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# Pseudo Cuntz Algebra and Recursive FP Ghost System in String Theory

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***Abstract***

Representation of the algebra of FP (anti)ghosts in string theory is studied by generalizing the recursive fermion system in the Cuntz algebra constructed previously. For that purpose, the pseudo Cuntz algebra, which is a $*$-algebra generalizing the Cuntz algebra and acting on indefinite-metric vector spaces, is introduced. The algebra of FP (anti)ghosts in string theory is embedded into the pseudo Cuntz algebra recursively in two different ways. Restricting a certain permutation representation of the pseudo Cuntz algebra, representations of these two recursive FP ghost systems are obtained. With respect to the zero-mode operators of FP (anti)ghosts, it is shown that one corresponds to the four-dimensional representation found recently by one of the present authors (M.A.) and Nakanishi, while the other corresponds to the two-dimensional one by Kato and Ogawa.

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## §1. Introduction

In our previous paper,[1] we have introduced the recursive fermion system (RFS) which gives embeddings of the fermion algebra (or CAR) into the Cuntz algebra $\mathcal{O}_2$ or $\mathcal{O}_{2^p}$ ($p \geqq 2$). We have shown how the representations of the fermion algebra are obtained by restricting those of the Cuntz algebra. According to embeddings, we can obtain unitarily inequivalent representations of the fermion algebra.[2] In that framework, however, we can not treat unphysical fermions such as FP (anti)ghosts, which are defined only on the basis of the indefinite-metric vector space, since the Cuntz algebra is represented on Hilbert space with the conventional positive-definite inner product. Since FP (anti)ghosts play quite an important role in gauge theories and quantum gravity, it is very desirable for us to have a similar formulation to manage them. In order to treat such unphysical fermion algebras in the same way we have done in the Cuntz algebra, we need to generalize the Cuntz algebra itself so that it acts on the indefinite-metric vector space.

As for the FP (anti)ghost fields in string theory, their mode-decomposed operators satisfy the anticommutation relations with the special structure owing to the Hermiticity of the FP (anti)ghost fields as follows:[c]

$$\{c_0,\, \bar{c}_0\} = -I, \quad c_0^* = c_0, \quad \bar{c}_0^* = \bar{c}_0, \tag{1.1}$$

$$\{c_m,\, \bar{c}_n^*\} = \{c_m^*,\, \bar{c}_n\} = -\delta_{m,n} I, \quad m,\, n = 1,\, 2\, \ldots, \tag{1.2}$$

and other anticommutation relations vanish. Diagonalizing (1.1) and (1.2), we rewrite them into the following

$$b_1 \equiv c_0 + \bar{c}_0 = b_1^*, \quad b_2 \equiv c_0 - \bar{c}_0 = b_2^*, \tag{1.3}$$

$$\{b_i,\, b_j\} = 2\eta_{i,j} I, \quad \eta_{i.j} = \mathrm{diag}(-1,\, +1), \quad i,\, j = 1,\, 2, \tag{1.4}$$

$$a_{2n+1} \equiv \frac{1}{\sqrt{2}}(c_n + \bar{c}_n), \quad a_{2(n+1)} \equiv \frac{1}{\sqrt{2}}(c_n - \bar{c}_n), \quad n = 1,\, 2,\, \ldots, \tag{1.5}$$

$$\{a_m,\, a_n^*\} = (-1)^m \delta_{m,n} I, \quad m,\, n = 3,\, 4,\, 5,\, \ldots, \tag{1.6}$$

and others vanish. Thus, the zero-mode operators satisfy the anticommutation relations isomorphic with the $(1+1)$-dimensional Clifford algebra. Therefore, we can not adopt the conventional Fock representation with regard to them. To overcome this difficulty, Kato and Ogawa[3] introduced the two-dimensional representation with respect to the zero-mode operators:[d]

$$\begin{aligned} &c_0\,|-\rangle = |+\rangle, \quad \bar{c}_0\,|+\rangle = -|-\rangle, \quad c_0\,|+\rangle = \bar{c}_0\,|-\rangle = 0, \\ &\langle \pm\,|\mp\rangle = 1, \quad \langle \pm\,|\pm\rangle = 0. \end{aligned} \tag{1.7}$$

Here, $|\pm\rangle$ is an eigenvector of the FP ghost number charge $iQ_c$ with eigenvalue $\pm\frac{1}{2}$, thus the FP ghost numbers are half-integers in Kato-Ogawa theory. Since this two-dimensional representation has the indefinite inner product with the off-diagonal metric

[c] Throughout this paper, we use $^*$ to denote the Hermitian conjugate instead of $^\dagger$ in conformity with the notation of $*$-algebra. The minus signs appearing in rhs of (1.1) and (1.2) are just for our convention. If one prefers to the plus sign, one only has to redefine $-\bar{c}_n$ as $\bar{c}_n$ for $n \geqq 0$.

[d] These equations are different from the original ones in Ref. 3) by an extra minus sign owing to our convention of the anticommutation relation (1.1).

structure, their vacuum with respect to the FP (anti)ghost operators is orthogonal with itself. Therefore, they also introduced the ad hoc metric operator in order to recover the vacuum expectation values in the conventional sense. However, it is not admissible to introduce such a metric operator since in general it violates the operator Hermitian conjugation at the representation level. More suitable representation for the zero-mode operators is the four-dimensional one which is recently obtained in the exact solution to the operator formalism of the conformal-gauge bosonic string theory.[4] The four vacuum vectors are denoted by

$$\begin{aligned} &|\,0\,\rangle, \quad c_0\,|\,0\,\rangle, \quad \bar{c}_0\,|\,0\,\rangle, \quad [\bar{c}_0,\, c_0]\,|\,0\,\rangle, \\ &\langle\,0\,|\,0\,\rangle = 1. \end{aligned} \tag{1.8}$$

In this formulation, we have the genuine vacuum (cyclic vector) $|\,0\,\rangle$ with the positive norm for the FP (anti)ghost operators, hence we do not need to introduce a metric operator as above. Since $|\,0\,\rangle$ is *not* an eigenvector of $iQ_c$, it is necessary to project out the zero-mode operators. Then, we have integer FP ghost numbers.

The purpose of this paper is to introduce the pseudo Cuntz algebra which acts on the indefinite-metric vector space and study the representation of the FP ghost algebra in string theory by restricting that of the pseudo Cuntz algebra. We construct two embeddings $\Phi_1$ and $\Phi_2$ of the FP ghost algebra $\mathcal{FP}$ into the pseudo Cuntz algebra $\mathcal{O}_{2,2}$ by extending our previous results for the recursive fermion system:

$$\Phi_i : \mathcal{FP} \hookrightarrow \mathcal{O}_{2,2} \quad i = 1,\, 2. \tag{1.9}$$

We show that one of them has a four-dimensional representation with respect to the zero-mode operators and the other a two-dimensional one, when a certain representation of the pseudo Cuntz algebra is restricted. These representations are unitarily inequivalent with each other.

The present paper is organized as follows. In Sec. 2, our previous results for the recursive fermion system in the Cuntz algebra are reviewed. In Sec. 3, the pseudo Cuntz algebra is introduced. In Sec. 4, the recursive construction of embeddings of the FP ghost algebra in string theory into the pseudo Cuntz algebra is presented. In Sec. 5, the representation of our FP ghost system is constructed. The final section is devoted to discussion. In Appendix, some embeddings among pseudo Cuntz algebras are presented in brief.

## §2. Recursive Fermion System in Cuntz Algebra

### §§2-1. Definition

We show a method of systematic construction of an embedding of the fermion algebra into the Cuntz algebra $\mathcal{O}_{2^p}$.

The Cuntz algebra[5] $\mathcal{O}_d$ is a $C^*$-algebra generated by $s_i$ $(i = 1,\, 2,\, \ldots,\, d)$ satisfying the following relations

$$s_i^*\, s_j = \delta_{i,j}\, I, \tag{2.1}$$

$$\sum_{i=1}^{d} s_i\, s_i^* = I, \tag{2.2}$$

where $I$ is the unit (or the identity operator). We often use the brief description such as $s_{i_1\cdots i_m} \equiv s_{i_1}\cdots s_{i_m}$, $s^*_{i_1\cdots i_m} \equiv s^*_{i_m}\cdots s^*_{i_1}$ and $s_{i_1\cdots i_m;j_n\cdots j_1} \equiv s_{i_1}\cdots s_{i_m}s^*_{j_n}\cdots s^*_{j_1}$.

The fermion algebra (or CAR) is a $C^*$-algebra generated by $a_n$ $(n = 1, 2, \ldots)$ satisfying

$$\{a_m, a_n\} = 0, \quad \{a_m, a_n^*\} = \delta_{m,n}I, \quad m, n = 1, 2, \ldots. \tag{2.3}$$

We construct embeddings of CAR into $\mathcal{O}_{2^p}$ as a $*$-subalgebra.[1)] For this purpose, we introduce $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_p \in \mathcal{O}_{2^p}$, a linear mapping $\zeta_p : \mathcal{O}_{2^p} \to \mathcal{O}_{2^p}$, and a unital (i.e., preserving $I$) endomorphism $\varphi_p$ on $\mathcal{O}_{2^p}$. A set $R_p = (\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_p; \zeta_p, \varphi_p)$ is called a *recursive fermion system of order p* ($\mathrm{RFS}_p$) in $\mathcal{O}_{2^p}$, if it satisfies

i) seed condition: $$\{\boldsymbol{a}_i, \boldsymbol{a}_j\} = 0, \quad \{\boldsymbol{a}_i, \boldsymbol{a}_j^*\} = \delta_{i,j}I, \tag{2.4}$$

ii) recursive condition: $$\{\boldsymbol{a}_i, \zeta_p(X)\} = 0, \quad \zeta_p(X)^* = \zeta_p(X^*), \quad X \in \mathcal{O}_{2^p}, \tag{2.5}$$

iii) normalization condition: $$\zeta_p(X)\zeta_p(Y) = \varphi_p(XY), \quad X, Y \in \mathcal{O}_{2^p}, \tag{2.6}$$

with none of $\boldsymbol{a}_i$ been expressed as $\zeta_p(X)$ with $X \in \mathcal{O}_{2^p}$. An embedding $\Phi_{R_p}$ of CAR into $\mathcal{O}_{2^p}$ associated with $R_p$ is defined by

$$\begin{gathered} \Phi_{R_p} : \mathrm{CAR} \hookrightarrow \mathcal{O}_{2^p}, \\ \Phi_{R_p}(a_{p(n-1)+i}) \equiv \zeta_p^{n-1}(\boldsymbol{a}_i), \quad i = 1, \ldots, p, \; n = 1, 2, \ldots. \end{gathered} \tag{2.7}$$

It is, indeed, straightforward to show that (2.7) satisfies the anticommutation relations isomorphic with (2.3).

In the following, we give examples called the *standard* $\mathrm{RFS}_p$ for the cases $p = 1, 2$.

(1) $p = 1$, the standard $\mathrm{RFS}_1$ in $\mathcal{O}_2$, $SR_1 = (\boldsymbol{a}; \zeta, \varphi)$:

$$\boldsymbol{a} \equiv s_1 s_2^*, \tag{2.8}$$

$$\zeta(X) \equiv s_1 X s_1^* - s_2 X s_2^*, \tag{2.9}$$

$$\varphi(X) \equiv \rho(X) \equiv s_1 X s_1^* + s_2 X s_2^*, \tag{2.10}$$

where $\rho$ is the canonical endomorphism of $\mathcal{O}_2$. The embedding $\Phi_{SR_1}$ of CAR into $\mathcal{O}_2$ associated with $SR_1$ is given by

$$\Phi_{SR_1}(a_n) = \zeta^{n-1}(\boldsymbol{a}), \quad n = 1, 2, \ldots. \tag{2.11}$$

(2) $p = 2$, the standard $\mathrm{RFS}_2$ in $\mathcal{O}_4$, $SR_2 = (\boldsymbol{a}_1, \boldsymbol{a}_2; \zeta_2, \varphi_2)$:

$$\boldsymbol{a}_1 \equiv s_1 s_2^* + s_3 s_4^*, \tag{2.12}$$

$$\boldsymbol{a}_2 \equiv s_1 s_3^* - s_2 s_4^*, \tag{2.13}$$

$$\zeta_2(X) \equiv s_1 X s_1^* - s_2 X s_2^* - s_3 X s_3^* + s_4 X s_4^*, \tag{2.14}$$

$$\varphi_2(X) \equiv \rho_4(X) \equiv \sum_{i=1}^{4} s_i X s_i^*, \tag{2.15}$$

where $\rho_4$ is the canonical endomorphism of $\mathcal{O}_4$. The embedding $\Phi_{SR_2}$ of CAR into $\mathcal{O}_4$ associated with $SR_2$ is given by

$$\Phi_{SR_2}(a_{2(n-1)+i}) = \zeta_2^{n-1}(\boldsymbol{a}_i), \quad i = 1, 2, \; n = 1, 2, \ldots. \tag{2.16}$$

In the same way, the standard $\mathrm{RFS}_p$ with a generic $p$ is explicitly constructed.[1)]

As for the standard $\mathrm{RFS}_1$ in $\mathcal{O}_2$, using mathematical induction, it is straightforward to see that $\Phi_{SR_1}(\mathrm{CAR})$ is identical with $\mathcal{O}_2^{U(1)}$, which is defined by a linear space generated by monomials of the form $s_{i_1\cdots i_k;\,j_k\cdots j_1}$, $k = 1, 2, \ldots$. Here, $\mathcal{O}_2^{U(1)}$ is nothing but the $U(1)$-invariant subalgebra of $\mathcal{O}_2$ with the $U(1)$ action been defined by $s_i \mapsto z s_i$, $z \in \mathbf{C}$, $|z| = 1$. From the one-to-one correspondence of $s_{i_1\cdots i_k;\,j_k\cdots j_1}$ with the matrix element $e_{i_1 j_1} \otimes \cdots \otimes e_{i_k j_k} \otimes I \otimes I \otimes \cdots$, $\mathcal{O}_2^{U(1)}$ is isomorphic with $\bigotimes_{\infty} M_2 \cong \mathrm{UHF}_2$, where $M_2$ denotes the algebra of all $2 \times 2$ complex matrices. In this correspondence, the embedding associated with $SR_1$ is transcribed into the form of infinite tensor products of matrices as follows:

$$\boldsymbol{a} \sim A \otimes I \otimes I \otimes \cdots, \qquad A \equiv \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{2.17}$$

$$\Phi_{SR_1}(a_n) \sim \underbrace{\sigma_3 \otimes \cdots \otimes \sigma_3}_{n-1} \otimes A \otimes I \otimes I \otimes \cdots, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.18}$$

Likewise, $\Phi_{SR_p}(\mathrm{CAR})$ is identical with $\mathcal{O}_p^{U(1)}$.

Substituting the homogeneous embedding $\Psi$ of $\mathcal{O}_4$ into $\mathcal{O}_2$ defined by[2)]

$$\begin{gathered} \Psi : \mathcal{O}_4 \hookrightarrow \mathcal{O}_2, \\ \Psi(s_1) \equiv t_{11}, \quad \Psi(s_2) \equiv t_{21}, \quad \Psi(s_3) \equiv t_{12}, \quad \Psi(s_4) \equiv t_{22}, \end{gathered} \tag{2.19}$$

where $t_{ij} \equiv t_i\, t_j$ with $t_1$ and $t_2$ denoting the generator of $\mathcal{O}_2$, into (2.12)–(2.14), we obtain

$$\Psi(\boldsymbol{a}_1) = t_{11;12} + t_{12;22} = t_{1;2}, \tag{2.20}$$

$$\Psi(\boldsymbol{a}_2) = t_{11;21} - t_{21;22} = \zeta(\Psi(\boldsymbol{a}_1)), \tag{2.21}$$

$$\begin{aligned} (\Psi \circ \zeta_2)(X) &= t_{11}\, \Psi(X)\, t_{11}^* - t_{21}\, \Psi(X)\, t_{21}^* - t_{12}\, \Psi(X)\, t_{12}^* + t_{22}\, \Psi(X)\, t_{22}^* \\ &= \zeta^2(\Psi(X)), \quad X \in \mathcal{O}_4, \end{aligned} \tag{2.22}$$

$$\zeta(Y) \equiv t_1 Y t_1^* - t_2 Y t_2^*, \quad Y \in \mathcal{O}_2, \tag{2.23}$$

where use has been made of (2.1) and (2.2) for $t_1$ and $t_2$. Therefore, (2.16) is rewritten as

$$(\Psi \circ \Phi_{SR_2})(a_{2(n-1)+i}) = \zeta^{2(n-1)+i-1}(\Psi(\boldsymbol{a}_1)), \quad i = 1, 2, \quad n = 1, 2, \ldots, \tag{2.24}$$

hence,

$$(\Psi \circ \Phi_{SR_2})(a_n) = \zeta^{n-1}(\Psi(\boldsymbol{a}_1)), \quad n = 1, 2, \ldots, \tag{2.25}$$

which is nothing but the embedding $\Phi_{SR}(a_n)$ associated with the standard $\mathrm{RFS}_1$ in $\mathcal{O}_2$ defined by (2.11). Likewise, the standard $\mathrm{RFS}_p$ is reduced to the standard $\mathrm{RFS}_1$ by the homogeneous embedding of $\mathcal{O}_{2^p}$ into $\mathcal{O}_2$.[2)]

### §§2-2. Representation

As the $*$-representation of $\mathcal{O}_d$, we consider the permutation representation.[6)] Let $\{e_n \,|\, n \in \mathbf{N}\}$ be an orthonormal basis of an infinite-dimensional Hilbert space $\mathcal{H}$. Let $\mu_i : \mathbf{N} \to \mathbf{N}$ $(i = 1, \ldots, d)$ be a branching function system defined by the following

conditions: (i) 1-to-1, (ii) $\mu_i(\mathbf{N}) \cap \mu_j(\mathbf{N}) = \emptyset$ $(i \neq j)$, (iii) $\bigcup_{i=1}^{d} \mu_i(\mathbf{N}) = \mathbf{N}$. For a given branching function system $\{\mu_i\}$, the permutation representation of $\mathcal{O}_d$ on $\mathcal{H}$ is defined by

$$s_i e_n = e_{\mu_i(n)}, \quad i = 1, 2, \ldots, d, \ n = 1, 2, \ldots. \tag{2.26}$$

Here, we identify $s_i$ and its representation on $\mathcal{H}$. As for the action of $s_i^*$ on $\mathcal{H}$, it is derived from the definition of the adjoint conjugation. Using the fact that any $n \in \mathbf{N}$ is uniquely expressed as $n = \mu_j(m)$ with appropriate $j$ and $m$, the result is written as

$$\begin{aligned} s_i^* e_n &= s_i^* e_{\mu_j(m)} \\ &= \delta_{i,j} e_m. \end{aligned} \tag{2.27}$$

An irreducible permutation representation is uniquely characterized by a label $(i_1, i_2, \ldots, i_k)$ $(i_1, i_2, \ldots, i_k = 1, 2, \ldots, d)$ which has no periodicity less than $k$. Here, the label $(i_1, i_2, \ldots, i_k)$ is called to have periodicity $\ell (< k)$, if $i_1 = i_{1+\ell}$, $i_2 = i_{2+\ell}$, $\ldots$, $i_{k-\ell} = i_k$, $i_{k-\ell+1} = i_1$, $\ldots$, $i_k = i_\ell$. Then, the irreducible permutation representation $\mathrm{Rep}(i_1, \ldots, i_k)$ is defined by the case that the product $s_{i_1} \cdots s_{i_k}$ (and its cyclic permutation) has the eigenvalue 1. We set the corresponding eigenvector of $s_{i_1} \cdots s_{i_k}$ on $e_1$. Especially, for Rep(1) or the standard representation of $\mathcal{O}_d$, we have

$$s_i \, e_n = e_{d(n-1)+i}, \qquad i = 1, 2, \ldots, d, \ n = 1, 2, \ldots, \tag{2.28}$$

$$s_i^* \, e_{d(n-1)+j} = \delta_{i,j} e_n, \quad i, j = 1, 2, \ldots, d, \ n = 1, 2, \ldots. \tag{2.29}$$

Restricting Rep(1) of $\mathcal{O}_2$ to the embedded image $\Phi_{SR_1}(\mathrm{CAR})$ associated with the standard RFS $SR_1$ in $\mathcal{O}_2$, we obtain

$$\Phi_{SR_1}(a_n) \, e_1 = s_1^n s_2^* \, e_1 = 0, \qquad n = 1, 2, \ldots, \tag{2.30}$$

$$\begin{aligned} \Phi_{SR_1}(a_{n_1})^* \Phi_{SR_1}(a_{n_2})^* \cdots \Phi_{SR_1}(a_{n_k})^* \, e_1 &= s_1^{n_1 - 1} s_2 s_1^{n_2 - n_1 - 1} s_2 \cdots s_1^{n_k - n_{k-1} - 1} s_2 \, e_1 \\ &= e_{N(n_1, \ldots, n_k)}, \quad 1 \leqq n_1 < \cdots < n_k, \end{aligned} \tag{2.31}$$

$$N(n_1, \ldots, n_k) \equiv 2^{n_1 - 1} + \cdots + 2^{n_k - 1} + 1. \tag{2.32}$$

From (2.30), $e_1$ of Rep(1) of $\mathcal{O}_2$ is the vacuum for the annihilation operators $a_n$ $(n = 1, 2, \ldots)$. Since any $n \in \mathbf{N}$ is uniquely expressed as $N(n_1, \ldots, n_k)$ with $1 \leqq n_1 < \cdots < n_k$ in (2.32), whole $\mathcal{H}$ is interpreted as the Fock space with the unique vacuum $e_1$ for $\Phi_{SR_1}(\mathrm{CAR})$.

In the case of the standard $\mathrm{RFS}_2$ in $\mathcal{O}_4$, restricting Rep(1) of $\mathcal{O}_4$, we obtain

$$\Phi_{SR_2}(a_{2(n-1)+i}) \, e_1 = s_1^{n-1} \boldsymbol{a}_i \, e_1 = 0, \quad i = 1, 2, \ n = 1, 2, \ldots. \tag{2.33}$$

Thus, $e_1$ of Rep(1) of $\mathcal{O}_4$ is a vacuum for the annihilation operators $\Phi_{SR_2}(a_n)$ $(n = 1, 2 \ldots)$, and the corresponding Fock space is generated by $\Phi_{SR_2}(a_{n_1})^* \Phi_{SR_2}(a_{n_2})^* \cdots \Phi_{SR_2}(a_{n_k})^* \, e_1$ with $1 \leqq n_1 < n_2 < \cdots < n_k$. As for $\Phi_{SR_2}(a_{2(n_1-1)+i_1})^* \Phi_{SR_2}(a_{2(n_2-1)+i_2})^* \cdots \Phi_{SR_2}(a_{2(n_k-1)+i_k})^* \, e_1$ with $1 \leqq n_1 < n_2 < \cdots < n_k$ and $i_1, \ldots, i_k = 1, 2$, it is expressed in terms of a monomial consisting only of $s_1$, $s_2$ (for $i_j = 1$) and $s_3$ (for $i_j = 2$) acting on $e_1$. On the other hand, in the case a product $\Phi_{SR_2}(a_{2(n_j-1)+i_j})^* \Phi_{SR_2}(a_{2(n_{j+1}-1)+i_{j+1}})^*$ with $n_j = n_{j+1}$ and $i_j = 1$, $i_{j+1} = 2$ is involved, it is expressed using a monomial

involving $s_4$. In this way, it is shown that $\Phi_{SR_2}(a_{2(n_1-1)+i_1})^* \cdots \Phi_{SR_2}(a_{2(n_k-1)+i_k})^* e_1$ with $1 \leqq 2(n_1-1)+i_1 < \cdots < 2(n_k-1)+i_k$ is expressed in the form of

$$s_1^{n'_1-1} s_{i'_1} s_1^{n'_2-n'_1-1} s_{i'_2} \cdots s_1^{n'_\ell-n'_{\ell-1}-1} s_{i'_\ell} e_1 = e_{N(n'_1,i'_1;\ldots;n'_\ell,i'_\ell)}, \quad i'_1, \ldots, i'_\ell = 2, 3, 4, \quad (2.34)$$

$$N(n'_1, i'_1; \ldots; n'_\ell, i'_\ell) \equiv \sum_{j=1}^{\ell} (i'_j - 1) 4^{n'_j-1} + 1, \quad (2.35)$$

where $n_1 = n'_1 < \cdots < n'_\ell$, and $k - \ell$ is equal to the number of pairs of $\Phi_{SR_2}(a_{2m-1})^* \Phi_{SR_2}(a_{2m})$ in $\Phi_{SR_2}(a_{2(n_1-1)+i_1})^* \cdots \Phi_{SR_2}(a_{2(n_k-1)+i_k})^*$. Since any $e_n \in \mathcal{H}$ is uniquely expressed in the form of (2.34), whole $\mathcal{H}$ is now interpreted as the Fock space with the unique vacuum $e_1$ for $\Phi_{SR_2}(\mathrm{CAR})$. One should note that it is possible to rewrite the expression for $\Phi_{SR_2}(a_{n_1})^* \Phi_{SR_2}(a_{n_2})^* \cdots \Phi_{SR_2}(a_{n_k})^* e_1$ with $1 \leqq n_1 < n_2 < \cdots < n_k$ into the same form as (2.31) with (2.32).[1)] Therefore, as a representation of CAR, the restriction of Rep(1) of $\mathcal{O}_4$ to $\Phi_{SR_2}(\mathrm{CAR})$ is exactly the same as that of Rep(1) of $\mathcal{O}_2$ to $\Phi_{SR_1}(\mathrm{CAR})$.

## §3. Pseudo Cuntz Algebra

We generalize the Cuntz Algebra $\mathcal{O}_d$ defined on Hilbert spaces to the pseudo Cuntz algebra $\mathcal{O}_{d,d'}$ on indefinite-metric vector spaces.[e)] We consider a $*$-algebra generated by $s_1, \ldots, s_{d+d'}$ $(d + d' \geqq 2)$ satisfying the following relations

$$s_i^* s_j = \eta_{ij} I, \quad (3.1)$$

$$\sum_{i,j=1}^{d+d'} \eta^{ij} s_i s_j^* = I, \quad (3.2)$$

where $\eta_{ij} = \eta^{ij} = \mathrm{diag}(\underbrace{+1, \cdots +1}_{d}, \underbrace{-1, \cdots, -1}_{d'})$. Then, $\mathcal{O}_{d,0}$ is identical with the dense subalgebra of $\mathcal{O}_d$.

For better understanding, let us introduce a vector space $\mathcal{V}$ called the Krein space,[7)] that is, a direct sum of two Hilbert spaces $\mathcal{V}_\pm$, where $\mathcal{V}_+$ has a positive definite inner product and $\mathcal{V}_-$ has a negative definite one. We set the orthonormal basis $\{e_n\}$ $(n = 1, 2, \ldots)$ of $\mathcal{V}$ in such a way that $e_{2n-1} \in \mathcal{V}_+$ and $e_{2n} \in \mathcal{V}_-$. Hence, we have

$$\langle e_m \,|\, e_n \rangle = (-1)^{m-1} \delta_{m,n}, \qquad m, n = 1, 2, \ldots, \quad (3.3)$$

where $\langle \cdot \,|\, \cdot \rangle$ denotes the inner product. We, next, define the operators $s_1$ and $s_2$ on $\mathcal{V}$ by

$$\begin{aligned} s_1\, e_{2n-1} &= e_{4n-3}, & s_1\, e_{2n} &= e_{4n}, \\ s_2\, e_{2n-1} &= e_{4n-2}, & s_2\, e_{2n} &= e_{4n-1}, \end{aligned} \qquad n = 1, 2, \ldots, \quad (3.4)$$

$$s_1 : \mathcal{V}_\pm \to \mathcal{V}_\pm, \qquad s_2 : \mathcal{V}_\pm \to \mathcal{V}_\mp. \quad (3.5)$$

From the definition of the adjoint conjugation, the operation of $s_i^*$ on $e_n$ is uniquely

[e)] We only consider to generalize the $*$-algebraic structure of the Cuntz algebra, since it seems difficult to generalize the $C^*$-norm structure with mathematical rigorous.

derived from (3.3) and (3.4) as follows:

$$\begin{aligned} s_1^* e_{4n-3} &= e_{2n-1}, & s_1^* e_{4n-2} &= 0, \\ s_1^* e_{4n-1} &= 0, & s_1^* e_{4n} &= e_{2n}, \\ s_2^* e_{4n-3} &= 0, & s_2^* e_{4n-2} &= -e_{2n-1}, \\ s_2^* e_{4n-1} &= -e_{2n}, & s_1^* e_{4n} &= 0, \end{aligned} \qquad n = 1, 2, \ldots. \tag{3.6}$$

Then, it is easy to see that $s_i$ and $s_j^*$ satisfy (3.1) and (3.2) with $d = d' = 1$. The above $*$-representation, which we call Rep(1), of $\mathcal{O}_{1,1}$ corresponds to Rep(1) of $\mathcal{O}_2$ because $s_1$ has an eigenvector $e_1$ with eigenvalue 1, there is no other monomial $s_{i_1 \cdots i_k}$ having eigenvector, and it is irreducible. In contrast to the case of $\mathcal{O}_2$, there exists no representation such as Rep(2), in which $s_2$ has an eigenvector, because of (3.5). In general, a permutation representation $(i_i, i_2, \ldots, i_k)$ of $\mathcal{O}_{1,1}$ is allowed only when the number of index 2 in $\{i_1, \ldots, i_k\}$ is even, since only in that case it is possible that $s_{i_1 \cdots i_k}$ has an eigenvector.

It is straightforward to construct permutation representations of $\mathcal{O}_{d,d'}$. Here, we give the results for a special case of $d = d'$. In this case, it is convenient to rearrange the generators of $\mathcal{O}_{d,d}$ in such a way that $\eta_{ij}$ in (3.1) and (3.2) is given by $\eta_{ij} = (-1)^{i-1}\delta_{ij}$ $(i, j = 1, \ldots, 2d)$. Then, Rep(1) of $\mathcal{O}_{d,d}$ on the Krein space $\mathcal{V}$ is given by

$$s_i e_{2n-1} = e_{4d(n-1)+i}, \qquad s_i e_{2n} = e_{4dn+1-i}, \qquad i = 1, \ldots, 2d, \tag{3.7}$$

$$s_i^* e_{4d(n-1)+j} = (-1)^{i-1}\delta_{i,j}\, e_{2n-1}, \quad s_i^* e_{4dn+1-j} = (-1)^{i-1}\delta_{i,j}\, e_{2n}, \quad i, j = 1, \ldots, 2d, \tag{3.8}$$

$$s_{2j-1} : \mathcal{V}_\pm \to \mathcal{V}_\pm, \qquad s_{2j} : \mathcal{V}_\pm \to \mathcal{V}_\mp, \qquad j = 1, \ldots, d. \tag{3.9}$$

## §4. Recursive FP Ghost System in String Theory

We denote the $*$-algebra generated by the FP (anti)ghosts in string theory by $\mathcal{FP}$. The generators of $\mathcal{FP}$ are FP ghost $c_n$ and FP antighost $\bar{c}_n$ $(n = 0, 1, 2, \ldots)$ with ${c_0}^* = c_0$ and ${\bar{c}_0}^* = \bar{c}_0$. They satisfy the following anticommutation relations

$$\{c_0, \bar{c}_0\} = -I, \tag{4.1}$$

$$\{c_m, \bar{c}_n^*\} = -\delta_{m,n}\, I, \qquad m, n = 1, 2, \ldots, \tag{4.2}$$

$$\{c_m, c_n\} = \{c_m, c_n^*\} = \{\bar{c}_m, \bar{c}_n\} = \{\bar{c}_m, \bar{c}_n^*\} = 0, \quad m, n = 0, 1, \ldots, \tag{4.3}$$

$$\{c_0, \bar{c}_n\} = \{c_m, \bar{c}_0\} = \{c_m, \bar{c}_n\} = 0, \qquad m, n = 1, 2, \ldots. \tag{4.4}$$

The purpose of this section is to give the recursive construction for embedding of $\mathcal{FP}$ into $\mathcal{O}_{2,2}$.

First, we introduce ICAR defined by the fermion algebra with indefinite signature, in which the generators $a_n$ $(n = 1, 2, \ldots)$ satisfy

$$\{a_m, a_n\} = 0, \quad \{a_m, a_n^*\} = (-1)^m \delta_{m,n} I, \quad m, n = 1, 2, \ldots. \tag{4.5}$$

We construct an embedding of ICAR into $\mathcal{O}_{2,2}$ with $\eta_{ij} = (-1)^{i-1}\delta_{ij}$ by generalizing $\mathrm{RFS}_2$ in $\mathcal{O}_4$. Let $\boldsymbol{a}_1, \boldsymbol{a}_2 \in \mathcal{O}_{2,2}$, $\zeta_{1+1} : \mathcal{O}_{2,2} \to \mathcal{O}_{2,2}$ be a linear mapping, and $\varphi_{1+1}$ a unital endomorphism of $\mathcal{O}_{2,2}$, respectively. A tetrad $R_{1+1} = (\boldsymbol{a}_1, \boldsymbol{a}_2; \zeta, \varphi)$ is called the

*recursive fermion system of* $(1,1)$*-type* ($\mathrm{RFS}_{1+1}$) in $\mathcal{O}_{2,2}$, if it satisfies[f)]

$$\text{i) seed condition:} \quad \{\boldsymbol{a}_i,\, \boldsymbol{a}_j\}=0, \quad \{\boldsymbol{a}_i,\, \boldsymbol{a}_j^*\}=(-1)^i\delta_{i,j}I, \quad i,\, j=1,\, 2, \tag{4.6}$$

$$\text{ii) recursive condition:} \quad \{\boldsymbol{a}_i,\, \zeta_{1+1}(X)\}=0,\ \zeta_{1+1}(X)^*=\zeta_{1+1}(X^*), \quad X\in\mathcal{O}_{2,2}, \tag{4.7}$$

$$\text{iii) normalization condition:} \quad \zeta_{1+1}(X)\zeta_{1+1}(Y)=\varphi_{1+1}(XY), \quad X,\, Y\in\mathcal{O}_{2,2}. \tag{4.8}$$

Then, the embedding $\Phi_{R_{1+1}}$ of ICAR into $\mathcal{O}_{2,2}$ associated with $R_{1+1}$ is defined by

$$\begin{gathered}\Phi_{R_{1+1}} : \mathrm{ICAR} \hookrightarrow \mathcal{O}_{2,2}, \\ \Phi_{R_{1+1}}(a_{2(n-1)+i}) \equiv \zeta_{1+1}^{n-1}(\boldsymbol{a}_i), \quad i=1,\, 2, \quad n=1,\, 2,\, \ldots.\end{gathered} \tag{4.9}$$

It is straightforward to reconfirm that (4.9) satisfy the anticommutation relation of (4.5). The simplest example of $\mathrm{RFS}_{1+1}$ in $\mathcal{O}_{2,2}$ is given by the *standard* $\mathrm{RFS}_{1+1}$, $SR_{1+1} = (\boldsymbol{a}_1,\, \boldsymbol{a}_2;\, \zeta_{1+1},\, \varphi_{1+1})$, which is defined by

$$\boldsymbol{a}_1 \equiv s_1 s_2^* + s_3 s_4^*, \quad \boldsymbol{a}_2 \equiv s_1 s_3^* + s_2 s_4^*, \tag{4.10}$$

$$\zeta_{1+1}(X) \equiv s_1 X s_1^* + s_2 X s_2^* - s_3 X s_3^* - s_4 X s_4^*, \tag{4.11}$$

$$\varphi_{1+1}(X) \equiv \rho_{2,2}(X) \equiv \sum_{i=1}^{4} (-1)^{i-1} s_i X s_i^*, \tag{4.12}$$

where $\rho_{2,2}$ should be called the canonical endomorphism of $\mathcal{O}_{2,2}$. Like the standard $\mathrm{RFS}_2$ in $\mathcal{O}_4$, $\Phi_{SR_{1+1}}(\mathrm{ICAR})$ is identical with the $\mathcal{O}_{2,2}^{U(1)} \subset \mathcal{O}_{2,2}$, which is a linear space spanned by monomials of the form $s_{i_1\cdots i_k;j_k\cdots j_1}$ $(k=1,\, 2,\, \ldots)$.

### §§4-1. RFPS1

As noted in Sec. 1, the subalgebra generated by the positive-mode operators of FP (anti)ghost $c_n$, $\bar{c}_n$ $(n = 1, 2, \ldots)$ is isomorphic with ICAR, while the subalgebra generated by the zero-mode operators $c_0$ and $\bar{c}_0$ is isomorphic with a $(1+1)$-dimensional Clifford algebra. The generators of the $(1 + 1)$-dimensional Clifford algebra $b_1$ and $b_2$ are written in terms of those in the 2-dimensional ICAR as follows

$$b_i = b_i^* = a_i + a_i^*, \quad i = 1,\, 2, \tag{4.13}$$

$$\{b_i,\, b_j\} = 2\eta_{i,j}I, \quad \eta_{i,j} = \mathrm{diag}(-1,\, +1), \tag{4.14}$$

where $a_1$ and $a_2$ satisfy (4.5). Therefore, we have a natural correspondence between generators of $\mathcal{FP}$ and those of ICAR as follows:

$$c_0 = \frac{1}{2}(b_1 + b_2) = \frac{1}{2}(a_1 + a_2 + a_1^* + a_2^*), \tag{4.15}$$

$$\bar{c}_0 = \frac{1}{2}(b_1 - b_2) = \frac{1}{2}(a_1 - a_2 + a_1^* - a_2^*), \tag{4.16}$$

$$c_n = \frac{1}{\sqrt{2}}(a_{2n+1} + a_{2n+2}), \quad n = 1,\, 2,\, \ldots, \tag{4.17}$$

---

[f)] In $\mathcal{O}_{2,2}$, there exists also the anticommutation relation algebra with negative-definite signature as will be shown in the last section.

$$\bar{c}_n = \frac{1}{\sqrt{2}}(a_{2n+1} - a_{2n+2}), \quad n = 1, 2, \ldots, \tag{4.18}$$

where $a_n$ $(n = 1, 2, \ldots)$ satisfy (4.5). From (4.9) and (4.15)–(4.18), it is straightforward to obtain the embedding $\Phi_{RFP1}$ of $\mathcal{FP}$ into $\mathcal{O}_{2,2}$ defined by

$$\Phi_{RFP_1} : \mathcal{FP} \hookrightarrow \mathcal{O}_{2,2},$$

$$\Phi_{RFP1}(c_0) \equiv \frac{1}{\sqrt{2}}(\boldsymbol{c} + \boldsymbol{c}^*), \qquad \Phi_{RFP1}(\bar{c}_0) \equiv \frac{1}{\sqrt{2}}(\bar{\boldsymbol{c}} + \bar{\boldsymbol{c}}^*), \tag{4.19}$$

$$\Phi_{RFP1}(c_n) \equiv \zeta^n_{1+1}(\boldsymbol{c}), \qquad \Phi_{RFP1}(\bar{c}_n) \equiv \zeta^n_{1+1}(\bar{\boldsymbol{c}}), \qquad n = 1, 2, \ldots, \tag{4.20}$$

$$\boldsymbol{c} \equiv \frac{1}{\sqrt{2}}(\boldsymbol{a}_1 + \boldsymbol{a}_2), \qquad \bar{\boldsymbol{c}} \equiv \frac{1}{\sqrt{2}}(\boldsymbol{a}_1 - \boldsymbol{a}_2), \tag{4.21}$$

where $\boldsymbol{c}$ and $\bar{\boldsymbol{c}}$ satisfy

$$\boldsymbol{c}^2 = \bar{\boldsymbol{c}}^2 = \{\boldsymbol{c}, \bar{\boldsymbol{c}}\} = 0, \tag{4.22}$$

$$\{\boldsymbol{c}, \bar{\boldsymbol{c}}^*\} = -I. \tag{4.23}$$

Substituting $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ of the standard $\mathrm{RFS}_{1+1}$ defined by (4.10) into (4.21) and (4.19), we obtain

$$\boldsymbol{c} = \frac{1}{\sqrt{2}}[s_1(s_2^* + s_3^*) + (s_2 + s_3)s_4^*], \tag{4.24}$$

$$\bar{\boldsymbol{c}} = \frac{1}{\sqrt{2}}[s_1(s_2^* - s_3^*) - (s_2 - s_3)s_4^*], \tag{4.25}$$

$$\Phi_{RFP1}(c_0) = \frac{1}{2}[(s_1 + s_4)(s_2^* + s_3^*) + (s_2 + s_3)(s_1^* + s_4^*)], \tag{4.26}$$

$$\Phi_{RFP1}(\bar{c}_0) = \frac{1}{2}[(s_1 - s_4)(s_2^* - s_3^*) + (s_2 - s_3)(s_1^* - s_4^*)]. \tag{4.27}$$

We call the tetrad $RFP1 = (\boldsymbol{c}, \bar{\boldsymbol{c}}; \zeta_{1+1}, \varphi_{1+1})$ the *recursive FP ghost system of the first type (RFPS1).*

### §§4-2. RFPS2

The embedding of $\mathcal{FP}$ into $\mathcal{O}_{2,2}$ is not uniquely given by RFPS1. To construct another one, let us note the existence of an embedding of the $(1+1)$-dimensional Clifford algebra generated by $b'_1$ and $b'_2$ into the 1-dimensional CAR with negative-definite signature, which is given by

$$b'_1 \equiv a_1 + a_1^*, \tag{4.28}$$

$$b'_2 \equiv \exp(i\pi a_1^* a_1) = I + 2a_1^* a_1 = a_1^* a_1 - a_1 a_1^*, \tag{4.29}$$

where $a_1$ constitutes a 1-dimensional $*$-subalgebra of ICAR (4.5). Here, $\exp(i\pi a_1^* a_1)$ is the Klein operator anticommuting with $a_1$. One should note that an identity $\exp(i2\pi a_1^* a_1) = I$ holds. Substituting the expressions for $\boldsymbol{a}_1$ of the standard $\mathrm{RFS}_{1,1}$ defined by (4.10) into $a_1$ in (4.28) and (4.29), we obtain another embedding $\Phi_{RFP2}$ of the zero-mode operators $c_0$ and $\bar{c}_0$ into $\mathcal{O}_{2,2}$ as follows:

$$
\begin{aligned}
\Phi_{RFP2} &: \mathcal{FP} \hookrightarrow \mathcal{O}_{2,2}, \\
\Phi_{RFP2}(c_0) &\equiv \frac{1}{2}(\boldsymbol{a}_1 + \boldsymbol{a}_1^* + \boldsymbol{a}_1^*\boldsymbol{a}_1 - \boldsymbol{a}_1\boldsymbol{a}_1^*) \\
&= \frac{1}{2}[(s_1+s_2)(s_1^*+s_2^*) + (s_3+s_4)(s_3^*+s_4^*)], \qquad (4.30) \\
\Phi_{RFS2}(\bar{c}_0) &\equiv \frac{1}{2}(\boldsymbol{a}_1 + \boldsymbol{a}_1^* - \boldsymbol{a}_1^*\boldsymbol{a}_1 + \boldsymbol{a}_1\boldsymbol{a}_1^*) \\
&= -\frac{1}{2}[(s_1-s_2)(s_1^*-s_2^*) + (s_3-s_4)(s_3^*-s_4^*)]. \qquad (4.31)
\end{aligned}
$$

In contrast with $b_2$ defined by (4.13) with $i = 2$, $b_2'$ no longer anticommutes with $a_n$ ($n = 3, 4, \ldots$), but commutes with them. In other words, since (4.29) is nonlinear in $a_1$, $\zeta_{1+1}(X)$ no longer anticommutes with it, hence the previous embedding $\Phi_{RFS1}(c_n)$ and $\Phi_{RFS1}(\bar{c}_n)$ ($n = 1, 2, \ldots$) do not anticommutes with $\Phi_{RFS2}(c_0)$ and $\Phi_{RFS2}(\bar{c}_0)$. In order to recover the anticommutativity, we introduce a new mapping $\zeta_0 : \mathcal{O}_{2,2} \to \mathcal{O}_{2,2}$ defined by

$$
\zeta_0(X) \equiv s_2 X s_1^* - s_1 X s_2^* + s_4 X s_3^* - s_3 X s_4^*, \quad X \in \mathcal{O}_{2,2}, \qquad (4.32)
$$

which satisfies

$$
\{\Phi_{RFP2}(c_0),\, \zeta_0(X)\} = \{\Phi_{RFP2}(\bar{c}_0),\, \zeta_0(X)\} = 0, \quad \zeta_0(X)^* = -\zeta_0(X^*), \qquad (4.33)
$$

$$
\zeta_0(X)\zeta_0(Y) = \varphi_{1+1}(XY), \qquad (4.34)
$$

where $\varphi_{1+1}$ is defined by (4.12). Since $\zeta_0(I) = s_{2;1} - s_{1;2} + s_{4;3} - s_{3;4} = \boldsymbol{a}_1^* - \boldsymbol{a}_1$, the anticommutativity in (4.33) is owing to $\{a_1 + a_1^*,\, a_1^* - a_1\} = \{\exp(i\pi a_1^* a_1),\, a_1^* - a_1\} = 0$. Then, we define another embedding $\Phi_{RFP2}$ of the positive-mode operators $c_n$ and $\bar{c}_n$ as follows

$$
\Phi_{RFP2}(c_n) \equiv \zeta_0(\zeta_{1+1}^{n-1}(\boldsymbol{c})), \quad \Phi_{RFP2}(\bar{c}_n) \equiv -\zeta_0(\zeta_{1+1}^{n-1}(\bar{\boldsymbol{c}})), \qquad n = 1, 2, \ldots, \qquad (4.35)
$$

where $\boldsymbol{c}$, $\bar{\boldsymbol{c}}$ and $\zeta_{1+1}$ are defined by (4.24), (4.25) and (4.11), respectively. It is straightforward to show that the above $\Phi_{RFP2}(c_n)$ and $\Phi_{RFP2}(\bar{c}_n)$ ($n \geqq 0$) indeed satisfy (4.1)–(4.4). We call a set $RFP2 = (c,\, \bar{c},\, c_0,\, \bar{c}_0;\, \zeta_{1+1},\, \zeta_0,\, \varphi_{1+1})$ the *recursive FP ghost system of the second type (RFPS2).*

The apparent difference between RFPS1 and RFPS2 is only that the degree of freedom corresponding to the generator $a_2$ of ICAR disappears in the latter. As shown in the next section, the most significant difference of them is that they correspond to two unitarily inequivalent representations of $\mathcal{FP}$.

## §5. Representation of RFPS

In this section, we consider restrictions of Rep(1) of $\mathcal{O}_{2,2}$ to $\Phi_{RFP1}(\mathcal{FP})$ and $\Phi_{RFP2}(\mathcal{FP})$. First, we recall the Rep(1) of $\mathcal{O}_{2,2}$ on the Krein space $\mathcal{V}$:

$$s_i\, e_{2n-1} = e_{8(n-1)+i}, \qquad s_i\, e_{2n} = e_{8n+1-i}, \tag{5.1}$$

$$s_i^*\, e_{8(n-1)+j} = (-1)^{i-1}\delta_{i,j}\, e_{2n-1}, \quad s_i^*\, e_{8n+1-j} = (-1)^{i-1}\delta_{i,j}\, e_{2n}, \tag{5.2}$$

where $i,\, j = 1,\, 2,\, 3,\, 4;\ n = 1,\, 2,\, \ldots$.

### §§5-1. Restriction of Rep(1) to RFPS1

In this subsection, we denote $C_n \equiv \Phi_{RFP1}(c_n)$, $\bar{C}_n \equiv \Phi_{RFP1}(\bar{c}_n)$ $(n \geqq 0)$ for simplicity of description.

Since $\zeta_{1,1}$ defined by (4.11) satisfies

$$\zeta_{1+1}(X)s_i = \epsilon_i s_i X, \quad \epsilon_i \equiv (-1)^{i-1+\left[\frac{i-1}{2}\right]}, \quad i = 1,\, 2,\, 3,\, 4, \tag{5.3}$$

where $[x]$ denotes the largest integer not greater than $x$, we obtain

$$C_n\, s_i\, e_1 = \epsilon_i s_i s_1^{n-1}\boldsymbol{c}\, e_1 = 0, \quad \bar{C}_n\, s_i\, e_1 = \epsilon_i s_i s_1^{n-1}\bar{\boldsymbol{c}}\, e_1 = 0, \quad n = 1,\, 2,\, \ldots. \tag{5.4}$$

Here, use has been made of $s_1^*\, e_1 = e_1$, $s_i^*\, e_1 = 0$ $(i = 2,\, 3,\, 4)$ and

$$\boldsymbol{c}\, e_1 = \bar{\boldsymbol{c}}\, e_1 = 0. \tag{5.5}$$

Hence, we have at least four vacuums $s_i\, e_1$ with respect to the positive-mode operators. In fact, there is no other vacuums annihilated by $C_n$ and $\bar{C}_n$ $(n = 1,\, 2,\, \ldots)$ because of the cyclicity of the representation as shown later.

As for the zero-mode operators defined by (4.26) and (4.27), we have

$$C_0\, s_1 = \frac{1}{2}(s_2 + s_3), \tag{5.6}$$

$$\bar{C}_0\, s_1 = \frac{1}{2}(s_2 - s_3), \tag{5.7}$$

hence,

$$s_2 = (C_0 + \bar{C}_0)\, s_1, \tag{5.8}$$

$$s_3 = (C_0 - \bar{C}_0)\, s_1. \tag{5.9}$$

Furthermore, since $C_0$ and $\bar{C}_0$ satisfy

$$[\bar{C}_0,\, C_0] = s_{4,1} - s_{1,4} + s_{2,3} - s_{3,2}, \tag{5.10}$$

we have

$$s_4 = [\bar{C}_0,\, C_0]\, s_1. \tag{5.11}$$

Thus, $\{s_1,\, s_2,\, s_3,\, s_4\}$ can be interpreted as a four-dimensional representation space of the zero-mode operators. Therefore, the four vacuums $s_i e_1$ with respect to the positive-mode operators are expressed by the zero-mode operators and $e_1$. From the anticommutativity between the positive-mode operators and the zero-mode operators, in order to construct Fock space based on the above four vacuums, it is sufficient to consider the action of positive-mode creation operators on $e_1$ only.

Because of the anticommutativity between two creation operators of different

modes, we only have to consider $\varphi^*_{n_1}\cdots\varphi^*_{n_k}\,e_1$ with $\varphi_{2m-1}\equiv\bar{C}_m$, $\varphi_{2m}\equiv C_m$, $1\leqq n_1<\cdots<n_k$. If $\varphi^*_{n_1}\cdots\varphi^*_{n_k}$ does not involve the product $\bar{C}^*_m C^*_m$ (i.e., $n_{i+1}>n_i+1$ for odd $n_i$), we obtain the following

$$\varphi^*_{n_1}\varphi^*_{n_2}\cdots\varphi^*_{n_k}\,e_1=\frac{1}{\sqrt{2^k}}s_1^{m_1}(s_2+(-1)^{n_1}s_3)s_1^{m_2-m_1-1}(s_2+(-1)^{n_2}s_3)\cdots \\ \cdots s_1^{m_k-m_{k-1}-1}(s_2+(-1)^{n_k}s_3)\,e_1, \tag{5.12}$$

where $m_i\equiv\left[\frac{n_i+1}{2}\right]$ with $[\,x\,]$ denoting the largest integer not greater than $x$. When $\bar{C}^*_{m_j}C^*_{m_j}$ is involved in $\varphi^*_{n_1}\cdots\varphi^*_{n_k}$ (i.e., $n_{i+1}=n_i+1$ for some odd $n_i$) in (5.12), the corresponding factors in rhs are replaced by $2s_1^{m_j-\cdots}s_4$. For example, we have

$$\bar{C}^*_m C^*_m\,e_1=s_1^m s_4\,e_1, \tag{5.13}$$

$$C^*_{m_1}\cdots\bar{C}^*_{m_j}C^*_{m_j}\cdots C^*_{m_k}\,e_1=\frac{1}{\sqrt{2^{k-1}}}s_1^{m_1}(s_2+s_3)s_1^{m_2-m_1-1}(s_2+s_3)\cdots \\ \cdots s_1^{m_j-m_{j-1}-1}s_4\cdots s_1^{m_k-m_{k-1}-1}(s_2+s_3)\,e_1. \tag{5.14}$$

Thus, the vector space $\mathcal{V}^{(0)}$ generated by action of the positive-mode creation operators on $e_1$ is a linear space spanned by $s_1^{n_1}s_{i_1}s_1^{n_2-n_1-1}s_{i_2}\cdots s_1^{n_\ell-n_{\ell-1}-1}s_{i_\ell}\,e_1$ with $\ell\in\mathbf{N}$, $1\leqq n_1<n_2<\cdots<n_\ell$ and $i_1,\ldots,i_\ell=2,3,4$. Since any $e_n\in\mathcal{V}$ is obtained by action of an appropriate monomial consisting of $s_i$ $(i=1,2,3,4)$ on $e_1$, and such a monomial is uniquely expressed by $s_1^{n_1-1}s_{i_1}s_1^{n_2-n_1-1}s_{i_2}\cdots s_1^{n_\ell-n_{\ell-1}-1}s_{i_\ell}$ ($\ell\in\mathbf{N}$, $1\leqq n_1<n_2<\cdots<n_\ell$; $i_1,\ldots,i_\ell=2,3,4$), the above results show that $\mathcal{V}^{(0)}=s_1\mathcal{V}$. Therefore, taking the contribution from the zero-mode operators into account, we obtain

$$\begin{aligned}\mathcal{V}&=s_1\mathcal{V}\oplus s_2\mathcal{V}\oplus s_3\mathcal{V}\oplus s_4\mathcal{V}\\&=\mathcal{V}^{(0)}\oplus C_0\mathcal{V}^{(0)}\oplus\bar{C}_0\mathcal{V}^{(0)}\oplus[\bar{C}_0,C_0]\mathcal{V}^{(0)},\end{aligned} \tag{5.15}$$

where use has been made of (5.8), (5.9) and (5.11). Hence, the restriction of Rep(1) of $\mathcal{O}_{2,2}$ to $\Phi_{RFP1}(\mathcal{FP})$ is cyclic with the cyclic vector $e_1$. This type of representation of the FP ghost algebra in string theory was found through the exact Wightman functions in the operator formalism by one of the present authors (M.A.) and Nakanishi.[4)]

From (5.15), we can express the subspace $\mathcal{V}^{(0)}$ irrelevant to the zero-mode FP (anti)ghost operators as follows

$$\begin{aligned}\mathcal{V}^{(0)}&=s_1s_1^*\mathcal{V}\\&=\{v\in\mathcal{V}\,|\,s_i^*v=0,\ i=2,3,4\},\end{aligned} \tag{5.16}$$

where the first line shows that $s_1s_1^*$ is the projection operator to $\mathcal{V}^{(0)}$ and the second one denotes the subsidiary condition to select $\mathcal{V}^{(0)}$.

### §§5-2. Restriction of Rep(1) to RFPS2

From (4.35), it is straightforward to have

$$\Phi_{RFP2}(c_n)s_i\,e_1=\Phi_{RFP2}(\bar{c}_n)s_i\,e_1=0,\quad i=1,2,3,4,\quad n=1,2,\ldots \tag{5.17}$$

in the same way as in RFPS1. Therefore, $s_i\,e_1$'s are vacuums for the positive-mode

operators also in this case. To specify the contribution from the zero-mode operators, we consider their action on $s_i$. From (4.30) and (4.31), it is easy to have

$$\Phi_{RFP2}(c_0)\,(s_{2i-1} - s_{2i}) = s_{2i-1} + s_{2i}, \quad \Phi_{RFP2}(\bar{c}_0)\,(s_{2i-1} + s_{2i}) = -(s_{2i-1} - s_{2i}), \quad i = 1, 2. \tag{5.18}$$

Thus, in contrast with RFPS1, $\{s_1, s_2, s_3, s_4\}$ is a direct sum of two two-dimensional representations of the zero-mode operators. Based on this features of RFPS2 and the similar consideration on the positive-mode operators in RFPS1, it is straightforward to obtain that the total space $\mathcal{V}$ is expressed as follows

$$\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2, \qquad \mathcal{V}_1 \perp \mathcal{V}_2, \tag{5.19}$$

$$\mathcal{V}_i \equiv \mathcal{V}_i^{(+)} \oplus \mathcal{V}_i^{(-)}, \qquad \mathcal{V}_i^{(\pm)} \equiv \frac{s_{2i-1} \pm s_{2i}}{\sqrt{2}}\mathcal{V}, \quad i = 1, 2, \tag{5.20}$$

$$\mathcal{V}_i^{(+)} = \Phi_{RFP2}(c_0)\mathcal{V}_i^{(-)}, \quad \mathcal{V}_i^{(-)} = -\Phi_{RFP2}(\bar{c}_0)\mathcal{V}_i^{(+)}, \tag{5.21}$$

where $\mathcal{V}_i^{(\pm)}$ is a Fock space with the vacuum $e_i^{(\pm)} \equiv \frac{s_{2i-1} \pm s_{2i}}{\sqrt{2}} e_1$ for the positive-mode operators. Therefore, each $\mathcal{V}_i$ is an invariant subspace for $\Phi_{RFP2}(\mathcal{FP})$. Here, $e_i^{(\pm)}$ satisfy the following:

$$e_i^{(+)} = \Phi_{RFP2}(c_0)\, e_i^{(-)}, \quad e_i^{(-)} = -\Phi_{RFP2}(\bar{c}_0)\, e_i^{(+)}, \quad i = 1, 2, \tag{5.22}$$

$$\langle\, e_i^{(\pm)} \,|\, e_j^{(\mp)} \,\rangle = \delta_{i,j}, \qquad \langle\, e_i^{(\pm)} \,|\, e_j^{(\pm)} \,\rangle = 0, \qquad i, j = 1, 2. \tag{5.23}$$

Therefore, $e_i^{(\pm)}$ for each $i$ correspond to the two-dimensional vacuums $|\pm\rangle$ introduced by Kato and Ogawa.[3] Since each $e_i^{(\pm)}$ is orthogonal with itself, the naive definition of vacuum expectation value is not appropriate in this representation of $\mathcal{FP}$. In order to recover the vacuum expectation value in the conventional sense, one is apt to introduce a metric operator $\eta$ satisfying $\eta\, e_i^{(\pm)} = e_i^{(\mp)}$. It should be noted, however, that the original definition of $*$-involution (or Hermitian conjugation) in the $*$-algebra is *not*, in general, respected in such an expectation value defined in terms of the metric operator. In the physical point of view, there is no reason to adhere to this kind of representation of $\mathcal{FP}$, in which there is *no* vacuum (or cyclic vector) having positive norm.

## §6. Discussion

In the present paper, we have introduced the pseudo Cuntz algebra and constructed two recursive FP ghost systems, RFPS1 and RFPS2 in $\mathcal{O}_{2,2}$, and their representations. As shown in the Appendix, there exists an embedding $\hat{\Psi}$ of $\mathcal{O}_{2,2}$ into $\mathcal{O}_{1,1}$ such as

$$\begin{gathered} \hat{\Psi} : \mathcal{O}_{2,2} \hookrightarrow \mathcal{O}_{1,1}, \\ \hat{\Psi}(s_1) \equiv t_{11}, \quad \hat{\Psi}(s_2) \equiv t_{21}, \quad \hat{\Psi}(s_3) \equiv t_{22}, \quad \hat{\Psi}(s_4) \equiv t_{12}, \end{gathered} \tag{6.1}$$

where $t_{ij} \equiv t_i\, t_j$ with $t_1$ and $t_2$ denoting generators of $\mathcal{O}_{1,1}$, hence, it is, of course, possible to discuss on the recursive FP ghost system in $\mathcal{O}_{1,1}$. However, we dare not to have done so because it would be rather complicated and not adequate to make clear-cut

descriptions. Indeed, substituting (6.1) into (4.10) and (4.11), the resultant expressions would not be so simple as (2.20)–(2.23). This is because the natural fermion subalgebra in $\mathcal{O}_{1,1}$ is not ICAR but NCAR considered bellow. In contrast with $\mathcal{O}_{1,1}$, we can treat embeddings of ICAR and NCAR in parallel in $\mathcal{O}_{2,2}$. Hence, it seems most transparent to discuss RFPS in $\mathcal{O}_{2,2}$.

As for the recursive fermion systems in the pseudo Cuntz algebra, there is a very impressive phenomenon as follows. We consider a fermion algebra NCAR with negative-definite signature, in which generators $a'_n$ $(n = 1, 2, \dots)$ satisfy the anticommutation relations as follows

$$\{a'_m, a'_n\} = 0, \quad \{a'_m, a'^*_n\} = -\delta_{m,n} I, \quad m, n = 1, 2, \dots . \tag{6.2}$$

It is possible to embed NCAR into $\mathcal{O}_{2,2}$ in the following way. Let us define a tetrad $SR_{0+2} = (\boldsymbol{a}'_1, \boldsymbol{a}'_2; \zeta_{0+2}, \varphi_{0+2})$, which is called the *standard recursive fermion system of* $(0,2)$*-type* (standard $\mathrm{RFS}_{0+2}$) in $\mathcal{O}_{2,2}$, by[g)]

$$\boldsymbol{a}'_1 \equiv s_1 s_2^* - s_4 s_3^*, \tag{6.3}$$

$$\boldsymbol{a}'_2 \equiv s_1 s_4^* + s_2 s_3^*, \tag{6.4}$$

$$\zeta_{0+2}(X) \equiv \sum_{i=1}^{4} s_i X s_i^*, \quad X \in \mathcal{O}_{2,2}, \tag{6.5}$$

$$\varphi_{0+2}(X) \equiv \rho_{2,2}(X) = \sum_{i=1}^{4} (-1)^{i-1} s_i X s_i^*, \quad X \in \mathcal{O}_{2,2}, \tag{6.6}$$

which satisfy

$$\text{i) seed condition:} \quad \{\boldsymbol{a}'_i, \boldsymbol{a}'_j\} = 0, \quad \{\boldsymbol{a}'_i, \boldsymbol{a}'^*_j\} = -\delta_{i,j} I, \quad i, j = 1, 2, \tag{6.7}$$

$$\text{ii) recursive condition:} \quad \{\boldsymbol{a}'_i, \zeta_{0+2}(X)\} = 0, \ \zeta_{0+2}(X)^* = \zeta_{0+2}(X^*), \ X \in \mathcal{O}_{2,2}, \tag{6.8}$$

$$\text{iii) normalization condition:} \quad \zeta_{0+2}(X)\zeta_{0+2}(Y) = \varphi_{0+2}(XY), \quad X, Y \in \mathcal{O}_{2,2}. \tag{6.9}$$

Then, an embedding $\Phi_{R_{0+2}}$ of NCAR into $\mathcal{O}_{2,2}$ associated with $SR_{0+2}$ is given by

$$\begin{aligned} &\Phi_{SR_{0+2}} : \mathrm{NCAR} \hookrightarrow \mathcal{O}_{2,2}, \\ &\Phi_{SR_{0+2}}(a'_{2(n-1)+i}) \equiv \zeta'^{n-1}(\boldsymbol{a}'_i), \quad i = 1, 2, \quad n = 1, 2, \dots . \end{aligned} \tag{6.10}$$

Furthermore, we can show that $\Phi_{SR_{0+2}}(\mathrm{NCAR})$ is identical with $\mathcal{O}_{2,2}^{U(1)}$ in the same way as $\Phi_{SR_{1+1}}(\mathrm{ICAR})$. This fact means that there exists a $*$-isomorphism between ICAR and NCAR in which metric structures are completely different from each other. Indeed, using (3.1) and (3.2) with $\eta_{ij} = (-1)^{i-1}\delta_{ij}$, we obtain the relation between $(\boldsymbol{a}_1, \boldsymbol{a}_2)$ in $SR_{1+1}$ and $(\boldsymbol{a}'_1, \boldsymbol{a}'_2)$ in $SR_{0+2}$ as follows

$$\boldsymbol{a}'_1 = \boldsymbol{a}_1 - (\boldsymbol{a}_1^* + \boldsymbol{a}_1)\boldsymbol{a}_2^* \boldsymbol{a}_2, \tag{6.11}$$

$$\boldsymbol{a}'_2 = (\boldsymbol{a}_1^* - \boldsymbol{a}_1)\boldsymbol{a}_2 \, ; \tag{6.12}$$

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[g)]Substituting (6.1) into (6.3)–(6.6), we obtain a RFS in $\mathcal{O}_{1,1}$ in the similar form of $SR_1$ in $\mathcal{O}_2$, which should be called the standard $\mathrm{RFS}_{0+1}$ since it gives an embedding of NCAR onto $\mathcal{O}_{1,1}^{U(1)}$.

$$\boldsymbol{a}_1 = \boldsymbol{a}'_1 + (\boldsymbol{a}'^*_1 + \boldsymbol{a}'_1)\boldsymbol{a}'^*_2\boldsymbol{a}'_2, \tag{6.13}$$

$$\boldsymbol{a}_2 = (\boldsymbol{a}'^*_1 - \boldsymbol{a}'_1)\boldsymbol{a}'_2. \tag{6.14}$$

Then, we obtain the following one-to-one correspondence of generators between ICAR and NCAR:

$$a'_{2n-1} \quad \Leftrightarrow \quad \exp(i\pi \sum_{k=1}^{n-1} a^*_{2k}\, a_{2k})\Big(a_{2n-1} - (a^*_{2n-1} + a_{2n-1})a^*_{2n}\, a_{2n}\Big), \tag{6.15}$$

$$a'_{2n} \quad \Leftrightarrow \quad \exp(i\pi \sum_{k=1}^{n-1} a^*_{2k-1}\, a_{2k-1})\Big(a^*_{2n-1} - a_{2n-1}\Big)a_{2n}\,; \tag{6.16}$$

$$a_{2n-1} \quad \Leftrightarrow \quad \exp(i\pi \sum_{k=1}^{n-1} a'^*_{2k}\, a'_{2k})\Big(a'_{2n-1} + (a'^*_{2n-1} + a'_{2n-1})a'^*_{2n}\, a'_{2n}\Big), \tag{6.17}$$

$$a_{2n} \quad \Leftrightarrow \quad \exp(i\pi \sum_{k=1}^{n-1} a'^*_{2k-1}\, a'_{2k-1})\Big(a'^*_{2n-1} - a'_{2n-1}\Big)a'_{2n}, \tag{6.18}$$

where $\exp(i\pi a^*_n a_n) = I - (-1)^n 2a^*_n a_n$ and $\exp(i\pi a'^*_n a'_n) = I + 2a'^*_n a'_n$ being the Klein operators anticommuting with $a_n$ and $a'_n$, respectively. Under this nonlinear transformation, the vacuum of the Fock representation is kept invariant, but neither the particle number nor the metric structure is preserved. Therefore, the difference between ICAR and NCAR is just by the choice of generators corresponding to unitarily inequivalent representations. Discovery of this kind of nonlinear transformation of the fermion algebra is greatly indebted to the expressions of fermion generators in terms of the pseudo Cuntz algebra.[8] The description of fermion algebras in the (pseudo) Cuntz algebra seems to play quite an important role in the study of fermion systems.

## Appendix. Embeddings among Pseudo Cuntz Algebras

Embeddings and endomorphisms considered in the ordinary Cuntz algebra may be easily generalized to the pseudo Cuntz algebra. In the similar way that an arbitrary Cuntz algebra $\mathcal{O}_d$ is embedded into $\mathcal{O}_2$ as its $*$-subalgebra,[1,5] we can show that an arbitrary pseudo Cuntz algebra $\mathcal{O}_{d,d'}$ is embedded into $\mathcal{O}_{1,1}$. As a special case, $\mathcal{O}_2$ is embedded into $\mathcal{O}_{1,1}$.[h]

First, let us note the existence of the following operator $J$ in $\mathcal{O}_{1,1}$:

$$J \equiv s_{2;1} - s_{1;2}, \tag{A.1}$$

$$J^* = -J, \quad J^*J = JJ^* = -J^2 = -I. \tag{A.2}$$

Then, it is straightforward to see that an embedding $\Psi_2$ of $\mathcal{O}_2$ into $\mathcal{O}_{1,1}$ is given by

$$\begin{aligned} &\Psi_2 : \mathcal{O}_2 \hookrightarrow \mathcal{O}_{1,1}, \\ &\Psi_2(S_1) \equiv s_1, \quad \Psi_2(S_2) \equiv s_2\, J, \end{aligned} \tag{A.3}$$

where $S_1$ and $S_2$ denote the generators of $\mathcal{O}_2$. Likewise, an embedding $\Psi_d$ of $\mathcal{O}_d$ into

[h] In precise, we consider only embeddings of the dense subalgebra of $\mathcal{O}_2$ into $\mathcal{O}_{1,1}$.

$\mathcal{O}_{1,1}$ is given by

$$\begin{aligned} &\Psi_d : \mathcal{O}_d \hookrightarrow \mathcal{O}_{1,1}, \\ &\Psi_d(S_i) = \begin{cases} s_2^{i-1}\, s_1\, J^{i-1} & \text{for } 1 \leqq i \leqq d-1, \\ s_2^{d-1}\, J^{d-1} & \text{for } i = d. \end{cases} \end{aligned} \tag{A.4}$$

Now, it is easy to see that an embedding $\Psi_{d,d'}$ of $\mathcal{O}_{d,d'}$ with $d' \geqq 1$ into $\mathcal{O}_{1,1}$ is given by

$$\begin{aligned} &\Psi_{d,d'} : \mathcal{O}_{d,d'} \hookrightarrow \mathcal{O}_{1,1}, \\ &\Psi_{d,d'}(S_i) = \begin{cases} s_2^{i-1}\, s_1\, J^{i-1} & \text{for } 1 \leqq i \leqq d, \\ s_2^{i-1}\, s_1\, J^{i} & \text{for } d+1 \leqq i \leqq d+d'-1, \\ s_2^{d+d'-1}\, J^{d+d'} & \text{for } i = d+d', \end{cases} \end{aligned} \tag{A.5}$$

with $S_i$'s being the generators of $\mathcal{O}_{d,d'}$.

As for $\mathcal{O}_{2^p,2^p}$ with $p \geqq 1$, there is an embedding into $\mathcal{O}_{1,1}$ in which each generator of $\mathcal{O}_{2^p,2^p}$ is mapped to an element in $\mathcal{O}_{1,1}$ homogeneously in $s_i$ without using $J$ as follows:

$$\begin{aligned} &\hat{\Psi}_{2^p,d^p} : \mathcal{O}_{2^p,2^p} \hookrightarrow \mathcal{O}_{1,1}, \\ &\hat{\Psi}_{2^p,2^p}(S_i) = s_{i_1} s_{i_2} \cdots s_{i_{p+1}}, \quad i = 1, \ldots, 2^{p+1},\ i_k = 1, 2,\ (k = 1, \ldots, p+1) \end{aligned} \tag{A.6}$$

with an appropriate one-to-one correspondence between the indices $i$ and $(i_1, i_2, \ldots, i_{p+1})$.

There is another type of embedding $\tilde{\Psi}_2$ of $\mathcal{O}_2$ into $\mathcal{O}_{1,1}$ as follows:

$$\begin{aligned} &\tilde{\Psi}_2 : \mathcal{O}_2 \hookrightarrow \mathcal{O}_{1,1}, \\ &\tilde{\Psi}_2(S_1) \equiv \rho(s_1), \quad \tilde{\Psi}_2(S_2) \equiv \xi(s_2), \\ &\rho(X) \equiv s_1\, X\, {s_1}^* - s_2\, X\, {s_2}^*, \\ &\xi(X) \equiv s_2\, X\, {s_1}^* - s_1\, X\, {s_2}^*, \end{aligned} \tag{A.7}$$

where $\rho$ is nothing but the canonical endomorphism of $\mathcal{O}_{1,1}$, and the mapping $\xi$ satisfies the following properties

$$\xi(X)^* = -\xi(X^*), \tag{A.8}$$

$$\xi(X)\xi(Y) = \rho(XY), \tag{A.9}$$

$$\xi(X)\rho(Y) = \rho(X)\xi(Y) = \xi(XY). \tag{A.10}$$

It is also straightforward to generalize $\tilde{\Psi}_2$ to the corresponding embedding of $\mathcal{O}_{d,d'}$ into $\mathcal{O}_{1,1}$.

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