{2}\right)}\ \tilde{C}_1, \qquad z\to 1.$$

Here $\tilde{G}_i^{-1}A_i^0\tilde{G}_i = \mathrm{diag}\left(\frac{\theta_i}{2},-\frac{\theta_i}{2}\right)$ and $\tilde{C}_0$, $\tilde{C}_1$ are connection matrices. As it is proved in [15] – and in [11] for the domain $D(\epsilon;\sigma)$ – we have:

$$\hat{Y}(z) = \hat{Y}_N(z)$$

$$\tilde{Y}(z) = \tilde{Y}_N(z)\ \hat{C}_0 \tag{110}$$

and, as a consequence of isomonodromicity, we have

$$M_1 = \hat{C}_1^{-1} e^{2\pi i\ \mathrm{diag}\left(\frac{\theta_1}{2},-\frac{\theta_1}{2}\right)}\hat{C}_1 \tag{111}$$

$$M_0 = \hat{C}_0^{-1}\tilde{C}_0^{-1} e^{2\pi i\ \mathrm{diag}\left(\frac{\theta_0}{2},-\frac{\theta_0}{2}\right)}\tilde{C}_0\hat{C}_0 \tag{112}$$

$$M_x = \hat{C}_0^{-1}\tilde{C}_1^{-1} e^{2\pi i\ \mathrm{diag}\left(\frac{\theta_x}{2},-\frac{\theta_x}{2}\right)}\tilde{C}_1\hat{C}_0 \tag{113}$$

The connection matrices $\hat{C}_0$, $\tilde{C}_0$, $\tilde{C}_1$ can be computed explicitly because the $2\times 2$ fuchsian systems (106) (107) can be reduced to the hyper-geometric equation.

**Lemma A3:** *The Gauss hyper-geometric equation*

$$z(1-z)\ \frac{d^2y}{dz^2} + [\gamma_0 - z(\alpha_0+\beta_0+1)]\ \frac{dy}{dz} - \alpha_0\beta_0\ y = 0 \tag{114}$$

*is equivalent to the system*

$$\frac{d\Psi}{dz} = \left[\frac{1}{z}\begin{pmatrix} 0 & 0 \\ -\alpha_0\beta_0 & -\gamma_0\end{pmatrix} + \frac{1}{z-1}\begin{pmatrix} 0 & 1 \\ 0 & \gamma_0-\alpha_0-\beta_0\end{pmatrix}\right]\ \Psi \tag{115}$$

*where* $\Psi = \begin{pmatrix} y \\ (z-1)\frac{dy}{dz}\end{pmatrix}$.

**Lemma A4:** *Let $B_0$ and $B_1$ be matrices of eigenvalues $0, 1-\gamma$, and $0, \gamma-\alpha-\beta-1$ respectively, such that*

$$B_0 + B_1 = \operatorname{diag}(-\alpha, -\beta), \quad \alpha \neq \beta$$

*Then*

$$B_0 = \begin{pmatrix} \frac{\alpha(\beta+1-\gamma)}{\alpha-\beta} & r_1 \\ \frac{\alpha\beta(\beta+1-\gamma)(\gamma-1-\alpha)}{(\alpha-\beta)^2}\ \frac{1}{r_1} & \frac{\beta(\gamma-\alpha-1)}{\alpha-\beta} \end{pmatrix}$$

$$B_1 = \begin{pmatrix} \frac{\alpha(\gamma-\alpha-1)}{\alpha-\beta} & -r_1 \\ -(B_0)_{21} & \frac{\beta(\beta+1-\gamma)}{\alpha-\beta} \end{pmatrix}$$

*for any $r_1 \neq 0$.*

We leave the proof as an exercise (note that $\alpha, \beta, \gamma$ are not the coefficients of PVI. We apologize for using the same symbols). The following lemma connects Lemmas A3 and A4:

**Lemma A5:** *The system (115) with*

$$\alpha_0 = \alpha, \quad \beta_0 = \beta + 1, \quad \gamma_0 = \gamma, \quad \alpha \neq \beta$$

*is gauge-equivalent to the system*

$$\frac{dX}{dz} = \left[\frac{B_0}{z} + \frac{B_1}{z-1}\right]\ X \tag{116}$$

*where $B_0$, $B_1$ are given in Lemma A4. This means that there exists a matrix*

$$G(z) = \begin{pmatrix} 1 & 0 \\ \frac{\alpha((\alpha-\beta)z+\beta+1-\gamma)}{\beta-\alpha}\ \frac{1}{r_1} & -\frac{z}{r_1} \end{pmatrix}$$

*such that $X(z) = G(z)\ \Psi(z)$. It follows that (116) and the corresponding hyper-geometric equation (114) have the same fuchsian singularities 0,1,$\infty$ and the same monodromy group.*

*Proof:* By direct computation. □

Note that the form of $G(z)$ ensures that if $y_1$, $y_2$ are independent solutions of the hyper-geometric equation, then a fundamental matrix of (116) may be chosen to be $X(z) = \begin{pmatrix} y_1(z) & y_2(z) \\ * & * \end{pmatrix}$.

Now we compute the monodromy matrices for the systems (106), (107) by reduction to an hyper-geometric equation. We assume

$$\sigma \notin \mathbf{Z}$$

Let us start with (106). With the gauge $Y^{(1)}(z) := z^{-\frac{\sigma}{2}}(z-1)^{-\frac{\theta_1}{2}}\hat{Y}_N(z)$ we transform (106) into

$$\frac{dY^{(1)}}{dz} = \left[\frac{A_1^0 - \frac{\theta_1}{2}}{z-1} + \frac{\Lambda - \frac{\sigma}{2}I}{z}\right]\ Y^{(1)} \tag{117}$$

We identify the matrices $B_0$, $B_1$ of Lemma A4 with $\Lambda - \frac{\sigma}{2}$ and $A_1^0 - \frac{\theta_1}{2}$. Therefore

$$\alpha = \frac{\theta_\infty + \theta_1 + \sigma}{2}, \quad \beta = \frac{-\theta_\infty + \theta_1 + \sigma}{2}, \quad \gamma = 1 + \sigma$$

(note that $\alpha - \beta = \theta_\infty \neq 0$. Also note that Lemma A2 for $\Lambda$ and $A_1^0$ follows from Lemma A4 applied to (117)). The hyper-geometric equation connected to the present system through Lemma A3 has coefficients

$$\begin{cases} \alpha_0 = \frac{\theta_\infty + \theta_1 + \sigma}{2} \\ \beta_0 = 1 + \frac{-\theta_\infty + \theta_1 + \sigma}{2} \\ \gamma_0 = \sigma + 1 \end{cases}$$

Therefore, according to the standard theory of hyper-geometric equations, the generic case occurs when

$$\sigma,\ \theta_1,\ \theta_\infty \notin \mathbf{Z}$$

Let $F(a,b,c,;z)$ be the hyper-geometric function. For $\sigma \notin \mathbf{Z}$ we have two independent solutions of the hyper-geometric equation

$$y_1^{(0)}(x) = F(\alpha_0,\beta_0,\gamma_0;z)$$

$$y_2^{(0)}(x) = z^{1-\gamma_0} F(\alpha_0-\gamma_0+1,\beta_0-\gamma_0+1,2-\gamma_0;z)$$

For $\theta_1 \notin \mathbf{Z}$ we have also solutions

$$y_1^{(1)}(z) = F(\alpha_0,\beta_0,\alpha_0+\beta_0+1-\gamma_0;1-z)$$

$$y_2^{(1)}(z) = (1-z)^{\gamma_0-\alpha_0-\beta_0} F(\gamma_0-\beta_0,\gamma_0-\alpha_0,1+\gamma_0-\alpha_0-\beta_0;1-z)$$

They are connected by

$$[y_1^{(0)},y_2^{(0)}] = [y_1^{(1)},y_2^{(1)}]C_{01}$$

$$C_{01} := \begin{pmatrix} \frac{\Gamma(\gamma_0-\alpha_0-\beta_0)\Gamma(\gamma_0)}{\Gamma(\gamma_0-\alpha_0)\Gamma(\gamma_0-\beta_0)} & \frac{\Gamma(\gamma_0-\alpha_0-\beta_0)\Gamma(2-\gamma_0)}{\Gamma(1-\alpha_0)\Gamma(1-\beta_0)} \\ \frac{\Gamma(\alpha_0+\beta_0-\gamma_0)\Gamma(\gamma_0)}{\Gamma(\alpha_0)\Gamma(\beta_0)} & \frac{\Gamma(\alpha_0+\beta_0-\gamma_0)\Gamma(2-\gamma_0)}{\Gamma(\alpha_0+1-\gamma_0)\Gamma(\beta_0+1-\gamma_0)} \end{pmatrix}$$

Note that

$$Y^{(1)}(z) = \hat{G}_0[I+O(z)]\ z^{\mathrm{diag}(0,-\sigma)}\ \hat{G}_0^{-1}\hat{C}_0, \quad z\to 0$$

$$= \hat{G}_1[I+O(z-1)]\ (z-1)^{\mathrm{diag}(0,-\theta_1)}\ \hat{C}_1, \quad z\to 1$$

From the behaviors of the hyper-geometric functions $y_i^{(0)}$ for $z\to 0$ and $y_i^{(1)}$ for $z\to 1$ and for a suitable choice of $\hat{G}_1$ we obtain

$$Y^{(1)} = \begin{pmatrix} y_1^{(0)} & y_2^{(0)} \\ * & * \end{pmatrix} \hat{G}_0^{-1}\hat{C}_0 = \begin{pmatrix} y_1^{(1)} & y_2^{(1)} \\ * & * \end{pmatrix} \hat{C}_1$$

Namely

$$\hat{C}_1(\hat{G}_0^{-1}\hat{C}_0)^{-1} = C_{01}$$

This will be enough for our purposes (note that we did not use the hypothesis $\theta_\infty \notin \mathbf{Z}$).

We turn to the system (107). With the gauge $Y^{(2)}(z) := z^{-\frac{\theta_0}{2}}(z-1)^{-\frac{\theta_x}{2}}\ \hat{G}_0^{-1}\tilde{Y}_N(z)\hat{G}_0$ we have

$$\frac{dY^{(2)}}{dz} = \left[\frac{B_0}{z}+\frac{B_1}{z-1}\right]Y^{(2)}$$

where

$$B_0 = \hat{G}_0^{-1}A_0^0\hat{G}_0 - \frac{\theta_0}{2}, \quad B_1 = \hat{G}_0^{-1}A_x^0\hat{G}_0 - \frac{\theta_x}{2}$$

$B_0$ and $B_1$ are as in Lemma A4, with

$$\alpha = \frac{-\sigma+\theta_0+\theta_x}{2}, \quad \beta = \frac{\sigma+\theta_0+\theta_x}{2}, \quad \gamma = 1+\theta_0$$

(Lemma A2 for $A_x^0$ and $A_0^0$ follows from Lemma A4 applied to the present case). The corresponding hyper-geometric equation has coefficients

$$\begin{cases} \alpha_0 = \frac{-\sigma+\theta_0+\theta_x}{2} \\ \beta_0 = 1+\frac{\sigma+\theta_0+\theta_x}{2} \\ \gamma_0 = 1+\theta_0 \end{cases}$$

Therefore, according to the standard theory of hyper-geometric equations, the generic case occurs for

$$\theta_0,\ \theta_x,\ \sigma \notin \mathbf{Z}$$

and we can choose three couples of independent solutions

$$\begin{cases} y_1^{(0)} = F(\alpha_0,\beta_0,\gamma_0;z) \\ y_2^{(0)} = z^{1-\gamma_0}F(1+\alpha_0-\gamma_0,1+\beta_0-\gamma_0,2-\gamma_0;z) \end{cases}$$

$$\begin{cases} y_1^{(1)} = F(\alpha_0,\beta_0,\alpha_0+\beta_0+1-\gamma_0;1-z) \\ y_2^{(1)} = (1-z)^{\gamma_0-\alpha_0-\beta_0}F(\gamma_0-\alpha_0,\gamma_0-\beta_0,\gamma_0+1-\alpha_0-\beta_0;1-z) \end{cases}$$

$$\begin{cases} y_1^{(\infty)} = z^{-\alpha_0}F(\alpha_0,\alpha_0-\gamma_0+1,\alpha_0+1-\beta_0;\frac{1}{z}) \\ y_2^{(\infty)} = z^{-\beta_0}F(\beta_0,\beta_0+1-\gamma_0,\beta_0+1-\alpha_0;\frac{1}{z} \end{cases}$$

They are connected by

$$[y_1^{(\infty)},y_2^{(\infty)}] = [y_1^{(0)},y_2^{(0)}]C'_{\infty 0}$$

$$C'_{\infty 0} = \begin{pmatrix} e^{-i\pi\alpha_0}\frac{\Gamma(1+\alpha_0-\beta_0)\Gamma(1-\gamma_0)}{\Gamma(1-\beta_0)\Gamma(1+\alpha_0-\gamma_0)} & e^{-i\pi\beta_0}\frac{\Gamma(1+\beta_0-\alpha_0)\Gamma(1-\gamma_0)}{\Gamma(1-\alpha_0)\Gamma(1+\beta_0-\gamma_0)} \\ e^{i\pi(\gamma_0-1-\alpha_0)}\frac{\Gamma(1+\alpha_0-\beta_0)\Gamma(\gamma_0-1)}{\Gamma(\alpha_0)\Gamma(\gamma_0-\beta_0)} & e^{i\pi(\gamma_0-1-\beta_0)}\frac{\Gamma(1+\beta_0-\alpha_0)\Gamma(\gamma_0-1)}{\Gamma(\beta_0)\Gamma(\gamma_0-\alpha_0)} \end{pmatrix}$$

$$[y_1^{(\infty)},y_2^{(\infty)}] = [y_1^{(1)},y_2^{(1)}]C'_{\infty 1}$$

$$C'_{\infty 1} = \begin{pmatrix} \frac{\Gamma(\gamma_0-\alpha_0-\beta_0)\Gamma(1+\alpha_0-\beta_0)}{\Gamma(1-\beta_0)\Gamma(\gamma_0-\beta_0)} & \frac{\Gamma(\gamma_0-\alpha_0-\beta_0)\Gamma(1+\beta_0-\alpha_0)}{\Gamma(1-\alpha_0)\Gamma(\gamma_0-\alpha_0)} \\ e^{i\pi(\gamma_0-\alpha_0-\beta_0)}\frac{\Gamma(\alpha_0+\beta_0-\gamma_0)\Gamma(1+\alpha_0-\beta_0)}{\Gamma(1+\alpha_0-\gamma_0)\Gamma(\alpha_0)} & e^{i\pi(\gamma_0-\alpha_0-\beta_0)}\frac{\Gamma(\alpha_0+\beta_0-\gamma_0)\Gamma(1+\beta_0-\alpha_0)}{\Gamma(1+\beta_0-\gamma_0)\Gamma(\beta_0)} \end{pmatrix}$$

Considering the asymptotic behavior of the hyper-geometric equations and of $Y^{(2)}(z)$ at $z=0,1,\infty$ we obtain

$$Y^{(2)} = \begin{pmatrix} y_1^{(\infty)} & \frac{s}{1+\sigma}y_2^{(\infty)} \\ * & * \end{pmatrix} = \begin{pmatrix} y_1^{(0)} & y_2^{(0)} \\ * & * \end{pmatrix}\tilde{C}_0\hat{G}_0 = \begin{pmatrix} y_1^{(1)} & y_2^{(1)} \\ * & * \end{pmatrix}\tilde{C}_1\hat{G}_0$$

Therefore

$$\tilde{C}_0\hat{G}_0\begin{pmatrix} 1 & 0 \\ 0 & \frac{1+\sigma}{s} \end{pmatrix} = C'_{\infty 0}, \quad \tilde{C}_1\hat{G}_0\begin{pmatrix} 1 & 0 \\ 0 & \frac{1+\sigma}{s} \end{pmatrix} = C'_{\infty 1}$$

We introduce the following matrices:

$$m_0 := \left[\left\{C'_{\infty 0}\begin{pmatrix} 1 & 0 \\ 0 & \frac{s}{1+\sigma} \end{pmatrix}\right\}^{-1} e^{2\pi i\ \mathrm{diag}\left(\frac{\theta_0}{2},-\frac{\theta_0}{2}\right)}\left\{C'_{\infty 0}\begin{pmatrix} 1 & 0 \\ 0 & \frac{s}{1+\sigma} \end{pmatrix}\right\}\right] \tag{118}$$

$$m_x := \left[\left\{C'_{\infty 1}\begin{pmatrix} 1 & 0 \\ 0 & \frac{s}{1+\sigma} \end{pmatrix}\right\}^{-1} e^{2\pi i\ \mathrm{diag}\left(\frac{\theta_x}{2},-\frac{\theta_x}{2}\right)}\left\{C'_{\infty 1}\begin{pmatrix} 1 & 0 \\ 0 & \frac{s}{1+\sigma} \end{pmatrix}\right\}\right] \tag{119}$$

$$m_1 := C_{01}^{-1}\ e^{2\pi i\ \mathrm{diag}\left(\frac{\theta_1}{2},-\frac{\theta_1}{2}\right)}\ C_{01} \tag{120}$$

From the above results, from the formulae (111), (112), (113), and from the cyclic properties of the trace we obtain

$$\mathrm{tr}(M_0M_x) \equiv \mathrm{tr}(m_0m_x), \quad \mathrm{tr}(M_1M_x) \equiv \mathrm{tr}(m_1m_x), \quad \mathrm{tr}(M_0M_1) \equiv \mathrm{tr}(m_0m_1)$$

We note that in order for the $\Gamma$-functions to be non-singular we have to require $(\pm\sigma\pm\theta_1\pm\theta_\infty)/2 \notin \mathbf{Z}$, $(\pm\sigma\pm\theta_0\pm\theta_x)/2 \notin \mathbf{Z}$. Therefore, to summarize we say that the *generic case* corresponds to

$$\sigma,\ \theta_0,\ \theta_x,\ \theta_1,\ \theta_\infty\ \notin \mathbf{Z}; \qquad \frac{\pm\sigma\pm\theta_1\pm\theta_\infty}{2},\ \frac{\pm\sigma\pm\theta_0\pm\theta_x}{2}\ \notin \mathbf{Z} \tag{121}$$

Computing the traces, we find

$$\mathrm{tr}(M_0M_x) = 2\cos(\pi\sigma) \tag{122}$$

$$\mathrm{tr}(M_1M_0) = F_1(\sigma,\theta_0,\theta_x,\theta_1,\theta_\infty)\ \frac{1}{s} + F_2(\sigma,\theta_0,\theta_x,\theta_1,\theta_\infty) + F_3(\sigma,\theta_0,\theta_x,\theta_1,\theta_\infty)\ s \tag{123}$$

$$\mathrm{tr}(M_xM_1) = -e^{-i\pi\sigma}F_1(\sigma,\theta_0,\theta_x,\theta_1,\theta_\infty)\ \frac{1}{s} + F_4(\sigma,\theta_0,\theta_x,\theta_1,\theta_\infty) - e^{i\pi\sigma}F_3(\sigma,\theta_0,\theta_x,\theta_1,\theta_\infty)\ s \qquad (124)$$

where $F_i(\sigma,\theta_0,\theta_x,\theta_1,\theta_\infty)$ are long expressions which can be explicitly computed from (118), (119), (120) in terms of $\Gamma$ functions ( and trigonometric functions). We omit the explicit formulae because they keep some space; in any case, they are easily computed with a computer from (118), (119), (120). The first equation determines $\sigma$ up to $\sigma \mapsto \pm\sigma + 2n$, $n \in \mathbf{Z}$. Once $\sigma$ is chosen, the last two equations determine

$$s = \frac{e^{i\pi\sigma}\mathrm{tr}(M_1M_0) + \mathrm{tr}(M_xM_1) - F_4 - e^{-i\pi\sigma}F_2}{F_1(e^{i\pi\sigma} - e^{-i\pi\sigma})} \qquad (125)$$

If we substitute $s$ in the coefficient $a$ given by (101) we get

$$a = a(\sigma;\theta_0,\theta_x,\theta_1,\theta_\infty,\mathrm{tr}(M_0M_x),\mathrm{tr}(M_0M_1),\mathrm{tr}(M_1M_x))$$

namely, we get the parameterization [2]

$$y(x;\sigma,a) = y\big(x;\ \theta_0,\theta_x,\theta_1,\theta_\infty,\mathrm{tr}(M_0M_x),\mathrm{tr}(M_0M_1),\mathrm{tr}(M_1M_x)\ \big) \qquad (126)$$

**Remark:** Note that we can fix $\sigma$ such that

$$0 \le \Re\sigma \le 1$$

and therefore, for the given monodromy data, the transcendent (126) has different representations

$$y\big(\ x;\ \ \pm\sigma + 2n,\ a(\pm\sigma + 2n,\ \theta_0,\theta_x,\theta_1,\theta_\infty,\mathrm{tr}(M_0M_x),\mathrm{tr}(M_0M_1),\mathrm{tr}(M_1M_x))\ \big)$$

on different domains $D(\epsilon_n;\pm\sigma + 2n)$ of theorem A1.

**Proposition A1:** *Let us consider the generic case (121) and let $\sigma \notin (-\infty,0]\cup[1,+\infty)$ and $a \neq 0$. Let $y(x)$ be a solution of PVI such that $y(x) = ax^{1-\sigma}$(1+higher order terms) as $x \to 0$ in an open domain contained in $D(\epsilon;\sigma)$ of theorem A1. Then, $y(x)$ coincides with $y(x;\sigma,a)$ of Theorem A1*

*Proof:* Observe that both $y(x)$ and $y(x;\sigma,a)$ have the same asymptotic behavior for $x \to 0$ in $D(\sigma)$. Let $A_0(x)$, $A_1(x)$, $A_x(x)$ be the matrices constructed from $y(x)$ and $A_0^*(x)$, $A_1^*(x)$, $A_x^*(x)$ constructed from $y(x;\sigma,a)$ by means of the formulae of the Appendix C of [17]. It follows that $A_i(x)$ and $A_i^*(x)$, $i = 0,1,x$, have the same asymptotic behavior as $x \to 0$. This is the behavior of Lemma A1. Therefore, according to the construction above (reduction to the hyper-geometric equation) $A_0(x)$, $A_1(x)$, $A_x(x)$ and $A_0^*(x)$, $A_1^*(x)$, $A_x^*(x)$ give the same monodromy. The solution of the Riemann-Hilbert problem for such monodromy is unique, up to diagonal conjugation of the fuchsian systems. Therefore $A_i(x)$ and $A_i^*(x)$, $i = 0,1,x$ are at most diagonally conjugated. Putting $[A(z;x)]_{12} = 0$ and $[A^*(z;x)]_{12} = 0$ we conclude that $y(x) \equiv y(x;\sigma,a)$. □

**CONNECTION PROBLEM:** We explain how to solve the connection problem for the transcendent $y(x;\sigma,a)$.

$[x = \infty]$: Let

$$t := \frac{1}{x}, \quad y(x) =: \frac{1}{t}\ \tilde{y}(t)$$

Then $y(x)$ solves PVI with parameters $\theta_0$, $\theta_x$, $\theta_1$, $\theta_\infty$ if and only if $\tilde{y}(t)$ solves PVI (with independent variable $t$) with parameters

$$(\tilde{\theta}_0,\tilde{\theta}_t,\tilde{\theta}_1,\tilde{\theta}_\infty) := (\theta_0,\theta_1,\theta_x,\theta_\infty) \qquad (127)$$

Namely, $\theta_1$ and $\theta_x$ are exchanged. From Theorem A1 if follows that there exists a solution $\tilde{y}(t;\sigma^{(\infty)},a^{(\infty)}) = a^{(\infty)}t^{1-\sigma^{(\infty)}}(1 + O(t^{\Delta}))$ for $t \to 0$. Therefore, PVI in variable $x$ has a solution

$$y(x) = a^{(\infty)}x^{\sigma^{(\infty)}}(1 + O(x^{-\Delta})), \quad x \to \infty$$

in

$$D(M;\sigma^{(\infty)};\vartheta_1,\vartheta_2,\tilde{\sigma}) := \{x \in \widetilde{\mathbf{C}\backslash\{\infty\}} \text{ s.t. } |x| > M,\ e^{-\vartheta_1\Im\sigma^{(\infty)}}|x|^{-\tilde{\sigma}} \le |x^{-\sigma^{(\infty)}}| \le e^{-\vartheta_2\Im\sigma^{(\infty)}}$$

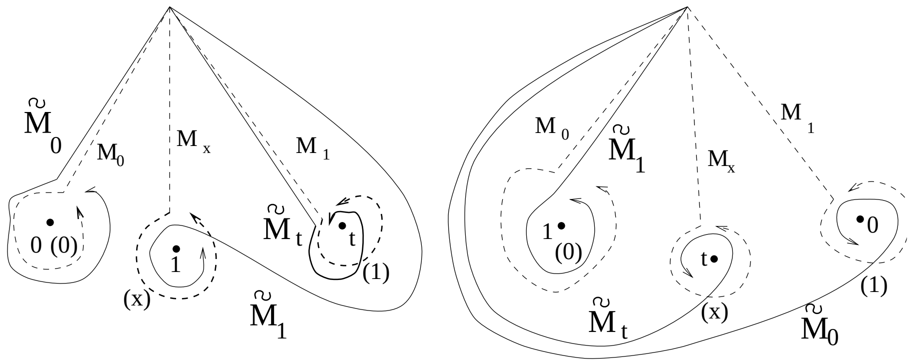


Figure 3: The basis of loops for the system in $z$ and for the system in $\tilde{z}$. In parenthesis are the poles of $z$, out of parenthesis the poles of $\tilde{z}$. The left figure corresponds to $t = 1/x$, $\tilde{z} = tz$. The right figure corresponds to $t = 1-x$, $\tilde{z} = 1-z$. The dotted loops are for the system in $z$, the others for the system in $\tilde{z}$

$$0 < \tilde{\sigma} < 1\}$$

where $M > 0$ is sufficiently big and $0 < \Delta < 1$ is small.

$[x = 1]$: Now let

$$t := 1 - x, \qquad y(x) =: 1 - \tilde{y}(t)$$

Then $y(x)$ satisfies PVI with parameters $\theta_0$, $\theta_x$, $\theta_1$, $\theta_\infty$ if and only if $\tilde{y}(t)$ solves PVI (with independent variable $t$) with parameters

$$(\tilde{\theta}_0, \tilde{\theta}_t, \tilde{\theta}_1, \tilde{\theta}_\infty) := (\theta_1, \theta_x, \theta_0, \theta_\infty) \tag{128}$$

Namely, $\theta_0$ and $\theta_1$ are exchanged. From Theorem A1 if follows that there exists a solution $\tilde{y}(t; \sigma^{(1)}, a^{(1)}) = a^{(1)} t^{1-\sigma^{(1)}}(1 + O(t^\Delta))$ for $t \to 0$. Therefore, PVI in variable $x$ has a solution

$$y(x) = 1 - a^{(1)}(1-x)^{1-\sigma^{(1)}} \left(1 + O((1-x)^\Delta)\right)$$

in

$$D(\epsilon; \sigma^{(1)}; \vartheta_1, \vartheta_2, \tilde{\sigma}) := \{x \in \widetilde{\mathbf{C}\backslash\{1\}} \text{ s.t. } |1-x| < \epsilon, \;\; e^{-\vartheta_1 \Im \sigma}|1-x|^{\tilde{\sigma}} \leq |(1-x)^{\sigma^{(1)}}| \leq e^{-\vartheta_2 \Im \sigma},$$

$$0 < \tilde{\sigma} < 1\}$$

Let us examine how the monodromy data change.

$[x = \infty]$: In the fuchsian system (104) we put $x = \frac{1}{t}$ and $\tilde{z} := tz$. We get

$$\frac{dY}{d\tilde{z}} = \left[\frac{A_0}{\tilde{z}} + \frac{A_x}{\tilde{z}-1} + \frac{A_1}{\tilde{z}-t}\right] Y$$

Therefore $z = 1, x$ have been exchanged and correspond to $\tilde{z} = t, 1$. The monodromy matrices $\tilde{M}_0$, $\tilde{M}_t$, $\tilde{M}_1$ corresponding to the loops ordered as in figure 3 with $t$ instead of $x$ are related to $M_0$, $M_1$, $M_x$ of (104) in the basis of figure 3:

$$M_0 = \tilde{M}_0, \quad M_x = \tilde{M}_t^{-1}\tilde{M}_1\tilde{M}_t, \quad M_1 = \tilde{M}_t$$

If we introduce the convenient notation

$$T_0 := \operatorname{tr}(M_0 M_x), \quad T_1 := \operatorname{tr}(M_1 M_x), \quad T_\infty := \operatorname{tr}(M_0 M_1)$$

we have

$$\begin{cases} T_0 = \operatorname{tr}(\tilde{M}_0 \tilde{M}_t^{-1}\tilde{M}_1\tilde{M}_t) \\ T_1 = \tilde{T}_1 \\ T_\infty = \tilde{T}_0 \end{cases}$$

where $\tilde{T}_1 = \mathrm{tr}(\tilde{M}_1\tilde{M}_t)$, etc. Now we observe that

$$\mathrm{tr}(\tilde{M}_0\tilde{M}_t^{-1}\tilde{M}_1\tilde{M}_t) = 4\big[\cos(\pi\tilde{\theta}_\infty)\cos(\pi\tilde{\theta}_t) + \cos(\pi\tilde{\theta}_0)\cos(\pi\tilde{\theta}_1)\big] - \mathrm{tr}(\tilde{M}_1\tilde{M}_0) - \mathrm{tr}(\tilde{M}_1\tilde{M}_t)\mathrm{tr}(\tilde{M}_0\tilde{M}_t)$$

This follows from the identity

$$\mathrm{tr}(AB) = \mathrm{tr}(A)\mathrm{tr}(B) - \mathrm{tr}(AB^{-1}), \quad A, B\ 2\times 2 \text{ matrices }, \quad \det(B) = 1$$

and from

$$\mathrm{tr}(\tilde{M}_1\tilde{M}_t\tilde{M}_0) = e^{i\pi\tilde{\theta}_\infty} + e^{-i\pi\tilde{\theta}_\infty}, \quad \mathrm{tr}(\tilde{M}_i) = e^{i\pi\tilde{\theta}_i} + e^{-i\pi\tilde{\theta}_i}, \ \ i = 0, x, 1$$

Therefore, recalling (127), we conclude that

$$\begin{cases} \tilde{T}_0 = T_\infty \\ \tilde{T}_1 = T_1 \\ \tilde{T}_\infty = 4\big[\cos(\pi\theta_\infty)\cos(\pi\theta_1) + \cos(\pi\theta_x)\cos(\pi\theta_0)\big] - (T_0 + T_1T_\infty) \end{cases}$$

Therefore the parameterization of $\sigma^{(\infty)}$, $a^{(\infty)}$ in terms of the monodromy data $\theta_\nu$ ($\nu = 0, x, 1, \infty$), $T_0$, $T_1$, $T_\infty$ is obtained as follows. As for $\sigma^{(\infty)}$ we get if from

$$2\cos(\pi\sigma^{(\infty)}) = T_\infty$$

As for $a^{(\infty)}$, we compute if from the formulae (125) (101) with the substitutions [3]

$$\sigma \mapsto \sigma^{(\infty)}$$

$$\theta_x \mapsto \theta_1, \quad \theta_1 \mapsto \theta_x$$

$$T_0 \mapsto T_\infty, \quad T_\infty \mapsto 4\big[\cos(\pi\theta_\infty)\cos(\pi\theta_1) + \cos(\pi\theta_x)\cos(\pi\theta_0)\big] - (T_0 + T_1T_\infty)$$

[$x = 1$]: If we put $x = 1 - t$, $z = 1 - \tilde{z}$, the system (104) becomes

$$\frac{dY}{d\tilde{z}} = \left[\frac{A_1}{\tilde{z}} + \frac{A_0}{\tilde{z}-1} + \frac{A_x}{\tilde{z}-t}\right]\ Y$$

Therefore $z = 0, 1$ have been exchanged. The monodromy matrices $\tilde{M}_0$, $\tilde{M}_t$, $\tilde{M}_1$ are

$$M_0 = \tilde{M}_1, \quad M_x = \tilde{M}_1\tilde{M}_t\tilde{M}_1^{-1}, \quad M_1 = \tilde{M}_1\tilde{M}_t\tilde{M}_0\tilde{M}_t^{-1}\tilde{M}_1^{-1}$$

As above, this implies that

$$\begin{cases} \tilde{T}_0 = T_1 \\ \tilde{T}_1 = T_0 \\ \tilde{T}_\infty = 4\left[\cos(\pi\theta_\infty)\cos(\pi\theta_x) + \cos(\pi\theta_0)\cos(\pi\theta_1)\right] - (T_\infty + T_0T_1) \end{cases}$$

Therefore, we obtain $\sigma^{(1)}$ from

$$2\cos(\pi\sigma^{(1)}) = T_1$$

and $a^{(1)}$ from the formulae (125) (101) with the substitutions [4]:

$$\sigma \mapsto \sigma^{(1)}$$

$$\theta_0 \mapsto \theta_1, \quad \theta_1 \mapsto \theta_0$$

$$T_0 \mapsto T_1, \quad T_1 \mapsto T_0, \quad T_\infty \mapsto 4\left[\cos(\pi\theta_\infty)\cos(\pi\theta_x) + \cos(\pi\theta_0)\cos(\pi\theta_1)\right] - (T_\infty + T_0T_1)$$

This solves the connection problem for $y(x;\sigma,a)$, which is the representation close to $x = 0$ in $D(\epsilon;\sigma)$ of $y(x;\theta_0,\theta_x,\theta_1,\theta_\infty;T_0,T_1,T_\infty)$ for monodromy data computed through (122), (123), (124). From the monodromy data we can compute $a^{(1)},\sigma^{(1)}$, $a^{(\infty)},\sigma^{(\infty)}$ as we just explained above. Therefore $y(x;\sigma,a)$ has representations in some $D(\epsilon;\sigma^{(1)})$ and $D(M;\sigma^{(\infty)})$ at $x = 1$ and $x = \infty$ respectively, with parameters $a^{(1)},\sigma^{(1)}$, $a^{(\infty)},\sigma^{(\infty)}$.

**Remark:** In the generic case, we proved the one-to-one correspondence between transcendents and monodromy data. However, the condition $\sigma \neq 1$ implies that we can not give the critical behaviors at $x=0$ for $T_0=\pm 2$, at $x=1$ for $T_1=\pm 2$, at $x=\infty$ for $T_\infty=\pm 2$ on the basis of Theorem A1. The connection problem for $\sigma=0$ is solved in [15] (in this case, the domain is $|x|<\epsilon$).

**NOTES**

NOTE 1: Frobenius manifolds are the geometrical setting for the WDVV equations and were introduced by Dubrovin in [3]. They are an important object in many branches of mathematics like singularity theory and reflection groups [32] [34] [6] [4], algebraic and enumerative geometry [19] [21].

NOTE 2: The limit of $a$, for $\theta_0, \theta_1, \theta_x \to 0$, $\theta_\infty = 2\mu$, computed from the formula (125), exists and it coincides with (77)!

NOTE 3: The substitutions we have obtained now reduce to $(x_0, x_1, x_\infty) \mapsto (x_\infty, -x_1, x_0 - x_1 x_\infty)$ in the non-generic case $\beta=\gamma=1-2\delta=0$.

NOTE 4: The substitutions we have obtained now reduce to $(x_0, x_1, x_\infty) \mapsto (x_1, x_0, x_0 x_1 - x_\infty)$ in the non-generic case $\beta=\gamma=1-2\delta=0$.

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