# ALGEBRAIC FIBER SPACES WHOSE GENERAL FIBERS ARE OF MAXIMAL ALBANESE DIMENSION

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ABSTRACT. The main purpose of this paper is to prove the Iitaka conjecture $C_{n,m}$ on the assumption that the sufficiently general fibers have maximal Albanese dimension.

## 0. INTRODUCTION

The main purpose of this paper is to prove the Iitaka conjecture $C_{n,m}$ on the assumption that the sufficiently general fibers have maximal Albanese dimension. For the definition of varieties of maximal Albanese dimension, see Definition 2.1 below. The proof of the main theorem was essentially given in [F, Section 7], which is a variant of [Ka2, Proof of Theorem 16].

The following is the main theorem of this paper.

**Theorem 0.1.** *Let $f : X \to Y$ be a proper surjective morphism between non-singular projective varieties with connected fibers. Assume that sufficiently general fibers are of maximal Albanese dimension. Then $\kappa(X) \geq \kappa(Y) + \kappa(X_\eta)$, where $X_\eta$ is the generic fiber of $f$.*

Iitaka's conjecture $C_{n,m}$ was proved on the assumption that the general fibers are of general type by Kollár. For the details, see [Ko] or Theorem 1.2. We use Kollár's theorem to prove Theorem 0.1. By the definition of general types, pluricanonical maps are birational. So, pluricanonical maps don't lose birational properties of varieties of general type. On the other hand, varieties of maximal Albanese dimension have Albanese maps that are generically finite on their images. Therefore, Albanese maps lose properties of varieties of maximal Albanese dimension little. So, it is not surprising that we can prove the above theorem.

We summarize the contents of this paper: Section 1 contains preliminaries. We recall Iitaka's conjecture $C_{n,m}$ and some known results. In Section 2, we define varieties of *maximal Albanese dimension* and

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collect some basic properties of them. Section 3 deals with a canonical bundle formula, which was obtained in [FM] and [F], *semistable parts* and so on. We investigate the relationship between semistable parts and *variations*. In Section 4, we will prove the main theorem. As stated above, the essential part of the proof was contained in [Ka2]. In Section 5, we treat some results about Abelian varieties which we need in the proof of the main theorem.

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We fix the notation used in this paper.

**Notation.** We will work over the complex number field $\mathbb{C}$ throughout this paper.

(i) A *sufficiently general point* $z$ (resp. *subvariety* $\Gamma$) of the variety $Z$ means that $z$ (resp. $\Gamma$) is not contained in the countable union of certain proper Zariski closed subsets.

Let $f : X \to Y$ be a morphism between varieties. A *sufficiently general fiber* $X_y = f^{-1}(y)$ of $f$ means that $y$ is a sufficiently general point in $Y$.

(ii) An *algebraic fiber space* $f : X \to Y$ is a proper surjective morphism between non-singular projective varieties $X$ and $Y$ with connected fibers.

(iii) Let $X$ be a smooth projective variety. If the Kodaira dimension $\kappa(X) > 0$, then we have the Iitaka fibration $f : X' \to Y$, where $X'$ and $Y$ are non-singular projective varieties, $X'$ is birationally equivalent to $X$, and $Y$ is of dimension $\kappa(X)$, such that the sufficiently general fiber of $f$ is smooth, irreducible with $\kappa = 0$. The Iitaka fibration is determined only up to birational equivalence. Since we are interested in questions of a birational nature, we usually assume that $X = X'$ and that $Y$ is smooth. Note that we often modify $f : X \to Y$ birationally without mentioning it. For the basic properties of the Kodaira dimension and the Iitaka fibration, see [Ue, Chapter III] or [Mo, Sections 1,2].

(iv) Let $B_+, B_-$ be the effective $\mathbb{Q}$-divisors on a variety $X$ without common irreducible components such that $B_+ - B_- = B$. They are called the *positive* and the *negative* parts of $B$.

Let $f : X \to Y$ be a surjective morphism. Let $B^h, B^v$ be the $\mathbb{Q}$-divisors on $X$ with $B^h + B^v = B$ such that an irreducible component of Supp$B$ is contained in Supp$B^h$ if and only if it is mapped onto $Y$. They are called the *horizontal* and the *vertical* parts of $B$ over $Y$. A divisor $B$ is said to be *horizontal* (resp. *vertical* ) over $Y$ if $B = B^h$ (resp. $B = B^v$). The phrase "over $Y$" might be suppressed if there is no danger of confusion.

(v) Let $\varphi : V \to W$ be a generically finite morphism between varieties. By the *exceptional locus* of $\varphi$, we mean the subset $\{v \in V | \dim \varphi^{-1}\varphi(v) \geq 1\}$ of $V$, and denote it by $\mathrm{Exc}(\varphi)$.

## 1. Preliminaries

We recall the Iitaka conjecture. The following is a famous conjecture by Iitaka [I1, p.26 Conjecture C]. For the details, see [Mo, Sections 6, 7].

**Conjecture 1.1** (Conjecture $C_{n,m}$). *Let $f : X \to Y$ be an algebraic fiber space with* $\dim X = n$ *and* $\dim Y = m$. *Then we have*

$$\kappa(X) \geq \kappa(Y) + \kappa(X_\eta),$$

*where $X_\eta$ is the generic fiber of $f$.*

We recall some known results about the above conjecture, which will be used in the proof of the main theorem. The following is a part of [Ko, p.363 Theorem]. We note that a simplified proof was obtained by Viehweg (see [V2, Theorem 1.20]).

**Theorem 1.2.** *Let $f : X \to Y$ be an algebraic fiber space such that the generic fiber of $f$ is of general type. Then we have;*

$$\kappa(X) \geq \kappa(Y) + \kappa(X_\eta).$$

Let us recall the notion of *variation* (cf. [V1, p.329]).

**Definition 1.3.** Let $f : X \to Y$ be an algebraic fiber space. The *variation* of $f$, which is denoted by $\mathrm{Var}(f)$, is defined to be the minimal number $k$, such that there exists a subfield $L$ of $\overline{\mathbb{C}(Y)}$ of transcendental degree $k$ over $\mathbb{C}$ and a variety $F$ over $L$ with $F \times_{\mathrm{Spec}(L)} \mathrm{Spec}(\overline{\mathbb{C}(Y)}) \sim X \times_Y \mathrm{Spec}(\overline{\mathbb{C}(Y)})$, where $\sim$ means "birational".

The next theorem is in [Ka2, Corollary 14]. It is also a special case of [Ka3, Corollary 1.2 (ii)]. See also [F, Corollary 7.3].

**Theorem 1.4.** *Let $f : X \to Y$ be an algebraic fiber space such that the geometric generic fiber is birationally equivalent to an Abelian variety. If $\kappa(Y) \geq 0$, then we have*

$$\kappa(X) \geq \max\{\kappa(Y), \mathrm{Var}(f)\}.$$

## 2. Varieties of maximal Albanese dimension

Let us recall the definition of the varieties of maximal Albanese dimension. I learned it from [CH] and [HP].

**Definition 2.1** (Varieties of maximal Albanese dimension)**.** Let $X$ be a smooth projective variety. Let $\mathrm{Alb}(X)$ be the Albanese variety of $X$ and $\mathrm{alb}_X : X \to \mathrm{Alb}(X)$ the corresponding Albanese map. We say that $X$ has *maximal Albanese dimension*, or is *of maximal Albanese dimension*, if $\dim(\mathrm{alb}_X(X)) = \dim X$.

**Remark 2.2.** A smooth projective variety $X$ has maximal Albanese dimension if and only if the cotangent bundle of $X$ is generically generated by its global sections, that is,

$$H^0(X, \Omega^1_X) \otimes_{\mathbb{C}} \mathcal{O}_X \to \Omega^1_X$$

is surjective at the generic point of $X$. It can be checked without any difficulty.

For the basic properties of Albanese mappings, see [Ue, Chapter IV §9].

**Proposition 2.3.** *The following properties are easy to check by the definition.*

(1) *The notion of maximal Albanese dimension is birationally invariant.*
(2) *Let $X$ be an Abelian variety. Then $X \simeq \mathrm{Alb}(X)$. Of course, it has maximal Albanese dimension.*
(3) *Let $X$ be a variety of maximal Albanese dimension. Let $Y$ be a smooth projective variety and $\varphi : Y \to X$ a morphism such that $\dim Y = \dim \varphi(Y)$. If $\varphi(Y) \not\subset \mathrm{Exc}(\mathrm{alb}_X)$, then $Y$ has maximal Albanese dimension.*

*Proof of (3).* By the universality of Albanese mappings, we have the following commutative diagram:

$$\begin{array}{ccc} Y & \xrightarrow{\mathrm{alb}_Y} & \mathrm{Alb}(Y) \\ {\scriptstyle\varphi}\downarrow & & {\scriptstyle\varphi_*}\downarrow \\ X & \xrightarrow{\mathrm{alb}_X} & \mathrm{Alb}(X). \end{array}$$

So, we have that $\varphi_*(\mathrm{alb}_Y(Y)) = \mathrm{alb}_X(\varphi(Y))$. By the assumption, $\dim(\mathrm{alb}_X(\varphi(Y))) = \dim Y$ holds. Therefore, we have that $\dim(\mathrm{alb}_Y(Y)) = \dim Y$. This means that $Y$ has maximal Albanese dimension. $\square$

(4) *Let $X$ and $Y$ be varieties of maximal Albanese dimension. Then, so is $X \times Y$. It is obvious since* $\mathrm{Alb}(X \times Y) \simeq \mathrm{Alb}(X) \times \mathrm{Alb}(Y)$.
(5) *Let $X$ be a variety of maximal Albanese dimension. Then the Kodaira dimension $\kappa(X) \geq 0$. If $\kappa(X) = 0$, then $X$ is birationally equivalent to* $\mathrm{Alb}(X)$ *by* [Ka1, Theorem 1].

The following proposition is [HP, Proposition 2.1].

**Proposition 2.4.** *Let $X$ be a smooth projective variety of maximal Albanese dimension, and let $f\colon X \to Y$ be the Iitaka fibration. We can assume that $Y$ is smooth by using Hironaka's desingularizaion theorem ( see Notation (iii) and Proposition 2.3 (1)). We have the following commutative diagram by the universal property of Albanese varieties:*

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{alb}_X} & \mathrm{Alb}(X) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_*} \\ Y & \xrightarrow{\mathrm{alb}_Y} & \mathrm{Alb}(Y). \end{array}$$

*Then we have:*
(a) *$Y$ has maximal Albanese dimension;*
(b) *$f_*$ is surjective and $\ker f_*$ is connected, of dimension* $\dim(X) - \kappa(X)$*;*
(c) *there exists an Abelian variety $P$ isogenous to $\ker f_*$ such that the sufficiently general fiber of $f$ is birationally equivalent to $P$.*

*Sketch of the proof.* By Proposition 2.3 (3) and (5), we can check that the sufficiently general fibers of $f$ are birationally equivalent to an Abelian variety. By easy dimension count, we can prove (a) and (b) without difficulty. For the details, see [HP, Proposition 2.1]. $\square$

## 3. Semistable parts and Variations

We review the basic definitions and properties of the semistable part $L^{ss}_{X/Y}$ without proof. For the details, we recommend the reader to see [FM, Sections 2, 4] and [F, Sections 3, 4].

**3.1.** Let $f : X \to Y$ be an algebraic fiber space such that the Kodaira dimension of the generic fiber of $f$ is zero, that is, $\kappa(X_\eta) = 0$. We fix the smallest $b \in \mathbb{N}$ such that the $b$-th plurigenus $P_b(X_\eta)$ is non-zero.

**Proposition 3.2** ([FM, Proposition 2.2]). *There exists one and only one $\mathbb{Q}$-divisor $D$ modulo linear equivalence on $Y$ with a graded $\mathcal{O}_Y$-algebra isomorphism*

$$\bigoplus_{i\geq 0} \mathcal{O}_Y(\lfloor iD \rfloor) \simeq \bigoplus_{i\geq 0} (f_*\mathcal{O}_X(ibK_{X/Y}))^{**},$$

*where $M^{**}$ denotes the double dual of $M$.*

*Furthermore, the above isomorphism induces the equality*

$$bK_X = f^*(bK_Y + D) + B,$$

*where $B$ is a $\mathbb{Q}$-divisor on $X$ such that $f_*\mathcal{O}_X(\lfloor iB_+ \rfloor) \simeq \mathcal{O}_Y$ $(\forall i > 0)$ and $\operatorname{codim}_Y f(\operatorname{Supp} B_-) \geq 2$. We note that for an arbitrary open set $U$ of $Y$, $D|_U$ and $B|_{f^{-1}(U)}$ depend only on $f|_{f^{-1}(U)}$.*

*If furthermore $b = 1$ and fibers of $f$ over codimension one points of $Y$ are all reduced, then the divisor $D$ is a Weil divisor.*

**Definition 3.3.** Under the notation of 3.2, we denote $D$ by $L_{X/Y}$. It is obvious that $L_{X/Y}$ depends only on the birational equivalence class of $X$ over $Y$.

The following definition is a special case of [FM, Definition 4.2] (see also [FM, Proposition 4.7]).

**Definition 3.4.** We set $s_P := b(1 - t_P)$, where $t_P$ is the log-canonical threshold of $f^*P$ with respect to $(X, -(1/b)B)$ over the generic point $\eta_P$ of $P$:

$$t_P := \max\{t \in \mathbb{R} \mid (X, -(1/b)B + tf^*P) \text{ is log-canonical over } \eta_P\}.$$

Note that $t_P \in \mathbb{Q}$ and that $s_P \neq 0$ only for a finite number of codimension one points $P$ because there exists a nonempty Zariski open set $U \subset Y$ such that $s_P = 0$ for every prime divisor $P$ with $P \cap U \neq \emptyset$. We note that $s_P$ depends only on $f|_{f^{-1}(U)}$ where $U$ is an open set containing $P$.

We set $L^{ss}_{X/Y} := L_{X/Y} - \sum_P s_P P$ and call it the *semistable part* of $K_{X/Y}$.

We note that $D$, $L_{X/Y}$, $s_P$, $t_P$ and $L^{ss}_{X/Y}$ are birational invariants of $X$ over $Y$.

Putting the above symbols together, we have *the canonical bundle formula* for $X$ over $Y$:

$$bK_X = f^*(bK_Y + L^{ss}_{X/Y}) + \sum_P s_P f^*P + B,$$

where $B$ is a $\mathbb{Q}$-divisor on $X$ such that $f_*\mathcal{O}_X(\lfloor iB_+ \rfloor) \simeq \mathcal{O}_Y$ $(\forall i > 0)$ and $\operatorname{codim}_Y f(\operatorname{Supp} B_-) \geq 2$.

**Definition 3.5** (Canonical cover of the generic fiber)**.** Under the notation of 3.1, consider the following construction. Since $\dim |bK_{X_\eta}| = 0$, there exists a Weil divisor $W$ on $X$ such that

(i) $W^h$ is effective and $f_*\mathcal{O}_X(iW^h) \simeq \mathcal{O}_Y$ for all $i > 0$, and
(ii) $bK_X - W$ is a principal divisor $(\psi)$ for some non-zero rational function $\psi$ on $X$.

Let $s : Z \to X$ be the normalization of $X$ in $\mathbb{C}(X)(\psi^{1/b})$. We call $Z \to X \to Y$ a *canonical cover* of $X \to Y$. We often call $Z' \to X$ a canonical cover after replacing $Z$ with its resolution $Z'$.

By using 3.5 and replacing $f : X \to Y$ birationally, we always make the situation as in 3.6 (see [Mo, (5.15.2)]).

**3.6.** Let $f : X \to Y$ and $h : W \to Y$ be algebraic fiber spaces such that

(i) the algebraic fiber space $f$ is as in 3.1,
(ii) $h$ factors as
$$h : W \xrightarrow{\ g\ } X \xrightarrow{\ f\ } Y,$$
where $g$ is generically finite,
(iii) there is a simple normal crossing divisor $\Sigma$ on $Y$ such that $f$ and $h$ are smooth over $Y_0 := Y \setminus \Sigma$,
(iv) the Kodaira dimension of the generic fiber $W_\eta$ is zero and the geometric genus $p_g(W_\eta) = 1$, where $\eta$ is the generic point of $Y$.

**3.7.** By the definition of $\mathrm{Var}(f)$, there are an algebraic fiber space $f' : X' \to Y'$ with $\overline{\mathbb{C}(Y')} = L$, a generically finite and generically surjective morphism $\pi : \overline{Y} \to Y$ and a generically surjective morphism $\rho : \overline{Y} \to Y'$ such that the induced algebraic fiber space $\overline{f} : \overline{X} \to \overline{Y}$ from $f$ by $\pi$ is birationally equivalent to that from $f'$ by $\rho$ as in the following commutative diagram:

$$\begin{array}{ccccc} X & \longleftarrow & \overline{X} & \longrightarrow & X' \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle \overline{f}} & & \big\downarrow{\scriptstyle f'} \\ Y & \underset{\pi}{\longleftarrow} & \overline{Y} & \underset{\rho}{\longrightarrow} & Y'. \end{array}$$

Furthermore, we can assume that $Y'$ and $\overline{Y}$ are smooth projective varieties. We also assume that there are simple normal crossing divisors $\Sigma$ on $Y$ and $\Sigma'$ on $Y'$ as in 3.6 such that both $\pi^{-1}(\Sigma)$ and $\rho^{-1}(\Sigma')$ are simple normal crossing divisors on $\overline{Y}$. Then by [F, Proposition 4.2], we obtain that
$$\pi^* L^{ss}_{X/Y} = L^{ss}_{\overline{X}/\overline{Y}} = \rho^* L^{ss}_{X'/Y'}.$$
Therefore, we have:

**Theorem 3.8.** *Let $f : X \to Y$ be an algebraic fiber space as in 3.6. Then*
$$\kappa(Y, L^{ss}_{X/Y}) = \kappa(Y', L^{ss}_{X'/Y'}) \leq \dim Y' = \mathrm{Var}(f).$$

**Remark 3.9.** By [Ka3, Theorem 1.1], $\kappa(Y, L^{ss}_{X/Y}) = \mathrm{Var}(f)$ if there exists a good minimal algebraic variety $X_{\overline{\eta},\mathrm{min}}$ defined over $\overline{\mathbb{C}(Y)}$ that is birationally equivalent to the geometric generic fiber $X_{\overline{\eta}}$ over $\overline{\mathbb{C}(Y)}$. For the details, see [Ka3].

**3.10.** In the same situation as in 3.6, we further assume that $\mathrm{Var}(f) = 0$. In this case, $Y'$ is a point and $\overline{X}$ is birationally equivalent to $\overline{Y} \times F$ for a non-singular projective variety $F$. Let $\widetilde{F}$ be a *canonical cover* of $F$ (see Definition 3.5). We note that we applied Definition 3.5 for $F \to \mathrm{Spec}\,\mathbb{C}$. Then

$$bL^{ss}_{\overline{Y}\times\widetilde{F}/\overline{Y}} = L^{ss}_{\overline{Y}\times F/\overline{Y}}$$

by [Mo, (5.15.8)] or [F, Lemma 4.1], where $b$ is the smallest positive integer such that the $b$-th plurigenus $P_b(F) \neq 0$. On the other hand, we can check easily that

$$L^{ss}_{\overline{Y}\times\widetilde{F}/\overline{Y}} \sim 0.$$

Thus, $mL^{ss}_{X/Y} \sim 0$ for some positive integer $m$. Therefore, we summarize;

**Theorem 3.11.** *Let $f : X \to Y$ be an algebraic fiber space as in 3.6. Assume that* $\mathrm{Var}(f) = 0$. *Then we obtain;*

$$mL^{ss}_{X/Y} \sim 0$$

*for some positive integer $m$.*

## 4. Main Theorem

The following is the main theorem of this paper. This says that the Iitaka conjecture $C_{n,m}$ is true on the assumption that the sufficiently general fibers have maximal Albanese dimension.

**Theorem 4.1.** *Let $f : X \to Y$ be an algebraic fiber space. Assume that sufficiently general fibers are of maximal Albanese dimension. Then $\kappa(X) \geq \kappa(Y) + \kappa(X_\eta)$, where $X_\eta$ is the generic fiber of $f$.*

*Proof of the theorem.* If $\kappa(Y) = -\infty$, then the inequality is obviously true. So, we can assume that $\kappa(Y) \geq 0$.

If $\kappa(X_\eta) = 0$, then the geometric generic fiber is birationally equivalent to an Abelian variety by Proposition 2.3 (5). Thus $\kappa(X) \geq \kappa(Y) = \kappa(Y) + \kappa(X_\eta)$ by Theorem 1.4. Therefore, we can assume that $\kappa(X_\eta) > 0$ from now on.

The following lemmas [I2, Lecture 4] are useful. We write it for the reader's convenience (see also [Ka2, Proposition 6]).

**Lemma 4.2** (Induction Lemma). *Under the same notation as in Theorem 4.1, it is sufficient to prove that $\kappa(X) > 0$ on the assumption that $\kappa(Y) \geq 0$ and $\kappa(X_\eta) > 0$.*

*Proof of the lemma.* We use the induction on the dimension of $X$. If $\dim X = 1$, then there is nothing to prove.

Let $\varphi : X \to Z$ be the Iitaka fibration associated to $X$. Since $\kappa(X) > 0$, we have $\dim Z = \kappa(X) > 0$. For a sufficiently general point $z \in Z$, the fiber $X_z = \varphi^{-1}(z)$ has Kodaira dimension zero. We define $f' := f|_{X_z} : X_z \to B = f(X_z)$. We note that the sufficiently general fiber of $f'$ is of maximal Albanese dimension. By induction hypothesis,

$$0 = \kappa(X_z) \geq \kappa(X_{z,y}) + \kappa(B),$$

where $y$ is a sufficiently general point of $B$. By Lemma 4.3 below, we have $\Gamma$, $\widetilde{W}$, $F$ and $G$. Since $\Gamma$ is sufficiently general, we can assume that $z \in \Gamma$. Furthermore, since $\dim \widetilde{W} = \dim Y$, $\kappa(Y) \leq \kappa(\widetilde{W})$ follows. And by the easy addition, we get

$$\begin{aligned} \kappa(\widetilde{W}) &\leq \kappa(F^{-1}(z)) + \dim \Gamma \\ &= \kappa(B) + \dim Y - \dim B. \end{aligned}$$

By hypothesis, $\kappa(Y) \geq 0$ and hence

$$0 \leq \kappa(Y) \leq \kappa(\widetilde{W}) \leq \kappa(B) + \dim Y - \dim B.$$

This implies $\kappa(B) \geq 0$.

On the other hand, $X_{z,y} = f^{-1}(y) \cap \varphi^{-1}(z)$ can be considered as a sufficiently general fiber of $\varphi|_{X_y} : X_y \to \varphi(X_y)$, where $y$ is also a sufficiently general point of $Y$. Thus, $\kappa(X_{z,y}) \geq 0$. More precisely, $X_{z,y}$ is of maximal Albanese dimension. Therefore, we get $\kappa(X_{z,y}) = \kappa(B) = 0$. By the easy addition,

$$\kappa(X_y) \leq \kappa(X_{z,y}) + \dim(\varphi(X_y)).$$

So, $\kappa(X_y) \leq \dim(\varphi(X_y))$. Clearly, we have

$$\begin{aligned} \dim(\varphi(X_y)) &= \dim X_y - \dim X_{z,y} \\ &= \dim X - \dim Y - (\dim X_z - \dim B) \\ &= \dim Z + \dim B - \dim Y \\ &= \kappa(X) + \dim B - \dim Y. \end{aligned}$$

Hence,

$$\begin{aligned} \kappa(X) &\geq \kappa(X_y) + \dim Y - \dim B \\ &\geq \kappa(Y) + \kappa(X_y). \end{aligned}$$

We note that $\kappa(X_\eta) = \kappa(X_y)$. We finish the proof of the lemma. $\square$

The next lemma was already used in the proof of Lemma 4.2. See [I2, p.46].

**Lemma 4.3** (Kawamata). *Let $f : X \to Y$ and $\varphi : X \to Z$ be proper surjective morphisms with connected fibers, where $X$, $Y$, and $Z$ are normal projective varieties. Then there exists a sufficiently general subvariety $\Gamma$ of $Z$, a variety $\widetilde{W}$ and morphisms $F : \widetilde{W} \to \Gamma$, $G : \widetilde{W} \to Y$ such that $F : \widetilde{W} \to \Gamma$ is a proper surjective morphisms with $F^{-1}(z) = f(\varphi^{-1}(z))$, and $G : \widetilde{W} \to Y$ is generically finite.*

*Proof.* Let $\Phi := (f, \varphi) : X \to Y \times Z$ and $V$ be the closure of $\mathrm{Im}\Phi$. Restricting the projection morphisms, we have $p : V \to Y$ and $q : V \to Z$. For $z \in Z$, $q^{-1}(z) = (f(\varphi^{-1}(z)), z) \simeq f(\varphi^{-1}(z))$, and for $y \in Y$, $p^{-1}(y) = (y, \varphi(f^{-1}(y))) \simeq \varphi(f^{-1}(y))$. Hence $p$ is surjective and let $r = \dim V - \dim Y$. If $r = 0$, then $\widetilde{W} := V$ has the required property. If $r > 0$, take a sufficiently general hyperplane section $Z_1$ of $Z$. $V_1 = q^{-1}(Z_1)$ is isomorphic to $V \times_Z Z_1$ and also $V_1 = V \cap (Y \times Z_1)$. Then $p_1 := p|_{V_1} : V_1 \to Y$ satisfies that $V_1$ is a variety and $p_1^{-1}(y) = (y, \varphi(f^{-1}(y)) \cap Z_1)$. Since $Z_1$ is sufficiently general, for a general point $y \in Y$, it follows that $\dim(\varphi(f^{-1}(y)) \cap Z_1) = r - 1$. Repeating this $r$ times, we have $\Gamma := Z_r$ and $\widetilde{W} = V_r$ have the required property. □

*Proof of the theorem continued.* We use the induction with respect to $\dim X$ to prove the main theorem. If the generic fiber $X_\eta$ is of general type, then $\kappa(X) \geq \kappa(Y) + \dim X_\eta > 0$ by Theorem 1.2. So we can assume that $\kappa(X_\eta) < \dim X_\eta$. Let $X \to Z \to Y$ be the relative Iitaka fibration. Then the geometric generic fiber of $g : X \to Z$ is birationally equivalent to an Abelian variety. So, $\kappa(X) \geq \max\{\kappa(Z), \mathrm{Var}(g)\}$ by Theorem 1.4. We note that the sufficiently general fiber of $h : Z \to Y$ has maximal Albanese dimension by Proposition 2.4 (a). Therefore, $\kappa(Z) \geq 0$ by the induction and we can apply Theorem 1.4 to $g : X \to Z$. If the Kodaira dimension of the sufficiently general fiber of $h$ is positive, then $\kappa(Z) > 0$ by the induction. Thus we have $\kappa(X) \geq \kappa(Z) > 0$. So, we can assume that the geometric generic fiber of $h$ is of Kodaira dimension zero. Therefore, $\kappa(Z) \geq \mathrm{Var}(h)$ since the geometric generic fiber is birationally equivalent to an Abelian variety (see Theorem 1.4).

Thus, we can assume that $\mathrm{Var}(h) = \mathrm{Var}(g) = 0$ and the geometric generic fiber of $h$ is birationally equivalent to an Abelian variety.

We shall prove that $\kappa(X, K_{X/Y}) > 0$ for a suitable birational model of $f : X \to Y$. Using [F, Lemma 7.8] and [Ka2, Theorems 8, 9], we reduce it to the case where $Z$ is birationally equivalent to a product $Y \times A$ for an Abelian variety $A$. Thus we come to the following situation:

$$f : X \xrightarrow{g} Z \xrightarrow{\nu} Y \times A \xrightarrow{h_1} Y,$$

where

(a) $A$ is an Abelian variety,
(b) $f$ is the given fiber space, $h_1$ is the projection, and $\nu$ is a proper birational morphism,
(c) there is a simple normal crossing divisor $D$ on $Z$ such that $g$ is smooth over $Z \setminus D$, and $f$ factors as
$$X \xrightarrow{\mu} \widetilde{X} \longrightarrow Y,$$
where $\mu$ is birational and $\widetilde{X}$ is a non-singular projective variety such that $B_-$ is an effective $\mu$-exceptional divisor by [F, Lemma 3.8]. We note that we can apply [F, Lemma 3.8] by the flattening theorem. We note that
$$K_X = g^*(K_Z + L^{ss}_{X/Z}) + \sum_{D_i} s_{D_i} g^* D_i + B,$$
where $D_i$ is an irreducible component of $D$ for every $i$ (see Section 3).

By the canonical bundle formula, we have
$$g_* K^{\otimes m}_{X/Z}(mB_-) \simeq \mathcal{O}_Z(\sum_i m s_{D_i} D_i),$$
where $m$ is a positive integer such that $ms_{D_i}$ are integers for every $i$. We note that we can assume that $mL^{ss}_{X/Z}$ is trivial by Theorem 3.11 since $\mathrm{Var}(g) = 0$. By restricting the canonical bundle formula to $X_y \to Z_y$, where $y$ is a sufficiently general point of $Y$, we obtain an irreducible component $D_0$ of $D$ such that $h_1\nu(D_0) = Y$ and $s_{D_0} \neq 0$ since $\kappa(X_y) = \dim Z_y \geq 1$.

Let $\overline{D}_0$ be the image of $D_0$ on $Y \times A$. Then $\kappa(Y \times A, \overline{D}_0) > 0$ by Corollary 5.4 below. On the other hand, every irreducible component of $\nu^*\overline{D}_0 - D_0$ is $\nu$-exceptional and
$$H^0(Z, \mathcal{O}_Z(ms_{D_0}(D_0 - \nu^*\overline{D}_0)) \otimes K^{\otimes mk}_{Z/Y}) \neq 0$$
for a sufficiently large integer $k$. We note that $K_{Y\times A} = h_1^* K_Y$ since $A$ is an Abelian variety. Combining the above, we obtain
$$H^0(Z, g_* K^{\otimes mk}_{X/Y}(kmB_-) \otimes \mathcal{O}_Z(-ms_{D_0}\nu^*\overline{D}_0)) \neq 0.$$
Therefore,
$$\kappa(X, K_{X/Y}) \geq \kappa(Z, \nu^*\overline{D}_0) = \kappa(Y \times A, \overline{D}_0) > 0.$$
We note that $B_-$ is effective and exceptional over $\widetilde{X}$. Thus, we finish the proof. □

## 5. Some remarks on Abelian varieties

The main purpose of this section is to prove Corollary 5.4, which was already used in the proof of the main theorem. The results below are variants of the theorem of cube.

**5.1.** Let $Y$ be a variety ($Y$ is not necessarily complete) and $A$ an Abelian variety. We define $Z := Y \times A$. Let $\mu : A \times A \to A$ be the multiplication. Then $A$ acts on $A$ naturally by the group law of $A$. This action induces a natural action on $Z$. We write it by $m : Z \times A \to Z$, that is,

$$m : ((y, a), b) \to (y, a + b),$$

where $(y, a) \in Y \times A = Z$ and $b \in A$. Let $p_{1i} : Z \times A \times A \to Z \times A$ be the projection onto the $(1, i)$-th factor for $i = 2, 3$ and $p_{23} : Z \times A \times A \to A \times A$ the projection onto the $(2, 3)$-factor. Let $p : Z \times A \times A \to Z$ be the first projection and $p_i : Z \times A \times A \to A$ the projection onto the $i$-th factor for $i = 2, 3$. We define the projection $\rho : Z = Y \times A \to A$. We fix a section $s : A \to Z$ such that $s(A) = \{y_0\} \times A$ for a point $y_0 \in Y$. We define the morphisms as follows;

$$\begin{aligned} \pi_i &:= p_i \circ (s \times id_A \times id_A) \ \text{ for } \ i = 2, 3, \\ \pi_{23} &:= p_{23} \circ (s \times id_A \times id_A), \\ \pi &:= \rho \times id_A \times id_A. \end{aligned}$$

Let $L$ be a line bundle on $Z$. We define a line bundle $\mathcal{L}$ on $Z \times A \times A$ as follows;

$$\begin{aligned} \mathcal{L} = \ & p^*L \otimes (id_Z \times \mu)^* m^* L \otimes (p_{12}^* m^* L)^{-1} \otimes (p_{13}^* m^* L)^{-1} \\ & \otimes \pi^*((\pi_{23}^* \mu^* s^* L)^{-1} \otimes \pi_2^* s^* L \otimes \pi_3^* s^* L). \end{aligned}$$

**Theorem 5.2.** *Under the above notation, we have that*

$$\mathcal{L} \simeq \mathcal{O}_{Z \times A \times A}.$$

*Proof.* It is not difficult to check that the restrictions $\mathcal{L}$ to each of $Z \times \{0\} \times A$ and $Z \times A \times \{0\}$ are trivial by the definition of $\mathcal{L}$, where $0$ is the origin of $A$. We can also check that the restriction onto $s(A) \times A \times A$ is trivial (cf. [Mu, p.58 Corollary 2]). In particular, $\mathcal{L}|_{\{z_0\} \times A \times A}$ is trivial for any point $z_0 \in s(A) \subset Z$. Therefore, by the theorem of cube [Mu, p.55 Theorem], we obtain that $\mathcal{L}$ is trivial. □

We write $T_a := m|_{Z \times \{a\}} : Z \simeq Z \times \{a\} \to Z$, that is,

$$T_a : (y, b) \to (y, b + a),$$

for $(y, b) \in Y \times A = Z$.

**Corollary 5.3.** *By restricting $\mathcal{L}$ to $Z \times \{a\} \times \{b\}$, we obtain;*

$$L \otimes T^*_{a+b}L \simeq T^*_a L \otimes T^*_b L,$$

*where $a, b \in A$.*

The following is a supplement and a generalization of [F, Lemma 7.11].

**Corollary 5.4.** *Let $D$ be a Cartier divisor on $Z$. Then $2D \sim T^*_a D + T^*_{-a} D$ for $a \in A$. In particular, if $Y$ is complete and $D$ is effective and not vertical with respect to $Y \times A \to Y$, then $\kappa(Z, D) > 0$.*

*Proof.* We put $L = \mathcal{O}_Z(D)$ and $y = -a$. Apply Corollary 5.3. We note that $\mathrm{Supp} D \neq \mathrm{Supp} T^*_a D$ if we choose $a \in A$ suitably. $\square$

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