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Nonlinear Transformation Group of CAR Fermion Algebra

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Abstract

Based on our previous work on the recursive fermion system in the Cuntz algebra, it is shown that a nonlinear transformation group of the CAR fermion algebra is induced from a $U(2^p)$ action on the Cuntz algebra $\mathcal{O}_{2^p}$ with an arbitrary positive integer p. In general, these nonlinear transformations are expressed in terms of finite polynomials in generators. Some Bogoliubov transformations are involved as special cases.

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In our previous paper,[1] we have introduced the recursive fermion system (RFS_p) which gives embeddings of CAR into the Cuntz algebra[2] $\mathcal{O}_{2^p}$ with an arbitrary positive integer p. As for the special case, which we call the *standard* RFS_p, CAR is embedded onto the $U(1)$-invariant $*$-subalgebra $\mathcal{O}_{2^p}^{U(1)}$ of $\mathcal{O}_{2^p}$. Here, the $U(1)$ action on $\mathcal{O}_d$ is defined by

$$\gamma_z : s_i \mapsto z\, s_i, \quad z \in \mathbf{C}, \quad |z| = 1 \tag{1}$$

with s_i $(i = 1, \ldots, d)$ being the generators of $\mathcal{O}_d$. Since an automorphism of $\mathcal{O}_{2^p}$, which is described by a $U(2^p)$ action, also keeps $\mathcal{O}_{2^p}^{U(1)}$ invariant, it induces a transformation of CAR. We find this type of transformations is *nonlinear* with respect to generators $\{a_m, a_n^* \,|\, m, n = 1, 2, \ldots\}$ in general, and includes some Bogoliubov transformations as special cases.

First, let us recall that the Cuntz algebra $\mathcal{O}_d$ is a C^*-algebra generated by s_i, $i = 1, 2, \ldots, d$, which satisfy the following relations:[2]

$$s_i^*\, s_j = \delta_{i,j} I, \tag{2}$$

$$\sum_{i=1}^{d} s_i\, s_i^* = I. \tag{3}$$

We often use the brief description such as $s_{i_1 i_2 \cdots i_m ; j_n \cdots j_2 j_1} \equiv s_{i_1} s_{i_2} \cdots s_{i_m} s_{j_n}^* \cdots s_{j_2}^* s_{j_1}^*$, $m + n \geqq 1$. The $U(1)$-invariant subalgebra $\mathcal{O}_d^{U(1)}$ of $\mathcal{O}_d$ is given by a linear space spanned by $s_{i_1 \cdots i_m ; j_m \cdots j_1}$, $m = 1, 2, \ldots$. We consider an automorphism α_u of $\mathcal{O}_d$ obtained from a natural $U(d)$ action α as follows:

$$\begin{gathered} \alpha : U(d) \curvearrowright \mathcal{O}_d, \\ \alpha_u(s_i) \equiv \sum_{k=1}^{d} s_k u_{ki}, \quad u \in U(d), \quad i = 1, 2, \ldots, d. \end{gathered} \tag{4}$$

Indeed, from the equality

$$\alpha_{u_1} \circ \alpha_{u_2} = \alpha_{u_1 u_2}, \quad u_1, u_2 \in U(d), \tag{5}$$

α becomes a $U(d)$ action on $\mathcal{O}_d$. Since $\alpha_u \circ \gamma_z = \gamma_z \circ \alpha_u$, restriction of α_u to $\mathcal{O}_d^{U(1)}$ gives an automorphism of $\mathcal{O}_d^{U(1)}$:

$$\begin{gathered} \alpha_u|_{O_d^{U(1)}} \in \mathrm{Aut}\, \mathcal{O}_d^{U(1)}, \\ \alpha_u(s_{i_1 \cdots i_m ; j_m \cdots j_1}) = \sum_{k_1, \ldots, \ell_m = 1}^{d} u_{k_1 i_1} \cdots u_{k_m i_m} u_{j_1 \ell_1}^* \cdots u_{j_m \ell_m}^* s_{k_1 \cdots k_m ; \ell_m \cdots \ell_1} \in O_d^{U(1)}. \end{gathered} \tag{6}$$

Using the homogeneous embedding Ψ of $\mathcal{O}_{d^r}$ $(r = 2, 3, \ldots)$ into $\mathcal{O}_d$ defined by[3]

$$\begin{gathered} \Psi : \mathcal{O}_{d^r} \hookrightarrow \mathcal{O}_d, \\ \Psi(S_i) \equiv s_{j_1} \cdots s_{j_r}, \quad i = 1, 2, \ldots, d^r;\ j_1, \ldots, j_r = 1, 2, \ldots, d, \\ i - 1 = \sum_{k=1}^{r} (j_k - 1) d^{k-1}, \end{gathered} \tag{7}$$

where $S_1, \ldots, S_{d^r}$ and $s_1, \ldots, s_d$ denote the generators of $\mathcal{O}_{d^p}$ and $\mathcal{O}_d$, respectively, it is straightforward to show that, for any $u \in U(d)$, there exists $v \in U(d^r)$ such that

$$(\alpha_u \circ \Psi)(X) = (\Psi \circ \alpha_v)(X), \quad X \in \mathcal{O}_{d^r}. \tag{8}$$

The generators of CAR are denoted by a_n, $n = 1, 2, \ldots$, which satisfy

$$\{a_m, a_n\} = 0, \quad \{a_m, a_n^*\} = \delta_{m,n} I, \quad m, n = 1, 2, \ldots. \tag{9}$$

We give a systematic embedding of CAR into $\mathcal{O}_{2^p}$ in the following. The *standard recursive fermion system of order p* (RFS_p) in $\mathcal{O}_{2^p}$ is given by a set $SR_p = (\boldsymbol{a}_1, \ldots, \boldsymbol{a}_p\,; \zeta_p, \varphi_p)$ defined by[1)]

$$\boldsymbol{a}_i \equiv \sum_{k=1}^{2^{p-i}} \sum_{\ell=1}^{2^{i-1}} (-1)^{\sum_{m=1}^{i-1} \left[\frac{\ell-1}{2^{m-1}}\right]} s_{2^i(k-1)+\ell} s^*_{2^{i-1}(2k-1)+\ell}, \quad i = 1, 2, \ldots, p, \tag{10}$$

$$\zeta_p(X) \equiv \sum_{i=1}^{2^p} (-1)^{\sum_{m=1}^{p} \left[\frac{i-1}{2^{m-1}}\right]} s_i X s_i^*, \quad X \in \mathcal{O}_{2^p}, \tag{11}$$

$$\varphi_p(X) \equiv \rho_{2^p}(X) \equiv \sum_{i=1}^{2^p} s_i X s_i^*, \quad X \in \mathcal{O}_{2^p}, \tag{12}$$

where s_i's ($i = 1, 2, \ldots, 2^p$) are generators of $\mathcal{O}_{2^p}$, $[x]$ denotes the largest integer not greater than x, and ρ_{2^p} is the canonical endomorphism of $\mathcal{O}_{2^p}$. Here, SR_p satisfies the following relations

$$\{\boldsymbol{a}_i, \boldsymbol{a}_j\} = 0, \quad \{\boldsymbol{a}_i, \boldsymbol{a}_j^*\} = \delta_{i,j} I, \quad i, j = 1, 2, \ldots, p, \tag{13}$$

$$\{\boldsymbol{a}_i, \zeta_p(X)\} = 0, \quad \zeta_p(X)^* = \zeta(X^*), \quad i = 1, 2, \ldots, p, \quad X \in \mathcal{O}_{2^p}, \tag{14}$$

$$\zeta_p(X)\zeta_p(Y) = \varphi_p(XY), \quad X, Y \in \mathcal{O}_{2^p}. \tag{15}$$

Then, the embedding Φ_{SR_p} of CAR into $\mathcal{O}_{2^p}$ associated with SR_p is defined by

$$\begin{aligned} &\Phi_{SR_p} : \mathrm{CAR} \hookrightarrow \mathcal{O}_{2^p}, \\ &\Phi_{SR_p}(a_{p(n-1)+i}) = \zeta_p^{n-1}(\boldsymbol{a}_i), \quad i = 1, 2, \ldots, p, \quad n = 1, 2, \ldots. \end{aligned} \tag{16}$$

For better understanding, we write explicitly $\boldsymbol{a}_i$ and ζ_p of SR_p for $p = 1, 2, 3$:

$$SR_1 \begin{cases} \boldsymbol{a}_1 \equiv s_{1;2}, \\ \zeta_1(X) \equiv s_1 X s_1^* - s_2 X s_2^*, \quad X \in \mathcal{O}_2\,; \end{cases} \tag{17}$$

$$SR_2 \begin{cases} \boldsymbol{a}_1 \equiv s_{1;2} + s_{3;4}, \\ \boldsymbol{a}_2 \equiv s_{1;3} - s_{2;4}, \\ \zeta_2(X) \equiv s_1 X s_1^* - s_2 X s_2^* - s_3 X s_3^* + s_4 X s_4^*, \quad X \in \mathcal{O}_4\,; \end{cases} \tag{18}$$

$$SR_3 \left\{ \begin{array}{l} \boldsymbol{a}_1 \equiv s_{1;2} + s_{3;4} + s_{5;6} + s_{7;8}, \\ \boldsymbol{a}_2 \equiv s_{1;3} - s_{2;4} + s_{5;7} - s_{6;8}, \\ \boldsymbol{a}_3 \equiv s_{1;5} - s_{2;6} - s_{3;7} + s_{4;8}, \\ \zeta_3(X) \equiv s_1 X s_1^* - s_2 X s_2^* - s_3 X s_3^* + s_4 X s_4^* \\ \qquad\quad -s_5 X s_5^* + s_6 X s_6^* + s_7 X s_7^* - s_8 X s_8^*, \quad X \in \mathcal{O}_8. \end{array} \right. \tag{19}$$

Using mathematical induction, it is shown that $\Phi_{SR_p}(\mathrm{CAR})$ is identical with $\mathcal{O}_{2^p}^{U(1)}$, that is, any $s_{i_1 \cdots i_m ; j_m \cdots j_1} \in \mathcal{O}_{2^p}^{U(1)}$ is expressed in terms of a monomial in $\Phi_{SR_p}(a_k)$ and $\Phi_{SR_p}(a_\ell^*)$ with $k, \ell \leqq pm$.

From (6) with $d = 2^p$ and using the fact $\mathcal{O}_{2^p}^{U(1)} = \Phi_{SR_p}(\mathrm{CAR})$, we have

$$\alpha_u|_{\Phi_{SR_p}(\mathrm{CAR})} \in \mathrm{Aut}\, \Phi_{SR_p}(\mathrm{CAR}), \quad u \in U(2^p). \tag{20}$$

Hence, the automorphism α_u of $\mathcal{O}_{2^p}$ induces a $U(2^p)$ action χ on CAR as follows

$$\begin{array}{c} \chi : U(2^p) \curvearrowright \mathrm{CAR}, \\ \chi_u(a_n) \equiv (\Phi_{SR_p}^{-1} \circ \alpha_u \circ \Phi_{SR_p})(a_n), \quad u \in U(2^p), \;\; n = 1, 2, \ldots . \end{array} \tag{21}$$

Here, the rhs of (21) is expressed in terms of a finite polynomial in a_k and a_ℓ^* with $k, \ell \leqq pm$ for $p(m-1) + 1 \leqq n \leqq pm$. Therefore, χ_u gives a nonlinear transformation of CAR associated with $u \in U(2^p)$. It is obvious that the whole set of χ_u with $u \in U(2^p)$ denoted by

$$\mathrm{Aut}_{U(2^p)}(\mathrm{CAR}) \equiv \{\chi_u \,|\, u \in U(2^p)\} \tag{22}$$

constitutes a kind of nonlinear realization of $U(2^p)$ on CAR. Restricting (8) with $d = 2^p$ to $\Phi_{SR_{rp}}(\mathrm{CAR})$, we obtain the following inclusion relation

$$\mathrm{Aut}_{U(2^p)}(\mathrm{CAR}) \subset \mathrm{Aut}_{U(2^{rp})}(\mathrm{CAR}), \quad r = 2, 3, \ldots . \tag{23}$$

Hence, it may be possible to define some kind of inductive limit as $\mathrm{Aut}_{U(2^\infty)}(\mathrm{CAR})$, but we will not proceed in detail here.

Although the $U(2^p)$ action χ on CAR is defined individually for each $p \in \mathbf{N}$, it is possible to consider a product $\chi_{u_1} \circ \chi_{u_2}$ even for $u_1 \in U(2^{p_1})$ and $u_2 \in U(2^{p_2})$ with $p_1 \neq p_2$ according to its definition. From (23), this product is also given explicitly in $\mathrm{Aut}_{U(2^q)}(\mathrm{CAR})$ with q being a common multiple of p_1 and p_2. Indeed, there exist v_1, $v_2 \in U(2^q)$ satisfying $\chi_{u_i} = \chi_{v_i}$, $i = 1, 2$, hence, we have $\chi_{u_1} \circ \chi_{u_2} = \chi_{v_1 v_2}$. In any way, we obtain an infinite-dimensional automorphism group of CAR by giving the whole set of such products as follows

$$\mathrm{Aut}_U(\mathrm{CAR}) \equiv \{\, \chi_{u_1} \circ \cdots \circ \chi_{u_k} \mid u_i \in U(2^{p_i}),\; i = 1, 2, \ldots, k\,;\; p_i, k = 1, 2, \ldots \}. \tag{24}$$

Hereafter, we consider the nonlinear transformation of CAR defined by (21) more concretely. We rewrite (11) as $\zeta_p(X) = \sum_{i \in \mathcal{I}_+} s_i X s_i^* - \sum_{i \in \mathcal{I}_-} s_i X s_i^*$ with $\mathcal{I}_\pm$ being a certain division of indices $\{1, 2, \ldots, 2^p\}$ satisfying $\sharp \mathcal{I}_\pm = 2^{p-1}$. Then, for any $u \in U(2^p) \bigcap U(2^{p-1}, 2^{p-1}) = U(2^{p-1}) \times U(2^{p-1})$, where each $U(2^{p-1})$ acts only on s_i with $i \in \mathcal{I}_+$

(or $i \in \mathcal{I}_-$), we have an equality $\alpha_u(\zeta_p(X)) = \zeta_p(\alpha_u(X))$, that is, ζ_p is *covariant* with respect to α_u for $u \in U(2^{p-1}) \times U(2^{p-1})$. On the other hand, if k is an odd positive integer, we have an identity $\zeta_p(X_1 \cdots X_k) = \zeta_p(X_1) \cdots \zeta_p(X_k)$ ($X_1, \ldots, X_k \in \mathcal{O}_{2^p}$). Therefore, if $u \in U(2^{p-1}) \times U(2^{p-1})$ and $\alpha_u(\boldsymbol{a}_i)$ involves only odd order monomials in $\boldsymbol{a}_i$ ($i = 1, \ldots, p$), $\{\alpha_u(\Phi_{SR_p}(a_{p(n-1)+i})) \mid i = 1, 2, \ldots, p\}$ for each n is expressed in terms of polynomials in the same form as $\{\alpha_u(\boldsymbol{a}_i) \mid i = 1, 2, \ldots, p\}$, that is, Φ_{SR_p} is *covariant* with respect to α_u.

For example, for $p \geqq 2$, let u be the transposition of $s_{2^{p-1}+1}$ and s_{2^p}. Then, from an identity $\boldsymbol{a}_1 \boldsymbol{a}_2 \cdots \boldsymbol{a}_p = (-1)^{\left[\frac{p}{2}\right]} s_{1,2^p}$, $\alpha_u(\boldsymbol{a}_p)$ is expressed in terms of a polynomial involving a term $(-1)^{\left[\frac{p}{2}\right]} \boldsymbol{a}_1 \cdots \boldsymbol{a}_p$. Furthermore, if p is odd, not only ζ_p but also Φ_{SR_p} is covariant with respect to this α_u.

In the following, we present some examples explicitly.

(1) $p = 1$, $u = u_1(\theta) \in SO(2) \subset U(2)$; $\alpha_{u_1(\theta)}(s_1) \equiv \cos\theta\, s_1 - \sin\theta\, s_2$, $\alpha_{u_1(\theta)}(s_2) \equiv \sin\theta\, s_1 + \cos\theta\, s_2$, $(0 \leqq \theta < 2\pi)$:

$$\begin{aligned} \chi_{u_1(\theta)}(a_1) &= \cos^2\theta\, a_1 - \sin^2\theta\, a_1^* + \sin\theta\cos\theta\,(I - 2a_1^* a_1) \\ &= \cos^2\theta\, a_1 - \sin^2\theta\, a_1^* + \sin\theta\cos\theta\exp(i\pi a_1^* a_1), \end{aligned} \tag{25}$$

$$\begin{aligned} \chi_{u_1(\theta)}(a_n) &= \prod_{k=1}^{n-1} \Big(\cos 2\theta - \sin 2\theta\, \exp\Big(i\pi \sum_{i=1}^{k-1} a_i^* a_i\Big)(a_k^* - a_k)\Big) \\ &\quad \times \left(\cos^2\theta\, a_n - \sin^2\theta\, a_n^* + \sin\theta\cos\theta \exp\Big(i\pi \sum_{j=1}^{n} a_j^* a_j\Big)\right), \quad n \geqq 2, \end{aligned} \tag{26}$$

where use has been made of an identity $(a_k^* a_k)^2 = a_k^* a_k$, and $\exp(i\pi a_k^* a_k)$ being the Klein operator anticommuting with a_k. One should note that an identity $\exp(i2\pi a_k^* a_k) = I$ holds. From (23), the same nonlinear transformation is involved in $\mathrm{Aut}_{U(2^p)}(\mathrm{CAR})$ for any $p \in \mathbf{N}$. Indeed, it is straightforward to show it for $p = 2$.

(2) $p = 2$, $u = u_2 \in \mathfrak{S}_4 \subset U(4)$; $\alpha_{u_2}(s_3) \equiv s_4$, $\alpha_{u_2}(s_4) \equiv s_3$, $\alpha_{u_2}(s_i) \equiv s_i$ for $i = 1, 2$:

$$\chi_{u_2}(a_1) = a_1 + (a_1^* - a_1) a_2^* a_2, \tag{27}$$

$$\chi_{u_2}(a_2) = -(a_1^* + a_1) a_2, \tag{28}$$

$$\chi_{u_2}(a_{2n-1}) = \exp\Big(i\pi \sum_{k=1}^{n-1} a_{2k}^* a_{2k}\Big)\Big(a_{2n-1} + (a_{2n-1}^* - a_{2n-1}) a_{2n}^* a_{2n}\Big), \quad n \geqq 2, \tag{29}$$

$$\chi_{u_2}(a_{2n}) = -\exp\Big(i\pi \sum_{k=1}^{n-1} a_{2k-1}^* a_{2k-1}\Big)(a_{2n-1}^* + a_{2n-1}) a_{2n}, \qquad n \geqq 2. \tag{30}$$

It is straightforward to confirm that the equality $(\alpha_{u_2} \circ \alpha_{u_2})(a_n) = a_n$ is satisfied, which is the result of $u_2 = u_2^{-1}$.

(3) $p=2$, $u=u_3\in\mathfrak{S}_4\subset U(4)$; $\alpha_{u_3}(s_2)\equiv s_3$, $\alpha_{u_2}(s_3)\equiv s_2$, $\alpha_{u_3}(s_i)\equiv s_i$ for $i=1,\,4$:

$$\chi_{u_3}(a_{2n-1}) = \exp\Big(i\pi a^*_{2n-1}a_{2n-1}\Big)a_{2n}, \tag{31}$$

$$\chi_{u_3}(a_{2n}) = \exp\Big(i\pi a^*_{2n}a_{2n}\Big)a_{2n-1}. \tag{32}$$

This transformation should be called a double Klein transformation which preserves anticommutativity between two annihilation (creation) operators.

(4) $p=2$, $u=u_4(\theta)\in SO(2)\subset U(4)$; $\alpha_{u_4(\theta)}(s_1)\equiv\cos\theta s_1-\sin\theta s_4$, $\alpha_{u_4(\theta)}(s_4)\equiv\sin\theta s_1+\cos\theta s_4$, $(0\leqq\theta<2\pi)$, $\alpha_{u_4(\theta)}(s_i)=s_i$ for $i=2,\,3$:

$$\chi_{u_4(\theta)}(a_{2n-1}) = \cos\theta\, a_{2n-1}+\sin\theta\, a^*_{2n}, \tag{33}$$

$$\chi_{u_4(\theta)}(a_{2n}) = -\sin\theta\, a^*_{2n-1}+\cos\theta\, a_{2n}. \tag{34}$$

This is nothing but an example of the conventional Bogoliubov transformation. Similar Bogoliubov transformations are obtained in $\mathrm{Aut}_{U(4^r)}(\mathrm{CAR})$ with $r\in\mathbf{N}$.

(5) $p=3$, $u=u_5\in\mathfrak{S}_8\subset U(8)$; $\alpha_{u_5}(s_5)\equiv s_8$, $\alpha_{u_5}(s_8)\equiv s_5$, $\alpha_{u_5}(s_i)\equiv s_i$ for $i\neq 5,\,8$:

$$\chi_{u_5}(a_{3n-2}) = a_{3n-2}-a_{3n-2}a^*_{3n}a_{3n}+(I-2a^*_{3n-2}a_{3n-2})a^*_{3n-1}a^*_{3n}a_{3n}, \tag{35}$$

$$\chi_{u_5}(a_{3n-1}) = a_{3n-1}-a_{3n-2}a^*_{3n}a_{3n}-a^*_{3n-2}(I-2a^*_{3n-1}a_{3n-1})a^*_{3n}a_{3n}, \tag{36}$$

$$\chi_{u_5}(a_{3n}) = -(a_{3n-2}a_{3n-1}+a^*_{3n-2}a_{3n-2}+a^*_{3n-1}a_{3n-1} +a^*_{3n-2}a^*_{3n-1}-2a^*_{3n-2}a_{3n-2}a^*_{3n-1}a_{3n-1})a_{3n}. \tag{37}$$

This is an example mentioned above for the case $p=3$. Here, $\chi_{u_5}(a_{3n})$ involves a term $-a_{3n-2}a_{3n-1}a_{3n}$, and Φ_{SR_3} is covariant with respect to α_{u_5}.

In conclusion, we summarize the properties of the above nonlinear transformations constituting $Aut_U(\mathrm{CAR})$:

(i) For any $\chi\in\mathrm{Aut}_U(\mathrm{CAR})$, $\chi(a_n)$ is expressed in terms of a finite polynomial in generators of CAR.

(ii) Action of any compact group on CAR is realized by certain elements of $\mathrm{Aut}_U(\mathrm{CAR})$.

(iii) Through a nonlinear transformation of CAR, particle numbers are not preserved in general.

(iv) A subset of $\mathrm{Aut}_U(\mathrm{CAR})$ consisting of all transformations, which keep the Fock vacuum invariant, constitutes a nontrivial subgroup of $\mathrm{Aut}_U(\mathrm{CAR})$.

(v) In general, $\chi\in\mathrm{Aut}_U(\mathrm{CAR})$ is an outer automorphism, that is, it is *not* expressed by the adjoint operation of a unitary element in CAR as seen by checking the action of χ on the Fock space. Restriction of the permutation representation of Cuntz algebra $\mathcal{O}_{2^p}$ to $\Phi_{SR_p}(\mathrm{CAR})$ determines the change of representations of CAR through the nonlinear transformation.

References

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