On Dufresne's relation between the probability laws of exponential functionals of Brownian motions with different drifts

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Abstract Denote by $\alpha_t^{(\mu)}$ the probability law of $A_t^{(\mu)} = \int_0^t \exp(2(B_s+\mu s))ds$ for a Brownian motion $\{B_s, s \geqq 0\}$. It is well known that $\alpha_t^{(\mu)}$ is of interest in a number of domains, e.g., mathematical finance, diffusion processes in random environments, stochastic analysis on the hyperbolic spaces and so on, but that it has complicated expressions. Recently, Dufresne [13] obtained some remarkably simple expressions for $\alpha_t^{(0)}$ and $\alpha_t^{(1)}$, as well as some equally remarkable relation between $\alpha_t^{(\mu)}$ and $\alpha_t^{(\nu)}$ for two different drifts μ and ν. In this paper, hinging on our previous results about $\alpha_t^{(\mu)}$, we give different proofs of Dufresne's results, and present extensions of them for the *processes* $\{A_t^{(\mu)}, t \geqq 0\}$.

1. Introduction

1.1. Let $B = \{B_t, t \geqq 0\}$ be a one-dimensional Brownian motion and, for $\mu \in \mathbf{R}$, $B^{(\mu)} = \{B_t^{(\mu)} \equiv B_t + \mu t, t \geqq 0\}$ a Brownian motion with a constant drift. We consider the Brownian exponential functional

$$A_t^{(\mu)} = \int_0^t \exp(2B_s^{(\mu)})\, ds, \qquad t > 0,$$

which plays an important role in a number of researches, e.g., mathematical finance ([14, 32]), diffusion processes in random environments ([8, 17]) and stochastic analysis on hyperbolic spaces ([15, 16]). Here, we cite only a very restricted number of related works. For a more complete list, see the references in these papers, as well as in [29], [33] and so on.

The original motivation for the present paper has been to compare several apparently different forms for the density of $A_t^{(\mu)}$ for fixed $t > 0$. The density (2.1) of $(A_t^{(\mu)}, B_t^{(\mu)})$ presented below has been obtained in Yor [32] and has been used in the works cited above; hence, the one-dimensional marginal of $A_t^{(\mu)}$ may be obtained by integrating both hand sides of (2.1) with respect to dx.

More recently, Dufresne [13] obtained other expressions for the density of $A_t^{(\mu)}$. One of the aims of this paper is to show that these two expressions are indeed equal, as it should be. This may be of some intrinsic interest; we also note that some authors, e.g., [3, 5, 20, 25], keep trying to obtain better expressions, e.g., for accurate implementation of Asian option prices.

1.2. On the other hand, we have studied, in previous works, the probability laws of the stochastic processes $A^{(\mu)} = \{A_t^{(\mu)}, t \geqq 0\}$, $\mu \in \mathbf{R}$, which led us, among other results, to an extension of the celebrated Pitman $2M-X$ theorem, replacing twice the maximum $2M_t \equiv 2\max_{0\leqq s\leqq t} X_s, t > 0$, by $\log\bigl(\int_0^t \exp(2X_s)ds\bigr)$. For precise statements and proofs, see [21, 22, 23, 24, 9]. See also [10] for a somewhat different summary and developments. Here, we give some short summary of our previous results which are related to the present work.

(i) An important relation between $A^{(-\mu)}$ and $A^{(\mu)}$ is the following identity: for $\mu > 0$,

$$\{\frac{1}{A_t^{(-\mu)}}, t > 0\} \overset{\text{(law)}}{=} \{\frac{1}{A_t^{(\mu)}} + 2\gamma_\mu, t > 0\}, \tag{1.1}$$

where γ_μ is a Gamma(μ) random variable, independent of $\{A_t^{(\mu)}, t \geqq 0\}$.

(ii) We then considered the stochastic process $Z^{(\mu)} = \{Z_t^{(\mu)} \equiv \exp(-B_t^{(\mu)})A_t^{(\mu)}, t \geqq 0\}$, which appears naturally from (1.1) since

$$\frac{d}{dt}\Bigl(\frac{1}{A_t^{(\pm\mu)}}\Bigr) = -\frac{1}{(Z_t^{(\pm\mu)})^2}.$$

In particular, it follows from (1.1) that $Z^{(-\mu)} \overset{\text{(law)}}{=} Z^{(\mu)}$. Moreover, both processes are Markovian with respect to their own respective filtration, with common infinitesimal generator

$$\mathcal{L}^{(\mu)} = \frac{1}{2}z^2\frac{d^2}{dz^2} + \Bigl\{\Bigl(\frac{1}{2}-\mu\Bigr)z + \Bigl(\frac{K_{1+\mu}}{K_\mu}\Bigr)\Bigl(\frac{1}{z}\Bigr)\Bigr\}\frac{d}{dz}.$$

Of particular interest to us is the fact that, for any fixed $t > 0$, the σ-field $\mathcal{F}_t^+ \equiv \sigma\{B_s^{(\mu)}, s \geqq t\}$ of the future of $B^{(\mu)}$ and the σ-field $\mathcal{Z}_t^- \equiv \sigma\{Z_s^{(\mu)}, s \leqq t\}$ of the past of $Z^{(\mu)}$ are conditionally independent given the variable $Z_t^{(\mu)}$.

Further extensions of this Markov property have been obtained by Baudoin [6].

(iii) In a subsequent work [9], we noticed that the stochastic process $Z^{(\mu)}$ is invariant under certain transforms on path space: precisely, letting $\alpha > 0$ and defining the transform T_α by

$$T_\alpha(\phi)_t = \phi(t) - \log(1 + \alpha A(\phi)_t),$$

we have $Z(T_\alpha(\phi))_t = Z(\phi)_t$, where

$$Z(\phi)_t = \exp(-\phi(t))A(\phi)_t \qquad \text{and} \qquad A(\phi)_t = \int_0^t \exp(2\phi(s))\, ds.$$

This prompted us ([9], Theorem 1.5) to prove an absolute continuity relationship between the laws of $\{T_\alpha(B^{(\mu)})_s, s \leqq t\}$ and $\{B_s^{(\mu)}, s \leqq t\}$. We also considered variants of T_α, which "anticipate" upon the Brownian filtration and which led us to another simple proof of the results mentioned in (ii).

Recently, Yano [28] has discussed these transforms in a general framework.

1.3. We now indicate how the rest of this paper is organized. In Section 2 we recall the expression of the density of $(A_t^{(\mu)}, B_t^{(\mu)})$ following Yor [32] and the expression by Dufresne [13] for the density of $A_t^{(\mu)}$. For clarity, we denote by $f_Y^{(\mu)}(a,t)$, respectively $f_D^{(\mu)}(a,t)$, the expressions of the density of $1/2A_t^{(\mu)}$ following Yor and Dufresne, and we simply write $f^{(\mu)}(a,t)$ when refering to the density of $1/2A_t^{(\mu)}$ without using explicit expression from either [32] or [13]. In subsection 2.3, we also gather a number of facts about generalized inverse Gaussian (GIG) laws, which are closely related to the Markov property of $Z^{(\mu)}$ mentioned in (ii) above and which will be very helpful in the sequel.

In Section 3 we show, using residue calculus, $f_Y^{(0)} = f_D^{(0)}$ and also $f_Y^{(1)} = f_D^{(1)}$. In the course of this study, we obtain some interesting identities involving the centered Gaussian law with variance t, which may be of interest by themselves and may be written as

$$E[H(B_t)\sin\big(\frac{\pi B_t}{t}\big)] = K(t)$$

for certain functions H and K. As particularly simple examples, we now present the following:

$$E[\tanh(B_t)\sin\big(\frac{\pi B_t}{t}\big)] = 2\pi\exp\big[-\frac{3\pi^2}{8t}\big],$$
$$E[(B_t)^{2n+1}\sin\big(\frac{\pi B_t}{t}\big)] = (-1)^n\sqrt{t}^{2n+1}H_{2n+1}\big(\frac{\pi}{\sqrt{t}}\big),$$

where H_{2n+1} is the $(2n+1)$-th Hermite polynomial. The first formula follows from (3.1) below. The second one follows by differentiating a number of times both hand sides of the elementary identity $E[\exp(i\alpha B_t)] = \exp(-\alpha^2 t/2)$ in α.

In Section 4, rather than trying to prove directly $f_Y^{(\mu)} = f_D^{(\mu)}$ for general $\mu \in \mathbf{R}$, we interpret a recurrence relation between $f_D^{(\mu)}$ and $f_D^{(\nu)}$, which was found by Dufresne [13], and we give an independent proof of this relation. In fact, we lift this Dufresne's recurrence formula for $f^{(\mu)}$ to the level of the processes $\{(A_t^{(\mu)})^{-1}, t > 0\}$ and $\{(A_t^{(\nu)})^{-1}, t > 0\}$, which represents one of our main contributions in this paper. The following result may be also considered as an extension of the main result, Theorem 1.1, in [24], where we considered the case $\mu+\nu=0$.

Theorem 1.1. *Let $\mu<\nu$ and set $\delta=(\nu-\mu)/2$ and $m=(\mu+\nu)/2$. Then, for every $t\geqq 0$ and for every non-negative functional F on $C([0,t]\to\mathbf{R})$, one has*

$$e^{\mu^2t/2}E\big[F\big(\frac{1}{A_s^{(\mu)}},s\leqq t\big)\big(\frac{1}{A_t^{(\mu)}}\big)^m\big]=e^{\nu^2t/2}E\big[F\big(\frac{1}{A_s^{(\nu)}}+2\gamma_\delta,s\leqq t\big)\big(\frac{1}{A_t^{(\nu)}}\big)^m\big], \tag{1.2}$$

where γ_δ is a Gamma(δ) *random variable independent of $\{B_s^{(\nu)}\}$.*

In Section 5 we seek a better understanding of Theorem 1.1 in two directions:
(a) For $\mu=0$, there is the simple and beautiful Bougerol identity in law,

$$\sinh(B_t)\overset{\text{(law)}}{=}\beta_{A_t^{(0)}}\qquad\text{for every fixed } t>0, \tag{1.3}$$

where, on the right hand side, $\{\beta_u,u\geqq 0\}$ is a Brownian motion independent of $A_t^{(0)}$. We show that some extensions of this identity are closely connected with the particular case of Theorem 1.1 when $\mu+\nu=1$.
(b) We consider the identity (1.2) in Theorem 1.1 and look for some extensions of it, which would feature some suitable pairs $\psi_\mu(A_t^{(\mu)},\exp(B_t^{(\mu)}))$ and $\psi_\nu(A_t^{(\nu)},\exp(B_t^{(\nu)}))$ instead of featuring the densities $(A_t^{(\mu)})^{-m}$ and $(A_t^{(\nu)})^{-m}$ on either side of (1.2). The result (Proposition 5.1) develops further our previous work in [9] already mentioned in (iii).

Section 6 consists of two further remarks. First we discuss in terms of GIG laws how Dufresne arrives at his recurrence formula. Next we show how the complexity of the different formulae for $f^{(\mu)}$, namely $f_Y^{(\mu)}$ and $f_D^{(\mu)}$, is related to Stieltjes' discussion in his celebrated mémoire [26] on the indeterminacy of the momemt problem for the log-normal distribution.

Our paper closes with two appendices. We devote Appendix A to show that Dufresne's identity, i.e., formula (1.2) written between $f((A_t^{(\mu)})^{-1})$ and $f((A_t^{(\nu)})^{-1}+2\gamma_\delta)$, may be obtained from the beta-gamma algebra;

$$\frac{\gamma_a}{\gamma_a+\gamma_b}\ \big(\overset{\text{(law)}}{=}\beta_{a,b}\big)\text{ is independent of }\ \gamma_a+\gamma_b\ \big(\overset{\text{(law)}}{=}\gamma_{a+b}\big), \tag{1.4}$$

where γ_a is a gamma random variable with parameter a, γ_a and γ_b are independent and $\beta_{a,b}$ is a beta variable with parameters (a,b); to achieve this, we make also use of either the representation (2.1) of the law of $(A_t^{(\mu)},B_t^{(\mu)})$ or the result in Yor [31] about the law of $A^{(\mu)}$ taken at an independent exponential random time.

In Appendix B, we develop some further extension of Theorem 1.1 and Proposition 5.1, by replacing the gamma variable γ_δ in (1.2) with a general random variable Y, independent of the underlying Brownian motion. We also show how these relationships are connected with certain independence properties of the GIG laws, which have already been discussed in [23] and [19].

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2. Preliminaries

2.1. Some results from Yor [32]**.** The law of $A_t^{(\mu)}$ may be obtained from the following expression for the density of $(A_t^{(\mu)}, B_t^{(\mu)})$:

$$P(A_t^{(\mu)} \in d\alpha, B_t^{(\mu)} \in dx) = e^{\mu x - \mu^2 t/2} \frac{1}{\alpha} \exp\Big[-\frac{1+e^{2x}}{2\alpha}\Big] \theta(e^x/\alpha, t)\, d\alpha\, dx, \quad \alpha > 0, x \in \mathbf{R}, \tag{2.1}$$

where

$$\theta(r,t) = \frac{r}{(2\pi^3 t)^{1/2}} \int_0^\infty e^{(\pi^2-\xi^2)/2t} e^{-r\cosh(\xi)} \sinh(\xi) \sin\Big(\frac{\pi\xi}{t}\Big)\, d\xi. \tag{2.2}$$

In fact, the function $\theta(r,t)$ is related, via the Laplace transform, to the usual modified Bessel functions $I_\nu(r)$: precisely,

$$\int_0^\infty e^{-\lambda t}\theta(r,t)\, dt = I_{\sqrt{2\lambda}}(r), \qquad \lambda \geqq 0,$$

and formula (2.2) follows from the integral representation of $I_\nu(r)$ along some contours in the complex plane (see Yor [30]).

It is more convenient for our purpose to consider $1/2A_t^{(\mu)}$ and we denote by $f^{(\mu)}(a,t)$ the density of $1/2A_t^{(\mu)}$ with respect to the Lebesgue measure da on $\mathbf{R}_+$. Then, using (2.1) and changing variables with $\sqrt{a}e^x = \eta$, we get

$$\begin{aligned} f^{(\mu)}(a,t) =& e^{-\mu^2 t/2} C_t e^{-a} a^{-(\mu+1)/2} \int_0^\infty \eta^\mu e^{-\eta^2}\, d\eta \\ & \times \int_0^\infty e^{-\xi^2/2t} \exp[-2\sqrt{a}\eta\cosh(\xi)] \sinh(\xi) \sin\Big(\frac{\pi\xi}{t}\Big)\, d\xi, \end{aligned} \tag{2.3}$$

where $C_t = 2(2\pi^3 t)^{-1/2} e^{\pi^2/2t}$. It is not hard to show, as a consequence of Stieltjes' remark (6.3) discussed in subsection 6.2 below, that, for any fixed $t > 0$, $\theta(r,t) = O(r^N)$ as $r \downarrow 0$ for any $N > 0$, since formula (6.3) yields

$$\frac{d^n}{dr^n}\theta(r,t)\bigg|_{r=0} = 0 \quad \text{for any } n = 1, 2, ...$$

Hence, the integral on the right hand side of (2.3) is convergent also when $\mu \leqq -1$.

To distinguish the above expression (2.3) from Dufresne's expression for $f^{(\mu)}$, given below in subsection 2.2, we denote temporarily the right hand side of (2.3) by $f_Y^{(\mu)}(a,t)$. Using the elementary formula

$$e^{-\sqrt{\lambda}\eta} = \frac{\eta}{\sqrt{2}} \int_0^\infty \exp\big[-\lambda s - \frac{\eta^2}{4s}\big] \frac{ds}{\sqrt{2\pi s^3}}, \qquad \lambda > 0, \eta > 0, \tag{2.4}$$

we also have

$$\begin{aligned} f_Y^{(\mu)}(a,t) =& e^{-\mu^2 t/2} \frac{C_t}{2\sqrt{\pi}} \Gamma\big(\frac{\mu+2}{2}\big) e^{-a} a^{-(\mu+1)/2} \int_0^\infty s^{(\mu-1)/2} e^{-as}\, ds \\ &\times \int_0^\infty e^{-\xi^2/2t} \frac{\sinh(\xi)\cosh(\xi)\sin(\pi\xi/t)}{(s + (\cosh(\xi))^2)^{(\mu+2)/2}}\, d\xi. \end{aligned} \tag{2.5}$$

It may be worth noting that, if we change the order of integrations on the right hand side of (2.5), we can rewrite the double integral in terms of the confluent hypergeometric functions (see, e.g., p.268, [18]).

2.2. Some results from Dufresne [13]**.** We next present the expressions $f_D^{(\mu)}(a,t)$ for the density of $1/2A_t^{(\mu)}$ due to Dufresne [13]. The expressions for $f_D^{(0)}$ and $f_D^{(1)}$ are simple enough. It should be noted that the expression (2.6) below has been obtained by Alili-Gruet [2] from Bougerol's identity (1.3) and Plancherel's formula:

$$f_D^{(0)}(a,t) = C_t' \frac{1}{\sqrt{a}} \int_0^\infty \exp[-a(\cosh(\zeta))^2] e^{-\zeta^2/2t} \cos\big(\frac{\pi\zeta}{2t}\big)\, d\zeta, \tag{2.6}$$

$$f_D^{(1)}(a,t) = e^{-t/2} C_t' \frac{1}{\sqrt{a}} \int_0^\infty \exp[-a(\cosh(\zeta))^2] e^{-\zeta^2/2t} \cosh(\zeta)\sinh(\zeta) \sin\big(\frac{\pi\zeta}{2t}\big)\, d\zeta, \tag{2.7}$$

where $C_t' = \sqrt{2}(\pi\sqrt{t})^{-1} e^{\pi^2/8t}$. Then, Dufresne gives the following expressions for general μ:
(i) ([13], formula (4.11)) if $\mu \neq -1, -3, ...,$

$$\begin{aligned} f_D^{(\mu)}(a,t) =& \frac{C_t'}{\sqrt{\pi}} e^{-\mu^2 t/2} a^{-(\mu+1)/2} \Gamma\big(\frac{\mu+1}{2}\big) \int_0^\infty \exp[-a(\cosh(\xi))^2] e^{-\xi^2/2t} \\ &\times \cosh(\xi)\cos\big(\frac{\pi\xi}{2t}\big)\ {}_1F_1\big(-\frac{\mu}{2}, \frac{1}{2}; a(\sinh(\xi))^2\big)\, d\xi, \end{aligned} \tag{2.8}$$

(ii) ([13], formula (4.13)) if $\mu \neq -2, -4, ...,$

$$
\begin{aligned}
f_D^{(\mu)}(a,t) =& \frac{2}{\sqrt{\pi}} C'_t e^{-\mu^2 t/2} a^{-\mu/2} \Gamma\big(\frac{\mu+1}{2}\big) \int_0^\infty \exp[-a(\cosh(\xi))^2] e^{-\xi^2/2t} \\
& \times \cosh(\xi)\sinh(\xi)\sin\big(\frac{\pi\xi}{2t}\big)\ {}_1F_1(-\frac{1-\mu}{2}, \frac{3}{2}; a(\sinh(\xi))^2)\, d\xi,
\end{aligned}
\tag{2.9}
$$

where ${}_1F_1$ is the (Kummer) confluent hypergeometric function (see [18]). To obtain (2.8) and (2.9), Dufresne proves some recurrence formula which allows to obtain $f_D^{(\mu)}$ for $\mu < \nu$ from $f_D^{(\nu)}$. Such recurrence formulae will be discussed in Sections 4 and 5 below and play an important role throughout this paper.

2.3. GIG laws. The generalized inverse Gaussian (GIG) laws constitute a three parameter family of probability measures on $\mathbf{R}_+$ which are given by

$$
\mathrm{GIG}(\mu; a, b)(dt) = \big(\frac{b}{a}\big)^\mu \frac{t^{\mu-1}}{2K_\mu(ab)} \exp\big[-\frac{1}{2}\big(\frac{a^2}{t} + b^2 t\big)\big]\, dt
$$

for $\mu \in \mathbf{R}, a, b > 0$. In the particular case, which is most important for us, where $\mu = 0$ and $a^2 = b^2 = 1/z$, we shall simply write $\mathrm{GIG}_z^{(0)}$ and speak of a $\mathrm{GIG}_z^{(0)}$ random variable. The GIG laws play an important role in our discussions and here we recall some of their properties. For details, see Barndorff-Nielsen [4], Vallois [27]. The reader should be aware of the different conventions in the parametrization of these laws; here, we follow [4] as we did in [23].

Let us denote by $I_{a,b}^{(\mu)}$ a $\mathrm{GIG}(\mu; a, b)$ random variable. Then one has, quite immediately,

$$
(I_{a,b}^{(\mu)})^{-1} \stackrel{\text{(law)}}{=} I_{b,a}^{(-\mu)}
\tag{2.10}
$$

and

$$
P(I_{a,b}^{(\nu)} \in dt) = \big(\frac{a}{b}\big)^{\mu-\nu} \frac{K_\mu(ab)}{K_\nu(ab)} t^{\nu-\mu} P(I_{a,b}^{(\mu)} \in dt) \quad \text{for every } \mu, \nu \in \mathbf{R}.
\tag{2.11}
$$

The following is a deeper identity in law between $I_{a,b}^{(\mu)}$ and $I_{a,b}^{(-\mu)}$, which involves an additional Gamma(μ) random variable, independent of $I_{a,b}^{(-\mu)}$: for $\mu > 0$,

$$
I_{a,b}^{(\mu)} \stackrel{\text{(law)}}{=} I_{a,b}^{(-\mu)} + \frac{2}{b^2}\gamma_\mu,
\tag{2.12}
$$

from which we deduce the following.

Proposition 2.1. *For $z > 0$ and $\delta > 0$, we let X_z be a $\mathrm{GIG}_z^{(0)}$ random variable and γ_δ be a Gamma(δ) random variable independent of X_z. Then, for any positive Borel function f on $\mathbf{R}_+$, one has*

$$
E[f(X_z)(X_z)^\delta] = E[f(X_z + 2z\gamma_\delta)(X_z)^{-\delta}].
$$

Proof. This is obtained by rewriting both laws of $I^{(\delta)}_{1/\sqrt{z},1/\sqrt{z}}$ and $I^{(-\delta)}_{1/\sqrt{z},1/\sqrt{z}}$ in terms of X_z. $\square$

Remark 2.1. In Appendix B, we shall give a more general identity involving the two-dimensional law of (X_z, γ_δ).

The next proposition gives a characterization of the subclass of the GIG laws, which consists of the $\mathrm{GIG}^{(0)}_z$ ones.

Proposition 2.2. *Let $z > 0$ be fixed and X be a positive random variable such that*

$$E\big[\psi\big(\frac{X}{1-s}\big)\exp\big(\frac{s}{2zX}\big)\big] = E\big[\psi\big(\frac{1}{X}\big)\exp\big(\frac{s}{2zX}\big)\big] \tag{2.13}$$

for any positive Borel function ψ on $\mathbf{R}_+$ and $s \in (0,1)$. Then, the distribution of X is $\mathrm{GIG}^{(0)}_z$. Conversely, if X is a $\mathrm{GIG}^{(0)}_z$ random variable, then (2.13) *holds.*

Proof. For simplicity, we assume that X has a strictly positive continuous density q, which we write in the form:

$$q(x) = \frac{1}{x}\exp\big[-\frac{1}{2z}\varphi(x)\big].$$

Then, (2.13) means

$$\varphi((1-s)x) = \varphi\big(\frac{1}{x}\big) - s\big(x - \frac{1}{(1-s)x}\big).$$

Now we set $x = (1-s)^{-1} \in (1,\infty)$. Then we get

$$\varphi(1-s) = \varphi(1) - 2 + \big(1 - s + \frac{1}{1-s}\big), \qquad s \in (0,1)$$

and

$$\varphi(x) = \varphi(1) - 2 + x + \frac{1}{x}, \qquad x \in (0,1). \tag{2.14}$$

We can also easily show that (2.14) holds for $x > 1$.

From the integral representation of $K_0(\xi)$:

$$K_0(\xi) = \frac{1}{2}\int_0^\infty \exp\big[-\frac{\xi}{2}\big(u + \frac{1}{u}\big)\big]\,\frac{du}{u},$$

we finally obtain the proposition.

The converse is obtained by straightforward computation. $\square$

As already noted in [22, 23], GIG laws are easily found from formula (2.1). In order to understand this fact better, we proceed in somewhat an abstract manner.

Let A be a positive, respectively B be a real valued, random variable. We assume that their joint distribution is given by

$$P(A \in da, B \in db) = \exp\big[-\frac{1+e^{2b}}{2a}\big]\widetilde{\theta}\big(\frac{e^b}{a}\big)\,\frac{da\,db}{a}, \quad a>0, b\in\mathbf{R},$$

for some positive function $\widetilde{\theta}$ on $\mathbf{R}_+$. Then we can easily prove the following, which shows that the GIG laws are important in our discussions.

Proposition 2.3. *Define $Z=\exp(-B)A$ and $X=\exp(B)$. Then one has*

$$P(Z\in dz) = 2K_0\big(\frac{1}{z}\big)\widetilde{\theta}\big(\frac{1}{z}\big)\,\frac{dz}{z}$$

and, conditionally on $Z=z$, X is distributed as a $\mathrm{GIG}_z^{(0)}$ random variable, i.e.,

$$P(X\in dx|Z=z) = \frac{1}{2K_0(1/z)}\exp\big[-\frac{1}{2z}\big(x+\frac{1}{x}\big)\big]\,\frac{dx}{x}.$$

Next consider random variables $A^{(\mu)}$ and $B^{(\mu)}, \mu\in\mathbf{R}$, whose joint distribution is given by

$$P(A^{(\mu)}\in da, B^{(\mu)}\in db) = \frac{1}{C_\mu}E[\mathbf{1}_{\{A\in da\}}\mathbf{1}_{\{B\in db\}}X^\mu],$$

where $C_\mu = E[X^\mu]$, and set $Z^{(\mu)}=\exp(-B^{(\mu)})A^{(\mu)}$. Then, by using Propositions 2.1 and 2.3, we obtain the following.

Proposition 2.4. *Assume $\mu<\nu$. Then, for any non-negative Borel functions φ and f on $\mathbf{R}_+$, one has*

$$C_\mu E[\varphi(Z^{(\mu)})f\big(\frac{1}{A^{(\mu)}}\big)\big(\frac{1}{A^{(\mu)}}\big)^m] = C_\nu E[\varphi(Z^{(\nu)})f\big(\frac{1}{A^{(\nu)}}+2\gamma_\delta\big)\big(\frac{1}{A^{(\nu)}}\big)^m], \tag{2.15}$$

where $m=(\nu+\mu)/2$ and $\delta=(\nu-\mu)/2$.

Proof. By Proposition 2.3, we obtain

$$\begin{aligned}C_\mu E[\varphi(Z^{(\mu)})f\big(\frac{1}{A^{(\mu)}}\big)\big(\frac{1}{A^{(\mu)}}\big)^m] &= E[\varphi(Z)f\big(\frac{1}{A}\big)\big(\frac{1}{A}\big)^m X^\mu]\\ &= \int_0^\infty \varphi(z)E[f\big(\frac{1}{zX}\big)\big(\frac{1}{zX}\big)^m X^\mu|Z=z]\,P(Z\in dz)\\ &= \int_0^\infty z^{-m}\varphi(z)E[f\big(\frac{1}{zX_z}\big)(X_z)^{-\delta}]\,P(Z\in dz),\end{aligned}$$

where X_z is a $\mathrm{GIG}_z^{(0)}$ random variable. Since $(X_z)^{-1}\overset{\text{(law)}}{=}X_z$ by (2.10), we also have

$$C_\mu E[\varphi(Z^{(\mu)})f\big(\frac{1}{A^{(\mu)}}\big)\big(\frac{1}{A^{(\mu)}}\big)^m] = \int_0^\infty z^{-m}\varphi(z)E[f\big(\frac{X_z}{z}+2\gamma_\delta\big)(X_z)^\delta]\,P(Z\in dz).$$

We now use Proposition 2.1. Then we obtain

$$C_\mu E[\varphi(Z^{(\mu)})f\big(\frac{1}{A^{(\mu)}}\big)\big(\frac{1}{A^{(\mu)}}\big)^m] = \int_0^\infty z^{-m}\varphi(z)E[f\big(\frac{X_z}{z}+2\gamma_\delta\big)(X_z)^{-\delta}]\,P(Z\in dz)$$
$$= \int_0^\infty z^{-m}\varphi(z)E[f\big(\frac{1}{zX_z}+2\gamma_\delta\big)(X_z)^{\delta}]\,P(Z\in dz),$$

where we have used (2.10) for the last equality.

Hence, using Proposition 2.3 again, we obtain

$$C_\mu E[\varphi(Z^{(\mu)})f\big(\frac{1}{A^{(\mu)}}\big)\big(\frac{1}{A^{(\mu)}}\big)^m] = \int_0^\infty z^{-m}\varphi(z)E[f\big(\frac{1}{A}+2\gamma_\delta\big)X^{\delta}|Z=z]P(Z\in dz)$$
$$= E[\varphi(Z)f\big(\frac{1}{A}+2\gamma_\delta\big)\big(\frac{1}{A}\big)^m X^\nu]$$
$$= C_\nu E[\varphi(Z^{(\nu)})f\big(\frac{1}{A^{(\nu)}}+2\gamma_\delta\big)\big(\frac{1}{A^{(\nu)}}\big)^m]. \qquad \square$$

3. Reconciliation of $f_Y^{(\mu)}$ and $f_D^{(\mu)}$ in the cases $\mu=0$ and $\mu=1$

In this section we show $f_Y^{(\mu)} = f_D^{(\mu)}$ when $\mu=0$ and $\mu=1$. In both instances, our strategy is based on some identities obtained by residue calculus.

When $\mu=0$, changing the variables with $s=(\sinh(\eta))^2$ on the right hand side of (2.5), we have

$$f_Y^{(0)}(a,t) = \frac{C_t}{\sqrt{\pi}}\frac{1}{\sqrt{a}}\int_0^\infty \exp[-a(\cosh(\eta))^2]\cosh(\eta)\,d\eta$$
$$\times\int_0^\infty e^{-\xi^2/2t}\frac{\sinh(2\xi)\sin(\pi\xi/t)}{\cosh(2\xi)+\cosh(2\eta)}\,d\xi.$$

Comparing with formula (2.6) for $f_D^{(0)}(a,t)$, we obtain the desired coincidence from the following:

Lemma 3.1. *For any $\eta>0$, one has the identity*

$$e^{\pi^2/2t}\int_0^\infty e^{-\xi^2/2t}\frac{\sinh(2\xi)\sin(\pi\xi/t)}{\cosh(2\xi)+\cosh(2\eta)}\,d\xi = \pi\exp\big[\frac{\pi^2}{8t}-\frac{\eta^2}{2t}\big]\cos\big(\frac{\pi\eta}{2t}\big). \tag{3.1}$$

Proof. We modify a method developed in Gruet [15] to show the coincidence of his expressions, which he derived from formula (2.1), for the heat kernels on the hyperbolic spaces with the classical ones: we consider the meromorphic function ϕ_η on $\mathbf{C}$ defined by

$$\phi_\eta(z) = \frac{\sinh(2z)}{\cosh(2z)+\cosh(2\eta)}e^{-z^2/2t}.$$

Then, noting that the set of poles of ϕ_η with $|\operatorname{Im} z|<\pi$ consists of four points $z=\pm\eta\pm\pi i/2$ and taking the two lines given by $\operatorname{Im} z=\pm\pi$ as a contour, standard residue calculus shows (3.1). $\square$

When $\mu = 1$, we start our computations from (2.3). At first, we note

$$\frac{1}{\sqrt{a}} e^{-a} \exp[-2\sqrt{a}\eta\cosh(\xi)] = e^{-a}\int_0^\infty \exp[-\sqrt{a}(p+2\eta\cosh(\xi))]\,dp$$

$$= e^{-a}\int_0^\infty \frac{p+2\eta\cosh(\xi)}{\sqrt{2}}\,dp\int_0^\infty \exp\bigl[-as - \frac{(p+2\eta\cosh(\xi))^2}{4s}\bigr]\,\frac{ds}{\sqrt{2\pi s^3}}$$

$$= \frac{1}{\sqrt{\pi}}\int_0^\infty (p+2\eta\cosh(\xi))\,dp\int_0^\infty \exp\bigl[-a(\cosh(\zeta))^2 - \frac{(p+2\eta\cosh(\xi))^2}{4(\sinh(\zeta))^2}\bigr]\frac{\cosh(\zeta)}{(\sinh(\zeta))^2}\,d\zeta,$$

where we have used (2.4) and have changed the variables with $s = (\sinh(\zeta))^2$. Inserting this into the right hand side of (2.3) and using the Fubini theorem, we get

$$\begin{aligned} f_Y^{(1)}(a,t) &= \frac{e^{-t/2}C_t}{\sqrt{\pi}\sqrt{a}}\int_0^\infty \exp[-a(\cosh(\zeta))^2]\frac{\cosh(\zeta)}{(\sinh(\zeta))^2}\,d\zeta \\ &\quad\times\int_0^\infty \eta e^{-\eta^2}\,d\eta\int_0^\infty e^{-\xi^2/2t}\sinh(\xi)\sin\bigl(\frac{\pi\xi}{t}\bigr)\,d\xi \\ &\quad\times\int_0^\infty (p+2\eta\cosh(\xi))\exp\bigl[-\frac{(p+2\eta\cosh(\xi))^2}{4(\sinh(\zeta))^2}\bigr]\,dp \\ &= \frac{2e^{-t/2}C_t}{\sqrt{\pi}\sqrt{a}}\int_0^\infty \exp[-a(\cosh(\zeta))^2](\sinh(\zeta))^2\cosh(\zeta)\,d\zeta \\ &\quad\times\int_0^\infty e^{-\xi^2/2t}\frac{\sinh(\xi)\sin(\pi\xi/t)}{\cosh(2\xi)+\cosh(2\zeta)}\,d\xi. \end{aligned}$$

Now, comparing with the expression (2.7), we see that the following lemma shows $f_Y^{(1)} = f_D^{(1)}$. The proof is the same as that of Lemma 3.1 and we omit it.

Lemma 3.2. *For any $\zeta > 0$, one has*

$$e^{\pi^2/2t}\sinh(\zeta)\int_0^\infty e^{-\xi^2/2t}\frac{\sinh(\xi)\sin(\pi\xi/t)}{\cosh(2\xi)+\cosh(2\zeta)}\,d\xi = \frac{\pi}{2}e^{\pi^2/8t}e^{-\zeta^2/2t}\sin\bigl(\frac{\pi\zeta}{2t}\bigr). \tag{3.2}$$

4. The general reconciliation

To give a proof of the coincidence of $f_Y^{(\mu)}$ and $f_D^{(\mu)}$ in the general case, it seems difficult to follow the same strategy as in the previous section since a priori we cannot apply residue calculus and cannot appeal to the injectivity of Laplace transform. Instead we develop another, perhaps easier, argument.

Assuming $\mu < \nu$ and setting $\delta = (\nu-\mu)/2$ and $m = (\mu+\nu)/2$, Dufresne [13], formula (4.14), has obtained

$$e^{\mu^2 t/2} f_D^{(\mu)}(a,t) = e^{\nu^2 t/2}\frac{a^{-m}e^{-a}}{\Gamma(\delta)}\int_0^a (a-b)^{\delta-1}b^m e^b f_D^{(\nu)}(b,t)\,db. \tag{4.1}$$

There ν is assumed to be a positive integer there, but it is easy to see that this formula also holds when ν is not an integer. We may rewrite (4.1) in the following equivalent form, which, if we rely on (2.1) instead of using Dufresne's result (4.1), may be seen as a consequence of Proposition 2.4.

Proposition 4.1. *For every positive Borel function ψ and for $t > 0$, one has*

$$e^{\mu^2 t/2} E\big[\psi\big(\frac{1}{A_t^{(\mu)}}\big)\big(\frac{1}{A_t^{(\mu)}}\big)^m\big] = e^{\nu^2 t/2} E\big[\psi\big(\frac{1}{A_t^{(\nu)}} + 2\gamma_\delta\big)\big(\frac{1}{A_t^{(\nu)}}\big)^m\big], \tag{4.2}$$

where γ_δ denotes a Gamma(δ) *variable, independent of $B^{(\nu)}$.*

Taking $\nu > 0$ and $\mu = -\nu$ in (4.2), one recovers

$$\frac{1}{A_t^{(-\nu)}} \overset{\text{(law)}}{=} \frac{1}{A_t^{(\nu)}} + 2\gamma_\nu \qquad \text{for fixed } t > 0.$$

This formula was obtained originally in Dufresne [12] and was extended as the identity in law (1.1) between processes by Matsumoto–Yor [24].

Now we are in a position to give a proof of Theorem 1.1, which is an extension of (4.2) at the process level. We will see that, again, the results and the setup of subsection 2.3 will play an important role, but there are further Markovian features involved.

Proof of Theorem 1.1. Setting $Z_s^{(\mu)} = \exp(-B_s^{(\mu)}) A_s^{(\mu)}$, we have

$$\frac{d}{ds}\big(\frac{1}{A_s^{(\mu)}}\big) = -\frac{1}{(Z_s^{(\mu)})^2}.$$

Hence, to show (1.2), it is equivalent to prove

$$e^{\mu^2 t/2} E\big[\widetilde{F}(Z_s^{(\mu)}, s \leqq t)\psi\big(\frac{1}{A_t^{(\mu)}}\big)\big(\frac{1}{A_t^{(\mu)}}\big)^m\big] = e^{\nu^2 t/2} E\big[\widetilde{F}(Z_s^{(\nu)}, s \leqq t)\psi\big(\frac{1}{A_t^{(\nu)}} + 2\gamma_\delta\big)\big(\frac{1}{A_t^{(\nu)}}\big)^m\big]$$

or, by the Cameron–Martin theorem,

$$E\big[\widetilde{F}(Z_s, s \leqq t)\psi\big(\frac{1}{A_t}\big)\frac{\exp(\mu B_t)}{(A_t)^m}\big] = E\big[\widetilde{F}(Z_s, s \leqq t)\psi\big(\frac{1}{A_t} + 2\gamma_\delta\big)\frac{\exp(\nu B_t)}{(A_t)^m}\big] \tag{4.3}$$

for every non-negative functional $\widetilde{F}$ and every non-negative function ψ on $\mathbf{R}_+$, where $Z_s = Z_s^{(0)}$ and $A_s = A_s^{(0)}$. Moreover, with the help of the conditional independence result recalled in the Introduction (ii), formula (4.3) is equivalent to

$$E\big[\psi\big(\frac{1}{A_t}\big)\frac{\exp(\mu B_t)}{(A_t)^m}|Z_t = z\big] = E\big[\psi\big(\frac{1}{A_t} + 2\gamma_\delta\big)\frac{\exp(\nu B_t)}{(A_t)^m}|Z_t = z\big],$$

for any non-negative function ψ on $\mathbf{R}_+$, or

$$E\big[\psi\big(\frac{e^{-B_t}}{z}\big)e^{-\delta B_t}|Z_t = z\big] = E\big[\psi\big(\frac{e^{-B_t}}{z} + 2\gamma_\delta\big)e^{\delta B_t}|Z_t = z\big], \tag{4.4}$$

which can be shown in the following way: by Proposition 2.3, the conditional distribution of $\exp(B_t)$ given $Z_t = z$ is $\mathrm{GIG}_z^{(0)}$, and we obtain from Proposition 2.1

$$E\big[\psi\big(\frac{e^{B_t}}{z}\big)e^{\delta B_t}|Z_t = z\big] = E\big[\psi\big(\frac{e^{B_t}}{z} + 2\gamma_\delta\big)e^{-\delta B_t}|Z_t = z\big].$$

Now, combining this equality with (2.10), we obtain (4.4) and (1.2). □

Remark 4.1. From formula (1.2), we obtain the following identity, which is more symmetric in μ and ν: if $\mu, \nu > 0$,

$$e^{\mu^2 t/2}E[F\big(\frac{1}{A_s^{(\mu)}} + 2\gamma_{\mu/2}, s \leqq t\big)\big(\frac{1}{A_t^{(\mu)}}\big)^m] = e^{\nu^2 t/2}E[F\big(\frac{1}{A_s^{(\nu)}} + 2\gamma_{\nu/2}, s \leqq t\big)\big(\frac{1}{A_t^{(\nu)}}\big)^m], \tag{4.5}$$

where γ_α ($\alpha = \mu/2$ and $\nu/2$) is independent of $A^{(2\alpha)}$. For the proof, we need only to recall the relationship, in the case $\mu < \nu$ with $\delta = (\nu - \mu)/2$, $\gamma_{\nu/2} \overset{\text{(law)}}{=} \gamma_\delta + \gamma_{\mu/2}$, where, on the right hand side, the two gamma random variables are independent.

We end this section by completing a proof of the coincidence of $f_D^{(\mu)}(a,t)$ and $f_Y^{(\mu)}(a,t)$.

Proof of $f_D^{(\mu)}(a,t) = f_Y^{(\mu)}(a,t)$ in full generality. We let q_δ be the density of a Gamma(δ) variable γ_δ and set

$$\widetilde{f}_D^{(\mu)}(a,t) = e^{\mu^2 t/2} f_D^{(\mu)}(a,t)a^m \qquad \text{and} \qquad \widetilde{f}_Y^{(\mu)}(a,t) = e^{\mu^2 t/2} f_Y^{(\mu)}(a,t)a^m.$$

Then Dufresne's recurrence formula (4.1) means

$$\widetilde{f}_D^{(\mu)}(a,t) = (\widetilde{f}_D^{(\nu)}(\cdot,t) * q_\delta)(a),$$

where $*$ denotes the usual convolution, while the arguments above in this section show that $\widetilde{f}_Y^{(\mu)}$ and $\widetilde{f}_Y^{(\nu)}$ satisfy the same identity.

Moreover we showed $\widetilde{f}_D^{(0)} = \widetilde{f}_Y^{(0)}$ (and $\widetilde{f}_D^{(1)} = \widetilde{f}_Y^{(1)}$) in the previous section. Therefore the injectivity of the Laplace transform yields $\widetilde{f}_D^{(\mu)} = \widetilde{f}_Y^{(\mu)}$ and $f_D^{(\mu)} = f_Y^{(\mu)}$ in general. □

5. Complements to the recurrence formula (1.2)

5.1. Some relations to Bougerol's identity. Bougerol [7] has obtained the identity

$$\sinh(B_t) \overset{\text{(law)}}{=} \beta_{A_t} \qquad \text{for fixed } t,$$

where $\{\beta_s, s \geqq 0\}$ is a Brownian motion independent of B and this identity has been extended in Alili–Dufresne–Yor [1] to

$$\sinh(Y_t^{(\mu)}) \overset{\text{(law)}}{=} \beta_{A_t^{(\mu)}} \qquad \text{for fixed } t, \tag{5.1}$$

where$\{Y_s^{(\mu)}, s \geqq 0\}$ is a diffusion process generated by the second order differential operator

$$\frac{1}{2}\frac{d^2}{dy^2} + \mu \tanh(y)\frac{d}{dy},$$

acting on $C_0^2(\mathbf{R})$. In the case $\mu = 1$, one has $\{Y_t^{(1)}, t \geqq 0\} \overset{\text{(law)}}{=} \{B_t + \varepsilon t, t \geqq 0\}$, where ε is a symmetric Bernoulli random variable valued in $\{-1, +1\}$ independent of B. Hence, from (5.1), we deduce

$$\beta_{A_s^{(1)}} \overset{\text{(law)}}{=} \sinh(B_t + \varepsilon t) \qquad \text{for fixed } t.$$

Moreover, letting $P^{(\mu)}$ be the probability law of $Y^{(\mu)}$, the following absolute continuity relationship has been obtained in [1]:

$$P^{(\mu)}|_{\mathcal{F}_t^Y} = (\cosh(Y_t))^{\mu} \exp\big[-\frac{\mu - \mu^2}{2}\int_0^t \frac{ds}{(\cosh(Y_s))^2} - \frac{1}{2}\mu^2 t\big] P^{(0)}|_{\mathcal{F}_t^Y},$$

where $(\mathcal{F}_t^Y)$ is the natural filtration of Y, the coordinate process on $C([0,\infty), \mathbf{R})$. We deduce from this, in the case $\mu + \nu = 1$,

$$P^{(\mu)}|_{\mathcal{F}_t^Y} = (\cosh(Y_t))^{\mu-\nu} e^{(\nu-\mu)t/2} P^{(\nu)}|_{\mathcal{F}_t^Y}. \tag{5.2}$$

Now, in (4.2), we choose μ and ν ($\mu < \nu$) such that $\mu + \nu = 1$. Then, letting $\varphi(y) = \exp(-x^2/2y)$, we get, with $\delta = (\nu - \mu)/2 = (\nu^2 - \mu^2)/2$,

$$\begin{aligned} E[\frac{1}{(2\pi A_t^{(\mu)})^{1/2}} e^{-x^2/2A_t^{(\mu)}}] &= e^{\delta t} E[\frac{1}{(2\pi A_t^{(\nu)})^{1/2}} e^{-x^2/2A_t^{(\nu)}}] E[e^{-x^2\gamma_\delta}] \\ &= e^{\delta t}\frac{1}{(1+x^2)^{\delta}} E[\frac{1}{(2\pi A_t^{(\nu)})^{1/2}} e^{-x^2/2A_t^{(\nu)}}] \end{aligned}$$

and so

$$E[\varphi(\beta_{A_t^{(\mu)}})] = e^{\delta t/2} E\big[\varphi(\beta_{A_t^{(\nu)}})\frac{1}{(1+(\beta_{A_t^{(\nu)}})^2)^{\delta}}\big] \quad \text{if } \mu + \nu = 1 \tag{5.3}$$

for every non-negative Borel function φ on $\mathbf{R}$ and a Brownian motion $\{\beta_s, s \geqq 0\}$ independent of $A^{(\mu)}$ and $A^{(\nu)}$. When $\mu = 0$ and $\nu = 1$, both hand sides are equal to $E[\varphi(\sinh(B_t))]$.

Formula (5.3) shows an identity between two probability laws for fixed t. It might be interesting to extend it to one at process level or to interpret (5.2) in terms of the processes $\{1/A_s^{(\mu)}, s \leqq t\}$ and $\{1/A_s^{(\nu)}, s \leqq t\}$, but we shall not pursue this here.

5.2. Some absolute continuity results. For $\alpha > 0$, let T_α be the transform on the path space $C(\mathbf{R}_+ \to \mathbf{R})$ defined by

$$T_\alpha(\phi)_s = \phi_s - \log(1 + \alpha A(\phi)_s), \qquad s \geqq 0,$$

where $A(\phi)_s = \int_0^s \exp(2\phi_u)du$. Note that, setting $e(\phi)_s = \exp(\phi(s))$,

$$\frac{1}{A(T_\alpha(\phi))_s} = \frac{1}{A(\phi)_s} + \alpha \qquad \text{and} \qquad e(T_\alpha(\phi))_s = \frac{e(\phi)_s}{1 + \alpha A(\phi)_s} \tag{5.4}$$

Setting $Z(\phi)_s = (e(\phi)_s)^{-1} A(\phi)_s$, Z is invariant under T_α, that is, $Z(T_\alpha(\phi))_s = Z(\phi)_s$, $s \geqq 0$. Moreover T_α, $\alpha > 0$, is a semigroup of transformations: $T_\alpha \circ T_\beta = T_{\alpha+\beta}, \alpha, \beta > 0$, and $T_{-\alpha}$, with obvious notations, may be defined on the set $T_\alpha(C(\mathbf{R}_+, \mathbf{R}))$ and $T_{-\alpha} \circ T_\alpha = Id$.

Then, in [9], the following formula has been obtained: for $\alpha > 0$,

$$E[F(T_\alpha(B)_s, s \leqq t)] = E[F(B_s, s \leqq t)\Delta_\alpha(e_t, \alpha A_t)], \tag{5.5}$$

where $e_t = \exp(B_t)$ and

$$\Delta_\alpha(v, b) = \mathbf{1}_{\{b<1\}}(1-b)^{-1} \exp\big[-\frac{\alpha}{2}\big(\frac{v^2}{1-b} - 1\big)\big].$$

By using (5.5), we shall generalize the recurrence relation (1.2), which may be rewritten by the Cameron–Martin theorem as

$$E\big[F(T_{2\gamma_\delta}(B)_s, s \leqq t)\frac{(e_t)^\nu}{(A_t)^m}\big] = E\big[F(B_s, s \leqq t)\frac{(e_t)^\mu}{(A_t)^m}\big]. \tag{5.6}$$

Our next result originated from our desire to understand how formula (5.6) would be changed if, instead of having $(A_t)^{-m}(e_t)^\nu$ on the left hand side, we considered a general function $\psi(A_t, e_t)$.

Proposition 5.1. *Assume $\mu < \nu$ and set $\delta = (\nu - \mu)/2, m = (\mu + \nu)/2$. Then, for any non-negative functional F and any non-negative function Φ on $\mathbf{R}_+^2$, one has*

$$\begin{aligned} &E[F(T_{2\gamma_\delta}(B)_s, s \leqq t)\frac{(e_t)^\nu}{(A_t)^m}\Phi(A_t, e_t)] \\ &\qquad = E[F(B_s, s \leqq t)\frac{(e_t)^\mu}{(A_t)^m}\Phi(2Z_t^2\gamma_\delta + A_t, 2Z_t\gamma_\delta + e_t)], \end{aligned} \tag{5.7}$$

where γ_δ is a Gamma(δ) *variable independent of $\{B_s, s \leqq t\}$.*

Before showing how (5.7) can be derived from (5.5), we present an easy variant of (5.7) by incorporating $(A_t)^{-m}(e_t)^\nu$ into $\Phi(A_t, e_t)$; a simple computation yields

$$\begin{aligned} &E[F(T_{2\gamma_\delta}(B)_s, s \leqq t)\Psi(A_t, e_t)] \\ &\qquad = E[F(B_s, s \leqq t)\frac{1}{(e_t^2 + 2A_t\gamma_\delta)^\delta}\Psi(2Z_t^2\gamma_\delta + A_t, 2Z_t\gamma_\delta + e_t)]. \end{aligned} \tag{5.8}$$

Proof of Proposition 5.1. The assertion (5.7) is equivalent to identity (5.8) for every $\delta > 0$ and, hence, to the following, thanks to the injectivity of Mellin transform: for any non-negative

Borel function φ on $\mathbf{R}_+$,

$$\int_0^\infty e^{-x}\varphi(x)E[F(T_{2x}(B)_s, s \leqq t)\Psi(A_t, e_t)]\,\frac{dx}{x}$$
$$= E[\int_0^\infty e^{-x}\varphi\big(\frac{x}{e_t^2+2A_tx}\big)F(B_s, s \leqq t)\Psi(2Z_t^2x+A_t, 2Z_tx+e_t)\,\frac{dx}{x}].$$

Moreover, changing variables with $y = x(e_t^2+2A_tx)^{-1}$, this is equivalent to

$$\text{(5.9)}\quad e^{-x}E[F(T_{2x}(X)_s, s \leqq t)\Psi(A_t, e_t)]$$
$$= E[\frac{\mathbf{1}_{\{x<1/2A_t\}}}{1-2A_tx}\exp\big[-\frac{e_t^2x}{1-2A_tx}\big]F(B_s, s \leqq t)\Psi\big(\frac{A_t}{1-2A_tx}, \frac{e_t}{1-2A_tx}\big)].$$

Clearly, if $\Psi \equiv 1$, formula (5.9) is precisely formula (5.5). But, conversely, we can also derive formula (5.9) from (5.5) by writing $A_t = A(T_{-2x}(T_{2x}(B)))_t$ and $e_t = e(T_{-2x}(T_{2x}(B)))_t$. In fact, we have

$$A(T_{-2x}(\phi))_t = \frac{A(\phi)_t}{1-2xA(\phi)_t} \qquad \text{and} \qquad e(T_{-2x}(\phi))_t = \frac{e(\phi)_t}{1-2xe(\phi)_t}$$

and, replacing $F(B_s, s \leqq t)$ by

$$F(B_s, s \leqq t)\Psi(A(T_{-2x}(B))_t, e(T_{-2x}(B))_t)$$

on the right hand side of (5.5), we obtain formula (5.9) from (5.5). □

We here comment upon the previous argument. We have shown that formulae (5.7) and (5.8) may be deduced by easy, although somewhat lengthy, manipulation from our original absolute continuity relationship (5.5). In particular, Theorem (1.1), i.e., extended Dufresne's recurrence formula, may be seen as a consequence of (5.5).

6. Further comments

6.1. Dufresne's approach to his recurrence formula. Dufresne, at the bottom of p.237 in [13], deduced his recurrence formula (4.1) from the following identity: for $\mu < \nu$, with $m = (\nu+\mu)/2$ and $\delta = (\nu-\mu)/2$, and $s \in (0,1)$,

$$\text{(6.1)}\qquad E[\frac{(e_t)^\mu}{(A_t)^m}\exp\big[\frac{s}{2A_t}\big]] = (1-s)^{-\delta}E[\frac{(e_t)^\nu}{(A_t)^m}\exp\big[\frac{s}{2A_t}\big]].$$

Indeed, writing $(1-s)^{-\delta} = E[\exp(s\gamma_\delta)]$ and using the injectivity of Laplace transform, we obtain Dufresne's recurrence formula in the form of (4.2). In this subsection, this identity (6.1) is obtained as a very simple consequence of Propositions 2.2 and 2.3. In fact we show it in the following extended form.

Proposition 6.1. *For any non-negative functional F and any non-negative Borel function ψ on $\mathbf{R}_+$, one has*

$$E[F(Z_s, s \leqq t)\psi\big(\frac{e_t}{1-s}\big)\exp\big[\frac{s}{2A_t}\big]] = E[F(Z_s, s \leqq t)\psi(e_t^{-1})\exp\big[\frac{s}{2A_t}\big]] \tag{6.2}$$

and, consequently, (6.1) *holds.*

Proof. By (2.1), Propositions 2.2 and 2.3, we have

$$E[\psi\big(\frac{e_t}{1-s}\big)\exp\big[\frac{s}{2ze_t}\big]|Z_t = z] = E[\psi(e_t^{-1})\exp\big[\frac{s}{2ze_t}\big]|Z_t = z],$$

which immediately implies (6.2), as a consequence of the conditional independence of e_t and $\{Z_s, s \leqq t\}$, given Z_t (see (ii) in the Introduction). □

6.2. Stieltjes's remark. Both the expressions of $f_Y^{(\mu)}$ and $f_D^{(\mu)}$ for the density of $1/2A_t^{(\mu)}$ present some difficulties for precise implementations due to a number of facts: in particular, the integral kernel $\theta(r,t)$ in (2.1), as well as $f_D^{(\mu)}$, are written as integrals involving the trigonometric functions $\sin(\pi\xi/t)$ and $\cos(\pi\xi/2t)$, respectively. As a consequence, it is not even clear a priori that $f_Y^{(\mu)}$ and $f_D^{(\mu)}$ are positive. As in Yor [32], we would like to relate this fact with some important remark made by Stieltjes in his fundamental mémoire [26]. In pp. J106–107 in the reprinted version, he showed

$$\int_{-\infty}^{\infty} e^{-\xi^2/2t} e^{n\xi} \sin\big(\frac{\pi\xi}{t}\big)\, d\xi = 0 \qquad \text{for any } n \in \mathbf{Z}. \tag{6.3}$$

From his remark (6.3), Stieltjes notes that, for the whole family $\mu_{t,\lambda}, -1 \leqq \lambda \leqq 1$, of the probability measures

$$\mu_{t,\lambda}(dx) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}\big(1 + \lambda \sin\big(\frac{\pi x}{t}\big)\big)\, dx,$$

$\langle \mu_{t,\lambda}, e_n \rangle$ is independent of λ, where $e_n(x) = \exp(nx)$. Hence, the moment problem for the log-normal distribution $\ell_t(dx) = \exp_*(\mu_t)$ is indeterminate.

But Stieltjes' considerations go much further, as his previously developed general theory then tells him that there exists two different measures on $\mathbf{R}_+$, which he denotes by $d\Phi$ and $d\Phi_1$ such that

$$\int_0^\infty \frac{d\Phi(u)}{x+u} \leqq \int_0^\infty \frac{u^{-\log u}}{x+u}\, du \leqq \int_0^\infty \frac{d\Phi_1(u)}{x+u} \tag{6.4}$$

(6.5) $\quad d\Phi(u)$ and $d\Phi_1(u)$ solve the moment problem for $\ell_{1/2}$

$$S(\Phi_1)(x) - S(\Phi)(x) \geqq \frac{2}{\sqrt{\pi}} \int_0^\infty \frac{\sin(2\pi \log u)}{x+u} u^{-\log u}\, du, \tag{6.6}$$

where $S(\Phi)(x) = \int_0^\infty (x+u)^{-1} d\Phi(u)$ is the Stieltjes transform of $d\Phi$. Thus, (6.6) gives some estimate for the degree of non-determinacy in the moment problem for $\ell_{1/2}$.

Moreover, Stieltjes computes the right hand side of (6.6) and yields

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-v^2} \frac{e^v}{e^y + e^v} \sin(2\pi v)\, dv = 2^{-\pi^2} \sqrt{\pi} e^{-y^2}, \tag{6.7}$$

a formula which, clearly, should be considered jointly with those in our above Lemmas 3.1 and 3.2.

Stieltjes' formula (6.7), as well as our formulae in the lemmas, seem to call for the explicit computation of the projection operator $L^2(d\mu_t) \ni f \mapsto \mathcal{P}(f)$, which is the orthogonal projection of f on $\mathcal{H}_t$, the subspace of $L^2(d\mu_t)$ generated by $\{e^{nx}, n \in \mathbf{Z}\}$. Note that (6.3) tells that the function $\sin(\pi x/t)$ belongs to $\mathcal{H}_t^\perp$, so that we have, for every $f \in L^2(d\mu_t)$,

$$\int_{-\infty}^{\infty} e^{-x^2/2t} f(x) \sin\left(\frac{\pi x}{t}\right) dx = \int_{-\infty}^{\infty} e^{-x^2/2t} (f(x) - \mathcal{P}(f)(x)) \sin\left(\frac{\pi x}{t}\right) dx. \tag{6.8}$$

We hope that, in some of the cases of interest for us, $f(x) - \mathcal{P}(f)(x)$ is simpler than the original function, thus leading to possibly a more friendly expression for $\theta(r,t)$.

Appendix A. The role of the beta-gamma algebra

A.1. Another proof of Proposition 4.1. In this part, we consider the setup of Proposition 4.1. We first explain the assertion as a consequence of the beta-gamma algebra (1.4);

$$\gamma_a \overset{\text{(law)}}{=} \beta_{a,b}\, \gamma_{a+b} \qquad \text{for} \qquad a, b > 0, \tag{A.1}$$

where γ_a,$\beta_{a,b}$ are Gamma(a),Beta(a,b) random variables, respectively, and, on the right hand side, $\beta_{a,b}$ and γ_{a+b} are independent. More generally, throughout this appendix, we assume that algebraic quantities featuring several random variables always involve independent variables.

We rewrite the result (4.2) in terms of the densities $f^{(\mu)}(a,t), \mu \in \mathbf{R}$, of $1/2A_t^{(\mu)}$: for $\mu < \nu$,

$$a^m f^{(\mu)}(a,t) = \frac{C_\nu}{C_\mu} E[\mathbf{1}_{\{a > \gamma_\delta\}} (a - \gamma_\delta)^m f^{(\nu)}(a - \gamma_\delta, t)], \tag{A.2}$$

where $m = (\nu+\mu)/2, \delta = (\nu-\mu)/2$ and $C_\mu = \exp(\mu^2 t/2) = E[\exp(\mu B_t)]$ for a Brownian motion $\{B_s, s \geqq 0\}$ with $B_0 = 0$. Setting $\widetilde{f}^{(\mu)}(a,t) = a^m f^{(\mu)}(a,t)$, we may write (A.2) as

$$\widetilde{f}^{(\mu)}(a,t) = \frac{C_\nu}{C_\mu} (\widetilde{f}^{(\nu)}(\cdot, t) * q_\delta)(a) \tag{A.3}$$

with q_δ denoting the density of γ_δ.

We now prove (A.2) and (A.3) from the expression for $f^{(\mu)}(a,t)$ in terms of the function $\theta(r,t)$ given by (2.1):

$$f^{(\mu)}(a,t) = \frac{e^{-a}}{C_\mu (2a)^\mu a} \int_0^\infty z^{\mu-1} e^{-z^2/4a} \theta(z,t)\, dz.$$

Since t is fixed throughout this part, we simply write $f^{(\mu)}(a)$ and $\theta(z)$ for $f^{(\mu)}(a,t)$ and $\theta(z,t)$, respectively.

For simplicity we assume $0 < \mu < \nu$ and consider the transform $S^{(\mu)}$ acting on non-negative functions on $\mathbf{R}_+$ by

$$S^{(\mu)}(\varphi)(a) = \int_0^\infty z^{\mu-1} e^{-z^2/4a} \varphi(z)\, dz.$$

Thus, we have

$$f^{(\mu)}(a) = \frac{e^{-a}}{C_\mu (2a)^\mu a} S^{(\mu)}(\theta)(a) = \frac{\Gamma(\mu/2) e^{-a}}{2C_\mu a^{1+\mu/2}} E[\theta(\sqrt{4a\gamma_{\mu/2}})]. \tag{A.4}$$

We now use (A.1). Then we obtain

$$\begin{aligned} E[\theta(\sqrt{4a\gamma_{\mu/2}})] &= E[\theta(\sqrt{4a\beta_{\mu/2,\delta}\ \gamma_{\nu/2}})] \\ &= \frac{1}{B(\mu/2,\delta)} \int_0^1 E[\theta(\sqrt{4ax\gamma_{\nu/2}})] x^{-1+\mu/2} (1-x)^{\delta-1}\, dx \\ &= \frac{1}{B(\mu/2,\delta)} \int_0^\infty E[\theta(\sqrt{4a(1+z)^{-1}\gamma_{\nu/2}})] \frac{z^{\delta-1}}{(1+z)^{\nu/2}}\, dz. \end{aligned}$$

Using (A.4) again, we obtain after some manipulations

$$f^{(\mu)}(a) = \frac{C_\nu}{C_\mu} a^{-m} \int_0^a (a-x)^m f^{(\nu)}(a-x) \frac{1}{\Gamma(\delta)} x^{\delta-1} e^{-x}\, dx,$$

which is exactly formula (A.2).

A.2. The law of $A^{(\mu)}$ at an independent exponential time and Dufresne's recurrence. We now replace the fixed time t by an exponential random time T_θ with parameter θ, independent of the underlying Brownian motion, in (4.2). For this purpose, we assume $m \equiv (\nu+\mu)/2 < 1$ and integrate both hand sides with respect to $\exp(-\lambda t)dt$ for $\lambda > \nu^2/2$. Thus we obtain

$$\begin{aligned} &\big(\lambda - \frac{\mu^2}{2}\big)^{-1} E[\psi\big(\frac{1}{A^{(\mu)}_{T_{\lambda-\mu^2/2}}}\big)\big(\frac{1}{A^{(\mu)}_{T_{\lambda-\mu^2/2}}}\big)^m] \\ &\qquad = \big(\lambda - \frac{\nu^2}{2}\big)^{-1} E[\psi\big(\frac{1}{A^{(\nu)}_{T_{\lambda-\nu^2/2}}} + 2\gamma_\delta\big)\big(\frac{1}{A^{(\nu)}_{T_{\lambda-\nu^2/2}}}\big)^m]. \end{aligned} \tag{A.5}$$

We shall use the result ([31]) that

$$A_{T_\theta}^{(\alpha)} \overset{\text{(law)}}{=} \frac{\beta_{1,a}}{2\gamma_b}, \tag{A.6}$$

where $a = (\sqrt{2\theta+\alpha^2}+\alpha)/2$ and $b = (\sqrt{2\theta+\alpha^2}-\alpha)/2$. Plugging this into equation (A.5) and writing both hand sides in the form of double integrals, we get

$$\begin{aligned}(\lambda - \frac{\mu^2}{2})^{-1}\frac{\Gamma(b_{\lambda,\mu}+m)}{\Gamma(b_{\lambda,\mu})}\frac{B(1-m,a_{\lambda,\mu})}{B(1,a_{\lambda,\mu})}E[\psi(\frac{2\gamma_{m+b_{\lambda,\mu}}}{\beta_{1-m,a_{\lambda,\mu}}})] \\ = (\lambda - \frac{\nu^2}{2})^{-1}\frac{\Gamma(b_{\lambda,\nu}+m)}{\Gamma(b_{\lambda,\nu})}\frac{B(1-m,a_{\lambda,\nu})}{B(1,a_{\lambda,\nu})}E[\psi(\frac{2\gamma_{m+b_{\lambda,\nu}}}{\beta_{1-m,a_{\lambda,\nu}}}+2\gamma_\delta)],\end{aligned} \tag{A.7}$$

where $a_{\lambda,\mu} = (\sqrt{2\lambda}+\mu)/2$ and $b_{\lambda,\mu} = (\sqrt{2\lambda}-\mu)/2$.

For (A.7) to hold, it is necessary and sufficient that

(i) the coefficients in front of the expectations are equal;

(ii) there is an identity in law:

$$\frac{\gamma_{m+b_{\lambda,\mu}}}{\beta_{1-m,a_{\lambda,\mu}}} \overset{\text{(law)}}{=} \frac{\gamma_{m+b_{\lambda,\nu}}}{\beta_{1-m,a_{\lambda,\nu}}} + \gamma_\delta. \tag{A.8}$$

The assertion of (i) is easily proven from the functional equation $\Gamma(z+1) = z\Gamma(z)$ once we note $a_{\lambda,\mu}b_{\lambda,\mu} = (\lambda - \mu^2/2)/2$. On the other hand, the identity (A.8) follows from the identity

$$\frac{\gamma_q}{\beta_{a-(p+q),p}} \overset{\text{(law)}}{=} \frac{\gamma_p}{\beta_{a-(p+q),q}} + \gamma_{q-p} \tag{A.9}$$

for $0 < p < q$ and $a > p+q$, which we shall also show by using the beta-gamma algebra (1.4). However, we note that (A.9) is precisely Dufresne's relation (11) on p.288 of [11], a paper devoted to studies on beta and gamma random variables.

Let us prove (A.9): first, we write from (1.4)

$$(\gamma_p, \gamma_{q-p}) \overset{\text{(law)}}{=} (\beta_{p,q-p}\,\gamma_q, (1-\beta_{p,q-p})\gamma_q).$$

Then (A.9) is equivalent to

$$\frac{1}{\beta_{a-(p+q),p}} \overset{\text{(law)}}{=} \frac{\beta_{p,q-p}}{\beta_{a-(p+q),q}} + (1-\beta_{p,q-p}). \tag{A.10}$$

Then, since

$$\beta_{a-(p+q),p} \overset{\text{(law)}}{=} \frac{\gamma_{a-(p+q)}}{\gamma_{a-(p+q)}+\gamma_p},$$

we see that (A.10) is equivalent to

$$\frac{\gamma_p}{\gamma_{a-(p+q)}} \overset{\text{(law)}}{=} \beta_{p.q-p}\frac{\gamma_p}{\gamma_{a-(p+q)}},$$

which easily follows from (1.4).

Now, by using the beta-gamma algebra, we have shown (A.5) from (A.6), and finally we have shown that Dufresne's recurrence relation (4.2) may be deduced from the identification of the law of $A^{(\mu)}_{T_\theta}$ as given in (A.6).

Appendix B. Further absolute continuity relationships

B.1. An extension of Proposition 5.1 and some properties of GIG and gamma laws. The purpose of this part is to show an extension ((B.2) below) of Proposition 5.1, where we consider a function $\widetilde{\Phi}$ of the three random variables A_t, e_t and γ_δ.

Let us denote, for simplicity, $T_{2\gamma_\delta}(B)_s$ by B'_s and, correspondingly, $A'_s = A(B')_s$ and so on. Recall from (5.4) that

$$\text{(B.1)} \qquad \frac{1}{A'_s} = \frac{1}{A_s} + 2\gamma_\delta \qquad \text{and} \qquad e'_s = \frac{e_s}{1 + 2\gamma_\delta A_s},$$

so that our previous proof of (5.7) and (5.8) "hides" or is closely related to the conditional law of $2\gamma_\delta$ given $\{B'_s, s \leqq t\}$. This motivated us to write the following more complete version of formula (5.8).

Proposition B.1. *Let $\delta > 0$. Then, for any non-negative functional F and any non-negative Borel function $\widetilde{\Phi}$ on $\mathbf{R}^3_+$, one has*

$$\text{(B.2)} \quad E[F(B'_s, s \leqq t)\widetilde{\Phi}(A_t, e_t, 2\gamma_\delta)]$$
$$= E[F(B_s, s \leqq t)\frac{1}{((e_t)^2 + 2A_t\gamma_\delta)^\delta}\widetilde{\Phi}\big(A_t + 2Z_t^2\gamma_\delta, e_t + 2Z_t\gamma_\delta, \frac{2\gamma_\delta}{(e_t)^2 + 2A_t\gamma_\delta}\big)],$$

where, as usual, γ_δ denotes a Gamma(δ) *variable independent of* $\{B_s, s \leqq t\}$.

The proof is quite similar to that of (5.8) and we do not repeat the arguments. We leave to the interested reader the derivation of the conditional law of $2\gamma_\delta$ given $\{B'_s, s \leqq t\}$, as an application of (B.2). It should be noted that, from (B.1), $2\gamma_\delta = 1/A'_\infty$, and that γ_δ is measurable with respect to $\{B'_s, s \geqq 0\}$.

However, it is of interest to consider various particular cases of (B.2), for instance, $\widetilde{\Phi} \equiv 1$ and $F \equiv 1$. Here, we discuss the latter case.

We have

$$\text{(B.3)} \qquad E[\widetilde{\Phi}(A_t, e_t, 2\gamma_\delta)] = E[\frac{1}{((e_t)^2 + 2A_t\gamma_\delta)^\delta}\widetilde{\Phi}\big(A_t + 2Z_t^2\gamma_\delta, e_t + 2Z_t\gamma_\delta, \frac{2\gamma_\delta}{(e_t)^2 + 2A_t\gamma_\delta}\big)]$$

Note that the different terms which appear with $\widetilde{\Phi}$ on the right hand side of (B.3) are, in succession, $Z_t(e_t+2Z_t\gamma_\delta)$, $e_t+2Z_t\gamma_\delta$ and $2\gamma_\delta(e_t)^{-1}(e_t+2Z_t\gamma_\delta)^{-1}$. Then, replacing $\widetilde{\Phi}(a,e,x)$ by $\widetilde{\Phi}(a/e,e,x)$ on both hand sides of (B.3), we obtain

$$E[\widetilde{\Phi}(Z_t,e_t,2\gamma_\delta)]=E[\frac{1}{((e_t)^2+2A_t\gamma_\delta)^\delta}\widetilde{\Phi}\big(Z_t,e_t+2Z_t\gamma_\delta,\frac{2\gamma_\delta}{e_t(e_t+2Z_t\gamma_\delta)}\big)]$$

so that we get the equivalent formulation

$$E[\Phi(e_t,2\gamma_\delta)|Z_t=z]=E[\frac{1}{(e_t(e_t+2z\gamma_\delta))^\delta}\Phi\big(e_t+2z\gamma_\delta,\frac{2\gamma_\delta}{e_t(e_t+2z\gamma_\delta)}\big)|Z_t=z]$$

for any non-negative function on $\mathbf{R}_+^2$.

We now use our notation in Proposition 2.1, where X_z denotes a $\mathrm{GIG}_z^{(0)}$ random variable, to obtain

$$E[\Phi(X_z,2\gamma_\delta)]=E[\frac{1}{(X_z(X_z+2z\gamma_\delta))^\delta}\Phi\big(X_z+2z\gamma_\delta,\frac{2\gamma_\delta}{X_z(X_z+2z\gamma_\delta}\big)]$$

or, changing $\Phi(z,x)$ into $z^\delta\Phi(z,x)$,

$$E[\Phi(X_z,2\gamma_\delta)(X_z)^\delta]=E[\Phi\big(X_z+2z\gamma_\delta,\frac{2\gamma_\delta}{X_z(X_z+2z\gamma_\delta)}\big)(X_z)^{-\delta}].$$

Moreover, by using (2.10) and (2.11), we obtain

$$E[\Phi\big(\frac{1}{X_z^{(-\delta)}},2z\gamma_\delta\big)]=E[\Phi\big(X_z^{(-\delta)}+2z\gamma_\delta,\frac{2z\gamma_\delta}{X_z^{(-\delta)}(X_z^{(-\delta)}+2z\gamma_\delta)}\big)],$$

which means

$$(X_z^{(-\delta)},2z\gamma_\delta)\overset{\text{(law)}}{=}\big(\frac{1}{X_z^{(-\delta)}+2z\gamma_\delta},\frac{1}{X_z^{(-\delta)}}-\frac{1}{X_z^{(-\delta)}+2z\gamma_\delta}\big),\tag{B.4}$$

where $X_z^{(-\delta)}$ is a $\mathrm{GIG}(-\delta,z^{-1/2},z^{-1/2})$ variable.

This identity in law and some equivalent form of it have already been discussed in Proposition 13.1 in [23] and shown to give a joint characterization of the GIG and the gamma laws by Letac and Wesolowski [19].

B.2. Replacing γ_δ by a general random variable. In this part, we show how to modify both Proposition 5.1 and the general absolute continuity relationship obtained in B.1 when we consider the quantity

$$E[F(T_{2Y}(B)_s,s\leqq t)\widetilde{\Phi}(A_t,e_t,2Y)]$$

for a non-negative function $\widetilde{\Phi}$ on $\mathbf{R}_+^3$ and a non-negative random variable Y, independent of B. We assume that the probability law of Y has a density with respect to the Lebesgue measure and write

$$P(Y \in dy) = (2y)^{-1} q(2y) e^{-y}\, dy.$$

Then, we get, by using (B.1) with $\delta = 1$:

$$E[F(T_{2Y}(B)_s, s \leqq t)\widetilde{\Phi}(A_t, e_t, 2Y)] = E[F(T_{2\gamma_1}(B)_s, s \leqq t)\widetilde{\Phi}(A_t, e_t, 2Y)\frac{1}{2\gamma_1} q(2\gamma_1)]$$
$$= E[F(B_s, s \leqq t)\widetilde{\Phi}\big(A_t + 2Z_t^2\gamma_1, e_t + 2Z_t\gamma_1, \frac{2\gamma_1}{(e_t)^2 + 2A_t\gamma_1}\big)\frac{1}{2\gamma_1} q\big(\frac{2\gamma_1}{(e_t)^2 + 2A_t\gamma_1}\big)$$
$$= E[F(B_s, s \leqq t)\widetilde{\Phi}\big(A_t + 2Z_t^2 Y, e_t + 2Z_tY, \frac{2Y}{(e_t)^2 + 2A_tY}\big)\Delta(e_t, A_t, Y)]$$

where

$$\Delta(e, a, y) = q(2y)^{-1} q\big(\frac{2y}{e^2 + 2ay}\big).$$

We now discuss briefly the modification of (B.3). We have

$$E[\widetilde{\Phi}(A_t, e_t, 2Y)] = E[\widetilde{\Phi}\big(A_t + 2Z_t^2Y, e_t + 2Z_tY, \frac{2Y}{(e_t)^2 + 2A_tY}\big)\Delta(e_t, A_t, Y)].$$

Conditioning with respect to $Z_t = z$ as in B.1 leads us to

$$E[\Phi(X_z, 2Y)] = E[\Phi\big(X_z + 2zY, \frac{2Y}{X_z(X_z + 2zY)}\big)\Lambda_z(X_z, Y)],$$

where $\Lambda_z(x, y) = q(2y)^{-1} q(x^{-1}(x + 2zy)^{-1} 2y)$. Using (2.10), we may write

(B.5) $$E[\Phi(X_z, 2Y)] = E[\Phi\big(\frac{1}{X_z + 2zY}, D_z(X_z, Y)\big)\frac{q(D_z(X_z, Y))}{q(2Y)}],$$

where

$$D_z(x, y) = \frac{1}{z}\big(\frac{1}{x} - \frac{1}{x + 2zy}\big).$$

From formula (B.5) which generalizes (B.4), one may appreciate the simplifying role of the gamma variables (for which the function q is a power) instead of the general variable Y.

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