Exact Solutions to the Two-dimensional BF and Yang-Mills Theories in the Light-cone Gauge

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Abstract

It is shown that the BRS-formulated two-dimensional BF theory in the light-cone gauge (coupled with chiral Dirac fields) is solved very easily in the Heisenberg picture. The structure of the exact solution is very similar to that of the BRS-formulated two-dimensional quantum gravity in the conformal gauge. In particular, the BRS Noether charge has anomaly. Based on this fact, a criticism is made on the reasoning of Kato and Ogawa, who derived the critical dimension $D = 26$ of string theory on the basis of the anomaly of the BRS Noether charge. By adding the $\widetilde{B}^2$ term to the BF-theory Lagrangian density, the exact solution to the two-dimensional Yang-Mills theory is also obtained.

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§1. Introduction

During the last decade, we have developed a new method of solving quantum field theory in the Heisenberg picture[1–5] and explicitly constructed exact solutions to various two-dimensional models.[6–12] Our method is different from the conventional covariant perturbation theory or the Feynman path-integral approach in the following respects:

1. In our approach, we first find the (nonlinear) infinite-dimensional Lie algebra of field operators and then construct its representation in terms of Wightman functions. On the other hand, path-integral approach directly gives the solution in terms of Green's functions without making any consideration at the operator level.

2. The Green's function, which is a vacuum expectation value of T*-product (= Lagrangian-formalism version of T-product) of operators, does *not* respect field equations and, therefore, Noether theorem, in contrast to the Wightman function (= vacuum expectation value of simple product of operators).

3. When there is a fundamental field having no canonical conjugate, the notion of T*-product becomes *indefinable* at the operator level. Then the qualitative argument based on Green's functions may become quite misleading, especially concerning conservation laws and anomaly. On the other hand, our approach gives the solution consistent with Noether theorem even in such a case.

4. In our approach, some *nonlinear* field equations may happen to be slightly violated at the representation level because singular products involved in them may not be faithfully represented. This phenomenon is called "field-equation anomaly".[10] The same one does not exist in path-integral approach, but, instead, some unusual loop diagrams arise from the difference between T*-product and T-product, and they cause anomaly.[14]

5. If field-equation anomaly is present for a field equation $\mathcal{F} = 0$, the existence or non-existence of anomaly for a particular symmetry G *can* depend on the expressions for the current of G which are different from each other by an operator multiple of $\mathcal{F}$. In path-integral approach, this phenomenon cannot be seen; what one can do is to convert the anomaly encountered in G into the anomaly of another symmetry.[15]

In our previous paper published already eight years ago,[4] we constructed the exact solution to the BRS-formulated two-dimensional BF theory in the Landau gauge. The analysis is rather complicated and does not encourage further investigation. In the present paper, we consider the same model in a non-covariant gauge. Since in the axial-gauge case, we encounter a trouble in quantizing FP ghosts, we discuss the BRS-formulated two-dimensional BF theory *in the light-cone gauge* (we abbreviate it as LGBF) coupled with D chiral Dirac fields. The analysis can easily be extended to the case of the two-dimensional Yang-Mills theory.

We find that the exact solution to LGBF can be obtained very simply and that its structure is quite similar to the BRS-formulated two-dimensional quantum gravity *in the* *conformal gauge*[12] (we abbreviate it as CGQG). In particular, in both models, all field equations except for the equation involving B-field are linear, and the B-field equation exhibits a field-equation anomaly (for $D \neq 26$ in CGQG and for $D \neq 0$ in LGBF). Although these models are formulated to be BRS-invariant (that is, an anomaly-free BRS generator exists), the BRS *Noether* charge exhibits BRS anomaly because of the mechanism explained above (Item 5).

In 1983, Kato and Ogawa,[16] who found BRS anomaly for $D \neq 26$ in CGQG (with boundary conditions corresponding to an open string), claimed that the critical dimension $D = 26$ of a string could be obtained in the BRS-formulated string theory. Recently, we have criticized their claim by pointing out that the reason why they encountered BRS anomaly is merely due to their adoption of the BRS *Noether* charge as the BRS generator, that is, the model itself is strictly BRS-invariant for any value of D.[13] The result found in the present paper strengthens our criticism: If the Kato-Ogawa reasoning of deriving the critical dimension were applied to LGBF, one would obtain $D = 0$, a result which is meaningless. Thus the adequacy of deriving the critical dimension of string theory on the basis of BRS anomaly is quite questionable.

In Sec. 2, we present the operator algebra of LGBF. In Sec. 3, we construct a complete set of Wightman functions and find the existence of field-equation anomaly for the B-field equation. In Sec. 4, the BRS invariance of the model is confirmed, but it is shown that the BRS Noether charge exhibits anomaly for $D \neq 0$. In Sec. 5, we solve the BRS-formulated two-dimensional Yang-Mills theory. The final section is devoted to discussion.

§2. Operator algebra

We discuss the BRS-formulated two-dimensional BF theory in the light-cone gauge. The fields considered are a non-abelian gauge field ${A_\mu}^a$, where Lie algebra $\mathfrak{g}$ is characterized by structure constants f^{abc}, the conjugate field $\widetilde{B}^a$, the B-field B^a, the FP-ghost C^a, the FP-antighost $\bar{C}^a$, and D chiral Dirac fields ψ_M. With light-cone coordinates $x^\pm = (x^0 \pm x^1)/\sqrt{2}$, the Lagrangian density is given by

$$\begin{aligned} \mathcal{L} \;=\; & \widetilde{B}^a(\partial_- {A_+}^a - \partial_+ {A_-}^a - f^{abc}{A_+}^b {A_-}^c) + B^a {A_-}^a \\ & + i\bar{C}^a(\delta^{ab}\partial_- + f^{acb}{A_-}^c)C^b + i{\psi_M}^\dagger(\partial_- - i{A_-}^a T^a)\psi_M, \end{aligned} \tag{2.1}$$

where T^a denotes the representation matrix of $\mathfrak{g}$, normalized as $\mathrm{tr}(T^a T^b) = \frac{1}{2}\delta^{ab}$.

Field equations derived from (2.1) are

$$ {A_-}^a = 0, \tag{2.2}$$

$$ \partial_- \Phi = 0 \quad \text{for} \quad \Phi = {A_+}^a,\ \widetilde{B}^a,\ C^a,\ \bar{C}^a,\ \psi_M, \tag{2.3}$$

$$ B^a + \partial_+ \widetilde{B}^a + f^{abc}({A_+}^b \widetilde{B}^c - i\bar{C}^b C^c) + {\psi_M}^\dagger T^a \psi_M = 0. \tag{2.4}$$

From (2.4) and (2.3), we obtain $\partial_- B^a = 0$. Thus all fields are functions of x^+ only.

Canonical quantization is carried out by taking ${A_+}^a$, C^a and ψ_M as canonical variables. We then obtain

$$[\widetilde{B}^a(x),\, A_+{}^b(y)] = -i\delta^{ab}\delta(x^+ - y^+), \tag{2.5}$$
$$\{\bar{C}^a(x),\, C^b(y)\} = \delta^{ab}\delta(x^+ - y^+), \tag{2.6}$$
$$\{\psi_M(x),\, \psi_N{}^\dagger(y)\} = \delta_{MN}\delta(x^+ - y^+). \tag{2.7}$$

Because of the x^--independence, (2.5)–(2.7) are already two-dimensional (anti)commutation relations. All other (anti)commutators vanish if the B-field B^a is not involved. Thus without B^a, the model is a free-field theory.

In the operator-level calculation, we can regard (2.4), which is a unique nonlinear equation, as the defining equation of B^a. By using it, we calculate commutation relations involving B^a:

$$[B^a(x),\, A_+{}^b(y)] = i(\delta^{ab}\partial_+ + f^{acb}A_+{}^c(x))\delta(x^+ - y^+), \tag{2.8}$$
$$[B^a(x),\, \Phi^b(y)] = -if^{abc}\Phi^c(x)\delta(x^+ - y^+) \quad \text{for} \quad \Phi^a = \widetilde{B}^a,\, C^a,\, \bar{C}^a,\, B^a, \tag{2.9}$$
$$[B^a(x),\, \psi_M(y)] = T^a\psi_M(x)\delta(x^+ - y^+). \tag{2.10}$$

From (2.8)–(2.10), we see that the local-gauge commutation relations, proposed by Kanno and Nakanishi in 1985,[17] are satisfied without non-gauge terms.

The δ'-term of (2.8) can be eliminated by introducing a field

$$B'^a \equiv B^a + \partial_+\widetilde{B}^a. \tag{2.11}$$

We then find

$$[\Phi^a(x),\, B'^b(y)] = -if^{abc}\Phi^c(y)\delta(x^+ - y^+) \quad \text{for} \quad \Phi^a = A_+{}^a,\, \widetilde{B}^a,\, C^a,\, \bar{C}^a,\, B'^a, \tag{2.12}$$
$$[\psi_M(x),\, B'^b(y)] = -T^b\psi_M(y)\delta(x^+ - y^+). \tag{2.13}$$

It is now easy to calculate all multiple commutators. For example, we have

$$[\,[A_+{}^a(x),\, B'^b(y)],\, \widetilde{B}^c(z)] = f^{abc}\delta(x^+ - y^+)\delta(y^+ - z^+). \tag{2.14}$$

§3. Wightman functions

As stated in the Introduction, we construct the representation of the operator algebra found in Sec. 2 in terms of Wightman functions, so as to be consistent with all vacuum expectation values of multiple commutators and to satisfy the energy-positivity condition.[8]

All 1-point functions are arbitrary in principle, but it is natural to set them equal to zero.

Nonvanishing 2-point and 3-point functions (apart from those obtained by field permutations) are as follows:

$$\langle\, A_+{}^a(x_1)\widetilde{B}^b(x_2)\,\rangle = \frac{1}{2\pi}\delta^{ab}\frac{1}{x_1{}^+ - x_2{}^+ - i0}, \tag{3.1}$$
$$\langle\, C^a(x_1)\bar{C}^b(x_2)\,\rangle = -\frac{i}{2\pi}\delta^{ab}\frac{1}{x_1{}^+ - x_2{}^+ - i0}, \tag{3.2}$$
$$\langle\, \psi_M(x_1)\psi_N{}^\dagger(x_2)\,\rangle = -\frac{i}{2\pi}\delta_{MN}\frac{1}{x_1{}^+ - x_2{}^+ - i0}; \tag{3.3}$$

and

$$\langle A_+^a(x_1)B'^b(x_2)\widetilde{B}^c(x_3)\rangle = -f^{abc}\varphi_3(x_1{}^+,x_2{}^+,x_3{}^+), \tag{3.4}$$

$$\langle C^a(x_1)B'^b(x_2)\bar{C}^c(x_3)\rangle = if^{abc}\varphi_3(x_1{}^+,x_2{}^+,x_3{}^+), \tag{3.5}$$

$$\langle \psi_M(x_1)B'^b(x_2)\psi_N{}^\dagger(x_3)\rangle = \delta_{MN}T^b\varphi_3(x_1{}^+,x_2{}^+,x_3{}^+) \tag{3.6}$$

with

$$\varphi_3(x_1{}^+,x_2{}^+,x_3{}^+) \equiv \frac{1}{(2\pi)^2}\cdot\frac{1}{(x_1{}^+-x_2{}^+-i0)(x_2{}^+-x_3{}^+-i0)}. \tag{3.7}$$

Generally, the nonvanishing truncated n-point functions[a] are

$$\begin{aligned}&\langle A_+{}^a(x_1)B'^{b_2}(x_2)\cdots B'^{b_{n-1}}(x_{n-1})\widetilde{B}^c(x_n)\rangle_{\mathrm{T}}\\ &\quad= i\langle C^a(x_1)B'^{b_2}(x_2)\cdots B'^{b_{n-1}}(x_{n-1})\bar{C}^c(x_n)\rangle_{\mathrm{T}}\\ &\quad= (-1)^n\sum_{P(j_2,\dots,j_{n-1})}^{(n-2)!} f(ab_{j_2}d_{j_2})f(d_{j_2}b_{j_3}d_{j_3})\cdots f(d_{j_{n-2}}b_{j_{n-1}}c)\\ &\qquad\qquad\times\varphi(x_1{}^+,x_{j_2}{}^+,\dots,x_{j_{n-1}}{}^+,x_n{}^+)\end{aligned} \tag{3.8}$$

with $f(abc)\equiv f^{abc}$ and

$$\begin{aligned}&\langle \psi_M(x_1)B'^{b_2}(x_2)\cdots B'^{b_{n-1}}(x_{n-1})\psi_N{}^\dagger(x_n)\rangle_{\mathrm{T}}\\ &\quad= -i^{n-1}\delta_{MN}\sum_{P(j_2,\dots,j_{n-1})}^{(n-2)!} T(b_{j_2})T(b_{j_3})\cdots T(b_{j_{n-1}})\varphi(x_1{}^+,x_{j_2}{}^+,\dots,x_{j_{n-1}}{}^+,x_n{}^+)\end{aligned} \tag{3.9}$$

with $T(b)\equiv T^b$, where $P(j_2,\dots,j_{n-1})$ denotes a permutation of $(j_2,\dots,j_{n-1})$ and

$$\begin{aligned}&\varphi(x_1{}^+,x_{j_2}{}^+,\dots,x_{j_{n-1}}{}^+,x_n{}^+)\\ &\quad\equiv \frac{1}{(2\pi)^{n-1}}\frac{1}{(x_1{}^+-x_{j_2}{}^+-i0)(x_{j_2}{}^+-x_{j_3}{}^+\mp i0)\cdots(x_{j_{n-1}}{}^+-x_n{}^+-i0)}\end{aligned} \tag{3.10}$$

with

$$\begin{aligned}x_j{}^+-x_k{}^+\mp i0 &= x_j{}^+-x_k{}^+-i0 \ \text{ if } \ j<k\\ &= x_j{}^+-x_k{}^++i0 \ \text{ if } \ j>k.\end{aligned} \tag{3.11}$$

We rewrite (2.4) into $\mathcal{F}^a\equiv B'^a-F^a=0$ with

$$F^a\equiv -f^{abc}(A_+{}^b\widetilde{B}^c-i\bar{C}^bC^c)-\psi_M{}^\dagger T^a\psi_M. \tag{3.12}$$

Then, using the generalized normal product rule, we obtain

$$\langle B'^a(x_1)B'^b(x_2)\rangle = 0, \tag{3.13}$$

$$\begin{aligned}\langle B'^a(x_1)F^b(x_2)\rangle &= \langle F^a(x_1)F^b(x_2)\rangle\\ &= -\frac{D}{2(2\pi)^2}\cdot\frac{\delta^{ab}}{(x_1{}^+-x_2{}^+-i0)^2}.\end{aligned} \tag{3.14}$$

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[a] As for 2-point and 3-point functions, truncated and nontruncated are the same because all 1-point functions vanish.

Thus we encounter field-equation anomaly for $D \neq 0$.

The perturbation-theoretical counterpart of this fact is as follows. In spite of (2.2), the Feynman propagator $\langle\, \mathrm{T}^* B^a(x_1) A_-{}^b(x_2)\,\rangle$ does *not* vanish but proportional to $\delta^2(x_1 - x_2)$. Therefore, the B' self-energy diagrams, which are loop diagrams of $A_+\widetilde{B}$, of $\bar{C}C$ and of $\psi_M \psi_M{}^\dagger$, are nonvanishing. The former two cancel, but the last one remains. This implies violation of BRS invariance in perturbation theory.

§4. BRS invariance

The BRS transformation is given by

$$\boldsymbol{\delta} A_\pm{}^a = \partial_\pm C^a + f^{acb} A_\pm{}^c C^b, \tag{4.1}$$

$$\boldsymbol{\delta} \widetilde{B}^a = -f^{abc} C^b \widetilde{B}^c, \tag{4.2}$$

$$\boldsymbol{\delta} C^a = -\frac{1}{2} f^{abc} C^b C^c, \tag{4.3}$$

$$\boldsymbol{\delta} \bar{C}^a = i B^a = i(B'^a - \partial_+ \widetilde{B}^a), \tag{4.4}$$

$$\boldsymbol{\delta} B^a = 0, \quad \boldsymbol{\delta} B'^a = -f^{abc} \partial_+ (C^b \widetilde{B}^c), \tag{4.5}$$

$$\boldsymbol{\delta} \psi_M = i C^a T^a \psi_M, \quad \boldsymbol{\delta} \psi_M{}^\dagger = -i \psi_M{}^\dagger C^a T^a. \tag{4.6}$$

Of course, field equations and (anti)commutation relations are consistent with BRS invariance.

In contrast to perturbation theory, our exact solution is consistent with BRS invariance. Indeed, explicit calculation based on Wightman functions shows

$$\langle\, \boldsymbol{\delta}(A_+{}^a(x_1) \bar{C}^b(x_2))\,\rangle = \partial_+{}^{x_1} \langle\, C^a(x_1) \bar{C}^b(x_2)\,\rangle - i \partial_+{}^{x_2} \langle\, A_+{}^a(x_1) \widetilde{B}^b(x_2)\,\rangle = 0, \tag{4.7}$$

$$\begin{aligned}
&\langle\, \boldsymbol{\delta}(A_+{}^a(x_1) \bar{C}^b(x_2) \widetilde{B}^c(x_3))\,\rangle \\
&\quad = f^{aed} \langle\, A_+{}^e(x_1) \widetilde{B}^c(x_3)\,\rangle \langle\, C^d(x_1) \bar{C}^b(x_2)\,\rangle + i \langle\, A_+{}^a(x_1) B'^b(x_2) \widetilde{B}^c(x_3)\,\rangle \\
&\qquad + f^{cde} \langle\, A_+{}^a(x_1) \widetilde{B}^e(x_3)\,\rangle \langle\, \bar{C}^b(x_2) C^d(x_3)\,\rangle \\
&\quad = 0,
\end{aligned} \tag{4.8}$$

$$\begin{aligned}
&\langle\, \boldsymbol{\delta}(C^a(x_1) \bar{C}^b(x_2) \bar{C}^c(x_3))\,\rangle \\
&\quad = \frac{1}{2} f^{ade} \Big[\langle\, C^d(x_1) \bar{C}^b(x_2)\,\rangle \langle\, C^e(x_1) \bar{C}^c(x_3)\,\rangle - (x_2 \leftrightarrow x_3,\ b \leftrightarrow c) \Big] \\
&\qquad - i \langle\, C^a(x_1) B'^b(x_2) \bar{C}^c(x_3)\,\rangle + i \langle\, C^a(x_1) \bar{C}^b(x_2) B'^c(x_3)\,\rangle \\
&\quad = 0,
\end{aligned} \tag{4.9}$$

$$\begin{aligned}
&\langle\, \boldsymbol{\delta}(\psi_M(x_1) \bar{C}^b(x_2) \psi_N{}^\dagger(x_3))\,\rangle \\
&\quad = i \langle\, C^a(x_1) \bar{C}^b(x_2)\,\rangle T^a \langle\, \psi_M(x_1) \psi_N{}^\dagger(x_3)\,\rangle + i \langle\, \psi_M(x_1) B'^b(x_2) \psi_N{}^\dagger(x_3)\,\rangle \\
&\qquad + i \langle\, \bar{C}^b(x_2) C^a(x_3)\,\rangle \langle\, \psi_M(x_1) \psi_N{}^\dagger(x_3)\,\rangle T^a \\
&\quad = 0.
\end{aligned} \tag{4.10}$$

The BRS Noether current $j_{\mathrm{B}}{}^\mu$ is given by

$$j_{\mathrm{B}}{}^+ = 0, \tag{4.11}$$

$$j_{\mathrm{B}}{}^- = \widetilde{B}^a \partial_+ C^a + f^{acb} \widetilde{B}^a A_+{}^c C^b + \frac{1}{2} i f^{abc} \bar{C}^a C^b C^c - C^a \psi_M{}^\dagger T^a \psi_M. \tag{4.12}$$

We define another BRS current $\widehat{j}_{\rm B}{}^\mu$ by

$$\widehat{j}_{\rm B}{}^+ \equiv j_{\rm B}{}^+, \tag{4.13}$$

$$\begin{aligned}\widehat{j}_{\rm B}{}^- &\equiv j_{\rm B}{}^- + C^a\mathcal{F}^a \\ &= B^aC^a - \frac{1}{2}if^{abc}\bar{C}^aC^bC^c + \partial_+(\widetilde{B}^aC^a),\end{aligned} \tag{4.14}$$

which is, of course, equal to $j_{\rm B}{}^\mu$ at the operator level because $\mathcal{F}^a = 0$. But $j_{\rm B}$ and $\widehat{j}_{\rm B}$ are not identical at the representation level because, as shown by (3.13) and (3.14), $\mathcal{F}^a \equiv B'^a - F^a = 0$ exhibits field-equation anomaly.

The BRS generators $Q_{\rm B}$ and $\widehat{Q}_{\rm B}$ are defined by

$$Q_{\rm B} = \int dx^+ j_{\rm B}{}^-, \tag{4.15}$$

$$\widehat{Q}_{\rm B} = \int dx^+ \widehat{j}_{\rm B}{}^-. \tag{4.16}$$

By using (anti)commutation relations but without using field equations, we can confirm that

$$i[\,\widehat{Q}_{\rm B},\, \Phi\,]_\mp = \boldsymbol{\delta}(\Phi) \tag{4.17}$$

for any fundamental field Φ. Likewise, we have

$$[\mathcal{F}^a(x),\, \Phi(y)] = 0 \quad \text{for} \quad \Phi = A_+{}^b,\, \widetilde{B}^b,\, C^b,\, \bar{C}^b,\, \psi, \tag{4.18}$$

but

$$[\mathcal{F}^a(x),\, B'^b(y)] = -if^{abc}\mathcal{F}^c(x)\delta(x^+ - y^+). \tag{4.19}$$

With the help of (2.6) and (2.12), (4.18) and (4.19) imply that $C^a\mathcal{F}^a$ (anti)commutes with any fundamental field except for $\bar{C}^b$:

$$\{C^a(x)\mathcal{F}^a(x),\, \bar{C}^b(y)\} = \mathcal{F}^b(x)\delta(x^+ - y^+). \tag{4.20}$$

We then find

$$\{Q_{\rm B} - \widehat{Q}_{\rm B},\, \bar{C}^b(y)\} = -\mathcal{F}^b(y). \tag{4.21}$$

From (4.17) and (4.21), we have[b)]

$$\begin{aligned}\langle\, \bar{C}^a(x_1)\widehat{Q}_{\rm B}{}^2\bar{C}^b(x_2)\,\rangle &= \langle\, \{\bar{C}^a(x_1),\, \widehat{Q}_{\rm B}\}\{\widehat{Q}_{\rm B},\, \bar{C}^b(x_2)\}\,\rangle \\ &= \langle\, B^a(x_1)B^b(x_2)\,\rangle = 0,\end{aligned} \tag{4.22}$$

$$\begin{aligned}\langle\, \bar{C}^a(x_1)Q_{\rm B}{}^2\bar{C}^b(x_2)\,\rangle &= \langle\, \{\bar{C}^a(x_1),\, Q_{\rm B}\}\{Q_{\rm B},\, \bar{C}^b(x_2)\}\,\rangle \\ &= \langle\, (\partial_+\widetilde{B}^a(x_1) - F^a(x_1))(\partial_+\widetilde{B}^b(x_2) - F^b(x_2))\,\rangle \\ &= -\frac{D}{2(2\pi)^2}\frac{\delta^{ab}}{(x_1{}^+ - x_2{}^+ - i0)^2}\end{aligned} \tag{4.23}$$

owing to (3.14).

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b) Note that we have not used $\mathcal{F}^a = 0$ in the above operator calculation.

Thus we find that $\widehat{Q}_{\rm B}$ is always nilpotent but the BRS Noether charge $Q_{\rm B}$ is *not* nilpotent for $D \neq 0$. The situation encountered here is quite similar to that of the bosonic string of Kato and Ogawa,[16] who analyzed the BRS-formulated CGQG and found that the BRS Noether charge is not nilpotent for $D \neq 26$. We criticized their claim by showing that the exact solution to CGQG is BRS invariant and that there is another BRS generator which is always nilpotent.[13] The result obtained in the present paper shows that similar phenomenon also occurs even *in a gauge theory.* Thus we should conclude that the derivation of the critical dimension based on BRS anomaly is almost meaningless.

Finally, we note that nothing anomalous happens for the FP-ghost number current in the present model.

§5. Yang-Mills theory

It is straightforward to generalize the results obtained in the previous sections to the case of the Yang-Mills theory. The Lagrangian density of Yang-Mills theory is given by (2.1) plus the $\widetilde{B}^2$ term as follows:

$$\begin{aligned} \mathcal{L} \;=\; & \widetilde{B}^a(\partial_- A_+{}^a - \partial_+ A_-{}^a - f^{abc} A_+{}^b A_-{}^c) - \frac{g^2}{2}\widetilde{B}^a \widetilde{B}^a \\ & + B^a A_-^a + i\bar{C}^a(\delta^{ab}\partial_- + f^{acb} A_-{}^c)C^b + i\psi_M{}^\dagger(\partial_- - iA_-{}^a T^a)\psi_M, \end{aligned} \tag{5.1}$$

where g is a coupling constant. Since the BF theory is trivially recovered as $g \to 0$ in (5.1), both the conventional Yang-Mills theory and the BF theory are described by (5.1). Indeed, as long as $g \neq 0$, we can make the field redefinition

$$\widetilde{B}^a \to \widetilde{B}^a + \frac{1}{g^2}(\partial_- A_+{}^a - \partial_+ A_-{}^a - f^{abc} A_+{}^b A_-{}^c) \tag{5.2}$$

to obtain the conventional Lagrangian density for Yang-Mills theory involving the $F_{-+}{}^2$ term with the extra $\widetilde{B}^2$ term which is completely decoupled with the other fields. One should note that in the light-cone gauge (5.2) is essentially a linear transformation, hence this field redefinition causes only a trivial modification to the theory.

Now, let us solve the model defined by (5.1). Field equations are the same as (2.2)–(2.4) except for (2.3) with $\Phi = A_+{}^a$, which is replaced by

$$\partial_- A_+{}^a - g^2 \widetilde{B}^a = 0. \tag{5.3}$$

From (5.3) and (2.3) with $\Phi = \widetilde{B}^a$, we obtain

$$\partial_-{}^2 A_+ = 0. \tag{5.4}$$

Since the canonical (anti)commutation relations are the same as in the BF theory, two-dimensional (anti)commutation relations (2.5)–(2.7) are preserved in the present theory.

To construct the two-dimensional commutation relation between ${A_+}^a(x)$ and ${A_+}^b(y)$, we set up the following Cauchy problem:

$$\begin{aligned} \partial_-{}^y \left[{A_+}^a(x),\, {A_+}^b(y)\right] &= ig^2\delta^{ab}\delta(x^+ - y^+), &(5.5)\\ \left[{A_+}^a(x),\, {A_+}^b(y)\right]\big|_{y^0=x^0} &= 0, &(5.6)\end{aligned}$$

where (5.5) is obtained from (5.3) with (2.5). Then, the solution to (5.5) with (5.6) is uniquely given by

$$[{A_+}^a(x),\, {A_+}^b(y)] = -ig^2\delta^{ab}(x^- - y^-)\delta(x^+ - y^+). \tag{5.7}$$

In the same way as LGBF, all other (anti)commutators vanish if the B-field B^a is not involved. Furthermore, it is straightforward to show that (2.9) for $\Phi^a = \widetilde{B}^a,\, C^a,\, \bar{C}^a$ and (2.10) remain unchanged. As for (2.8), it is modified as follows:

$$[B^a(x),\, {A_+}^b(y)] = i(\delta^{ab}\partial_+ + f^{acb}{A_+}^c(x))\delta(x^+ - y^+) - ig^2 f^{acb}\widetilde{B}^c(x)(x^- - y^-)\delta(x^+ - y^+). \tag{5.8}$$

It is possible to rewrite (5.8) into

$$[B^a(x),\, {A_+}^b(y)] = i(\delta^{ab}\partial_+ + f^{acb}{A_+}^c(y))\delta(x^+ - y^+) \tag{5.9}$$

which is trivially consistent with the field equation $\partial_- B^a = 0$. Here, we have used the operator identity

$$\Big({A_+}^a(x) - {A_+}^a(y) - g^2\widetilde{B}^a(x)(x^- - y^-)\Big)\delta(x^+ - y^+) = 0, \tag{5.10}$$

which is obtained as a trivial solution to the Cauchy problem $\partial_-{}^x f(x,y) = 0$ with $f(x,y)|_{x^0=y^0} = 0$. Using (5.9), we can show that (2.9) for $\Phi^a = B^a$ is satisfied.

It is, now, straightforward to obtain the operator subalgebra consisting of fields other than $\widetilde{B}$. For that purpose, we have only to substitute $\frac{1}{g^2}\partial_- A_+$ to $\widetilde{B}$ in (2.4), (5.8) and (5.10), and omit two-dimensional commutation relations involving $\widetilde{B}$. Then, we find the resultant operator algebra is exactly the same as that in the ${F_{-+}}^2$ theory.

Next, we consider Wightman functions. In order to make results useful also in the ${F_{-+}}^2$ theory, we avoid to use B'^a defined by (2.11). In addition to (3.1)–(3.6), in which B'^b is replaced by B^b, we obtain the following nonvanishing 2-point and 3-point functions

$$\langle\, {A_+}^a(x_1){A_+}^b(x_2)\,\rangle = -\frac{g^2}{2\pi}\delta^{ab}\frac{{x_1}^- - {x_2}^-}{{x_1}^+ - {x_2}^+ - i0}, \tag{5.11}$$

$$\langle\, {A_+}^a(x_1)B^b(x_2)\,\rangle = -\frac{1}{2\pi}\delta^{ab}\frac{1}{({x_1}^+ - {x_2}^+ - i0)^2}, \tag{5.12}$$

$$\langle\, A_+^a(x_1)B^b(x_2)A_+^c(x_3)\,\rangle = g^2 f^{abc}({x_1}^- - {x_3}^-)\varphi_3({x_1}^+, {x_2}^+, {x_3}^+). \tag{5.13}$$

$$\langle\, A_+^a(x_1)B^b(x_2)B^c(x_3)\,\rangle = \frac{1}{(2\pi)^2}f^{abc}\frac{1}{({x_1}^+ - {x_2}^+ - i0)({x_2}^+ - {x_3}^+ - i0)({x_1}^+ - {x_3}^+ - i0)}. \tag{5.14}$$

Likewise, additional nonvanishing truncated n-point functions ($n \geqq 4$) are given by

$$\langle\, A_+{}^a(x_1) B^{b_2}(x_2) \cdots B^{b_{n-1}}(x_{n-1}) A_+{}^c(x_n)\,\rangle_{\rm T}$$
$$= (-1)^{n-1} g^2\,(x_1{}^- - x_n{}^-) \sum_{P(j_2,\ldots,j_{n-1})}^{(n-2)!} f(ab_{j_2}d_{j_2}) f(d_{j_2}b_{j_3}d_{j_3}) \cdots f(d_{j_{n-2}}b_{j_{n-1}}c)$$
$$\times \varphi(x_1{}^+, x_{j_2}{}^+, \ldots, x_{j_{n-1}}{}^+, x_n{}^+), \tag{5.15}$$

$$\langle\, A_+{}^a(x_1) B^{b_2}(x_2) \cdots B^{b_n}(x_n)\,\rangle_{\rm T}$$
$$= \frac{(-1)^n}{(2\pi)^{n-1}} \cdot \frac{1}{2} \sum_{P(j_2,\ldots,j_n)}^{(n-1)!} f(ab_{j_2}d_{j_2}) f(d_{j_2}b_{j_3}d_{j_3}) \cdots f(d_{j_{n-2}}b_{j_{n-1}}b_{j_n})$$
$$\times \frac{1}{(x_1{}^+ - x_{j_2}{}^+ - i0)(x_{j_2}{}^+ - x_{j_3}{}^+ \mp i0) \cdots (x_{j_{n-1}}{}^+ - x_{j_n}{}^+ \mp i0)(x_{j_n}{}^+ - x_{j_2}{}^+ \mp i0)} \tag{5.16}$$

with (3.11).[c)] Since the above Wightman functions affect neither $\langle\, B^a(x_1) B^b(x_2)\,\rangle$ nor $\langle\, B^a(x_1) \mathcal{F}^b(x_2)\,\rangle$, where $\mathcal{F}^a$ is defined by the lhs of (2.4) with substituting $\widetilde{B} = \frac{1}{g^2}\partial_- A_+$, the field-equation anomaly in the BF theory remains unchanged in the Yang-Mills theory. So does the discussion on the BRS invariance made in Sec. 4.

§6. Discussion

In the present paper, we have constructed the exact solutions to the BRS-formulated two-dimensional BF and Yang-Mills theories in the light-cone gauge. We have found that the solution to LGBF is quite similar to that of the BRS-formulated two-dimensional quantum gravity in the conformal gauge. In particular, it exhibits field-equation anomaly for the B-field equation. Previously, we constructed the exact solution to the BRS-formulated two-dimensional BF theory *in the Landau gauge.*[4)] Although in that work we did not discuss field-equation anomaly, we can check, by using the explicit expressions for the Wightman functions presented there, that field-equation anomaly is *not* encountered even if Dirac fields are taken into account. Furthermore, it was found that there is no field-equation anomaly in two-dimensional nonlinear abelian gauge models,[18)] though they are not BRS-formulated and without matter fields. Thus our result is the first finding of field-equation anomaly *in the gauge theory.*

In the local-gauge commutation relations (2.8)–(2.10), we have set the variables of fields appearing in the rhs to coincide with that of B in the lhs, since any fields in LGBF is independent of x^-. These forms are convenient in calculating truncated n-point functions. To be precise, however, this choice of variables is different from the original expression for the local gauge-commutation relation defined by Kanno and Nakanishi.[17)] In Yang-Mills theory, in which A_+ is dependent on x^-, the local-gauge commutation relation for A_+ is correctly described by (5.9) without non-gauge term, but not by (5.8).

Throughout this paper, we have restricted ourselves to considering only the left-handed chiral Dirac field ψ_M ($M = 1, \ldots, D$) as matter fields for simplicity of description.

[c)] In (5.16), the factor $\frac{1}{2}$ arises because (j_2, j_3, j_4, ... , j_n) and (j_2, j_n, ... , j_4, j_3) give the same contribution to the sum.

In LGBF, it is possible to add the right-handed chiral Dirac field $\chi_{M'}$ $(M' = 1, ..., D')$ coupled with A_+ without any fundamental difficulty. Even in this case, the B-field B is independent on x^-, hence the local-gauge commutation relations are satisfied without non-gauge terms. Therefore, we have $\langle\, B^a(x_1)B^b(x_2)\,\rangle = 0$ and $\langle\, B^a(x_1)\mathcal{F}^b(x_2)\,\rangle \neq 0$ unless $D = 0$ in the same way as in Sec. 3. Truncated n-point function consisting only of $\widetilde{B}$'s no longer vanishes, but is proportional to D'. Since it satisfies $\partial_+{}^{x_2}\langle\, \widetilde{B}^a(x_1)\widetilde{B}^b(x_2)\,\rangle_{\mathrm{T}} = 0$, we have $\langle\, \widetilde{B}^a(x_1)\mathcal{F}^b(x_2)\,\rangle = 0$, that is, no extra field-equation anomaly appears.

We emphasize that the BRS structure of LGBF is almost completely parallel to that of CGQG. The latter is essentially nothing but the bosonic string. Although Kato and Ogawa[16] claimed, by analyzing CGQG with string boundary conditions, that the critical dimension $D = 26$ could be derived from the nilpotency condition of the BRS Noether charge, the solution to CGQG itself is BRS invariant and a BRS generator nilpotent for any value of D can be constructed owing the the existence of field-equation anomaly. In the present paper, we have found also in LGBF that the solution is BRS invariant and a BRS generator nilpotent for any value of D exists while the BRS Noether charge is not nilpotent for $D \neq 0$. We have to conclude, therefore, that the logical basis of deriving the critical dimension from the nilpotency condition for the BRS Noether charge is quite questionable.

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