NOTES ON TORIC VARIETIES FROM MORI THEORETIC VIEWPOINT (NEW VERSION)

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Abstract. The main purpose of this notes is to supplement the paper [Re], which treated Minimal Model Program (also called Mori's Program) on toric varieties. We calculate lengths of negative extremal rays of toric varieties. As an application, we obtain a generalization of Fujita's conjecture for singular toric varieties. We also prove that every toric variety has a small projective toric $\mathbb{Q}$-factorialization.

0. Introduction

The main purpose of this notes is to supplement the paper [Re], which treated Minimal Model Program (also called Mori's Program) on toric varieties. We calculate lengths of negative extremal rays of toric varieties. It is an easy exercise once we understand [Re]. As a corollary, we obtain a strong version of Fujita's conjecture for singular toric varieties. Related topics are [Ft], [Ka], [L] and [Mu, Section 4]. We will freely use the notation in [Fl], [Re] and work over an algebraically closed field k of arbitrary characteristic throughout this paper.

The following is the main theorem of this paper.

Theorem 0.1 (Cone Theorem). *Let X be an n-dimensional (not necessarily $\mathbb{Q}$-factorial) projective toric variety over k. We can write the cone of curves as follows:*

$$NE(X) = \sum \mathbb{R}_{\geq 0}[C].$$

Let $D = \sum_j d_j D_j$ be a $\mathbb{Q}$-divisor, where D_j is an irreducible torus invariant divisor and $0 \leq d_j \leq 1$ for every j. Assume that $K_X + D$ is $\mathbb{Q}$-Cartier. Then, for each extremal ray $\mathbb{R}_{\geq 0}[C]$, there exists an $(n-1)$-dimensional cone τ such that $[V(\tau)] \in \mathbb{R}_{>0}[C]$ and

$$-(K_X + D) \cdot V(\tau) \leq n + 1.$$

Date: 2001/12/8.

1991 *Mathematics Subject Classification.* Primary 14M25; Secondary 14E30.

Moreover, we can choose τ such that $-(K_X+D)\cdot V(\tau)\leq n$ unless $X\simeq \mathbb{P}^n$ and $\sum_j d_j<1$.

Section 1 deals with the proof of Theorem 0.1. We will treat an application of this theorem in Section 2. Professor Kajiwara informed me of [Mu] in Kinosaki after I finished the preliminary version of this paper. The following is Mustaţă's formulation of Fujita's conjecture for toric varieties. He proved it on the assumption that X is non-singular as an application of his vanishing theorem (see [Mu, Theorem 0.3]). Our proof doesn't use vanishing theorems.

Corollary 0.2 (Strong version of Fujita's conjecture)**.** *Let X be an n-dimensional (not necessarily $\mathbb{Q}$-factorial) projective toric variety over k and $D=\sum_j d_jD_j$ be a $\mathbb{Q}$-divisor, where D_j is an irreducible torus invariant divisor and $0\leq d_j\leq 1$ for every j. Assume that K_X+D is $\mathbb{Q}$-Cartier. Let L be a line bundle on X.*

(1) *Suppose that $(L\cdot C)\geq n$ for every torus invariant integral curve $C\subset X$. Then K_X+D+L is nef unless $X\simeq\mathbb{P}^n$, $\sum_j d_j<1$ and $L\simeq\mathcal{O}_{\mathbb{P}^n}(n)$.*
(2) *Suppose that $(L\cdot C)\geq n+1$ for every torus invariant integral curve $C\subset X$. Then K_X+D+L is ample unless $X\simeq\mathbb{P}^n$, $D=0$ and $L\simeq\mathcal{O}_{\mathbb{P}^n}(n+1)$.*

Of course, we can recover [Mu, Theorem 0.3] easily if we assume that X is non-singular. See also Remark 1.13.

In section 3, we collect some results obtained by Minimal Model Program on toric varieties. We need Lemma 3.5 for the proof of Theorem 0.1. We prove that every toric variety has a small projective toric $\mathbb{Q}$-factorialization. For related topics, see [OP, Section 3].

Comment. After I circulated [F], I found [L]. In [L], Laterveer proved Fujita's conjecture for $\mathbb{Q}$-Gorenstein projective toric varieties by the similar argument to mine.

Acknowledgements. Some parts of this paper were obtained in 1999, when I was a Research Fellow of the Japan Society for the Promotion of Science. I would like to express my gratitude to Professors Masanori Ishida, Shigefumi Mori, Tadao Oda, Takeshi Kajiwara, and Hiromichi Takagi, who gave me various advice and useful comments. I like to thank Doctor Hiroshi Sato, who gave me various advice and answered my questions. I also like to thank Doctor Takeshi Abe, who led me to this problem.

1. Proof of the theorem

1.1. First, let us recall *weighted projective spaces*. We adopt toric geometric descriptions. This helps the readers to understand Theorem 0.1. However, it is not necessary for the proof of Theorem 0.1.

1.2 (c.f. [Fl, p.35]). Let $\mathbb{P}(d_1, \cdots, d_{n+1})$ be a *weighted projective space*. To construct this as a toric variety, start with the fan whose cones generated by proper subsets of $\{v_1, \cdots, v_{n+1}\}$, where any n of these vectors are linearly independent, and their sum is zero. The lattice N is taken to be generated by the vectors $e_i = (1/d_i) \cdot v_i$ for $1 \leq i \leq n+1$. The resulting toric variety is in fact $\mathbb{P} = \mathbb{P}(d_1, \cdots, d_{n+1})$. We note that $\operatorname{Pic} \mathbb{P} \simeq \mathbb{Z}$. Let f_i be a unique primitive lattice point in the cone $\langle e_i \rangle$ with $e_i = u_i f_i$ for $u_i \in \mathbb{Z}_{>0}$. We put $d = \gcd(u_1 d_1, \cdots, u_{n+1} d_{n+1})$ and define $c_i = (1/d) u_i d_i$ for every i. Then we obtain that $\mathbb{P}(d_1, \cdots, d_{n+1}) \simeq \mathbb{P}(c_1, \cdots, c_{n+1})$ and $\sum c_i f_i = 0$. By changing the order, we can assume that $c_1 \leq c_2 \leq \cdots \leq c_{n+1}$. We note that $-K_{\mathbb{P}} = \sum V(f_i)$. Let τ be the $(n-1)$-dimensional cone $\langle f_1, \cdots, f_{n-1} \rangle$. Then we have

$$-K_{\mathbb{P}} \cdot V(\tau) = \sum_{i=1}^{n+1} V(f_i) \cdot V(\tau) = \frac{c}{c_n c_{n+1}} (\sum_{i=1}^{n+1} c_i) \leq n+1,$$

where $c = \gcd(c_n, c_{n+1})$. We note that

$$V(f_i) \cdot V(\tau) = \frac{c\ c_i}{c_n c_{n+1}}\ .$$

For calculations of intersection numbers, we recommend the readers to see [Fl, p.100] and [Re, (2.7)]. If the equality holds in the above equation, then $c_i = 1$ for every i. Thus, we obtain $\mathbb{P} \simeq \mathbb{P}^n$.

Remark 1.3. Suppose that $\gcd(d_1, \cdots, d_{n+1}) = 1$. Then, e_i is primitive in $\langle e_i \rangle \cap N$ if and only if $\gcd(d_1, \cdots, d_{i-1}, d_{i+1}, \cdots, d_{n+1}) = 1$.

1.4. If $n = 2$, then $c = 1$ since f_1 is primitive and $\sum c_i f_i = 0$. Therefore, we have

$$-K_{\mathbb{P}} \cdot V(\tau) = \frac{1}{c_2 c_3} (\sum_{i=1}^{3} c_i) \leq \frac{1}{2} + \frac{1}{2} + 1 \leq 2 = n$$

when $\mathbb{P} \not\simeq \mathbb{P}^2$. So, we have that $-K_{\mathbb{P}} \cdot V(\tau) \leq n$ if $n = 2$ and $\mathbb{P} \not\simeq \mathbb{P}^2$. If $-K_{\mathbb{P}} \cdot V(\tau) = 2$, then $\mathbb{P} \simeq \mathbb{P}(1, 1, 2)$.

1.5 (c.f. Proposition 1.10 below). When $n \geq 3$, the above inequality in 1.4 is not true. Assume that $n \geq 3$. Let $\mathbb{P}$ be an n-dimensional weighted projective space $\mathbb{P}(l-1, l-1, l, \cdots, l)$, where $l \geq 2$. Then we

obtain

$$-K_{\mathbb{P}} \cdot V(\tau) = n + 1 - \frac{2}{l}.$$

So, we have $-K_{\mathbb{P}} \cdot V(\tau) > n$ when $l \geq 3$. If we make l large, then $-K_{\mathbb{P}} \cdot V(\tau)$ becomes close to $n+1$.

1.6. Let $\mathbb{P} = \mathbb{P}(1, \cdots, 1, l-1, l)$ be an n-dimensional weighted projective space with $l \geq 2$ and $n \geq 2$. Then we have

$$-K_{\mathbb{P}} \cdot V(\tau) = \frac{n + 2l - 2}{l(l-1)}.$$

Thus, if we make l large, then $-K_{\mathbb{P}} \cdot V(\tau)$ becomes close to zero.

1.7. Next, we treat $\mathbb{Q}$-factorial toric Fano varieties with Picard number one. This type of varieties plays an important role for the analysis of extremal contractions. Here, we adopt the following description 1.9 for the definition of $\mathbb{Q}$-factorial toric Fano varieties with Picard number one. By this, it is easy to see that every extremal contraction contains them in the fibers (see Proof of the theorem below). Of course, weighted projective spaces are in this class.

Remark 1.8. In [Re, (0.1)], it is stated that any fiber of an extremal contraction is a weighted projective space. However, it is not true since there exists a $\mathbb{Q}$-factorial toric Fano variety with Picard number one that is not a weighted projective space.

1.9 ($\mathbb{Q}$-factorial toric Fano varieties with Picard number one)**.** Now we fix $N \simeq \mathbb{Z}^n$. Let $\{v_1, \cdots, v_{n+1}\}$ be a set of primitive vectors such that $N_{\mathbb{R}} = \sum_i \mathbb{R}_{\geq 0} v_i$. We define n-dimensional cones

$$\sigma_i := \langle e_1, \cdots, e_{i-1}, e_{i+1}, \cdots, e_{n+1} \rangle$$

for $1 \leq i \leq n+1$. Let Δ be the complete fan generated by n-dimensional cones σ_i for every i. Then we obtain a complete toric variety $X = X(\Delta)$ with Picard number $\rho(X) = 1$. We call it a *$\mathbb{Q}$-factorial toric Fano variety with Picard number one*. We define $(n-1)$-dimensional cones $\mu_{i,j} = \sigma_i \cap \sigma_j$ for $i \neq j$. We can write $\sum_i a_i v_i = 0$, where $a_i \in \mathbb{Z}_{>0}$, $\gcd(a_1, \cdots, a_{n+1}) = 1$, and $a_1 \leq a_2 \leq \cdots \leq a_{n+1}$ by changing the order. Then we obtain

$$0 < V(v_{n+1}) \cdot V(\mu_{n,n+1}) = \frac{\mathrm{mult}(\mu_{n,n+1})}{\mathrm{mult}(\sigma_n)} \leq 1,$$

$$V(v_i) \cdot V(\mu_{n,n+1}) = \frac{a_i}{a_{n+1}} \cdot \frac{\mathrm{mult}(\mu_{n,n+1})}{\mathrm{mult}(\sigma_n)},$$

and

$$\begin{aligned} -K_X \cdot V(\mu_{n,n+1}) &= \sum_{i=1}^{n+1} V(v_i) \cdot V(\mu_{n,n+1}) \\ &= \frac{1}{a_{n+1}} (\sum_{i=1}^{n+1} a_i) \frac{\mathrm{mult}(\mu_{n,n+1})}{\mathrm{mult}(\sigma_n)} \leq n+1. \end{aligned}$$

For "mult" in the above equations, see [Fl, p.48 and p.100]. If $-K_X \cdot V(\mu_{n,n+1}) = n+1$, then $a_i = 1$ for every i and $\mathrm{mult}(\mu_{n,n+1}) = \mathrm{mult}(\sigma_n)$.

Proposition 1.10. *If $X \not\cong \mathbb{P}^n$, then there exists some pair (l,m) such that $-K_X \cdot V(\mu_{l,m}) \leq n$.*

Proof. Assume the contrary. Then we obtain

$$-K_X \cdot V(\mu_{k,n+1}) = \frac{1}{a_{n+1}} (\sum_{i=1}^{n+1} a_i) \frac{\mathrm{mult}(\mu_{k,n+1})}{\mathrm{mult}(\sigma_k)} > n$$

for $1 \leq k \leq n$. Thus

$$(n+1)a_{n+1} \geq \sum_{i=1}^{n+1} a_i > \frac{\mathrm{mult}(\sigma_k)}{\mathrm{mult}(\mu_{k,n+1})} n a_{n+1}$$

for every k. Since

$$\frac{\mathrm{mult}(\sigma_k)}{\mathrm{mult}(\mu_{k,n+1})} \in \mathbb{Z}_{>0},$$

we have that $\mathrm{mult}(\sigma_k) = \mathrm{mult}(\mu_{k,n+1})$ for every k. This implies that a_k divides a_{n+1} for all k.

Claim. $a_1 = \cdots = a_{n+1} = 1$.

Proof of Claim. If $a_1 = a_{n+1}$, then we obtain the required results. So, we assume that $a_1 \neq a_{n+1}$. On this assumption, we have that $a_2 \neq a_{n+1}$ since v_1 is primitive and $\sum_i a_i v_i = 0$. In this case, we have

$$-K_X \cdot V(\mu_{k,n+1}) = \frac{1}{a_{n+1}} (\sum_{i=1}^{n+1} a_i) \leq n.$$

We note that

$$\frac{a_i}{a_{n+1}} \leq \frac{1}{2}$$

for $i = 1, 2$. This is a contradiction. So we obtain that $a_1 = \cdots = a_{n+1} = 1$. □

In this case, $-K_X \cdot V(\mu_{i,j}) > n$ implies $-K_X \cdot V(\mu_{i,j}) = n+1$ for every pair (i,j). Then $\mathrm{mult}(\mu_{i,j}) = \mathrm{mult}(\sigma_i)$ for $i \neq j$. So, we have that $\mathrm{mult}(\sigma_i) = 1$ for every i. Therefore, we obtain $X \simeq \mathbb{P}^n$. This is a contradiction. □

Remark 1.11. The usual definition of Fano varieties is the following: X is Fano if $-K_X$ is an ample $\mathbb{Q}$-Cartier divisor. It is easy to check that the notion of $\mathbb{Q}$-factorial toric Fano varieties with Picard number one by the usual definition coincides with ours.

1.12. From now on, we freely use the notation in [Re], especially, [Re, (2.2)] (see also [Re, (1.10)]).

Proof of the theorem. Step 1. We assume that X is $\mathbb{Q}$-factorial. Let $R = \mathbb{R}_{\geq 0}[C]$ be an extremal ray. There exists an elementary contraction $\varphi_R : X \to Y$, which is corresponding to the extremal ray R. The $\mathbb{Q}$-factorial toric Fano variety $P \subset X$ with Picard number $\rho(P) = 1$, which is corresponding to the cone $\sigma = \langle e_1, \cdots, e_\beta \rangle$, that is, $P = V(\sigma) \subset X$, is a fiber of $\varphi_R|_A : A \to B$ (c.f. [Re, (2.5)]). We note that

$$K_P = -\sum_{i=\beta+1}^{n+1} V(\widetilde{\rho}_i),$$

where $\widetilde{\rho}_i = \langle e_1, \cdots, e_\beta, e_i \rangle$ for $\beta + 1 \leq i \leq n+1$. On the other hand, $V(\widetilde{\rho}_i) = b_i V(e_i) \cdot V(\sigma)$ for some $b_i \in \mathbb{Z}_{>0}$ since the cones are simplicial (see [Fl, p.100]). Let $\widetilde{\tau}$ be an $(n-1)$-dimensional cone containing σ. We have that

$$\begin{aligned}
K_P \cdot V(\widetilde{\tau}) &= -\sum_{i=\beta+1}^{n+1} V(\widetilde{\rho}_i) \cdot V(\widetilde{\tau}) \\
&= -V(\widetilde{\tau}) \cdot \Big(\sum_{i=\beta+1}^{n+1} b_i V(e_i) \cdot V(\sigma) \Big) \\
&= V(\widetilde{\tau}) \cdot \Big(K_X + \sum_{\text{every ray}} V(e_i) - \sum_{i=\beta+1}^{n+1} b_i V(e_i) \Big) \\
&= V(\widetilde{\tau}) \cdot \Big(K_X + \sum_{i=\beta+1}^{n+1} (1-b_i) V(e_i) + \sum_{\text{others}} V(e_i) \Big) \\
&\leq (K_X + D) \cdot V(\widetilde{\tau}).
\end{aligned}$$

We note that

$$K_X + \sum_{\text{every ray}} V(e_i) \sim 0$$

and D can be written as $\sum_j d_j V(e_j)$ with $0 \leq d_j \leq 1$ by the assumption, and that $V(\widetilde{\tau}) \cdot V(e_i) > 0$ if and only if $\beta + 1 \leq i \leq n+1$ by [Re, (2.2)] (see also [Re, (2.4), (2.7), (2.10)]). We choose $\widetilde{\tau}$ as in the above argument 1.9, that is, $-K_P \cdot V(\widetilde{\tau}) \leq n - \beta + 1$, where $\dim P = n - \beta$. Then, by the above argument and the choice of $\widetilde{\tau}$,

$$-(K_X + D) \cdot V(\widetilde{\tau}) \leq -K_P \cdot V(\widetilde{\tau}) \leq n - \beta + 1.$$

Therefore, if the minimal length of a $(K_X + D)$-negative extremal ray is greater than n, then $\beta = \alpha = 0$. Thus we have $X \simeq \mathbb{P}^n$ and $\sum_j d_j < 1$ by Proposition 1.10. Therefore, we obtain the required result when X is $\mathbb{Q}$-factorial.

Step 2 (c.f. [L, (2.4) Lemma]). We assume that X is not $\mathbb{Q}$-factorial. Let $f : (\widetilde{X}, \widetilde{D}) \to (X, D)$ be a projective modification constructed in Lemma 3.5 below. We note that $X \not\simeq \mathbb{P}^n$. Let $R = \mathbb{R}_{\geq 0}[C]$ be a $(K_X + D)$-negative extremal ray. We take $V(\tau) \in \mathbb{R}_{>0}[C]$ such that $-(K_X + D) \cdot V(\tau)$ is minimal. We take $V(\widetilde{\tau})$ on $\widetilde{X}$ such that $f_* V(\widetilde{\tau}) = V(\tau)$. We can write $V(\widetilde{\tau}) = \sum a_i V(\widetilde{\tau}_i)$ in $NE(\widetilde{X})$ for $a_i \in \mathbb{R}_{>0}$ such that $V(\widetilde{\tau}_i)$ is extremal and $-(K_{\widetilde{X}} + \widetilde{D}) \cdot V(\widetilde{\tau}_i) \leq n$ for every i by Theorem 0.1 since $\widetilde{X}$ is not a projective space. Since $\sum_i a_i f_* V(\widetilde{\tau}_i) = V(\tau) \in R$, we have that $f_* V(\widetilde{\tau}_i) \in R$ for every i. So, there exists some i such that $0 \neq f_* V(\widetilde{\tau}_i) = bV(\tau)$ in R for $b \geq 1$ since $-(K_X + D) \cdot V(\tau)$ is minimal. Therefore,

$$-(K_X + D) \cdot V(\tau) = -\frac{1}{b}(K_{\widetilde{X}} + \widetilde{D}) \cdot V(\widetilde{\tau}_i) \leq n.$$

Thus we finished the proof. □

Remark 1.13. In Step 1 in the proof of the theorem, we assume that X is non-singular. Then we obtain that $b_i = 1$ and $V(\widetilde{\tau}) \cdot V(e_i) \in \mathbb{Z}$. We note that $V(\widetilde{\tau}) \cdot V(e_i) > 0$ if and only if $\beta + 1 \leq i \leq n+1$. It can be checked easily that P is an $(n - \beta)$-dimensional projective space $\mathbb{P}^{n-\beta}$ and $K_P \cdot V(\widetilde{\tau}) = -(n - \beta + 1)$. Thus, Proposition 4.3, Lemma 4.4, and Propositions 4.5, 4.6 in [Mu] can be checked easily by the above computation (see also [Re, (2.10)(i)]). Therefore, we can recover [Mu, Section 4] without using vanishing theorems.

2. Applications to Fujita's conjecture on toric varieties

In this section, we treat some applications of Theorem 0.1. Corollary 0.2 follows from Theorem 0.1 directly.

Proof of Corollary 0.2. It is obvious by Theorem 0.1, [Od, §2.3 Theorem 2.18] and [Mu, Theorem 3.2]. □

Corollary 2.1. *In Corollary 0.2 (1), we further assume that $K_X + D$ is Cartier. Then $K_X + D + L$ is generated by global sections unless $X \simeq \mathbb{P}^n$, $D = 0$, and $L \simeq \mathcal{O}_{\mathbb{P}^n}(n)$.*

Proof. It is obvious by Corollay 0.2 (1). We note that every nef line bundle is generated by its global sections on a complete toric variety. It is well-known (see, for example, [Mu, Theorem 3.1]). □

By combining Corollary 0.2 with Demazur's theorem ([Od, §2.3 Corollary 2.15]), we obtain the following result. This is the original version of Fujita's conjecture on toric varieties.

Corollary 2.2 (Fujita's conjecture for toric varieties)**.** *Let X be a nonsingular projective toric variety over k and L an ample line bundle on X. Then $K_X+(n+1)L$ is generated by global sections and $K_X+(n+2)L$ is very ample, where $n = \dim X$. Moreover, if $(X, L) \not\simeq (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$, then $K_X + nL$ is generated by global sections and $K_X + (n+1)L$ is very ample.*

Remark 2.3. For very ampleness on singular toric varieties, see [L, 3. Very ampleness].

3. Remarks on Minimal Model Program for toric varieties

In this section, we use the notation in [KM] and [Ut]. The following lemma is well-known to specialists. It may help the readers to understand this section.

Lemma 3.1. *Let X be a complete toric variety over k and D the complement of the big torus in X as a reduced divisor. Then the pair (X, D) is log-canonical. Furthermore, if K_X is $\mathbb{Q}$-Cartier, then the pair $(X, 0)$ is log-terminal.*

Proof. Let $g : Y \to X$ be a toric resolution of singularities. Then we have

$$K_Y + E = g^*(K_X + D),$$

where E is the complement of the big torus in Y as a reduced divisor. Thus, the pair (X, D) is log-canonical by the definition (see [KM, Definition 2.34]). If K_X is $\mathbb{Q}$-Cartier, then D is $\mathbb{Q}$-Cartier since $K_X + D \sim 0$. Note that $\mathrm{Supp} g^*D = \mathrm{Supp} E$ and g^*D is an effective $\mathbb{Q}$-divisor. Therefore, the pair $(X, 0)$ is log-terminal (see [KM, Definition 2.34]). □

The following is a variant of [Ut, 17.10 Theorem] for toric varieties. We recommend the readers who are not familiar with Minimal Model Program to see [KM, §3.7].

Theorem 3.2. *Let X be a complete toric variety over k and $g : Y \to X$ a projective birational toric morphism from a $\mathbb{Q}$-factorial toric variety Y. Let $\mathcal{E}$ be a subset of the exceptional divisors. Then there is a factorization*

$$g : Y \dashrightarrow \widetilde{X} \to X$$

with the following properties:

(1) *$h : Y \dashrightarrow \widetilde{X}$ is a local isomorphism at every generic point of the divisor that is not in $\mathcal{E}$;*
(2) *h contracts every exceptional divisor in $\mathcal{E}$;*
(3) *$\widetilde{X}$ is projective over X and $\mathbb{Q}$-factorial. Of course, the pair $(\widetilde{X}, 0)$ is log-terminal by Lemma 3.1.*

In particular, if $\mathcal{E}$ is the set of all the g-exceptional divisors, then $f : \widetilde{X} \to X$ is small, that is, an isomorphism in codimension one. We call this a small projective toric $\mathbb{Q}$-factorialization.

Proof. Let $g : Y \to X$ be as above and $E = \sum E_i$ the complement of the big torus in Y. We note that

$$K_Y + E = g^*(K_X + D) \sim 0.$$

Apply $(K_Y + \sum_{E_i \notin \mathcal{E}} E_i + \sum_{E_j \in \mathcal{E}} 2E_j)$-log minimal model program over X. We note that divisorial contractions and log-flips always exist by [Re, (0.1)] (see also [KMM, §5-2]). Here, a log-flip means an *elementary transformation* with respect to a $(K_Y + \sum_{E_i \notin \mathcal{E}} E_i + \sum_{E_j \in \mathcal{E}} 2E_j)$-negative extremal ray in the terminology of [Re]. Since the relative Picard number $\rho(Y/X)$ is finite, divisorial contractions can occur finite times. So it is enough to check the termination of log-flips. Assume that there exists an infinite sequence of log-flips:

$$Y_0 \dashrightarrow Y_1 \dashrightarrow \cdots \dashrightarrow Y_m \dashrightarrow \cdots .$$

Let Δ be the fan corresponding to Y_0. Since the log-flips don't change one-dimensional cones of Δ, there are numbers $k < l$ such that $Y_k \simeq Y_l$ over X. This is a contradiction because there is a valuation v such that the discrepancies satisfy

$$a(v, Y_k, \sum_{E_i \notin \mathcal{E}} E_i + \sum_{E_j \in \mathcal{E}} 2E_j) < a(v, Y_l, \sum_{E_i \notin \mathcal{E}} E_i + \sum_{E_j \in \mathcal{E}} 2E_j)$$

(see [KM, Lemma 3.38]), where $\sum_{E_i \notin \mathcal{E}} E_i + \sum_{E_j \in \mathcal{E}} 2E_j$ means the proper transform of it on Y_k or Y_l. Therefore, we obtain $f : \widetilde{X} \to X$ with the above mentioned properties by [KM, Lemma 3.39]. □

Remark 3.3. Since we can take a projective toric desingularization as $g : Y \to X$ in Theorem 3.2, there exists at least one small projective toric $\mathbb{Q}$-factorialization for X.

Remark 3.4 (c.f. [KM, Theorem 6.38])**.** Let X be a complete toric variety and $f_i : X_i \to X$ be small projective toric $\mathbb{Q}$-factorializations for $i = 1, 2$. Then X_1 and X_2 can be obtained from each other by a finite succession of *elementary transformations*[1]. It can be checked by the log Minimal Model Program over X as in the proof of Theorem 3.2 and [KM, Lemma 6.39]. Details are left to the readers.

By Theorem 3.2, we obtain the next lemma, which was already used in the proof of Corollary 0.2.

Lemma 3.5. *Let X be a projective toric variety over k and $D = \sum_j d_j D_j$ be a $\mathbb{Q}$-divisor, where D_j is an irreducible torus invariant divisor and $0 \leq d_j \leq 1$ for every j. Assume that $K_X + D$ is $\mathbb{Q}$-Cartier. Then there exists a projective birational toric morphism $f : \widetilde{X} \to X$ such that $\widetilde{X}$ has only $\mathbb{Q}$-factorial singularities and $K_{\widetilde{X}} + \widetilde{D} = f^*(K_X + D)$, where $\widetilde{D} = \sum_i \widetilde{d}_i \widetilde{D}_i$ is a $\mathbb{Q}$-divisor such that $\widetilde{D}_i$ is an irreducible torus invariant divisor and $0 \leq \widetilde{d}_i \leq 1$ for every i.*

By Sumihiro's equivariant embedding theorem, we can remove the assumption that X is complete.

Corollary 3.6 (Small projective toric $\mathbb{Q}$-factorialization)**.** *Let X be a toric variety over k. Then there exists a small projective toric morphism $f : \widetilde{X} \to X$ such that $\widetilde{X}$ is $\mathbb{Q}$-factorial.*

Proof. We can compactify X by Sumihiro's theorem [Od, §1.4]. So, this corollary follows from Theorem 3.2 and Remark 3.3 easily. □

The existence of a small projective toric $\mathbb{Q}$-factorialization means the following corollary.

Corollary 3.7. *Let Δ be a fan. Then there exists a projective simplicial subdivision $\widetilde{\Delta}$ of Δ, that is, the morphism $X(\widetilde{\Delta}) \to X(\Delta)$ is projective and $X(\widetilde{\Delta})$ is $\mathbb{Q}$-factorial, such that the set of one-dimensional cones of $\widetilde{\Delta}$ coincides with that of Δ.*

Remark 3.8. The above corollary seems to follow from the theory of Gelfand-Kapranov-Zelevinskij decompositions. For details about GKZ-decompositions, see [OP, Section 3], especially, [OP, Corollary 3.8]. We note that [OP] generalized and reformulated results on [Re].

[1]This *elementary transformation* was called *flop* in [OP] (see [OP, p.397 Remark]). However, it might be better to call it *log-canonical flop* from the log Minimal Model Theoretic viewpoint (c.f. Lemma 3.1). See also [Ut, 6.8 Definition].

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