On infra-red singularities associated with QC photons

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1 Introduction

Block and Nordsieck [1] showed in 1937 how to remove the infra-red divergences from quantum electrodynamics: compute the low frequency classical part to all orders in the fine structure constant, compute a probability, exploit a cancellation in the classical part when a probability is computed, and use perturbation theory only on the surviving remainder. Many applications were made, but in the period from 1965 to 1975 workers such as Kibble [2], Chung [3], Storrow [4], and Zwanziger [5] showed that the momentum-space applications of this method gave incorrect behaviors at large distances, or equivalently in the singularity structure. In particular, the pole-factorization

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property associated with the large-distance fall-off of the propagator of a charged stable particle was disrupted. This creates difficulties with the reduction formulas and with the interpretation of the theory, because the defining characteristics of a physical particle are disrupted.

In a 1983 paper [6] the second-named author showed how this problem could be overcome by going to a coordinate-space treatment. In coordinate space one can center the emissions of the classically describable soft-photon bremsstrahlung radiation associated with a deflection of a charged particle on the coordinate-space vertex at which the deflection occurs. This improves the accuracy of the classical part of the calculation. It also allows the photon interaction to be separated into two parts, a classical part and a quantum part, and allows all the contributions of "classical" photons, which are photons that interact only via the classical part of the interaction, to be gathered into a single exactly definable unitary operator that effectively drops out of computations of probabilities. The remainder is expected, by the Block-Nordsieck argument, to be nicely convergent in the infra-red regime, because the soft-photon classical part has been treated exactly.

One unexpected feature of this treatment pertains to the propagators of the QC photons, which are the photons that connect onto the charged-particle lines by a quantum coupling at one end and a classical coupling at the other: these photons are propagated by a retarded propagator, rather than the usual Feynman propagator.

In 1995 we published a set of three papers [7] that examined in depth the remainder terms involving the soft quantum couplings, and concluded that they were all finite, and gave no contribution to the singularity structure that was as strong as the leading pole contributions. However, in this treatment we used, without explanation, the Feynman propagators for all lines. The reason was that the second-named author had argued informally that we could go back to an earlier form of the result in which only Feynman propagators appeared. However, this author has recently re-examined that argument and has concluded that the final form given in the 1983 paper [6] must be used. The present paper addresses the resulting problem of incorporating the retarded propagators.

A key causality property that was available before, namely that positive energy flows always forward in time, is no longer available for QC photons, and the earlier arguments now fail. We have not encountered any reason to believe that the desired analyticity properties will actually fail, but have not so far been able to construct, as we did before, a general proof that these good properties will always hold. In the present paper we examine some simple cases and find that the expected properties do hold. We also find, and will report, some interesting mathematical properties of these functions that emerge from the microlocal analysis.

The fact that retarded propagators should appear in an expression that is the result of just summing and rearranging the usual Feynman series may seem strange. The proof was given in the 1983 paper [6], and will not be reconstructed here. But the reason is easily described.

Consider a Feynman diagram D consisting of one or more charged-particle closed loops, plus a set of hard-photon external lines connected to the diagram at a set of coordinate points $x = (x_1, x_2, \dots, x_n)$, plus a set of soft-photon lines. [Diagrams with external charged-particle lines are treated by exploiting the proved pole-factorization properties associated with the internal charged-particle lines.] Let $\tilde{F}^D_{op}(x)$ be the scattering operator in photon space that corresponds to this coordinate-space diagram, but with no classical photons. Then the result of adding to it the contribution associated with all numbers of classical photons, both internal and external, is

$$\begin{aligned}\hat{F}^D_{op}(x) &= \exp(a^* \cdot J(L(x)))\tilde{F}^D_{op}\exp(-(a \cdot J(L(x)))) \\ &\quad \times \exp(i\Phi(L(x)) - \langle J^*(L(x) \cdot J(L(x))\rangle) \\ &= U(L(x))\tilde{F}^D_{opr}(x),\end{aligned}$$

where $U(L(x))$ is the unitary operator formed by bringing together the three exponential factors from the preceding lines, and the subscript "opr" signifies that the Feynman propagators for QC particles have been changed to retarded propagators. The change in the propagator type is caused by the fact that the creation and annihilation operators for photons are no longer normal ordered; the factor $\exp(-a \cdot J(L(x)))$ that annihilates classical (C-type) photons has been moved to the left of the operator $\tilde{F}^D_{op}$ that creates quantum (Q-type) photons. This change introduces an extra mass-shell delta function for negative-energy photons flowing from a C-type vertex to a Q-type vertex. This changes the Feynman propagator to a retarded propagator, for QC-type photons (cf. (2.3) below). We need the unitary operator $U(L(x))$ on the left so that it will effectively drop out when a probability is computed, as explained in detail in the 1983 paper [6]. The problem is then to show that the remaining factor has for the dominant analytic structure the same structure found for theories of massive particles.

In our 1995 papers [7] we studied the simplest case with six external lines and six internal lines, with the external momenta arranged so as to put three of the internal particles far away from the mass shell, and the other three close to the mass shell. We then need to verify that the dominant singularity structure along the usual triangle diagram singularity surface is exactly the one associated with the triangle diagram surface when all relevant particles are massive. That singularity type is logarithmic. Our purpose here is to show that this result continues to hold, at least for the simplest cases, when retarded propagators, rather than Feynman propagators, are used for the QC photons.

2 Characteristic features of the problem

In this section we explain some characteristic features of the problem that we encounter by using the retarded propagator

$$\frac{1}{(k_0+i0)^2-\vec{k}^2} \tag{2.1}$$

instead of the Feynman propagator

$$\frac{1}{k^2+i0}\ , \tag{2.2}$$

where k is the momentum from the C-vertex to the Q-vertex, flowing always from right to left. Note that they are related in the following manner:

$$\frac{1}{(k_0+i0)^2-\vec{k}^2} = \frac{1}{k^2+i0} + (2\pi i)\delta^-(k^2), \tag{2.3}$$

where $\delta^-(k^2)$ stands for $\theta(-k_0)\delta(k^2)$ (cf. e.g. [8]). Here, and throughout this paper, we use the same symbols and notations used in our 1995 papers [7]. In our formalism each right-hand end-point C lies on one of the "hard" vertices v_1, v_2 and v_3. Hence our conclusion in [7, p.2510 ff.] concerning the right-most vertex V_R is unaffected by the inclusion of C-vertices; each V_R must coincides with some v_i, $i \in \{1,2,3\}$. However, the arguments pertaining to the left-most vertex V_L are disrupted. A typical example is shown in Figure 1.

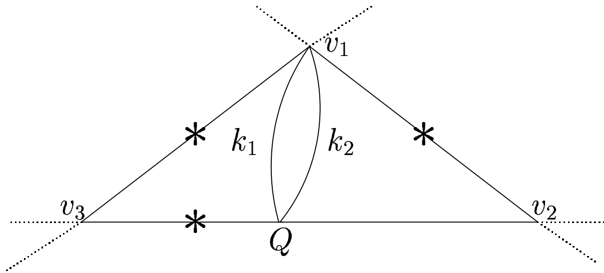

Figure 1. A diagram with $V_R = v_1$ and $V_L = Q$. A starred line stands for a pole factor in the sense of [7].

Diagrams of this sort raise the following question: what is the effect of the singularities associated with the inside triangles formed by v_1, v_2 and Q? An important point to note is that points where $k_1 + k_2 = 0$ with $k_1^2 = k_2^2 = 0$ may be relevant to the resulting singularities; such points are irrelevant if all photon propagators are Feynman propagators, i.e., if only QQ photons are considered. As we will show in Section 3, the study of such singularities is an interesting new issue in microlocal analysis. The vanishing of $k_1 + k_2$ leads to a singularity in the residue factor $1/p_2(k_1 + k_2)$. To supply appropriate $\pm i0$'s to the denominators in the residue factors we need to decompose the domain of integration according to the relative magnitudes of $|k_l|$'s so that $|k_1| = |k_2|$ does not touch the boundary of each integral (considered in polar coordinates). (Cf. [7, p.2497].) Since such a decomposition is not unique, we have to consider all relevant terms simultaneously to assert that the net contribution is not stronger than the logarithmic singularities along the Landau surface associated with the triangle diagram. That is, we have to simultaneously consider diagrams in Figure 2 if we want to discuss the effect of the diagram in Figure 1.

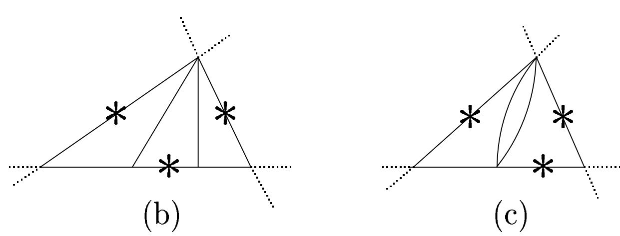

Figure 2. Diagrams accompanying the diagram in Fig. 1.

We note that we have to analyze each diagram in Figures 1 and 2 separately making use of different techniques. (Cf. Section 3.)

3 Study of some basic examples

In this section we study some basic examples of QC couplings and confirm that the resulting singularities are not stronger than the ordinary triangle diagram singularities, i.e., the logarithmic singularity. Actually we confirm they are strictly weaker than the logarithmic singularity.

Let us first study the diagrams in Figures 1 and 2. For the sake of uniformity the diagram in Figure 1 is labeled as (3.1.a) and diagrams (b) and (c) in Figure 2 are respectively labeled as (3.1.b) and (3.1.c). In what follows we often omit the dotted external lines for the sake of simplicity.

Using the power-counting result ([7, p.2496 ff.]) and assigning $+i0$ uniformly to each residue factor ([7, p.2507]) the integral F_c associated with the diagram (3.1.c), which is the simplest to analyze, is given in the polar coordinate system for k_l's (cf. [7]) by (3.2) below. Here and in what follows we use the following labeling and orientation of lines of triangle diagram:

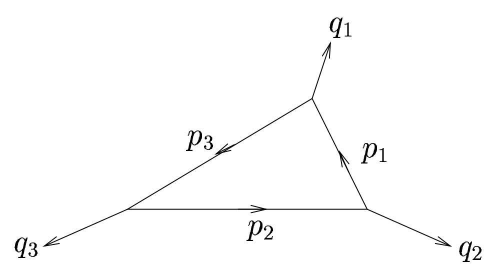

Figure 3.

$$F_c(q_1,q_2) = \int d^4p_3 \frac{1}{p_1^2 - m^2 + i0} \frac{1}{p_2^2 - m^2 + i0} G \tag{3.2}$$

with

$$\begin{aligned} G = & \int_0^\kappa dr_1 \int_0^c dr_2 \int d^4\Omega_1 \delta(\Omega_1 \widetilde{\Omega}_1 - 1) \\ & \times \int d^4\Omega_2 \delta(\Omega_2 \widetilde{\Omega}_2 - 1) \frac{1}{(p_3 - r_1(\Omega_1 + r_2\Omega_2))^2 - m^2 + i0} \\ & \times \frac{1}{2p_2(\Omega_1 + r_2\Omega_2) + i0} \frac{1}{(\Omega_{1,0} + i0)^2 - \overrightarrow{\Omega}_1^2} \frac{1}{(\Omega_{2,0} + i0)^2 - \overrightarrow{\Omega}_2^2} \end{aligned} \tag{3.3}$$

where $\widetilde{\Omega}_j = (\Omega_{j,0}, -\overrightarrow{\Omega}_j)$ for $\Omega_j = (\Omega_{j,0}, \overrightarrow{\Omega}_j)$, $\kappa > 0$ and $c > 1$. Here κ denotes a cut-off parameter, c designates the domain of integration, and we have dropped small and unimportant terms in the denominators and ignored numerator factors. By ignoring physically unimportant contribution from $r_1 = \kappa$, we find

$$G = (\log(p_3^2 - m^2 + i0)) \times H, \tag{3.4}$$

where

$$\begin{aligned} H = &\int_0^c dr_2 \int d^4\Omega_1 \delta(\Omega_1 \widetilde{\Omega}_1 - 1) \int d^4\Omega_2 \delta(\Omega_2 \widetilde{\Omega}_2 - 1) \\ &\times \frac{1}{2p_2(\Omega_1 + r_2\Omega_2) + i0} \frac{1}{2p_3(\Omega_1 + r_2\Omega_2) - i0} \\ &\times \frac{1}{(\Omega_{1,0} + i0)^2 - \vec{\Omega}_1^2} \frac{1}{(\Omega_{2,0} + i0)^2 - \vec{\Omega}_2^2} \, . \end{aligned} \tag{3.5}$$

We now verify that H is non-singular if we ignore the non-physical contribution from $r_2 = c$. If H is non-singular, (3.4) immediately entails that F_c has the form

$$A\varphi \log(\varphi + i0) + B \tag{3.6}$$

with A and B being analytic near the Landau surface $\{\varphi = 0\}$. As we show below, microlocal analysis summarized in the form of Landau table is effective in confirming the analyticity of H. The Landau table for the integral H is as follows, if we ignore the contribution from $r_2 = c$.

	dp_2	dp_3	$d\Omega_1$	$d\Omega_2$
1	$\Omega_1 + r_2\Omega_2$	0	p_2	$r_2 p_2$
2	0	$-(\Omega_1 + r_2\Omega_2)$	$-p_3$	$-r_2 p_3$
3	0	0	$\sigma_1\Omega_1$	0
4	0	0	0	$\sigma_2\Omega_2$
5	0	0	$\pm\widetilde{\Omega}_1$	0

Here $\sigma_j = \operatorname{sign} \Omega_{j,0}$. Note that the sixth row corresponding to $\delta(\Omega_2\widetilde{\Omega}_2 - 1)$ has been omitted in the above table as in [7]; the closed loop condition for the dr_2-column, which is also omitted in the above table, guarantees that the row is irrelevant to the singularity of (3.5) (cf. [7, Appendix A]). Needless to say, the closed loop condition for the dr_2-column originates from the r_2-integration in (3.5). On the other hand, the integral (3.5) does not contain

r_1-integration. Hence we have to include the fifth row in the above table. Fortunately, however, there is no net contribution from the inclusion of the fifth row. In fact, the closed loop conditions are:

$$\alpha_1 p_2 - \alpha_2 p_3 + \alpha_3 \sigma_1 \Omega_1 + \beta_5 \widetilde{\Omega}_1 = 0 \ , \tag{3.7}$$

$$\alpha_1 r_2 p_2 - \alpha_2 r_2 p_3 + \alpha_4 \sigma_2 \Omega_2 = 0 \tag{3.8}$$

with α_j non-negative and β_5 real. Multiplying (3.7) by Ω_1 and (3.8) by Ω_2 respectively, and then summing them up we find

$$\beta_5 \Omega_1 \widetilde{\Omega}_1 = 0, \quad \textit{i.e.}, \quad \beta_5 = 0 \tag{3.9}$$

by noting that α_1 (resp., α_2) vanishes unless $p_2(\Omega_1 + r_2\Omega_2)$ (resp., $p_3(\Omega_1 + r_2\Omega_2)$) vanishes. The relation (3.9) means that we may ignore the fifth row.

By the mass-shell constraint on p_2 and $-p_3$, $\alpha_1 p_2 - \alpha_2 p_3$ cannot be a light cone vector unless $(\alpha_1, \alpha_2) = 0$. Thus (3.7) and (3.8) imply $\alpha_j = 0$ ($j = 1, 2, 3, 4$). This means that H is well-defined and non-singular. Therefore F_c has the form (3.6).

Next we study the integral associated with the diagram (3.1.b). In this case the residue factors are non-singular when $\Omega_1^2 = \Omega_2^2 = 0$. Hence nothing peculiar to QC-couplings can occur; the analysis of the integral is basically the same as that for non-separable diagrams in QQ couplings ([7, p.2514-p.2515]). For the convenience of the reader we briefly discuss how the computation is done. Let us concentrate our attention to a point near $r_1 = 0$ and $r_2 = 1$. Since the closed loop condition for $d\Omega_j$-column ($j = 1, 2$) in the Landau table forces the Landau constant associated with Ω_j^2 ($j = 1, 2$, respectively) to vanish, it suffices for us to consider the following integral:

$$\begin{aligned} F_b(q_2, q_3) = & \int_0^{\kappa} dr_1 \int_{1-\kappa'}^{1+\kappa'} dr_2 \\ & \times \int_{|\Omega_1|=1} d^4\Omega_1 \int_{|\Omega_2|=1} d^4\Omega_2 \log(\varphi(q_2 - r_1 r_2 \Omega_2, q_3 - r_1\Omega_1) + i0), \end{aligned} \tag{3.10}$$

where κ and κ' are sufficiently small positive constants. Here we have used the fact that the singularity of the integral associated with the diagram in

Figure 3 is a logarithmic one near the triangle diagram singularity surface $\{\varphi = 0\}$. Since it follows from the Landau equation that

$$\frac{\partial\varphi}{\partial q_2} = -\alpha_1 p_1, \quad \frac{\partial\varphi}{\partial q_3} = \alpha_3 p_3 \ , \tag{3.11}$$

where α_j is the Landau constant associated with p_j, we find

$$\left.\frac{\partial\varphi(q_2 - r_1 r_2\Omega_2, q_3 - r_1\Omega_1)}{\partial r_1}\right|_{r_1 = 0} = \alpha_1 p_1 r_2 \Omega_2 - \alpha_3 p_3 \Omega_1 \ . \tag{3.12}$$

If we consider the problem in a neighborhood of $\{r_1 = 0, \Omega_1^2 = \Omega_2^2 = 0, \Omega_1 /\!\!/ \Omega_2\}$, we can readily confirm that the right-hand side of (3.12) does not vanish. (Cf. the remark at the end of this paragraph.) With this non-vanishing property of $\partial\varphi/\partial r_1$ we can easily compute the integral (3.10) to find it has again the singularity of the form (3.6). Thus the singularity structure of F_b is again described by (3.6). (The non-vanishing property of $\partial\varphi/\partial r_1$ can be verified as follows: if $\Omega_1 = -\Omega_2$ and $r_2 = 1$, the right-hand side of (3.12) is equal to

$$(\alpha_1 p_1 + \alpha_3 p_3)\Omega_2 \ . \tag{3.13}$$

Using the closed loop condition for the ordinary triangle diagram we find it is equal to

$$-\alpha_2 p_2 \Omega_2 \ . \tag{3.14}$$

Since we consider the problem at a point where $\alpha_2 \gneqq 0$, this is different from 0. If $\Omega_1 = \Omega_2$ and $r_2 = 1$, then, as $\alpha_1 p_1 - \alpha_3 p_3$ is a massive vector, $(\alpha_1 p_1 - \alpha_3 p_3)\Omega_1$ cannot vanish either. Thus the right-hand side of (3.12) is shown to be different from 0 at the point in question.)

Let us now study the integral F_a associated with the diagram (3.1.a). By the same computation done for F_c, we find that the singularity of F_a is again of the form (3.6) if the following integral $\widetilde{H}$ is well-defined and analytic ignoring the non-physical contributions from $r_2 = 1 \pm \kappa$.

$$\widetilde{H} = \int_{1-\kappa}^{1+\kappa} dr_2 \int_{|\Omega_1|=1} d^4\Omega_1 \int_{|\Omega_2|=1} d^4\Omega_2 \times \frac{1}{2p_2(\Omega_1 + r_2\Omega_2) + i0} \frac{1}{2p_1(\Omega_1 + r_2\Omega_2) + i0} \times \frac{1}{(\Omega_{1,0} + i0)^2 - \vec{\Omega}^2} \frac{1}{(\Omega_{2,0} + i0)^2 - \vec{\Omega}_2^2}; \tag{3.15}$$

The trouble is, however, that the usual reasoning based on microlocal analysis cannot guarantee the well-definedness of $\widetilde{H}$; in the Landau table for the integral $\widetilde{H}$ all columns may sum up to 0 with some non-zero Landau constants. (The so-called $u = 0$ problem.) This means the product of several factors in the integrand of (3.15) is not guaranteed to be well-defined by the general theory of microlocal analysis [9]. If such a point "naturally" appears, we usually find that the singularities of relevant factors are rather tame and that the tameness (such as continuity) makes their product well-defined. Fortunately we can find some "tameness" in this case, although the "tameness" we encounter below is a quite novel one. To find the tameness we have to compute the integral explicitly. To perform the explicit computation we replace the propagator by $\delta(\Omega_j^2)$ and use the frame where $p_2 = (m, 0, 0, 0)$. Letting x_j denote the component of $\vec{\Omega}_j$ in the $\vec{p}_1$ direction, and normalizing $m = 1$ for simplicity, we can then rewrite the integral (3.15) on a neighborhood of $\{\Omega_1 = -\Omega_2\}$ in the following form up to a constant factor:

$$I = \int_{1-\kappa}^{1+\kappa} dr_2 \int_{-1}^{1} dx_1 \int_{-1}^{1} dx_2 \times \frac{1}{1 - r_2 + i0} \frac{1}{1 - ax_1 - r_2(1 - ax_2) - i0}, \tag{3.16}$$

where a is a non-zero and small positive analytic function of p_1. Here the $-i0$ in the second factor is due to the fact that $p_{1,0}p_{2,0} < 0$. The Landau table for (3.16) is as follows:

dr_2	dx_1	dx_2	da
-1	0	0	0
$1-ax_2$	a	$-ar_2$	$x_1-r_2x_2$
0	$\pm 1_{(\text{end})}$	0	0
0	0	$\pm 1_{(\text{end})}$	0

Here $1_{(\text{end})}$ indicates that the component survives only at the end point $x_j = +1$ or -1. It is evident that $u = 0$ points appear when $x_1 = x_2 = \pm 1$ and $r_2 = 1$. However a straightforward computation shows

$$
\begin{aligned}
(3.17) \qquad & \int_{-1}^{1} dx_1 \int_{-1}^{1} dx_2 \frac{1}{1-ax_1-r_2(1-ax_2)-i0} \\
&= \frac{1}{a^2r_2}[(1+a)(1-r_2)\log\{(1+a)(1-r_2)-i0\} \\
&\quad - (1-a)(1-r_2)\log\{(1-a)(1-r_2)-i0\} \\
&\quad + g] \ ,
\end{aligned}
$$

where

$$
\begin{aligned}
(3.18) \quad g =& (2ar_2+(1+a)(1-r_2))\log\{2ar_2+(1+a)(1-r_2)-i0\} \\
&+ (-2ar_2+(1-a)(1-r_2))\log\{-2ar_2+(1-a)(1-r_2)-i0\}
\end{aligned}
$$

is non-singular near $r_2 = 1$ (as a is small and non-zero). Thus we have clearly found the origin of the $u = 0$ problem, and at the same time, understood why it is not a real problem. First the first two terms in the right-hand side of (3.17) respectively come from $x_1 = x_2 = -1$ and $x_1 = x_2 = 1$, and they are boundary values taken from the domain $\{\mathrm{Im}(1-r_2) < 0\}$, while the integrand of I contains a factor $(1-r_2+i0)^{-1}$. Fortunately the singular part of (3.17) contains a factor that kills this singularity $(1-r_2+i0)^{-1}$. Thus the integrand of I is well-defined in spite of the existence of a $u = 0$ problem. Then it is clear that the resulting function is analytic in a. Therefore $\widetilde{H}$ is well-defined and analytic; hence the singularity of F_a associated with the diagram (3.1.a) is again of the form (3.6).

Remark. The argument given above shows that, if we assign $-i0$ instead of $+i0$ to each residue factor in diagrams (3.1.a), (3.1.b) and (3.1.c), then a $u = 0$ problem appears in the diagram (3.1.c) and the computation of F_a

becomes simple. It is also worth mentioning that the resulting singularities of F_a, F_b and F_c are all of the form (3.6), which is much weaker than the ordinary triangle diagram singularity.

To see what occurs in more complicated diagrams let us examine the following diagrams given in Figure 4; each of them contains two QC photon lines and one QQ photon line.

As we will see below, we can analyze the functions associated with these diagrams by the same method as that used to analyze integrals associated with diagrams (3.1.a), (3.1.b) or (3.1.c); the difference is just a combinatorial complexity.

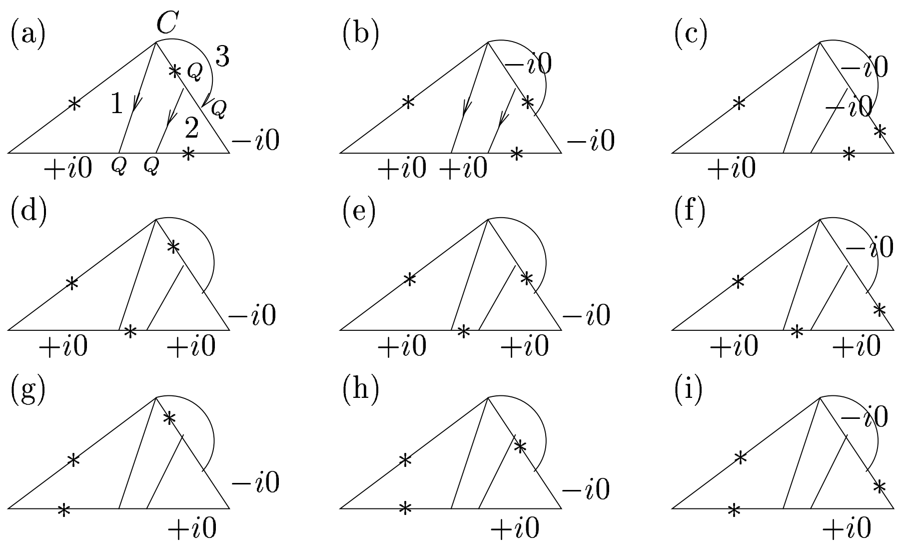

Figure 4. Diagrams with two QC photon lines and one QQ photon line.

The reason we treat these 9 diagrams in Figure 4 at the same time is that the uniform assignment of $\pm i0$ to the residue factors on the bottom segment (i.e. $+i0$) and those on the right slope (i.e., $-i0$) forces us to use several different techniques to analyze each diagram. For the sake of simplicity we discuss the problem in the region where $|k_j|$ $(j = 1, 2, 3)$ are of the same magnitude, i.e. $r_2 \neq 0$, $r_3 \neq 0$ with $r_1 \geq 0$ so that both $(p_1(\Omega_2 - r_3\Omega_3) - i0)^{-1}$ and $(p_2(\Omega_1 + r_2\Omega_2) + i0)^{-1}$ may become singular.

Let us now study the singularity structure of the function associated with each diagram in Figure 4. In what follows we freely use the power-counting result obtained in [7, p.2496 ff.] in rewriting the integration over k-variables to that over the (r, Ω)-variables.

Case(a): Let us study the integral associated with the diagram given in Figure 4(a). The Landau table is:

	$d\Omega_1$	$d\Omega_2$	$d\Omega_3$	dp
1	$r_1(p_1+r_1(\Omega_1+r_2r_3\Omega_3))$	0	$r_1r_2r_3(p_1+r_1(\Omega_1+r_2r_3\Omega_3))$	$p_1+r_1(\Omega_1+r_2r_3\Omega_3)$
2	0	$-p_1$	r_3p_1	$-\Omega_2+r_3\Omega_3$
3	$r_1(p_2+r_1(\Omega_1+r_2\Omega_2))$	$r_1r_2(p_2+r_1(\Omega_1+r_2\Omega_2))$	0	$p_2+r_1(\Omega_1+r_2\Omega_2)$
4	p_2	r_2p_2	0	$\Omega_1+r_2\Omega_2$
5	0	0	0	p_3
6	$\sigma\Omega_1$	0	0	0
7	0	Ω_2	0	0
8	0	0	Ω_3	0

Here p denotes a loop momentum of the triangle diagram and we omitted the rows corresponding to non-singular residue factors such as $(p_1\Omega_2 - i0)^{-1}$ etc. The symbol σ indicates the line 1 is with a retarded propagator, as usual. The symbol $\sigma_3\Omega_3$ may be used, but for simplicity we omitted σ_3. Since $\alpha_2 p_1 + \alpha_4 r_2 p_2$ $(\alpha_2, \alpha_4 \geq 0)$ cannot be a light-cone vector unless it vanishes, the closed loop condition for the $d\Omega_2$ column implies $\alpha_2 = \alpha_4 = 0$. Hence we may detour the singularity at $p_1(\Omega_2 - r_3\Omega_3) = 0$ and that at $p_2(\Omega_1 + r_2\Omega_2) = 0$. Hence by integrating over the triangle loop momentum p first, we are to calculate the following integral:

$$
\begin{aligned}
(3.19) \qquad F_a = & \iiint_{|\Omega_1|=|\Omega_2|=|\Omega_3|=1} d^4\Omega_1 d^4\Omega_2 d^4\Omega_3 \int_0^{\kappa} dr_1 \int_{r_2\approx 1} dr_2 \int_{r_3\approx 1} dr_3 \\
& \times f_\Delta(q_2 - r_1r_2r_3\Omega_3 + r_1r_2\Omega_2, q_3 - r_1\Omega_1 - r_1r_2\Omega_2),
\end{aligned}
$$

where f_Δ denotes an analytic function multiple of the triangle singularity, i.e., $\log(\varphi(q_2, q_3) + i0)$. Here we have used the power-counting result for Q-couplings to rewrite d^4k to $drd\Omega$ without extra divergent factor like r_1^{-1}. We now calculate

$$
\begin{aligned}
(3.20) \qquad & \frac{\partial\varphi}{\partial r_1}(q_2 - r_1r_2r_3\Omega_3 + r_1r_2\Omega_2, q_3 - r_1\Omega_1 - r_1r_2\Omega_2)|_{r_1=0} \\
= & \frac{\partial\varphi}{\partial q_2}(-r_2r_3\Omega_3 + r_2\Omega_2) - \frac{\partial\varphi}{\partial q_3}(\Omega_1 + r_2\Omega_2) \\
= & a_1p_1(r_2r_3\Omega_3 - r_2\Omega_2) - a_3p_3(\Omega_1 + r_2\Omega_2),
\end{aligned}
$$

where a_j denotes the Landau constant associated with $(p_j^2 - m^2 + i0)^{-1}$ in the triangle singularity. By choosing the detours so that $\operatorname{Im} p_1(r_2\Omega_2 - r_2r_3\Omega_3)$

may become much bigger than $\operatorname{Im} p_3(\Omega_1 + r_2\Omega_2)$ we find $\frac{\partial\varphi}{\partial r_1}|_{r_1=0}$ does not vanish. Hence the integral (3.19) can be readily computed to produce the singularity of the form (3.6).

Case(b): In studying the integral associated with the diagram in Figure 4(b), it is easier to analyze the associated integral by choosing a new loop momentum $p_3 + k_1 = \tilde{p}$ as the integration variable. Then the Landau table is as follows:

	$d\Omega_1$	$d\Omega_2$	$d\Omega_3$	$d\tilde{p}$
1	0	$r_1r_2(\tilde{p}_1+r_1r_2(\Omega_2+r_3\Omega_3))$	$r_1r_2r_3(\tilde{p}_1+r_1r_2(\Omega_2+r_3\Omega_3))$	$\tilde{p}_1+r_1r_2(\Omega_2+r_3\Omega_3)$
2	0	$r_1r_2(\tilde{p}_2+r_1r_2\Omega_2)$	0	$\tilde{p}_2+r_1r_2\Omega_2$
3	$\tilde{p}_2$	$r_2\tilde{p}_2$	0	$\Omega_1+r_2\Omega_2$
4	$-r_1\tilde{p}_3$	0	0	$\tilde{p}_3-r_1\Omega_1$
5	$\sigma\Omega_1$	0	0	0
6	0	Ω_2	0	0
7	0	0	Ω_3	0

Here $\tilde{p}_j$ denotes $p_j + k_1$. Since $\alpha_3\tilde{p}_2 - \alpha_4 r_1\tilde{p}_3$ cannot be a light cone vector unless it vanishes, the closed loop condition for the $d\Omega_1$-column implies $\alpha_3 = 0$. Hence we can detour the singularity $\tilde{p}_2(\Omega_1 + r_2\Omega_2) = 0$. This time the integral F_b that corresponds to F_a is obtained by replacing the integrand of F_a by

$$f_\triangle(q_2 - r_1r_2r_3\Omega_3, q_3 - r_1(\Omega_1 + r_2\Omega_2)). \tag{3.21}$$

On the other hand we find

$$\begin{aligned}&\frac{\partial\varphi}{\partial r_1}(q_2 - r_1r_2r_3\Omega_3, q_3 - r_1(\Omega_1 + r_2\Omega_2))|_{r_1=0}\\ =&a_1p_1r_2r_3\Omega_3 - a_3p_3(\Omega_1 + r_2\Omega_2).\end{aligned} \tag{3.22}$$

Hence by choosing the detour so that $\operatorname{Im} p_3(\Omega_1 + r_2\Omega_2)$ is much greater than $\operatorname{Im} p_1\Omega_3$ we find the same result as for the diagram in Fig. 4(a).

Case (c): To study the integral associated with the diagram in Fig. 4(c), we use a method different from that used for analyzing the integral associated with the diagram in Fig. 4(b) (although the same method may be employed). Choosing $p_3 + k_1 + k_2 = \tilde{p}_3$ as a new variable, we integrate $(\tilde{p}_3^2 - m^2 + i0 - 2r_1(\tilde{p}_3(\Omega_1 + r_2\Omega_2) - i0))^{-1}$ over dr_1. The contribution from $r_1 = 0$ is then

$$[\log(\tilde{p}_3^2 - m^2 + i0)]\left(\frac{1}{2\tilde{p}_3(\Omega_1 + r_2\Omega_2) - i0}\right). \tag{3.23}$$

Then the "Landau table" for the product

$$(3.24)\quad (\tilde{p}_1(\Omega_2 - r_3\Omega_3) - i0)^{-1}(\tilde{p}_2(\Omega_1 + r_2\Omega_2) + i0)^{-1}(\tilde{p}_3(\Omega_1 + r_2\Omega_2) - i0)^{-1} \times ((\Omega_{1,0} + i0)^2 - \vec{\Omega}_1^2)^{-1}(\Omega_2^2 + i0)^{-1}((\Omega_{3,0} + i0)^2 - \vec{\Omega}_3^3)^{-1}$$

is given as follows:

	$d\Omega_1$	$d\Omega_2$	$d\Omega_3$	$d\tilde{p}_1$	$d\tilde{p}_2$	$d\tilde{p}_3$
1	0	$-\tilde{p}_1$	$r_3\tilde{p}_1$	$-\Omega_2 + r_3\Omega_3$	0	0
2	$\tilde{p}_2$	$r_2\tilde{p}_2$	0	0	$\Omega_1 + r_2\Omega_2$	0
3	$-\tilde{p}_3$	$-r_2\tilde{p}_3$	0	0	0	$-(\Omega_1 + r_2\Omega_2)$
4	$\sigma_1\Omega_1$	0	0	0	0	0
5	0	Ω_2	0	0	0	0
6	0	0	$\sigma_3\Omega_3$	0	0	0

Since $-\alpha_1\tilde{p}_1 + \alpha_2 r_2\tilde{p}_3 - \alpha_3 r_2\tilde{p}_3$ cannot be a light-cone vector, the closed loop condition for the $d\Omega_2$-column implies $\alpha_1 = \alpha_2 = \alpha_3 = 0$ (as $r_2 \neq 0$); then we also find $\alpha_4 = \alpha_5 = \alpha_6 = 0$. This means that the product (3.24) is well-defined and its integration over $d^4\Omega_1 d^4\Omega_2 d^4\Omega_3$ gives an analytic function. (Strictly speaking we have to take into account of $\delta(\Omega_1\tilde{\Omega}_1 - 1)$ as r_1-integration has been done to get the factor $(2\tilde{p}_3(\Omega_1 + r_2\Omega_2) - i0)^{-1}$. But, the argument is exactly the same as that for the integral H given by (3.5) and we do not give the detailed argument here. This remark applies also to the discussion in Case (g) and Case (h) below.) Thus, by combining (3.23) with other two poles and integrating them over $\tilde{p}_1$, we find the function associated with Fig. 4(c) is again of the form (3.6).

Case(d): The Landau table for the integral associated with the diagram in Fig. 4(d) is as follows:

	$d\Omega_1$	$d\Omega_2$	$d\Omega_3$	dp
1	$r_1(p_1+r_1(\Omega_1+r_2r_3\Omega_3))$	0	$r_1r_2r_3(p_1+r_1(\Omega_1+r_2r_3\Omega_3))$	$p_1+r_1(\Omega_1+r_2r_3\Omega_3)$
2	0	$-p_1$	r_3p_1	$-\Omega_2 + r_3\Omega_3$
3	$r_1(p_2 + r_1\Omega_1)$	0	0	$p_2 + r_1\Omega_1$
4	0	0	0	p_3
5	$\sigma_1\Omega_1$	0	0	0
6	0	Ω_2	0	0
7	0	0	$\sigma_3\Omega_3$	0

We then obtain $\alpha_2 = 0$ by the closed loop condition for the $d\Omega_2$-column. Hence we can find a distortion avoiding $p_1(\Omega_2 - r_3\Omega_3) = 0$. The integral corresponding to (3.19) is obtained by replacing the integrand by $f_\Delta(q_2 - r_1r_2r_3\Omega_3, q_3 - r_1\Omega_1)$. Since

$$\begin{aligned}\tag{3.25} &\frac{\partial\varphi}{\partial r_1}(q_2 - r_1r_2r_3\Omega_3, q_3 - r_1\Omega_1)|_{r_1=0} \\ =&(a_1p_1)(r_2r_3\Omega_3) - (a_3p_3)\Omega_1.\end{aligned}$$

Thus by choosing a detour so that $\operatorname{Im} p_1\Omega_3$ is much bigger than $\operatorname{Im} p_3\Omega_1$, we find $\partial\varphi/\partial r_1|_{r_1=0}$ does not vanish, and hence the integral in question has the form (3.6).

Case (e): The residue factors in the integrand of the integral associated with the diagram in Fig. 4(e) are all non-singular. Hence it suffices to confirm $\frac{\partial\varphi}{\partial r_1}(q_2 - r_1r_2(\Omega_2 + r_3\Omega_3), q_3 - r_1\Omega_1)|_{r_1=0}$ is different from 0. In fact

$$\begin{aligned}\tag{3.26} &\frac{\partial\varphi}{\partial r_1}(q_2 - r_1r_2(\Omega_2 + r_3\Omega_3), q_3 - r_1\Omega_1)|_{r_1=0} \\ =&a_1r_2r_3p_1(\Omega_2 + r_3\Omega_3) - a_3p_3\Omega_1.\end{aligned}$$

Since we are considering the problem near $\{\Omega_1 = -\Omega_2 = -\Omega_3\}$, this is close to $(a_1p_1 + a_3p_3)\omega - a_1p_1\omega = a_3p_3\omega$ $(\omega = \Omega_2)$. Thus, reflecting the fact that no residue factor is singular in this case, the condition $\partial\varphi/\partial r_1|_{r_1=0} \neq 0$ is automatically satisfied. Hence the resulting singularity is again of the form (3.6).

Case (f): The Landau table for the integral associated with the diagram in Fig. 4(f) is as follows:

	$d\Omega_1$	$d\Omega_2$	$d\Omega_3$	dp
1	$r_1(p_1+r_1(\Omega_1+r_2\Omega_2))$	$r_1r_2(p_1+r_1(\Omega_1+r_2\Omega_2))$	0	$p_1+r_1(\Omega_1+r_2\Omega_2)$
2	0	$-p_1$	r_3p_1	$-\Omega_2+r_3\Omega_3$
3	$r_1(p_2+r_1\Omega_1)$	0	0	$p_2+r_1\Omega_1$
4	0	0	0	p_3
5	$\sigma_1\Omega_1$	0	0	0
6	0	Ω_2	0	0
7	0	0	$\sigma_3\Omega_3$	0

The closed loop condition for the $d\Omega_3$-column then implies $\alpha_2 = 0$. (Note that, if $r_3 = 0$, the residue factor corresponding to the second row is non-singular and hence $\alpha_2 = 0$.) Hence we can detour $p_1(\Omega_2 - r_3\Omega_3) = 0$. In this case the integral that corresponds to (3.19) has $f_\Delta(q_2 - r_1 r_2 \Omega_2, q_3 - r_1\Omega_1)$ as its integrand. Since

$$\begin{aligned}\frac{\partial\varphi}{\partial r_1}&(q_2 - r_1 r_2\Omega_2, q_3 - r_1\Omega_1)|_{r_1=0}\\ =&a_1 r_2 p_1 \Omega_2 - a_3 p_3 \Omega_1,\end{aligned} \tag{3.27}$$

at the point in question this is close to $a_1 p_1 \omega + a_3 p_3 \omega = -a_2 p_2 \omega$ ($\omega = \Omega_2$). In this case again $\partial\varphi/\partial r_1|_{r_1=0}$ does not vanish (without the help of the distortion). Thus we conclude again the singularity structure (3.6).

Case (g): To analyze the integral associated with the diagram in Fig. 4(g), we use the same technique as for Fig. 4(c). Then we need to consider the product

$$\begin{aligned}&(p_1(\Omega_1 + r_2 r_3 \Omega_3) + i0)^{-1}(p_1(\Omega_2 - r_3\Omega_3) - i0)^{-1}\\ \times&(p_2(\Omega_1 + r_2\Omega_2) + i0)^{-1}((\Omega_{1,0} + i0)^2 - \vec{\Omega}_1^2)^{-1}\\ \times&(\Omega_2^2 + i0)^{-1}((\Omega_{3,0} + i0)^2 - \vec{\Omega}_3^2)^{-1}.\end{aligned} \tag{3.28}$$

The "Landau table" for this product is:

	$d\Omega_1$	$d\Omega_2$	$d\Omega_3$	dp_1	dp_2
1	p_1	0	$r_2 r_3 p_1$	$\Omega_1 + r_2 r_3 \Omega_3$	0
2	0	$-p_1$	$r_3 p_1$	$-\Omega_2 + r_3\Omega_3$	0
3	p_2	$r_2 p_2$	0	0	$\Omega_1 + r_2\Omega_2$
4	$\sigma_1\Omega_1$	0	0	0	0
5	0	Ω_2	0	0	0
6	0	0	$\sigma_3\Omega_3$	0	0

It is now clear that the product is well-defined and the final integral is again of the form (3.6).

Case (h): To analyze the integral associated with the diagram in Fig. 4(h), we use the same technique as that used in Case (g). This time the product to be considered after performing the integration

$$\int_0^\kappa dr_1 (p_1^2 - m^2 + i0 + 2r_1(p_1(\Omega_1 + r_2\Omega_2 + r_2 r_3 \Omega_3) + i0))^{-1}$$

is

$$\begin{aligned}&(p_1(\Omega_1 + r_2\Omega_2 + r_2 r_3\Omega_3) + i0)^{-1}\\ \times&(p_2(\Omega_1 + r_2\Omega_2) + i0)^{-1}((\Omega_{1,0} + i0)^2 - \vec{\Omega}_1^2)^{-1}\\ \times&(\Omega_2^2 + i0)^{-1}((\Omega_{3,0} + i0)^2 - \vec{\Omega}_3^2)^{-1},\end{aligned} \tag{3.29}$$

and the associated “Landau table” is

	$d\Omega_1$	$d\Omega_2$	$d\Omega_3$	dp_1	dp_2
1	p_1	r_2p_1	$r_2r_3p_1$	$\Omega_1 + r_2\Omega_2 + r_2r_3\Omega_3$	0
2	p_2	r_2p_2	0	0	$\Omega_1 + r_2\Omega_2$
3	$\sigma_1\Omega_1$	0	0	0	0
4	0	Ω_2	0	0	0
5	0	0	$\sigma_3\Omega_3$	0	0

Since we are considering the problem in a region where $r_2, r_3 \approx 1$, it is clear that no $u = 0$ problem arises, and thus the resulting integral is of the form (3.6). If we allow $r_3 = 0$, then a $u = 0$ problem arises. Inclusion of the point $r_3 = 0$ would result in $\varphi(\log(\varphi + i0))^2$-singularity, instead of (3.6).

Case (i): To analyze the integral associated with the diagram in Fig. 4(i), again we perform the integration

$$\int_0^\kappa dr_1(p_1^2 - m^2 + i0 + 2r_1(p_1(\Omega_1 + r_2\Omega_2) + i0))^{-1}$$

first and pick up the contribution from $r_1 = 0$. Then the product to be considered is

$$\begin{aligned}(3.30)\qquad &(p_1(\Omega_1 + r_2\Omega_2) + i0)^{-1}\\ &\times(p_1(\Omega_2 - r_3\Omega_3) - i0)^{-1}(p_2(\Omega_1 + r_2\Omega_2) + i0)^{-1}\\ &\times((\Omega_{1,0} + i0)^2 - \vec{\Omega}_1^2)^{-1}(\Omega_2^2 + i0)^{-1}((\Omega_{3,0} + i0)^2 - \vec{\Omega}_3^2)^{-1}.\end{aligned}$$

Writing down the Landau table for the product (3.30), one can readily see that it admits a $u = 0$ point just as in the case of the integral associated with the diagram (3.1.a). To analyze this troublesome product we first consider the (r_3, Ω_3)-dependent factor and integrate it over $dr_3 d^4\Omega_3$, i.e., we first consider the following integral:

$$(3.31)\qquad \int_{1-\kappa}^{1+\kappa} dr_3 \int_{|\Omega_3|=1} d^4\Omega_3 (p_1(\Omega_2 - r_3\Omega_3) - i0)^{-1}((\Omega_{3,0} + i0)^2 - \vec{\Omega}_3^2)^{-1}.$$

The associated “Landau table” is as follows.

	$d\Omega_3$	dp_1	$d\Omega_2$
1	r_3p_1	$-\Omega_2 + r_3\Omega_3$	$-p_1$
2	$\sigma_3\Omega_3$	0	0

Since the closed loop condition for the $d\Omega_3$-column implies $\alpha_1 = \alpha_2 = 0$ near $r_3 = 1$, (3.31) is well-defined and analytic in (p_1, Ω_2). The part of (3.30) which is irrelevant to (r_3, Ω_3) is, i.e.,

$$(3.32) \qquad \begin{aligned} &(p_1(\Omega_1 + r_2\Omega_2) + i0)^{-1}(p_2(\Omega_1 + r_2\Omega_2) + i0)^{-1} \\ &\times((\Omega_{1,0} + i0)^2 - \vec{\Omega}_1^2)^{-1}(\Omega_2^2 + i0)^{-1}, \end{aligned}$$

is the integrand we encounted in (3.14). (The difference between $(\Omega_2^2 + i0)^{-1}$ and $((\Omega_{2,0} + i0)^2 - \vec{\Omega}_2^2)^{-1}$ is not important.) Since the $u = 0$ problem for the integral (3.14) has been resolved, we can perform the integration of (3.30) over $(r_2, r_3, \Omega_1, \Omega_2, \Omega_3)$ near $r_2 = r_3 = 1$, and we then obtain an analytic function of (p_1, p_2). Thus the integral associated with the diagram in Fig. 4(i) is again of the form (3.6).

The study done in this section indicates that the effect of the QC problem discussed in Section 2 should be strictly weaker than the ordinary triangle diagram singularity, although we have not yet proved the fact in general; since the infra-red finiteness has been confirmed in general ([7, p.2496 ff.]), the problem of confirming the weakness of the resulting integrals should be an interesting mathematical problem.

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