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# Recursive Fermion System in Cuntz Algebra. I

## — Embeddings of Fermion Algebra into Cuntz Algebra —

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***Abstract***

Embeddings of the CAR (canonical anticommutation relations) algebra of fermions into the Cuntz algebra $\mathcal{O}_2$ (or $\mathcal{O}_{2d}$ more generally) are presented by using recursive constructions. As a typical example, an embedding of CAR onto the $U(1)$-invariant subalgebra of $\mathcal{O}_2$ is constructed explicitly. Generalizing this construction to the case of $\mathcal{O}_{2^p}$, an embedding of CAR onto the $U(1)$-invariant subalgebra of $\mathcal{O}_{2^p}$ is obtained. Restricting a permutation representation of the Cuntz algebra, we obtain the Fock representation of CAR. We apply the results to embed the algebra of parafermions of order $p$ into $\mathcal{O}_{2^p}$ according to the Green's ansatz.

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## §1. Introduction

It is well understood that the $C^*$-algebra[c] of CAR (canonical anticommutation relations)[1] of fermions, in which the generators $a_m$ and $a_n^*$ ($m, n = 1, 2, \ldots$) satisfy

$$\{a_m, a_n\} = \{a_m^*, a_n^*\} = 0, \tag{1.1}$$

$$\{a_m, a_n^*\} = \delta_{m,n} I, \tag{1.2}$$

is isomorphic with the $\mathrm{UHF}_2$ (i.e., the uniformly hyperfinite algebra of Glimm type $2^\infty$), which is defined by the $C^*$-algebra isomorphic with (the norm closure of) $\bigcup_{n=1}^{\infty} M_{2^n}$ or $\bigotimes^{\infty} M_2$ with $M_k$ denoting the algebra of all $k \times k$ complex matrices. The Cuntz algebra[2] $\mathcal{O}_2$ is a simple $C^*$-algebra generated by $s_1$ and $s_2$ satisfying

$$s_i^* s_j = \delta_{i,j}\, I, \qquad i, j = 1, 2, \tag{1.3}$$

$$s_1 s_1^* + s_2 s_2^* = I. \tag{1.4}$$

We define a $U(1)$ action on $\mathcal{O}_2$ by

$$s_i \mapsto z s_i, \quad z \in \mathbf{C}, \quad |z| = 1, \quad i = 1, 2. \tag{1.5}$$

Then, the subalgebra $\mathcal{O}_2^{U(1)}$ consisting of $U(1)$-invariant elements of $\mathcal{O}_2$ is a linear space generated by monomials of the form

$$s_{i_1} \cdots s_{i_k} s_{j_k}^* \cdots s_{j_1}^*, \tag{1.6}$$

where $i_1, \ldots, i_k, j_1, \ldots, j_k = 1, 2$. From the one-to-one correspondence between the matrix element $e_{i_1 j_1} \otimes e_{i_2 j_2} \otimes \cdots \otimes e_{i_k j_k} \otimes I \otimes I \otimes \cdots$ and (1.6), we have an isomorphism[2] between $\mathrm{UHF}_2$ and $\mathcal{O}_2^{U(1)}$. Therefore, we have an isomorphism between CAR and $\mathcal{O}_2^{U(1)} \subset \mathcal{O}_2$. In other words, there exists an embedding of CAR into $\mathcal{O}_2$. Although this fact may be well-known, we could find neither its explicit expression nor systematic study about it in literature. Furthermore, since the Cuntz algebra is finitely generated, its algebraic structures seem more manageable than that of CAR. Indeed, we can make many nontrivial unital $*$-endomorphisms in the Cuntz algebra explicitly as will be shown in our succeeding paper.[6] Thus, if we have explicit expressions for the embedding of CAR into the Cuntz algebra, we can study various properties of CAR by restricting those of the Cuntz algebra.

The parafermion algebra,[4,5] in which the generators satisfy a certain set of double commutation relations, is not yet studied at all as a $C^*$-subalgebra of the Cuntz algebra. According to the Green's ansatz, we decompose each generator of the parafermion algebra into a sum of $p$ fermionic generators *commuting* with each other, where $p$ is the order of parastatistics. Then, the parafermion algebra of order $p$ is a subset of a tensor product of $p$ CAR's, which is isomorphic with $\mathrm{UHF}_{2^p} \cong \mathcal{O}_{2^p}^{U(1)} \subset \mathcal{O}_{2^p}$. Therefore, we may also have an embedding of a parafermion algebra into the Cuntz algebra.

The purpose of this paper is to present the explicit expressions for the above embedding of the algebra of (para)fermions into the Cuntz algebra in a recursive way and to

[c] In this paper, we discuss only on the $*$-algebraic structure and avoid considering the $C^*$-norm structure.

construct representations of the former by restricting those of the latter. We call a fundamental ingredient for our recursive construction of embeddings of CAR into $\mathcal{O}_2$ (or $\mathcal{O}_{2d}$ with $d = 1, 2, \ldots$, more generally) a *recursive fermion system (RFS)*. It is shown that there exists a map $\zeta$ on $\mathcal{O}_{2d}$ such that

$$a_n = \zeta^{n-1}(\boldsymbol{a}), \quad n = 1, 2, \ldots \tag{1.7}$$

satisfy (1.1) and (1.2) for a suitable element $\boldsymbol{a} \in \mathcal{O}_{2d}$. In $\mathcal{O}_2$, we introduce the simplest example of RFS in which the image of CAR is identical with $\mathcal{O}_2^{U(1)}$. It should be noted, however, that RFS in $\mathcal{O}_2$ is *not* unique, that is, an image of CAR is not necessarily identical with $\mathcal{O}_2^{U(1)}$. We also consider a generalization of RFS in $\mathcal{O}_{2^p}$, which we call $\mathrm{RFS}_p$, and find that in a special case the image of CAR is identical with $\mathcal{O}_{2^p}^{U(1)} \cong \mathrm{UHF}_{2^p}$. As an explicit representation of the Cuntz algebra, we consider the permutation representation.[3] Restricting a certain permutation representation of $\mathcal{O}_2$ (or $\mathcal{O}_{2^p}$) to the images of CAR, we obtain the Fock representation of CAR. We apply the above results to obtain a recursive construction of the algebra of parafermions of order $p$ in $\mathcal{O}_{2^p}$. We call its fundamental ingredient a *recursive parafermion system of order p ($\mathrm{RPFS}_p$)*. The relation between $\mathrm{RFS}_p$ and $\mathrm{RPFS}_p$ is described by the Klein transformation. What is discussed in this paper is the most fundamental aspect of the recursive fermion system. Importance of using the Cuntz algebra to describe fermion algebra will be shown in our succeeding papers.

The present paper is organized as follows. In Sec.2, we make brief introduction of the Cuntz algebra necessary for our discussion in the succeeding sections. In Sec.3, the recursive fermion system is presented and the restriction of a permutation representation to the image of CAR is studied. In Sec.4, an embedding of parafermion algebra into the Cuntz algebra is recursively constructed by using the Green's ansatz. The final section is devoted to discussion.

## §2. Brief Introduction of Cuntz Algebra

The Cuntz algebra[2] $\mathcal{O}_d$ ($d \geqq 2$) is a simple $C^*$-algebra generated by $s_1, s_2, \ldots, s_d$ satisfying the following relations:

$$s_i^* s_j = \delta_{i,j} I, \tag{2.1}$$

$$\sum_{i=1}^{d} s_i s_i^* = I, \tag{2.2}$$

where $^*$ is a $*$-involution (or an adjoint operation), $I$ being the unit (or the identity operator). We often use the brief description such as $s_{i_1 i_2 \cdots i_m} \equiv s_{i_1} s_{i_2} \cdots s_{i_m}$, $s^*_{i_1 i_2 \cdots i_m} \equiv s^*_{i_m} \ldots s^*_{i_2} s^*_{i_1}$ and $s_{i_1 \cdots i_m ; j_n \cdots j_1} \equiv s_{i_1} \cdots s_{i_m} s^*_{j_n} \cdots s^*_{j_1}$. From the relation (2.1), $\mathcal{O}_d$ is a linear space generated by monomials of the form $s_{i_1 \cdots i_m ; j_n \cdots j_1}$ with $m + n \geqq 1$. One should note that $*$-representations of $\mathcal{O}_d$ are inevitably infinite dimensional, because (2.1) means that $s_i$'s are unitary in finite dimension while (2.2) prevents them from being so.

Obviously, a $d \times d$ matrix algebra $M_d$ is isomorphic with the subalgebra of $\mathcal{O}_d$ generated by $s_{i;j}$. Likewise, a tensor product $M_d \otimes M_d$ is isomorphic with the subalgebra generated by $s_{i_1 i_2 ; j_2 j_1}$. In general, $\mathrm{UHF}_d \cong \bigotimes^{\infty} M_d$ is isomorphic with $\mathcal{O}_d^{U(1)}$, which is a linear

space generated by monomials of the form $s_{i_1 \cdots i_k ; j_k \cdots j_1}$, $k \geqq 1$.

A unital $*$-endomorphism $\varphi$ of $\mathcal{O}_d$ is defined by a mapping $\varphi : \mathcal{O}_d \to \mathcal{O}_d$ satisfying

$$\varphi(\alpha X + \beta Y) = \alpha\varphi(X) + \beta\varphi(Y), \quad \alpha, \beta \in \mathbf{C},\ X, Y \in \mathcal{O}_d, \tag{2.3}$$

$$\varphi(XY) = \varphi(X)\varphi(Y), \quad X, Y \in \mathcal{O}_d, \tag{2.4}$$

$$\varphi(X^*) = \varphi(X)^*, \quad X \in \mathcal{O}_d, \tag{2.5}$$

$$\varphi(I) = I. \tag{2.6}$$

A typical example of unital $*$-endomorphisms of $\mathcal{O}_d$ is the canonical endomorphism $\rho$ defined by

$$\rho(X) = \sum_{i=1}^{d} s_i X s_i^*, \quad X \in \mathcal{O}_d. \tag{2.7}$$

Indeed, from (2.1), $\rho$ satisfies $\rho(X)\rho(Y) = \rho(XY)$. From (2.2), $\rho$ is unital, that is, $\rho(I) = I$, hence $S_i \equiv \rho(s_i)$, $i = 1, 2, \ldots, d$ satisfy the relations (2.1) and (2.2).

In the present paper, we consider the following $*$-representation $\pi_s$ of $\mathcal{O}_d$ on a countable infinite-dimensional Hilbert space $\mathcal{H}$. Let $\{e_n\}_{n=1}^{\infty}$ be a complete orthonormal basis of $\mathcal{H}$. We define the action of $\pi_s(s_i)$ $(i = 1, \ldots, d)$ on $\mathcal{H}$ by

$$\pi_s(s_i) e_n \equiv e_{\mu_i(n)}, \quad \mu_i(n) \equiv d(n-1) + i, \quad i = 1, \ldots, d, \quad n \in \mathbf{N}. \tag{2.8}$$

By this definition, $\pi_s(s_i)$ is defined on whole $\mathcal{H}$ linearly as a bounded operator. Then, the action of $\pi_s(s_i)^*$ on $e_n$ is determined by the definition of the adjoint operation. Since for any $n \in \mathbf{N}$ there exists a pair $\{j,\ m\}$ which satisfy $\mu_j(m) = n$, we consider $\pi_s(s_i)^*$ on $e_{\mu_j(m)}$ as follows

$$\begin{aligned} \langle \pi_s(s_i)^* e_{\mu_j(m)} | e_\ell \rangle &= \langle e_{\mu_j(m)} | \pi_s(s_i) e_\ell \rangle \\ &= \langle e_{\mu_j(m)} | e_{\mu_i(\ell)} \rangle \\ &= \delta_{i,j}\delta_{m,\ell} \\ &= \delta_{i,j} \langle e_m | e_\ell \rangle \quad \text{for any } \ell, \end{aligned} \tag{2.9}$$

where $\langle \cdot \,|\, \cdot \rangle$ denotes the inner product on $\mathcal{H}$. In the third equality of (2.9), use has been made of that $\mu_i$'s $(i = 1, 2, \ldots, d)$ are injections and $\mu_i(\mathbf{N}) \bigcap \mu_j(\mathbf{N}) = \emptyset$ for each $i \neq j$ by (2.8). Hence, we obtain

$$\pi_s(s_i)^* e_{\mu_j(m)} = \delta_{i,j} e_m. \tag{2.10}$$

It is, now, straightforward to show that $\pi_s(s_i)$ and $\pi_s(s_i)^*$ defined by (2.8) and (2.10) satisfy the relations (2.1) and (2.2) on any $e_n$. The above representation $\pi_s$ of $\mathcal{O}_d$ is an example of the permutation representations[3] with a branching function system $\{\mu_i\}_{i=1}^{d}$. One should note that $\pi_s(s_1)$ has eigenvalue 1 with eigenvector $e_1$. An irreducible representation which satisfies this property is uniquely determined up to unitary equivalence. We call this permutation representation *Rep(1)* or the *standard representation* of $\mathcal{O}_d$.

## §3. Recursive Fermion System (RFS)

In this section, we consider how to construct embeddings of CAR into $\mathcal{O}_2$ (or $\mathcal{O}_{2d}$ with $d = 1, 2, \ldots$) in a systematic way.

### §§3-1. RFS in $\mathcal{O}_{2d}$

Let $\boldsymbol{a} \in \mathcal{O}_{2d}$, $\zeta : \mathcal{O}_{2d} \to \mathcal{O}_{2d}$ be a linear mapping, and $\varphi$ a unital $*$-endomorphism on $\mathcal{O}_{2d}$, respectively. A triad $R = (\boldsymbol{a}, \zeta, \varphi)$ is called a *recursive fermion system (RFS)* in $\mathcal{O}_{2d}$, if it satisfies the following conditions:[d)]

$$\text{i) seed condition:} \qquad \boldsymbol{a}^2 = 0, \quad \{\boldsymbol{a}, \boldsymbol{a}^*\} = I, \tag{3.1}$$

$$\text{ii) recursive condition:} \qquad \{\boldsymbol{a}, \zeta(X)\} = 0, \quad \zeta(X)^* = \zeta(X^*), \quad X \in \mathcal{O}_{2d}, \tag{3.2}$$

$$\text{iii) normalization condition:} \quad \zeta(X)\zeta(Y) = \varphi(XY), \quad X, Y \in \mathcal{O}_{2d}. \tag{3.3}$$

Here, $\boldsymbol{a}$ and $\zeta$ are called the *seed* and the *recursive map,* respectively, of the RFS $R$. For a given RFS $R = (\boldsymbol{a}, \zeta, \varphi)$, the embedding associated with $R$,

$$\Phi_R : \mathrm{CAR} \hookrightarrow \mathcal{O}_{2d}, \tag{3.4}$$

is defined by specifying images of generators $a_n$ $(n = 1, 2, \ldots)$ of CAR as follows

$$\Phi_R(a_n) \equiv \zeta^{n-1}(\boldsymbol{a}) \equiv (\underbrace{\zeta \circ \zeta \circ \cdots \circ \zeta}_{n-1})(\boldsymbol{a}), \quad n = 1, 2, \ldots. \tag{3.5}$$

Indeed, from (3.1)–(3.5), we obtain

$$\{\Phi_R(a_m), \Phi_R(a_n)\} = \varphi^{m-1}(\{\boldsymbol{a}, \zeta^{n-m}(\boldsymbol{a})\}) = \varphi^{m-1}(0) = 0, \quad m \leqq n, \tag{3.6}$$

$$\{\Phi_R(a_m), \Phi_R(a_n)^*\} = \varphi^{m-1}(\{\boldsymbol{a}, \varphi^{n-m}(\boldsymbol{a}^*)\}) = \varphi^{m-1}(0) = 0, \quad m < n, \tag{3.7}$$

$$\{\Phi_R(a_n), \Phi_R(a_n)^*\} = \varphi^{n-1}(\{\boldsymbol{a}, \boldsymbol{a}^*\}) = \varphi^{n-1}(I) = I. \tag{3.8}$$

We denote $\mathcal{A}_R \equiv \Phi_R(\mathrm{CAR}) \subset \mathcal{O}_{2d}$. $\mathcal{A}_R$ is called the CAR subalgebra associated with $R$.

A typical example of RFS in $\mathcal{O}_2$ is given by the *standard RFS* $SR = (\boldsymbol{a}, \zeta, \varphi)$, which is defined by

$$\boldsymbol{a} \equiv s_1 s_2^*, \tag{3.9}$$

$$\zeta(X) \equiv s_1 X s_1^* - s_2 X s_2^*, \quad X \in \mathcal{O}_2, \tag{3.10}$$

$$\varphi(X) \equiv \rho(X) = s_1 X s_1^* + s_2 X s_2^*, \quad X \in \mathcal{O}_2, \tag{3.11}$$

where $\rho$ is the canonical endomorphism (2.7) of $\mathcal{O}_2$. Indeed, it is easy to show that (3.9)–(3.11) satisfy (3.1)–(3.3). For the standard RFS $SR$, we denote $\mathcal{A}_S \equiv \Phi_{SR}(\mathrm{CAR})$. Then, we have $\mathcal{A}_S = \mathcal{O}_2^{U(1)}$ as is easily proved by mathematical induction: It is obvious that $\mathcal{A}_S \subset \mathcal{O}_2^{U(1)}$, while $\mathcal{A}_S \supset \mathcal{O}_2^{U(1)}$ is also satisfied since any $s_{i_1 \cdots i_k; j_k \cdots j_1} \in \mathcal{O}_2^{U(1)}$, $k \geqq 1$ is expressed in terms of $\Phi_{SR}(a_n)$ $(n \leqq k)$.

The standard RFS $SR$ in $\mathcal{O}_2$ can be easily extrapolated to the corresponding one

[d)] It is possible to define a RFS satisfying the conditions similar to (3.1)–(3.3) in any unital $C^*$-algebra apart from its existence.

$R = (\boldsymbol{a}, \zeta, \varphi)$ in $\mathcal{O}_{2d}$ $(d = 1, 2, \ldots)$ as follows:

$$\boldsymbol{a} \equiv \sum_{k=1}^{d} \epsilon_k \, s_{i_k} s_{j_k}^*, \tag{3.12}$$

$$\zeta(X) \equiv \sum_{k=1}^{d} \epsilon_k' \, (s_{i_k} X s_{i_k}^* - s_{j_k} X s_{j_k}^*), \quad X \in \mathcal{O}_{2d}, \tag{3.13}$$

$$\varphi(X) \equiv \rho(X) = \sum_{i=1}^{2d} s_i X s_i^*, \quad X \in \mathcal{O}_{2d}, \tag{3.14}$$

$$\Phi_R(a_n) \equiv \zeta^{n-1}(\boldsymbol{a}), \tag{3.15}$$

where an arbitrary division of indices $\{1, \ldots, 2d\}$ into two ordered parts $\{i_1 \equiv 1, i_2, \ldots, i_d\}$ and $\{j_1, j_2, \ldots, j_d\}$ is introduced in (3.12); $\epsilon_1 = \epsilon_1' = +1$, $\epsilon_k, \epsilon_k' = \pm 1$ $(k \geqq 2)$. The subalgebra of $\mathcal{O}_{2d}$ generated by $\Phi_R(a_n)$ $(n \leqq k)$ is isomorphic with $\overset{k}{\bigotimes}(I_d \otimes M_2) \cong \overset{k}{\bigotimes} M_2$, where $I_d$ is the $d \times d$ unit matrix. Thus, $\mathcal{A}_R$ determined by (3.12)–(3.15) is isomorphic with $\mathrm{UHF}_2$.

### §§3-2. Generalization of RFS

As an example of RFS in $\mathcal{O}_4$, we may set

$$\boldsymbol{a} = s_1 s_2^* + s_3 s_4^*, \tag{3.16}$$

$$\zeta(X) = s_1 X s_1^* - s_2 X s_2^* \pm (s_3 X s_3^* - s_4 X s_4^*). \tag{3.17}$$

It is interesting to consider whether there are nontrivial elements in the form of (3.12) which anticommute with $\boldsymbol{a}$ defined by (3.16). We find such an element is uniquely given by

$$\tilde{\boldsymbol{a}} = s_1 s_3^* - s_2 s_4^*. \tag{3.18}$$

Furthermore, if we require that $\zeta(X)$ anticommute with $\tilde{\boldsymbol{a}}$ as well as with $\boldsymbol{a}$, $\zeta$ is uniquely determined as

$$\zeta(X) = s_1 X s_1^* - s_2 X s_2^* - s_3 X s_3^* + s_4 X s_4^*. \tag{3.19}$$

Thus, we can extend the RFS $R = (\boldsymbol{a}, \zeta, \varphi)$ in $\mathcal{O}_4$ defined by (3.16), (3.19) and (3.14) with $d = 2$ to the tetrad $SR_2 = (\boldsymbol{a}_1 \equiv \boldsymbol{a}, \boldsymbol{a}_2 \equiv \tilde{\boldsymbol{a}}; \zeta, \varphi)$ in such a way that

$$\begin{gathered} \Phi_{SR_2} : \mathrm{CAR} \hookrightarrow \mathcal{O}_4, \\ \Phi_{SR_2}(a_{2(n-1)+i}) \equiv \zeta^{n-1}(\boldsymbol{a}_i), \quad i = 1, 2, \ n = 1, 2, \ldots \end{gathered} \tag{3.20}$$

gives an embedding of CAR into $\mathcal{O}_4$. In contrast with the ordinary RFS given by a triad, we call this kind of tetrad $R_2 = (\boldsymbol{a}_1, \boldsymbol{a}_2; \zeta, \varphi)$ $\mathrm{RFS}_2$. Although we have defined a $\mathrm{RFS}_2$ with specifying the explicit expressions for $R_2 = SR_2$ in the above, $\mathrm{RFS}_2$ itself can be, of course, defined abstractly without such a specific expression. We call the above $\mathrm{RFS}_2$ denoted by $SR_2$ the *standard RFS*$_2$ since it is the special one in which $\mathcal{A}_{S_2} \equiv \Phi_{SR_2}(\mathrm{CAR}) = \mathcal{O}_4^{U(1)}$: It is obvious that $\mathcal{A}_{S_2} \subset \mathcal{O}_4^{U(1)}$, while $\mathcal{A}_{S_2} \supset \mathcal{O}_4^{U(1)}$ is also satisfied since any $s_{i_1 \cdots i_k ; j_k \cdots j_1} \in \mathcal{O}_4$, $k \geqq 1$ is expressed in terms of $\Phi_{SR_2}(a_n)$ $(n \leqq 2k)$.

Likewise, in $\mathcal{O}_{2^p}$, we can generalize RFS to RFS$_p$ $R_p = (\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_p, \zeta, \varphi)$ in such a way that

$$\begin{aligned} &\Phi_{R_p} : \mathrm{CAR} \hookrightarrow \mathcal{O}_{2^p}, \\ &\Phi_{R_p}(a_{p(n-1)+i}) \equiv \zeta^{n-1}(\boldsymbol{a}_i), \quad i = 1, \ldots, p,\ n = 1, 2, \ldots \end{aligned} \tag{3.21}$$

gives an embedding of CAR into $\mathcal{O}_{2^p}$ and none of $\boldsymbol{a}_i$ $(i = 1, \ldots, p)$ is expressed as $\zeta(X)$ with $X \in \mathcal{O}_{2^p}$. Here $\zeta$ and $\varphi$ are a linear mapping and a unital endomorphism on $\mathcal{O}_{2^p}$, respectively; $(\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_p; \zeta, \varphi)$ should satisfy

i) seed condition: $\{\boldsymbol{a}_i, \boldsymbol{a}_j\} = 0, \quad \{\boldsymbol{a}_i, \boldsymbol{a}_j^*\} = \delta_{i,j} I,$ (3.22)

ii) recursive condition: $\{\boldsymbol{a}_i, \zeta(X)\} = 0, \quad \zeta(X)^* = \zeta(X^*), \quad X \in \mathcal{O}_{2^p},$ (3.23)

iii) normalization condition: $\zeta(X)\zeta(Y) = \varphi(XY), \quad X, Y \in \mathcal{O}_{2^p}.$ (3.24)

The *standard RFS*$_p$ $SR_p = (\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_p; \zeta, \varphi)$, in which $\mathcal{A}_{S_p} \equiv \Phi_{SR_p}(\mathrm{CAR}) = \mathcal{O}_{2^p}^{U(1)}$, is given by

$$\boldsymbol{a}_i = \sum_{k=1}^{2^{p-i}} \sum_{\ell=1}^{2^{i-1}} (-1)^{\sum_{m=1}^{i-1} \left[\frac{\ell-1}{2^{m-1}}\right]} s_{2^i(k-1)+\ell} s^*_{2^{i-1}(2k-1)+\ell}, \quad i = 1, \ldots, p, \tag{3.25}$$

$$\zeta(X) = \sum_{i=1}^{2^p} (-1)^{\sum_{m=1}^{p} \left[\frac{i-1}{2^{m-1}}\right]} s_i X s_i^*, \tag{3.26}$$

$$\varphi(X) = \rho(X) \equiv \sum_{i=1}^{2^p} s_i X s_i^*, \tag{3.27}$$

where $[x]$ denotes the largest integer not greater than $x$. We have constructed (3.25) in the *bootstrap* way: For any $\boldsymbol{a}_i$, each term $s_k s_\ell^*$ in $\boldsymbol{a}_i$ requires that $\boldsymbol{a}_j$ $(j \neq i)$ should involve either $s_{m_j} s_k^* \mp s_{n_j} s_\ell^*$ or $s_k s_{m_j}^* \mp s_\ell s_{n_j}^*$ with some $m_j$ and $n_j$, and then $\boldsymbol{a}_i$ is conversely required to involve $\pm s_{m_j} s_{n_j}^*$ for all $j$, so that $\{\boldsymbol{a}_i \,|\, i = 1, 2, \ldots, p\}$ satisfy the $p$-dimensional canonical anticommutation relations. In this way, for a given $\boldsymbol{a}_1 \equiv \sum_{k=1}^{2^{p-1}} s_{2k-1} s_{2k}^*$, the other $\boldsymbol{a}_i$'s are uniquely constructed with the normalization that the sign factor for the term $s_1 s_k^*$ in $\boldsymbol{a}_i$ is plus. As for $\zeta$ in (3.26), it is uniquely determined from (3.23) and from the normalization that the sign factor for $s_1 X s_1^*$ is plus.

### §§3-3. Representation of RFS

In this subsection and the next, we consider the representation of CAR, which is obtained by restricting the standard representation $\pi_s$ on $\mathcal{H}$ defined by (2.8) of $\mathcal{O}_2$ (or $\mathcal{O}_{2^p}$) to the standard RFS (or the standard RFS$_p$):

$$\mathrm{CAR} \overset{\Phi_{SR}}{\hookrightarrow} \mathcal{O}_2 \xrightarrow{\pi_s} \mathcal{L}(\mathcal{H}). \tag{3.28}$$

Hereafter, we identify $s_i$ and $a_n$ with $\pi_s(s_i)$ and $(\pi_s \circ \Phi_{SR})(a_n)$ in $\mathcal{O}_2$ (or $(\pi_s \circ \Phi_{SR_p})(a_n)$ in $\mathcal{O}_{2^p}$), respectively, for simplicity of description.

For the standard representation Rep(1) of $\mathcal{O}_2$ defined by (2.8), we find that $e_1$ is a vacuum with respect to the annihilation operators $a_n$ ($n = 1, 2, \ldots$) defined by (3.5) with (3.9) and (3.10), that is, it satisfies

$$a_n e_1 = 0 \quad \text{for } n = 1, 2, \ldots. \tag{3.29}$$

The proof of (3.29) is the following: From (2.8) and (2.10), we obtain

$$\begin{aligned} s_1^* e_{2n-1} &= e_n, & s_1^* e_{2n} &= 0, \\ s_2^* e_{2n-1} &= 0, & s_2^* e_{2n} &= e_n. \end{aligned} \qquad n = 1, 2, \ldots. \tag{3.30}$$

Hence, $e_1$ is the simultaneous eigenvector both for $s_1^*$ with eigenvalue 1 and for $s_2^*$ with 0. Thus, from (3.5) with (3.9) and (3.10), we have

$$a_n\, e_1 = \zeta^{n-1}(\boldsymbol{a})\, e_1 = s_1^{n-1}\, \boldsymbol{a}\, e_1 = 0. \tag{3.31}$$

The (antisymmetric) Fock space with the vacuum $e_1$ is generated by operation of monomials $a_{n_1}^* \cdots a_{n_k}^*$ on $e_1$. Since $a_m^*$ and $a_n^*$ anticommute with each other, we have only to consider the case $n_1 < \cdots < n_k$. For $k = 1$, from (2.8) and (3.30), we have

$$\begin{aligned} a_{n_1}^*\, e_1 &= \zeta^{n_1-1}(\boldsymbol{a}^*)\, e_1 \\ &= s_1^{n_1-1} \boldsymbol{a}^*\, e_1 \\ &= s_1^{n_1-1} s_2\, e_1 \\ &= e_{N(n_1)}, \qquad N(n_1) \equiv 2^{n_1-1} + 1. \end{aligned} \tag{3.32}$$

For $k = 2$, from (2.1), (2.8), (3.5), (3.9), (3.10) and (3.32), we have

$$\begin{aligned} a_{n_1}^* a_{n_2}^*\, e_1 &= \zeta^{n_1-1}(\boldsymbol{a}^*)\, s_1^{n_2-1}\, s_2\, e_1 \\ &= s_1^{n_1-1}\, \boldsymbol{a}^*\, s_1^{n_2-n_1}\, s_2\, e_1 \\ &= s_1^{n_1-1}\, s_2\, s_1^{n_2-n_1-1}\, s_2\, e_1 \\ &= e_{N(n_1,n_2)}, \qquad N(n_1, n_2) \equiv 2^{n_1-1} + 2^{n_2-1} + 1. \end{aligned} \tag{3.33}$$

Likewise, in general, we obtain

$$a_{n_1}^* a_{n_2}^* \cdots a_{n_k}^*\, e_1 = s_1^{n_1-1}\, s_2\, s_1^{n_2-n_1-1}\, s_2\, s_1^{n_3-n_2-1}\, s_2 \cdots s_1^{n_k-n_{k-1}-1}\, s_2\, e_1$$

$$= e_{N(n_1,\cdots,n_k)}, \tag{3.34}$$

$$N(n_1, \cdots, n_k) \equiv 2^{n_1-1} + 2^{n_2-1} + \cdots + 2^{n_k-1} + 1. \tag{3.35}$$

Since it is obvious that any $n \in \mathbf{N}$ is expressible in the form of $N(n_1, \cdots, n_k) - 1$ (binary expression), any $e_n$ is uniquely given in the form of (3.34), that is, $e_1$ is a cyclic vector in $\mathcal{H}$ and there is no vacuum annihilated by $a_n$ ($n = 1, 2, \ldots$) other than $e_1$. Therefore, the restriction of Rep(1) of $\mathcal{O}_2$ to $\mathcal{A}_S$ is the Fock representation with the unique vacuum.

One should note that any vector $e_N$ instead of $e_1$ of the basis in Rep(1) of $\mathcal{O}_2$ can be set as the vacuum of Fock representation by the Bogoliubov transformation exchanging *finite* number of the annihilation/creation operators such as

$$a_n \mapsto a'_n = \begin{cases} a_n^* & \text{for } n \in \{n_1, n_2, \ldots, n_k\}, \\ a_n & \text{otherwise}, \end{cases} \tag{3.36}$$

where the set of indices $\{n_1, n_2, \ldots, n_k\}$ corresponds to $N$ through the relation (3.35).

### §§3-4. Representation of RFS$_p$

As for the restriction of Rep(1) of $\mathcal{O}_{2^p}$ to the standard RFS$_p$, it is straightforward to generalize the previous results. From (2.8) and (2.10) with $d = 2^p$, we have

$$s_i^* e_{2^p(n-1)+j} = \delta_{i,j} e_n, \quad i, j = 1, 2, \ldots, 2^p, \; n = 1, 2, \ldots, \tag{3.37}$$

hence, for (3.21) with (3.25) and (3.26), we obtain

$$a_{p(m-1)+i}\, e_1 = s_1^{m-1} \boldsymbol{a}_i\, e_1 = 0, \quad i = 1, 2, \ldots, p, \; m = 1, 2, \ldots. \tag{3.38}$$

Thus, $e_1$ of Rep(1) of $\mathcal{O}_p$ is a vacuum for the annihilation operators $a_n$ $(n = 1, 2 \ldots)$ of the standard RFS$_p$, and the corresponding Fock space is generated by $a_{n_1}^* a_{n_2}^* \cdots a_{n_k}^*\, e_1$ with $1 \leqq n_1 < n_2 < \cdots < n_k$, $k \geqq 1$. Their explicit expressions are obtained in the following.

In case $1 \leqq m_1 < m_2 < \cdots < m_k$ for $n_j = p(m_j - 1) + i_j$, $i_j = 1, \ldots, p$, $j = 1, \ldots, k$, likewise in (3.34), we have

$$\begin{aligned} &a_{p(m_1-1)+i_1}^* a_{p(m_2-1)+i_2}^* \cdots a_{p(m_k-1)+i_k}^*\, e_1 \\ &\qquad = s_1^{m_1-1} s_{M_1+1} s_1^{m_2-m_1-1} s_{M_2+1} \cdots s_1^{m_k-m_{k-1}-1} s_{M_k+1}\, e_1, \end{aligned} \tag{3.39}$$

$$M_j \equiv 2^{i_j - 1}, \quad j = 1, 2, \ldots, k. \tag{3.40}$$

On the other hand, in case $(m_{j-1} <)\, m_j = m_{j+1} = \cdots = m_{j+\ell}\, (< m_{j+\ell+1})$ and $i_j < i_{j+1} < \cdots < i_{j+\ell}$ with some $j$ and $\ell$ for $n_j = p(m_j - 1) + i_j$, $i_j = 1, \ldots, p$, $j = 1, \ldots, k$, the corresponding factor $s_1^{m_j - m_{j-1} - 1} s_{M_j+1} \cdots s_1^{m_{j+\ell} - m_{j+\ell-1} - 1} s_{M_{j+\ell}+1}$ in rhs of (3.39) is replaced by

$$s_1^{m_j - m_{j-1} - 1} s_{M_{j,\ell}+1}, \quad M_{j,\ell} \equiv \sum_{i=j}^{j+\ell} M_i. \tag{3.41}$$

Therefore, using the formula

$$s_1^{n_1-1} s_{i_1+1} s_1^{n_2-n_1-1} s_{i_2+1} \cdots s_1^{n_k-n_{k-1}-1} s_{i_k+1}\, e_1 = e_{N(n_1, i_1; \ldots; n_k, i_k)}, \tag{3.42}$$

$$N(n_1, i_1; \ldots; n_k, i_k) \equiv \sum_{j=1}^{k} i_j\, 2^{p(n_j-1)} + 1 \tag{3.43}$$

for $1 \leqq n_1 < \cdots < n_k$ and $i_1, \ldots, i_k = 1, 2, \ldots, 2^p - 1$, it is straightforward to obtain

$$a_{p(m_1-1)+i_1}^* \cdots a_{p(m_k-1)+i_k}^*\, e_1 = e_{N(m_1, i_1; \cdots; m_k, i_k)}, \tag{3.44}$$

$$N(m_1, i_1; \cdots; m_k, i_k) \equiv \sum_{j=1}^{k} 2^{p(m_j-1)+i_j-1} + 1, \tag{3.45}$$

for $p(m_1 - 1) + i_1 < \cdots < p(m_k - 1) + i_k$, hence,

$$a_{n_1}^* \cdots a_{n_k}^*\, e_1 = e_{N(n_1, \cdots, n_k)}, \quad n_1 < \cdots < n_k, \tag{3.46}$$

$$N(n_1, \cdots, n_k) \equiv \sum_{j=1}^{k} 2^{n_j - 1} + 1, \tag{3.47}$$

which is exactly the same as (3.34) with (3.35). Thus, in the same way as the standard RFS, the restriction of Rep(1) of $\mathcal{O}_{2^p}$ to $\mathcal{A}_{S_p}$ is the Fock representation with the unique vacuum. It should be noted that this Fock representation is *strictly common to all the standard RFS$_p$.* This result may be understood by the fact that the standard RFS$_p$ can be reduced to the standard RFS through a certain embedding of $\mathcal{O}_{2^p}$ into $\mathcal{O}_2$.[6)]

## §4. Recursive Parafermion System

In this section, we apply RFS to embed the algebra of parafermions into the Cuntz algebra. According to the Green's ansatz, the algebra of parafermions of order $p$ is described in terms of $p$ fermion algebras in which any element in one fermion algebra *commute* with those in others. Hence, we can embed the algebra of parafermions into $\mathcal{O}_{2^p}$ by generalizing RFS$_p$ so that they satisfy the suitable properties.

### §§4-1. Parafermion Algebra

To begin with, we summarize the property of the algebra of parafermions.[4,5)]

Let the *parafermion algebra of order* $p\,(=1,\,2,\,\ldots)$ *PFA$_p$* be the $*$-algebra generated by $a_n$ $(n=1,\,2,\,\ldots)$ satisfying

$$[a_\ell,\,[a_m,\,a_n]\,]=0, \tag{4.1}$$

$$[a_\ell,\,[a_m^*,\,a_n]\,]=2\delta_{\ell,m}a_n, \tag{4.2}$$

$$\prod_{k=0}^{p}\Big(N_n+\big(k-\frac{p}{2}\big)I\Big)=0,\quad N_n\equiv\frac{1}{2}[a_n^*,\,a_n], \tag{4.3}$$

where we have omitted other relations obtained from (4.1) and (4.2) by using the $*$-involution (or the adjoint operation) and the Jacobi identity. The case $p=1$, PFA$_1$, is nothing but CAR. Here, we have explicitly introduced the order $p$ dependence of the algebra in (4.3). In the conventional standpoint in studying parastatistical algebra, starting with the double (anti)commutation relations such as (4.1) and (4.2) only, one introduces the positive integer $p$ to specify the vacuum as in (4.5) below in considering its representation. Then, the additional algebraic relation describing spectrum of the number operator $N_n$ in (4.3) is obtained. However, it seems more natural to introduce the order $p$ at the very beginning to make discussions with distinguishing manifestly the algebra and its representation. Thus, we have defined the parafermion algebra by (4.1)–(4.3).

The Fock representation of PFA$_p$ is characterized by the unique vacuum $|\,0\,\rangle$ which satisfies

$$a_n|\,0\,\rangle=0,\qquad n=1,\,2,\,\ldots \tag{4.4}$$

$$a_m a_n^*|\,0\,\rangle=p\,\delta_{m,n}|\,0\,\rangle. \tag{4.5}$$

The Fock space for parafermion is spanned by $|\,0\,\rangle$ and of $a_{n_1}^*\cdots a_{n_k}^*|\,0\,\rangle$ $(k=1,\,2,\,\ldots)$. The positivity of the inner product of this Fock space is due to the fact that $p$ is a positive integer.

It is possible to embed PFA$_p$ into $\bigotimes^{p}$CAR. Here, it should be noted that any element of CAR$\otimes I\otimes\cdots\otimes I$ *commutes* with that of $I\otimes$ CAR$\otimes I\otimes\cdots\otimes I$, and likewise for any other combinations. This kind of description of parastatistical algebra is introduced by

Green[4]. Let $a_n^{(\alpha)}$ $(\alpha = 1, \ldots, p;\ n = 1, 2, \ldots)$, which are called Green components, be generators of $\overbrace{I \otimes \cdots \otimes I}^{\alpha-1} \otimes \mathrm{CAR} \otimes \overbrace{I \otimes \cdots \otimes I}^{p-\alpha}$. Then, they satisfy the unconventional anticommutation relations as follows:

$$\{a_m^{(\alpha)},\, (a_n^{(\alpha)})^*\} = \delta_{m,n} I, \tag{4.6}$$

$$\{a_m^{(\alpha)},\, a_n^{(\alpha)}\} = 0, \tag{4.7}$$

$$[a_m^{(\alpha)},\, a_n^{(\beta)}] = [a_m^{(\alpha)}, (a_n^{(\beta)})^*] = 0 \quad \text{for } \alpha \neq \beta. \tag{4.8}$$

We denote the $*$-algebra generated by the Green components $\{a_n^{(\alpha)} \,|\, \alpha = 1, \ldots, p,\ n = 1, 2, \ldots\}$ by $\mathrm{GCA}_p$ (*Green-component algebra of order p*) $(\cong \bigotimes^p \mathrm{CAR})$. Then, an embedding $\iota$ of $\mathrm{PFA}_p$ into $\mathrm{GCA}_p$ is given by a linear combination of the Green components as follows

$$\begin{aligned} &\iota : \mathrm{PFA}_p \hookrightarrow \mathrm{GCA}_p, \\ &\iota(a_n) \equiv \sum_{\alpha=1}^{p} a_n^{(\alpha)}. \end{aligned} \tag{4.9}$$

It is straightforward to show that (4.9) satisfies (4.1)–(4.3). The vacuum satisfying (4.4) and (4.5) is given by the vacuum of Fock representation of $\mathrm{GCA}_p$:

$$a_n^{(\alpha)} \,|\, 0 \,\rangle = 0, \quad \alpha = 1, 2, \ldots, p, \ \ n = 1, 2, \ldots. \tag{4.10}$$

## §§4-2. Embedding of Parafermion Algebra

We, now, consider a recursive construction of embedding of $\mathrm{PFA}_p$ into $\mathcal{O}_{2^p}$. Let $\boldsymbol{a}^{(\alpha)} \in \mathcal{O}_{2^p}$, $\zeta_\alpha : \mathcal{O}_{2^p} \to \mathcal{O}_{2^p}$ be a linear mapping and $\varphi_\alpha$ a unital $*$-endomorphism of $\mathcal{O}_{2^p}$ for $\alpha = 1, 2, \ldots, p$. Then, a set of $p$ triads $RP_p = \{\, (\boldsymbol{a}^{(\alpha)};\, \zeta_\alpha,\, \varphi_\alpha) \,|\, \alpha = 1, \ldots, p \,\}$ is called a *recursive parafermion system of oder p* $\mathrm{RPFS}_p$, if it satisfies the following conditions:

i) seed condition:

$$(\boldsymbol{a}^{(\alpha)})^2 = 0, \quad \{\boldsymbol{a}^{(\alpha)},\, \boldsymbol{a}^{(\alpha)*}\} = I, \tag{4.11}$$

$$[\boldsymbol{a}^{(\alpha)},\, \boldsymbol{a}^{(\beta)}] = 0, \quad [\boldsymbol{a}^{(\alpha)},\, \boldsymbol{a}^{(\beta)*}] = 0 \quad \text{for } \alpha \neq \beta, \tag{4.12}$$

ii) recursive condition:

$$\{\boldsymbol{a}^{(\alpha)},\, \zeta_\alpha(X)\} = 0, \quad X \in \mathcal{O}_{2^p}, \tag{4.13}$$

$$[\boldsymbol{a}^{(\alpha)},\, \zeta_\beta(X)] = 0 \quad \text{for } \alpha \neq \beta, \tag{4.14}$$

iii) normalization condition:

$$\zeta_\alpha(X)\zeta_\alpha(Y) = \varphi_\alpha(XY), \quad X, Y \in \mathcal{O}_{2^p}, \tag{4.15}$$

$$[\zeta_\alpha(X),\, \zeta_\beta(Y)] = 0 \quad \text{if } \ [X, Y] = 0, \ \ \text{for } {}^\forall\alpha,\, \beta. \tag{4.16}$$

Then, an embedding $\Phi_{RP_p}$ of $\mathrm{GCA}_p$ into $\mathcal{O}_{2^p}$ is determined as

$$\begin{aligned} &\Phi_{RP_p} : \mathrm{GCA}_p \hookrightarrow \mathcal{O}_{2^p}, \\ &\Phi_{RP_p}(a_n^{(\alpha)}) \equiv \zeta_\alpha^{n-1}(\boldsymbol{a}^{(\alpha)}), \quad \alpha = 1, 2, \ldots, p, \ n = 1, 2, \ldots, \end{aligned} \tag{4.17}$$

which satisfy (4.6)–(4.8) as shown by straightforward calculation. Therefore, an embedding of $\mathrm{PFA}_p$ into $\mathcal{O}_{2^p}$ is obtained by restricting (4.17) to $\mathrm{PFA}_p \subset \mathrm{GCA}_p$ as follows

$$(\Phi_{RP_p} \circ \iota)(a_n) = \sum_{\alpha=1}^{p} \zeta_\alpha^{n-1}(\boldsymbol{a}^{(\alpha)}), \quad n = 1, 2, \ldots. \tag{4.18}$$

First, let us consider the case $p = 2$. We apply (3.12)–(3.15) to $\mathcal{O}_4$. According to the way of dividing indices $\{1, 2, 3, 4\}$ into two parts in (3.12), we obtain some $\boldsymbol{a}$'s, among which we can easily find a pair of $\boldsymbol{a}$'s commuting with each other. Here, we give an example, what we call the *standard RPFS*$_2$, $SRP_2 = \{ (\boldsymbol{a}^{(\alpha)}; \zeta_\alpha, \varphi_\alpha) \,|\, \alpha = 1, 2 \}$:

$$\boldsymbol{a}^{(1)} \equiv s_1 s_2^* + s_3 s_4^*, \tag{4.19}$$

$$\zeta_1(X) \equiv s_1 X s_1^* - s_2 X s_2^* + s_3 X s_3^* - s_4 X s_4^*, \tag{4.20}$$

$$\boldsymbol{a}^{(2)} \equiv s_1 s_3^* + s_2 s_4^*, \tag{4.21}$$

$$\zeta_2(X) \equiv s_1 X s_1^* + s_2 X s_2^* - s_3 X s_3^* - s_4 X s_4^*, \tag{4.22}$$

$$\varphi_1(X) = \varphi_2(X) = \rho(X) \equiv s_1 X s_1^* + s_2 X s_2^* + s_3 X s_3^* + s_4 X s_4^*. \tag{4.23}$$

As in the case of the standard $\mathrm{RFS}_2$, it is shown that $\Phi_{SRP_2}(\mathrm{GCA}_2)$ is identical with $\mathcal{O}_4^{U(1)}$.

One should note that (4.19) and (4.21) can be obtained from $(\boldsymbol{a}_1, \boldsymbol{a}_2)$ of the standard $\mathrm{RFS}_2$ defined by (3.16) and (3.18) through the Klein transformation as follows:

$$\boldsymbol{a}^{(1)} = \boldsymbol{a}_1, \tag{4.24}$$

$$\boldsymbol{a}^{(2)} = (I - 2\boldsymbol{a}_1^* \boldsymbol{a}_1)\, \boldsymbol{a}_2 = \left[\exp(i\pi \boldsymbol{a}_1^* \boldsymbol{a}_1)\right] \boldsymbol{a}_2. \tag{4.25}$$

Furthermore, $\zeta_\alpha$, $\alpha = 1, 2$ defined by (4.20) and (4.22) satisfy the following

$$\zeta_1^{n-1}(X) = \left[\exp\left(i\pi \sum_{k=1}^{n-1} \Phi_{SR_2}(a_{2k})^* \Phi_{SR_2}(a_{2k})\right)\right] \zeta^{n-1}(X), \tag{4.26}$$

$$\zeta_2^{n-1}(X) = \left[\exp\left(i\pi \sum_{k=1}^{n-1} \Phi_{SR_2}(a_{2k-1})^* \Phi_{SR_2}(a_{2k-1})\right)\right] \zeta^{n-1}(X), \tag{4.27}$$

where $\zeta$ is defined by (3.19) and use has been made of an identity $\zeta(XYZ) = \zeta(X)\zeta(Y)\zeta(Z)$. Thus, the Green components $\{\Phi_{SRP_2}(a_n^{(\alpha)})\}$ are rewritten in terms of generators of CAR of the standard $\mathrm{RFS}_2$ $\{\Phi_{SR_2}(a_n)\}$ (3.20) through the Klein transformation defined by

$$\Phi_{SRP_2}(a_n^{(1)}) = \begin{cases} \Phi_{SR_2}(a_1) & \text{for } n = 1, \\ \left[\exp\left(i\pi \sum_{k=1}^{n-1} \Phi_{SR_2}(a_{2k})^* \Phi_{SR_2}(a_{2k})\right)\right] \Phi_{SR_2}(a_{2n-1}) & \text{for } n \geqq 2, \end{cases} \tag{4.28}$$

$$\Phi_{SRP_2}(a_n^{(2)}) = \left[\exp\left(i\pi \sum_{k=1}^{n} \Phi_{SR_2}(a_{2k-1})^* \Phi_{SR_2}(a_{2k-1})\right)\right] \Phi_{SR_2}(a_{2n}), \tag{4.29}$$

which gives an automorphism of $\mathcal{O}_4^{U(1)}$. It should be noted that it is impossible to lift the

above Klein transformation to a unital $*$-endomorphism of $\mathcal{O}_4$.

For the case of generic $p$, we construct the standard RPFS$_p$ $SRP_p = \{(\boldsymbol{a}^{(\alpha)}; \zeta_\alpha, \phi_\alpha) \,|\, \alpha = 1, \ldots, p\}$ as follows: Let $s_i$'s be generators of $\mathcal{O}_{2p}$. For an arbitrarily fixed $\alpha$, each term $s_i s_j^*$ in $\boldsymbol{a}^{(\alpha)}$ requires that $\boldsymbol{a}^{(\beta)}$ $(\beta \neq \alpha)$ should involve either $s_i s_{k_\beta}^* + s_j s_{\ell_\beta}^*$ or $s_{k_\beta} s_i^* + s_{\ell_\beta} s_j^*$ with some $k_\beta$ and $\ell_\beta$, and then $\boldsymbol{a}^{(\alpha)}$ is conversely required to involve $s_{k_\beta} s_{\ell_\beta}^*$ for all $\beta$. In this way, by setting $\boldsymbol{a}^{(1)} \equiv \sum_{k=1}^{2^{p-1}} s_{2k-1} s_{2k}^*$, the explicit expressions for $(\boldsymbol{a}^{(\alpha)}, \zeta_\alpha, \varphi_\alpha)$, $\alpha = 1, 2, \ldots, p$, are given by

$$\boldsymbol{a}^{(\alpha)} = \sum_{k=1}^{2^{p-\alpha}} \sum_{\ell=1}^{2^{\alpha-1}} s_{2^\alpha(k-1)+\ell} s^*_{2^{\alpha-1}(2k-1)+\ell}, \tag{4.30}$$

$$\zeta_\alpha(X) = \sum_{i=1}^{2^p} (-1)^{\left[\frac{i-1}{2^{\alpha-1}}\right]} s_i X s_i^*, \tag{4.31}$$

$$\varphi_\alpha(X) = \rho(X) \equiv \sum_{i=1}^{2^p} s_i X s_i^*, \tag{4.32}$$

where $[x]$ denotes the largest integer not greater than $x$. It is shown that $(\Phi_{SRP_p} \circ \iota)(\mathrm{PFA}_p)$ is a proper subset of $\mathcal{O}_{2^p}^{U(1)} = \Phi_{SRP_p}(\mathrm{GCA}_p)$. In the same ways as in RPFS$_2$, it is possible to rewrite the Green components of RPFS$_p$ $\{\Phi_{SRP_p}(a_n^{(\alpha)})\}$ in terms of generators of the standard RFS$_p$ $\{\Phi_{SR_p}(a_n)\}$ using the Klein transformation.

## §5. Discussion

In the present paper, we have introduced the most fundamental aspect of the recursive fermion system (RFS) in the Cuntz algebra $\mathcal{O}_2$ and its generalization (RFS$_p$) in $\mathcal{O}_{2^p}$, and apply it to construct the recursive parafermion system of order $p$ (RPFS$_p$). As explicit examples, we have presented the standard RFS and the standard RFS$_p$ which give embeddings of CAR onto the $U(1)$-invariant subalgebra $O_2^{U(1)} \cong \mathrm{UHF}_2$ and $\mathcal{O}_{2^p}^{U(1)} \cong \mathrm{UHF}_{2^p}$, respectively. Although they are the simplest and most important examples, we can construct other ones explicitly using unital $*$-endomorphisms of the Cuntz algebra. As for the canonical endomorphism, it commutes with the $U(1)$ action, hence its restriction to the standard RFS yields another RFS which gives an embedding onto a proper subset of $\mathcal{O}_2^{U(1)}$. Since, however, a generic unital $*$-endomorphism does *not* necessarily commute with the $U(1)$ action, there exists a RFS $R$ such that $\Phi_R(\mathrm{CAR}) \not\subseteq \mathcal{O}_2^{U(1)}$. For example, the following unital $*$-endomorphisms $\varphi_i$ $(i = 1, 2)$ of $\mathcal{O}_2$ do not commute with the $U(1)$ action:

$$\varphi_1(s_1) = s_{1;1} + s_{21;2}, \quad \varphi_1(s_2) = s_{22}, \tag{5.1}$$

$$\varphi_2(s_1) = s_{2;1} + s_{12;2}, \quad \varphi_2(s_2) = s_{11}. \tag{5.2}$$

In our construction of embeddings of the CAR algebra into the Cuntz algebra, the indices of fermion operators $\{\Phi_R(a_n)\}$ $(n = 1, 2, \ldots)$ denote the ordering of generations yielded recursively by the recursive map $\zeta : \Phi_R(a_{n-1}) \mapsto \Phi_R(a_n)$ $(n = 1, 2, \ldots)$. However,

in the physical point of view, they should be interpreted as modes which distinguish momentum of particle or other physical degrees of freedom. The relation between the recursive map $\zeta$ and such physical meaning of indices of fermion operators is still an open problem.

Description of the CAR algebra of fermions in terms of the Cuntz algebra is *not* just a rewriting of well-known things. In our succeeding paper[6], we will discuss on various applications of RFS and $\mathrm{RFS}_p$ by restricting the properties of the Cuntz algebra. In the conventional viewpoint in the $C^*$-algebra, the structure of the Cuntz algebra has been studied through that of the UHF algebra ($\mathrm{UHF}_2 \cong \mathrm{CAR}$), since the latter is believed to be understood enough. However, inverting the way of consideration, quite a new viewpoint will open and show us various novel structures of CAR. For example, by restricting endomorphisms and automorphisms of the Cuntz algebra to RFS or $\mathrm{RFS}_p$, we can explicitly construct nontrivial endomorphisms of CAR as mentioned above and automorphisms which are expressed in terms of *nonlinear* transformations.[7] The Bogoliubov transformations, which are expressed in terms of linear transformations of the annihilation/creation operators, are no more than almost trivial examples in this context. We will also find that infinite branching of vacuum of fermions occurs through a certain type of endomorphisms including (5.1).

In gauge theories and quantum gravity, fermions called Faddeev-Popov (FP) ghosts play quite an important role at the fundamental level of theory. In our another succeeding paper,[8] we will consider recursive construction of FP ghost algebra in string theory. Since it is possible to formulate the FP ghosts only on the basis of the indefinite-metric state vector space, we need to generalize the Cuntz algebra in such a way that it acts on the indefinite-metric vector space. The resultant Cuntz-like algebra is called the *pseudo Cuntz algebra.* We will construct two embeddings of the FP ghost algebra in string theory into the pseudo Cuntz algebra and discuss on restricted representations. The special attention will be payed for the zero-mode FP ghost operators.

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