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A constructive approach to traveling waves in chemotaxis

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Abstract

In this paper we study the existence of one and multidimensional traveling wave solutions for chemotaxis or so-called Keller-Segel models. We present a constructive approach to give modelers a choice of sensitivity, production and decay functionals at hand.

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Contents

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1 Introduction and Motivation

In this paper we study the existence of traveling wave solutions for Keller-Segel models of the form

$$\left.\begin{array}{rcl} u_t &=& \nabla(k(u)\nabla u - u\phi(v)\nabla v) \\ v_t &=& k_c\Delta v + g(u,v) \end{array}\right\} \tag{1}$$

in one and higher space dimensions. This system is known as one of the classical models to describe chemotaxis, the active motion of a population u towards a chemical signal v, a phenomenon which is well known in microbiology. Many different patterns can be observed in chemotactic species. One of them are traveling waves or pulses which spread trough the population. These can occur due to growth of the respective population but also due to the dynamics of motion of populations especially when they are in a non-reproductive stage. Here we are concerned with the later phenomenon.

For simplified versions of the above given system several results on the existence of one-dimensional traveling wave solutions are known (see for instance [2, 3, 4, 6, 11, 13] and [15]). None of these references deals with the case where the evolution of the chemical includes diffusion, production and decay. But there are hints for the existence of such solutions in some of these papers (see for example [14, Figure 3, page 212]). In [11] E. F. Keller discussed several functional forms for which one might expect a traveling wave solution. The dynamics of the chemical signal includes diffusion and different decay terms, but no production. Furthermore the connection between the chemotactic sensitivity and the decay term are discussed. In [2, 3, 4] a phase plane analysis is done for the bifurcation of traveling waves in chemotaxis models without diffusion of the chemical. As far as we know there are no existence results concerning traveling wave solutions for the above given Keller-Segel type model (1) in space dimensions larger than one. In [7] traveling wave patterns in 2-dimensional chemotaxis systems are considered which are mainly caused by population growth.

In the present paper we show the existence of traveling wave solutions for (1) with homogeneous Neumann boundary conditions in case of space dimension one. Our results allow a general class of nonlinearities which include some of the systems studied in previous papers like [6, 13] and [15] as special cases.

Furthermore we will prove for space dimensions higher than one that there exist traveling wave solutions for (1) in the cylindrical domain $\Omega = \mathbb{R} \times \Gamma$, where Γ is a bounded domain in $\mathbb{R}^{N-1}$ $(N \geq 1)$, with no-flux-boundary conditions for the first equation and homogeneous Dirichlet boundary conditions on $\partial\Gamma$.

For homogeneous Neumann boundary data mass is conserved for $u(x,t)$. Thus one expects traveling pulse solutions for $u(t,x)$, whereas the second equation might generate a traveling front for $v(x,t)$.

Before we start our calculations we give a short motivation why multidimensional traveling wave solutions for (1) can be expected.

Consider

$$\left.\begin{array}{rcl} 0 &=& \nabla(\nabla u + u\nabla(1/v)) \\ v_t &=& \Delta v + u - uv \end{array}\right\} \tag{2}$$

with homogeneous Neumann boundary conditions. This is a stationary chemotaxis problem where mass is not conserved. Thus looking for a traveling front solution for u makes sense. First we express u in terms of v.

$$u = \lambda e^{-1/v}. \tag{3}$$

Suppose $\lambda = 1$. Then substituting (3) into the second equation we obtain

$$v_t = \Delta v + e^{-1/v}(1-v) \tag{4}$$

with homogeneous Neumann boundary data. For (4) existence of a traveling front solution was proved by Berestycki and Nirenberg, [1]. Here the result is cited from ([19, Theorem 3.4, p. 22]).

Theorem 1.1 (citation from [19]) *Consider* (4) *on a cylindrical domain* $\Omega = \mathbb{R}\times\Gamma$, *where* Γ *is a bounded domain in* $\mathbb{R}^{N-1}$. *Then there exists a critical speed* $c^* > 0$ *such that solutions on* (4) *exist, satisfying* $v(x,t) = v(x_1 - ct, x_2, ..., x_N)$, $v(-\infty, x_2, ..., x_N) = 0$, $v(\infty, x_2, ..., x_N) = 1$, *and zero Neumann boundary conditions on* $\mathbb{R}\times\partial\Gamma$, *if and only if* $-c \geq c^*$. *For every* $-c \geq c^*$, *there is a solution with* $v_s > 0$ $(s = x_1 - ct)$.

In the present paper we will give a constructive approach to show the existence of traveling pulse, respectively traveling wave solutions for (1). For the derivation and description of this model (1) and results about the time asymptotic behavior of the solutions we refer to [9, 10, 12, 16, 18] and especially the references therein.

2 One dimensional traveling wave and pulse solutions

In this section we consider (1) in $\mathbb{R}\times\{t \geq 0\}$. With the traveling wave ansatz $u(x,t) = u(x-ct)$, $v(x,t) = v(x-ct)$ and no-flux boundary conditions for the first equation we obtain

$$-cu = k(u)u_z - u\Phi(v)_z + \text{const}, \tag{5}$$

where

$$\Phi(v) = \int_0^v \phi(s)\, ds.$$

Because of the conservation of mass we can only expect traveling pulse solutions for u. This results in the condition that

$$u(z) \to 0,\ z \to \pm\infty. \tag{6}$$

Together with Neumann boundary condition this implies that the constant in (5) equals zero. Dividing by u we obtain

$$-c = \frac{k(u)}{u} u_z - \Phi(v)_z. \tag{7}$$

If $\frac{k(u)}{u}$ is integrable with respect to u and $K(u)$ is invertible with $K'(u) = \frac{k(u)}{u}$ then

$$u(z) = K^{-1}\left(\Phi(v(z)) - cz\right). \tag{8}$$

If $k(u) = 1$, respectively $k(u)/u = 1/u$, then $K(u) = \log u$ and $K^{-1}(y) = e^y$. If $\Phi(v(z)) = \log(v(z))$ then

$$v(z) = u(z)e^{cz}. \tag{9}$$

Therefore the traveling wave ansatz allows us to solve the first equation of the Keller-Segel model explicitly.

2.1 A short survey on known results

In [13] E.F. Keller and L.A. Segel studied the following system:

$$\left.\begin{array}{rcl} u_t & = & \mu u_{xx} - (\delta u v^{-1} v_x)_x \\ v_t & = & -ku \end{array}\right\} \tag{10}$$

with homogeneous Neumann boundary data.

The traveling wave ansatz yields

$$u(z) = \mathrm{const} \cdot (v(z))^{\delta/\mu} e^{-cz/\mu}. \tag{11}$$

Substitution into the second equation of (10) and integration results in

$$v(z) = \left(\mathrm{const} \cdot kc^{-2}(\delta - \mu)e^{-cz/\mu} + v_\infty^{1-(cz/\mu)}\right)^{-1/((\delta/\mu)-1)}, \tag{12}$$

where $-\infty < v_\infty := v(\infty) < \infty$. If one assumes

$$v_\infty^{1-(\delta/\mu)} = \mathrm{const} \cdot kc^{-2}(\delta - \mu).$$

Then

$$\frac{v(z)}{v_\infty} = \left(1 + e^{-cz/\mu}\right)^{-1/((\delta/\mu)-1)} \tag{13}$$

and

$$\frac{u(z)}{c^2 v_\infty (\mu k)^{-1}} = \frac{1}{(\delta/\mu) - 1} e^{-cz/\mu} \left(1 + e^{-cz/\mu}\right)^{-\delta/(\delta-\mu)}. \tag{14}$$

Since we are looking for bounded solutions we suppose

$$\delta > \mu, \tag{15}$$

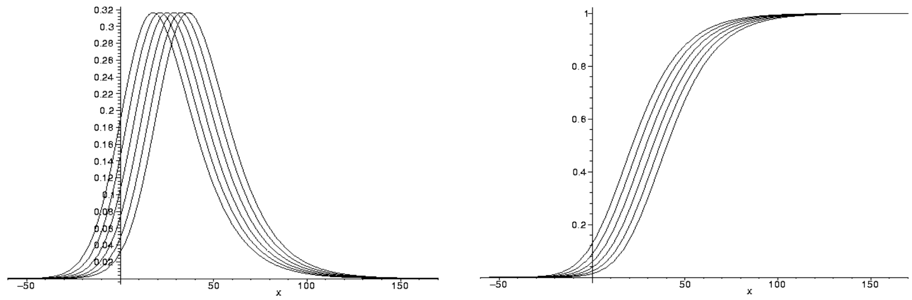

Figure 1: The shape of the functions $u/(c^2 v_\infty (\mu k)^{-1})$ and v/v_∞ given by the formulas (14) & (13) for $\delta = 4,\ \mu = 3,\ c = \frac{3}{16}$ and $t = 0, 20, 40, 60, 80, 100$.

which implies

$$\lim_{cz/\mu \to -\infty} v = 0, \quad \lim_{cz/\mu \to -\infty} u = 0.$$

The functions u and v are plotted in Figure 2.1.

In [15], T. Nagai and T. Ikeda studied the existence and stability of traveling wave solutions of the system

$$\left.\begin{array}{rcl} u_t &=& \mu u_{xx} - (\delta u v^{-1} v_x)_x \\ v_t &=& \varepsilon v_{xx} - ku \end{array}\right\} \tag{16}$$

with homogeneous Neumann boundary data. They considered both cases $\varepsilon = 0$ and $\varepsilon > 0$. It was proved that traveling wave solutions $(u_\varepsilon, v\varepsilon)$ for (16) with $\varepsilon \geq 0$ satisfy

$$||v_\varepsilon - v_0||_{L^\infty(\mathbb{R})} = O(\varepsilon) \text{ and } ||u_\varepsilon - u_0||_{L^\infty(\mathbb{R})} = O(\sqrt{\varepsilon}) \text{ as } \varepsilon \to 0. \tag{17}$$

Here $u_\varepsilon(z)$ is given by $e^{-cz} v_\varepsilon^\delta$. Furthermore it was shown that traveling wave solutions are linearly unstable for perturbations in the sets

$$X := \{(u,v) \mid u, v \in L^1(\mathbb{R}),\ \int_{-\infty}^{\infty} u(z)\ dz = 0\}$$

and

$$X_w := \{(u,v) \in X \mid wu, wv \in L^1(\mathbb{R})\}$$

where the weight function $w(z)$ equals $e^{-\varrho z}$ for $z < 1$ and $e^{\omega z}$ for $z > 1$. Here $\omega \geq c$, which is the wave speed, and $\varrho \geq c/(\delta - 1)$.

To our knowledge no further results on the existence of traveling wave solutions for the full system (1) are known.

2.2 Construction of traveling waves for chemotaxis without reproduction

In this section we develop a constructive approach to find traveling wave solutions for the Keller-Segel model.

First we consider the situation when $k(u, v) = 1$. Using the traveling wave ansatz we conclude from the first equation that

$$u(z) = G(v)e^{-cz} = G(v)a(z).$$

Similar to the approach in [17] we assume

$$\frac{d}{dz}v = F(v)b(z)$$

which implies

$$v_{zz} = F'(v)F(v)b^2(z) + F(v)b'(z).$$

Using this for the second equation of the Keller-Segel model we get after some rearrangements

$$F'(v) = -\frac{c}{k_c b(z)} - \frac{b'(z)}{b^2(z)} - \frac{g(v, G(v)a(z))}{k_c F(v)b^2(z)}. \tag{18}$$

If b solves

$$b'(z) = -\frac{c}{k_c}b(z) \tag{19}$$

which means $b(z) = \tilde{C}e^{-\frac{c}{k_c}z}$ then

$$F(v) = \sqrt{-\frac{2\tilde{C}e^{2\frac{c}{k_c}z}}{k_c}\int^v g(s, G(s)e^{-cz})ds + const.} \tag{20}$$

All equations are supposed to hold for general v. For our constructive ansatz we first make the following assumption on $g(v, G(v)e^{-cz})$

$$g(v, G(v)e^{-cz}) = \sum_{j=-\infty}^{\infty} e^{-cjz}G^j(v) \sum_{i=-\infty}^{\infty} a_{i,j}v^i \tag{21}$$

where

$$G(v) = \sum_{l=-\infty}^{\infty} g_l v^l. \tag{22}$$

Let

$$b(z) = \sum_{j=0}^{\infty} b_j e^{-cjz} \text{ and } F(v) = \sum_{i=-\infty}^{\infty} f_i v^i. \tag{23}$$

Then we have to solve

$$\begin{aligned} & k_c \sum_{i=-\infty}^{\infty} i f_i v^{i-1} \sum_{j=-\infty}^{\infty} f_j v^j \sum_{k=0}^{\infty} b_k e^{-ckz} \sum_{l=0}^{\infty} b_l e^{-clz} \\ = & -c \sum_{j=-\infty}^{\infty} f_j v^j \sum_{k=0}^{\infty} b_k e^{-ckz} + c k_c \sum_{j=-\infty}^{\infty} f_j v^j \sum_{k=0}^{\infty} k b_k e^{-ckz} \\ & - \sum_{i,j=-\infty}^{\infty} a_{ij} v^i G^j e^{-cjz} \end{aligned} \tag{24}$$

Comparison of the coefficients for $v^m e^{-cnz}$ results in

$$k_c \sum_{i=-\infty}^{\infty} i f_i f_{m-i+1} \sum_{l=0}^{\infty} b_l b_{n-l} = -c f_m b_n + c k_c n f_m b_n - A_m \tag{25}$$

where A_m is the overall coefficient of v^m in $\sum_{i=-\infty}^{\infty} a_{in} v^i G^n$.

<u>For $F(v) = f_1 v$</u>, which means $F'(v) = f_1$, we get the following identities:

$$\frac{G^j(v)}{F(v)} \sum_{i=-\infty}^{\infty} a_{ij} v^i = \left(\sum_{l=-\infty}^{\infty} b_l v^l \right)^j \left(\sum_{i=-\infty}^{\infty} a_{i,j} v^{i-1} \right) =: k_j = const., \ \forall j.$$

As a consequence $b(z)$ must solve

$$b'(z) = -f_1 b^2(z) - \frac{cb(z)}{k_c} - \frac{1}{k_c f_1} \sum_{j=-\infty}^{\infty} k_j e^{-cjz}. \tag{26}$$

Thus we get the following system of equations for the coefficients b_j.

$$\begin{aligned} 0 &= -f_1 b_0^2 - \frac{c}{k_c} b_0 - \frac{k_0}{k_c f_1} \\ -b_j j c &= -f_1 \sum_{l=0}^{\infty} b_l b_{j-l} - \frac{c}{k_c} b_j - \frac{k_j}{k_c f_1}. \end{aligned} \tag{27}$$

Now going back to our first assumption on the shape of $\frac{d}{dz} v$ we see that v equals

$$v(z) = \text{const} \cdot e^{f_1 b_0 z - \frac{f_1}{c} \sum_{j=1}^{\infty} \frac{b_j}{j} e^{-cjz}} \tag{28}$$

Since we are interested in traveling wave solutions, the following conditions have to be satisfied:

$$v(-\infty) = 0, \ v(\infty) = \text{const}.$$

Thus

$$b_0 \quad = \quad 0 \text{ in (28) and therefore } k_0 = \sum_{i=-\infty}^{\infty} a_{i,0} v^{i-1} = 0.$$

Consequently $a_{i,0} = 0 \; \forall i$.

In the examples given in the next section we will see how to use the above method to find traveling waves and pulses.

2.2.1 Examples

In this section we give some explicit examples for the existence of traveling wave and pulse solutions by using the above given methods. First we consider the following u-equation

$$u_t = u_{xx} - \chi \left(u \left(\log v \right)_x \right)_x \tag{29}$$

with Neumann boundary conditions and $\chi \in \mathbb{N}$. This is because our constructions favor polynomial functionals ($\chi \log v = \log v^\chi$). Later we discuss also $\chi \in \mathbb{R}$. After applying the traveling wave ansatz $z = x - ct$ we get

$$u = v^\chi e^{-cz}. \tag{30}$$

So in our notation $G(v) = v^\chi$. Now we want to construct suitable equations for v to get a traveling pulse for u and a traveling wave for v. Let us assume $F(v) = f_1 v$. Since $b_0 = 0$ in this case we start with conditions (27):

$$-k_c f_1 b_1 c = -c f_1 b_1 - k_1 \quad \text{or} \quad k_1 = v^\chi \sum_{i=-\infty}^{\infty} a_{i1} v^{i-1} = (k_c - 1) b_1 c f_1. \tag{31}$$

For simplicity we assume that $b(z) = e^{-cz}$, so $b_1 = 1$ and $b_j = 0$ for $j \neq 1$. With this we have

$$k_2 = v^{2\chi} \sum_{i=-\infty}^{\infty} a_{i2} v^{i-1} = -k_c f_1^2. \tag{32}$$

So

$$a_{1-\chi,1} = (k_c - 1) c f_1 \quad \text{and} \quad a_{1-2\chi,2} = -k_c f_1^2. \tag{33}$$

Therefore the second equation has the form

$$v_t = k_c v_{xx} + (k_c - 1) c f_1 v^{1-\chi} u - k_c f_1^2 v^{1-2\chi} u^2. \tag{34}$$

For $\chi = 1$ this results in

$$v_t = k_c v_{xx} + (k_c - 1) c f_1 u - k_c f_1^2 \frac{u^2}{v}. \tag{35}$$

The general system has the traveling wave solutions

$$u(x,t) = e^{-\frac{\chi f_1}{c} e^{-c(x-ct)}} e^{-c(x-ct)}, \;\; v(x,t) = e^{-\frac{f_1}{c} e^{-c(x-ct)}} \tag{36}$$

Furthermore we see from (34) and (36) that the wave speed c is uniquely determined and we have the equality $c = f_1$.

For general coefficients $\tilde{\chi} \in \mathbb{R}$ we can always construct an example by comparing with a model for $\chi \in \mathbb{N}$. Let u, v be traveling pulse and wave solutions for

$$u = v^{\chi} e^{-cz} \quad \text{and} \quad v_t = k_c v_{xx} + g(u,v). \tag{37}$$

Then $\tilde{u} = \tilde{v}^{\tilde{\chi}} e^{-cz}$ and $\tilde{v}$ solve

$$\tilde{v}_t = k_c \tilde{v}_{xx} + g(\tilde{u}\tilde{v}^{\chi - \tilde{\chi}}, \tilde{v}) \tag{38}$$

More general we see,

Corollary 2.1 *Let us assume that there is a traveling wave solution for the system*

$$\left.\begin{array}{rcl} u_t &=& (u_x - u(\log(v))_x)_x \\ v_t &=& k_c v_{xx} + g(u,v) \end{array}\right\} \tag{39}$$

with homogeneous Neumann boundary data. Then there exists also a traveling wave solution for the system

$$\left.\begin{array}{rcl} u_t &=& (u_x - u(\log(\Phi(v))_x)_x \\ v_t &=& k_c v_{xx} + g\left(\frac{vu}{\Phi(v)}, v\right). \end{array}\right\} \tag{40}$$

Remark 2.1 *Whether the solution of the second equation satisfies also the Neumann boundary conditions and is therefore a reasonable solution or not depends on the function $\Phi(v)$.*

The proof of this corollary is obvious.

In a second approach we again consider

$$u_t = \left(u_x - \chi u \left(\log v\right)_x\right)_x \tag{41}$$

but assume that $F'(v)$ depends on v. The traveling wave ansatz gives, as before, $u = v^{\chi} e^{-cz}$ or $G(v) = v^{\chi}$. The simplest assumption for b is that $b(z) = e^{-\frac{c}{k_c} z}$. In any case formula (20) holds. For $\frac{1}{k_c} = n \in \mathbb{N}$ the comparison of the coefficients of the series expansions yields

$$\begin{array}{rcl} 0 &=& -cf_m b_n + cf_m k_c n b_n - a_{m-\chi n, n} \\ \frac{1}{n} b_n^2 \sum\limits_{i=-\infty}^{\infty} i f_i f_{m-i+1} &=& -a_{m-\chi 2n, 2n} \end{array} \tag{42}$$

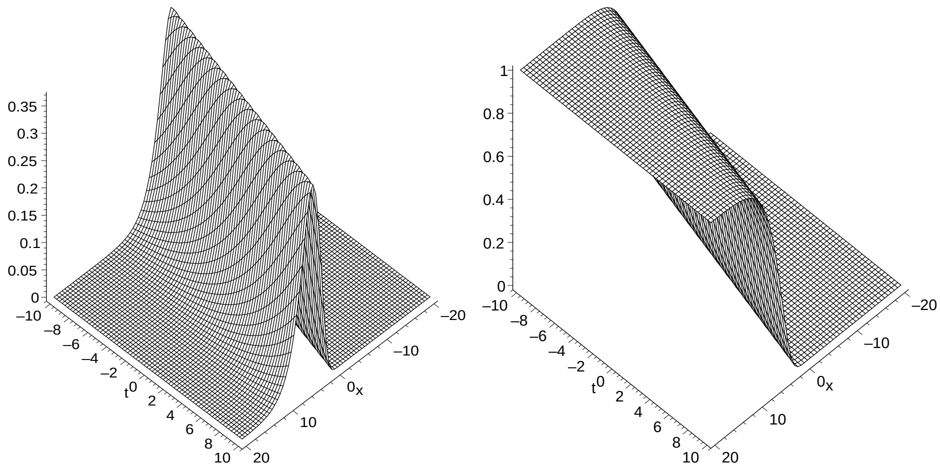

Figure 2: The functions u and v given by the formulas (36) for $f_1 = 0.5$ and $\chi = 1$.

where $b_n = 1$. Assume $k_c = 1$. Then $a_{m-\chi,1} = 0$ for all m. Since $F(v)$ was supposed to be nonlinear we start to look for a suitable functional g for the case $F(v) = f_2 v^2$. So $f_j = 0$ for $j \neq 2$. We have

$$2f_2^2 = a_{3-2\chi,2} \tag{43}$$

This results in the following equation for v

$$v_t = v_{xx} - 2f_2^2 u^2 v^{3-2\chi}, \tag{44}$$

which for $\chi = 2$ is

$$v_t = v_{xx} - 2f_2^2 \frac{u^2}{v}. \tag{45}$$

Here the traveling wave and - pulse solutions are given by

$$u(z) = \frac{e^{-cz}}{(1+e^{-cz})^{\gamma\chi}},\ v(z) = \frac{1}{(1+e^{-cz})^{\gamma}} \tag{46}$$

where $\gamma = 1/(\chi - 1), \chi > 1$ and $c = f_2\sqrt{\frac{2}{\gamma(\gamma+1)}}$. For $\chi = 2$ we obtain $\gamma = 1$ and $c = f_2$. Once the solutions for u and v are known one can add additional birth and death terms into the equation, but which finally disappear, e.g.

$$u_t = \left(u_x - u\left(\log v^2\right)_x\right)_x \tag{47}$$

$$v_t = v_{xx} + f_2^2\left(v^3 - v^2 + uv - 2\frac{u^2}{v}\right). \tag{48}$$

For general $\chi \in \mathbb{R}$ we can again argue with Corollary 2.1.

As a third example we consider the following version of (1) with nonlinear diffusion:

$$\left.\begin{array}{rcl} u_t & = & (u^m u_x - u(\Phi(v))_x)_x \ (m \geq 1) \\ v_t & = & v_{xx} + g(u,v) \end{array}\right\} \tag{49}$$

on $\mathbb{R}$ with homogeneous Neumann boundary data. With the traveling wave ansatz $u(x,t) = u(x-ct),\ v(x,t) = v(x-ct)$, we are led to the identity

$$u(z) = (m(\Phi(v(z)) - cz))^{1/m}. \tag{50}$$

Since we are looking for a traveling pulse solution for the function $u(z)$, we have to assume

$$\Phi(v(z)) - cz \to 0 \text{ as } z \to \pm\infty. \tag{51}$$

For simplicity we consider the case $m = 2$. Though it seems to be extremely unlikely we assume also in this case

$$u(z) = G(v)a(z) = G(v)e^{-\gamma z}. \tag{52}$$

The last equality is a simplified ansatz. Again

$$\frac{d}{dz}v = F(v)e^{-cz} \tag{53}$$

and we consider the situation that

$$F'(v) = -\frac{c}{b(z)} - \frac{b'(z)}{b^2(z)} - \frac{g(v, G(v)e^{-\gamma z})}{F(v)b^2(z)} \tag{54}$$

depends on v. Here k_c was supposed to be equal to 1. Doing the usual expansion and since $b_1 = 1$ and $b_j = 0$ for $j \neq 1$ we end up with

$$\sum_{i=-\infty}^{\infty} i f_i v^{i-1} \sum_{j=-\infty}^{\infty} f_j v^j e^{-2cz} = - \sum_{i,j=-\infty}^{\infty} a_{ij} v^i G^j e^{-\gamma j z} \tag{55}$$

Since we have assumed $m = 2$, a first good guess, after comparing the expression and conditions for u and v is $\gamma = \frac{c}{2}$. Of course this is just a try. With this we have

$$\sum_{i=-\infty}^{\infty} i f_i v^{i-1} \sum_{j=-\infty}^{\infty} f_j v^j = - \sum_{i=-\infty}^{\infty} a_{i4} v^i G^4. \tag{56}$$

Since $F'(v)$ depends on v, the simplest ansatz is $F(v) = f_2 v^2$. Then for e.g. $G(v) = \beta v$ we get

$$2f_2^2 = -\beta^4 a_{-1,4} \tag{57}$$

which gives the following v-equation

$$v_t = v_{xx} - \frac{2f_2^2}{\beta^4}\frac{u^4}{v}. \tag{58}$$

Now the conditions on u have to be checked

$$\sqrt{2(\Phi(v) - cz)} = \beta v e^{-cz/2} \tag{59}$$

We know already that

$$v = \frac{1}{1 + e^{-cz}} \tag{60}$$

due to our choice for $F(v)$. To cancel cz in (59) the functional $\Phi = \Phi_1 + \Phi_2$ has to include a suitable term. For instance $\Phi_1(v) = \log(\frac{1}{v} - 1) = \log(\frac{1-v}{v})$ is a good choice. So we are left with the condition

$$2\Phi_2(v) = \beta^2 v^2 e^{-cz}. \tag{61}$$

This is fulfilled for e.g.

$$\Phi_2(v) = v - v^2 \text{ and } \beta = \sqrt{2}. \tag{62}$$

Again additional birth and death terms which cancel can be included into the equation for v, e.g.

$$v_t = v_{xx} + f_2^2 \left(v^3 - v^2 + \frac{u^2 v}{2} - \frac{u^4}{2v} \right). \tag{63}$$

We see that

$$u(z) = \frac{\sqrt{2} e^{-cz/2}}{(1 + e^{-cz})}, \; v(z) = \frac{1}{1 + e^{-cz}}$$

is a traveling wave solution of (49). Also in this example the unique wave speed is given by $c = f_2$. In all three cases a traveling wave solution with a traveling pulse $u(x - ct)$ exists. Obviously there seem to be a large number of possible systems of type (1), where traveling wave solutions exist, as can be seen from the construction principle. Although the special ansatz for the first two examples allows us to construct also traveling wave solutions for the third example, it is not reasonable to always expect that

$$u(z) = (m(\Phi(v(z)) - cz))^{1/m} = G(v) e^{-\gamma z}. \tag{64}$$

To get rid of this condition we have to use a different approach. This will be described in the next section.

2.2.2 An alternative approach

We consider system (1) where $k(u)/u$ is integrable with respect to u and its integral $K(u)$ is assumed to be invertible, on $\mathbb{R}$, so

$$K^{-1}(\phi(v) - cz) = u. \tag{65}$$

Then we are left to analyze

$$-cv' = k_c v'' + g\left(v, K^{-1}(\phi(v) - cz)\right) \tag{66}$$
$$v(-\infty) = 0, v(\infty) = const. \tag{67}$$

where the condition $K^{-1}(\phi(v) - cz) \to 0$ for $z \to \pm\infty$ has to be fulfilled.

Now we choose $v(z) = w(e^{-cz}) = w(y)$ where c is the traveling wave speed and $w(\infty) = 0, w(0) = const$. Then we obtain

$$c^2 y w' = k_c c^2 y w' + k_c c^2 y^2 w'' + g\left(w, K^{-1}(\phi(w) + \log y)\right). \tag{68}$$

We define

$$g\left(w, K^{-1}(\phi(w) + \log y)\right) = \tilde{g}(w, y) = a_1(w)y + a_2(w)y^2 + a_3(w, y) \tag{69}$$

and consider the following three cases:

(a) $k_c \neq 1$:

In this case we look for a solution w which satisfies:

$$(1 - k_c)c^2 w' = a_1(w) \tag{70}$$
$$k_c c^2 w'' = -a_2(w) \tag{71}$$
$$a_3(w, y) = 0. \tag{72}$$

So

$$-a_2(w) = \frac{k_c}{1 - k_c}\frac{d}{dy}a_1(w) = \frac{k_c}{(1 - k_c)^2 c^2}a_1(w)a_1'(w). \tag{73}$$

Therefore

$$\int_0^y ds = \frac{k_c}{k_c - 1}\int_{w(0)}^{w(y)} \frac{\frac{d}{dr}a_1(r)}{a_2(r)} dr =: A(w). \tag{74}$$

We assume that $A(w)$ is invertible. Then

$$w = A^{-1}(y) \tag{75}$$

with $a_3(w, y) = 0$ solves our problem.

(b) $k_c = 1$:

Here we define

$$g\left(w, K^{-1}(\phi(w) + \log y)\right) = \tilde{g}(w, y) = a_2(w)y^2 + \tilde{a}(w, y). \tag{76}$$

And we assume that

$$c^2 w'' = -a_2(w) \tag{77}$$

solves $\tilde{a}(w,y)=0$. Then

$$c^2\int_0^y w''(s)ds = c^2 w'(y) = \int_{w(0)}^{w(y)} -a_2(q)dq. \tag{78}$$

So in this case

$$y = -\int_{w(0)}^{w(y)} \frac{c^2}{\int_{r(0)}^{r(y)} a_2(q)dq} dr =: A(w) \tag{79}$$

has to be invertible. And $w = A^{-1}(y)$ has to solve $\tilde{a}(w,y)=0$.

(c) $k_c = 0$:

Here we define

$$g\left(w, K^{-1}(\phi(w)+\log y)\right) = \tilde{g}(w,y) = a_1(w)y + \tilde{a}(w,y) \tag{80}$$

And we assume that

$$c^2 w' = a_1(w) \tag{81}$$

solves $\tilde{a}(w,y)=0$. Then

$$c^2\int_0^y w'(s)ds = c^2(w(y)-w(0)) = \int_{w(0)}^{w(y)} a_1(q)dq. \tag{82}$$

This identity gives us the function w, which has to solve $\tilde{a}(w,y)=0$.

The conditions given in this section are more general and also allow models and nonlinearities like in our third example, but where our special choice (64) does not work.

Since we are looking for traveling front solutions for the chemical with a “classical shape” we can look for a function $w(s)$ that satisfies $w'(s) < 0$ on $[0,\infty)$. Now we see in case (a) that $a_1(s)$ has to be a positive function for all $s \in \mathbb{R}^+$ if $k_c > 1$. This shows that in such a case a production term is necessary to guarantee the existence of a traveling wave solution. Furthermore this implies for $k_c \neq 0$ that $a_2(w)$ does not vanish, which is obvious in case (b) if one looks for nontrivial solutions. This however implies that we need a u^2-term in the second equation to construct examples for traveling wave solutions. For further remarks on non vanishing functions $a_2(w)$ we refer to the concluding section.

In case (c) we see that $a_1(s)$ has to be negative for all $s \in \mathbb{R}^+$ to guarantee the existence of a traveling front solution. Thus a decay term is the crucial term in this case. Furthermore we see that it has to be a u-term of order one in contrast to case (a) where we need a u-term of order 2 to get traveling wave solutions.

Remark 2.2 *The system studied by Keller and Segel* [13] *fits into the setting of case (c). In their case we have that $a_1(w) = w$ and $\tilde{a}(w,y) \equiv 0$. However this setting does not include the systems studied by Nagai and Ikeda* [15] *with $\varepsilon \neq 0$.*

3 Traveling pulse solutions with multiple peaks

An interesting question is whether the chemotaxis-system generates traveling wave solutions with multiple peaks. Corollary 2.1 allows to construct examples where multiple peak traveling wave exist by making appropriate choices for the sensitivity functional. Here we only consider two different classes. The first one is given by the system

$$\left.\begin{array}{rcl} u_t & = & (u_x - u(\log(\Phi(v))_x)_x \\ v_t & = & v_{xx} - 6\alpha^2 \frac{v^2 u^2}{\Phi(v)} \end{array}\right\} \tag{83}$$

where the chemotactic sensitivity function has to have suitable properties. We will be more precise below. Using the traveling wave ansatz we get that

$$u(z) = \Phi(v)e^{-cz}.$$

Now the second equation implies that the wave speed $c = \alpha$ and

$$v(z) = \frac{1}{(1 + e^{-\alpha z})^2}.$$

Thus we see that possible multiple peak traveling waves for the function u are generated by the chemotactic sensitivity. Numerical experiments show that multiple peak traveling waves are generated if $\Phi(v)$ is a small perturbation of the traveling front v. To give a concrete example we look at the system

$$\left.\begin{array}{rcl} u_t & = & (u_x - u(\log(4v^2 - 4v^{3/2} + v))_x)_x \\ v_t & = & v_{xx} - 6\alpha^2 \frac{v^2 u^2}{4v^2 - 4v^{3/2} + v} \end{array}\right\} \tag{84}$$

A solution of the problem is given by

$$u(x - \alpha t) := \frac{(1 - e^{-\alpha(x-\alpha t)})^2}{(1 + e^{-\alpha(x-\alpha t)})^4} e^{-\alpha(x-\alpha t)} \text{ and } v(x - \alpha t) := \frac{1}{(1 + e^{-\alpha(x-\alpha t)})^2}. \tag{85}$$

As we can see in Figure 3 we have two traveling peaks which follow the chemoattractant. The same arguments also imply that the system

$$\left.\begin{array}{rcl} u_t & = & (u_x - u(\log(\Phi(v))_x)_x \\ v_t & = & 2v_{xx} + \alpha^2 \left(\frac{vu}{\Phi(v)} - 2\frac{vu^2}{(\Phi(v))^2}\right) \end{array}\right\} \tag{86}$$

has traveling wave solutions, which are

$$u(x - \alpha t) = \Phi(v(x - \alpha t))e^{-\alpha(x-\alpha t)} \text{ and } v(x - \alpha t) = e^{-e^{-\alpha(x-\alpha t)}}.$$

For appropriate $\Phi(v)$ the function $u(x, t)$ might become a traveling pulse solution with multiple peaks. One possible function $\Phi(v)$ that would generate a multiple peak traveling wave is

$$\Phi(v) = v - v^{5/4} + v^4. \tag{87}$$

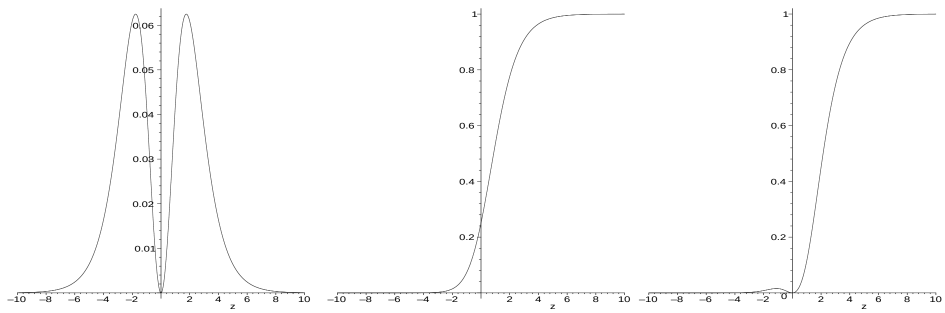

Figure 3: The functions $u(z) = \frac{(1-e^{-\alpha z})^2}{(1+e^{-\alpha z})^4} e^{-\alpha z}$ (left) and $v := \frac{1}{(1+e^{-\alpha z})^2}$ (middle) and the sensitivity function $\Phi(v) = 4v^2 - 4v^{3/2} + v$ (right) for the speed $c = \alpha = 1$.

The second traveling peak in the population is caused by the sensitivity function. This becomes, as mentioned above, after substituting the solution into the functional form, a perturbation of the traveling front of the chemical distribution. More then two traveling peaks are also possible to construct. These systems might be of interest to describe spiral patterns during chemotactic movement of populations. Once again one can add arbitrary many growth and decay terms to this example.

4 Multidimensional traveling waves and pulses in cylinders

Now we look for multidimensional traveling wave solutions for (1). We know from the results in [7] that 2-dimensional traveling wave solutions for the system

$$\left.\begin{array}{rcl} u_t &=& \nabla(\nabla u - u\nabla(\log(\Phi(v)))) + u(u-a)(1-u) \\ v_t &=& \eta\Delta v - \gamma v + \alpha u. \end{array}\right\} \tag{88}$$

exist. As mentioned in the introduction, we are interested in traveling waves that are due to the dynamics in populations which are in a non-reproductive stage and which therefore are not initialized by a growth term of the respective population. Here we only give some examples where multidimensional traveling wave solutions exist. Consider the following system

$$\left.\begin{array}{rcl} u_t &=& \nabla(\nabla u - u\nabla(\log(\Phi(v)))) \\ v_t &=& \eta\Delta v + \gamma v + g(u,v) \end{array}\right\} \tag{89}$$

on $\Omega = \mathbb{R} \times \Gamma$ with no-flux boundary conditions for the first equation and where the function v satisfies homogeneous Dirichlet boundary data on $\partial\Gamma$. Assume

$$\log(\Phi(s \cdot r)) = \Psi_1(s) + \Psi_2(r) \tag{90}$$

and

$$g\left(e^{\Psi_1(s)+\Psi_2(r)-c\xi}, s\cdot r\right) = rg_1(e^{\Psi_1(s)-c\xi}, s) \tag{91}$$

for arbitrary $s,\ r \in \mathbb{R}$. Also suppose that there is a one-dimensional traveling wave solution for the system

$$\left.\begin{array}{rcl} u_t &=& \nabla(\nabla u - u\nabla(\log(\Phi(v)))) \\ v_t &=& \eta\Delta v + g_1(u,v). \end{array}\right\} \tag{92}$$

and a positive solution of the Dirichlet problem

$$\begin{array}{rcl} -\eta\Delta_y v &=& \gamma v \text{ in } \Gamma \subset \mathbb{R}^{N-1} \\ v &=& 0 \text{ on } \partial\Gamma. \end{array}$$

Then there exists a positive traveling wave solution for (89) with $u(x,t) = u(x_1 - ct, x_2, ..., x_N) = u_1(x_1 - ct)u_2(x_2, ..., x_N)$ and $v(x,t) = v(x_1 - ct, x_2, ..., x_N) = v_1(x_1 - ct)v_2(x_2, ..., x_N)$ such that u is a traveling pulse.

It is not difficult to check our previous claim. To simplify the notation we set $u(x_1, ..., x_N, t) = u(x, y, t)$ and $v(x_1, ..., x_N) = v(x, y, t)$. Using the ansatz $u(x,y,t) = u_1(x-ct)u_2(y)$ and $v(x,y,t) = v_1(x-ct)v_2(y)$ and the properties (90) and (91) we see that one is led to the equations

$$(u_1)_t = (u_1)_{xx} - (u_1)_x(\Psi_1(v_1))_x - u_1(\Psi_1(v_1))_{xx} \tag{93}$$

for u_1 and

$$0 = (u_2)_{yy} - (u_2)_y(\Psi(v_2))_y - u_2(\Psi(v_2))_{yy} \tag{94}$$

for u_2. We therefore get

$$u_1(x-ct) = e^{\Psi_1(v_1(x-ct))-c(x-ct)},\ u_2(y) = e^{\psi_2(v_2(y))}. \tag{95}$$

Our assumption on $g(u,v)$ now allows us to separate variables in the second equation of the system and to derive equations for v_1 and v_2. So we get a parabolic equation for v_1

$$(v_1)_t = \eta(v_1)_{xx} + g_1(e^{\Psi_1(v_1)-c(x-ct)}, v_1) \tag{96}$$

and for v_2 an $(N-1)$-dimensional Dirichlet problem:

$$\begin{array}{rcccl} -\eta v_{yy} &=& -\eta\Delta_y v &=& \gamma v \text{ in } \Gamma \\ && v &=& 0 \text{ on } \partial\Gamma, \end{array}$$

These equations are solvable independently from each other and we find the explicit traveling wave solution

$$u = e^{\Psi_1(v_1(\xi))+\Psi_2(v_2(y))-c\xi},\ v = v_2(y)v_1(\xi). \tag{97}$$

An example of a system where our assumptions are fulfilled is

$$\left.\begin{array}{rcl} u_t &=& \nabla(\nabla u - u\nabla(\log(v))) \\ v_t &=& 2\Delta v + 2v + \alpha^2 u - 2\alpha^2\frac{u^2}{v} \end{array}\right\} \tag{98}$$

where the domain and boundary conditions are as described at the beginning of this section. However we specify Γ to be the interval $[0, \pi]$. This system has the solution

$$u(x,y,t) = sin(y)e^{-\alpha(x-\alpha t)}e^{-e^{-\alpha(x-\alpha t)}}, \; v(x,y,t) = sin(y)e^{-e^{-\alpha(x-\alpha t)}}. \quad (99)$$

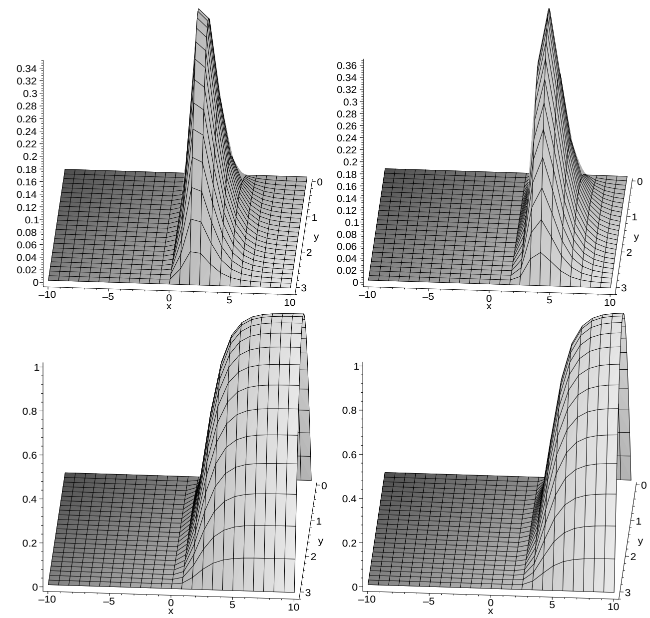

Figure 4: The two dimensional traveling wave solution functions $u(x, y, t)$ (first and second on top) and $v(x, y, t)$ (first and second below) given by the formula (99) for $\alpha = 1$ and $t = 2$ and $t = 4$.

5 Conclusion and Discussion of the results

In this paper we presented two different methods how to can find examples for chemotaxis models which have traveling wave solution which are not caused by growth of the respective population. In our results the traveling waves occur due to the dynamics of motion in a population which is not reproducing but reacting to an attractive chemical. Our approach allows diffusion, production and decay of the chemical. Furthermore we described explicit methods to construct systems with multiple peak traveling waves. These solutions might be interesting to study spiral wave patterns during chemotactic movement of populations. Most often such patterns are modeled by the well-known FitzHugh-Nagumo equations. However, the mechanisms which produce the waves in our case is completely different. In the models presented here the number of peaks is generated by the chemotactic sensitivity functional and the dynamics of the chemoattractant. Thus the underlying dynamics are completely different from those that result from a growth term of the respective population.

The problem of the existence of traveling wave solution can be reduced to a nonlinear boundary value on the half-line $[0, \infty)$. There are several existence results for the possibly singular ODE on the half-line

$$-w'' = \frac{(k_c - 1)}{k_c y} w' + \tilde{g}(w, y) \tag{100}$$

with boundary values $w(\infty) = 0$ and $w(0) = const$. For example one could use the results by Chen and Zhang [5] to formulate conditions on $\tilde{g}(w, y)$ that would guarantee the existence of a solution. These would give properties for the sensitivity function and the production and decay rates of the chemical. However, most results of this type are based on the assumption that the existence of a sub- and a supersolution is known. But these are usually as difficult to find as the exact solution.

As we have seen in our examples and our different approaches, we need at least a power of u in the consumption and production rates that is higher than 2 to include a diffusion term in the second equation of the system. There is an interesting connection between these observations and a comment in [16]. Nanjundiah suggested as reasonable production rates of cAMP during the aggregation of the cellular slime mold *Dictyostelium discoideum*: "*At high cell-densities, one can expect a fall-off in the rate proportional (say) to the number of cell-pairs in a region: then* $g(u, v) = g_0(v)u - g_1(v)u^2$. *Similarly, one can expect a fall-off at high* v." These are exactly the production and decay rates that we have found in examples.

In [15] Nagai and Ikeda studied a model which was first analyzed by Keller and Segel [13] and is covered by our second constructive approach. They proved that the traveling wave solution of their model is unstable. Thus one is led to the question, what happens with unstable traveling wave solutions? The global existence of a solution of the Keller-Segel model on bounded domains Ω is guaranteed in one spatial dimension if $g(u, v)$ is uniformly in $L^2(\Omega)$ and in two dimension if g belongs uniformly to $L^4(\Omega)$. This gives uniformly $H^2(\Omega)$ in one and $W^{2,4}(\Omega)$ in two dimensions, according to parabolic regularity theory (compare for example [8] for more details on the two dimensional case). We do not have a proof for the global existence of solutions in our situation. So one might ask whether a unstable traveling wave solution might lead to a blow-up solution or not.

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