Form factors of $SU(N)$ invariant Thirring model

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Abstract. We obtain a new integral formula for solutions of the rational quantum Knizhnik-Zamolodchikov equation associated with Lie algebra sl_N at level zero. Our formula contains the integral representation of form factors of $SU(N)$ invariant Thirring model constructed by F. Smirnov. We write down recurrence relations arising from the construction of the form factors. We check that the recurrence relations hold for the form factors of the energy momentum tensor.

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1 Introduction

In this paper we study solutions of the quantum Knizhnik-Zamolodchikov (qKZ) equation satisfied by form factors of the $SU(N)$ invariant Thirring model ($SU(N)$ ITM), and give recurrence relations for the solutions to be form factors of $SU(N)$ ITM.

In the study of integrable quantum field theories it is an important problem to determine all local operators in the theory. To study this problem the form factor bootstrap approach is an appropriate method. The form factors of a local operator $\mathcal{O}$ are functions $\{f^{\mathcal{O}}(\beta_1, \cdots, \beta_n)\}_n$ satisfying certain axioms written by certain difference equations and recurrence relations. Thus the problem of determining all local operators is reduced to giving all the solutions to the equations.

In this paper we consider the $SU(N)$ ITM. Form factors of some local operators in this model constructed by Smirnov [S1] are not sufficient in order to determine all local operators. To construct a large family of form factors one of suitable methods is to use the hypergeometric solutions of the qKZ equation [TV1, TV2, NPT].

Now let us recall some results in [NT] for form factors in the $N = 2$ case. In [NT] sufficiently many form factors have been constructed for the $SU(2)$ ITM using the hypergeometric solutions as follows. Form factors $\{f(\beta_1, \cdots, \beta_n)\}$ of $SU(2)$ ITM are functions taking values in the tensor product of the n-copies of the vector representation V of $SU(2)$, and they satisfy the following axioms:

$$\begin{aligned}
&\text{(I) } P_{j,j+1}S_{j,j+1}(\beta_j - \beta_{j+1})f(\cdots, \beta_j, \beta_{j+1}, \cdots) = f(\cdots, \beta_{j+1}, \beta_j, \cdots),\\
&\text{(II) } P_{n-1,n} \cdots P_{1,2} f(\beta_1 - 2\pi i, \beta_2, \cdots, \beta_n) = (-1)^{-\frac{n}{2}} f(\beta_2, \cdots, \beta_n, \beta_1),\\
&\text{(III) } 2\pi i \operatorname{res}_{\beta_n=\beta_{n-1}+\pi i} f(\beta_1, \cdots, \beta_n)\\
&\quad = \left(I - (-1)^{\frac{n}{2}-1} S_{n-1,n-2}(\beta_{n-1} - \beta_{n-2}) \cdots S_{n-1,1}(\beta_{n-1} - \beta_1)\right) \left(f(\beta_1, \cdots, \beta_{n-2}) \otimes \mathbf{e}_0\right),
\end{aligned}$$

where $S(\beta)$ is the S-matrix of the model, $P_{i,j}$ is the permutation of the i-th and j-th components and $\mathbf{e}_0$ is the suitably normalized sl_2 singlet vector in $V^{\otimes 2}$. A family of solutions to (I), (II) and (III) is constructed in the following way. First we note that (I) and (II) imply the following system of difference equations:

$$\begin{aligned}
f(\beta_1, \cdots, \beta_j - 2\pi i, \cdots, \beta_n) = (-1)^{-\frac{n}{2}} & S_{j,j-1}(\beta_j - \beta_{j-1} - 2\pi i) \cdots S_{j,1}(\beta_j - \beta_1 - 2\pi i)\\
& \times S_{j,n}(\beta_j - \beta_n) \cdots S_{j,j+1}(\beta_j - \beta_{j+1}) f(\beta_1, \cdots, \beta_j, \cdots, \beta_n).
\end{aligned}$$

This is nothing but the qKZ equation associated with sl_2 at level zero. In [NPT], the integral formulae for solutions of the qKZ equation were given. These solutions take values in the space of singular vectors of sl_2. Moreover these solutions span the subspace of singular vectors over the field of appropriate periodic functions [T]. Any solution is obtained by applying sl_2 successively to these solutions. Thus we have the complete description of the solutions of the sl_2 qKZ equation at level zero. Next let us consider the axiom (III). In the hypergeometric description a solution of (I) and (II) is specified by a certain function P_n called deformed cycle [S2] (or "p-function" in [BK]). Then the axiom (III) is derived from a recurrence relation for P_n and P_{n-2}. Each local operator corresponds to a sequence of deformed cycles $\{P_n\}_n$ satisfying this recurrence relation. A large family of solutions to the recurrence relation has been constructed extending Smirnov's construction of the chargeless (or weight zero) local operators for the sine-Gordon model [S3]. This construction was extended to charged local operators and a new abelian symmetry was found [NT].

In this paper we consider form factors in the $SU(N)$ ITM. The form factors of a local operator $\mathcal{O}$ are functions $\{f^{\mathcal{O},(l_1,\cdots,l_n)}(\beta_1,\cdots,\beta_n)\}$ such that

$$f^{\mathcal{O},(l_1,\cdots,l_n)}(\beta_1,\cdots,\beta_n) \in V^{(l_1)} \otimes \cdots \otimes V^{(l_n)},$$

where $V^{(l)}$ is the l-th fundamental representation of $SU(N)$. Now let us recall the axioms satisfied by form factors of $SU(N)$ ITM [S1]:

$$\begin{aligned}
&(\mathrm{I})\, P_{j,j+1} S_{j,j+1}^{(l_j,l_{j+1})}(\beta_j - \beta_{j+1}) f^{(\cdots,l_j,l_{j+1},\cdots)}(\cdots,\beta_j,\beta_{j+1},\cdots) = f^{(\cdots,l_{j+1},l_j,\cdots)}(\cdots,\beta_{j+1},\beta_j,\cdots),\\
&(\mathrm{II})\, P_{n-1,n}\cdots P_{1,2} f^{(l_1,l_2,\cdots,l_n)}(\beta_1 - 2\pi i,\beta_2,\cdots,\beta_n) = e^{-\frac{N-1}{N}\pi i \sum_{j=1}^{n} l_j} f^{(l_2,\cdots,l_n,l_1)}(\beta_2,\cdots,\beta_{n-1},\beta_1),\\
&(\mathrm{III})\, 2\pi i\, \mathrm{res}_{\beta_n=\beta_{n-1}+\pi i} f^{(l_1,\cdots,l_n)}(\beta_1,\cdots,\beta_n)\\
&\quad = \delta_{l_{n-1}+l_n,N}\left(I + e^{-\frac{2\pi i}{N}+\frac{N-1}{N}\pi i \sum_{j=1}^{n-2} l_j} S_{n-1,n-2}^{(l_{n-1},l_{n-2})}(\beta_{n-1}-\beta_{n-2})\cdots S_{n-1,1}^{(l_{n-1},l_1)}(\beta_{n-1}-\beta_1)\right)\\
&\qquad\quad \times f^{(l_1,\cdots,l_{n-2})}(\beta_1,\cdots,\beta_{n-2}) \otimes \mathbf{e}_0,
\end{aligned}$$

where $S^{(l,l')}(\beta)$ is the S-matirx acting on $V^{(l)} \otimes V^{(l')}$ and $\mathbf{e}_0$ is the suitably normalized singlet vector in $V^{(l_{n-1})} \otimes V^{(l_n)}$. Moreover, they satisfy a number of formulae for residues corresponding to bound states. The most fundamental one is the following. If $l_{n-1}+l_n < N$ the residue of $f^{(l_1,\cdots,l_{n-1},l_n)}(\beta_1,\cdots,\beta_n)$ at $\beta_n = \beta_{n-1} + \frac{l_{n-1}+l_n}{N}\pi i$ is given by

$$(\mathrm{IV})\, 2\pi i\, \mathrm{res} f^{(l_1,\cdots,l_{n-1},l_n)}(\beta_1,\cdots,\beta_n) = a_{l_{n-1},l_n} f^{(l_1,\cdots,l_{n-2},l_{n-1}+l_n)}(\beta_1,\cdots,\beta_{n-2},\beta_{n-1}+\frac{l_n \pi i}{N}),$$

where $a_{l,l'}$ is a certain constant. In [S1] the form factors of some local operators are constructed.

We study the problem to give the solutions of (I)-(IV) in a similar approach to the case of $N = 2$. Again the first step is to solve the qKZ equation derived from (I) and (II):

$$\begin{aligned}
f^{(l_1,\cdots,l_n)}(\beta_1,\cdots,\beta_j - 2\pi i,\cdots,\beta_n) &= e^{-\frac{N-1}{N}\pi i \sum_j l_j} S_{j,j-1}(\beta_j - \beta_{j-1} - 2\pi i)\cdots S_{j,1}(\beta_j - \beta_1 - 2\pi i)\\
&\times S_{j,n}(\beta_j - \beta_n)\cdots S_{j,j+1}(\beta_j - \beta_{j+1}) f^{(l_1,\cdots,l_n)}(\beta_1,\cdots,\beta_j,\cdots,\beta_n).
\end{aligned}$$

However, it is difficult to construct solutions of the qKZ equation above for general $l_1,\cdots,l_n$. Some representations of solutions were constructed in [TV3, BKZ] in terms of Jackson integrals, that is, formal infinite sums. It seems difficult to prove the convergence of these sums. In this paper we give a new integral formula for solutions of the qKZ equation taking values in the product of the vector representations, that is , $l_1 = \cdots = l_n = 1$, and consider the form factors of chargeless local operators of type

$$f^{(1,\cdots,1,k)}(\beta_1,\cdots,\beta_{n-k},\beta_{n-k+1}) \in (V^{(1)})^{\otimes(n-k)} \otimes V^{(k)}.$$

The conditions (I)-(IV) are closed conditions among these functions. In fact we suppose that $f^{(1,\cdots,1)}(\beta_1,\cdots,\beta_n)$ is a form factor. Then, by taking the residue as in the axiom (IV) successively, we obtain form factors $f^{(1,\cdots,1,k)}(\beta_1,\cdots,\beta_{n-k},\beta_{n-k+1})$, $(k = 2,\cdots,N-1)$. At last, we obtain a form factor $f^{(1,\cdots,1)}(\beta_1,\cdots,\beta_{n-N})$ from $f^{(1,\cdots,1,N-1)}$ by the axiom (III). In this way the form factor on $(V^{(1)})^{\otimes(n-N)}$ is given by one on $(V^{(1)})^{\otimes n}$. Moreover we consider form factors of chargeless local operators. Then form factors are of weight zero, and hence the number of the components of the tensor product that $f^{(1,\cdots,1)}(\beta_1,\cdots,\beta_n)$ takes values in is a multiple of N, say $n=mN$. Then the axioms imply some relation between $f_m:=f^{(1,\cdots,1)}(\beta_1,\cdots,\beta_n)$ and $f_{m-1}:=f^{(1,\cdots,1)}(\beta_1,\cdots,\beta_{n-N})$. We write down this relation by using our integral formula for solutions of the qKZ equation taking values in the product of the vector representations. As a result we get recurrence relations as in the case of $N=2$.

First we start from a certain integral formula for solutions on the tensor product of the vector representations, that is, $(V^{(1)})^{\otimes n}$. This integral formula is obtained as the limit $q\to 1$ of the hypergeometric solutions of the trigonometric qKZ equation associated with the quantum affine algebra $U_q(sl_N)$ at $|q|=1$, which was constructed in [MTT]. In [MTT] it is proved that, if the parameters in the qKZ equation are generic, the set of the solutions become a basis of the weight subspace that the solutions take values in. Nevertheless, in the case of $N>2$, it is not easy to calculate the residues of solutions in the axioms (III) and (IV) from this integral formula. One reason for this is that this integral formula contains much more integrations than the integral representation of form factors of $SU(N)$ ITM constructed by Smirnov. In order to avoid this difficulty we simplify the integral formula in the following way. The integral formula for solutions of the sl_N qKZ equation at level zero contains as the integrand solutions of the sl_{N-1} qKZ equation at level one. Substituting the sl_{N-1} part with a special solution, we get a simplified integral formula for solutions of the sl_N qKZ equation at level zero. The method above of rewriting the integral formula was used by A. Nakayashiki in the case of the differential KZ equation [N1].

The special solution mentioned above of the sl_{N-1} qKZ equation is obtained as the limit $q\to 1$ of a solution of the trigonometric qKZ equation associated with $U_q(\widehat{sl}_{N-1})$ at $|q|<1$. The highest-to-highest matrix element of the product of intertwining operators

$$\langle\Lambda_i|\Phi(z_1)\cdots\Phi(z_\ell)|\Lambda_i\rangle=\sum_{\epsilon_1,\cdots,\epsilon_\ell}\langle\Lambda_i|\Phi_{\epsilon_1}(z_1)\cdots\Phi_{\epsilon_\ell}(z_\ell)|\Lambda_i\rangle v_{\epsilon_1}\otimes\cdots\otimes v_{\epsilon_\ell}$$

satisfies the qKZ equation [FR]. In the case that representations are at level one, we can calculate this matrix element by using the bosonization of intertwining operators [K]. The coefficients in this matrix element are given by some integral formulae. However, the coefficients in the rhs above are determined from functional relations arising from the commutation relation of intertwining operators [DO] and the coefficient of the extremal component calculated explicitly in [N2]. We choose the limit $q\to 1$ of this solution as the special solution of the rational sl_{N-1} qKZ equation at level one.

Note that we can get a certain integral formula for solutions of the qKZ equation by generalizing suitably the integral representation of form factors constructed by Smirnov. We show that this integral formula is obtained from our simplified integral formula in the following way. The simplified integral formula still contains one more integration than Smirnov's formula. However, we can carry out one-time integration of the simplified integral formula in a similar way to the case of sl_2 [NPT]. After this integration, Smirnov's formula is obtained.

Now let us return to the construction of form factors $\{f_m\}$. The solutions of the qKZ equation given by the simplified integral formula are parameterized by functions called deformed cycles as in the case of $N=2$. Fix m and let P_m be the deformed cycle associated with f_m. By calculating the dimension of the space of deformed cycles, we can find that the space spanned by these solutions is quite smaller than the weight subspace of weight zero (See

[N3] for a similar argument in the case of the differential KZ equation). Hence, even if f_m is given in terms of the simplified integral formula, the form factor f_{m-1}, which is obtained by calculating residues of f_m successively, may not be represented by the simplified integral formula. However, under some conditions for the deformed cycle P_m, the form factor f_{m-1} is also given in terms of the simplified integral formula. Then we obtain recurrence relations for P_m and P_{m-1} (see Proposition 7.2), where P_{m-1} is the deformed cycle associated with f_{m-1}. We check that the recurrence relations hold for the form factors of the energy momentum tensor presented by Smirnov [S1]. It is still an open problem to construct solutions of the recurrence relations different from the deformed cycles associated with the form factors constructed by Smirnov.

The plan of this paper is as follows. In Section 2 we give the qKZ equation studied in this paper. The integral formula obtained by taking the limit $q \to 1$ of the hypergeometric solutions of the $U_q(sl_N)$ qKZ equation at $|q| = 1$ is given in Section 3. In Section 4 we construct a special solution of the sl_{N-1} qKZ equation at level one. By using this special solution we rewrite the integral formula obtained in Section 3 and get the simplified integral formula for the sl_N qKZ equation in Section 5. In Section 6 we see that the formula in Section 5 contains Smirnov's formula. We study form factors of $SU(N)$ ITM in Section 7 by using the simplified integral formula and write down recurrence relations for deformed cycles. We check that the deformed cycle associated with the energy momentum tensor satisfies the recurrence relations. In Section 8 we give some supplements about the special solution in Section 4 and proofs of lemmas and propositions in the previous sections.

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2 The qKZ equation

Let $V_N := \oplus_{j=0}^{N-1} \mathbb{C}\, v_j$ be the vector representation of sl_N with the highest weight vector v_0. We denote by $R(\beta)$ the rational R-matrix given by

$$R(\beta) := \frac{\beta + \hbar P}{\beta + \hbar} \in \mathrm{End}((V_N)^{\otimes 2}). \tag{2.1}$$

Here $\hbar$ is a nonzero complex number and P is the permutation operator: $P(u \otimes v) := v \otimes u$.

Fix a nonzero complex number p. We consider the (rational) qKZ equation:

$$\psi(\beta_1, \cdots, \beta_j + p, \cdots, \beta_n) = K_j(\beta_1, \cdots, \beta_n)\psi(\beta_1, \cdots, \beta_j, \cdots, \beta_n), \quad (j = 1, \cdots, n), \tag{2.2}$$

where

$$\begin{aligned} K_j(\beta_1, \cdots, \beta_n) \ &:= \ R_{j,j-1}(\beta_j - \beta_{j-1} + p) \cdots R_{j,1}(\beta_j - \beta_1 + p) \\ &\quad \times R_{j,n}(\beta_j - \beta_n) \cdots R_{j,j+1}(\beta_j - \beta_{j+1}). \end{aligned} \tag{2.3}$$

Here ψ is a $(V_N)^{\otimes n}$-valued unknown function and $R_{i,j}(\beta)$ is the operator acting on the tensor product of i-th and j-th components as $R(\beta)$. The number $-N + p/\hbar$ is called the *level* of this qKZ equation.

Let e_k be the generator of sl_N associated with the simple root α_k. The action of e_k on V_N is given by $e_k v_j = \delta_{k,j} v_{j-1}$.

In the following, we consider the qKZ equation at level zero, that is, the case of

$$p = N\hbar \tag{2.4}$$

and solutions of (2.2) satisfying the highest weight condition:

$$E_k \psi(\beta_1, \cdots, \beta_n) = 0, \quad \text{where} \quad E_k := \sum_{j=1}^{n} 1 \otimes \cdots \otimes \overset{j-\text{th}}{e_k} \otimes \cdots \otimes 1, \quad (k = 1, \cdots, N-1). \tag{2.5}$$

Hereafter we assume that $\mathrm{Im}\hbar < 0$.

3 General solution at level zero

Let us write down an integral formula for solutions of (2.2). We can obtain this formula by taking the limit $q \to 1$ of solutions to the qKZ equation associated with $U_q(sl_N)$ at $|q| = 1$ [MTT].

First we introduce some notations. For non-negative integers $\nu_1, \cdots, \nu_{N-1}$ satisfying

$$\nu_0 := n \geqslant \nu_1 \geqslant \cdots \geqslant \nu_{N-1} \geqslant \nu_N := 0, \tag{3.1}$$

we denote by $\mathcal{Z}_{\nu_1,\cdots,\nu_{N-1}}$ the set of all n-tuples $J = (J_1, \cdots, J_n) \in (\mathbb{Z}_{\geqslant 0})^n$ such that

$$\#\{r; J_r \geqslant j\} = \nu_j. \tag{3.2}$$

For $J = (J_1, \cdots, J_n) \in \mathcal{Z}_{\nu_1,\cdots,\nu_{N-1}}$, we set

$$v_J := v_{J_1} \otimes \cdots \otimes v_{J_n} \in (V_N)^{\otimes n}. \tag{3.3}$$

We set

$$\mathcal{N}_j^J := \{r; J_r \geqslant j\} \tag{3.4}$$

and define integers $r_{j,m}^J, (0 \leqslant j \leqslant N-1, 1 \leqslant m \leqslant \nu_j)$ by

$$\mathcal{N}_j^J = \{r_{j,1}^J, \cdots, r_{j,\nu_j}^J\}, \quad r_{j,1}^J < \cdots < r_{j,\nu_j}^J. \tag{3.5}$$

Note that $r_{0,m}^J = m$. For example, for $J = (1,2,0,1,0,2) \in \mathcal{Z}_{4,2}$, we have $\mathcal{N}_1^J = \{1,2,4,6\}$ and $\mathcal{N}_2^J = \{2,6\}$.

For $J \in \mathcal{Z}_{\nu_1,\cdots,\nu_{N-1}}$, we define sets $M_k^J, (k = 1, \cdots N-1)$ as follows. The set M_k^J satisfies

$$M_k^J \subset \{1, 2, \cdots, \nu_{k-1}\}, \quad \#M_k^J = \nu_k. \tag{3.6}$$

The elements of $M_k^J := \{m_{k,1}^J, \cdots, m_{k,\nu_k}^J\}$, $(m_{k,1}^J < \cdots < m_{k,\nu_k}^J)$ are defined by the following rule:

$$r_{k,j}^J = r_{k-1,m_{k,j}^J}^J. \tag{3.7}$$

For example, for $J = (1,2,0,1,0,2) \in \mathcal{Z}_{4,2}$, we have $M_1^J = \{1,2,4,6\}$ and $M_2^J = \{2,4\}$.

Let us introduce some functions. For a subset $K = \{k_1, \cdots, k_l\} \subset \{1, \cdots, m\}, (k_1 < \cdots < k_l)$, we define the rational function g_K and the trigonometric function P_K by

$$g_K(t_1, \cdots, t_l | z_1, \cdots, z_m) := \prod_{a=1}^{l} \left(\frac{1}{t_a - z_{k_a}} \prod_{j=1}^{k_a - 1} \frac{t_a - z_j - \hbar}{t_a - z_j} \right) \prod_{1 \leqslant a < b \leqslant l} (t_a - t_b - \hbar), \tag{3.8}$$

$$P_K(e^{\frac{2\pi i}{p} t_1}, \cdots, e^{\frac{2\pi i}{p} t_l} | e^{\frac{2\pi i}{p} z_1}, \cdots, e^{\frac{2\pi i}{p} z_m}) := \prod_{a=1}^{l} \left(\prod_{j=1}^{k_a - 1} (1 - e^{\frac{2\pi i}{p}(t_a - z_j - \hbar)}) \prod_{j=k_a+1}^{m} (1 - e^{\frac{2\pi i}{p}(t_a - z_j)}) \right). \tag{3.9}$$

Introduce a set of variables $\{\gamma_{j,m}\}$, $(1 \leqslant j \leqslant N-1,\ 1 \leqslant m \leqslant \nu_j)$. For $J \in \mathcal{Z}_{\nu_1, \cdots, \nu_{N-1}}$, we set

$$w_J^{(N)}(\{\gamma_{j,m}\} | \beta_1, \cdots, \beta_n) := \mathrm{Skew}_{N-1} \circ \cdots \circ \mathrm{Skew}_1 \left(\prod_{k=1}^{N-1} g_{M_k^J}(\{\gamma_{k,m}\} | \{\gamma_{k-1,m'}\}) \right), \tag{3.10}$$

where $\gamma_{0,m} := \beta_m$ and the operator Skew_k is the skew-symmetrization with respect to the variables $\gamma_{k,m}$, $(1 \leqslant m \leqslant \nu_k)$:

$$\mathrm{Skew}_k X(\gamma_{k,1}, \cdots, \gamma_{k,\nu_k}) := \sum_{\sigma \in S_{\nu_k}} (\mathrm{sgn}\sigma) X(\gamma_{k,\sigma(1)}, \cdots, \gamma_{k,\sigma(\nu_k)}). \tag{3.11}$$

Next we set

$$P_J(\{e^{\frac{2\pi i}{p}\gamma_{j,m}}\} | e^{\frac{2\pi i}{p}\beta_1}, \cdots, e^{\frac{2\pi i}{p}\beta_n}) := \prod_{k=1}^{N-1} P_{M_k^J}(\{e^{\frac{2\pi i}{p}\gamma_{k,m}}\} | \{e^{\frac{2\pi i}{p}\gamma_{k-1,m'}}\}), \tag{3.12}$$

and define the space $\mathcal{P}_{\nu_1, \cdots, \nu_{N-1}}$ by

$$\mathcal{P}_{\nu_1, \cdots, \nu_{N-1}} := \sum_{J \in \mathcal{Z}_{\nu_1, \cdots, \nu_{N-1}}} \mathbb{C}\, P_J. \tag{3.13}$$

For $J \in \mathcal{Z}_{\nu_1, \cdots, \nu_{N-1}}$ and $P \in \mathcal{P}_{\nu_1, \cdots, \nu_{N-1}}$, we define a function $I_J[P] = I_J[P](\beta_1, \cdots, \beta_n)$ by

$$\begin{aligned} I_J[P] \ :=& \left(\prod_{j=1}^{N-1} \prod_{m=1}^{\nu_j} \int_{C_j} d\gamma_{j,m} \right) \prod_{k=1}^{N-1} \Big(\phi(\{\gamma_{k,m}\} | \{\gamma_{k-1,m'}\}) \, \varphi(\{\gamma_{k,m}\}) \Big) \\ & \times w_J^{(N)}(\{\gamma_{j,m}\} | \{\beta_m\}) \prod_{j=1}^{N-1} \frac{\prod_{1 \leqslant a < b \leqslant \nu_j} \mathrm{sh}\frac{\pi i}{p}(\gamma_{j,a} - \gamma_{j,b} - \hbar)}{\prod_{m=1}^{\nu_j} \prod_{m'=1}^{\nu_{j-1}} (1 - e^{\frac{2\pi i}{p}(\gamma_{j,m} - \gamma_{j-1,m'})})} P(\{e^{\frac{2\pi i}{p}\gamma_{j,m}}\}), \end{aligned} \tag{3.14}$$

where

$$\phi(t_1,\cdots,t_l|z_1,\cdots,z_m) := \prod_{a=1}^{l}\prod_{j=1}^{m}\frac{\Gamma\left(\frac{t_a-z_j-\hbar}{p}\right)}{\Gamma\left(\frac{t_a-z_j}{p}\right)}, \tag{3.15}$$

$$\varphi(t_1,\cdots,t_l) := \prod_{1\leqslant a<b\leqslant l}\Gamma\left(\tfrac{t_a-t_b+\hbar}{p}\right)\Gamma\left(\tfrac{t_b-t_a+\hbar}{p}\right). \tag{3.16}$$

The contour C_j for $\gamma_{j,m}$, $(1\leqslant m\leqslant \nu_j)$ is a deformation of the real axis $(-\infty,\infty)$ such that the poles at

$$\gamma_{j-1,m'}+\hbar-p\mathbb{Z}_{\geqslant 0},\ (1\leqslant m'\leqslant \nu_{j-1}),\quad \gamma_{j,a}-\hbar-p\mathbb{Z}_{\geqslant 0},\ (a\neq m) \tag{3.17}$$

are above C_j and the poles at

$$\gamma_{j-1,m'}+p\mathbb{Z}_{\geqslant 0},\ (1\leqslant m'\leqslant \nu_{j-1}),\quad \gamma_{j,a}+\hbar+p\mathbb{Z}_{\geqslant 0},\ (a\neq m) \tag{3.18}$$

are below C_j. These conditions are not compatible if all the poles really exist. However, we can define $I_J[P_{J'}]$ for each $P_{J'}\in\mathcal{P}_{\nu_1,\cdots,\nu_{N-1}}$ because $P_{J'}$ has zeroes at some points of (3.17) and (3.18), and we can deform the real axis such that the conditions above are satisfied for the actual poles of the integrand of (3.14). Then we define $I_J[P]$ for $P\in\mathcal{P}_{\nu_1,\cdots,\nu_{N-1}}=\sum_{J'\in\mathcal{Z}_{\nu_1,\cdots,\nu_{N-1}}}\mathbb{C}\,P_{J'}$ as a linear combination of $I_J[P_{J'}]$ (See [MTT] for details).

Set

$$\psi_P(\beta_1,\cdots,\beta_n) := \sum_{J\in\mathcal{Z}_{\nu_1,\cdots,\nu_{N-1}}} I_J[P](\beta_1,\cdots,\beta_n)v_J \tag{3.19}$$

Theorem 3.1 *If $\nu_1,\cdots,\nu_{N-1}$ satisfy*

$$\nu_{j-1}+\nu_{j+1}\geqslant 2\nu_j,\quad \text{for all } j=1,\cdots,N-1, \tag{3.20}$$

then the integral (3.14) converges and ψ_P is a solution of the qKZ equation (2.2) satisfying the highest weight condition (2.5).

Remark. In the case of $N=2$, (3.19) is nothing but the integral formula for solutions of the sl_2 qKZ equation at level zero constructed in [NPT].

Proof. The convergence of the integral (3.14) under the condition (3.20) can be proved in a similar way to the proof of Proposition 2 in [MT].

Set

$$R(\beta)v_{\epsilon_1}\otimes v_{\epsilon_2} = \sum_{\epsilon_1',\epsilon_2'} R(\beta)_{\epsilon_1',\epsilon_2'}^{\epsilon_1,\epsilon_2} v_{\epsilon_1'}\otimes v_{\epsilon_2'}. \tag{3.21}$$

For $J\in\mathcal{Z}_{\nu_1,\cdots,\nu_{N-1}}$, we abbreviate $w^{(N)}_{J_1,\cdots,J_n}(\{\gamma_{j,m}\}|\beta_1,\cdots,\beta_n)$ to $w_{J_1,\cdots,J_n}(\beta_1,\cdots,\beta_n)$, and we write down dependence on $\beta_1,\cdots,\beta_n$ of the integrand $w^{(N)}_J$ and P in $I_J[P]$ as follows:

$$I_J[P] = I(w_{J_1,\cdots,J_n}(\beta_1,\cdots,\beta_n), P(\beta_1,\cdots,\beta_n)). \tag{3.22}$$

Then we can show the following formulae in the same way as the proof of Lemma 1 and Lemma 3 in [MT]:

$$\begin{aligned} & w_{J_1,\cdots,J_{k+1},J_k,\cdots,J_n}(\beta_1,\cdots,\beta_{k+1},\beta_k,\cdots,\beta_n) \\ & = \sum_{J'_k,J'_{k+1}} R(\beta_k-\beta_{k+1})_{J_k,J_{k+1}}^{J'_k,J'_{k+1}} w_{J_1,\cdots,J'_k,J'_{k+1},\cdots,J_n}(\beta_1,\cdots,\beta_k,\beta_{k+1},\cdots,\beta_n), \qquad (3.23) \\ & I(w_{J_n,J_1,\cdots,J_{n-1}}(\beta_n,\beta_1,\cdots,\beta_{n-1}),P(\beta_1,\cdots,\beta_n))|_{\beta_n\to\beta_n+p} \\ & = I(w_{J_1,\cdots,J_n}(\beta_1,\cdots,\beta_{n-1},\beta_n),P(\beta_1,\cdots,\beta_n)). \qquad (3.24) \end{aligned}$$

It is easy to see that ψ_P is a solution of the qKZ equation from (3.23) and (3.24).

Let us prove that ψ_P satisfies the highest weight condition. Note that

$$E_k\psi_P = \sum_{J'\in\mathcal{Z}_{\nu_1,\cdots,\nu_k-1,\cdots,\nu_{N-1}}} \left(\sum_{\substack{j=1 \\ J'_j=k-1}}^{n} I_{J'_1,\cdots,J'_j+1,\cdots,J'_n}[P] \right) v_{J'}. \qquad (3.25)$$

Hence it suffices to prove that

$$\sum_{\substack{j=1 \\ J'_j=k-1}}^{n} I_{J'_1,\cdots,J'_j+1,\cdots,J'_n}[P] = 0 \qquad (3.26)$$

for $J'\in\mathcal{Z}_{\nu_1,\cdots,\nu_k-1,\cdots,\nu_{N-1}}$.

First we prove (3.26) in the case of $N=3$. The proof for the highest weight condition with $k=2$, that is, $E_2\psi_P=0$, is similar to the proof for the case of sl_2 in [NPT]. Let us prove the case of $N=3$ and $k=1$.

In this proof we set $\gamma_{1,a} =: \alpha_a$ and $\gamma_{2,m} =: \gamma_m$ and

$$\Phi(\{\gamma_m\}|\{\alpha_a\}|\{\beta_j\}) := \prod_{k=1}^{2} \left(\phi(\{\gamma_{k,m}\}|\{\gamma_{k-1,m}\})\varphi(\{\gamma_{k,m}\}) \prod_{1\leqslant m<m'\leqslant\nu_k} \mathrm{sh}\frac{\pi i}{p}(\gamma_{k,m}-\gamma_{k,m'}-\hbar) \right) \qquad (3.27)$$

for simplicity's sake. Then the following equality holds.

Lemma 3.2

$$\begin{aligned} \hbar \sum_{\substack{j=1 \\ J'_j=0}}^{n} w^{(3)}_{J'_1,\cdots,J'_j+1,\cdots,J'_n} = & \ \mathrm{Skew}\Big(g_{M_1^{J'}}(\{\alpha_a\}_{2\leqslant a\leqslant\nu_1}|\{\beta_j\}) g_{M_2^{J'}}(\{\gamma_m\}|\{\alpha_a\}_{2\leqslant a\leqslant\nu_1}) \qquad (3.28) \\ & \times \left(\prod_{a=2}^{\nu_1}(\alpha_1-\alpha_a-\hbar) \prod_{m=1}^{\nu_2} \frac{\gamma_m-\alpha_1-\hbar}{\gamma_m-\alpha_1} - \prod_{j=1}^{n} \frac{\alpha_1-\beta_j-\hbar}{\alpha_1-\beta_j} \prod_{a=2}^{\nu_1}(\alpha_1-\alpha_a+\hbar) \right) \Big). \end{aligned}$$

Here $J'=(J'_1,\cdots,J'_n)\in\mathcal{Z}_{\nu_1-1,\nu_2}$ and Skew *is the skew-symmetrizations with respect to $\gamma_1,\cdots,\gamma_{\nu_1}$ and $\alpha_1,\cdots,\alpha_{\nu_2}$.*

This lemma is proved in Section 8.2.

From Lemma 3.2, we have

$$\hbar \sum_{\substack{j=1 \\ J'_j=0}}^{n} I_{J'_1,\cdots,J'_j+1,\cdots,J'_n}[P] = \left(\prod_{a=1}^{\nu_1}\int_{C_1} d\alpha_a \prod_{m=1}^{\nu_2}\int_{C_2} d\gamma_m\right) \Phi(\{\gamma_m\}|\{\alpha_a\}|\{\beta_j\}) \tag{3.29}$$

$$\times\,(\text{the rhs of }(3.28))\frac{P(\{e^{\frac{2\pi i}{p}\gamma_m}, e^{\frac{2\pi i}{p}\alpha_a}\})}{\prod_{a=1}^{\nu_1}\left(\prod_{m=1}^{\nu_2}(1-e^{\frac{2\pi i}{p}(\gamma_m-\alpha_a)})\prod_{j=1}^{n}(1-e^{\frac{2\pi i}{p}(\alpha_a-\beta_j)})\right)}.$$

Note that

$$\frac{\Phi|_{\alpha_1\to\alpha_1+p}}{\Phi} = \prod_{j=1}^{n}\frac{\alpha_1-\beta_j-\hbar}{\alpha_1-\beta_j}\prod_{m=1}^{\nu_2}\frac{\gamma_m-\alpha_1-p}{\gamma_m-\alpha_1-\hbar-p}\prod_{a=2}^{\nu_1}\frac{\alpha_1-\alpha_a+\hbar}{\alpha_1-\alpha_a-\hbar+p}. \tag{3.30}$$

Hence the integration in (3.29) with respect to α_1 is given by

$$\left(\int_{C_1} - \int_{C_1+p}\right)d\alpha_1 \Phi(\{\gamma_m\}|\alpha_1|\{\beta_j\})\prod_{a=2}^{\nu_1}(\alpha_1-\alpha_a-\hbar)\prod_{m=1}^{\nu_2}\frac{\gamma_m-\alpha_1-\hbar}{\gamma_m-\alpha_1}$$

$$\times\frac{P(\{e^{\frac{2\pi i}{p}\gamma_{j,m}}\})}{\prod_{m=1}^{\nu_2}(1-e^{\frac{2\pi i}{p}(\gamma_m-\alpha_1)})\prod_{j=1}^{n}(1-e^{\frac{2\pi i}{p}(\alpha_1-\beta_j)})}. \tag{3.31}$$

It is easy to see that the contour C_1 can be deformed to C_1+p without crossing the poles of the integrand in (3.31). Hence (3.31) equals zero, and this completes the proof for the case of $N=3$ and $k=1$.

The proof for the cases $N>3$ is similar. The case of $k=N-1$ can be proved in a similar manner to the proof for the sl_2 case in [NPT], and the other case can be proved from Lemma 3.2 and the calculation (3.31) for $\beta_j=\gamma_{k-1,j}, \alpha_a=\gamma_{k,a}$ and $\gamma_m=\gamma_{k+1,m}$. $\square$

Now let us see that the formula (3.14) contains as the integrand the integral representation of solutions to the sl_{N-1} qKZ equation at level one. Set $\alpha_a := \gamma_{1,a}$ and $\ell := \nu_1$. Let us consider $I_J[P]$, where $P\in\mathcal{P}_{\nu_1,\cdots,\nu_{N-1}}$ is in the following form:

$$P(\{e^{\frac{2\pi i}{p}\alpha_a}\},\{e^{\frac{2\pi i}{p}\gamma_{k,m}}\}_{k\geqslant 2}) = P_1(\{e^{\frac{2\pi i}{p}\alpha_a}\}|\{e^{\frac{2\pi i}{p}\beta_j}\})\bar{P}(\{e^{\frac{2\pi i}{p}\gamma_{k,m}}\}_{k\geqslant 2}|\{e^{\frac{2\pi i}{p}\alpha_a}\}). \tag{3.32}$$

We write down the skew-symmetrization in $w_J^{(N)}$ with respect to α_a's. Then we get

$$I_J[P] = \left(\prod_{a=1}^{\ell}\int_{C_1} d\alpha_a\right)\phi(\{\alpha_a\}|\{\beta_j\})\varphi(\{\alpha_a\}) \tag{3.33}$$

$$\times\sum_{\sigma\in S_\ell}(\mathrm{sgn}\sigma)g_{M_1^J}(\{\alpha_{\sigma(a)}\}|\{\beta_j\})I_{\bar{J}}^{(\sigma)}[\bar{P}](\{\alpha_a\})\frac{\prod_{1\leqslant a<b\leqslant\ell}\mathrm{sh}\frac{\pi i}{p}(\alpha_a-\alpha_b-\hbar)}{\prod_{a=1}^{\ell}\prod_{j=1}^{n}(1-e^{\frac{2\pi i}{p}(\alpha_a-\beta_j)})}P_1(\{e^{\frac{2\pi i}{p}\alpha_a}\}|\{e^{\frac{2\pi i}{p}\beta_j}\}).$$

The notation in the above formula is as follows. For $J=(J_1,\cdots,J_n)\in\mathcal{Z}_{\ell,\nu_2,\cdots,\nu_{N-1}}$, we define $\bar{J}:=(\bar{J}_1,\cdots,\bar{J}_\ell)\in\mathcal{Z}_{\nu_2,\cdots,\nu_{N-1}}$ by

$$\bar{J}_a := J_{r^J_{1,a}} - 1. \tag{3.34}$$

In other words, $\bar{J}$ is obtained by picking up non-zero components of $J = (J_1, \cdots, J_n)$ and adding (-1) to each component. For example, for $J = (1,2,0,1,0,2) \in \mathcal{Z}_{4,2}$, we have $\bar{J} = (0,1,0,1) \in \mathcal{Z}_2$. For $\bar{J} = (\bar{J}_1, \cdots, \bar{J}_\ell) \in \mathcal{Z}_{\nu_2,\cdots,\nu_{N-1}}$, we define sets $M_k^{\bar{J}}, (k = 1, \cdots, N-2)$ in the same way as (3.7) and the function $w_{\bar{J}}^{(N-1)}$ from $M_k^{\bar{J}}$'s. Then we set

$$I_{\bar{J}}^{(\sigma)}[\bar{P}](\alpha_1, \cdots, \alpha_\ell) := \left(\prod_{k=2}^{N-1} \prod_{m=1}^{\nu_k} \int_{C_k} d\gamma_{k,m}\right) \prod_{k=2}^{N-1} \Big(\phi(\{\gamma_{k,m}\}|\{\gamma_{k-1,m'}\})\, \varphi(\{\gamma_{k,m}\})\Big) \tag{3.35}$$

$$\times\, w_{\bar{J}}^{(N-1)}(\{\gamma_{k,m}\}_{k\geqslant 2}|\{\alpha_{\sigma(a)}\}) \prod_{k=2}^{N-1} \frac{\prod_{1\leqslant a<b\leqslant \nu_k} \mathrm{sh}\frac{\pi i}{p}(\gamma_{k,a} - \gamma_{k,b} - \hbar)}{\prod_{m=1}^{\nu_k} \prod_{m'=1}^{\nu_{k-1}} (1 - e^{\frac{2\pi i}{p}(\gamma_{k,m} - \gamma_{k-1,m'})})} \bar{P}(\{e^{\frac{2\pi i}{p}\gamma_{k,m}}\}_{k\geqslant 2}|\{e^{\frac{2\pi i}{p}\alpha_a}\}).$$

Set

$$v_{\bar{J}} := v_{\bar{J}_1} \otimes \cdots \otimes v_{\bar{J}_\ell} \in (V_{N-1})^{\otimes \ell}. \tag{3.36}$$

and

$$\bar{\psi}_{\bar{P}}(\alpha_1, \cdots, \alpha_\ell) := \sum_{\bar{J} \in \mathcal{Z}_{\nu_2,\cdots,\nu_{N-1}}} I_{\bar{J}}^{(\mathrm{id})}[\bar{P}](\alpha_1, \cdots, \alpha_\ell) v_{\bar{J}}. \tag{3.37}$$

Recall that $p = N\hbar$. Then we see that $\bar{\psi}_{\bar{P}}$ is a solution of the qKZ equation associated with sl_{N-1} at level $-(N-1) + p/\hbar = 1$ satisfying the highest weight condition. In the next section we construct a special solution to this sl_{n-1} qKZ eqaution at level one without any integration.

4 Special solution at level one

In the following we fix a positive integer m and assume that

$$\nu_j = (N-j)m, \ (0 \leqslant j \leqslant N), \tag{4.1}$$

that is, we consider singlet solutions in $(V_N)^{\otimes Nm}$ at level zero.

Let us construct a special solution of the qKZ equation associated with sl_{N-1} at level one. Note that $\ell = \nu_1 = (N-1)m$.

Lemma 4.1 *There exists a set of rational functions $\{H_{\epsilon_1,\cdots,\epsilon_\ell}(\alpha_1, \cdots, \alpha_\ell)\}_{(\epsilon_1,\cdots,\epsilon_\ell) \in \mathcal{Z}_{(N-2)m,\cdots 2m,m}}$ uniquely determined by the following conditions:*

$$\begin{aligned} H_{\cdots,\epsilon_{p+1},\epsilon_p,\cdots}(\cdots, \alpha_{p+1}, \alpha_p, \cdots) &= \frac{\alpha_p - \alpha_{p+1}}{\alpha_p - \alpha_{p+1} + \hbar} H_{\cdots,\epsilon_p,\epsilon_{p+1},\cdots}(\cdots, \alpha_p, \alpha_{p+1}, \cdots) \\ &\quad + \frac{\hbar}{\alpha_p - \alpha_{p+1} + \hbar} H_{\cdots,\epsilon_{p+1},\epsilon_p,\cdots}(\cdots, \alpha_p, \alpha_{p+1}, \cdots), \end{aligned} \tag{4.2}$$

$$H_{\epsilon_1,\cdots,\epsilon_\ell}(\alpha_1, \cdots, \alpha_{\ell-1}, \alpha_\ell - N\hbar) = \prod_{a=1}^{\ell-1} \frac{\alpha_a - \alpha_\ell + \hbar}{\alpha_a - \alpha_\ell + (N-1)\hbar} H_{\epsilon_\ell,\epsilon_1,\cdots,\epsilon_{\ell-1}}(\alpha_\ell, \alpha_1, \cdots, \alpha_{\ell-1}). \tag{4.3}$$

Moreover,

$$H_{\underbrace{0\cdots0}_{m}\underbrace{1\cdots1}_{m}\cdots\underbrace{(N-2)\cdots(N-2)}_{m}}(\alpha_1, \cdots, \alpha_\ell) = \prod_{\substack{a,b \\ (\epsilon_a < \epsilon_b)}} \frac{1}{\alpha_a - \alpha_b - \hbar}. \tag{4.4}$$

Remark. From (4.2) and (4.4), it is easy to see that

$$\prod_{1\leqslant a<b\leqslant \ell}(\alpha_a-\alpha_b-\hbar)\, H_{\epsilon_1,\cdots,\epsilon_\ell}(\alpha_1,\cdots,\alpha_\ell) \tag{4.5}$$

is a polynomial in $\alpha_1,\cdots,\alpha_\ell$ of order less than or equal to $m-1$ in each variable.

In the following we construct $\{H_{\epsilon_1,\cdots,\epsilon_\ell}\}$ from a solution of the qKZ equation associated with $U_q(\widehat{sl}_{N-1})$ with $0<q<1$ at level one.

Let Λ_i, $(0\leqslant i\leqslant N-2)$ be the fundamental weights of $\widehat{sl}_{N-1}$ and $V(\Lambda_i)$ the level one irreducible highest weight $U_q(\widehat{sl}_{N-1})$ module with the highest weight Λ_i and the highest weight vector $|\Lambda_i\rangle$. Then there exist the type I intertwining operators $\tilde{\Phi}^{(i)}(z)$ [DO]:

$$\tilde{\Phi}^{(i)}(z): V(\Lambda_{i+1})\longrightarrow V(\Lambda_i)\otimes V_z, \qquad \tilde{\Phi}^{(i)}(z)|\Lambda_{i+1}\rangle = |\Lambda_i\rangle\otimes v_i+\cdots, \tag{4.6}$$

where V_z is the homogeneous evaluation module associated with the vector representation V_{N-1}.

Set

$$\Phi^{(i)}(z) := z^{\Delta_i-\Delta_{i+1}}\tilde{\Phi}^{(i)}(z), \qquad \Delta_i := \frac{(\Lambda_i+2\rho|\Lambda_i)}{2N}, \tag{4.7}$$

and

$$G(z_1,\cdots,z_\ell) := \langle\Lambda_i|\Phi(z_1)\cdots\Phi(z_\ell)|\Lambda_i\rangle \in V_{z_1}\otimes\cdots\otimes V_{z_\ell}. \tag{4.8}$$

Then G satisfies the (trigonometric) qKZ equation at level one [FR]:

$$\begin{aligned} G(z_1,\cdots,q^{2N}z_j,\cdots,z_\ell) &= R^q_{j,j-1}(q^{2N}z_j/z_{j-1})\cdots R^q_{j,1}(q^{2N}z_j/z_1)\cdot(q^{-2\Lambda_i-2\rho})_j \\ &\times \left(R^q_{\ell,j}(z_\ell/z_j)\right)^{-1}\cdots\left(R^q_{j+1,j}(z_{j+1}/z_j)\right)^{-1} G(z_1,\cdots,z_j,\cdots,z_\ell). \end{aligned} \tag{4.9}$$

Here $R^q(z)$ is the trigonometric R-matrix given by

$$R^q(z) = q^{\frac{1}{N-1}-1}\frac{(q^2z;q^{2(N-1)})_\infty(q^{2N-4}z;q^{2(N-1)})_\infty}{(z;q^{2(N-1)})_\infty(q^{2N-2}z;q^{2(N-1)})_\infty}\bar{R}^q(z), \quad \bar{R}^q(z)(v_0\otimes v_0)=v_0\otimes v_0. \tag{4.10}$$

Now we set

$$H^q(z_1,\cdots,z_\ell) := \prod_{a=1}^{\ell} z^{(1-\frac{1}{N-1})a}\prod_{1\leqslant a<b\leqslant\ell}\frac{(q^{2(N-1)}z_b/z_a;q^{2(N-1)})_\infty}{(q^2z_b/z_a;q^{2(N-1)})_\infty}G(z_1,\cdots,z_\ell). \tag{4.11}$$

From the commutation relation

$$\begin{aligned} &\Phi(z_2)\Phi(z_1) \\ &= \left(\frac{z_1}{z_2}\right)^{\frac{1}{N-1}-1}\frac{(q^{2(N-1)}z_2/z_1;q^{2(N-1)})_\infty(q^2z_1/z_2;q^{2(N-1)})_\infty}{(q^{2(N-1)}z_1/z_2;q^{2(N-1)})_\infty(q^2z_2/z_1;q^{2(N-1)})_\infty}\bar{R}^q(z_1/z_2)\Phi(z_1)\Phi(z_2), \end{aligned} \tag{4.12}$$

we find

$$P_{p,p+1}\bar{R}^q(z_p/z_{p+1})H^q(\cdots,z_p,z_{p+1},\cdots)=H^q(\cdots,z_{p+1},z_p,\cdots). \tag{4.13}$$

Expand H^q as

$$H^q(z_1,\cdots,z_\ell)=\sum_{\epsilon_1,\cdots,\epsilon_\ell}H^q_{\epsilon_1,\cdots,\epsilon_\ell}(z_1,\cdots,z_\ell)v_{\epsilon_1}\otimes\cdots\otimes v_{\epsilon_\ell}. \tag{4.14}$$

From (4.9) and (4.13), we have

$$H^q_{\epsilon_1,\cdots,\epsilon_\ell}(z_1,\cdots,z_{\ell-1},q^{-2N}z_\ell)=q^{c_{N,\ell}}\prod_{a=1}^{\ell-1}\frac{z_\ell-q^2z_a}{z_\ell-q^{2(N-1)}z_a}H^q_{\epsilon_\ell,\epsilon_1,\cdots,\epsilon_{\ell-1}}(z_\ell,z_1,\cdots,z_{\ell-1}), \tag{4.15}$$

where $c_{N,\ell}$ is a certain constant.

The extremal component is calculated in [N2]:

$$H^q_{\underbrace{0\cdots0}_{m}\underbrace{1\cdots1}_{m}\cdots\underbrace{(N-2)\cdots(N-2)}_{m}}=\prod_{a=1}^{\ell}z_a^{(1-\frac{1}{N-1})(\ell+\frac12)+\frac{i}{N-1}}\prod_{a=1}^{(i+1)m}z_a^{-1}\prod_{\substack{a<b\\(\epsilon_a<\epsilon_b)}}\frac{1}{z_a-q^2z_b}. \tag{4.16}$$

Now we consider the scaling limit of H^q as

$$q=e^{\lambda\frac{\hbar}{2}},\quad z_a=e^{\lambda\alpha_a},\qquad \lambda\to\infty. \tag{4.17}$$

Then $\bar{R}(z)$ goes to the rational R-matrix (2.1).

Set

$$H_{\epsilon_1,\cdots,\epsilon_\ell}(\alpha_1,\cdots,\alpha_\ell):=\lim_{\lambda\to\infty}\lambda^{\frac{(\ell-m)\ell}{2}}H^q_{\epsilon_1,\cdots,\epsilon_\ell}(z_1,\cdots,z_\ell). \tag{4.18}$$

It is easy to see that $\{H_{\epsilon_1,\cdots,\epsilon_\ell}\}$ satisfies (4.2), (4.3) and (4.4) from (4.13), (4.15) and (4.16), respectively.

Remark. We have a more explicit formula for the scaling limit of H^q. See Proposition 8.3.

By using $\{H_{\epsilon_1,\cdots,\epsilon_\ell}\}$, we have a special solution of the qKZ equation at level one.

Proposition 4.2 *Set*

$$\begin{aligned}&\bar{\psi}(\alpha_1,\cdots,\alpha_\ell) && (4.19)\\ &:=\prod_{1\leqslant a<b\leqslant\ell}\left(\frac{\Gamma(\frac{\alpha_a-\alpha_b-\hbar}{p})}{\Gamma(\frac{\alpha_a-\alpha_b+\hbar}{p})}(\alpha_a-\alpha_b-\hbar)\right)\sum_{\epsilon_1,\cdots,\epsilon_\ell}H_{\epsilon_1,\cdots,\epsilon_\ell}(\alpha_1,\cdots,\alpha_\ell)v_{\epsilon_1}\otimes\cdots\otimes v_{\epsilon_\ell}.\end{aligned}$$

Then $\bar{\psi}$ is a solution of the qKZ equation (2.2) associated with sl_{N-1} at level one satisfying the highest weight condition.

Proof. It is easy to see that $\bar{\psi}$ is a solution of the qKZ equation from (4.2) and (4.3). The highest weight condition is proved in Section 8.1. □

5 Simplified integral formula at level zero

By using the special solution (4.19), we can find a simpler integral formula in the case of level zero as follows.

Suppose that there exists $\bar{P} \in \mathcal{P}_{(N-2)m,\cdots,2m,m}$ such that

$$I^{(\mathrm{id})}_{\bar{J}}[\bar{P}](\alpha_1,\cdots,\alpha_\ell) = \prod_{1\leqslant a<b\leqslant \ell} \left(\frac{\Gamma(\frac{\alpha_a-\alpha_b-\hbar}{p})}{\Gamma(\frac{\alpha_a-\alpha_b+\hbar}{p})}(\alpha_a-\alpha_b-\hbar) \right) H_{\bar{J}}(\alpha_1,\cdots,\alpha_\ell) \tag{5.1}$$

for all $\bar{J} \in \mathcal{Z}_{(N-2)m,\cdots,2m,m}$. In the following we omit $\bar{P}$ and abbreviate $I^{(\sigma)}_{\bar{M}}[\bar{P}]$ to $I^{(\sigma)}_{\bar{M}}$. In order to rewrite (3.33) using $H_{J\bar{M}}$, we need a formula for $I^{(\sigma)}_{\bar{M}}$ for any $\sigma \in S_\ell$. This formula is given by

$$I^{(\sigma)}_{\bar{J}} = \prod_{1\leqslant a<b\leqslant \ell} \left(\frac{\Gamma(\frac{\alpha_a-\alpha_b-\hbar}{p})}{\Gamma(\frac{\alpha_a-\alpha_b+\hbar}{p})}(\alpha_a-\alpha_b-\hbar) \right) T_\sigma(H_{\bar{J}}), \quad \text{for} \quad \text{all} \quad \sigma \in S_\ell, \tag{5.2}$$

where T_σ is the permutation of variables defined by

$$T_\sigma(X)(\alpha_1,\cdots,\alpha_\ell) := X(\alpha_{\sigma(1)},\cdots,\alpha_{\sigma(\ell)}) \tag{5.3}$$

for a function X.

We can see (5.2) by induction on the length of σ. In the case of $\sigma = \mathrm{id}$, (5.2) is nothing but (5.1). Assume that (5.2) holds for $\sigma \in S_\ell$. Set $\tau := (k, k+1) \in S_\ell$. From (3.23), we find

$$\frac{\alpha_{\sigma(k)}-\alpha_{\sigma(k+1)}+\hbar}{\alpha_{\sigma(k)}-\alpha_{\sigma(k+1)}} I^{(\sigma\tau)}_{\cdots,\bar{J}_k,\bar{J}_{k+1},\cdots} = I^{(\sigma)}_{\cdots,\bar{J}_{k+1},\bar{J}_k,\cdots} + \frac{\hbar}{\alpha_{\sigma(k)}-\alpha_{\sigma(k+1)}} I^{(\sigma)}_{\cdots,\bar{J}_k,\bar{J}_{k+1},\cdots}. \tag{5.4}$$

Then we have

$$\begin{aligned}
&\left\{ \prod_{1\leqslant a<b\leqslant \ell} \left(\frac{\Gamma(\frac{\alpha_a-\alpha_b-\hbar}{p})}{\Gamma(\frac{\alpha_a-\alpha_b+\hbar}{p})}(\alpha_a-\alpha_b-\hbar) \right) \right\}^{-1} \left(I^{(\sigma)}_{\cdots,\bar{J}_{k+1},\bar{J}_k,\cdots} + \frac{\hbar}{\alpha_{\sigma(k)}-\alpha_{\sigma(k+1)}} I^{(\sigma)}_{\cdots,\bar{J}_k,\bar{J}_{k+1},\cdots} \right) \\
&= T_\sigma \left(H_{\cdots,\bar{J}_{k+1},\bar{J}_k,\cdots} + \frac{\hbar}{\alpha_k-\alpha_{k+1}} H_{\cdots,\bar{J}_k,\bar{J}_{k+1},\cdots} \right) \quad \text{from} \quad (5.2) \\
&= T_\sigma \left(\frac{\alpha_k-\alpha_{k+1}+\hbar}{\alpha_k-\alpha_{k+1}} T_\tau(H_{\cdots,\bar{J}_k,\bar{J}_{k+1},\cdots}) \right) \quad \text{from} \quad (4.2) \\
&= \frac{\alpha_{\sigma(k)}-\alpha_{\sigma(k+1)}+\hbar}{\alpha_{\sigma(k)}-\alpha_{\sigma(k+1)}} T_{\sigma\tau}\left(H_{\cdots,\bar{J}_k,\bar{J}_{k+1},\cdots}\right).
\end{aligned} \tag{5.5}$$

Hence we have (5.2) for $\sigma\tau \in S_\ell$.

By substituting (3.33) with (5.2), we get

$$\begin{aligned}
I_J[P] = (-p\pi i)^{\frac{\ell(\ell-1)}{2}} \left(\prod_{a=1}^{\ell} \int_{C_1} d\alpha_a \right) \phi(\{\alpha_a\}|\{\beta_j\}) \frac{P_1(\{e^{\frac{2\pi i}{p}\alpha_a}\}|\{e^{\frac{2\pi i}{p}\beta_j}\})}{\prod_{a=1}^{\ell}\prod_{j=1}^{n}(1-e^{\frac{2\pi i}{p}(\alpha_a-\beta_j)})} \\
\times \sum_{\sigma\in S_\ell} (\mathrm{sgn}\sigma) T_\sigma(g_{M_1^J} H_{J\bar{M}})(\alpha_1,\cdots,\alpha_\ell).
\end{aligned} \tag{5.6}$$

Here we used

$$\Gamma(\frac{x-\hbar}{p})\Gamma(\frac{-x+\hbar}{p})(x-\hbar)\mathrm{sh}\frac{\pi i}{p}(x-\hbar) = -p\pi i \tag{5.7}$$

and note that the function $\varphi(\{\alpha_a\})$ is canceled out. In this way we find a simplified representation (5.6) of the integral (3.14).

Now let us prove that the formula (5.6) gives really an integral formula for solutions of the sl_N qKZ equation at level zero. First we set

$$\begin{aligned} w_J(\{\alpha_a\}|\{\beta_j\}) &:= \sum_{\sigma\in S_\ell}(\mathrm{sgn}\sigma)T_\sigma(g_{M_1^J}H_{\bar{J}})(\alpha_1,\cdots,\alpha_\ell) \\ &= \mathrm{Skew}(g_{M_1^J}(\{\alpha_a\}|\{\beta_j\})H_{\bar{J}}(\{\alpha_a\})), \end{aligned} \tag{5.8}$$

where Skew is the skew-symmetrization with respect to $\alpha_1,\cdots,\alpha_\ell$. Note that w_J is a rational function of $\alpha_1,\cdots,\alpha_\ell$ with at most simple poles at points $\beta_1,\cdots,\beta_n$ from Remark (4.5).

Next we consider the part of P_1 in (5.6). Let us define the space of "deformed cycles" as follows. Let $\mathcal{C}_n$ be the space of p-periodic entire functions of $\beta_1,\cdots,\beta_n$. We denote by $\widehat{\mathcal{P}}_n^{\otimes\ell}$ the space of polynomials in $e^{\frac{2\pi i}{p}\alpha_1},\cdots,e^{\frac{2\pi i}{p}\alpha_\ell}$ of order less than or equal to n in each vartiable $e^{\frac{2\pi i}{p}\alpha_a}$ with the coefficients in $\mathcal{C}_n$. Then we set

$$\widehat{\mathcal{F}}_q^{\otimes\ell} := \left\{ \frac{P(\{e^{\frac{2\pi i}{p}\alpha_a}\})}{\prod_{a=1}^{\ell}\prod_{j=1}^{n}(1-e^{\frac{2\pi i}{p}(\alpha_a-\beta_j)})}; P\in\widehat{\mathcal{P}}_n^{\otimes\ell} \right\}. \tag{5.9}$$

We call the elements of $\widehat{\mathcal{F}}_q^{\otimes\ell}$ deformed cycles.

For a deformed cycle $W\in\widehat{\mathcal{F}}_q^{\otimes\ell}$, we set

$$F_J[W] := \left(\prod_{a=1}^{\ell}\int_C d\alpha_a\right)\phi(\{\alpha_a\}|\{\beta_j\})w_J(\{\alpha_a\}|\{\beta_j\})W(\{e^{\frac{2\pi i}{p}\alpha_a}\}|\{\beta_j\}) \tag{5.10}$$

where the contour C is a deformation of the real axis $(-\infty,\infty)$ such that the poles at $\beta_j+\hbar-p\mathbb{Z}_{\geqslant 0}$ are above C and the poles at $\beta_j+p\mathbb{Z}_{\geqslant 0}$ are below C. Note that, unlike the integral (3.14), the integrand in (5.10) does not have poles at the points $\alpha_a=\alpha_b\pm\hbar\pm p\mathbb{Z}_{\geqslant 0}, (a\neq b)$ because the function $\varphi(\{\alpha_a\})$ was canceled out.

The function $F_J[W]$ gives the simplified integral formula of solutions to the sl_N qKZ equation as follows. First we have that

Lemma 5.1 *The integral (5.10) converges for any deformed cycle W.*

Proof. From the Stirling formula, we have

$$\phi(\alpha|\beta_1,\cdots,\beta_n) = (\alpha/p)^{-n\hbar/p}(1+o(1)), \quad \text{as} \quad \alpha\to\pm\infty. \tag{5.11}$$

Note that $n\hbar/p=m$.

Recall Remark (4.5). By using (5.11), we see that there exist two constants C and $M > 0$ such that

$$\phi(\{\alpha_a\})w_J(\{\alpha_a\})W(\{e^{\frac{2\pi i}{p}\alpha_a}\}) \leqslant C\prod_{a=1}^{\ell}|\alpha_a|^{-2} \quad \text{for} \quad |\alpha_a| > M \ (a = 1, \cdots, \ell). \tag{5.12}$$

This completes the proof. □

Theorem 5.2 *For $W \in \widehat{\mathcal{F}}_q^{\otimes \ell}$, set*

$$\psi_W(\beta_1, \cdots, \beta_n) := \sum_{J \in \mathcal{I}_{(N-1)m,\cdots,2m,m}} F_J[W](\beta_1, \cdots, \beta_n)v_J. \tag{5.13}$$

Then ψ_W is a solution of the qKZ equation (2.2) satisfying the highest weight condition.

Remark. In the case of $N = 2$ we have $H_{\bar{J}} = 1$ for all J. Then (5.13) is the integral formula constructed in [NPT].

From the definition (5.10), we have that $\psi_{\mathrm{Skew}W} = \ell!\psi_W$. Hence the dimension of the space spanned by the solutions (5.13) is at most that of $\bigwedge^\ell \widehat{\mathcal{F}}_q$, where $\bigwedge^\ell \widehat{\mathcal{F}}_q$ is the subspace of deformed cycles skew-symmetric with respect to $\alpha_1, \cdots, \alpha_\ell$. In the case of $N > 2$ and $m > 1$, the dimension of $\bigwedge^\ell \widehat{\mathcal{F}}_q$ is much less than that of the subspace of singlet vectors in $(V_N)^{\otimes n}$ (This can be shown by a similar argument to Discussions in [N3]). Therefore the space of solutions given by the simplified integral formula is quite smaller than the space of singlet vectors.

Proof. We abbreviate $w_{J_1,\cdots,J_n}(\alpha_1, \cdots, \alpha_\ell|\beta_1, \cdots, \beta_n)$ to $w_{J_1,\cdots,J_n}(\beta_1, \cdots, \beta_n)$. In order to see that ψ_W is a solution, it suffices to prove (3.23) and (3.24) for w_J and $F_J[W]$ as in the proof of Theorem 3.1.

We can prove (3.24) for $F_J[W]$ in a similar way to the proof of Lemma 3 in [MT] by using (4.3). Here let us prove (3.23). If $J_k = 0$ or $J_{k+1} = 0$, it is easy to see (3.23) in the same way as the proof in the case of sl_2 (see [NPT]). Here we consider the case of $J_k > 0$ and $J_{k+1} > 0$.

Let α_a be the integral variable attached to the k-th component of $(V_N)^{\otimes n}$, that is, $r^J_{1,a} = k$. For two functions f_1 and f_2 we write $f_1 \sim f_2$ if $f_1 - f_2$ is symmetric with respect to α_a and α_{a+1}. We use the following abbreviation:

$$H_{\bar{J}_1,\cdots,\bar{J}_a,\bar{J}_{a+1},\cdots,\bar{J}_\ell}(\alpha_1, \cdots, \alpha_a, \alpha_{a+1}, \cdots, \alpha_\ell) = H_{\bar{J}_a,\bar{J}_{a+1}}(\alpha_a, \alpha_{a+1}). \tag{5.14}$$

The rhs of (3.23) for w_J in (5.10) is the skew-symmetrization of

$$\begin{aligned} &Q(\alpha_a, \alpha_{a+1})\frac{1}{\alpha_a - \beta_k}\frac{1}{\alpha_{a+1} - \beta_{k+1}}\frac{\alpha_{a+1} - \beta_k - \hbar}{\alpha_{a+1} - \beta_k}(\alpha_a - \alpha_{a+1} - \hbar) \\ &\times \left\{\frac{\beta_k - \beta_{k+1}}{\beta_k - \beta_{k+1} + \hbar}H_{\bar{J}_a,\bar{J}_{a+1}}(\alpha_a, \alpha_{a+1}) + \frac{\hbar}{\beta_k - \beta_{k+1} + \hbar}H_{\bar{J}_{a+1},\bar{J}_a}(\alpha_a, \alpha_{a+1})\right\}, \end{aligned} \tag{5.15}$$

where $Q(\alpha_a,\alpha_{a+1})$ is a certain symmetric function with respect to α_a and α_{a+1}. From (4.2), we have

$$\begin{aligned}
(5.15) &= Q(\alpha_a,\alpha_{a+1})\frac{1}{\alpha_a-\beta_k}\frac{1}{\alpha_{a+1}-\beta_{k+1}}\frac{\alpha_{a+1}-\beta_k-\hbar}{\alpha_{a+1}-\beta_k}(\alpha_a-\alpha_{a+1}-\hbar)\\
&\times\left\{\frac{\beta_k-\beta_{k+1}}{\beta_k-\beta_{k+1}+\hbar}H_{\bar{J}_a,\bar{J}_{a+1}}(\alpha_a,\alpha_{a+1})\right.\\
&\left.+\frac{\hbar}{\beta_k-\beta_{k+1}+\hbar}\left(\frac{\alpha_a-\alpha_{a+1}+\hbar}{\alpha_a-\alpha_{a+1}}H_{\bar{J}_a,\bar{J}_{a+1}}(\alpha_{a+1},\alpha_a)-\frac{\hbar}{\alpha_a-\alpha_{a+1}}H_{\bar{J}_a,\bar{J}_{a+1}}(\alpha_a,\alpha_{a+1})\right)\right\}\\
&\sim Q(\alpha_a,\alpha_{a+1})H_{\bar{J}_a,\bar{J}_{a+1}}(\alpha_a,\alpha_{a+1})\\
&\times\left\{\frac{1}{\alpha_a-\beta_k}\frac{1}{\alpha_{a+1}-\beta_{k+1}}\frac{\alpha_{a+1}-\beta_k-\hbar}{\alpha_{a+1}-\beta_k}(\alpha_a-\alpha_{a+1}-\hbar)\frac{\beta_k-\beta_{k+1}}{\beta_k-\beta_{k+1}+\hbar}\right.\\
&-\frac{\alpha_{a+1}-\alpha_a+\hbar}{\alpha_{a+1}-\alpha_a}\frac{1}{\alpha_{a+1}-\beta_k}\frac{1}{\alpha_a-\beta_{k+1}}\frac{\alpha_a-\beta_k-\hbar}{\alpha_a-\beta_k}(\alpha_{a+1}-\alpha_a-\hbar)\frac{\hbar}{\beta_k-\beta_{k+1}+\hbar}\\
&\left.-\frac{\hbar}{\alpha_a-\alpha_{a+1}}\frac{1}{\alpha_a-\beta_k}\frac{1}{\alpha_{a+1}-\beta_{k+1}}\frac{\alpha_{a+1}-\beta_k-\hbar}{\alpha_{a+1}-\beta_k}(\alpha_a-\alpha_{a+1}-\hbar)\frac{\hbar}{\beta_k-\beta_{k+1}+\hbar}\right\}\\
&= Q(\alpha_a,\alpha_{a+1})H_{\bar{J}_a,\bar{J}_{a+1}}(\alpha_a,\alpha_{a+1})\\
&\times\frac{(\alpha_a-\alpha_{a+1}-\hbar)\left\{(\alpha_a-\beta_{k+1})(\alpha_{a+1}-\beta_k)-\hbar(\alpha_{a+1}-\beta_{k+1})+\hbar^2\right\}}{(\alpha_a-\beta_k)(\alpha_a-\beta_{k+1})(\alpha_{a+1}-\beta_k)(\alpha_{a+1}-\beta_{k+1})}.
\end{aligned} \tag{5.16}$$

On the other hand, the lhs of (3.23) is the skew-symmetrization of

$$\begin{aligned}
&Q(\alpha_a,\alpha_{a+1})H_{\bar{J}_{a+1},\bar{J}_a}(\alpha_{a+1},\alpha_a)\\
&\times\frac{1}{\alpha_a-\beta_{k+1}}\frac{1}{\alpha_{a+1}-\beta_k}\frac{\alpha_{a+1}-\beta_{k+1}-\hbar}{\alpha_{a+1}-\beta_{k+1}}(\alpha_a-\alpha_{a+1}-\hbar),
\end{aligned} \tag{5.17}$$

where $Q(\alpha_a,\alpha_{a+1})$ is the same function as $Q(\alpha_a,\alpha_{a+1})$ in (5.15). From (4.2), we have

$$\begin{aligned}
(5.17) &= Q(\alpha_a,\alpha_{a+1})\frac{1}{\alpha_a-\beta_{k+1}}\frac{1}{\alpha_{a+1}-\beta_k}\frac{\alpha_{a+1}-\beta_{k+1}-\hbar}{\alpha_{a+1}-\beta_{k+1}}(\alpha_a-\alpha_{a+1}-\hbar)\\
&\times\left\{\frac{\alpha_a-\alpha_{a+1}+\hbar}{\alpha_a-\alpha_{a+1}}H_{\bar{J}_a,\bar{J}_{a+1}}(\alpha_{a+1},\alpha_a)-\frac{\hbar}{\alpha_a-\alpha_{a+1}}H_{\bar{J}_a,\bar{J}_{a+1}}(\alpha_a,\alpha_{a+1})\right\}\\
&\sim Q(\alpha_a,\alpha_{a+1})H_{\bar{J}_a,\bar{J}_{a+1}}(\alpha_a,\alpha_{a+1})\\
&\times\left\{-\frac{1}{\alpha_{a+1}-\beta_{k+1}}\frac{1}{\alpha_a-\beta_k}\frac{\alpha_a-\beta_{k+1}-\hbar}{\alpha_a-\beta_{k+1}}(\alpha_{a+1}-\alpha_a-\hbar)\frac{\alpha_a-\alpha_{a+1}+\hbar}{\alpha_a-\alpha_{a+1}}\right.\\
&\left.-\frac{1}{\alpha_a-\beta_{k+1}}\frac{1}{\alpha_{a+1}-\beta_k}\frac{\alpha_{a+1}-\beta_{k+1}-\hbar}{\alpha_{a+1}-\beta_{k+1}}(\alpha_a-\alpha_{a+1}-\hbar)\frac{\hbar}{\alpha_a-\alpha_{a+1}}\right\}\\
&= (5.16).
\end{aligned} \tag{5.18}$$

Hence (3.23) holds.

Let us prove the highest weight condition. In the same way as (3.25), it suffices to prove that

$$\sum_{\substack{a=1\\ J'_a=k-1}}^{n} F_{J'_1,\cdots,J'_a+1,\cdots,J'_n}[W]=0,\ (k=1,\cdots,N-1), \tag{5.19}$$

where $(J'_1, \cdots, J'_n) \in \mathcal{Z}_{\nu_1,\cdots,\nu_k-1,\cdots,\nu_N}$.

First consider the case of $k > 1$. From the highest weight condition for $\bar{\psi}$, we can see that

$$\sum_{\substack{a=1\\ J'_a=k-1}}^{n} w_{J'_1,\cdots,J'_a+1,\cdots,J'_n}[W] = 0. \tag{5.20}$$

Hence (5.19) holds in the case of $k > 1$.

Now we consider the case of $k = 1$.

Lemma 5.3 *For $J' = (J'_1, \cdots, J'_N) \in \mathcal{Z}_{\ell-1,\nu_2,\cdots,\nu_N}$, the following equality holds:*

$$\begin{aligned}\hbar \sum_{\substack{a=1\\ J'_a=0}}^{n} w_{J'_1,\cdots,J'_a+1,\cdots,J'_n} &= \mathrm{Skew}\Big(g_{M_1^{J'}}(\{\alpha_a\}_{2\leqslant a\leqslant \ell}|\{\beta_j\}) \\ &\times \Big\{\prod_{a=2}^{\ell}(\alpha_1-\alpha_a-\hbar)H(\alpha_1) - \prod_{j=1}^{n}\frac{\alpha_1-\beta_j-\hbar}{\alpha_1-\beta_j}\prod_{a=2}^{\ell}(\alpha_1-\alpha_a+(N-1)\hbar)H(\alpha_1+N\hbar)\Big\}\Big),\end{aligned} \tag{5.21}$$

Here we used the following abbreviation:

$$H(\alpha_1) = H_{0,\bar{J}_1,\cdots,\bar{J}_{\ell-1}}(\alpha_1, \alpha_2, \cdots, \alpha_\ell). \tag{5.22}$$

This lemma is proved in Section 8.2.

From (5.21), we can get (5.19) for $k = 1$ by the same calculation as (3.31). □

6 Modification of the integral formula

6.1 One-time integration

Recall that $n = Nm$. Let $\{\omega_{\epsilon_1,\cdots,\epsilon_n}(\beta_1, \cdots, \beta_n)\}_{(\epsilon_1,\cdots,\epsilon_n)\in\mathcal{Z}_{(N-1)m,\cdots,2m,m}}$ be the set of vectors in $(V_N)^{\otimes Nm}$ uniquely defined by the following conditions:

$$\omega_{\cdots,\epsilon_{j+1},\epsilon_j,\cdots}(\cdots, \beta_{j+1}, \beta_j, \cdots) = P_{j,j+1}R_{j,j+1}(\beta_j - \beta_{j+1})\omega_{\cdots,\epsilon_j,\epsilon_{j+1},\cdots}(\cdots, \beta_j, \beta_{j+1}, \cdots), \tag{6.1}$$

and

$$\begin{aligned}&\omega_{\underbrace{0\cdots0}_{m}\underbrace{1\cdots1}_{m}\cdots\underbrace{(N-1)\cdots(N-1)}_{m}}(\beta_1, \cdots, \beta_n) \\ &= \underbrace{v_0\otimes\cdots\otimes v_0}_{m}\otimes\underbrace{v_1\otimes\cdots\otimes v_1}_{m}\otimes\cdots\otimes\underbrace{v_{N-1}\otimes\cdots\otimes v_{N-1}}_{m}.\end{aligned} \tag{6.2}$$

Here $P_{j,j+1}$ is the permutation operator acting on the tensor product of j-th and $(j + 1)$-th components.

For $J \in \mathcal{Z}_{(N-1)m,\cdots,2m,m}$, we set

$$K_r^J := \mathcal{N}_r^J \setminus \mathcal{N}_{r+1}^J =: \{k_{r,1}^J, \cdots, k_{r,m}^J\}, \quad k_{r,1}^J < \cdots < k_{r,m}^J. \tag{6.3}$$

Note that $\{1, \cdots, n\} = \sqcup_{r=0}^{N-1} K_r^J$.

We define rational functions $\mu_J^{(a)}$ and $\tilde{w}_J$ by

$$\mu_J^{(a)}(\alpha|\{\beta_j\}) := \frac{1}{\alpha - \beta_a} \prod_{\substack{k \in K_r^J \\ k \neq a}} \frac{\alpha - \beta_k - \hbar}{\beta_a - \beta_k - \hbar} \prod_{\substack{j \in K_p^J \\ p > r}} \frac{\alpha - \beta_j - \hbar}{\alpha - \beta_j}, \quad (a \in K_r^J), \tag{6.4}$$

$$\tilde{w}_J(\{\alpha_a\}|\{\beta_j\}) := \mathrm{Skew}\left(\prod_{s=1}^{N-1} \prod_{j=1}^{m} \mu_J^{(k^J_{N-s,j})}(\alpha_{j+(s-1)m}|\{\beta_j\}) \right), \tag{6.5}$$

where Skew is the skew-symmetrization with respect to $\alpha_1, \cdots, \alpha_\ell$. Note that $\ell = (N-1)m$.

Proposition 6.1

$$\sum_{J \in \mathcal{Z}_{(N-1)m, \cdots, 2m, m}} w_J v_J \tag{6.6}$$
$$= (-1)^{\frac{\ell(\ell-1)}{2}} \sum_{J \in \mathcal{Z}_{(N-1)m, \cdots, 2m, m}} \tilde{w}_J \omega_J \prod_{r=1}^{N-1} \prod_{\substack{a,b \in K_r^J \\ a<b}} \frac{(\beta_b - \beta_a - \hbar)(\beta_a - \beta_b - \hbar)}{\beta_a - \beta_b} \prod_{\substack{a \in K_s^J, b \in K_r^J \\ 0 \leqslant r < s \leqslant N-1}} \frac{\beta_a - \beta_b - \hbar}{\beta_a - \beta_b}.$$

This proposition is proved in Section 8.1.

By using Proposition 6.1, we rewrite ψ_W in terms of $\tilde{w}_J$ and ω_J. Then we can carry out the integration once as follows.

Recall the definition of ϕ (3.15):

$$\phi(\alpha|\beta_1, \cdots, \beta_n) := \prod_{j=1}^{n} \frac{\Gamma(\frac{\alpha - \beta_j - \hbar}{p})}{\Gamma(\frac{\alpha - \beta_j}{p})}. \tag{6.7}$$

For a function $f(\alpha)$, we define a function Df by

$$(Df)(\alpha) := f(\alpha) - f(\alpha + p)\frac{\phi(\alpha + p)}{\phi(\alpha)} = f(\alpha) - f(\alpha + p) \prod_{j=1}^{n} \frac{\alpha - \beta_j - \hbar}{\alpha - \beta_j}. \tag{6.8}$$

Set

$$L_J^{(0)}(\alpha) := \prod_{k \in k_0^J} (\alpha - \beta_k - N\hbar). \tag{6.9}$$

Proposition 6.2

$$(DL_J^{(0)})(\alpha) = \hbar \sum_{r=1}^{N-1} \sum_{k \in K_r^J} \prod_{j \in K_0^J} (\beta_k - \beta_j - r\hbar) \prod_{\substack{j \in K_r^J \\ j \neq k}} \frac{\beta_k - \beta_j - \hbar}{\beta_k - \beta_j} \mu_J^{(k)}(\alpha). \tag{6.10}$$

This proposition is proved in Section 8.2.

From Proposition 6.2, we have

$$\tilde{w}_J(\{\alpha_a\}) = \frac{\hbar}{\prod_{j\in K_0^J}(\beta_{k^J_{1,m}}-\beta_j-\hbar)}\prod_{\substack{j\in K_1^J\\ j\neq k^J_{1,m}}}\frac{\beta_{k^J_{1,m}}-\beta_j}{\beta_{k^J_{1,m}}-\beta_j-\hbar}$$
$$\times\,\mathrm{Skew}\left(\mu_J^{(k^J_{N-1,1})}(\alpha_1)\cdots\mu_J^{(k^J_{1,m-1})}(\alpha_{\ell-1})(DL_J^{(0)})(\alpha_\ell)\right) \qquad (6.11)$$

Take the deformed cycle of the following form:

$$W(\{e^{\frac{2\pi i}{p}\alpha_a}\}) = \prod_{a=1}^{\ell}\frac{P_a(e^{\frac{2\pi i}{p}\alpha_a})}{\prod_{j=1}^{n}(1-e^{\frac{2\pi i}{p}(\alpha_a-\beta_j)})} \in \widehat{\mathcal{F}}_q^{\otimes\ell}. \qquad (6.12)$$

Let us consider the following integral:

$$\left(\prod_{a=1}^{\ell}\int_C d\alpha_a\right)\phi(\{\alpha_a\}|\{\beta_j\})\tilde{w}_J(\{\alpha_a\})W(\{e^{\frac{2\pi i}{p}\alpha_a}\}) \qquad (6.13)$$

Using (6.11), we can carry out the integration once in (6.13) by using the following formula:

$$\int_C d\alpha\phi(\alpha)(DL_J^{(0)})(\alpha)\frac{P_a(e^{\frac{2\pi i}{p}\alpha})}{\prod_{j=1}^{n}(1-e^{\frac{2\pi i}{p}(\alpha-\beta_j)})}$$
$$=\left(\int_C-\int_{C+p}\right)d\alpha\phi(\alpha)L_J^{(0)}(\alpha)\frac{P_a(e^{\frac{2\pi i}{p}\alpha})}{\prod_{j=1}^{n}(1-e^{\frac{2\pi i}{p}(\alpha-\beta_j)})} = p^m(P_a^{-\infty}-P_a^{+\infty}), \qquad (6.14)$$

where

$$P_a^{\pm\infty} := \lim_{\alpha\to\pm\infty}\frac{P_a(e^{\frac{2\pi i}{p}\alpha})}{\prod_{j=1}^{n}(1-e^{\frac{2\pi i}{p}(\alpha-\beta_j)})}. \qquad (6.15)$$

The formula (6.14) can be obtained from (5.11).

Especially, if $P_a^{\pm\infty}=0$, $(1\leqslant a\leqslant \ell-1)$, then we have

$$(6.13) = p^m(P_\ell^{-\infty}-P_\ell^{+\infty})\frac{\hbar}{\prod_{j\in K_0^J}(\beta_{k^J_{1,m}}-\beta_j-\hbar)}\prod_{\substack{j\in K_1^J\\ j\neq k^J_{1,m}}}\frac{\beta_{k^J_{1,m}}-\beta_j}{\beta_{k^J_{1,m}}-\beta_j-\hbar}$$
$$\times\left(\prod_{a=1}^{\ell-1}\int_C d\alpha_a\right)\phi(\{\alpha_a\}_{1\leqslant a\leqslant \ell-1}|\{\beta_j\})\frac{\prod_{a=1}^{\ell-1}P_a(e^{\frac{2\pi i}{p}\alpha_a})}{\prod_{a=1}^{\ell-1}\prod_{j=1}^{n}(1-e^{\frac{2\pi i}{p}(\alpha_a-\beta_j)})}$$
$$\times\,\mathrm{Skew}\left(\mu_J^{(k^J_{N-1,1})}(\alpha_1)\cdots\mu_J^{(k^J_{1,m-1})}(\alpha_{\ell-1})\right). \qquad (6.16)$$

6.2 Smirnov's formula

From (6.16), we can get the integral formula for solutions of the qKZ equation constructed by Smirnov [S1] as follows.

For a rational function $f(\alpha)$, let $[f(\alpha)]_+ \in \mathbb{C}[\alpha]$ be its polynomial part:

$$f(\alpha) = [f(\alpha)]_+ + o(1), \quad \text{as} \quad \alpha \to \infty. \tag{6.17}$$

Denote by $T_\hbar$ the difference operator defined by $T_\hbar f(\alpha) := f(\alpha) - f(\alpha + \hbar)$.

Set

$$L_J^{(r)}(\alpha) := \prod_{j \in K_r^J} (\alpha - \beta_j - N\hbar), \quad (r = 0, \cdots, N-1) \tag{6.18}$$

and

$$Q_J^{(k)}(\alpha) := \sum_{r=0}^{N-1} L_J^{(r)}(\alpha + r\hbar) T_\hbar \left(\left[\frac{\prod_{s=o}^{r-1} L_J^{(s)}(\alpha + (r-1)\hbar) \prod_{s=r+1}^{N-1} L_J^{(s)}(\alpha + r\hbar)}{(\alpha + r\hbar)^k} \right]_+ \right) \tag{6.19}$$

for $k = 1, \cdots, \ell$. Here we note that $\ell = (N-1)m$ and hence $Q_J^{(\ell)} = 0$.

Proposition 6.3 *For $a \in K_r^J, (r > 0)$, the following equality holds:*

$$\hbar^{-1} \left(D\Big(\prod_{\substack{j=1 \\ j \neq a}}^{n} (\alpha - \beta_j - N\hbar)\Big) - \sum_{k=1}^{\ell-1} (\beta_a + N\hbar)^{k-1} Q_J^{(k)}(\alpha) \right) \tag{6.20}$$

$$= \prod_{\substack{j \in K_t^J \\ t < r}} (\beta_a - \beta_j - \hbar) \prod_{\substack{j \in K_r^J \\ j \neq a}} (\beta_a - \beta_j - \hbar) \prod_{\substack{j \in K_t^J \\ t > r}} (\beta_a - \beta_j) \mu_J^{(a)}(\alpha)$$

$$+ \sum_{u=1}^{N-r-1} \sum_{b \in K_{r+u}^J} \prod_{\substack{j \in K_{r+u}^J \\ j \neq b}} \frac{\beta_b - \beta_j - \hbar}{\beta_b - \beta_j}$$

$$\times \sum_{s=0}^{u-1} \prod_{j \in K_{r+s}^J} (\beta_b - \beta_j - (u-s)\hbar) \frac{\prod_{\substack{j \in K_t^J \\ t < r+s}} (\beta_a - \beta_j - \hbar) \prod_{\substack{j \in K_t^J \\ t > r+s}} (\beta_a - \beta_j)}{(\beta_b - \beta_a - (u-s-1)\hbar)(\beta_b - \beta_a - (u-s)\hbar)} \mu_J^{(b)}(\alpha).$$

This proposition is proved in Section 8.2.

From the same calculation as (6.14), it is easy to see that, if $P = P(e^{\frac{2\pi i}{p}\alpha})$ satisfies

$$\lim_{\alpha \to \pm\infty} \frac{P(e^{\frac{2\pi i}{p}\alpha})}{\prod_{j=1}^{n} (1 - e^{\frac{2\pi i}{p}(\alpha - \beta_j)})} = 0, \tag{6.21}$$

then we have

$$\int_C d\alpha \phi(\alpha | \{\beta_j\}) D\Big(\prod_{\substack{j=1 \\ j \neq a}}^{n} (\alpha - \beta_j - N\hbar)\Big) \frac{P(e^{\frac{2\pi i}{p}\alpha})}{\prod_{j=1}^{n} (1 - e^{\frac{2\pi i}{p}(\alpha - \beta_j)})} = 0. \tag{6.22}$$

From Proposition 6.3 we see that, for $a \in K_r^J, (r > 0)$, by adding the linear sum of $\mu_J^{(b)}$'s $(b \in K_s^J, s > r)$ to $\mu_J^{(a)}$ as in the rhs of (6.20), we get the lhs of (6.20). Moreover, the first term in the lhs of (6.20) vanishes after the integral over C from (6.22). Therefore, in (6.16), we can replace

$$\mathrm{Skew}\left(\mu_J^{(k^J_{N-1,1})}(\alpha_1)\cdots\mu_J^{(k^J_{1,m-1})}(\alpha_{\ell-1})\right) \tag{6.23}$$

by

$$\det[(\beta_a + N\hbar)^{k-1}]_{\substack{a\notin K_0^J,\, a\neq k^J_{1,m} \\ 1\leqslant k\leqslant \ell-1}} \det[Q_J^{(k)}(\alpha_b)]_{1\leqslant k,b\leqslant \ell-1} \tag{6.24}$$

multiplied by a certain rational function of $\beta_1, \cdots, \beta_n$ determined from (6.20).

Finally we get the following formula for solutions:

Theorem 6.4 *Suppose that W is a deformed cycle of the form (6.12) with $P_a^{\pm\infty} = 0$, $(1 \leqslant a \leqslant \ell - 1)$. Then*

$$\begin{aligned} \psi_W &= (-1)^{\frac{N}{2}m(m+1)+m^2}(P_\ell^{-\infty} - P_\ell^{+\infty}) \\ &\quad \times \sum_{J\in\mathcal{Z}_{(N-1)m,\cdots,2m,m}} \omega_J \prod_{\substack{a\in K_r^J, b\in K_s^J \\ 0\leqslant r<s\leqslant N-1}} \frac{1}{\beta_a - \beta_b} \\ &\quad \times \left(\prod_{a=1}^{\ell-1}\int_C d\alpha_a\right) \phi(\{\alpha_a\}|\{\beta_j\}) \det[Q_J^{(b)}(\alpha_a)]_{a,b=1}^{\ell-1} \frac{\prod_{a=1}^{\ell-1} P_a(e^{\frac{2\pi i}{p}\alpha_a})}{\prod_{a=1}^{\ell-1}\prod_{j=1}^{n}(1 - e^{\frac{2\pi i}{p}(\alpha_a - \beta_j)})}, \end{aligned} \tag{6.25}$$

Corollary 6.5 *Suppose that W is a deformed cycle satisfying the assumption in Theorem 6.4. If it also holds that $P_\ell^{\pm\infty} = 0$, then $\psi_W = 0$.*

Remark. The formula (6.25) is nothing but the integral formula constructed by Smirnov [S1]. Note that indices for basis of the vector representation in [S1] are reverse to that of V_N, that is, e_j in [S1] is equal to v_{N-j}. Let $A_k(\alpha|B^{(1)}|\cdots|B^{(N)})$ be the polynomial defined in [S1], page 185. Then

$$Q_J^{(k)}|_{\hbar=-\frac{2\pi i}{N}, p=-2\pi i} = A_{\ell-k}\left(\alpha - \pi i\left|\{\beta_j - 2\pi i - \frac{\pi i}{N}\}_{j\in K^J_{N-1}}\right|\cdots\left|\{\beta_j - 2\pi i - \frac{\pi i}{N}\}_{j\in K^J_0}\right.\right). \tag{6.26}$$

7 Form factors of $SU(N)$ invariant Thirring model

7.1 Axioms for form factors

In the following we assume that

$$\hbar = -\frac{2\pi i}{N}, \qquad p = N\hbar = -2\pi i. \tag{7.1}$$

Consider the l-th fundamental representation of $SU(N)$:

$$V^{(l)} \simeq \wedge^l V_N. \tag{7.2}$$

This space is realized as the subspace of $(V_N)^{\otimes l}$ spanned by the following vectors

$$(V_N)^{\otimes l} \ni v_{[\epsilon_1,\cdots,\epsilon_l]} := \sum_{\sigma\in S_l} (\mathrm{sgn}\sigma) v_{\epsilon_{\sigma(1)}} \otimes \cdots \otimes v_{\epsilon_{\sigma(l)}}, \quad (0 \leqslant \epsilon_1 < \cdots < \epsilon_l \leqslant N-1). \tag{7.3}$$

In the following we denote by $V^{(l)}$ this subspace.

Fix a positive integer m and assume that $n = Nm$. As mentioned in Introduction we consider form factors of type

$$f^{(1,\cdots,1,k)}(\beta_1,\cdots,\beta_{n-k},\beta_{n-k+1}) \in (V^{(1)})^{\otimes(n-k)} \otimes V^{(k)}. \tag{7.4}$$

In the following we abbreviate $f^{(1,\cdots,1,k)}$ to $f^{(k)}$ for $k = 2,\cdots,N-1$.

The form factor associated with n rank-1 particles $f_m(\beta_1,\cdots,\beta_n) := f^{(1,\cdots,1)}(\beta_1,\cdots,\beta_n)$ takes values in $(V^{(1)})^{\otimes n}$ and satisfies the following conditions:

$$P_{j,j+1}S_{j,j+1}(\beta_j - \beta_{j+1}) f_m(\cdots,\beta_j,\beta_{j+1},\cdots) = f_m(\cdots,\beta_{j+1},\beta_j,\cdots), \tag{7.5}$$

$$P_{n-1,n}\cdots P_{1,2} f_m(\beta_1 - 2\pi i,\beta_2,\cdots,\beta_n) = e^{-\frac{(N-1)n}{2N}\pi i} f_m(\beta_2,\cdots,\beta_{n-1},\beta_1), \tag{7.6}$$

where $S(\beta)$ is the S-matrix defined by

$$S(\beta) := S_0(\beta)R(\beta), \qquad S_0(\beta) := \frac{\Gamma(\frac{N-1}{N} + \frac{\beta}{2\pi i})\Gamma(-\frac{\beta}{2\pi i})}{\Gamma(\frac{N-1}{N} - \frac{\beta}{2\pi i})\Gamma(\frac{\beta}{2\pi i})}. \tag{7.7}$$

The function f_m has a simple pole at the point $\beta_n = \beta_{n-1} - \hbar$ with the following residue:

$$2\pi i\mathrm{res}_{\beta_n=\beta_{n-1}-\hbar} f_m(\beta_1,\cdots,\beta_n) = f^{(2)}\left(\beta_1,\cdots,\beta_{n-2},\beta_{n-1} - \frac{\hbar}{2}\right). \tag{7.8}$$

where $f^{(2)}$ is the form factor associated with $(n-2)$ rank-1 particles and one rank-2 particle, that is, a vector in $(V^{(1)})^{\otimes(n-2)} \otimes V^{(2)} \subset (V^{(1)})^{\otimes n}$. Generally, for $2 \leqslant k \leqslant N-2$, the form factor

$$f^{(k)}(\beta_1,\cdots,\beta_{n-k+1}) \in (V^{(1)})^{\otimes(n-k)} \otimes V^{(k)} \tag{7.9}$$

has a simple pole at the point $\beta_{n-k+1} = \beta_{n-k} - \frac{k+1}{2}\hbar$ with the residue

$$\begin{aligned}
&2\pi i\mathrm{res} f^{(k)}(\beta_1,\cdots,\beta_{n-k+1}) \\
&= f^{(k+1)}\left(\beta_1,\cdots,\beta_{n-k-1},\beta_{n-k} - \frac{k}{2}\hbar\right) \in (V^{(1)})^{\otimes(n-k-1)} \otimes V^{(k+1)}.
\end{aligned} \tag{7.10}$$

In the case of $k = N-1$, the residue at $\beta_{n-N+2} = \beta_{n-N+1} - \frac{N}{2}\hbar = \beta_{n-N+1} + \pi i$ is given by

$$\begin{aligned}
&2\pi i\mathrm{res} f^{(N-1)}(\beta_1,\cdots,\beta_{n-N+2}) \\
&= \left(I + e^{-\frac{2\pi i}{N} + \frac{N-1}{N}(n-N)\pi i} S_{n-N+1,n-N}(\beta_{n-N+1} - \beta_{n-N}) \cdots S_{n-N+1,1}(\beta_{n-N+1} - \beta_1)\right) \\
&\quad \times f_{m-1}(\beta_1,\cdots,\beta_{n-N}) \otimes v_{[0,1,\cdots,N-1]},
\end{aligned} \tag{7.11}$$

where $f_{m-1}(\beta_1,\cdots,\beta_{n-N}) := f^{(1,\cdots,1)}(\beta_1,\cdots,\beta_{n-N})$ is the form factor associated with $(n - N) = (m-1)N$ rank-1 particles satisfying (7.5) and (7.6).

7.2 Recurrence relations for deformed cycles

Define a function $\zeta(\beta)$ by

$$\zeta(\beta) := \frac{\Gamma_2(-i\beta + \frac{2(2N-1)}{N}\pi)\Gamma_2(i\beta + \frac{2(N-1)}{N}\pi)}{\Gamma_2(-i\beta + 2\pi)\Gamma_2(i\beta)}, \quad \Gamma_2(x) = \Gamma_2(x|2\pi, 2\pi). \tag{7.12}$$

Here $\Gamma_2(x|\omega_1, \omega_2)$ is the double gamma function satisfying

$$\frac{\Gamma_2(x + \omega_1|\omega_1, \omega_2)}{\Gamma_2(x|\omega_1, \omega_2)} = \frac{1}{\Gamma_1(x|\omega_2)}, \tag{7.13}$$

where

$$\Gamma_1(x|\omega) := \frac{\omega^{\frac{x}{\omega} - \frac{1}{2}}}{\sqrt{2\pi}}\Gamma(\frac{x}{\omega}). \tag{7.14}$$

We refer the reader to [JM] for other properties of the double gamma function. From the definiton of $\zeta(\beta)$, we can see that

$$\zeta(\beta - 2\pi i) = \zeta(-\beta). \tag{7.15}$$

For $P \in \widehat{\mathcal{P}}_n^{\otimes \ell}$, we set

$$f_P := e^{\frac{(N-1)n}{2N}\sum_{j=1}^n \beta_j} \prod_{1 \leqslant j < j' \leqslant n} \zeta(\beta_j - \beta_{j'})\Psi_P. \tag{7.16}$$

Here Ψ_P is the solution (5.13) of the qKZ equation given by

$$\Psi_P := \psi_W, \quad \text{where} \quad W(\{e^{-\alpha_a}\}) := \frac{P(\{e^{-\alpha_a}\})}{\prod_{a=1}^{\ell}\prod_{j=1}^{n}(1 - e^{-(\alpha_a - \beta_j)})} \in \widehat{\mathcal{F}}_q^{\otimes \ell}. \tag{7.17}$$

It is easy to see the following proposition from (3.23) and (3.24) for $F_J[W]$.

Proposition 7.1 *If P is symmetric with respect to $\beta_1, \cdots, \beta_n$, then f_P satisfies (7.5) and (7.6).*

Suppose that the form factor $f_m \in (V^{(1)})^{\otimes n}$ is parametrized by $P_m \in \widehat{\mathcal{P}}^{\otimes \ell}$ as (7.16): $f_m = f_{P_m}$. Similarly, suppose that $f_{m-1} = f_{P_{m-1}}$ for $P_{m-1} \in \widehat{\mathcal{P}}^{\otimes(\ell - N + 1)}$. Now we give a sufficient condition for P_m and P_{m-1} to satisfy (7.8), (7.10) and (7.11) for certain functions $f^{(k)} \in (V^{(1)})^{\otimes(n-k)} \otimes V^{(k)}$, $(k = 2, \cdots, N-1)$.

For two polynomials P_1 and P_2 of $e^{-\alpha_1}, \cdots, e^{-\alpha_r}$, we write

$$P_1 \sim P_2 \quad \text{if} \quad \mathrm{Skew}(P_1 - P_2) = 0, \tag{7.18}$$

where Skew is the skew-symmetrization with respect to $\alpha_1, \cdots, \alpha_r$.

Proposition 7.2 *For $P_m \in \widehat{\mathcal{P}}_n^{\otimes \ell}$ and $P_{m-1} \in \widehat{\mathcal{P}}_{n-N}^{\otimes(\ell-N+1)}$, suppose that there exists a set of polynomials of $e^{-\alpha_a}$'s*

$$\widehat{P}^{(k)}(\alpha_1,\cdots,\alpha_{\ell-k+1}|\beta_1,\cdots,\beta_{n-k-1}|\beta_{n-k}) \in \widehat{\mathcal{P}}_{n-k}^{\otimes(\ell-k+1)}, \quad \text{and} \tag{7.19}$$

$$P^{(k)}(\alpha_1,\cdots,\alpha_{\ell-k+1}|\beta_1,\cdots,\beta_{n-k}|\beta_{n-k+1}) \in \widehat{\mathcal{P}}_{n-k+1}^{\otimes(\ell-k+1)}, \quad (k=1,\cdots,N-1) \tag{7.20}$$

satisfying the following conditions:

$$P^{(1)} = P_m(\alpha_1,\cdots,\alpha_\ell|\beta_1,\cdots,\beta_n), \tag{7.21}$$

$$P^{(k)}|_{\beta_{n-k+1}=\beta_{n-k}-\frac{k+1}{2}\hbar} \sim \prod_{a=1}^{\ell-k}(1-e^{-(\alpha_a-\beta_{n-k})})\widehat{P}^{(k)}, \quad (k=1,\cdots,N-2), \tag{7.22}$$

$$P^{(k+1)} = \widehat{P}^{(k)}(\alpha_1,\cdots,\alpha_{\ell-k},\beta_{n-k}+\frac{k}{2}\hbar|\beta_1,\cdots,\beta_{n-k-1}|\beta_{n-k}+\frac{k}{2}\hbar), \quad (k=1,\cdots,N-2), \tag{7.23}$$

$$P^{(N-1)}|_{\beta_{n-N+2}=\beta_{n-N+1}-\frac{N}{2}\hbar} \sim \prod_{a=1}^{\ell-N+2}(1-e^{-(\alpha_a-\beta_{n-N+1})})(1-e^{-(\alpha_a-\beta_{n-N+1}-\hbar)})\widehat{P}^{(N-1)}, \tag{7.24}$$

$$\begin{aligned}&\widehat{P}^{(N-1)}|_{\alpha_{\ell-N+2}=\beta_{n-N+1}-\delta(N-1)\hbar}\\ &\quad = d_m^{-1} e^{-\frac{(N-1)(2n-N)}{2}(\beta_{n-N+1}-\delta(N-1)\hbar)} P_{m-1}(\alpha_1,\cdots,\alpha_{\ell-N+1}|\beta_1,\cdots,\beta_{n-N}),\end{aligned} \tag{7.25}$$

where $\delta=0,1$ and d_m is a constant defined by (7.88).

Then there exists a set of functions

$$f^{(k)}(\beta_1,\cdots,\beta_{n-k+1}) \in (V^{(1)})^{\otimes(n-k)} \otimes V^{(k)}, \quad (k=2,\cdots,N-1) \tag{7.26}$$

satisfying (7.8), (7.10) and (7.11).

In the rest of this subsection, we prove this proposition.

Recall that $\ell=(N-1)m$. We denote by $\mathcal{Z}_{N-1}^{(m)}$ the additive group freely generated by the elements $(\epsilon_1,\cdots,\epsilon_\ell) \in \mathcal{Z}_{(N-2)m,\cdots,2m,m}$. We set

$$\begin{aligned}&(\epsilon_1,\cdots,\epsilon_{\ell-a},[\epsilon_{\ell-a+1},\cdots,\epsilon_{\ell-b}],\epsilon_{\ell-b+1},\cdots,\epsilon_\ell)\\ &:= \sum_{\sigma\in S_{a-b}}(\mathrm{sgn}\sigma)(\epsilon_1,\cdots,\epsilon_{\ell-a},\epsilon_{\ell-a+\sigma(1)},\cdots,\epsilon_{\ell-a+\sigma(a-b)},\epsilon_{\ell-b},\cdots,\epsilon_\ell) \in \mathcal{Z}_{N-1}^{(m)}.\end{aligned} \tag{7.27}$$

Set

$$G_{\epsilon_1,\cdots,\epsilon_\ell}(\alpha_1,\cdots,\alpha_\ell) := \prod_{1\leqslant a<b\leqslant \ell}(\alpha_a-\alpha_b-\hbar)H_{\epsilon_1,\cdots,\epsilon_\ell}(\alpha_1,\cdots,\alpha_\ell). \tag{7.28}$$

Note that $G_{\epsilon_1,\cdots,\epsilon_\ell}$ is a polynomial of $\alpha_1,\cdots,\alpha_\ell$ from Remark (4.5). For $\epsilon\in\mathcal{Z}_{N-1}^{(m)}$, we define G_ϵ by (7.28) and $G_{\epsilon+\epsilon'} := G_\epsilon + G_{\epsilon'}$.

Here we note that the function w_J defined in (5.8) is given by

$$w_J = \mathrm{Skew}\Big(\prod_{a=1}^{\ell}\left(\frac{1}{\alpha_a-\beta_{m_a}}\prod_{j=1}^{m_a-1}\frac{\alpha_a-\beta_j-\hbar}{\alpha_a-\beta_j}\right) G_{\bar{J}_1,\cdots,\bar{J}_\ell}(\alpha_1,\cdots,\alpha_\ell)\Big), \tag{7.29}$$

where $\{m_1,\cdots,m_\ell\} := M_1^J, (m_1 < \cdots < m_\ell)$ and $(\bar{J}_1,\cdots,\bar{J}_\ell)$ is defined in (3.34). Here we recall that the set

$$M_1^J = \{r; J_r \geqslant 1\} = \{r; J_r \neq 0\} \tag{7.30}$$

parametrizes the position of non-zero components in $J = (J_1,\cdots,J_n)$, and the values on these componentes are determined from $\bar{J} = (\bar{J}_1,\cdots,\bar{J}_\ell)$ by (3.34).

From (4.2), (4.3) and (4.4), we see that

$$\begin{aligned} G_{\cdots,\epsilon_{k+1},\epsilon_k,\cdots}(\cdots,\alpha_{k+1},\alpha_k,\cdots) = &-\frac{\alpha_k-\alpha_{k+1}}{\alpha_k-\alpha_{k+1}-\hbar}G_{\cdots,\epsilon_k,\epsilon_{k+1},\cdots}(\cdots,\alpha_k,\alpha_{k+1},\cdots) \\ &-\frac{\hbar}{\alpha_k-\alpha_{k+1}-\hbar}G_{\cdots,\epsilon_{k+1},\epsilon_k,\cdots}(\cdots,\alpha_k,\alpha_{k+1},\cdots), \end{aligned} \tag{7.31}$$

$$G_{\epsilon_1,\cdots,\epsilon_\ell}(\alpha_1,\cdots,\alpha_{\ell-1},\alpha_\ell-N\hbar) = (-1)^{\ell-1}G_{\epsilon_\ell,\epsilon_1,\cdots,\epsilon_{\ell-1}}(\alpha_\ell,\alpha_1,\cdots,\alpha_{\ell-1}), \tag{7.32}$$

$$G_{0,\cdots,0,1,\cdots,1,\cdots,N-2,\cdots,N-2}(\alpha_1,\cdots,\alpha_\ell) = \prod_{s=0}^{N-2}\prod_{\substack{a<b \\ (\epsilon_a=s=\epsilon_b)}}(\alpha_a-\alpha_b-\hbar). \tag{7.33}$$

Lemma 7.3 *The following formulae hold:*

$$G_{\cdots,\epsilon_k,\epsilon_{k+1},\cdots}(\cdots,\alpha,\alpha-\hbar,\cdots) = -G_{\cdots,\epsilon_{k+1},\epsilon_k,\cdots}(\cdots,\alpha,\alpha-\hbar,\cdots), \tag{7.34}$$

$$G_{\cdots,[\epsilon_k,\epsilon_{k+1}],\cdots}(\cdots,\alpha,\alpha-\hbar,\cdots) = G_{\cdots,[\epsilon_k,\epsilon_{k+1}],\cdots}(\cdots,\alpha-\hbar,\alpha,\cdots), \tag{7.35}$$

$$\begin{aligned} &G_{\epsilon_1,\cdots,\epsilon_{\ell-N+1},0,1,\cdots,N-2}(\alpha_1,\cdots,\alpha_{\ell-N+1},\beta,\beta-\hbar,\cdots,\beta-(N-2)\hbar) \\ &= (-1)^{\frac{(N-1)(N-2)}{2}m}\prod_{a=1}^{\ell-N+1}(\alpha_a-\beta-\hbar)G_{\epsilon_1,\cdots,\epsilon_{\ell-N+1}}(\alpha_1,\cdots,\alpha_{\ell-N+1}). \end{aligned} \tag{7.36}$$

Proof. It is easy to see (7.34) and (7.35) from (7.31).

Let us prove (7.36). Note that both sides of (7.36) satisfy (7.31) as functions of $\alpha_1,\cdots,\alpha_{\ell-N+1}$. Hence it sufficies to prove that (7.36) holds for

$$(\epsilon_1,\cdots,\epsilon_{\ell-N+1}) = (\underbrace{0,\cdots,0}_{m-1},\cdots,\underbrace{N-2,\cdots,N-2}_{m-1}). \tag{7.37}$$

In the case of $N=2$ this is trivial. In the case of $N=3$ we can prove this from (7.34) and the following formula:

$$\begin{aligned} &G_{\underbrace{0,\cdots,0}_{m-1},\underbrace{1,\cdots,1}_{m},0}(\alpha_1,\cdots,\alpha_{2m}) \\ &= (-1)^{m-1}\prod_{a=1}^{m-1}(\alpha_a-\alpha_{2m}-2\hbar)\prod_{a=m}^{2m-2}(\alpha_a-\alpha_{2m-1}-\hbar)G_{\underbrace{0,\cdots,0}_{m-1},\underbrace{1,\cdots,1}_{m-1}}(\alpha_1,\cdots,\alpha_{2m-2}). \end{aligned} \tag{7.38}$$

This formula can be proved easily from (7.32) and (7.33).

In the case of $N > 3$, from (7.38), we have

$$G_{\underbrace{0,\cdots,0}_{m-1},\underbrace{1,\cdots,1}_{m},0,2,\cdots,2,\cdots,N-2,\cdots,N-2}(\alpha_1,\cdots,\alpha_\ell) \tag{7.39}$$
$$= (-1)^{m-1} \prod_{\substack{a,b \\ (a<b,1\leqslant\epsilon_a=\epsilon_b)}} (\alpha_a - \alpha_b - \hbar)\, G_{\underbrace{0,\cdots,0}_{m},\underbrace{1,\cdots,1}_{m}}(\alpha_1,\cdots,\alpha_{m-1},\alpha_{2m}+\hbar,\alpha_{2m+1},\cdots,\alpha_{3m}).$$

Repeating this calculation, we find

$$G_{\underbrace{0,\cdots,0}_{m-1},\cdots,\underbrace{N-2,\cdots,N-2}_{m-1},N-2,N-1,\cdots,0}(\alpha_1,\cdots,\alpha_\ell)$$
$$= (-1)^{\frac{(N-1)(N-2)}{2}(m-1)} \prod_{s=0}^{N-2} \prod_{\substack{a \\ (\epsilon_a=s,a\leqslant\ell-N+1)}} (\alpha_a - \alpha_{\ell-s} - (N-1-s)\hbar)$$
$$\times G_{\underbrace{0,\cdots,0}_{m-1},\cdots,\underbrace{N-2,\cdots,N-2}_{m-1}}(\alpha_1,\cdots,\alpha_{\ell-N+1}). \tag{7.40}$$

By setting $\alpha_{\ell-s} = \beta - (N-2-s)\hbar$, $(0 \leqslant s \leqslant N-2)$ and using (7.34), we see (7.36) for (7.37). $\square$

Now let us calculate residues of f_{P_m} for P_m satisfying the assumption of Proposition 7.2. It is easy to see that, at each point of taking residues (7.8), (7.10) and (7.11), the coefficient part $e^{\frac{N-1}{2N}n\pi i\sum_j\beta_j}\prod\zeta(\beta_j-\beta_{j'})$ is regular. Hence it suffices to consider residues of ψ_W.

Set

$$\mathrm{RES}_k(F) := \left(2\pi i \mathrm{res}_{\beta_{n-k+1}=\beta_{n-k}-\frac{k+1}{2}\hbar} F\right)|_{\beta_{n-k}\to\beta_{n-k}+\frac{k}{2}\hbar} \tag{7.41}$$

for a function $F = F(\beta_1,\cdots,\beta_{n-k+1})$ and $k = 1,\cdots,N-2$. Then we have

$$f^{(k+1)}(\beta_1,\cdots,\beta_{n-k}) = \mathrm{RES}_k f^{(k)}(\beta_1,\cdots,\beta_{n-k+1}) \tag{7.42}$$

from (7.10).

Lemma 7.4 *Let P_m be a polynomial satisfying the assumption in Proposition 7.2 and W_m is the deformed cycle determined from P_m by (7.17). Suppose that $J \in \mathcal{Z}_{(N-1)m,\cdots,2m,m}$ satisfies $J_a \neq 0$, $(a = n-k,\cdots,n)$ for some k, $(1 \leqslant k \leqslant N-2)$. Then the following formula holds:*

$$\mathrm{RES}_k \circ\cdots\circ \mathrm{RES}_1\,(F_J[W_m]) \tag{7.43}$$
$$= \frac{1}{k!}\prod_{j=1}^{k} a_{m,j} \prod_{j=1}^{n-k-1}\left(e^{\frac{k}{2}(\beta_{n-k}-\beta_j)+\frac{k}{4}\hbar}\prod_{t=0}^{k-1}\Gamma(\frac{\beta_{n-k}-\beta_j+(t-\frac{k}{2})\hbar}{-2\pi i}+1)\Gamma(\frac{\beta_{n-k}-\beta_j+(\frac{k}{2}-t)\hbar}{2\pi i})\right)$$
$$\times\left(\prod_{a=1}^{\ell-k}\int_{C^{(k)}} d\alpha_a\right)\phi^{(k)}(\{\alpha_a\}|\{\beta_j\}_{j\leqslant n-k-1}|\beta_{n-k})w_J^{(k)}(\{\alpha_a\}|\{\beta_j\}_{j\leqslant n-k-1}|\beta_{n-k})W^{(k+1)}(\{e^{-\alpha_a}\}).$$

In the formula above, the functions $\phi^{(k)}, w_J^{(k)}$ and $W^{(k+1)}$ are defined by

$$\phi^{(k)}(\{\alpha_a\}|\{\beta_j\}_{j\leqslant n-k-1}|\beta_{n-k}) := \prod_{a=1}^{\ell-k}\left(\prod_{j=1}^{n-k-1}\frac{\Gamma(\frac{\alpha_a-\beta_j-\hbar}{-2\pi i})}{\Gamma(\frac{\alpha_a-\beta_j}{-2\pi i})}\frac{\Gamma(\frac{\alpha_a-\beta_{n-k}-(\frac{k}{2}+1)\hbar}{-2\pi i})}{\Gamma(\frac{\alpha_a-\beta_{n-k}+\frac{k}{2}\hbar}{-2\pi i})}\right), \tag{7.44}$$

$$w_J^{(k)}(\{\alpha_a\}|\{\beta_j\}_{j\leqslant n-k-1}|\beta_{n-k}) := \mathrm{Skew}\left(g_J^{(k)}(\{\alpha_a\}|\{\beta_j\}_{j\leqslant n-k-1}|\beta_{n-k})\right)$$

$$g_J^{(k)}(\{\alpha_a\}|\{\beta_j\}_{j\leqslant n-k-1}|\beta_{n-k}) := g_{M_1^J\backslash\{n-k+1,\cdots,n\}}(\{\alpha_a\}|\beta_1,\cdots,\beta_{n-k-1},\beta_{n-k}-\frac{k}{2}\hbar) \tag{7.45}$$

$$\times G_{\bar{J}_1,\cdots\bar{J}_{\ell-k-1},[\bar{J}_{\ell-k},\cdots,\bar{J}_\ell]}(\alpha_1,\cdots,\alpha_{\ell-k},\beta_{n-k}+\frac{k}{2}\hbar,\beta_{n-k}+\frac{k-2}{2}\hbar,\cdots,\beta_{n-k}-\frac{k-2}{2}\hbar),$$

$$W^{(k+1)}(\{e^{-\alpha_a}\}) := \frac{P^{(k+1)}(\{e^{-\alpha_a}\})}{\prod_{a=1}^{\ell-k}\left(\prod_{j=1}^{n-k-1}(1-e^{-(\alpha_a-\beta_j)})(1-e^{-(\alpha_a-\beta_{n-k}+\frac{k}{2}\hbar)})\right)}. \tag{7.46}$$

The contour $C^{(k)}$ is a deformation of the real axis $(-\infty,\infty)$ such that the poles at

$$\beta_j+\hbar+2\pi i\mathbb{Z}_{\geqslant 0},\ (1\leqslant j\leqslant n-k-1),\quad \beta_{n-k}+(\frac{k}{2}+1)\hbar+2\pi i\mathbb{Z}_{\geqslant 0} \tag{7.47}$$

are above $C^{(k)}$ and the poles at

$$\beta_j-2\pi i\mathbb{Z}_{\geqslant 0},\ (1\leqslant j\leqslant n-k-1),\quad \beta_{n-k}-\frac{k}{2}\hbar-2\pi i\mathbb{Z}_{\geqslant 0} \tag{7.48}$$

are below $C^{(k)}$. The constant a_r is defined by

$$a_{m,r} := (2\pi i)^{-(Nm-r-2)}e^{-\frac{r(r-3)}{4}\hbar}\Gamma(-\frac{1}{N})\Gamma(\frac{N-r}{N})\prod_{j=1}^{r-1}\Gamma(\frac{N-j-1}{N})\Gamma(\frac{j}{N}). \tag{7.49}$$

Remark. Note that, under the assumption in Proposition 7.4, the residue (7.43) is skew-symmetric with respect to $J_{n-k},\cdots,J_n$ from the definition of $w_J^{(k)}$.

Proof. Let us calculate $\mathrm{RES}_1 F_J[W_m]$. It can be shown that the point $\beta_n=\beta_{n-1}-\hbar$ is a simple pole of $F_J[W]$ for any $W\in\widehat{\mathcal{F}}_q^{\otimes\ell}$ in the same way as the proof of Proposition 3 in [NT]. Hence, in the calculation of the residue, we can replace P_m by $P_m|_{\beta_n=\beta_{n-1}-\hbar}=\prod_{a=1}^{\ell-1}(1-e^{-(\alpha_a-\beta_{n-1})})\widehat{P}^{(1)}$.

Then we consider the integral

$$\left(\prod_{a=1}^{\ell}\int_C d\alpha_a\right)\phi(\{\alpha_a\})w_J(\{\alpha_a\})\frac{\prod_{a=1}^{\ell-1}(1-e^{-(\alpha_a-\beta_{n-1})})\widehat{P}^{(1)}(\{e^{-\alpha_a}\})}{\prod_{a=1}^{\ell}\prod_{j=1}^{n}(1-e^{-(\alpha_a-\beta_j)})}. \tag{7.50}$$

The singularity of this integral at $\beta_n\to\beta_{n-1}-\hbar$ comes from the pinch of the contour C by the poles of the integrand at $\alpha_a=\beta_{n-1}$ and $\alpha_a=\beta_n+\hbar$. Note that the integrand of (7.50) is regular at $\alpha_a=\beta_{n-1}$, $(1\leqslant a\leqslant \ell-1)$. Hence only the contour for α_ℓ may be pinched. The singularity at $\beta_n=\beta_{n-1}-\hbar$ comes from the residue at $\alpha_\ell=\beta_{n-1}$.

Let us rewrite the integrand of (7.50) as follows. First we have

$$\phi(\alpha_\ell|\{\beta_j\})\frac{1}{\prod_{j=1}^n(1-e^{-(\alpha_\ell-\beta_j)})}$$
$$=\prod_{j=1}^n\left((-2\pi i)^{-1}e^{\frac{1}{2}(\alpha_\ell-\beta_j)}\Gamma(\frac{\alpha_\ell-\beta_j-\hbar}{-2\pi i})\Gamma(\frac{\alpha_\ell-\beta_j}{2\pi i}+1)\right). \tag{7.51}$$

When we put $\alpha_\ell=\beta_{n-1}$ in the rhs of (7.51), the factor $\Gamma(\frac{\beta_{n-1}-\beta_n-\hbar}{-2\pi i})$ appears. The singularity of (7.50) at $\beta_n=\beta_{n-1}-\hbar$ comes from this factor.

Expand w_J in (7.50) as follows:

$$w_J=\sum_{\sigma\in S_\ell}(\mathrm{sgn}\sigma)\prod_{a=1}^{\ell}\left(\frac{1}{\alpha_{\sigma(a)}-\beta_{m_a}}\prod_{j=1}^{m_a-1}\frac{\alpha_{\sigma(a)}-\beta_j-\hbar}{\alpha_{\sigma(a)}-\beta_j}\right)G_{\bar{J}}(\{\alpha_{\sigma(a)}\}), \tag{7.52}$$

where $M_1^J=:\{m_1,\cdots,m_\ell\}$, $m_1<\cdots<m_\ell$. It is easy to see that the pinch of the contour for α_ℓ occurs only when $\sigma(\ell-1)=\ell$ or $\sigma(\ell)=\ell$. For such terms, we deform the contour by taking the residue at $\alpha_\ell=\beta_{n-1}$, that is,

$$\int_C(*)d\alpha_\ell=(\text{regular term})+(-2\pi i)\mathrm{res}_{\alpha_\ell=\beta_{n-1}}(*). \tag{7.53}$$

Because the function (7.51) is regular at $\alpha_\ell=\beta_{n-1}$, it suffices to calculate the residue of the rational function (7.52).

Consider the case of $\sigma(\ell-1)=\ell$. Then the residue of (7.52) at $\alpha_\ell=\beta_{n-1}$ is given by

$$(-1)\prod_{j=1}^{n-1}\frac{\beta_{n-1}-\beta_j-\hbar}{\beta_{n-1}-\beta_j} \tag{7.54}$$
$$\times\sum_{\tau\in S_{\ell-1}}(\mathrm{sgn}\tau)\prod_{a=1}^{\ell-2}\left(\frac{1}{\alpha_{\tau(a)}-\beta_{m_a}}\prod_{j=1}^{m_a-1}\frac{\alpha_{\tau(a)}-\beta_j-\hbar}{\alpha_{\tau(a)}-\beta_j}\right)\frac{1}{\alpha_{\tau(\ell-1)}-\beta_n}\prod_{j=1}^{n-1}\frac{\alpha_{\tau(\ell-1)}-\beta_j-\hbar}{\alpha_{\tau(\ell-1)}-\beta_j}$$
$$\times G_{\bar{J}}(\alpha_{\tau(1)},\cdots,\alpha_{\tau(\ell-2)},\beta_{n-1},\alpha_{\tau(\ell-1)}).$$

Here we set $\tau:=\sigma\cdot(\ell-1,\ell)\in S_\ell$. Similarly, we find that the residue in the case of $\sigma(\ell)=\ell$ is given by

$$\prod_{j=1}^{n-1}\frac{\beta_{n-1}-\beta_j-\hbar}{\beta_{n-1}-\beta_j}\sum_{\sigma\in S_{\ell-1}}(\mathrm{sgn}\sigma)\prod_{a=1}^{\ell-1}\left(\frac{1}{\alpha_{\sigma(a)}-\beta_{m_a}}\prod_{j=1}^{m_a-1}\frac{\alpha_{\sigma(a)}-\beta_j-\hbar}{\alpha_{\sigma(a)}-\beta_j}\right)\frac{-\hbar}{\beta_n-\beta_{n-1}}$$
$$\times G_{\bar{J}}(\alpha_{\sigma(1)},\cdots,\alpha_{\sigma(\ell-2)},\alpha_{\sigma(\ell-1)},\beta_{n-1}). \tag{7.55}$$

Then the limit as $\beta_n\to\beta_{n-1}-\hbar$ of the sum of (7.54) and (7.55) is given by

$$(-1)\prod_{j=1}^{n-1}\frac{\beta_{n-1}-\beta_j-\hbar}{\beta_{n-1}-\beta_j}\times \tag{7.56}$$

$$\times \sum_{\sigma\in S_{\ell-1}} (\mathrm{sgn}\sigma) \prod_{a=1}^{\ell-2} \left(\frac{1}{\alpha_{\sigma(a)}-\beta_{m_a}} \prod_{j=1}^{m_a-1} \frac{\alpha_{\sigma(a)}-\beta_j-\hbar}{\alpha_{\sigma(a)}-\beta_j} \right)$$
$$\times \frac{1}{\alpha_{\sigma(\ell-1)}-\beta_{n-1}} \prod_{j=1}^{n-2} \frac{\alpha_{\sigma(\ell-1)}-\beta_{n-1}-\hbar}{\alpha_{\sigma(\ell-1)}-\beta_{n-1}}$$
$$\times \Big(\frac{\alpha_{\sigma(\ell-1)}-\beta_{n-1}-\hbar}{\alpha_{\sigma(\ell-1)}-\beta_{n-1}+\hbar} G_{\bar{J}}(\alpha_{\sigma(1)},\cdots,\alpha_{\sigma(\ell-2)},\beta_{n-1},\alpha_{\sigma(\ell-1)})$$
$$+ G_{\bar{J}}(\alpha_{\sigma(1)},\cdots,\alpha_{\sigma(\ell-2)},\alpha_{\sigma(\ell-1)},\beta_{n-1})\Big).$$

Using (7.31), we have

$$\begin{aligned}
&\frac{\alpha_{\sigma(\ell-1)}-\beta_{n-1}-\hbar}{\alpha_{\sigma(\ell-1)}-\beta_{n-1}+\hbar} G_{\epsilon_1,\cdots,\epsilon_\ell}(\alpha_{\sigma(1)},\cdots,\alpha_{\sigma(\ell-2)},\beta_{n-1},\alpha_{\sigma(\ell-1)}) \\
&+ G_{\epsilon_1,\cdots,\epsilon_\ell}(\alpha_{\sigma(1)},\cdots,\alpha_{\sigma(\ell-2)},\alpha_{\sigma(\ell-1)},\beta_{n-1}) \\
&= \frac{\alpha_{\sigma(\ell-1)}-\beta_{n-1}}{\alpha_{\sigma(\ell-1)}-\beta_{n-1}+\hbar} G_{\epsilon_1,\cdots,\epsilon_{\ell-2},[\epsilon_{\ell-1},\cdots,\epsilon_\ell]}(\alpha_{\sigma(1)},\cdots,\alpha_{\sigma(\ell-2)},\alpha_{\sigma(\ell-1)},\beta_{n-1}). \qquad (7.57)
\end{aligned}$$

From this calculation, it is easy to get the formula (7.43) in the case of $k=1$.

We can prove (7.43) in the case of $k>1$ by a similar calculation. Then we use the following formula

$$\begin{aligned}
&\frac{\alpha_{\ell-k}-\beta-\hbar}{\alpha_{\ell-k}-\beta+k\hbar} G_{\epsilon_1,\cdots,\epsilon_{\ell-k},[\epsilon_{\ell-k+1},\cdots,\epsilon_\ell]}(\alpha_1,\cdots,\alpha_{\ell-k-1},\beta,\alpha_{\ell-k},\beta-\hbar,\cdots,\beta-(k-1)\hbar) \\
&+ \frac{1}{k} G_{\epsilon_1,\cdots,\epsilon_{\ell-k},[\epsilon_{\ell-k+1},\cdots,\epsilon_\ell]}(\alpha_1,\cdots,\alpha_{\ell-k-1},\alpha_{\ell-k},\beta,\beta-\hbar,\cdots,\beta-(k-1)\hbar) \qquad (7.58) \\
&= \frac{1}{k}\frac{\alpha_{\ell-k}-\beta}{\alpha_{\ell-k}-\beta+k\hbar} G_{\epsilon_1,\cdots,\epsilon_{\ell-k-1},[\epsilon_{\ell-k},\cdots,\epsilon_\ell]}(\alpha_1,\cdots,\alpha_{\ell-k-1},\alpha_{\ell-k},\beta,\beta-\hbar,\cdots,\beta-(k-1)\hbar).
\end{aligned}$$

instead of (7.57). This formula can be proved from (7.31) and (7.34). $\square$

Lemma 7.5 *Let P_m be a polynomial satisfying the assumption in Proposition 7.2. Suppose that $J\in\mathcal{Z}_{(N-1)m,\cdots,2m,m}$ satisfies $J_a\neq 0$, $(a=n-k+1,\cdots,n)$ and $J_{n-k}=0$ for some k, $(0\leqslant k\leqslant N-2)$. Fix s such that $1\leqslant s\leqslant N-k-2$. Then*

$$\mathrm{RES}_{k+s}\circ\cdots\circ\mathrm{RES}_1\left(F_J[W_m]\right)=0 \qquad (7.59)$$

if $J_a=0$ for some $a,(n-k-s\leqslant a\leqslant n-k-1)$.

If $J_a\neq 0$, $(a=n-k-s,\cdots,n-k-1)$, the following formula holds:

$$\begin{aligned}
&\mathrm{RES}_{k+s}\circ\cdots\circ\mathrm{RES}_1\left(F_J[W_m]\right) \\
&= \frac{(-1)^s}{k!}\prod_{j=1}^{k+s} a_{m,j} \prod_{j=1}^{n-k-s}\Big(e^{\frac{k+s}{2}(\beta_{n-k-s}-\beta_j)+\frac{k+s}{4}\hbar} \\
&\qquad \times \prod_{t=0}^{k+s-1}\Gamma\Big(\frac{\beta_{n-k-s}-\beta_j+(t-\frac{k+s}{2})\hbar}{-2\pi i}+1\Big)\Gamma\Big(\frac{\beta_{n-k-s}-\beta_j+(\frac{k+s}{2}-t)\hbar}{2\pi i}\Big)\Big)\times
\end{aligned}$$

$$\times \left(\prod_{a=1}^{\ell-k-s} \int_{C^{(k+s)}} d\alpha_a \right) \phi^{(k+s)}(\{\alpha_a\}|\{\beta_j\}_{j\leqslant n-k-s-1}|\beta_{n-k-s})$$
$$\times w_J^{(k,k+s)}(\{\alpha_a\}|\{\beta_j\}_{j\leqslant n-k-s-1}|\beta_{n-k-s}) W^{(k+s+1)}(\{e^{-\alpha_a}\}). \tag{7.60}$$

Here $\psi^{(k+s)}$ and $W^{(k+s+1)}$ are given by (7.44) and (7.46), respectively. The function $w_J^{(k,k')}$ is defined by

$$\begin{aligned} w_J^{(k,k')}(\{\alpha_a\}|\{\beta_j\}_{j\leqslant n-k'-1}|\beta_{n-k'}) &:= \mathrm{Skew}\left(g_J^{(k,k')}(\{\alpha_a\}|\{\beta_j\}_{j\leqslant n-k'-1}|\beta_{n-k'}) \right) \\ g_J^{(k,k')}(\{\alpha_a\}|\{\beta_j\}_{j\leqslant n-k'-1}|\beta_{n-k'}) &:= g_{M_1^J \setminus \{n-k',\cdots,n\}}(\{\alpha_a\}|\beta_1,\cdots,\beta_{n-k'-1}) \\ \times G_{\bar{J}_1,\cdots\bar{J}_{\ell-k-1},[\bar{J}_{\ell-k},\cdots,\bar{J}_\ell]}&(\alpha_1,\cdots,\alpha_{\ell-k'},\beta_{n-k'}+\frac{k'}{2}\hbar,\cdots,\beta_{n-k'}-\frac{k'-2}{2}\hbar). \end{aligned} \tag{7.61}$$

Remark. As in Lemma 7.4, we see that the residue (7.60) is skew-symmetric with respect to $J_{n-k-s},\cdots,J_n$.

Proof. Let us consider the case $J_{n-k-1} \neq 0$ and calculate the residue for $s=1$. From Lemma 7.4, it sufficies to calculate the residue at $\beta_{n-k+1} = \beta_{n-k} - \frac{k+1}{2}\hbar$ of the following integral:

$$\left(\prod_{a=1}^{\ell-k+1} \int_{C^{(k)}} d\alpha_a \right) \phi^{(k-1)}(\{\alpha_a\}|\{\beta_j\}_{j\leqslant n-k}|\beta_{n-k+1})$$
$$\times w_J^{(k-1)}(\{\alpha_a\}|\{\beta_j\}_{j\leqslant n-k}|\beta_{n-k+1}) W^{(k)}(\{e^{-\alpha_a}\}). \tag{7.62}$$

As in the proof of Lemma 7.4, we can replace $W^{(k)}$ by

$$\frac{\prod_{a=1}^{\ell-k}(1-e^{-(\alpha_a-\beta_{n-k})})\widehat{P}^{(k)}(\{e^{-\alpha_a}\})}{\prod_{a=1}^{\ell-k+1}\left(\prod_{j=1}^{n-k}(1-e^{-(\alpha_a-\beta_j)})(1-e^{-(\alpha_a-\beta_{n-k+1}+\frac{k+1}{2}\hbar)})\right)}. \tag{7.63}$$

Then the calculation of the residue is quite similar to that in the proof of Lemma 7.4. The singularity of the integral at $\beta_{n-k+1} = \beta_{n-k} - \frac{k+1}{2}\hbar$ comes from the pinch of the contour by the poles of the integrand at $\alpha_a = \beta_{n-k}$ and $\alpha_a = \beta_{n-k+1} + \frac{k+1}{2}\hbar$. Since (7.63) is regular at $\alpha_a = \beta_{n-k}$, $(1 \leqslant a \leqslant \ell-k)$, only the contour for $\alpha_{\ell-k+1}$ may be pinched.

Expand $w_J^{(k-1)}$ in (7.62) as follows:

$$\begin{aligned} w_J^{(k-1)} = &\sum_{\sigma\in S_{\ell-k+1}} (\mathrm{sgn}\sigma) \prod_{a=1}^{\ell-k} \left(\frac{1}{\alpha_{\sigma(a)}-\beta_{m_a}} \prod_{j=1}^{m_a-1} \frac{\alpha_{\sigma(a)}-\beta_j-\hbar}{\alpha_{\sigma(a)}-\beta_j} \right) \\ &\times \frac{1}{\alpha_{\sigma(\ell-k+1)}-\beta_{n-k+1}+\frac{k-1}{2}\hbar} \prod_{j=1}^{n-k} \frac{\alpha_{\sigma(a)}-\beta_j-\hbar}{\alpha_{\sigma(a)}-\beta_j} \\ &\times G_{\bar{J}_1,\cdots,\bar{J}_{\ell-k},[\bar{J}_{\ell-k+1},\cdots,\bar{J}_\ell]}(\alpha_{\sigma(1)},\cdots,\alpha_{\sigma(\ell-k+1)},\beta_{n-k+1}+\frac{k-1}{2}\hbar,\cdots,\beta_{n-k+1}-\frac{k-3}{2}\hbar). \end{aligned} \tag{7.64}$$

It is easy to see that the pinch of the contour for $\alpha_{\ell-k+1}$ occurs only when $\sigma(\ell-k+1) = \ell-k+1$. By calculating the residue of (7.64) at $\alpha_{\ell-k+1} = \beta_{n-k}$ and taking the limit as $\beta_{n-k+1} \to \beta_{n-k} - \frac{k+1}{2}\hbar$, we get (7.60) for $s=1$.

Repeating this calculation, we find (7.60) for $s > 1$. Note that, if $J_a = 0$ for some a, $(n-k-s \leqslant a \leqslant n-k-1)$, the pinch of the contour does not occur in the limit $\beta_{a+1} \to \beta_a - \frac{n-a+1}{2}\hbar$, and hence (7.59) holds. $\square$

Now we set

$$f^{(k)}(\beta_1, \cdots, \beta_{n-k+1}) := \mathrm{RES}_k \circ \cdots \circ \mathrm{RES}_1 f_{P_m}. \tag{7.65}$$

From Remarks in Lemma 7.4 and Lemma 7.5, we see that

Corollary 7.6

$$f^{(k)}(\beta_1, \cdots, \beta_{n-k+1}) \in (V^{(1)})^{\otimes(n-k)} \otimes V^{(k)}, \quad (k = 2, \cdots, N-1). \tag{7.66}$$

Let us calculate the residue of $f^{(N-1)}$ at $\beta_{n-N+2} = \beta_{n-N+1} - \frac{N}{2}\hbar$. From (7.34) and (7.36), we see that the point $\beta_{n-N+2} = \beta_{n-N+1} - \frac{N}{2}\hbar$ is a simple pole of $f^{(N-1)}$ in the same way as the proof of Proposition 3 in [NT]. Hence it sufficies to calculate the residue of

$$F_J^*[\widehat{W}^{(N-1)}] := \left(\prod_{a=1}^{\ell-N+2} \int_{C^{(N-2)}} d\alpha_a\right) \phi^{(N-2)}(\{\alpha_a\}) w_J^*(\{\alpha_a\}) \widehat{W}^{(N-1)}(\{e^{-\alpha_a}\}), \tag{7.67}$$

where

$$\begin{aligned} * &= (N-2), \quad \text{if} \quad J_{n-a} \neq 0, \ (a = 0, \cdots, N-2), \\ * &= (k, N-2), \quad \text{if} \quad J_{n-k} = 0 \text{ and } J_a \neq 0, \ (1 \leqslant a \leqslant N-2, a \neq k), \end{aligned}$$

and

$$\begin{aligned} \widehat{W}^{(N-1)} &:= \frac{\prod_{a=1}^{\ell-N+2}(1 - e^{-(\alpha_a - \beta_{n-N+1})})(1 - e^{-(\alpha_a - \beta_{n-N+1} - \hbar)})\widehat{P}^{(N-1)}}{\prod_{a=1}^{\ell-N+2}\left(\prod_{j=1}^{n-N+1}(1 - e^{-(\alpha_a - \beta_j)})(1 - e^{-(\alpha_a - \beta_{n-N+2} + \frac{N-2}{2}\hbar)})\right)} \\ &= \prod_{a=1}^{\ell-N+2} \frac{1 - e^{-(\alpha_a - \beta_{n-N+1} - \hbar)}}{\prod_{j=1}^{n-N+2}(1 - e^{-(\alpha_a - \beta_j)})(1 - e^{-(\alpha_a - \beta_{n-N+2} + \frac{N-2}{2}\hbar)})} \widehat{P}^{(N-1)}. \end{aligned} \tag{7.68}$$

Consider the decomposition

$$\widehat{W}^{(N-1)} = \widehat{W}_0 + \widehat{W}_1, \quad \widehat{W}_\delta := \frac{e^{\delta\hbar}(1 - e^{-(\alpha_{\ell-N+2} - \beta_{n-N+2} + (\frac{N}{2} + \delta - 1)\hbar)})}{1 - e^{\hbar}} \widehat{W}^{(N-1)}. \tag{7.69}$$

Set

$$\widehat{\psi} := \widehat{\psi}_0 + \widehat{\psi}_1, \quad \widehat{\psi}_\delta(\beta_1, \cdots, \beta_{n-N+1} | \beta_{n-N+2}) := \sum_J F_J^*[\widehat{W}_\delta] v_J. \tag{7.70}$$

First let us calculate the residue of $\widehat{\psi}_0$. The result is the following.

Lemma 7.7 *The residue of $\widehat{\psi}_0$ is given by*

$$
\begin{aligned}
&2\pi i\mathrm{res}_{\beta_{n-N+2}=\beta_{n-N+1}-\frac{N}{2}\hbar}\widehat{\psi}_0 \\
&= (-1)^{\frac{(N-1)(N-2)}{2}m}(-2\pi i)^{\ell-N+1}a_{m,N-1} \\
&\times \prod_{j=1}^{n-N}\Big(e^{\frac{1}{2}(\beta_{n-N+1}-\beta_j)}\Gamma(\frac{\beta_{n-N+1}-\beta_j-\hbar}{-2\pi i}+1)\Gamma(\frac{\beta_{n-N+1}-\beta_j}{2\pi i})\Big) \\
&\times d_m^{-1}e^{-\frac{(N-1)(2n-N)}{2}\beta_{n-N+1}}\Psi_{P_{m-1}}(\beta_1,\cdots,\beta_{n-N})\otimes v_{[0,1,\cdots,N-1]}.
\end{aligned}
\tag{7.71}
$$

Here $\Psi_{P_{m-1}}$ is the solution of the qKZ equation defined by

$$
\Psi_{P_{m-1}} := \psi_{W_{m-1}}, \quad \text{where} \quad W_{m-1} := \frac{P(\{e^{-\alpha_a}\})}{\prod_{a=1}^{\ell-N+1}\prod_{j=1}^{n-N}(1-e^{-(\alpha_a-\beta_j)})}. \tag{7.72}
$$

Proof. The singularity of $F_J^*[\widehat{W}_0]$ at the point $\beta_{n-N+2} = \beta_{n-N+1} - \frac{N}{2}\hbar$ comes from the pinch of the contour C by poles at $\alpha_a = \beta_{n-N+1} - (N-1)\hbar, \beta_{n-N+1}, \beta_{n-N+1}+\hbar$ and $\alpha_a = \beta_{n-N+2}-(\frac{N}{2}-1)\hbar, \beta_{n-N+2}+\frac{N}{2}\hbar, \beta_{n-N+2}+(\frac{N}{2}+1)\hbar$, respectively. On the other hand we can see that the integrand $\phi^{(N-2)}w_J^*\widehat{W}_0$ is regular at $\alpha_a = \beta_{n-N+1}-(N-1)\hbar, \beta_{n-N+1}, \beta_{n-N+1}+\hbar$, $(1 \leqslant a \leqslant \ell-N+1)$, hence only the contour for $\alpha_{\ell-N+2}$ may be pinched. Moreover, the integrand is regular at $\alpha_{\ell-N+2} = \beta_{n-N+2}-(\frac{N}{2}-1)\hbar, \beta_{n-N+2}+(\frac{N}{2}+1)\hbar$. Therefore the contour for $\alpha_{\ell-N+2}$ may be pinched only by the poles at β_{n-N+1} and $\beta_{n-N+2}+\frac{N}{2}\hbar$. In order to avoid this pinch, we deform the contour C by taking the residue at $\alpha_{\ell-N+2} = \beta_{n-N+1}$ in the same way as the proof of Lemma 7.4.

Then, after the similar calculation to that in the proof of Lemma 7.4, we get the following integral:

$$
\begin{aligned}
2\pi i\mathrm{res}F_J^*[\widehat{W}_0] &= (\text{a certain function of } \beta_1,\cdots,\beta_{n-N+1}) \\
&\times \left(\prod_{a=1}^{\ell-N+1}\int_C d\alpha_a\right)\phi(\{\alpha_a\}|\{\beta_j\}_{1\leqslant j\leqslant n-N})\prod_{a=1}^{\ell-N+1}\frac{1}{\alpha_a-\beta_{n-N+1}-\hbar} \\
&\times w_J^{(k,N-1)}(\{\alpha_a\}|\{\beta_j\}_{j\leqslant n-N}|\beta_{n-N+1})\frac{P_{m-1}(\{e^{-\alpha_a}\})}{\prod_{a=1}^{\ell-N+1}\prod_{j=1}^{n-N}(1-e^{-(\alpha_a-\beta_j)})}.
\end{aligned}
\tag{7.73}
$$

By using (7.34) and (7.36), we get (7.71). □

Next we write down the formula for the residue of $\widehat{\psi}_1$.

Lemma 7.8 *The residue of $\widehat{\psi}_1$ is given by*

$$
\begin{aligned}
&2\pi i\mathrm{res}_{\beta_{n-N+2}=\beta_{n-N+1}-\frac{N}{2}\hbar}\widehat{\psi}_1 \\
&= (-1)^{\frac{(N-1)(N-2)}{2}m}(-2\pi i)^{\ell-N+1}a_{m,N-1} \\
&\times \prod_{j=1}^{n-N}\Big(e^{\frac{1}{2}(\beta_{n-N+1}-\beta_j-(N-1)\hbar)}\Gamma(\frac{\beta_{n-N+1}-\beta_j-N\hbar}{-2\pi i}+1)\Gamma(\frac{\beta_{n-N+1}-\beta_j-(N-1)\hbar}{2\pi i})\Big) \\
&\times R_{n-N+1,n-N}(\beta_{n-N+1}-\beta_{n-N})\cdots R_{n-N+1,1}(\beta_{n-N+1}-\beta_1) \\
&\times e^{(1-\frac{N}{2})\hbar}d_m^{-1}e^{-\frac{(N-1)(2n-N)}{2}(\beta_{n-N+1}-(N-1)\hbar)}\Psi_{P_{m-1}}(\beta_1,\cdots,\beta_{n-N})\otimes v_{[0,1,\cdots,N-1]}.
\end{aligned}
\tag{7.74}
$$

Proof. Note that $\widehat{\psi}_1$ satisfies

$$\widehat{\psi}_1(\cdots,\beta_{i+1},\beta_i,\cdots|\beta_{n-N+2}) = P_{i,i+1}R_{i,i+1}(\beta_i-\beta_{i+1})\widehat{\psi}_1(\cdots,\beta_i,\beta_{i+1},\cdots|\beta_{n-N+2}). \quad (7.75)$$

Hence we have

$$\begin{aligned}\widehat{\psi}_1(\beta_1,\cdots,\beta_{n-N+1}|\beta_{n-N+2}) &= R_{n-N+1,n-N}(\beta_{n-N+1}-\beta_{n-N})\cdots R_{n-N+1,1}(\beta_{n-N+1}-\beta_1)\\ &\quad\times P_{n-N+1,n-N}\cdots P_{2,1}\widehat{\psi}_1(\beta_{n-N+1},\beta_1,\cdots,\beta_{n-N}|\beta_{n-N+2}). \quad (7.76)\end{aligned}$$

The second line of the rhs above is given by

$$\begin{aligned}&P_{n-N+1,n-N}\cdots P_{2,1}\widehat{\psi}_1(\beta_{n-N+1},\beta_1,\cdots,\beta_{n-N}|\beta_{n-N+2}) \quad (7.77)\\ &=\sum_M F^*_{J_{n-N+1},J_1,\cdots,J_{n-N},J_{n-N+2},\cdots,J_n}[\widehat{W}_1](\beta_{n-N+1},\beta_1,\cdots,\beta_{n-N}|\beta_{n-N+2})v_{J_1}\otimes\cdots\otimes v_{J_n}.\end{aligned}$$

Now set $\beta'_{n-N+1}:=\beta_{n-N+1}+2\pi i$ and consider the singularlity of $F^*_{J_{n-N+1},J_1,\cdots,J_{n-N},J_{n-N+2},\cdots,J_n}$ at $\beta_{n-N+2}=\beta'_{n-N+1}+\frac{N}{2}\hbar$. In the same way as the proof of Lemma 7.7, this singularity comes from the pinch of the contour, and we see that only the contour for $\alpha_{\ell-N+2}$ may be pinched by the poles at $\alpha_{\ell-N+2}=\beta'_{n-N+1}+\hbar$ and $\alpha_{\ell-N+2}=\beta_{n-N+2}-(\frac{N}{2}-1)\hbar$. In order to avoid this pinch, we deform the contour by taking the residue at $\alpha_{\ell-N+2}=\beta_{n-N+2}-(\frac{N}{2}-1)\hbar$ in the following way. We rewrite the integrand in $F^*_{J_{n-N+1},J_1,\cdots,J_{n-N},J_{n-N+2},\cdots,J_n}$ in terms of $\beta_1,\cdots,\beta_{n-N},\beta'_{n-N+1}$ and β_{n-N+2} by using

$$\phi(\alpha|\beta_1,\cdots,\beta_{n-N},\beta_{n-N+1}) = \phi(\alpha|\beta_1,\cdots,\beta_{n-N},\beta'_{n-N+1})\frac{\alpha-\beta'_{n-N+1}-N\hbar}{\alpha-\beta'_{n-N+1}-(N+1)\hbar}. \quad (7.78)$$

Then the integrand $\phi^{(N-2)}w^*_J\widehat{W}_1$ is given by

$$\begin{aligned}&\phi^{(N-2)}(\{\alpha_a\}|\{\beta_j\}_{j\leqslant n-N},\beta'_{n-N+1}|\beta_{n-N+2})\widehat{W}_1(\{e^{-\alpha_a}\})\prod_{a=1}^{\ell-N+2}\frac{\alpha_a-\beta'_{n-N+1}-N\hbar}{\alpha_a-\beta'_{n-N+1}-(N+1)\hbar}\\ &\times\sum_{\sigma\in S_{\ell-N+2}}(\mathrm{sgn}\sigma)g^*_J(\{\alpha_{\sigma(a)}\}|\beta'_{n-N+1}+N\hbar,\beta_1,\cdots,\beta_{n-N}|\beta_{n-N+2}). \quad (7.79)\end{aligned}$$

In the case of $J_{n-N+2}>0$, we also change the integration variable $\alpha_{\sigma(1)}\to\alpha_{\sigma(1)}-2\pi i$ and set $\tau:=\sigma\cdot(1,2,\cdots,\ell-N+2)\in S_{\ell-N+2}$. Then we get the following integral:

$$\begin{aligned}&\phi^{(N-2)}(\{\alpha_a\}|\{\beta_j\}_{j\leqslant n-N},\beta'_{n-N+1}|\beta_{n-N+2})\widehat{W}_1(\{e^{-\alpha_a}\}) \quad (7.80)\\ &\times\sum_{\tau\in S_{\ell-N+2}}(-1)^{\ell-N+1}(\mathrm{sgn}\tau)\prod_{a=1}^{\ell-N+1}\frac{\alpha_{\tau(a)}-\beta'_{n-N+1}-N\hbar}{\alpha_{\tau(a)}-\beta_{n-N+1}-(N+1)\hbar}\prod_{j=1}^{n-N}\frac{\alpha_{\tau(\ell-N+2)}-\beta_j-\hbar}{\alpha_{\tau(\ell-N+2)}-\beta_j}\\ &\times\frac{\alpha_{\tau(\ell-N+2)}-\beta_{n-N+2}-\frac{N}{2}\hbar}{\alpha_{\tau(\ell-N+2)}-\beta_{n-N+2}+(\frac{N}{2}-1)\hbar}g^*_J(\{\alpha_{\tau(a)}\}|\beta'_{n-N+1}+N\hbar,\beta_1,\cdots,\beta_{n-N}|\beta_{n-N+2}).\end{aligned}$$

Then we see that the pinch of the contour for $\alpha_{\ell-N+2}$ occurs only when $\sigma(\ell-N+2)=\ell-N+2$ in (7.79), and $\tau(\ell-N+1)=\ell-N+2$ or $\tau(\ell-N+2)=\ell-N+2$ in (7.80). Hence it suffices to calculate the residue at $\alpha_{\ell-N+2}=\beta_{n-N+2}-(\frac{N}{2}-1)\hbar$ for such terms. In this calculation, we use the following formula

$$
\begin{aligned}
&\frac{\alpha-\beta}{\alpha-\beta+N\hbar}G_{\epsilon_1,\cdots,\epsilon_{\ell-N},[\epsilon_{\ell-N+1},\cdots,\epsilon_{\ell-1}],\epsilon_\ell}(\alpha_1,\cdots,\alpha_{\ell-N},\beta-(N-1)\hbar,\beta-\hbar,\cdots,\beta-(N-2)\hbar,\alpha)\\
&\quad+(N-1)G_{\epsilon_1,\cdots,\epsilon_{\ell-N},[\epsilon_{\ell-N+1},\cdots,\epsilon_{\ell-1}],\epsilon_\ell}(\alpha_1,\cdots,\alpha_{\ell-N},\alpha,\beta-\hbar,\cdots,\beta-(N-1)\hbar) \qquad (7.81)\\
&\quad=\frac{\alpha-\beta+(N-1)\hbar}{\alpha-\beta}G_{\epsilon_1,\cdots,\epsilon_{\ell-N},[\epsilon_{\ell-N+1},\cdots,\epsilon_{\ell-1},\epsilon_\ell]}(\alpha_1,\cdots,\alpha_{\ell-N},\alpha,\beta-\hbar,\cdots,\beta-(N-1)\hbar)
\end{aligned}
$$

instead of (7.58). This formula can be obtained from (7.31), (7.34) and (7.35).

After this calculation, we get the following integral:

$$
\begin{aligned}
&\left(\prod_{a=1}^{\ell-N+1}\int_C d\alpha_a\right)\phi(\{\alpha_a\}|\{\beta_j\}_{1\leqslant j\leqslant n-N})\prod_{a=1}^{\ell-N+1}\frac{1}{\alpha_a-\beta_{n-N+1}} \qquad (7.82)\\
&\qquad\times w_J^{(k,N-1)}(\{\alpha_a\}|\{\beta_j\}_{j\leqslant n-N}|\beta_{n-N+1}-\hbar)\frac{P_{m-1}(\{e^{-\alpha_a}\})}{\prod_{a=1}^{\ell-N+1}\prod_{j=1}^{n-N}(1-e^{-(\alpha_a-\beta_j)})}.
\end{aligned}
$$

From (7.34) and (7.36), we get (7.74). $\square$

Note that

$$
\prod_{j=1}^{n-N}\frac{\Gamma(\frac{\beta_{n-N+1}-\beta_j-N\hbar}{-2\pi i}+1)\Gamma(\frac{\beta_{n-N+1}-\beta_j-(N-1)\hbar}{2\pi i})}{\Gamma(\frac{\beta_{n-N+1}-\beta_j-\hbar}{-2\pi i}+1)\Gamma(\frac{\beta_{n-N+1}-\beta_j}{2\pi i})}=\prod_{j=1}^{n-N}S_0(\beta_{n-N+1}-\beta_j). \qquad (7.83)
$$

Therefore we get

$$
\begin{aligned}
&2\pi i\,\mathrm{res}_{\beta_{n-N+2}=\beta_{n-N+1}-\frac{N}{2}\hbar}\widehat{\psi}\\
&=(-1)^{\frac{(N-1)(N-2)}{2}m}a_{m,N-1}(-2\pi i)^{\ell-N+1}\\
&\times\prod_{j=1}^{n-N}\left(e^{\frac{1}{2}(\beta_{n-N+1}-\beta_j)}\Gamma(\frac{\beta_{n-N+1}-\beta_j-\hbar}{-2\pi i}+1)\Gamma(\frac{\beta_{n-N+1}-\beta_j}{2\pi i})\right)\\
&\times d_m^{-1}e^{-\frac{(N-1)(2n-N)}{2}\beta_{n-N+1}}\\
&\times\left(I+e^{-\frac{2\pi i}{N}+\frac{N-1}{N}(n-N)\pi i}S_{n-N+1,n-N}(\beta_{n-N+1}-\beta_{n-N})\cdots S_{n-N+1,1}(\beta_{n-N+1}-\beta_1)\right)\\
&\quad\times\Psi_{P_{m-1}}(\beta_1,\cdots,\beta_{n-N})\otimes v_{[0,1,\cdots,N-1]}. \qquad (7.84)
\end{aligned}
$$

At last we write down the formula for $\mathrm{res}f^{(N-1)}$. Note that, for any regular function $F(\beta_1,\cdots,\beta_n)$, we have

$$
\begin{aligned}
&2\pi i\,\mathrm{res}_{\beta_{n-N+2}=\beta_{n-N+1}-\frac{N}{2}\hbar}\circ\mathrm{RES}_{N-2}\circ\cdots\circ\mathrm{RES}_1(F\psi_W)\\
&=F(\beta_1,\cdots,\beta_{n-N},\beta_{n-N+1},\beta_{n-N+1}-\hbar,\cdots,\beta_{n-N+1}-(N-1)\hbar)\\
&\quad\times 2\pi i\,\mathrm{res}_{\beta_{n-N+2}=\beta_{n-N+1}-\frac{N}{2}\hbar}\circ\mathrm{RES}_{N-2}\circ\cdots\circ\mathrm{RES}_1(\psi_W). \qquad (7.85)
\end{aligned}
$$

By using

$$\prod_{k=0}^{N-1}\zeta(\beta+k\hbar)=\prod_{s=0}^{N-2}\left\{\Gamma_1(\frac{-i\beta+\frac{2(s+1)}{N}\pi}{2\pi})\Gamma_1(\frac{\beta+\frac{2s}{N}\pi}{2\pi})\right\}^{-1}$$

$$=(2\pi)^{\frac{(N-1)(N+1)}{N}}\prod_{s=0}^{N-2}\left\{\Gamma(\frac{\beta+(N-s-1)\hbar}{2\pi i}+1)\Gamma(\frac{\beta+s\hbar}{-2\pi i})\right\}^{-1}, \qquad (7.86)$$

we get

$$2\pi i\mathrm{res}_{\beta_{n-N+2}=\beta_{n-N+1}-\frac{N}{2}\hbar}f^{(N-1)}$$

$$=d_m^{-1}(-i)^{\frac{(N-1)(N-2)}{2}m}e^{\frac{(N-1)m}{2}\pi i}(-2\pi i)^{(N-1)(m-1)}(2\pi)^{(N-1)(N+1)(m-1)}\prod_{s=1}^{N-1}\zeta(s\hbar)^{N-s}\prod_{k=1}^{N-1}a_{m,k}$$

$$\times\Big(I+e^{-\frac{2\pi i}{N}+\frac{N-1}{N}(n-N)\pi i}S_{n-N+1,n-N}(\beta_{n-N+1}-\beta_{n-N})\cdots S_{n-N+1,1}(\beta_{n-N+1}-\beta_1)\Big)$$

$$\times f_{P_{m-1}}(\beta_1,\cdots,\beta_{n-N})\otimes v_{[0,1,\cdots,N-1]}. \qquad (7.87)$$

Hence, if d_m is given by

$$d_m=(-i)^{\frac{(N-1)(N-2)}{2}m}e^{\frac{(N-1)m}{2}\pi i}(-2\pi i)^{(N-1)(m-1)}(2\pi)^{(N-1)(N+1)(m-1)}$$

$$\times\prod_{s=1}^{N-1}\zeta(s\hbar)^{N-s}\prod_{k=1}^{N-1}a_{m,k}, \qquad (7.88)$$

then (7.11) holds. This completes the proof of Proposition 7.2.

7.3 Deformed cycles associated with energy momentum tensor

Hereafter we use the following notation:

$$A_a:=e^{-\alpha_a},\quad B_j:=e^{-\beta_j},\quad\text{and}\quad\omega:=e^{\hbar}=e^{-\frac{2\pi i}{N}}. \qquad (7.89)$$

The n rank-1 particle form factor $f_{\mu\nu}$ of the energy momentum tensor $T_{\mu\nu}$ was determined in [S1]. In terms of our formula, it is given by

$$f_{\mu\nu}(\beta_1,\cdots,\beta_n)=C_0f_{P_{\mu\nu}}(\beta_1,\cdots,\beta_n), \qquad (7.90)$$

where C_0 is a constant independent of n,μ,ν, and $P_{\mu\nu}$ is given by

$$P_{\mu\nu}(A_1,\cdots,A_\ell):=c_m\left(\sum_{j=1}^n B_j^{-1}-(-1)^\mu\sum_{j=1}^n B_j\right) \qquad (7.91)$$

$$\times\left((-1)^{\nu+\frac{(N-1)(N-2)}{2}m}\omega^{-\frac{m(m-1)}{2}}(\prod_{a=1}^{\ell}A_a)^n\,w(A_2^{-1},\cdots,A_\ell^{-1})+w(A_2,\cdots,A_\ell)\right).$$

Note that $n = Nm$ and $\ell = (N-1)m$. Here

$$c_m := \omega^{-\frac{N(N-1)(N-2)}{3}m} \prod_{j=1}^{m} d_j^{-1}, \qquad w(A_2, \cdots, A_\ell) := \prod_{a=2}^{\ell} A_a^{a+[\frac{a-1}{N-1}]}, \tag{7.92}$$

where $[\,\cdot\,]$ is Gauss' symbol.

In this subsection, we prove that $f_{\mu\nu}$ satisfies (7.8), (7.10) and (7.11).

First we consider the case of $m > 1$. Fix m such that $m > 1$, and set

$$P^- := c_m(-1)^{\frac{(N-1)(N-2)}{2}m} \omega^{-\frac{m(m-1)}{2}} (\prod_{a=1}^{\ell} A_a)^n \, w^-(A_1^{-1}, \cdots, A_\ell^{-1}), \quad \text{and} \tag{7.93}$$

$$P^+ := c_m w^+(A_1, \cdots, A_\ell), \tag{7.94}$$

where

$$w^{\pm}(A_1, \cdots, A_\ell) := \prod_{j=0}^{N-1} (1 - \omega^j B_{n-1}^{\mp 1} A_1) w(A_2, \cdots, A_\ell). \tag{7.95}$$

From Corollary 6.5, we have $f_{w^+(\{A_a\})} = f_{w(\{A_a\})}$ and $f_{(\prod_a A_a)^n w^-(\{A_a^{-1}\})} = f_{(\prod_a A_a)^n w(\{A_a^{-1}\})}$. Hence we have

$$f_{P_{\mu\nu}} = f_{P'_{\mu\nu}}, \quad P'_{\mu\nu} := \left(\sum_{j=1}^{n} B_j^{-1} - (-1)^\mu \sum_{j=1}^{n} B_j \right) \left((-1)^\nu P^- + P^+ \right). \tag{7.96}$$

Proposition 7.9 *Set $P_m := P^{\pm}$ and*

$$P_{m-1} := c_{m-1} w(A_2, \cdots, A_{\ell-N+1}), \quad \text{for} \quad P_m = P^+, \tag{7.97}$$

$$\begin{aligned} P_{m-1} := {} & c_{m-1}(-1)^{\frac{(N-1)(N-2)}{2}(m-1)} \omega^{-\frac{(m-1)(m-2)}{2}} \\ & \times (\prod_{a=1}^{\ell-N+1} A_a)^{n-N} \, w(A_2^{-1}, \cdots, A_{\ell-N+1}^{-1}), \quad \text{for} \quad P_m = P^-. \end{aligned} \tag{7.98}$$

Then P_m and P_{m-1} satisfy the assumption in Proposition 7.2 for certain polynomials $\widehat{P}^{(k)}$ and $P^{(k)}$, $(k = 1, \cdots, N-1)$.

Note that

$$\sum_{j=0}^{N-1} (\omega^j B_{n-N+1})^{\pm 1} = 0. \tag{7.99}$$

Therefore Proposition 7.9 implies that $f_{P_{\mu\nu}}$ satisfies (7.8), (7.10) and (7.11) for $m > 1$.

In the proof of Proposition 7.9, we use the following lemmas:

Lemma 7.10 *Set*

$$P_k(A_1,\cdots,A_N|B) := \sum_{j=0}^{k} \omega^{-\frac{j(j-1)}{2}} \begin{bmatrix} k \\ j \end{bmatrix} B^{-j} \prod_{a=1}^{N-j-1} A_a^a \prod_{a=N-j}^{N-1} A_a^{a+1} \cdot A_N^{N+1}. \tag{7.100}$$

for $k=0,\cdots,N-2$. *Here* $\left[\begin{smallmatrix} k \\ j \end{smallmatrix}\right]$ *is defined by*

$$\begin{bmatrix} k \\ j \end{bmatrix} := \frac{(1-\omega^{-k})(1-\omega^{-(k-1)})\cdots(1-\omega^{-(k-j+1)})}{(1-\omega^{-j})(1-\omega^{-(j-1)})\cdots(1-\omega^{-1})}, \tag{7.101}$$

that is, the q-binomial coefficient with $q=\omega^{-1}$. *Then*

$$\mathrm{Skew}\left(P_k|_{B\to\omega B}\right) = \mathrm{Skew}\left(\prod_{a=1}^{N-1}(1-B^{-1}A_a)P_{k+1}\right), \tag{7.102}$$

where Skew *is the skew-symmetrization with respect to* $A_1,\cdots,A_N$.

We can prove this lemma easily by using

$$\begin{bmatrix} k \\ j \end{bmatrix} = \begin{bmatrix} k-1 \\ j-1 \end{bmatrix} + \omega^{-j}\begin{bmatrix} k-1 \\ j \end{bmatrix}. \tag{7.103}$$

It is also easy to see that the following lemma holds:

Lemma 7.11 *Suppose that*

$$c_1B^{-2}\omega + c_2B^{-1}(1+\omega) + c_3 = 0 \tag{7.104}$$

for three constants c_1, c_2 *and* c_3. *Then*

$$\begin{aligned} &\mathrm{Skew}\left(c_1A_2A_3^3 + c_2A_2^2A_3^3 + c_3A_1A_2^2A_3^3\right) \\ &= \mathrm{Skew}\left((1-B^{-1}A_1)(1-\omega B^{-1}A_1)(c_1A_2A_3^3 + c_2A_2^2A_3^3)\right), \end{aligned} \tag{7.105}$$

where Skew *is the skew-symmetrization with respect to* A_1, A_2 *and* A_3.

From Lemma 7.11 we find the following formula:

Lemma 7.12

$$\begin{aligned} &\mathrm{Skew}\left(P_{N-2}(A_1,\cdots,A_N|\omega B)\right) \\ &= \mathrm{Skew}\left(\prod_{a=1}^{N-1}(1-B^{-1}A_a)(1-\omega B^{-1}A_a)P_0(A_1,\cdots,A_N|B)\right). \end{aligned} \tag{7.106}$$

Proof. The lhs is given by

$$P_{N-2}(A_1,\cdots,A_N|\omega B)=\sum_{j=0}^{N-2}\omega^{-\frac{j(j+1)}{2}}\begin{bmatrix}N-2\\ j\end{bmatrix}B^{-j}\prod_{a=1}^{N-j-1}A_a^a\prod_{a=N-j}^{N-1}A_a^{a+1}\cdot A_N^{N+1}. \quad (7.107)$$

It is easy to see that

$$\begin{bmatrix}N-2\\ j\end{bmatrix}\omega+\omega^{-(j+1)}\begin{bmatrix}N-2\\ j+1\end{bmatrix}(1+\omega)+\omega^{-(j+1)-(j+2)}\begin{bmatrix}N-2\\ j+2\end{bmatrix}=0, \quad (7.108)$$

for $j=0,\cdots,N-4$. Hence, from Lemma 7.11, we get

$$\begin{aligned}&P_{N-2}(A_1,\cdots,A_N|\omega B) &(7.109)\\ &\sim\prod_{a=1}^{N-2}(1-B^{-1}A_a)(1-\omega B^{-1}A_a)\prod_{a=1}^{N-1}A_a^a\left(1+\omega^{-1}\begin{bmatrix}N-2\\ 1\end{bmatrix}B^{-1}A_{N-1}\right)A_N^{N+1}.\end{aligned}$$

Here we write $f\sim g$ if $\mathrm{Skew}(f-g)=0$.

Note that

$$1+\omega^{-1}\begin{bmatrix}N-2\\ 1\end{bmatrix}B^{-1}A_{N-1}=(1-B^{-1}A_{N-1})(1-\omega B^{-1}A_{N-1})-\omega B^{-2}A_{N-1}^2. \quad (7.110)$$

Therefore (7.106) holds. □

Set

$$P'_k(A_1,\cdots,A_N|B):=\prod_{j=0}^{N-k-1}(1-\omega^jB^{-1}A_1)P_k(1,A_2,\cdots,A_N|B) \quad (7.111)$$

for $k=0,\cdots,N-2$. Then we can also see that

$$\mathrm{Skew}\left(P'_k|_{B\to\omega B}\right)=\mathrm{Skew}\left(\prod_{a=1}^{N-1}(1-B^{-1}A_a)P'_{k+1}\right),\quad\text{and} \quad (7.112)$$

$$\begin{aligned}&\mathrm{Skew}\left(P'_{N-2}(A_1,\cdots,A_N|\omega B)\right)\\ &=\mathrm{Skew}\left(\prod_{a=1}^{N-1}(1-B^{-1}A_a)(1-\omega B^{-1}A_a)P_0(1,A_2,\cdots,A_N|B)\right).\end{aligned} \quad (7.113)$$

Proof of Proposition 7.9

First let us prove for $P_m=P^+$. Note that

$$\begin{aligned}P^+&=c_mP'_0(A_1,\cdots,A_{N-1},1) &(7.114)\\ &\times\prod_{s=1}^{m-2}\left(\prod_{a=s(N-1)+1}^{(s+1)(N-1)}A_a\right)^{sN}P_0(A_{s(N-1)+1},\cdots,A_{(s+1)(N-1)},1)\prod_{a=\ell-N+2}^{\ell}A_a^{n-\ell-1+a}.\end{aligned}$$

We set

$$\widehat{P}^{(k)} := c_m \prod_{j=0}^{n-k+1+j} (\omega^j B_{n-k})^{n-k+1+j} P'_k(A_1, \cdots, A_{N-1}, 1|B_{n-k}) \tag{7.115}$$
$$\times \prod_{s=1}^{m-2} \left(\prod_{a=s(N-1)+1}^{(s+1)(N-1)} A_a \right)^{sN} P_k(A_{s(N-1)+1}, \cdots, A_{(s+1)(N-1)}, 1|B_{n-k}) \prod_{a=\ell-N+2}^{\ell-k+1} A_a^{n-\ell-1+a},$$

for $k = 1, \cdots, N-2$, and

$$\widehat{P}^{(N-1)} := c_m \prod_{j=0}^{N-3} (\omega^j B_{n-N+1})^{n-N+2+j} P_0(A_1, \cdots, A_{N-1}, 1) \tag{7.116}$$
$$\times \prod_{s=1}^{m-2} \left(\prod_{a=s(N-1)+1}^{(s+1)(N-1)} A_a \right)^{sN} P_0(A_{s(N-1)+1}, \cdots, A_{(s+1)(N-1)}, 1) A_{\ell-N+2}^{n-N+1}.$$

We define $P^{(k)}$ from $\widehat{P}^{(k)}$ by (7.23). Then we can check that $\widehat{P}^{(k)}$ and $P^{(m)}$, $(k = 1, \cdots, N-1)$ satisfy the assumption in Proposition 7.2 by using Lemma 7.10, Lemma 7.12 and

$$P_k(A_1, \cdots, A_N|B) = A_1 P_k(1, A_2, \cdots, A_N|B) = P_k(A_1, \cdots, A_{N-1}, 1|B) A_N^{N+1}. \tag{7.117}$$

We can prove the case of $P_m = P^-$ in a similar way by using

$$\begin{aligned} &\mathrm{Skew}\left(P_k(A_1^{-1}, \cdots, A_N^{-1}|\omega^{-1}B^{-1})\right) \\ &= \mathrm{Skew}\left(\prod_{a=1}^{N-1} (-BA_a^{-1})(1 - B^{-1}A_a) \cdot P_{k+1} \right), \end{aligned} \tag{7.118}$$

and

$$\begin{aligned} &\mathrm{Skew}\left(P_{N-2}(A_1^{-1}, \cdots, A_N^{-1}|\omega^{-1}B^{-1})\right) \\ &= \mathrm{Skew}\left(\prod_{a=1}^{N-1} (\omega^{-1}B^2A_a^{-2})(1 - B^{-1}A_a)(1 - \omega B^{-1}A_a) \cdot P_0(A_1^{-1}, \cdots, A_N^{-1}|B^{-1}) \right) \end{aligned} \tag{7.119}$$

instead of (7.102) and (7.106), respectively. □

At last we show that $f_{\mu\nu}$ satisfies (7.8), (7.10) and (7.11) in the case of $m = 1$. In this case, we set

$$\widehat{P}^{(k)} := P'_k(A_1, \cdots, A_{N-k}, \omega B_{N-k}, \cdots, \omega^{k-1} B_{N-k}, 1|B_{N-k}) \tag{7.120}$$

for $k = 1, \cdots, N-2$. Then the assumption in Proposition 7.2 is satisfied for $P_1 = P^+$ except (7.24) and (7.25). Similarly, we can see that the assumption except (7.24) and (7.25) holds for $P_1 = P^-$. Hence in the same way as the proofs of Lemma 7.4 and Lemma 7.5 we can calculate the residue

$$\mathrm{RES}_{N-2} \circ \cdots \circ \mathrm{RES}_1 f_{P_\mu^\pm}, \tag{7.121}$$

where

$$P_\mu^\pm := \left(\sum_{j=1}^n B_j^{-1} - (-1)^\mu \sum_{j=1}^n B_j \right) P^\pm. \tag{7.122}$$

Then we see that the residue of (7.121) at $\beta_2 = \beta_1 + \pi i$ equals zero because of (7.99).

Therefore, the form factor $f_{P_{\mu\nu}}$ satisfies (7.8), (7.10) and (7.11) for all $m > 0$.

8 Supplements and proofs

8.1 Properties of Smirnov's basis

First we extend the definition of $\omega_{\epsilon_1,\cdots,\epsilon_n}(\beta_1,\cdots,\beta_n)$ in Section 6.1 as follows. For $\epsilon = (\epsilon_1,\cdots,\epsilon_n) \in \mathcal{Z}_{\nu_1,\cdots,\nu_{N-1}}$, we set

$$v_\epsilon := v_{\epsilon_1} \otimes \cdots \otimes v_{\epsilon_n}. \tag{8.1}$$

Define a partial order in $\mathcal{Z}_{\nu_1,\cdots,\nu_{N-1}}$:

$$(\epsilon_1,\cdots,\epsilon_n) \leqslant (\epsilon_1',\cdots,\epsilon_n') \quad \text{if and only if} \quad \sum_{i=1}^r \epsilon_i \leqslant \sum_{i=1}^r \epsilon_i' \quad \text{for all} \quad r. \tag{8.2}$$

We define two elements $\epsilon^{\max}$ and $\epsilon^{\min}$ of $\mathcal{Z}_{\nu_0,\cdots,\nu_{N-1}}$ by

$$\epsilon^{\max} := (\underbrace{N-1,\cdots,N-1}_{\nu_{N-1}},\cdots,\underbrace{1,\cdots,1}_{\nu_1-\nu_2},\underbrace{0,\cdots,0}_{n-\nu_1}), \tag{8.3}$$

$$\epsilon^{\min} := (\underbrace{0,\cdots,0}_{n-\nu_1},\underbrace{1,\cdots,1}_{\nu_1-\nu_2},\cdots,\underbrace{N-1,\cdots,N-1}_{\nu_{N-1}}). \tag{8.4}$$

We define $\{\omega_{\epsilon_1,\cdots,\epsilon_n}(\beta_1,\cdots,\beta_n)\}_{(\epsilon_1,\cdots,\epsilon_n)\in\mathcal{Z}_{\nu_1,\cdots,\nu_{N-1}}}$ by the conditions (6.1) and $\omega_{\epsilon^{\min}} := v_{\epsilon^{\min}}$. Then we see that

$$\omega_{\epsilon_1,\cdots,\epsilon_n}(\beta_1,\cdots,\beta_n) = \prod_{\substack{a<b \\ (\epsilon_a>\epsilon_b)}} \frac{\beta_a-\beta_b}{\beta_a-\beta_b-\hbar} v_{\epsilon_1,\cdots,\epsilon_n} + (\text{lower term}). \tag{8.5}$$

Lemma 8.1 *For $(\epsilon_1,\cdots,\epsilon_n) \in \mathcal{Z}_{\nu_1,\cdots,\nu_{N-1}}$, the following formula holds:*

$$E_k \omega_{\epsilon_1,\cdots,\epsilon_n} = \sum_{\substack{a \\ (\epsilon_a=k)}} \prod_{\substack{b \\ (\epsilon_b=k, b\neq a)}} \frac{\beta_b-\beta_a-\hbar}{\beta_b-\beta_a} \omega_{\epsilon_1,\cdots,\epsilon_a-1,\cdots,\epsilon_n}, \quad (k=1,\cdots N-1). \tag{8.6}$$

Proof. The proof in the case of $N=2$ is given in [S1]. Here let us prove the case of $N>2$.

Note that the action of E_k commutes with that of $R_{i,i+1}$ for all i. Hence we see that both sides satisfy (6.1). Moreover, it can be checked that (8.6) holds for $(\epsilon_1,\cdots,\epsilon_\ell) = \epsilon^{\min}$ by using (8.6) in the case of $N=2$. Therefore, (8.6) holds for all $(\epsilon_1,\cdots,\epsilon_n)$. □

In the rest of this subsection, we use the following simple lemma.

Lemma 8.2 *Suppose that a* $(V_N)^{\otimes n}$*-valued function*

$$F(x_1,\cdots,x_n)=\sum_{(\epsilon_1,\cdots,\epsilon_n)\in\mathcal{Z}_{\nu_1,\cdots,\nu_{N-1}}} F_{\epsilon_1,\cdots,\epsilon_n}(x_1,\cdots,x_n)v_{\epsilon_1}\otimes\cdots\otimes v_{\epsilon_n} \tag{8.7}$$

satisfies

$$F(\cdots,x_{j+1},x_j,\cdots)=P_{j,j+1}R_{j,j+1}(x_j-x_{j+1})F(\cdots,x_j,x_{j+1},\cdots). \tag{8.8}$$

and $F_{\epsilon_1,\cdots,\epsilon_n}=0$ *for some* $(\epsilon_1,\cdots,\epsilon_n)$. *Then* $F=0$.

By using $\{\omega_{\epsilon_1,\cdots,\epsilon_n}\}$, we can get another formula for the special solution at level one (4.19), and prove the highest weight condition as follows.

Proposition 8.3

$$\begin{aligned}&\sum_{(\epsilon_1,\cdots,\epsilon_\ell)\in\mathcal{Z}_{(N-2)m,\cdots,2m,m}} H_{\epsilon_1,\cdots,\epsilon_\ell}(\alpha_1,\cdots,\alpha_\ell)v_{\epsilon_1}\otimes\cdots\otimes v_{\epsilon_\ell}\\ &=\sum_{(\epsilon_1,\cdots,\epsilon_\ell)\in\mathcal{Z}_{(N-2)m,\cdots,2m,m}}\prod_{\substack{a,b\\(\epsilon_a<\epsilon_b)}}\frac{1}{\alpha_a-\alpha_b}\omega_{\epsilon_1,\cdots,\epsilon_\ell}(\alpha_1,\cdots,\alpha_\ell),\end{aligned} \tag{8.9}$$

where $\ell=(N-1)m$. *The function (8.9) satisfies the highest weight condition.*

Proof. From (4.2) and (6.1), it is easy to see that both sides satisfy (8.8).

Now we consider the coefficients of $v_{\epsilon^{\max}}$ of both sides. From (8.5), we can calculate the coefficient of the rhs easily, and see that it suffices to prove that

$$H_{\epsilon^{\max}}=\prod_{\substack{a,b\\(\epsilon_a^{\max}<\epsilon_b^{\max})}}\frac{1}{\alpha_a-\alpha_b+\hbar}. \tag{8.10}$$

First consider the case of $N=3$. Then we can calculate $H_{\epsilon^{\max}}$ explicitly from (4.3) and (4.4), and get (8.10). In the case of $N>3$, we have the following from (8.10) for $N=3$:

$$H_{1,\cdots,1,0,\cdots,0,2,\cdots,2,\cdots,N-2,\cdots,N-2}=\prod_{\substack{a,b\\(\epsilon_a=0,\epsilon_b=1)}}\frac{1}{\alpha_a-\alpha_b+\hbar}\prod_{\substack{a,b\\(\epsilon_a<\epsilon_b,2m<b)}}\frac{1}{\alpha_a-\alpha_b-\hbar}. \tag{8.11}$$

Repeating this calculation, we get (8.10) for $N>3$. In this way we find (8.9) from Lemma 8.2.

Let us prove the highest weight condition for the rhs of (8.9). From (8.1), we have

$$\begin{aligned}E_k(\text{the rhs of }(8.9))=&\sum_{\epsilon\in\mathcal{Z}_{(N-2)m,\cdots,(N-k-1)m-1,\cdots,m}}\omega_\epsilon\prod_{\substack{a,b\\(\epsilon_a<\epsilon_b,(\epsilon_a,\epsilon_b)\neq(k,k+1))}}\frac{1}{\alpha_a-\alpha_b}\\ &\times\sum_{\substack{a\\(\epsilon_a=k-1)}}\prod_{\substack{j\\(\epsilon_j=k-1,j\neq a)}}\frac{1}{\alpha_j-\alpha_a}\prod_{\substack{j,b\\(\epsilon_j=k-1,j\neq a,\epsilon_b=k-1)}}\frac{1}{\alpha_j-\alpha_b}\prod_{\substack{j\\(\epsilon_j=k)}}\frac{\alpha_j-\alpha_a-\hbar}{\alpha_j-\alpha_a}.\end{aligned} \tag{8.12}$$

The second line above eqauls zero from the following lemma. □

Lemma 8.4 *For $x_1, \cdots, x_{r+1}$ and $y_1, \cdots, y_{r-1}$, the following equality holds:*

$$\sum_{s=1}^{r+1} \prod_{j \neq s} \left(\frac{1}{x_j - x_s} \prod_{t=1}^{r-1} \frac{1}{x_j - y_t} \right) \prod_{t=1}^{r-1} \frac{y_t - x_s - \hbar}{y_t - x_s} = 0. \tag{8.13}$$

This lemma is easy to prove by induction on r.

Proof of Proposition 6.1

The case of $N = 2$ is proved in [NPT].

From (3.23) for w_M and (6.1), we see that both sides of (6.6) satisfy (8.8). Hence it suffices to check the coefficients of $v_{\epsilon^{\max}}$ of both sides are equal, that is

$$w_{\epsilon^{\max}} = (-1)^{\frac{\ell(\ell-1)}{2}} \prod_{r=1}^{N-1} \prod_{\substack{a<b \\ (\epsilon_a = r = \epsilon_b)}} \frac{(\beta_b - \beta_a - \hbar)(\beta_a - \beta_b - \hbar)}{\beta_a - \beta_b} \tilde{w}_{\epsilon^{\max}}. \tag{8.14}$$

This equality can be proved by using (8.14) with $N = 2$. $\square$

8.2 Proofs of equalities of rational functions

Proof of Lemma 3.2

Here we set $r_{1,m} := r^J_{1,m-1}$, $(2 \leqslant m \leqslant \nu_1)$ and $r_{2,m} := r^J_{2,m}$, $(1 \leqslant m \leqslant \nu_2)$. We set $r_{1,1} = 0 = r_{2,0}$ and $r_{1,\nu_1+1} = n + 1 = r_{2,\nu_2+1}$.

Define functions f_a, $(1 \leqslant a \leqslant n)$ as follows.

For $r_{1,t} < a < r_{1,t+1}$ such that $r_{2,q} < a < r_{2,q+1}$, we set

$$\begin{aligned} f_a \; &:= \; (-1)^{t-1} g_{M_1^J \cup \{a\}}(\alpha_2, \cdots, \alpha_t, \alpha_1, \alpha_{t+1}, \cdots, \alpha_{\nu_1} | \{\beta_j\}) \\ &\quad \times g_{M_2^J}(\{\gamma_m\} | \alpha_2, \cdots, \alpha_{\nu_1}) \prod_{k=q+1}^{\nu_2} \frac{\gamma_k - \alpha_1 - \hbar}{\gamma_k - \alpha_1}. \end{aligned} \tag{8.15}$$

Note that $\mathrm{Skew} f_a = w^{(3)}_{J_1, \cdots, J_a+1, \cdots, J_n}$.

For $a = r_{1,t}$ such that $r_{2,q} < r_{1,t} < r_{2,q+1}$, set

$$\begin{aligned} f_{r_{1,t}} &:= (-1)^t g_{M_1^J}(\alpha_2, \cdots, \alpha_{\nu_1} | \{\beta_j\}) g_{M_2^J}(\{\gamma_m\} | \alpha_2, \cdots, \alpha_{\nu_1}) \prod_{k=q+1}^{\nu_2} \frac{\gamma_k - \alpha_1 - \hbar}{\gamma_k - \alpha_1} \\ &\times \prod_{j < r_{1,t}} \frac{\alpha_1 - \beta_j - \hbar}{\alpha_1 - \beta_j} \left\{ (\alpha_1 - \alpha_t - \hbar) + \frac{\alpha_1 - \beta_{r_{1,t}} - \hbar}{\alpha_1 - \beta_{r_{1,t}}} (\alpha_t - \alpha_1 - \hbar) \right\} \\ &\times \prod_{b=2}^{t-1} (\alpha_b - \alpha_1 - \hbar) \prod_{b=t+1}^{\nu_1} (\alpha_1 - \alpha_b - \hbar). \end{aligned} \tag{8.16}$$

Note that $f_{r_{1,t}}$ is symmetric with respect to α_1 and α_t.

For $a = r_{2,q} = r_{1,t}$, we set

$$
\begin{aligned}
f_{r_{2,q}} &:= (-1)^t g_{M_1^J}(\alpha_2, \cdots, \alpha_{\nu_1} | \{\beta_j\}) g_{M_2^J}(\{\gamma_m\} | \alpha_2, \cdots, \alpha_{\nu_1}) \\
&\times \prod_{k=q+1}^{\nu_2} \frac{\gamma_k - \alpha_1 - \hbar}{\gamma_k - \alpha_1} \prod_{j<r_{2,q}} \frac{\alpha_1 - \beta_j - \hbar}{\alpha_1 - \beta_j} \prod_{b=2}^{t-1} (\alpha_b - \alpha_1 - \hbar) \prod_{b=t+1}^{\nu_1} (\alpha_1 - \alpha_b - \hbar) \\
&\times \left\{ (\alpha_1 - \alpha_t - \hbar) \frac{\gamma_q - \alpha_1 - \hbar}{\gamma_q - \alpha_1} + \frac{\alpha_1 - \beta_{r_{2,q}} - \hbar}{\alpha_1 - \beta_{r_{2,q}}} (\alpha_t - \alpha_1 - \hbar) \right\}. \qquad (8.17)
\end{aligned}
$$

Note that $f_{r_{2,q}}$ is symmetric with respect to α_1 and α_t.

It is easy to check that

$$
\begin{aligned}
\hbar \sum_{a=1}^{n} f_a &= g_{M_1^J}(\{\alpha_a\}_{2 \leqslant a \leqslant \nu_1} | \{\beta_j\}) g_{M_2^J}(\{\gamma_m\} | \{\alpha_a\}_{2 \leqslant a \leqslant \nu_1}) \qquad (8.18) \\
&\times \left(\prod_{a=2}^{\nu_1} (\alpha_1 - \alpha_a - \hbar) \prod_{m=1}^{\nu_2} \frac{\gamma_m - \alpha_1 - \hbar}{\gamma_m - \alpha_1} - \prod_{j=1}^{n} \frac{\alpha_1 - \beta_j - \hbar}{\alpha_1 - \beta_j} \prod_{a=2}^{\nu_1} (\alpha_1 - \alpha_a + \hbar) \right).
\end{aligned}
$$

By skew-symetrizing both sides above, we have (3.28). $\square$

Proof of Lemma 5.3

The proof is quite similar to that of Lemma 3.2.

We set $M_1^J = \{m_2, \cdots, m_\ell\}$, $m_2 < \cdots m_\ell$, $m_1 = 0, m_{\ell+1} = n+1$ and $\epsilon_r := \bar{J}_{r-1}$, $(2 \leqslant r \leqslant \ell)$.

Define functions f_a, $(1 \leqslant a \leqslant n)$ as follows.

For $m_r < a < m_{r+1}$, we set

$$
\begin{aligned}
f_a := (-1)^{r-1} & g_{M_1^J \cup \{a\}}(\alpha_2, \cdots, \alpha_r, \alpha_1, \alpha_{r+1}, \cdots, \alpha_\ell | \{\beta_j\}) \\
& \times H_{\epsilon_2, \cdots, \epsilon_r, 0, \epsilon_{r+1}, \cdots, \epsilon_\ell}(\alpha_2, \cdots, \alpha_r, \alpha_1, \alpha_{r+1}, \cdots, \alpha_\ell). \qquad (8.19)
\end{aligned}
$$

Note that $\mathrm{Skew} f_a = w_{J_1, \cdots, J_a+1, \cdots, J_n}$.

For $a = m_r$, we set

$$
\begin{aligned}
f_{m_r} &:= (-1)^{r-2} g_{M_1^J}(\alpha_2, \cdots, \alpha_\ell | \{\beta_j\}) \prod_{j<m_r} \frac{\alpha_1 - \beta_j - \hbar}{\alpha_1 - \beta_j} \prod_{b=2}^{r-1} (\alpha_b - \alpha_1 - \hbar) \prod_{b=r+1}^{\ell} (\alpha_1 - \alpha_b - \hbar) \\
&\times \Big\{ (\alpha_1 - \alpha_r - \hbar) H_{\epsilon_2, \cdots, \epsilon_{r-1}, 0, \epsilon_r, \cdots, \epsilon_\ell}(\alpha_2, \cdots, \alpha_{r-1}, \alpha_1, \alpha_r, \cdots, \alpha_\ell) \qquad (8.20) \\
&+ (\alpha_r - \alpha_1 - \hbar) \frac{\alpha_1 - \beta_{m_r} - \hbar}{\alpha_1 - \beta_{m_r}} H_{\epsilon_2, \cdots, \epsilon_r, 0, \epsilon_{r+1}, \cdots, \epsilon_\ell}(\alpha_2, \cdots, \alpha_r, \alpha_1, \alpha_{r+1}, \cdots, \alpha_\ell) \Big\}.
\end{aligned}
$$

From (4.2), it can be checked that f_{m_r} is symmetric with respect to α_1 and α_r.

We can see that

$$
\hbar \sum_{a=1}^{n} f_a = g_{M_1^J}(\{\alpha_a\}_{2 \leqslant a \leqslant \ell} | \{\beta_j\}) \times
$$

$$\times \left\{ \prod_{a=2}^{\ell-1}(\alpha_1 - \alpha_a - \hbar) H_{0,\epsilon_2,\cdots,\epsilon_\ell}(\alpha_1, \alpha_2, \cdots, \alpha_\ell) \right.$$
$$\left. + (-1)^\ell \prod_{a=2}^{\ell}(\alpha_a - \alpha_1 - \hbar) \prod_{j=1}^{n} \frac{\alpha_1 - \beta_j - \hbar}{\alpha_1 - \beta_j} H_{\epsilon_2,\cdots,\epsilon_\ell,0}(\alpha_2, \cdots, \alpha_\ell, \alpha_1) \right\}. \tag{8.21}$$

By using (4.3) and skew-symmetrizing both sides above, we have (5.21). $\square$

Lemma 8.5 *Let $I_s, (s = 0, \cdots, r)$ be sets of indices such that $\# I_s = m$ for all s. Then the following equality holds:*

$$\hbar \sum_{s=1}^{r} \prod_{j \in I_s}(x - y_j - \hbar) \prod_{t=s+1}^{r} \prod_{j \in I_t} \frac{x - y_j - \hbar}{x - y_j} \sum_{k \in I_s} \frac{\prod_{j \in I_0}(y_k - y_j - s\hbar)}{(x - y_k - \hbar)(x - y_k) \prod_{\substack{j \in I_s \\ j \neq k}}(y_k - y_j)}$$
$$+ \frac{\prod_{j \in I_0 \cup \cdots \cup I_r}(x - y_j - \hbar)}{\prod_{j \in I_1 \cup \cdots \cup I_r}(x - y_j)} = \prod_{j \in I_0}(x - y_j - (r+1)\hbar). \tag{8.22}$$

This lemma can be proved by induction on r.

Proof of Proposition 6.2.

Note that both sides of (6.10) are rational functions of α with at most simple poles at points $\beta_j, (j \in K_r, r > 0)$, and have the same growth $O(\alpha^{m-2})$ as $\alpha \to \infty$.

We can see that both sides have the same residue at points $\alpha = \beta_b, (b \in K_r^J, r > 0)$ from Lemma 8.5 with

$$x = \beta_b, \quad I_s = K_s^J \quad \text{and} \quad y_j = \beta_j. \tag{8.23}$$

Moreover, it can be checked that both sides have the same value at points $\alpha = \beta_b + \hbar, (b \in K_q, q > 0)$ from Lemma 8.5 with

$$\begin{aligned} &x = \beta_b + \hbar, \quad I_0 = K_0^J, \quad I_s = K_{q+s}^J, \ (s > 0), \quad r = N - 1, \\ &y_j = \beta_j + r\hbar, \ (j \in K_0^J) \quad \text{and} \quad y_j = \beta_j, \ (j \in K_s^J, s > q). \end{aligned} \tag{8.24}$$

Hence (6.10) holds. $\square$

Lemma 8.6 *Let $I_s, (s = 0, \cdots, d)$ be sets of indices such that $\# I_s = m$ for all s. For $a \in I_0$, the following equality holds:*

$$\sum_{s=0}^{d} \prod_{j \in I_s}(x - y_j - (d+1-s)\hbar) \frac{\prod_{\substack{j \in I_t \\ t < s}}(y_a - y_j - \hbar) \prod_{\substack{j \in I_t \\ t > s}}(y_a - y_j)}{(x - y_a - (d-s)\hbar)(x - y_a - (d+1-s))}$$
$$= \prod_{\substack{j \in I_0 \\ j \neq a}}(y_a - y_j - \hbar) \prod_{\substack{j \in I_t \\ t > 0}}(y_a - y_j) \frac{1}{x - y_a} \prod_{\substack{j \in I_0 \\ j \neq a}} \frac{x - y_j - \hbar}{y_a - y_j - \hbar} \prod_{\substack{j \in I_t \\ t > 0}} \frac{x - y_j - \hbar}{x - y_j} +$$

$$+\hbar\sum_{q=1}^{d}\sum_{k\in I_q}\prod_{\substack{j\in I_q\\ j\neq k}}\frac{y_k-y_j-\hbar}{y_k-y_j}\frac{1}{x-y_k}\prod_{\substack{j\in I_q\\ j\neq k}}\frac{x-y_j-\hbar}{y_k-y_j-\hbar}\prod_{\substack{j\in I_t\\ t>q}}\frac{x-y_j-\hbar}{x-y_j}$$

$$\times\sum_{s=0}^{q-1}\prod_{j\in I_s}(y_k-y_j-(q-s)\hbar)\frac{\prod_{\substack{j\in I_t\\ t<s}}(y_a-y_j-\hbar)\prod_{\substack{j\in I_t\\ t>s}}(y_a-y_j)}{(y_k-y_j-(q-s-1)\hbar)(y_k-y_j-(q-s))}. \quad (8.25)$$

Proof. Let us prove (8.25) by induction on d.

It is easy to see that (8.25) holds in the case of $d=0$.

Suppose that (8.25) holds for $0,1,\cdots,d-1$. First note that the the singularity of the lhs is only the simple pole at $x=y_a$. Hence both sides are rational functions of x with simple poles at points y_a and y_j, $(j\in I_u, u>0)$, and have the same growth $O(x^{m-2})$ as $x\to\infty$. It is easy to see that residues of both sides at $x=y_a$ are equal. We can check that both sides have the same residue also at $x=y_j$, $(j\in I_u, u>0)$ from (8.25) with $d=u-1$. Moreover, both sides have the same value at $x=y_j+\hbar$, $(j\in I_d)$. Therefore (8.25) holds also for d. $\square$

Proof of Proposition 6.3.

Consider the following function $f(\alpha,y)$:

$$f(\alpha,y):=\sum_{s=0}^{N-1}L_J^{(s)}(\alpha+s\hbar)\,T_\hbar^\alpha\left(\frac{U_J^{(s)}(\alpha)-U_J^{(s)}(y-s\hbar)}{\alpha-y+s\hbar}\right), \quad (8.26)$$

where $T_\hbar^\alpha$ is the difference operator $T_\hbar$ with respect to α, and

$$U_J^{(s)}(\alpha):=\prod_{k=0}^{s-1}L_J^{(k)}(\alpha+(s-1)\hbar)\prod_{k=s+1}^{N-1}L_J^{(k)}(\alpha+s\hbar). \quad (8.27)$$

For $a\in K_r^J$, we have

$$\begin{aligned} f(\alpha,\beta_a+N\hbar) &= D\Big(\prod_{\substack{j=1\\ j\neq a}}^{n}(\alpha-\beta_j-N\hbar)\Big) \quad (8.28)\\ &\quad -\hbar\sum_{s=r}^{N-1}L_J^{(s)}(\alpha+s\hbar)\frac{\prod_{k=0}^{s}L_J^{(k)}(\beta_a+(N-1)\hbar)\prod_{k=s+1}^{N-1}L_J^{(k)}(\beta_a+N\hbar)}{(\alpha-\beta_a+(s-N)\hbar)(\alpha-\beta_a+(s-N+1)\hbar)}. \end{aligned}$$

We find that the sum in the rhs of (8.28) equals the rhs of (6.20) by using Lemma 8.6 with

$$I_s=K_{r+s}^J,\quad d=N-r-1,\quad x=\alpha\quad\text{and}\quad y_j=\beta_j. \quad (8.29)$$

On the other hand, we have

$$\frac{U_J^{(s)}(\alpha)-U_J^{(s)}(y-s\hbar)}{\alpha-y+s\hbar}=\sum_{k=1}^{(N-1)m}\left[\frac{U_J^{(s)}(\alpha)}{(\alpha+s\hbar)^k}\right]_+y^{k-1}. \quad (8.30)$$

Hence we get

$$f(\alpha,\beta_a+N\hbar)=\sum_{k=1}^{(N-1)m}(\beta_a+N\hbar)^{k-1}Q_J^{(k)}(\alpha). \tag{8.31}$$

This completes the proof. $\square$

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