

# MULTIPLE ZETA VALUES AND GROTHENDIECK-TEICHMÜLLER GROUPS

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ABSTRACT. We get a canonical embedding from the spectrum of the algebra of multiple zeta values modulo  $\pi^2$  into the graded version of the Grothendieck-Teichmüller group by using relations of the Drinfel'd associator. On the other hand, it is known that the rational pro- $l$  Galois image algebraic group is embedded into the Grothendieck-Teichmüller pro-algebraic group for each prime  $l$ . Via these embeddings, we get two kinds of relationship between the spectrum of the algebra of multiple zeta values and the pro- $l$  Galois image algebraic group.

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## 0. INTRODUCTION

The aim of this article is to establish a certain relationship among Grothendieck-Teichmüller groups, multiple zeta values and certain Galois image implicitly building in Drinfel'd's great paper in 1991 [Dr]. In the pro-finite group case on Galois Side, Y. Ihara showed explicitly that the *absolute Galois group*  $G_{\mathbf{Q}} := \text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$  can be embedded into the Grothendieck-Teichmüller (pro-finite) group  $\widehat{GT}$  in [Ih94] following the lines suggested in [Dr] (another group theoretical proof can be found in [Na]). But here we work on the unipotent pro-algebraic group case, especially on Hodge Side. We concentrate on the construction of an embedding from the spectrum of the algebra of multiple zeta values modulo  $\pi^2$  into one of Grothendieck-Teichmüller unipotent pro-algebraic groups (§3) following the lines of [Dr], from which we shall get two kinds of interesting relationship (§5 and §6) between Hodge Side (§3) and Galois Side (§4).

In connection with the study of Galois representations on the algebraic fundamental group of the projective line minus 3 points, the Grothendieck-Teichmüller (pro-finite) group  $\widehat{GT}$  recently appeared in consecutive paper [HS], [Ih90]~[Ih00], [LNS], [NS], [S] and [SL]. This pro-finite group  $\widehat{GT}$  was constructed by V.G. Drinfel'd in [Dr] and it is said that it may (or may not?) coincide with a certain combinatorial pro-finite group predicted by A. Grothendieck in [Gr]Ch 2. But, in fact, Drinfel'd originally invented and studied the pro-algebraic group  $\underline{GT}$  instead of the pro-finite group  $\widehat{GT}$  in his study of the deformation of quasi-triangular quasi-Hopf quantized universal enveloping algebras. In this paper, we shall call this pro-algebraic group  $\underline{GT}$

the *Grothendieck-Teichmüller* (pro-algebraic) *group*. There he also introduced another pro-algebraic group  $\underline{GRT}$ , which we call the *graded Grothendieck-Teichmüller group* as a graded version of  $\underline{GT}$ . In this paper, their unipotent parts  $\underline{GT}_1$  and  $\underline{GRT}_1$  play a role in *Galois Side* and *Hodge Side* respectively.

In §4.2, from the pro- $l$  Galois representation

$$p_1^{(l)} : Gal(\overline{\mathbf{Q}}/\mathbf{Q}(\mu_{l^\infty})) \rightarrow Aut \pi_1^{(l)}(\mathbf{P}_{\mathbf{Q}}^1 - \{0, 1, \infty\}) \quad ,$$

we shall associate the embedding of pro-algebraic group

$$\Phi_{\mathbf{Q}}^{(l)} : \underline{Gal}_{\mathbf{Q}}^{(l)} \hookrightarrow \underline{GT}_1$$

for each prime  $l$ , where  $\underline{Gal}_{\mathbf{Q}}^{(l)}$  is the associated pro-algebraic group over  $\mathbf{Q}$  (for definition, see §4.2). On this embedding  $\Phi_{\mathbf{Q}}^{(l)}$ , we shall see that it is natural to conjecture that  $\Phi_{\mathbf{Q}}^{(l)}$  gives an isomorphism in §4.2. Namely

**Conjecture B** .  $\underline{Gal}_{\mathbf{Q}}^{(l)} \cong \underline{GT}_1$  for all prime  $l$ .

In contrast, we get a Hodge counterpart of the embedding  $\Phi_{\mathbf{Q}}^{(l)}$ , which is our main result. Let  $Z_.$  be the graded  $\mathbf{Q}$ -algebra generated by all multiple zeta values and  $Z_./(\pi^2)$  be its quotient algebra modulo the graded principle ideal generated by  $\pi^2$  (see §3.1). In §3.1, we shall associate a scheme  $Spec Z_./(\pi^2)$  and show that

**Theorem 3.2.5.** *There is a canonical embedding of schemes*

$$\Phi_{DR} : Spec Z_./(\pi^2) \hookrightarrow \underline{GRT}_1 \quad .$$

In §3.4, we shall also see that it is natural to conjecture that  $\Phi_{DR}$  gives an isomorphism. Namely

**Conjecture A** .  $Spec Z_./(\pi^2) \cong \underline{GRT}_1$ .

In §5 and §6, we will discuss two kinds of relationship between  $\underline{GT}_1$  and  $\underline{GRT}_1$ . The first one is on the relationship between  $\mathbf{C}$ -structures (cf. *Notation 4.2.1*) of  $\underline{GT}_1$  and  $\underline{GRT}_1$ .

**Proposition 5.1.2.** *There exists a natural isomorphism  $p : \underline{GT}_1 \times_{\mathbf{Q}} \mathbf{C} \xrightarrow{\sim} \underline{GRT}_1 \times_{\mathbf{Q}} \mathbf{C}$ . If we identify their groups of  $\mathbf{C}$ -valued points by this isomorphism, two groups of rational points  $\underline{GT}_1(\mathbf{Q})$  and  $\underline{GRT}_1(\mathbf{Q})$  become inner conjugate to each other in  $\underline{GRT}_1(\mathbf{C})$ .*

By combining this proposition with two embeddings  $\Phi_{\mathbf{Q}}^{(l)}$  and  $\Phi_{DR}$ , we get Figure 1. Here  $Hom_{\mathbf{Q}\text{-alg}}(Z_./(\pi^2), \mathbf{Q})$  stands for the set of  $\mathbf{Q}$ -algebra homomorphisms from  $Z_./(\pi^2)$  to  $\mathbf{Q}$ .

The second one is

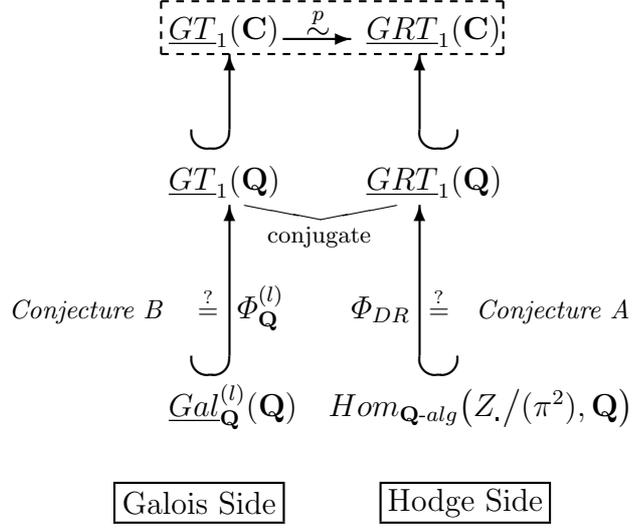


FIGURE 1

**Theorem 6.3.2.** *The pro-algebraic group  $\underline{GrGT}_1$  (for definition, see (6.3.2)) is isomorphic to  $\underline{GRT}_1$ , i.e.*

$$\underline{GrGT}_1 \cong \underline{GRT}_1 .$$

By combining this theorem with two embeddings  $\Phi_{\mathbf{Q}}^{(l)}$  and  $\Phi_{DR}$ , we get Figure 2.

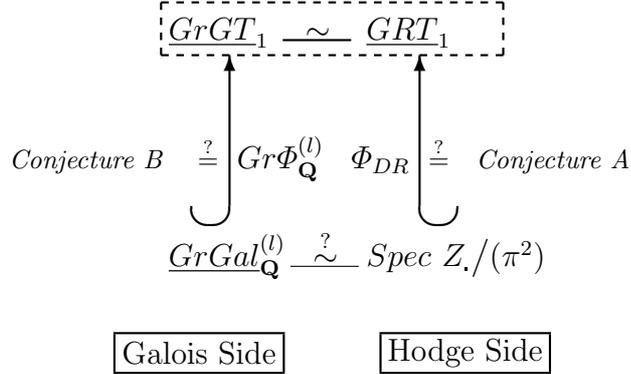


FIGURE 2

In *Remark* 6.5.1, we shall see that the main result of our previous article [F] can be deduced from Figure 2. As a corollary of *Theorem* 6.3.2, we get an interesting correspondence between the  $l$ -adic Ihara associator  $\Phi_{Ih}^{(l)}$  (§6.6.1) and the Drinfel'd associator  $\Phi_{KZ}$  (§6.6.1).

**Proposition 6.6.1.**

$$Gr\Phi_{\mathbf{Q}\langle\langle A, B \rangle\rangle}^{(l)\sharp}(\Phi_{GRT}) = Gr\Phi_{Ih}^{(l)} \quad , \quad \Phi_{DR\langle\langle A, B \rangle\rangle}^{\sharp}(\Phi_{GRT}) = \Phi_{KZ} \pmod{\pi^2},$$

from which we get Figure 3.

$$\begin{array}{ccc}
 \boxed{\text{Galois Side}} & & \boxed{\text{Hodge Side}} \\
 Gr^W \mathcal{O}(\underline{Gal}_{\mathbf{Q}}^{(l)})\langle\langle A, B \rangle\rangle & \xleftarrow{Gr\Phi_{\mathbf{Q}\langle\langle A, B \rangle\rangle}^{(l)\sharp}} \mathcal{O}(\underline{GRT}_1)\langle\langle A, B \rangle\rangle \xrightarrow{\Phi_{DR\langle\langle A, B \rangle\rangle}^{\sharp}} & \left(\mathbb{Z}/(\pi^2)\right)\langle\langle A, B \rangle\rangle \\
 \underbrace{Gr\Phi_{Ih}^{(l)}}_{l\text{-adic Ihara associator}} & \xleftarrow{\Phi_{GRT}} & \underbrace{\Phi_{KZ} \pmod{\pi^2}}_{\text{Drinfel'd associator}}
 \end{array}$$

FIGURE 3

The pictures in *Corollary 6.6.4* and *Corollary 6.6.6* should be better called cases of meta-abelian quotient of Figure 3.

This paper is organized as follows. In §1, we shall make a brief review of the notion of weight filtrations of negatively weighted extensions [HM]. §2 is devoted to a (long) review and detailed description of  $\underline{GT}$ ,  $\underline{GRT}$  and  $\underline{M}$  constructed by Drinfel'd [Dr]. In §3, we shall recall the definition of multiple zeta values and construct an embedding  $\Phi_{DR}$  in Theorem 3.2.5, which is our main result. In §4, we shall discuss the embedding  $\Phi_{\mathbf{Q}}^{(l)}$ . Finally in §5 and §6, we shall give two kinds of interesting relationship between  $\underline{GT}_1$  and  $\underline{GRT}_1$  and then make a interesting comparison between Hodge Side (§3) and Galois Side (§4) in §6.6.

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## 1. ON WEIGHT FILTRATIONS BY HAIN-MATSUMOTO

This section is devoted to a brief review of the notion of weight filtrations on modules of negatively weighted pro-algebraic groups, which is introduced in [HM]. In §1.3, we will introduce a helpful proposition (*Proposition 1.3.2*), which will be used later.

**1.1. Negatively weighted extension.** We recall the definition of negatively weighted extension in [HM]§3.

**Notation 1.1.1.** Let  $k$  be a field with characteristic 0. Let  $S$  be a reductive algebraic group over  $k$ . Let  $\varpi : \mathbb{G}_m \rightarrow S$  be a *central cocharacter*, which means a homomorphism whose image is contained in the

center of  $S$ . Let  $G$  be an algebraic group over  $k$  which is an extension of  $S$  by a unipotent algebraic group  $U$ ;  $1 \rightarrow U \rightarrow G \rightarrow S \rightarrow 1$ . By [HM] *Proposition 2.3*, the maximal abelian quotient  $H_1(U)$  of this algebraic group  $U$  becomes an  $S$ -module. Therefore it becomes a  $\mathbb{G}_m$ -module via  $\varpi$ . Thus we can decompose *uniquely* as  $H_1(U) = \bigoplus_{k \in \mathbf{Z}} W_k$ , where  $W_k$  is the  $\mathbb{G}_m$ -module whose  $\mathbb{G}_m$ -action is given by the  $k$ -th power multiplication.

**Definition 1.1.2.** A *pro-algebraic group* over a field  $k$  is a projective limit of algebraic groups over  $k$ . A pro-algebraic group  $\mathcal{G} = \varprojlim_{\alpha} G_{\alpha}$  is called *unipotent* if each  $G_{\alpha}$  is unipotent.

**Definition 1.1.3** ([HM]§3). The extension algebraic group  $G$  in *Notation 1.1.1* is called **negatively weighted** with respect to  $\varpi$  if  $W_k = 0$  for all  $k \geq 0$ . A *pro-algebraic group*  $\mathcal{G}$  which is an extension of a reductive algebraic group  $S$  by a unipotent *pro-algebraic group*  $\mathcal{U}$  is also called **negatively weighted** with respect to  $\varpi$  if it is a projective limit of algebraic groups which are negatively weighted extensions of  $S$  with respect to  $\varpi$ .

**1.2. Weight filtration.** We will review the fact shown in [HM]§3 that each representation of negatively weighted pro-algebraic group is equipped with a natural weight filtration.

**Notation 1.2.1.** Let  $\mathcal{G}$  be a pro-algebraic group over a field  $k$  with characteristic 0 which is a negatively weighted extension of a reductive algebraic group  $S$  over  $k$  by a unipotent pro-algebraic group  $\mathcal{U}$  over  $k$  with respect to a central cocharacter  $\varpi : \mathbb{G}_m \rightarrow S$ . By [HM] *Lemma 3.1*, there exists a homomorphism  $\tilde{\varpi} : \mathbb{G}_m \rightarrow \mathcal{G}$  which is a lift of  $\varpi$ . Let  $V$  be a finite dimensional  $k$ -vector space equipped with  $\mathcal{G}$ -action. Then this  $\mathcal{G}$ -module  $V$  becomes a  $\mathbb{G}_m$ -module via  $\tilde{\varpi}$ . So we can decompose as  $V = \bigoplus_{a \in \mathbf{Z}} V_a$ , where  $V_a$  is the  $\mathbb{G}_m$ -module whose  $\mathbb{G}_m$ -action is given by the  $a$ -th power multiplication.

In this subsection, we assume *Notation 1.2.1*.

**Definition 1.2.2** ([HM]§3). The **weight filtration** of  $\mathcal{G}$ -module  $V$  is the ascending filtration  $W = \{W_n V\}_{n \in \mathbf{Z}}$  of  $k$ -linear subspaces defined by  $W_n V = \bigoplus_{a \leq n} V_a$  for each  $n \in \mathbf{Z}$ .

In [HM] *Proposition 3.8*, it was proved that this weight filtration is natural in the sense that it does not depend on the choice of lift  $\tilde{\varpi}$  above and it was also shown that  $W_n V$  is stable by the  $\mathcal{G}$ -action. Moreover,

**Proposition 1.2.3** ([HM\*] *Proposition 4.10*). *The weight filtration  $W$  of a finite dimensional  $\mathcal{G}$ -module  $V$  is the unique ascending filtration  $W = \{W_n V\}_{n \in \mathbf{Z}}$  of  $\mathcal{G}$ -submodules which is characterized by the following properties:*

- (a)  $\bigcap_{n \in \mathbf{Z}} W_n V = 0, \quad \bigcup_{n \in \mathbf{Z}} W_n V = V.$
- (b) *The action of  $\mathcal{U}$  on  $Gr_n^W V$  ( $:= W_n V / W_{n-1} V$ ) is trivial for all  $n \in \mathbf{Z}$ .*
- (c) *The action of  $\mathbb{G}_m$  on  $Gr_n^W V$  via  $\varpi$  is given by the  $n$ -th power multiplication for all  $n \in \mathbf{Z}$ .*

**1.3. Filtered Hopf algebras.** Assume *Notation 1.2.1*.

**Notation 1.3.1.** Let  $\mathcal{U} = \varprojlim_{\alpha} U_{\alpha}$  denote a projective limit of unipotent algebraic groups  $U_{\alpha}$ . The regular function ring  $\mathcal{O}(\mathcal{U})$  of  $\mathcal{U}$  is defined to be the inductive limit  $\mathcal{O}(\mathcal{U}) := \varinjlim_{\alpha} \mathcal{O}(U_{\alpha})$  of those of  $U_{\alpha}$ . This  $k$ -algebra  $\mathcal{O}(\mathcal{U})$  is equipped with a structure of Hopf algebra over  $k$ . Let  $A$  be an arbitrary  $k$ -algebra. Take any element  $g \in \mathcal{G}(A)$ . Then the inner automorphism  $\tau_g^{-1} : \mathcal{U} \times A \rightarrow \mathcal{U} \times A$  defined by  $x \mapsto g^{-1} x g$  induces the isomorphism  $(\tau_g^{-1})^{\#} : \mathcal{O}(\mathcal{U})_A \rightarrow \mathcal{O}(\mathcal{U})_A$  of Hopf algebras. By the correspondence  $\tau : \mathcal{G} \rightarrow \underline{Aut} \mathcal{O}(\mathcal{U})$  defined by  $g \mapsto (\tau_g^{-1})^{\#}$ , we regard  $\mathcal{O}(\mathcal{U})$  as a left module of  $\mathcal{G}$ .

Although  $\mathcal{O}(\mathcal{U})$  is infinite-dimensional, it is equipped with the following weight filtration.

**Proposition 1.3.2.** *The regular function ring  $\mathcal{O}(\mathcal{U})$  is equipped with a weight filtration, which is an ascending filtration of finite dimensional  $k$ -linear sub-spaces:*

$$W : \dots = W_{-2} \mathcal{O}(\mathcal{U}) = W_{-1} \mathcal{O}(\mathcal{U}) = 0 \subseteq W_0 \mathcal{O}(\mathcal{U}) \subseteq W_1 \mathcal{O}(\mathcal{U}) \subseteq \dots \subseteq W_n \mathcal{O}(\mathcal{U}) \subseteq W_{n+1} \mathcal{O}(\mathcal{U}) \subseteq \dots$$

*It satisfies the following properties as in Proposition 1.2.3:*

- (a)  $\bigcap_{n \in \mathbf{Z}} W_n \mathcal{O}(\mathcal{U}) = 0, \quad \bigcup_{n \in \mathbf{Z}} W_n \mathcal{O}(\mathcal{U}) = \mathcal{O}(\mathcal{U}).$
- (b) *The action of  $\mathcal{U}$  on  $Gr_n^W \mathcal{O}(\mathcal{U})$  by  $\tau$  is trivial for all  $n \in \mathbf{Z}$ .*
- (c) *The action of  $\mathbb{G}_m$  on  $Gr_n^W \mathcal{O}(\mathcal{U})$  via  $\tau$  and  $\varpi$  is the  $n$ -th power multiplication for all  $n \in \mathbf{Z}$ .*

*Moreover this filtration is compatible with all structure morphisms of Hopf algebras, i.e.  $(\mathcal{O}(\mathcal{U}), W)$  becomes a filtered Hopf algebra.*

**Proof .** Regard  $Lie \mathcal{U}$  as a left  $\mathcal{G}$ -module by its adjoint representation. In [HM] *Proposition 4.5*, it is shown that  $Lie \mathcal{U}$  is equipped with a

weight filtration of finite dimensional  $k$ -linear subspaces

$$\begin{aligned} W : \cdots \cdots \subseteq W_{-n-1} \text{Lie}\mathcal{U} \subseteq W_{-n} \text{Lie}\mathcal{U} \subseteq \cdots \cdots \\ \cdots \subseteq W_{-1} \text{Lie}\mathcal{U} = W_0 \text{Lie}\mathcal{U} = W_1 \text{Lie}\mathcal{U} = \cdots \cdots = \text{Lie}\mathcal{U} \end{aligned}$$

which satisfies properties (a)~(c) in *Proposition 1.2.3*. Since  $\mathcal{U}$  is unipotent,  $\mathcal{O}(\mathcal{U})$  is isomorphic to the dual of the universal enveloping algebra  $\mathbb{U}\text{Lie}\mathcal{U}$  of  $\text{Lie}\mathcal{U}$  as left  $\mathcal{G}$ -modules. Thus  $\mathcal{O}(\mathcal{U})$  is equipped with the induced filtration from the above one on  $\text{Lie}\mathcal{U}$ , from which the first half of our statement can be deduced. The second half is immediate since the  $\mathcal{G}$ -action on  $\mathcal{O}(\mathcal{U})$  by  $\tau$  is consistent with all structure morphisms of Hopf algebras, i.e. the product map, the unit map, the co-product map, the co-unit map and the antipode map.  $\square$

## 2. A DETAILED ANALYSIS ON DRINFEL'D'S GT'S

This section is devoted to a long review and detailed analysis on  $\underline{GT}$ ,  $\underline{GRT}$  and  $\underline{M}$  constructed by Drinfel'd [Dr] in terms of weight filtration (§1) by Hain-Matsumoto. In §2.1 (resp. §2.2), we shall recall the definition of  $\underline{GRT}$  (resp.  $\underline{GT}$ ) [Dr] and discuss the weight filtration of the regular function ring  $\mathcal{O}(\underline{GRT}_1)$  (resp.  $\mathcal{O}(\underline{GT}_1)$ ). §2.3 will be devoted to a review of the definition of the Drinfel'd associator  $\varphi_{KZ}$  and the pro-torsor  $\underline{M}$  [Dr].

**2.1. On  $\underline{GRT}$ .** §2.1.1 is devoted to reviewing the definition of the pro-algebraic group  $\underline{GRT}$  that appeared in [Dr]. In §2.1.2, we will endow the regular function ring  $\mathcal{O}(\underline{GRT}_1)$  of the unipotent part  $\underline{GRT}_1$  of  $\underline{GRT}$  with a grading and show that it becomes a graded Hopf algebra.

**2.1.1. The graded Grothendieck-Teichmüller group.**

**Notation 2.1.1.** Let  $k$  be any  $\mathbf{Q}$ -algebra. Let  $k\langle\langle A, B \rangle\rangle$  be the graded non-commutative formal power series ring over  $k$  with 2 variables  $A$  and  $B$  with degrees given by  $\deg A = \deg B = 1$ . Denote the subset of  $k\langle\langle A, B \rangle\rangle$  consisting of formal Lie series in  $k\langle\langle A, B \rangle\rangle$  by  $\mathbb{L}_k^\wedge$ . Let  $R$  be a completed non-commutative  $k$ -algebra and  $\phi : k\langle\langle A, B \rangle\rangle \rightarrow R$  be a completed  $k$ -algebra homomorphism uniquely defined by  $\phi(A) = a$ ,  $\phi(B) = b$  for certain elements  $a, b \in R$ . For  $g \in k\langle\langle A, B \rangle\rangle$ , we denote the image  $\phi(g) \in R$  by  $g(a, b)$ .

**Definition 2.1.2** ([Dr]§5). The graded<sup>1</sup> Grothendieck-Teichmüller (pro-algebraic) group  $\underline{GRT}$  is the pro-linear algebraic group over  $\mathbf{Q}$  whose set of  $k$ -valued points is defined as follows:

$$\underline{GRT}(k) = \{(c, g) \in k^\times \times k\langle\langle A, B \rangle\rangle \mid g \text{ satisfies (0)~(iii) below.}\}$$

<sup>1</sup>The word *graded* means the natural grading (see §2.1.2).

$$\left\{ \begin{array}{l} (0) \quad g \in \exp[\mathbb{L}_k^\wedge, \mathbb{L}_k^\wedge] \\ (i) \quad g(A, B)g(B, A) = 1 \\ (ii) \quad g(C, A)g(B, C)g(A, B) = 1 \quad \text{for } A + B + C = 0 \\ (iii) \quad g(X_{1,2}, X_{2,3})g(X_{3,4}, X_{4,5})g(X_{5,1}, X_{1,2})g(X_{2,3}, X_{3,4})g(X_{4,5}, X_{5,1}) = 1 \\ \qquad \text{in } \underline{U\mathfrak{P}}^{(5)}(k) \quad (\text{ see Note 2.1.3}). \end{array} \right.$$

The multiplication map <sup>2</sup>  $m'$  of the graded Grothendieck-Teichmüller group is given as follows:

$$\begin{aligned} m' : \underline{GRT}(k) \times \underline{GRT}(k) &\longrightarrow \underline{GRT}(k) \\ (c_2, g_2) \times (c_1, g_1) &\longmapsto (c_2, g_2) \circ (c_1, g_1) \quad , \end{aligned}$$

$$\text{where } (c_2, g_2) \circ (c_1, g_1) := \left( c_1 c_2, g_2(A, B) g_1\left(\frac{A}{c_2}, g_2(A, B)^{-1} \frac{B}{c_2} g_2(A, B)\right) \right).$$

**Note 2.1.3.** (1) The defining relation (i) (resp. (ii), (iii)) is sometimes called 2 (resp. 3, 5)-cycle relation.

(2) On (0), for any element  $h$  (in the topological commutator  $[\mathbb{L}_k^\wedge, \mathbb{L}_k^\wedge]$ ) of  $\mathbb{L}_k^\wedge$ , we define  $\exp h := 1 + \frac{h}{1!} + \frac{h^2}{2!} + \frac{h^3}{3!} + \cdots \in k\langle\langle A, B \rangle\rangle$ . This series converges since  $h$  has no coefficient (degree 0) term.

(3) On (iii),  $\underline{U\mathfrak{P}}^{(5)}(k)$  means the completion by degree (i.e. that is respected to the filtration induced by degree) of the universal enveloping algebra of the pure sphere 5-braid graded Lie algebra  $\mathfrak{P}^{(5)}$  tensored with  $k$  and  $X_{i,j}$  ( $1 \leq i, j \leq 5$ ) stand for the standard generators of  $\mathfrak{P}^{(5)}$  (for definitions, see [Ih90]~[Ih92] and also [F] §2.1.).

(4) It can be checked easily that (iii) implies (i).

We remark that the same pro-algebraic group also appeared and was studied by Z. Wojtkowiak [Wo]. In this paper, especially we examine its unipotent part  $\underline{GRT}_1$ .

**Definition 2.1.4** ([Dr]§5). The unipotent graded Grothendieck-Teichmüller (pro-algebraic) group  $\underline{GRT}_1$  is the unipotent sub-pro-algebraic group of  $\underline{GRT}$  whose set of  $k$ -valued points is

$$\begin{aligned} \underline{GRT}_1(k) &= \{ (1, g) \in \underline{GRT}(k) \} \\ &= \{ g \in k\langle\langle A, B \rangle\rangle \mid g \text{ satisfies (0)~(iii) in Definition 2.1.2.} \} . \end{aligned}$$

Note that we get the following exact sequence of pro-algebraic groups

$$(2.1.1) \quad 1 \longrightarrow \underline{GRT}_1 \longrightarrow \underline{GRT} \longrightarrow \mathbb{G}_m \longrightarrow 1$$

$$(c, g) \longmapsto c \quad .$$

<sup>2</sup>For our convenience, we reverse the original definition of the multiplication of  $\underline{GRT}$  in [Dr]

**Remark 2.1.5.** The above exact sequence is equipped with a standard section

$$\begin{aligned} s_0 : \mathbb{G}_m &\rightarrow \underline{GRT} \\ c &\mapsto (c, 1) \quad . \end{aligned}$$

This property may distinguish  $\underline{GRT}$  from  $\underline{GT}$  (§2.2).

**Lemma 2.1.6.** *The pro-algebraic group  $\underline{GRT}$  is a negatively weighted extension (Definition 1.1.3) of  $\mathbb{G}_m$  by the unipotent pro-algebraic group  $\underline{GRT}_1$  with respect to the central cocharacter  $\varpi : \mathbb{G}_m \rightarrow \mathbb{G}_m$  defined by  $x \mapsto x$ .*

**Proof .** It follows from

$$(c, 1) \circ (1, g(A, B)) \circ (c^{-1}, 1) = \left(1, g\left(\frac{A}{c}, \frac{B}{c}\right)\right).$$

□

2.1.2. *The graded Hopf algebra  $\mathcal{O}(\underline{GRT}_1)$ .* In this subsection, we will see that the regular function ring  $\mathcal{O}(\underline{GRT}_1)$  of  $\underline{GRT}_1$  is naturally equipped with a structure of graded Hopf algebra.

**Notation 2.1.7.** By a *word* we mean a monic and monomial element  $W$  of  $\mathbf{Q}\langle\langle A, B \rangle\rangle$  (§2.1.1) whose degree is greater than 0. Note that we *do not* include 1 among words. For each word  $W$ , we define  $wt W := deg W$  and  $wt 1 := 0$ . Suppose that  $R$  is an arbitrary  $\mathbf{Q}$ -algebra. Then each element  $g \in R\langle\langle A, B \rangle\rangle$  can be expanded uniquely as  $g = x_1(g) + \sum_{W:\text{words}} x_W(g)W$  where  $x_1(g)$  and  $x_W(g) \in R$ . Let  $W$  be a word or 1. We define the map  $x_W : \underline{GRT}_1(R) \rightarrow R$  which is determined by  $g \mapsto x_W(g)$ . Then these  $x_W$ 's generate the algebra  $\mathcal{O}(\underline{GRT}_1)$  i.e.  $\mathcal{O}(\underline{GRT}_1) = \mathbf{Q}[x_1, x_W]_{W:\text{words}}$ . We remark that  $x_1(g) = 1$  for all  $g \in \underline{GRT}_1(R)$  because of Definition 2.1.2 (0).

By Proposition 1.3.2,  $\mathcal{O}(\underline{GRT}_1)$  is equipped with a weight filtration  $W = \{W_n \mathcal{O}(\underline{GRT}_1)\}_{n \in \mathbf{Z}}$  satisfying properties (a)~(c) in Proposition 1.3.2 with respect to  $\varpi$  (Lemma 2.1.6) and the pair  $(\mathcal{O}(\underline{GRT}_1), W)$  becomes a filtered Hopf algebra. Recall that  $s_0$  (Remark 2.1.5) is the standard section of the exact sequence (2.1.1). By imitating the prescription described in §1.2,  $\mathcal{O}(\underline{GRT}_1)$  and  $W_n \mathcal{O}(\underline{GRT}_1)$  can be decomposed into  $\mathcal{O}(\underline{GRT}_1) = \bigoplus_{0 \leq a} V_a$  and  $W_n \mathcal{O}(\underline{GRT}_1) = \bigoplus_{0 \leq a \leq n} V_a$  as  $\mathbb{G}_m$ -modules with respect to  $s_0$ , where  $V_a$  is the  $\mathbb{G}_m$ -module whose  $\mathbb{G}_m$ -action (given by  $\tau \circ s_0$ ) is the  $a$ -th power multiplication. Since this grading on  $\mathcal{O}(\underline{GRT}_1)$ ,  $:= \bigoplus_{0 \leq a} V_a$  is natural, it provides a natural

isomorphism

$$(2.1.2) \quad \tilde{s}_0 : Gr^W \mathcal{O}(\underline{GRT}_1) \rightarrow \mathcal{O}(\underline{GRT}_1).$$

Therefore it follows that

**Proposition 2.1.8.** (a) *The Hopf algebra  $\mathcal{O}(\underline{GRT}_1)$  is equipped with a weight filtration  $W$ .*

(b) *The pair  $(\mathcal{O}(\underline{GRT}_1), W)$  becomes a filtered Hopf algebra.*

(c) *By (2.1.2),  $\mathcal{O}(\underline{GRT}_1)$  is equipped with a structure of graded Hopf algebra.*

**Note 2.1.9.** This grading on  $\mathcal{O}(\underline{GRT}_1) = \mathbf{Q}[x_W]_{W:\text{words}}$  is given by  $\deg x_W = wt W$ .

2.1.3. *The stable derivation algebra.* We shall review the definition of the stable derivation algebra  $\mathfrak{D}$ , which was constructed by Y. Ihara in his successive works on the Galois representation on the pro- $l$  fundamental group  $\pi_1^{(l)}(\mathbf{P}_{\mathbf{Q}}^1 - \{0, 1, \infty\})$  (see [Ih90]~[Ih92] and [Ih99]).

**Notation 2.1.10.** Let  $\mathbb{L}_w$  ( $w \geq 1$ ) denote the degree  $w$ -part of the free completed Lie algebra  $\mathbb{L}_{\mathbf{Q}}^{\wedge}$  (see §2.1.1) of rank 2 and let  $\mathbb{L}_{\bullet} = \bigoplus_{w \geq 1} \mathbb{L}_w$

be the free (non-completed) graded Lie algebra over  $\mathbf{Q}$ . For  $f$  in  $\mathbb{L}_{\bullet}$ , we define the *special derivation*  $D_f : \mathbb{L}_{\bullet} \rightarrow \mathbb{L}_{\bullet}$  which is the derivation determined by  $D_f(A) = 0$  and  $D_f(B) = [B, f]$ . It can be checked easily that  $[D_f, D_g] = D_h$  with  $h = [f, g] + D_f(g) - D_g(f)$ .

**Definition 2.1.11** ([Ih90] and [Ih92]). The **stable derivation algebra**  $\mathfrak{D}$  is the graded Lie subalgebra  $\mathfrak{D}$  of  $Der \mathbb{L}_{\bullet}$  which has the following presentation:  $\mathfrak{D}_{\bullet} = \bigoplus_{w \geq 1} \mathfrak{D}_w$ , where

$$\mathfrak{D}_w = \{D_f \in Der \mathbb{L}_{\bullet} \mid f \in \mathbb{L}_w \text{ satisfies (0) } \sim \text{(iii) below.}\}$$

$$\left\{ \begin{array}{l} (0) \quad f \in [\mathbb{L}_{\bullet}, \mathbb{L}_{\bullet}] \quad (:= \bigoplus_{a=2}^{\infty} \mathbb{L}_a) \\ (i) \quad f(A, B) + f(B, A) = 0 \\ (ii) \quad f(A, B) + f(B, C) + f(C, A) = 0 \quad \text{for } A + B + C = 0 \\ (iii) \quad \sum_{i \in \mathbf{Z}/5} f(X_{i, i+1}, X_{i+1, i+2}) = 0 \quad \text{in } \mathfrak{P}_{\bullet}^{(5)} \quad (\text{see Note 2.1.3}). \end{array} \right.$$

Here, for any Lie algebra  $H$  and  $\alpha, \beta \in H$ ,  $f(\alpha, \beta)$  denotes the image of  $f \in \mathbb{L}_{\bullet}$  by the homomorphism  $\mathbb{L}_{\bullet} \rightarrow H$  defined by  $A \mapsto \alpha$  and  $B \mapsto \beta$ .

**Remark 2.1.12.** (1) Each derivation  $D \in \mathfrak{D}_{\bullet}$  determines a unique element  $f \in [\mathbb{L}_{\bullet}, \mathbb{L}_{\bullet}]$  such that  $D = D_f$ .

(2) The relation (iii) implies (i).

- (3) The completion by degree  $\mathfrak{D}^\wedge = \widehat{\bigoplus_{w \geq 1} \mathfrak{D}_w}$  of Ihara's stable derivation algebra is equal to the pro-Lie algebra  $\mathbf{grr}_1(\mathbf{Q})$  [Dr] which is the Lie algebra  $Lie \underline{GRT}_1$  of the graded Grothendieck-Teichmüller group.

On the structure of  $\mathfrak{D}$ , there is a standard conjecture in [Ih99] which is related to conjectures on the associated graded Lie algebra of the image of the Galois representation on  $\pi_1^{(l)}(\mathbf{P}_{\mathbf{Q}}^1 - \{0, 1, \infty\})$  by Ihara ([Ih90]) and P. Deligne ([De]).

**Conjecture 2.1.13** ([Ih99]).  $\mathfrak{D}$  is a free Lie algebra generated by  $f_m$ , where  $f_m$  is a suitable element of  $\mathfrak{D}_m$  ( $m = 3, 5, 7, \dots$ ).

Furthermore, Ihara proposed the following problem:

**Problem 2.1.14** ([Ih99]). Construct  $f_m$  explicitly. Is there any canonical choice?

By his consideration of  $f_m$  ( $m = 3, 5, 7, \dots$ ), each  $f_m$  must be of *depth* 1 (see [Ih99]Ch II).

**Remark 2.1.15.** M. Matsumoto and H. Tsunogai ([Tsu]) calculated the dimensions of graded pieces of the stable derivation algebra on the lower weights. Especially Tsunogai verified *Conjecture 2.1.13* up to weight 14.

Tsunogai posed the following problem (which arises from his computation table).

**Problem 2.1.16** (H. Tsunogai). On the defining relations (*Definition 2.1.11*) of the stable derivation algebra, does 5-cycle relation (iii) imply 3-cycle relation (ii)?

2.2. **On  $\underline{GT}$ .** §2.2.1 is devoted to reviewing the definition of the pro-algebraic group  $\underline{GT}$  [Dr]§4. In §2.2.2, we will endow the regular function ring  $\mathcal{O}(\underline{GT}_1)$  of its unipotent part  $\underline{GT}_1$  with a weight filtration (§1.2) and show that it is equipped with a structure of filtered Hopf algebra.

2.2.1. *The Grothendieck-Teichmüller group.* Let  $F_2$  be the free group of rank 2 generated by  $X$  and  $Y$  and denote its Malcev completion (see, for example [HM] §A.1) by  $\underline{F}_2$ . Let  $k$  be any  $\mathbf{Q}$ -algebra.

**Definition 2.2.1** ([Dr]§4). The Grothendieck-Teichmüller (pro-algebraic) group is the pro-linear algebraic group  $\underline{GT}$  defined over  $\mathbf{Q}$  whose set of  $k$ -valued points is the subset of that of  $\mathbb{G}_m \times \underline{F}_2$  defined as follows:

$$\underline{GT}(k) = \{(\lambda, f) \in k^\times \times \underline{F}_2(k) \mid (\lambda, f) \text{ satisfies (0)~(iii) below.}\}$$

$$\left\{ \begin{array}{l} \text{(0)} \quad f \in [\underline{F}_2, \underline{F}_2](k) \\ \text{(i)} \quad f(X, Y)f(Y, X) = 1 \\ \text{(ii)} \quad f(Z, X)Z^m f(Y, Z)Y^m f(X, Y)X^m = 1 \quad \text{for } XYZ = 1, m = \frac{\lambda-1}{2} \\ \text{(iii)} \quad f(x_{1,2}, x_{2,3})f(x_{3,4}, x_{4,5})f(x_{5,1}, x_{1,2})f(x_{2,3}, x_{3,4})f(x_{4,5}, x_{5,1}) = 1 \\ \hspace{10em} \text{in } \underline{P}_5(k) \quad (\text{ see Note 2.2.2.}) . \end{array} \right.$$

The multiplication map<sup>3</sup>  $m$  of the Grothendieck-Teichmüller group is given as follows:

$$\begin{aligned} m : \underline{GT}(k) \times \underline{GT}(k) &\longrightarrow \underline{GT}(k) \\ (\lambda_2, f_2) \times (\lambda_1, f_1) &\longmapsto (\lambda_2, f_2) \circ (\lambda_1, f_1) , \end{aligned}$$

where  $(\lambda_2, f_2) \circ (\lambda_1, f_1) := \left( \lambda_1 \lambda_2, f_2(X, Y) f_1(X^{\lambda_2}, f_2(X, Y)^{-1} Y^{\lambda_2} f_2(X, Y)) \right)$ .

**Note 2.2.2.** Here for any unipotent group scheme  $\underline{H}$  over  $\mathbf{Q}$  and morphism of group schemes  $\Phi : \underline{F}_2 \rightarrow \underline{H}$  with  $\Phi(x) := \alpha$  and  $\Phi(y) := \beta \in \underline{H}(\mathbf{Q})$ , we denote the image of  $f \in \underline{F}_2(\mathbf{Q})$  by  $\Phi(f) \in \underline{H}(\mathbf{Q})$ . We note that in (ii),  $X^m$ ,  $Y^m$  and  $Z^m$  also make sense since  $\underline{F}_2$  is unipotent. In the condition (iii),  $\underline{P}_5$  means the Malcev completion of the pure sphere 5-braid group  $P_5$  and  $x_{i,j}$  ( $1 \leq i, j \leq 5$ ) stand for standard generators of  $P_5$  (cf. [Ih91]). The relation (iii) implies (i).

V. G. Drinfel'd constructed  $\underline{GT}$  and  $\underline{GRT}$  as deformation groups of quasi-triangular quasi-Hopf quantized universal enveloping algebras. They act in a different way on the classifying space of these algebras. In [Dr], it is shown that their actions are commutative to each other.

**Definition 2.2.3** ([Dr]§5). The unipotent Grothendieck-Teichmüller (pro-algebraic) group  $\underline{GT}_1$  is the unipotent sub-pro-algebraic group of  $\underline{GT}$  whose set of  $k$ -valued points is

$$\underline{GT}_1(k) := \{f \in \underline{F}_2(k) \mid (1, f) \in \underline{GT}(k)\}.$$

Note that we get the following exact sequence of pro-algebraic group

$$(2.2.1) \quad \begin{array}{ccccccc} 1 & \longrightarrow & \underline{GT}_1 & \longrightarrow & \underline{GT} & \longrightarrow & \mathbb{G}_m \longrightarrow 1 \\ & & & & (\lambda, f) & \longmapsto & \lambda \end{array} .$$

**Lemma 2.2.4.** *The pro-algebraic group  $\underline{GT}$  is a negatively weighted extension (Definition 1.1.3) of  $\mathbb{G}_m$  by the unipotent pro-algebraic group  $\underline{GT}_1$  with respect to the central cocharacter<sup>4</sup>  $\frac{1}{\varpi} : \mathbb{G}_m \rightarrow \mathbb{G}_m$  defined by  $x \mapsto x^{-1}$ .*

<sup>3</sup>For our convenience, we reverse the original definition of the multiplication of  $\underline{GT}$  in [Dr]

<sup>4</sup>On the contrary,  $\underline{GRT}$  was a negatively weighted extension with respect to  $\varpi$ .

**Proof .** It follows from a calculation slightly more complicated than *Lemma 2.1.6* (however, by combining *Lemma 2.1.6* with *Proposition 5.1.2*, we can find another easier proof.)  $\square$

**Remark 2.2.5.** By [HM]*Proposition A.8*, there exists the pro-linear algebraic group  $\underline{AutF}_2$  defined over  $\mathbf{Q}$  which represents the functor  $K \mapsto \underline{AutF}_2 \times K$  from field extensions over  $\mathbf{Q}$  to groups (for the definition of  $\underline{AutF}_2 \times K$ , see [HM]). By the correspondence  $X \mapsto X^\lambda$  and  $Y \mapsto f^{-1}Y^\lambda f$ , where  $(\lambda, f) \in \underline{GT}(K)$ ,  $\underline{GT}$  can be regarded as a sub-algebraic group of  $\underline{AutF}_2$ .

2.2.2. *The filtered Hopf algebra*  $(\mathcal{O}(\underline{GT}_1), W)$ . In this subsection, we will see that the regular function ring  $\mathcal{O}(\underline{GT}_1)$  of  $\underline{GT}_1$  is naturally equipped with a structure of filtered Hopf algebra.

As in the case of  $\mathcal{O}(\underline{GT}_1)$  (§2.1.2) we can show that  $\mathcal{O}(\underline{GT}_1)$  is equipped with a weight filtration  $W = \{W_n \mathcal{O}(\underline{GT}_1)\}_{n \in \mathbf{Z}}$  satisfying properties (a)~(c) (*Proposition 1.3.2*) with respect to  $\frac{1}{\varpi}$  (*Lemma 2.2.4*). By *Proposition 1.3.2*,

**Proposition 2.2.6.** (1) *The Hopf algebra  $\mathcal{O}(\underline{GT}_1)$  is equipped with a weight filtration  $W$ .*  
 (2) *The pair  $(\mathcal{O}(\underline{GT}_1), W)$  becomes a filtered Hopf algebra.*

Remark that we do not know any natural structure of graded Hopf algebra instead of that of filtered Hopf algebras on  $\mathcal{O}(\underline{GT}_1)$ , since we do not know any natural section of the exact sequence (2.2.1) like *Remark 2.1.5*.

2.3. **On  $\underline{M}$  and  $\varphi_{KZ}$ .** We shall review the definition of the pro-torsor  $\underline{M}$  ([Dr]) in §2.3.1 and the original definition of the Drinfel'd associator  $\varphi_{KZ}$  ([Dr]) in §2.3.2 which plays a role to prove the main theorem in §3.

2.3.1. *The middle Grothendieck-Teichmüller torsor.* Let  $k$  be any  $\mathbf{Q}$ -algebra.

**Definition 2.3.1** ([Dr]§4). The middle Grothendieck-Teichmüller torsor is the pro-variety  $\underline{M}$  defined over  $\mathbf{Q}$  whose set of  $k$ -valued points is defined as follows:

$$\underline{M}(k) = \{(\mu, \varphi) \in k^\times \times k\langle\langle A, B \rangle\rangle \mid (\mu, \varphi) \text{ satisfies (0)~(iii) below.}\}$$

$$\left\{ \begin{array}{l} (0) \quad \varphi \in \exp[\mathbb{L}_k^\wedge, \mathbb{L}_k^\wedge] \\ (i) \quad \varphi(A, B)\varphi(B, A) = 1 \\ (ii) \quad e^{\frac{\mu}{2}A}\varphi(C, A)e^{\frac{\mu}{2}C}\varphi(B, C)e^{\frac{\mu}{2}B}\varphi(A, B) = 1 \quad \text{for } A + B + C = 0 \\ (iii) \quad \varphi(X_{1,2}, X_{2,3})\varphi(X_{3,4}, X_{4,5})\varphi(X_{5,1}, X_{1,2})\varphi(X_{2,3}, X_{3,4})\varphi(X_{4,5}, X_{5,1}) = 1 \\ \qquad \qquad \qquad \text{in } \underline{U\mathfrak{P}}^{(5)}(k) \quad (\text{ see Note 2.1.3}). \end{array} \right.$$

The right  $\underline{GT}$ -action and the left  $\underline{GRT}$ -action on  $\underline{M}$  are defined as follows <sup>5</sup>:

**The right  $\underline{GT}$ -action:**

$$\begin{aligned} \underline{M}(k) \times \underline{GT}(k) &\longrightarrow \underline{M}(k) \\ (\mu, \varphi) \times (\lambda, f) &\longmapsto (\mu, \varphi) \circ (\lambda, f) \quad , \end{aligned}$$

$$\text{where } (\mu, \varphi) \circ (\lambda, f) := \left( \mu\lambda, \varphi(A, B) f(e^A, \varphi(A, B)^{-1} e^B \varphi(A, B)) \right).$$

**The left  $\underline{GRT}$ -action:**

$$\begin{aligned} \underline{GRT}(k) \times \underline{M}(k) &\longrightarrow \underline{M}(k) \\ (c, g) \times (\mu, \varphi) &\longmapsto (c, g) \circ (\mu, \varphi) \quad , \end{aligned}$$

$$\text{where } (c, g) \circ (\mu, \varphi) := \left( \frac{\mu}{c}, g(A, B) \varphi\left(\frac{A}{c}, g(A, B)^{-1} \cdot \frac{B}{c} \cdot g(A, B)\right) \right).$$

**Remark 2.3.2.** It can be checked easily that the relation (iii) implies (i).

Drinfel'd showed the followings:

**Proposition 2.3.3** ([Dr] *Proposition 5.1*). *The right action of  $\underline{GT}(k)$  on  $\underline{M}(k)$  is free and transitive.*

**Proposition 2.3.4** ([Dr] *Proposition 5.5*). *The left action of  $\underline{GRT}(k)$  on  $\underline{M}(k)$  is free and transitive.*

Therefore  $\underline{M}$  has a structure of bi-torsor by the right  $\underline{GT}$ -action and the left  $\underline{GRT}$ -action. In [Dr]§4, Drinfel'd considered the following sub-bi-torsor.

**Definition 2.3.5** ([Dr]§4). The pro-variety  $\underline{M}_1$  is the pro-subvariety of  $\underline{M}$  defined over  $\mathbf{Q}$  whose set of  $k$ -valued points is as follows:

$$\underline{M}_1(k) = \{ \varphi \in k\langle\langle A, B \rangle\rangle \mid (1, \varphi) \in \underline{M}(k) \}$$

<sup>5</sup>The right and left are opposite to the original ones in [Dr]. Because we turned over the direction of its multiplication of  $\underline{GRT}(k)$  (resp.  $\underline{GT}(k)$ ) in *Definition 2.1.2* (resp. *Definition 2.2.1*).

By restricting the the right action of  $\underline{GT}$  and the left action of  $\underline{GRT}$  to their unipotent parts,  $\underline{GT}_1$  and  $\underline{GRT}_1$ , respectively, we see that  $\underline{M}_1$  has a structure of bi-torsor. The following was one of the main theorems in [Dr], which was proved in [Dr] §5.

**Proposition 2.3.6** ([Dr] Theorem A').  $\underline{M}_1(\mathbf{Q}) \neq \emptyset$ .

**Remark 2.3.7.** Z. Wojtkowiak constructed a pro-algebraic group  $G$  and a  $G$ -torsor  $TG$  in [Wo]Appendix §A. In fact, his  $G$ -torsor  $TG$  is isomorphic to Drinfel'd's  $\underline{GRT}$ -torsor  $\underline{M}$ .

2.3.2. *The Drinfel'd associator.* Consider the *Knizhnik-Zamolodchikov equation* (*KZ equation* for short)

$$(KZ) \quad \frac{\partial g}{\partial u}(u) = \frac{1}{2\pi i} \left( \frac{A}{u} + \frac{B}{u-1} \right) \cdot g(u) \quad ,$$

where  $g(u)$  is an analytic function in complex variable  $u$  with values in  $\mathbf{C}\langle\langle A, B \rangle\rangle$ , where ‘analytic’ means each coefficient is analytic. The equation (KZ) has singularities only at  $0, 1$  and  $\infty$ . Let  $\mathbf{C}'$  be the complement of the union of the real half-lines  $(-\infty, 0]$  and  $[1, +\infty)$  in the complex plane, which is a simply-connected domain. The equation (KZ) has a unique analytic solution on  $\mathbf{C}'$  having a specified value at any given point on  $\mathbf{C}'$ . Moreover, at the singular points  $0$  and  $1$ , there exist unique solutions  $g_0(u)$  and  $g_1(u)$  of (KZ) such that

$$g_0(u) \approx u^{\frac{A}{2\pi i}} \quad (u \rightarrow 0), \quad g_1(u) \approx (1-u)^{\frac{B}{2\pi i}} \quad (u \rightarrow 1),$$

where  $\approx$  means that  $g_0(u) \cdot u^{-\frac{A}{2\pi i}}$  (resp.  $g_1(u) \cdot (1-u)^{-\frac{B}{2\pi i}}$ ) has an analytic continuation in a neighborhood of  $0$  (resp.  $1$ ) in  $\mathbf{C}$  with value  $1$  at  $0$  (resp.  $1$ ). Here,  $u^a := \exp(a \cdot \log u) := 1 + \frac{(a \cdot \log u)}{1!} + \frac{(a \cdot \log u)^2}{2!} + \frac{(a \cdot \log u)^3}{3!} + \dots$  and  $\log u := \int_1^u \frac{dt}{t}$  in  $\mathbf{C}'$ . In the same way,  $(1-u)^b$  can be defined on  $\mathbf{C}'$ . Since  $g_0(u)$  and  $g_1(u)$  are both invertible unique solutions of (KZ) with the specified asymptotic behaviors, they must coincide with each other up to multiplication from the right by an invertible element of  $\mathbf{C}\langle\langle A, B \rangle\rangle$ .

**Definition 2.3.8.** The Drinfel'd associator is the element  $\varphi_{KZ}(A, B)$  of  $\mathbf{C}\langle\langle A, B \rangle\rangle$  which is defined by

$$g_0(u) = g_1(u) \cdot \varphi_{KZ}(A, B) \quad .$$

In [Dr], the following is shown:

**Proposition 2.3.9** ([Dr]). *The pair  $(1, \varphi_{KZ})$  satisfies (0)~(iii) in Definition 2.3.1, i.e.  $\varphi_{KZ} \in \underline{M}_1(\mathbf{C})$  (for definition, see Definition 2.3.5).*

### 3. HODGE SIDE

We shall make a brief review on MZV's (multiple zeta values) in §3.1. We shall construct a canonical embedding from the spectrum of the  $\mathbf{Q}$ -algebra generated by all MZV's modulo the ideal generated by  $\pi^2$  into the graded Grothendieck-Teichmüller group  $\underline{GRT}_1$  in §3.2, which is one of our main results in this paper. In §3.3, we shall make another analogous embedding into the middle Grothendieck-Teichmüller torsor  $\underline{M}_1$ .

3.1. **MZV's.** We make a short review on MZV's.

**Definition 3.1.1.** For each (multi-)index  $\mathbf{k} = (k_1, k_2, \dots, k_m)$  of positive integers with  $k_1, \dots, k_{m-1} \geq 1, k_m > 1$ , the corresponding **multiple zeta value** (MZV for short)  $\zeta(\mathbf{k})$  is, by definition, the real number defined by the convergent series:

$$\zeta(\mathbf{k}) = \sum_{\substack{0 < n_1 < \dots < n_m \\ n_i \in \mathbf{N}}} \frac{1}{n_1^{k_1} \dots n_m^{k_m}} .$$

The *weight* of  $\mathbf{k} : wt(\mathbf{k})$  is defined as  $wt(\mathbf{k}) = k_1 + \dots + k_m$ . For each natural number  $w$ , let  $Z_w$  be the  $\mathbf{Q}$ -vector subspace of  $\mathbf{R}$  generated by all MZV's of indices with weight  $w : Z_w = \langle \zeta(\mathbf{k}) | wt(\mathbf{k}) = w \rangle_{\mathbf{Q}} \subseteq \mathbf{R}$ , and put  $Z_0 = \mathbf{Q}$ . Define  $Z_{\bullet}$  as the *formal direct sum* of  $Z_w$  for all  $w \geq 0$ :  $Z_{\bullet} = \bigoplus_{w \geq 0} Z_w$ .

On the dimension of the  $\mathbf{Q}$ -vector space of MZV's at each weight, we have the following conjecture.

**Dimension Conjecture .** ([Za])  $dim_{\mathbf{Q}} Z_w$  is equal to  $d_w$ , which is given by the Fibonacci-like recurrence  $d_w = d_{w-2} + d_{w-3}$ , with initial values  $d_0 = 1, d_1 = 0, d_2 = 1$ .

More details on MZV's and the above conjecture were discussed in the author's previous article [F]. Recently T. Terasoma [Te] and A. Goncharov [Gon] showed its upper-bound;  $dim Z_w \leq d_w$  for all  $w \geq 0$ , by the theory of mixed Tate motives. On the contrary, to show its lower bound;  $dim Z_w \geq d_w$  ( $w \geq 1$ ), seems to be quite difficult because we need to show their linear independency, which might be a difficult problem in transcendental number theory.

3.2. **Main result.** We shall construct a canonical embedding from  $Spec Z_{\bullet}/(\pi^2)$  into  $\underline{GRT}_1$ .

**Property 3.2.1.** *The graded  $\mathbf{Q}$ -vector space  $Z_{\bullet}$  has a structure of graded  $\mathbf{Q}$ -algebra, i.e.  $Z_a \cdot Z_b \subseteq Z_{a+b}$  for  $a, b \geq 0$ .*

This follows from definitions of MZV's (for example, see [F] *Property 1.2.2.*). We call  $Z$ , the MZV algebra.

**Notation 3.2.2.** Let  $Z./(\pi^2) = Z./\pi^2 Z$ , be the MZV-algebra  $Z$ , modulo the principal homogeneous ideal  $(\pi^2) := \pi^2 Z$ , generated by  $\pi^2 = 6\zeta(2) \in Z_2$ . This is the graded  $\mathbf{Q}$ -algebra whose grading is given by  $Z./(\pi^2) = \bigoplus_{w \geq 0} (Z./(\pi^2))_w$ , where

$$(3.2.1) \quad (Z./(\pi^2))_w := \begin{cases} \mathbf{Q} & w = 0 \\ 0 & w = 1 \\ Z_w/(\pi^2 \cdot Z_{w-2}) & w \geq 2 \end{cases} .$$

**Remark 3.2.3.** As far as the author knows, no point is known in  $\text{Spec } Z./(\pi^2)$  except one which is determined by the maximal ideal  $Z_{>0}/\pi^2 Z$ , since it is related to transcendental problems in transcendental number theory.

**Notation 3.2.4.** Let  $\mathbb{A}_\bullet = \bigoplus_{w \geq 0} \mathbb{A}_w = \mathbf{Q}\langle A, B \rangle$  be the non-commutative graded polynomial ring over  $\mathbf{Q}$  with two variables  $A$  and  $B$  with  $\deg A = \deg B = 1$ , where  $\mathbb{A}_w$  is the homogeneous degree  $w$  part of  $\mathbb{A}_\bullet$ .

**Theorem 3.2.5.** *There is a surjection*

$$(3.2.2) \quad \Phi_{DR}^\sharp : \mathcal{O}(\underline{GRT}_1)_\bullet \twoheadrightarrow Z./(\pi^2)$$

*of graded  $\mathbf{Q}$ -algebras, which associates an embedding of schemes*

$$(3.2.3) \quad \Phi_{DR} : \text{Spec } Z./(\pi^2) \hookrightarrow \underline{GRT}_1 .$$

**Proof .** Put

$$(3.2.4) \quad \Phi_{KZ}(A, B) = 1 + \sum_{W: \text{words}} I(W)W := \varphi_{KZ}(2\pi i A, 2\pi i B) \in \mathbf{C}\langle\langle A, B \rangle\rangle,$$

where  $\varphi_{KZ}$  is the Drinfel'd associator (*Definition 2.3.8*). For each word  $W$  with  $wt(W) = w$ ,  $I(W)$  lies in  $Z_w$  (see [F] *Property I §3.2.*), i.e.  $\Phi_{KZ} \in \widehat{\bigoplus}_{w \geq 0} (\mathbb{A}_w \otimes_{\mathbf{Q}} Z_w)$ . By *Proposition 2.3.9*, it satisfies

$$\left\{ \begin{array}{l} \text{(0)} \log \Phi_{KZ}(A, B) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \{\sum I(W)W\}^n \in [\mathbb{L}_{\mathbf{C}}^\wedge, \mathbb{L}_{\mathbf{C}}^\wedge] \left( = \widehat{\bigoplus}_{a \geq 2} (\mathbb{L}_a \otimes_{\mathbf{Q}} \mathbf{C}) \right) \\ \text{(I)} \Phi_{KZ}(A, B)\Phi_{KZ}(B, A) = 1 \\ \text{(II)} e^{\pi i A} \Phi_{KZ}(C, A) e^{\pi i C} \Phi_{KZ}(B, C) e^{\pi i B} \Phi_{KZ}(A, B) = 1 \\ \quad \text{for } A + B + C = 0 \\ \text{(III)} \Phi_{KZ}(X_{1,2}, X_{2,3}) \Phi_{KZ}(X_{3,4}, X_{4,5}) \Phi_{KZ}(X_{5,1}, X_{1,2}) \\ \quad \Phi_{KZ}(X_{2,3}, X_{3,4}) \Phi_{KZ}(X_{4,5}, X_{5,1}) = 1 \quad \text{in } \underline{U\mathfrak{P}}^{(5)}(\mathbf{C}) . \end{array} \right.$$

For each word  $W$  with  $wt(W) = w$ , denote the quotient class of  $I(W) \in Z_w$  in  $(Z/(\pi^2))_w$  by  $\overline{I(W)}$  and put

$$\overline{\Phi_{KZ}}(A, B) := 1 + \sum_{W:\text{words}} \overline{I(W)}W \in \widehat{\bigoplus}_{w \geq 0} \left( \mathbb{A}_w \otimes_{\mathbf{Q}} (Z/(\pi^2))_w \right).$$

Then the above four formulae imply

$$\left\{ \begin{array}{l} \text{(0) } \log \overline{\Phi_{KZ}}(A, B) \in \widehat{\bigoplus}_{w \geq 2} \left( \mathbb{L}_w \otimes_{\mathbf{Q}} (Z/(\pi^2))_w \right) \\ \text{(I) } \overline{\Phi_{KZ}}(A, B) \overline{\Phi_{KZ}}(B, A) = 1 \quad \text{in } \widehat{\bigoplus}_{w \geq 0} \left( \mathbb{A}_w \otimes_{\mathbf{Q}} (Z/(\pi^2))_w \right) \\ \text{(II) } \overline{\Phi_{KZ}}(C, A) \overline{\Phi_{KZ}}(B, C) \overline{\Phi_{KZ}}(A, B) = 1 \\ \quad \text{for } A + B + C = 0 \quad \text{in } \widehat{\bigoplus}_{w \geq 0} \left( \mathbb{A}_w \otimes_{\mathbf{Q}} (Z/(\pi^2))_w \right) \\ \text{(III) } \overline{\Phi_{KZ}}(X_{1,2}, X_{2,3}) \overline{\Phi_{KZ}}(X_{3,4}, X_{4,5}) \overline{\Phi_{KZ}}(X_{5,1}, X_{1,2}) \\ \quad \overline{\Phi_{KZ}}(X_{2,3}, X_{3,4}) \overline{\Phi_{KZ}}(X_{4,5}, X_{5,1}) = 1 \quad \text{in } \widehat{\bigoplus}_{w \geq 0} \left( U\mathfrak{P}_w^{(5)} \otimes_{\mathbf{Q}} (Z/(\pi^2))_w \right) \end{array} \right. .$$

So  $\overline{\Phi_{KZ}}(A, B)$  determines a  $Z/(\pi^2)$ -valued point of  $\underline{GRT}_1$ , i.e.  $\overline{\Phi_{KZ}}(A, B) \in \underline{GRT}_1(Z/(\pi^2))$ . Thus we obtain the algebra homomorphism (3.2.2) by sending each  $x_W$  (*Notation 2.1.7*) to  $\overline{I(W)}$ . Since  $x_W$ 's (resp.  $\overline{I(W)}$ 's) are algebraic generators of the graded algebra  $\mathcal{O}(\underline{GRT}_1)$  (resp.  $Z/(\pi^2)$ ) whose degree is equal to  $wt W$ ,  $\Phi_{DR}^\sharp$  is a surjective algebra homomorphism preserving their degrees. From this surjective algebra homomorphism  $\Phi_{DR}^\sharp$ , we obtain the embedding (3.2.3) of schemes.  $\square$

**3.3. Related embedding into  $\underline{M}_1$ .** In §3.2, we get an embedding from  $\text{Spec } Z/(\pi^2)$  into the graded Grothendieck-Teichmüller group  $\underline{GRT}_1$ . But on the contrary, in this subsection, we get a related embedding from the spectrum of a modified algebra of the MZV algebra into the middle Grothendieck-Teichmüller torsor  $\underline{M}_1$ .

**Definition 3.3.1.** For each index  $\mathbf{k} = (k_1, k_2, \dots, k_m)$  of positive integers with  $k_1, \dots, k_{m-1} \geq 1$ ,  $k_m > 1$ , we define the corresponding **modified multiple zeta value** by

$$\tilde{\zeta}(\mathbf{k}) := \frac{1}{(2\pi i)^{wt \mathbf{k}}} \zeta(\mathbf{k}).$$

For each natural number  $w$ , let  $\mathfrak{Z}_{\leq w}$  be the  $\mathbf{Q}$ -vector subspace of  $\mathbf{C}$  generated by all  $\tilde{\zeta}(\mathbf{k})$ 's with  $wt(\mathbf{k}) \leq w$ :  $\mathfrak{Z}_{\leq w} := \langle \zeta(\mathbf{k}) \mid wt(\mathbf{k}) \leq w \rangle_{\mathbf{Q}} \subseteq \mathbf{C}$ , and put  $\mathfrak{Z}_{\leq 0} := \mathbf{Q}$ . Define  $\mathfrak{Z}$  to be the  $\mathbf{Q}$ -vector subspace of  $\mathbf{C}$  generated by all  $\tilde{\zeta}(\mathbf{k})$ 's.

**Notation 3.3.2.** By *Property 3.2.1*,  $\mathfrak{Z}$  becomes a filtered  $\mathbf{Q}$ -algebra with ascending filtration  $W = \{\mathfrak{Z}_{\leq a}\}_{a \geq 0}$ , i.e.  $\mathfrak{Z}_{\leq a} \cdot \mathfrak{Z}_{\leq b} \subseteq \mathfrak{Z}_{\leq a+b}$  ( $a, b \geq 0$ ). Let  $Gr_*^W \mathfrak{Z}$  denote the associated graded  $\mathbf{Q}$ -algebra of  $\mathfrak{Z}$ :  $Gr_*^W \mathfrak{Z} = \bigoplus_{a \geq 0} V_a$  where  $V_a = \mathfrak{Z}_{\leq a} / \mathfrak{Z}_{\leq a-1}$  for  $a \geq 1$  and  $V_0 = \mathfrak{Z}_{\leq 0} = \mathbf{Q}$ .

For each  $a \geq 0$ , let  $f_a : \mathfrak{Z}_{\leq a} \rightarrow Z_a$  denote the  $\mathbf{Q}$ -linear map defined by sending each  $\tilde{\zeta}(\mathbf{k})$  with  $wt(\mathbf{k}) \leq a$ , to  $Re\{(2\pi i)^a \tilde{\zeta}(\mathbf{k})\} \in Z_a \subset \mathbf{R}$ , where  $Re$  stands for the real parts. If  $a \geq 2$  and  $wt(\mathbf{k}) \leq a-1$ , then  $f_a(\tilde{\zeta}(\mathbf{k})) \in \pi^2 Z_{a-2} \subset \mathbf{R}$ . Thus  $f_a$  induces a  $\mathbf{Q}$ -linear map  $g_a : Gr_a^W \mathfrak{Z} \rightarrow (Z_*/(\pi^2))_a$ .

**Proposition 3.3.3.** *The  $\mathbf{Q}$ -linear maps  $\{g_a\}_{a \geq 0}$  induce the following canonical isomorphism of graded  $\mathbf{Q}$ -algebras:*

$$g := \bigoplus_{a \geq 0} g_a : Gr_*^W \mathfrak{Z} \xrightarrow{\sim} Z_*/(\pi^2) \quad .$$

**Proof .** The surjectivity of  $g_a$  ( $a \geq 0$ ) is trivial. The injectivity of  $g_a$  is trivial for  $a = 0, 1$ . Suppose that  $a \geq 2$  and  $g_a(\sum_{i=1}^m r_i \tilde{\zeta}(\mathbf{k}_i)) \equiv 0$  for  $m \in \mathbf{N}$ ,  $r_i \in \mathbf{Q}$  and  $wt(\mathbf{k}_i) = a$  ( $1 \leq i \leq m$ ). Then

$$\begin{aligned} f_a\left(\sum_{i=1}^m r_i \tilde{\zeta}(\mathbf{k}_i)\right) &\in \pi^2 Z_{a-2} \quad . \\ \therefore \sum_{i=1}^m r_i \zeta(\mathbf{k}_i) &\in \pi^2 Z_{a-2} \quad . \\ \therefore \sum_{i=1}^m r_i \tilde{\zeta}(\mathbf{k}_i) &\in \mathfrak{Z}_{\leq a-2} \quad . \end{aligned}$$

Therefore  $g_a$  is injective for  $a \geq 2$ . To check that the linear map  $g$  is a homomorphism of graded  $\mathbf{Q}$ -algebras is immediate.  $\square$

The following proposition is an analogue of *Theorem 3.2.5*.

**Proposition 3.3.4.** *There is a surjection*

$$(3.3.1) \quad \Phi_{Hod}^\sharp : \mathcal{O}(\underline{M}_1) \twoheadrightarrow \mathfrak{Z}$$

*of  $\mathbf{Q}$ -algebras, which associates an embedding of schemes*

$$(3.3.2) \quad \Phi_{Hod} : Spec \mathfrak{Z} \hookrightarrow \underline{M}_1 \quad .$$

**Proof .** By imitating the proof of *Theorem 3.2.5*, we can construct the surjection  $\Phi_{Hod}^\sharp$  thanks to *Proposition 2.3.9*.  $\square$

It can be verified directly that, in fact, the surjection  $\Phi_{Hod}^\sharp : \mathcal{O}(\underline{M}_1) \rightarrow \mathfrak{Z}$  is strictly compatible with the weight filtration of  $\mathcal{O}(\underline{M}_1)$  (§6.1) and that of  $\mathfrak{Z}$  (*Notation* 3.3.2), i.e.  $\Phi_{Hod}^\sharp(W_a \mathcal{O}(\underline{M}_1)) = \mathfrak{Z}_{\leq a}$  for  $a \geq 0$ .

Here *Hod* stands for Hodge. The relationship between *Theorem* 3.2.5 and *Proposition* 3.3.4 will be discussed in *Proposition* 6.2.4.

3.4. *Spec*  $Z/(\pi^2) = \underline{GRT}_1$ ? We will discuss the conjecture that the embedding  $\Phi_{DR}$  in §3.2 might be an isomorphism.

**Notation 3.4.1.** For a graded vector space  $V = \bigoplus_{a \in \mathbf{Z}} V_a$ , we denote its completion by degree by  $V^\wedge := \widehat{\bigoplus}_{a \in \mathbf{Z}} V_a$  and its graded dual vector space by  $V_* := \bigoplus_{a \in \mathbf{Z}} V_a^*$ , where  $V_a^*$  is the dual vector space of  $V_a$ .

For any two formal power series  $P(t), Q(t)$  in  $\mathbf{Q}[[t]]$ , we express  $P(t) \geq Q(t)$  when the formal power series  $P(t) - Q(t)$  has all coefficients non-negative.

**Proposition 3.4.2.** *Assume the generatedness part of Conjecture 2.1.13 which is equivalent to saying that  $\mathfrak{D}$  is a Lie algebra generated by one element in each degree  $m$  ( $m = 3, 5, 7, \dots$ ) and assume the lower bound part of Dimension conjecture (§3.1), which is equivalent to saying that  $\dim Z_w \geq d_w$  holds for all  $w \geq 0$ . Then the embedding  $\Phi_{DR} : \text{Spec } Z/(\pi^2) \hookrightarrow \underline{GRT}_1$  must be an isomorphism.*

**Proof .** By taking the differential of  $\Phi_{DR}$  at the point  $e$  which corresponds to the unit element of  $\underline{GRT}_1$ , we get an embedding

$$(d\Phi_{DR})_e : (NZ^*)^\wedge \hookrightarrow \text{Lie } \underline{GRT}_1 \quad .$$

Here  $(NZ^*)^\wedge$  is the completion by degree of the dual vector space of the *new-zeta space*  $NZ = \bigoplus_{w \geq 2} NZ_w := (Z_{>2}/Z_{>0}^2)$  (see [F]§1.3.). Recall that  $\mathfrak{D}^\wedge = \widehat{\bigoplus}_{w \geq 1} \mathfrak{D}_w \simeq \text{Lie } \underline{GRT}_1$  (see *Remark* 2.1.12). Thus we get an embedding of graded vector spaces

$$(3.4.1) \quad (d\Phi_{DR})_e : (NZ^*)^\wedge = \widehat{\bigoplus}_{w \geq 1} NZ_w^* \hookrightarrow \mathfrak{D}^\wedge = \widehat{\bigoplus}_{w \geq 1} \mathfrak{D}_w \quad .$$

Then we have

$$\sum_{w=0}^{\infty} d_w t^w \leq \sum_{w=0}^{\infty} \dim_{\mathbf{Q}} Z_w \cdot t^w$$

by the above second assumption,

$$\leq \frac{1}{1-t^2} \prod_{w=1}^{\infty} \frac{1}{(1-t^w)^{\dim_{\mathbf{Q}} NZ_w}}$$

by the definition of  $NZ$ ,

$$\leq \frac{1}{1-t^2} \prod_{w=1}^{\infty} \frac{1}{(1-t^w)^{\dim_{\mathbf{Q}} \mathfrak{D}_w}}$$

by (3.4.1) and

$$\leq \sum_{w=0}^{\infty} d_w t^w$$

by [F] *Lemma 4.3.6* combined with the first assumption.

Therefore all above inequalities must be equalities, which implies that  $Z$  is a polynomial algebra and  $(d\Phi_{DR})_e : (Z_{>2}/Z_{>0}^2)^{\wedge} \hookrightarrow \mathfrak{D}_\bullet^{\wedge}$  is an isomorphism. Thus

$$(3.4.2) \quad \sum_{w=0}^{\infty} \dim_{\mathbf{Q}}(Z/(\pi^2))_w \cdot t^w = \prod_{w=1}^{\infty} \frac{1}{(1-t^w)^{\dim_{\mathbf{Q}} \mathfrak{D}_w}} \quad .$$

On the other hand, from the surjectivity (*Theorem 3.2.5*) of  $\Phi_{DR}^{\sharp} : \mathcal{O}(\underline{GRT}_1)_\bullet \rightarrow Z/(\pi^2)$ , we get

$$(3.4.3) \quad \dim_{\mathbf{Q}}(Z/(\pi^2))_w \leq \dim_{\mathbf{Q}} \mathcal{O}(\underline{GRT}_1)_w \quad \text{for all } w.$$

Put  $\mathfrak{m}_e = \bigoplus_{w>0} \mathcal{O}(\underline{GRT}_1)_w$ . Then this graded vector space  $\mathfrak{m}_e$  is the defining ideal of the unit element  $e$  in  $\underline{GRT}_1$ . Since  $\mathfrak{D}_\bullet^{\wedge} \simeq \text{Lie } \underline{GRT}_1$ ,  $\mathfrak{D}_\bullet^* = \bigoplus_{w \geq 0} \mathfrak{D}_w^*$  is canonically isomorphic to the graded vector space  $\mathfrak{m}_e / (\mathfrak{m}_e)^2$ . Thus

$$(3.4.4) \quad \sum_{w=0}^{\infty} \dim_{\mathbf{Q}} \mathcal{O}(\underline{GRT}_1)_w \cdot t^w \leq \prod_{w=1}^{\infty} \frac{1}{(1-t^w)^{\dim_{\mathbf{Q}} \mathfrak{D}_w}} \quad .$$

From (3.4.2)  $\sim$  (3.4.4), it follows that

$$\dim_{\mathbf{Q}}(Z/(\pi^2))_w = \dim_{\mathbf{Q}} \mathcal{O}(\underline{GRT}_1)_w \quad \text{for all } w.$$

Thus  $\Phi_{DR}^{\sharp}$  must be an isomorphism, which means that the embedding  $\Phi_{DR}$  (3.2.2) of the pro-algebraic group must be an isomorphism.  $\square$

Therefore it may be natural to pose that

**Conjecture A** . The above embedding  $\Phi_{DR}$  of the affine scheme is an isomorphism, i.e.

$$\Phi_{DR} : \text{Spec } Z/(\pi^2) \cong \underline{GRT}_1$$

- Remark 3.4.3.** (1) This conjecture claims that  $\text{Spec } Z./(\pi^2)$  is naturally equipped with a structure of non-commutative group scheme, which is equivalent to saying that the quotient algebra  $Z./(\pi^2)$  has a structure of non-co-commutative Hopf algebra.
- (2) A unipotent algebraic group is isomorphic to its Lie algebra (as varieties), hence to an affine space. Together with this, the above conjecture would imply that  $Z./(\pi^2)$  must be a polynomial algebra. By taking *Remark 3.2.3* into account, it looks difficult to show this last statement, due to problems in transcendental number theory.
- (3) In §6.2, we shall see that *Conjecture A* is equivalent to saying that  $\Phi_{\text{Hod}}$  is an isomorphism (*Conjecture A'*).

#### 4. GALOIS SIDE

We will introduce pro-algebraic groups  $\underline{\text{Gal}}_{\mathbf{Q}}^{(l)}$  and  $\underline{\text{Gal}}_{\mathbf{Q}_l}^{(l)}$ , and discuss their relationship to the Grothendieck-Teichmüller pro-algebraic group  $\underline{GT}_1$ .

**4.1. The pro- $l$  Galois representation.** The *absolute Galois group*  $\text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$  acts on the algebraic fundamental group  $\pi_1(\mathbf{P}_{\mathbf{Q}}^1 - \{0, 1, \infty\}, \overrightarrow{01})$  of the projective line minus 3 points, where  $\overrightarrow{01}$  means the tangential base point (see [De]§15). Let  $l$  be a prime. With this representation, we can associate the following continuous group homomorphism into the automorphism group of the free pro- $l$  group  $\widehat{F}_2^{(l)}$  of rank 2

$$p_1^{(l)} : \text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}(\mu_{l^\infty})) \rightarrow \text{Aut } \widehat{F}_2^{(l)}$$

for each prime  $l$ , where  $\mu_{l^\infty}$  stands for the group of all  $l$ -powerth roots of unity. By [HM] *Corollary A.10*, there exists a natural topological group homomorphism into  $\underline{\text{Aut}}F_2(\mathbf{Q}_l)$  (cf. *Remark 2.2.5*)

$$p_2^{(l)} : \text{Aut } \widehat{F}_2^{(l)} \rightarrow \underline{\text{Aut}}F_2(\mathbf{Q}_l) .$$

By combining these two homomorphisms, we get the following Galois representation

$$\varphi_l := p_2^{(l)} \circ p_1^{(l)} : \text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}(\mu_{l^\infty})) \rightarrow \underline{\text{Aut}}F_2(\mathbf{Q}_l) .$$

By imitating the construction of the embedding  $\text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \hookrightarrow \widehat{GT}$  in [Ih94], we can show that its image is contained in the following pro- $l$  group version of the Grothendieck-Teichmüller group  $\widehat{GT}_1^{(l)}$ .

**Lemma 4.1.1.**

$$\text{Im } p_1^{(l)} \subseteq \widehat{GT}_1^{(l)} .$$

Here  $\widehat{GT}_1^{(l)} := \left\{ \sigma \in \text{Aut } \widehat{F}_2^{(l)} \mid \begin{array}{l} \sigma(x) = x, \sigma(y) = f^{-1}yf \text{ for } \exists f \in \widehat{F}_2^{(l)} \\ \text{which satisfies (0) } \sim \text{ (iii) below.} \end{array} \right\}$ .

$$\left\{ \begin{array}{l} \text{(0)} \quad f \in [\widehat{F}_2^{(l)}, \widehat{F}_2^{(l)}] \\ \text{(i)} \quad f(X, Y)f(Y, X) = 1 \\ \text{(ii)} \quad f(Z, X)f(Y, Z)f(X, Y) = 1 \text{ for } XYZ = 1 \\ \text{(iii)} \quad f(x_{1,2}, x_{2,3})f(x_{3,4}, x_{4,5})f(x_{5,1}, x_{1,2})f(x_{2,3}, x_{3,4})f(x_{4,5}, x_{5,1}) = 1 \\ \hspace{15em} \text{in } \widehat{P}_5^{(l)} \quad (\text{see Note 4.1.2}) \end{array} \right. .$$

**Note 4.1.2.** Here  $[\widehat{F}_2^{(l)}, \widehat{F}_2^{(l)}]$  means the topological commutator subgroup of  $\widehat{F}_2^{(l)}$  and  $\widehat{P}_5^{(l)}$  is the pro- $l$  completion of the pure sphere braid group  $P_5$  and  $x_{i,j}$ 's ( $1 \leq i, j \leq 5$ ) are its standard generators ([Ih91]). Note that  $\sigma \in \widehat{GT}_1^{(l)}$  determines  $f \in \widehat{F}_2^{(l)}$  uniquely because of the condition (0).

Since there exists a natural topological group homomorphism  $\widehat{F}_2^{(l)} \rightarrow F_2(\mathbf{Q}_l)$  with Zariski dense image, it follows from definitions

**Lemma 4.1.3.**

$$p_2^{(l)}(\widehat{GT}_1^{(l)}) \subseteq \underline{GT}_1(\mathbf{Q}_l) \quad (\subseteq \underline{\text{Aut}}F_2(\mathbf{Q}_l)) \quad .$$

But it seems not so clear that

**Problem 4.1.4.** Is  $p_2^{(l)}(\widehat{GT}_1^{(l)})$  Zariski dense in  $\underline{GT}_1(\mathbf{Q}_l)$  or not?

By *Lemma 4.1.1* and *Lemma 4.1.3*, it follows

**Proposition 4.1.5.**

$$\text{Im } \varphi_l \subseteq \underline{GT}_1(\mathbf{Q}_l) \quad .$$

4.2.  $\underline{\text{Gal}}_{\mathbf{Q}}^{(l)} = \underline{GT}_1$ ?. In this subsection, we will construct a pro-algebraic group version of the Galois image  $\text{Im } \varphi_l$ .

**Notation 4.2.1.** Let  $k$  be a field extension of  $\mathbf{Q}$ . For any pro-algebraic group  $\mathcal{G} = \varprojlim_r G^{(r)}$  over  $\mathbf{Q}$ , we define the  $k$ -**structure** of  $\mathcal{G}$  to be the pro-algebraic group  $\mathcal{G} \times_{\mathbf{Q}} k := \varprojlim_r (G^{(r)} \times_{\mathbf{Q}} k)$  over  $k$ .

**Definition 4.2.2.** The pro-algebraic group  $\underline{\text{Gal}}_{\mathbf{Q}}^{(l)}$  over  $\mathbf{Q}$  is the smallest pro-sub-variety of  $\underline{GT}_1$  defined over  $\mathbf{Q}$  whose set of  $\mathbf{Q}_l$ -rational points contains the image of  $\text{Im } \varphi_l$ .

Note that  $\underline{Gal}_{\mathbf{Q}}^{(l)}$  has a structure of pro-linear algebraic group defined over  $\mathbf{Q}$ .

**Definition 4.2.3.** The pro-algebraic group  $\underline{Gal}_{\mathbf{Q}_l}^{(l)}$  over  $\mathbf{Q}_l$  is the smallest pro-sub-variety of  $\underline{GT}_1 \times_{\mathbf{Q}} \mathbf{Q}_l$  defined over  $\mathbf{Q}_l$  whose set of  $\mathbf{Q}_l$ -rational points contains the image of  $Im \varphi_l$ .

Note that  $\underline{Gal}_{\mathbf{Q}_l}^{(l)}$  has a structure of pro-linear algebraic group defined over  $\mathbf{Q}_l$ .

**Notation 4.2.4.** From these definitions, we get the following embeddings of pro-algebraic groups.

$$(4.2.1) \quad T_l : \underline{Gal}_{\mathbf{Q}_l}^{(l)} \hookrightarrow \underline{Gal}_{\mathbf{Q}}^{(l)} \times_{\mathbf{Q}} \mathbf{Q}_l$$

$$(4.2.2) \quad \Phi_{\mathbf{Q}}^{(l)} : \underline{Gal}_{\mathbf{Q}}^{(l)} \hookrightarrow \underline{GT}_1$$

$$(4.2.3) \quad \Phi_{\mathbf{Q}_l}^{(l)} : \underline{Gal}_{\mathbf{Q}_l}^{(l)} \hookrightarrow \underline{GT}_1 \times_{\mathbf{Q}} \mathbf{Q}_l$$

Note that  $\Phi_{\mathbf{Q}_l}^{(l)} = \left( \Phi_{\mathbf{Q}}^{(l)} \times id_{\mathbf{Q}_l} \right) \circ T_l$ .

On these embeddings it may be natural to state the following two conjectures which are related to those of [De] and [Ih99].

**Conjecture B .** The embedding  $\Phi_{\mathbf{Q}}^{(l)}$  (4.2.2) of the pro-algebraic group is an isomorphism, i.e.

$$\Phi_{\mathbf{Q}}^{(l)} : \underline{Gal}_{\mathbf{Q}}^{(l)} \cong \underline{GT}_1 \quad \text{for all prime } l$$

**Conjecture C .** The embedding  $T_l$  (4.2.1) of the pro-algebraic group is an isomorphism, i.e.

$$T_l : \underline{Gal}_{\mathbf{Q}_l}^{(l)} \cong \underline{Gal}_{\mathbf{Q}}^{(l)} \times_{\mathbf{Q}} \mathbf{Q}_l$$

It seems possible to deduce from the unpublished result in [DG] (announced in [Gon]) that there exists a sub-algebraic groups  $\mathcal{M}$  of  $\underline{AutF}_2$  defined over  $\mathbf{Q}$  such that

$$\underline{Gal}_{\mathbf{Q}_l}^{(l)} \cong \mathcal{M} \times_{\mathbf{Q}} \mathbf{Q}_l$$

for all prime  $l$ . We remark that especially this implies the validity of *Conjecture C*.

**4.3. The  $l$ -adic Galois image Lie algebra.** In this subsection, we make a short review of the  $l$ -adic Galois image Lie algebra  $\mathfrak{g}^{(l)}$  [Ih90] and give a relationship among *Conjecture B*, *Conjecture C* and Ihara's conjecture (see *Conjecture 4.3.2* below) [Ih99].

**Notation 4.3.1.** Let  $\{\widehat{F}_2^{(l)}(m)\}_{m \in \mathbf{N}}$  be the lower central series of the free pro- $l$  group  $\widehat{F}_2^{(l)}$  of rank 2 which is defined inductively by  $\widehat{F}_2^{(l)}(1) := \widehat{F}_2^{(l)}$ ,  $\widehat{F}_2^{(l)}(m+1) := [\widehat{F}_2^{(l)}, \widehat{F}_2^{(l)}(m)]$  ( $m \geq 1$ ), where  $[\cdot, \cdot]$  means the topological commutator.

For each filtered vector space  $(V, \{W_n V\}_{n \in \mathbf{Z}})$  where  $\{W_n V\}_{n \in \mathbf{Z}}$  is the ascending family of sub-vector spaces of  $V$ , we define the graded vector space  $Gr_{\cdot}^W V := \bigoplus_{n \in \mathbf{Z}} Gr_n^W V$ , where  $Gr_n^W V := W_n V / W_{n-1} V$ .

Let

$$p_1^{(l)}(m) : Gal(\overline{\mathbf{Q}}/\mathbf{Q}(\mu_{l^\infty})) \rightarrow Aut\left(\widehat{F}_2^{(l)} / \widehat{F}_2^{(l)}(m+1)\right)$$

be the induced homomorphisms from  $p_1^{(l)}$  (§4.1) ( $m \geq 1$ ). We denote  $\mathbf{Q}^{(l)}(m)$  by the field corresponding to  $Ker p_1^{(l)}(m)$ . The formal direct sum

$$\mathfrak{g}^{(l)} := \bigoplus_{m \geq 1} \mathfrak{g}_m^{(l)} \quad \text{where} \quad \mathfrak{g}_m^{(l)} := Gal(\mathbf{Q}^{(l)}(m+1)/\mathbf{Q}^{(l)}(m)) \otimes_{\mathbf{Z}_l} \mathbf{Q}_l$$

has a structure of the graded Lie algebra over  $\mathbf{Q}_l$  by taking the commutator as the bracket (see [Ih90]). This  $l$ -adic graded Lie algebra  $\mathfrak{g}^{(l)}$  is called the  **$l$ -adic Galois image Lie algebra**. In [Ih90], it was shown that there is a natural (graded) embedding from the  $l$ -adic Galois image Lie algebra  $\mathfrak{g}^{(l)}$  into the  $l$ -adic stable derivation (Lie-)algebra  $\mathfrak{D}_{\cdot} \otimes_{\mathbf{Q}} \mathbf{Q}_l$  (*Definition 2.1.11*)

(4.3.1)

$$\Psi_l \left( = \bigoplus_{w \geq 2} \Psi_{(l),w} \right) : \mathfrak{g}^{(l)} \left( = \bigoplus_{w \geq 2} \mathfrak{g}_w^{(l)} \right) \hookrightarrow \mathfrak{D}_{\cdot} \otimes_{\mathbf{Q}} \mathbf{Q}_l \left( = \bigoplus_{w \geq 2} \mathfrak{D}_w \otimes_{\mathbf{Q}} \mathbf{Q}_l \right) \quad .$$

On this embedding of  $l$ -adic graded Lie algebras, we have the following

**Conjecture 4.3.2** ([Ih99] *Conjecture 1*). The above embedding  $\Psi_l$  of  $l$ -adic graded Lie algebras is an isomorphism, i.e.

$$\Psi_l : \mathfrak{g}^{(l)} \cong \mathfrak{D}_{\cdot} \otimes_{\mathbf{Q}} \mathbf{Q}_l \quad \text{for all prime } l.$$

In fact, the validity of *Conjecture 4.3.2* is equivalent to the validity of *Conjecture B* and *Conjecture C*:

**Proposition 4.3.3.** *The embedding  $\Phi_{\mathbf{Q}_l}^{(l)} : Gal_{\mathbf{Q}_l}^{(l)} \hookrightarrow \underline{GT}_1 \times_{\mathbf{Q}} \mathbf{Q}_l$  (4.2.3) of pro-algebraic groups is an isomorphism if and only if the embedding  $\Psi_l : \mathfrak{g}^{(l)} \hookrightarrow \mathfrak{D}_{\cdot} \otimes_{\mathbf{Q}} \mathbf{Q}_l$  (4.3.1) of  $l$ -adic graded Lie algebras is an isomorphism.*

**Proof .** Recall that  $\underline{GT}_1$  is a unipotent pro-algebraic group. Therefore

(4.3.2)

the embedding  $\Phi_{\mathbf{Q}_l}^{(l)} : \underline{Gal}_{\mathbf{Q}_l}^{(l)} \hookrightarrow \underline{GT}_1 \times \mathbf{Q}_l$  is an isomorphism  $\iff$   
the embedding of Lie algebras  $d\Phi_{\mathbf{Q}_l}^{(l)} : \text{Lie } \underline{Gal}_{\mathbf{Q}_l}^{(l)} \hookrightarrow \text{Lie } (\underline{GT}_1 \times \mathbf{Q}_l)$   
is an isomorphism.

We regard  $\text{Lie } \underline{GT}_1$  as a  $\underline{GT}$ -module by the adjoint action. Since  $\underline{GT}$  is a negatively weighted extension with respect to  $\frac{1}{\varpi} : \mathbb{G}_m \rightarrow \mathbb{G}_m$  (*Lemma 2.2.4*), the pro-Lie algebra  $\text{Lie } \underline{GT}_1$  is naturally equipped with a weight filtration ([HM] *Proposition 4.5*). With this filtration,  $\text{Lie } \underline{GT}_1$  becomes a filtered Lie algebra ([HM] *Proposition 3.4*). Since the weight grade functor  $Gr_{\cdot}^W$  is exact ([HM] *Theorem 3.12*), it follows that

(4.3.3)

the embedding  $d\Phi_{\mathbf{Q}_l}^{(l)} : \text{Lie } \underline{Gal}_{\mathbf{Q}_l}^{(l)} \hookrightarrow \text{Lie } (\underline{GT}_1 \times \mathbf{Q}_l)$  is an isomorphism  $\iff$   
its associated embedding  $Gr_{\cdot}^W d\Phi_{\mathbf{Q}_l}^{(l)} : Gr_{\cdot}^W \text{Lie } \underline{Gal}_{\mathbf{Q}_l}^{(l)} \hookrightarrow Gr_{\cdot}^W \text{Lie } (\underline{GT}_1 \times \mathbf{Q}_l)$   
is an isomorphism

It was shown that  $Gr_{\cdot}^W \text{Lie } \underline{Gal}_{\mathbf{Q}_l}^{(l)} \cong \mathfrak{g}_{\cdot}^{(l)}$  in [HM] *Theorem 8.4* and that  $Gr_{\cdot}^W \text{Lie } \underline{GT}_1 \cong \mathfrak{D}_{\cdot}$  in [Dr] *Theorem 5.6*. Together with these two isomorphisms, the embedding  $d\Phi_{\mathbf{Q}_l}^{(l)}$  yields  $\Psi_l$ . Namely

$$(4.3.4) \quad Gr_{\cdot}^W d\Phi_{\mathbf{Q}_l}^{(l)} = \Psi_l.$$

From (4.3.2)  $\sim$  (4.3.4), the statement of the proposition follows.  $\square$

## 5. $\underline{GT}_1 = \underline{GRT}_1$ PART I

In §5 and §6, we shall discuss a relationship between  $\underline{GT}_1$  and  $\underline{GRT}_1$  and then compare “Galois Side” (§4) and “Hodge Side” (§3) via these two Grothendieck-Teichmüller pro-algebraic groups.

In this section, we prove two kinds of isomorphism between  $\underline{GT}$  and  $\underline{GRT}$  in §5.1. In §5.3, we shall explain Figure 1 (§0).

**5.1. Comparison between  $\underline{GT}$  and  $\underline{GRT}$ .** Here we shall prove a few propositions, which may be regarded as corollaries of Drinfel’d’s results.

**5.1.1. Non-canonical isomorphism between their  $\mathbf{Q}$ -structures.** Let  $k$  be any field of characteristic 0.

**Proposition 5.1.1.** *Between  $k$ -structures (Notation 4.2.1) of  $\underline{GT}$  and  $\underline{GRT}$ , there exists an isomorphism of group schemes*

$$(5.1.1) \quad S_\varphi : \underline{GT} \times k \xrightarrow{\sim} \underline{GRT} \times k$$

which arises from each point of  $\varphi$  of  $\underline{M}(k)$ .

**Proof .** Take any point  $\varphi$  on  $\underline{M}(k)$ . Recall that its existence follows from *Proposition 2.3.6*. Assume that  $R$  is an arbitrary  $k$ -algebra. Then it is immediate that each  $\varphi$  determines an isomorphism  $S_\varphi : \underline{GT}(R) \xrightarrow{\sim} \underline{GRT}(R)$  of groups such that  $\varphi \circ f = S_\varphi(f) \circ \varphi$  for all  $f \in \underline{GT}(R)$ , by regarding  $\varphi \in \underline{M}(R)$ .  $\square$

Especially by taking a rational point  $\varphi$  of  $\underline{M}$ , we obtain an isomorphism between their  $\mathbf{Q}$ -structures. This isomorphism is non-canonical since it depends on the choice of  $\varphi$ .

5.1.2. *Standard isomorphism over  $\mathbf{C}$ .* Recall that  $\varphi_{KZ}$  is a standard point on  $\underline{M}_1(\mathbf{C})$  (*Proposition 2.3.9*). Especially by taking the above (*Proposition 5.1.1*) point  $\varphi$  by  $\varphi_{KZ} \in \underline{M}_1(\mathbf{C})$ , we get

**Proposition 5.1.2.** *Between  $\mathbf{C}$ -structures of  $\underline{GT}$  and  $\underline{GRT}$ , there exists a standard isomorphism*

$$p = S_{\varphi_{KZ}} : \underline{GT} \times_{\mathbf{Q}} \mathbf{C} \xrightarrow{\sim} \underline{GRT} \times_{\mathbf{Q}} \mathbf{C}.$$

If we identify their groups of  $\mathbf{C}$ -valued points by  $p$ , their subgroups of  $\mathbf{Q}$ -rational points are conjugate to each other in this common  $\mathbf{C}$ -structure. Namely,  $p(\underline{GT}(\mathbf{Q})) = a \circ (\underline{GRT}(\mathbf{Q})) \circ a^{-1} \subseteq \underline{GRT}(\mathbf{C})$  for some  $a \in \underline{GRT}(\mathbf{C})$ . Moreover, we can take this  $a$  as an element of  $\underline{GRT}_1(\mathbf{C})$ .

Thus by restricting  $p$  to unipotent parts, we get Figure 4.

$$\begin{array}{ccc}
 \boxed{\underline{GT}_1(\mathbf{C}) \xrightarrow{p} \underline{GRT}_1(\mathbf{C})} & & \\
 \uparrow & & \uparrow \\
 \underline{GT}_1(\mathbf{Q}) & & \underline{GRT}_1(\mathbf{Q}) \\
 \underbrace{\hspace{10em}} & & \\
 \text{conjugate} & & 
 \end{array}$$

FIGURE 4

**Proof .** Choose any rational point  $\varphi \in \underline{M}_1(\mathbf{Q})$  (cf. *Proposition 2.3.6*). By *Proposition 2.3.4*, there should exist a unique  $\mathbf{C}$ -valued point  $a \in$

$\underline{GRT}_1(\mathbf{C})$  such that  $\varphi_{KZ} = a \circ \varphi$ . Thus

$$\begin{aligned}\varphi_{KZ} \circ f &= p(f) \circ \varphi_{KZ} = (p(f) \circ a) \circ \varphi \\ \varphi_{KZ} \circ f &= a \circ \varphi \circ f = (a \circ s_\varphi(f)) \circ \varphi.\end{aligned}$$

for each  $f \in \underline{GT}(\mathbf{Q})$ . This implies that  $p(f) = a \circ s_\varphi(f) \circ a^{-1}$  for all  $f \in \underline{GT}(\mathbf{Q})$ . Therefore  $p(\underline{GT}(\mathbf{Q})) = a \circ s_\varphi(\underline{GT}(\mathbf{Q})) \circ a^{-1}$ . By combining it with  $s_\varphi(\underline{GT}(\mathbf{Q})) = \underline{GRT}(\mathbf{Q})$ , we get  $p(\underline{GT}(\mathbf{Q})) = a \circ \underline{GRT}(\mathbf{Q}) \circ a^{-1}$ .  $\square$

We remark that this isomorphism  $p$  does not descend to that of  $\mathbf{Q}$ -structure, since there appear periods in each coefficient of  $\varphi_{KZ}(A, B)$  such as  $\frac{\zeta(3)}{(2\pi i)^3}$ ,  $\frac{\zeta(2,3)}{(2\pi i)^5}$ , ....., which do not belong to  $\mathbf{Q}$ .

**5.2. Digression on a candidate of a canonical free basis of the stable derivation algebra.** In this subsection, we make a few extra remarks on Ihara's Problem 2.1.14.

**Notation 5.2.1.** Denote the logarithmic isomorphism of  $\underline{GRT}_1(\mathbf{C})$  by  $Log : \underline{GRT}_1(\mathbf{C}) \xrightarrow{\sim} (\mathfrak{D} \otimes_{\mathbf{Q}} \mathbf{C})^\wedge$ , where  $(\mathfrak{D} \otimes_{\mathbf{Q}} \mathbf{C})^\wedge$  means the completion by degree of the stable derivation algebra  $\mathfrak{D} = \bigoplus_{w \geq 1} \mathfrak{D}_w$  (§2.1.3) tensored with  $\mathbf{C}$ .

Since  $\underline{M}_1$  is defined over  $\mathbf{Q}$ , the complex conjugate  $\overline{\varphi_{KZ}}$  of  $\varphi_{KZ}$  also lies in  $\underline{M}_1(\mathbf{C})$ . Thus by *Proposition 2.3.4*, there should exist a unique element  $g \in \underline{GRT}_1(\mathbf{C})$  such that  $g \circ \varphi_{KZ} = \overline{\varphi_{KZ}}$ . In the proof of [Dr] *Proposition 6.3*, Drinfel'd get an element  $\tilde{\psi} := Log g \in (\mathfrak{D} \otimes_{\mathbf{Q}} \mathbf{C})^\wedge$ , which has the following presentation:

$$\tilde{\psi} = \sum_{m \geq 3: \text{ odd}} \psi_m \quad \text{where} \quad \psi_m = \frac{2\zeta(m)}{(2\pi i)^m} (adA)^m(B) + \dots \in \mathfrak{D}_m \otimes \mathbf{C}.$$

The essentially same element was also obtained in [Ra]. It can be checked that  $\psi_m$  ( $m \geq 3 : \text{ odd}$ ) lie on  $\mathfrak{D}_m \otimes_{\mathbf{Q}} \frac{1}{(2\pi i)^m} Z_m$ . Since each  $\psi_m$  is an element of  $\mathfrak{D}_m \otimes \mathbf{C}$  with depth 1 in the sense of [Ih99] *Lecture II* §2, these  $\psi_m$ 's might be a candidate of canonical free basis of  $\mathfrak{D} \otimes \mathbf{C}$  asked by Ihara in *Problem 2.1.14*.

**Problem 5.2.2.** These  $\psi_m$  ( $m \geq 3 : \text{ odd}$ ) generate a free Lie subalgebra of  $\mathfrak{D} \otimes \mathbf{C}$  ?

But Ihara asked for the free basis of the  $\mathbf{Q}$ -structure of the stable derivation algebra. On the above elements, we cannot expect that all  $\psi'_m := \frac{(2\pi i)^m}{2\zeta(m)} \psi_m$  lies in  $\mathfrak{D}_m$ . Since, for example, to show that  $\psi'_{11}$  lies in

$\mathfrak{D}_{11}$ , we must show the *linear dependency* of 9 MZV's on the list of  $Z_{11}$  on [F] *Example* 4.3.4, which are predicted to be linearly independent by Dimension conjecture (§3.1).

**5.3. Comparison between Galois Side and Hodge Side.** By combining *Proposition* 5.1.2 with the embedding  $\Phi_{DR}$  (3.2.3) and  $\Phi_{\mathbf{Q}}^{(l)}$  (4.2.2), we get Figure 1 in §0.

Validities of *Conjecture A* (§3.4), *Conjecture B* (§4.2) and *Conjecture C* (§4.2) would imply the following relationship between Galois Side and Hodge Side.

*“In Galois Side, there should exist a common  $\mathbf{Q}$ -structure of  $\underline{Gal}_{\mathbf{Q}_l}^{(l)}$  for all prime  $l$ , i.e.  $\underline{Gal}_{\mathbf{Q}_l}^{(l)} = \underline{GT}_1 \times_{\mathbf{Q}} \mathbf{Q}_l$ . On the other hand, in Hodge Side,  $\text{Spec } Z_./(\pi^2)$  should be equipped with a structure of pro-algebraic group and it should provide a different  $\mathbf{Q}$ -structure of the above common  $\mathbf{Q}$ -structure. After tensored with  $\mathbf{C}$ , these two  $\mathbf{Q}$ -structures should be isomorphic. Moreover their two  $\mathbf{Q}$ -structures should be conjugate with each other in this common  $\mathbf{C}$ -structure.”*

## 6. $\underline{GT}_1 = \underline{GRT}_1$ PART II

In §6.1, we shall introduce a weight filtration on the regular function ring  $\mathcal{O}(\underline{M}_1)$ . §6.2 will be devoted to a relationship between  $\underline{M}_1$  and  $\underline{GRT}_1$ . In §6.3, we shall discuss a relationship between  $\underline{GT}_1$  and  $\underline{GRT}_1$  which is different from that of §5 and then compare “Galois Side” and “Hodge Side” via these two Grothendieck-Teichmüller pro-algebraic groups again in §6.5 and §6.6. We shall also explain Figure 2 (§0) in §6.5.

**6.1. Weight filtration of  $\mathcal{O}(\underline{M}_1)$ .** We introduce two kinds of weight filtration on the regular function ring  $\mathcal{O}(\underline{M}_1)$  of  $\underline{M}_1$  (*Definition* 2.3.5)  $W_{r_\varphi}$  and  $W_{l_\varphi}$  by the right  $\underline{GT}_1$ -action and the left  $\underline{GRT}_1$ -action respectively for each element  $\varphi \in \underline{M}_1(\mathbf{Q})$ . And then in *Proposition* 6.1.1 and *Proposition* 6.1.2, we shall show that their filtration are canonical (do not depend on the choice of  $\varphi \in \underline{M}_1(\mathbf{Q})$ ) and coincide with each other.

Fix a point  $\varphi$  on  $\underline{M}_1(\mathbf{Q})$ . By *Proposition* 2.3.3 and *Proposition* 2.3.4, we get isomorphisms of schemes  $r_\varphi : \underline{GT} \rightarrow \underline{M}$  and  $l_\varphi : \underline{GRT} \rightarrow \underline{M}$  such that  $r_\varphi(f) = \varphi \circ f$  for all  $f \in \underline{GT}(A)$  and  $l_\varphi(g) = g \circ \varphi$  for all  $g \in \underline{GRT}(A)$ , where  $A$  is an arbitrary  $\mathbf{Q}$ -algebra. Note that  $S_\varphi = l_\varphi^{-1} \circ r_\varphi$  (5.1.1). Especially by restricting  $r_\varphi$  (resp.  $l_\varphi$ ) to  $\underline{GT}_1$  (resp.  $\underline{GRT}_1$ ), we get an isomorphism of  $\mathbf{Q}$ -algebras  $r_\varphi^\# : \mathcal{O}(\underline{M}_1) \rightarrow \mathcal{O}(\underline{GT}_1)$  (resp.  $l_\varphi^\# : \mathcal{O}(\underline{M}_1) \rightarrow \mathcal{O}(\underline{GRT}_1)$ ). Denote  $r_\varphi^\# \circ (l_\varphi^{-1})^\# : \mathcal{O}(\underline{GRT}_1) \rightarrow \mathcal{O}(\underline{GT}_1)$  by  $S_\varphi^\#$ .

By transporting the weight filtration  $W = \{W_n \mathcal{O}(\underline{GT}_1)\}_{n \in \mathbf{Z}}$  (§2.2.2) on  $\mathcal{O}(\underline{GT}_1)$  via  $r_\varphi^\sharp$ , we get a weight filtration  $W_{r_\varphi} = \{W_{n,r_\varphi} \mathcal{O}(\underline{M}_1)\}_{n \in \mathbf{Z}}$  on  $\mathcal{O}(\underline{M}_1)$  such that  $r_\varphi^\sharp(W_{n,r_\varphi} \mathcal{O}(\underline{M}_1)) = W_n \mathcal{O}(\underline{GT}_1)$  for  $n \in \mathbf{Z}$ .

Similarly we get another filtration  $W_{l_\varphi} = \{W_{n,l_\varphi} \mathcal{O}(\underline{M}_1)\}_{n \in \mathbf{Z}}$  on  $\mathcal{O}(\underline{M}_1)$  from  $W = \{W_n \mathcal{O}(\underline{GRT}_1)\}_{n \in \mathbf{Z}}$  (§2.1.2) by  $l_\varphi^\sharp$ .

**Proposition 6.1.1.**  $W_{r_\varphi} = W_{l_\varphi}$  .

**Proof .** Recall that the weight filtration on  $\mathcal{O}(\underline{GRT}_1)$  (resp. on  $\mathcal{O}(\underline{GT}_1)$ ) was intrinsically defined by its adjoint action in §1.3. Since  $\underline{GT}$  and  $\underline{GRT}$  are isomorphic via  $S_\varphi$  (5.1.1) and the isomorphism  $S_\varphi$  is consistent with their central cocharacter (i.e.  $S_\varphi^{-1} \circ s_0 \circ \varpi$  (*Remark 2.1.5* and *Lemma 2.1.6*) becomes a lift of  $\frac{1}{\varpi}$  (*Lemma 2.2.4*)), we get  $r_\varphi^\sharp \circ (l_\varphi^{-1})^\sharp(W_n \mathcal{O}(\underline{GRT}_1)) = S_\varphi^\sharp(W_n \mathcal{O}(\underline{GRT}_1)) = W_n \mathcal{O}(\underline{GT}_1)$  for each  $n \in \mathbf{Z}$ . Therefore  $W_{n,r_\varphi} \mathcal{O}(\underline{M}_1) = W_{n,l_\varphi} \mathcal{O}(\underline{M}_1)$  for  $n \in \mathbf{Z}$ .  $\square$

**Proposition 6.1.2.** *The filtration  $W_{l_\varphi}$  on  $\mathcal{O}(\underline{M}_1)$  does not depend on the choice of  $\varphi \in \underline{M}_1(\mathbf{Q})$ .*

**Proof .** Take any two points  $\varphi, \varphi'$  of  $\underline{M}_1(\mathbf{Q})$ . By *Proposition 2.3.4*, there should exist a unique element  $a \in \underline{GRT}_1(\mathbf{Q})$  such that  $\varphi' = a \circ \varphi$ . Denote  $R_a : \underline{GRT}_1 \rightarrow \underline{GRT}_1$  to be the right action, deduced from  $x \mapsto x \circ a$  for  $x \in \underline{GRT}_1(A)$ , where  $A$  is an arbitrary  $\mathbf{Q}$ -algebra. Then  $l_{\varphi'} = l_\varphi \circ R_a$ . Thus for each  $n \in \mathbf{Z}$ ,

$$(6.1.1) \quad (l_{\varphi'}^{-1})^\sharp(W_n \mathcal{O}(\underline{GRT}_1)) = (l_\varphi^{-1})^\sharp(R_{a^{-1}})^\sharp(W_n \mathcal{O}(\underline{GRT}_1)).$$

Recall that  $\mathcal{O}(\underline{GRT}_1)$  is a graded Hopf algebra whose grading is given by  $\deg x_W = wt W$  for each word  $W$  (*Note 2.1.9*), from which we can deduce that

$$(6.1.2) \quad (R_{a^{-1}})^\sharp(W_n \mathcal{O}(\underline{GRT}_1)) = W_n \mathcal{O}(\underline{GRT}_1)$$

for each  $n \in \mathbf{Z}$ . Thus by (6.1.1) and (6.1.2)

$$\begin{aligned} W_{n,l_{\varphi'}} \mathcal{O}(\underline{M}_1) &= (l_{\varphi'}^{-1})^\sharp(W_n \mathcal{O}(\underline{GRT}_1)) \\ &= (l_\varphi^{-1})^\sharp(W_n \mathcal{O}(\underline{GRT}_1)) = W_{n,l_\varphi} \mathcal{O}(\underline{M}_1) \end{aligned}$$

for  $n \in \mathbf{Z}$ .  $\square$

**Notation 6.1.3.** From now on, we denote briefly this filtration  $W_{l_\varphi}$  (or equivalently  $W_{r_\varphi}$ ) by  $W$ .

From the algebra isomorphism  $l_\varphi^\sharp$  and the construction of this filtration  $W$ , it immediately follows that  $W$  is compatible with the structure of  $\mathbf{Q}$ -algebra of  $\mathcal{O}(\underline{M}_1)$ . Namely  $(\mathcal{O}(\underline{M}_1), W)$  becomes a filtered  $\mathbf{Q}$ -algebra.

**6.2. Comparison between  $\underline{M}_1$  and  $\underline{GRT}_1$ .** In this subsection, we introduce a relationship among filtered algebras  $\mathcal{O}(\underline{GRT}_1)$  and  $\mathcal{O}(\underline{M}_1)$  in *Theorem 6.2.2* and make a remark on a relationship between *Theorem 3.2.5* and *Proposition 3.3.4*.

**Notation 6.2.1.** We denote by  $Gr_{\cdot}^W \mathcal{O}(\underline{M}_1) := \bigoplus_{n \geq 0} Gr_n^W \mathcal{O}(\underline{M}_1)$  the graded algebra of the regular function ring  $\mathcal{O}(\underline{M}_1)$  with respect to the filtration  $W$  (*Notation 6.1.3*).

Since  $(\mathcal{O}(\underline{M}_1), W)$  is a filtered  $\mathbf{Q}$ -algebra,  $Gr_{\cdot}^W \mathcal{O}(\underline{M}_1)$  is a graded  $\mathbf{Q}$ -algebra. Recall that  $\mathcal{O}(\underline{GRT}_1)$  is also a graded Hopf algebra over  $\mathbf{Q}$  (§2.1.2).

**Theorem 6.2.2.** *Between two  $\mathbf{Q}$ -structures of graded algebra  $Gr_{\cdot}^W \mathcal{O}(\underline{M}_1)$  and  $\mathcal{O}(\underline{GRT}_1)$ , there exists a canonical isomorphism*

$$(6.2.1) \quad r^{\sharp} : Gr_{\cdot}^W \mathcal{O}(\underline{M}_1) \xrightarrow{\sim} \mathcal{O}(\underline{GRT}_1) .$$

**Proof .** At first, fix any rational point  $\varphi \in \underline{M}_1(\mathbf{Q})$ . By the definition of weight filtration  $W$  on  $\mathcal{O}(\underline{M}_1)$  and  $l_{\varphi}^{\sharp}$  (§6.1), we obtain the isomorphism of graded  $\mathbf{Q}$ -algebras

$$Gr_{\cdot}^W l_{\varphi}^{\sharp} : Gr_{\cdot}^W \mathcal{O}(\underline{M}_1) \xrightarrow{\sim} Gr_{\cdot}^W \mathcal{O}(\underline{GRT}_1) .$$

We put  $r^{\sharp} = \tilde{s}_0 \circ Gr_{\cdot}^W l_{\varphi}^{\sharp}$  (for  $\tilde{s}_0$ , see (2.1.2)). Then the theorem follows from the following lemma.

**Lemma 6.2.3.** *The above isomorphism  $Gr_{\cdot}^W l_{\varphi}^{\sharp}$  is canonical in the sense that it does not depend on the choice of rational points  $\varphi \in \underline{M}_1(\mathbf{Q})$ .*

**Proof .** We proceed the way as in the proof on *Proposition 6.1.2*. Take another rational point  $\varphi' \in \underline{M}_1(\mathbf{Q})$ . Then there exists a unique element  $a \in \underline{GRT}_1(\mathbf{Q})$  such that  $l_{\varphi'} = l_{\varphi} \circ R_a$ . Thus it is enough to show that the graded algebra homomorphism  $Gr_{\cdot}^W R_a^{\sharp} : Gr_{\cdot}^W \mathcal{O}(\underline{GRT}_1) \rightarrow Gr_{\cdot}^W \mathcal{O}(\underline{GRT}_1)$  does not depend on the choice of  $a \in \underline{GRT}_1(\mathbf{Q})$ , where  $Gr_{\cdot}^W R_a^{\sharp}$  means the associated graded quotient map of  $R_a^{\sharp} : \mathcal{O}(\underline{GRT}_1) \rightarrow \mathcal{O}(\underline{GRT}_1)$  (cf. (6.1.2)). Recall that for all  $a \in \underline{GRT}_1(\mathbf{Q})$ , the first term  $x_1(a)$  of its word expansion is always equal to 1 (*Notation 2.1.7*). By combining it with the fact that  $\mathcal{O}(\underline{GRT}_1) = \mathbf{Q}[x_1, x_W]_{W:\text{words}}$  is equipped with a structure of graded Hopf algebra whose grading is given by  $\deg x_1 = 0$  and  $\deg x_w = wt W$  for each word  $W$ , we see that  $Gr_{\cdot}^W R_a^{\sharp}$  induces the map on each graded component of  $Gr_{\cdot}^W \mathcal{O}(\underline{GRT}_1)$ , which is independent from  $a \in \underline{GRT}_1(\mathbf{Q})$ .  $\square$

Put  $\underline{GrM}_1 := \text{Spec } Gr_{\cdot}^W \mathcal{O}(\underline{M}_1)$ . Then the map (6.2.1) induces the following isomorphism of schemes over  $\mathbf{Q}$ .

$$(6.2.2) \quad r : \underline{GRT}_1 \xrightarrow{\sim} \underline{GrM}_1 .$$

We relate the two embeddings  $\Phi_{DR} : Spec Z./(\pi^2) \hookrightarrow \underline{GRT}_1$  (3.2.3) and  $\Phi_{Hod} : Spec \mathfrak{Z} \hookrightarrow \underline{M}_1$  (3.3.2). It is easy to see that the surjective algebra homomorphism  $\Phi_{Hod}^\sharp : \mathcal{O}(\underline{M}_1) \twoheadrightarrow \mathfrak{Z}$  (3.3.1) is strictly compatible with the filtrations of  $\mathcal{O}(\underline{M}_1)$  (§6.1) and  $\mathfrak{Z}$  (Notation 3.3.2). Therefore we obtain the surjective graded  $\mathbf{Q}$ -algebra homomorphism

$$(6.2.3) \quad Gr_*^W \Phi_{Hod}^\sharp : Gr_*^W \mathcal{O}(\underline{M}_1) \twoheadrightarrow Gr_*^W \mathfrak{Z} .$$

Thus by Proposition 3.3.3, we get the associated embedding of schemes

$$(6.2.4) \quad Gr \Phi_{Hod} : Spec Z./(\pi^2) \hookrightarrow \underline{GrM}_1 .$$

It is easy to see the following proposition.

**Proposition 6.2.4.**  $\Phi_{DR} = r^{-1} \circ Gr \Phi_{Hod}$  .

Therefore Conjecture A (§3.4) is equivalent to the following:

**Conjecture A'** . The embedding  $\Phi_{Hod}$  of affine schemes is an isomorphism, i.e.

$$\Phi_{Hod} : Spec \mathfrak{Z} \cong \underline{M}_1$$

**Remark 6.2.5.** (1) This conjecture claims that all  $\mathbf{Q}$ -linear relations among modified MZV's (Definition 3.3.1) must be deduced from the defining equations (0)~(iii) of  $\underline{M}_1$  (Definition 2.3.5) (Remark 2.3.1 and Tsunogai's problem 2.1.16 more strongly suggest that (0) and (iii) should be enough). In particular, it may be our interesting problem to try to check whether all various algebraic relations among (modified) MZV's, which was found by many mathematicians (for example, see [HO], [Ihk-K] and [O]) could be deduced from (0)~(iii) or not.

(2) In contrast, G. Racinet introduced a certain pro-algebraic group  $\underline{DMR}_0$  over  $\mathbf{Q}$  and a  $\underline{DMR}_0$ -torsor  $\underline{DMR}_1$  in [Ra]. With his result, we also associate the embedding  $\Phi_{Rac} : Spec \mathfrak{Z} \hookrightarrow \underline{DMR}_1$ . The conjecture ([Ra]) by D. Zagier and M. Kontsevich can be reformulated that  $\Phi_{Rac}$  is an isomorphism. But it is not clear whether his  $\underline{DMR}_1$  is isomorphic to  $\underline{M}_1$  or not, which is left to our future research.

### 6.3. Comparison between $\underline{GT}_1$ and $\underline{GRT}_1$ .

**Notation 6.3.1.** Denote by  $Gr_*^W \mathcal{O}(\underline{GT}_1) := \bigoplus_{n \geq 0} Gr_n^W \mathcal{O}(\underline{GT}_1)$  the graded algebra of the regular function ring  $\mathcal{O}(\underline{GT}_1)$  by the filtration  $W$  (§2.2.2).

Since  $(\mathcal{O}(\underline{GT}_1), W)$  has a structure of filtered Hopf algebra over  $\mathbf{Q}$  (Proposition 2.2.6),  $Gr_*^W \mathcal{O}(\underline{GT}_1)$  becomes a graded Hopf algebra over  $\mathbf{Q}$ . Recall that  $\mathcal{O}(\underline{GRT}_1)$  is also a graded Hopf algebra over  $\mathbf{Q}$  (Proposition 2.1.8).

**Theorem 6.3.2.** *Between two  $\mathbf{Q}$ -structures of graded Hopf algebra  $Gr_{\bullet}^W \mathcal{O}(\underline{GT}_1)$  and  $\mathcal{O}(\underline{GRT}_1)$ , there exists a canonical isomorphism of graded Hopf algebras*

$$(6.3.1) \quad q^{\sharp} : Gr_{\bullet}^W \mathcal{O}(\underline{GT}_1) \xrightarrow{\sim} \mathcal{O}(\underline{GRT}_1). \quad .$$

**Proof .** At first, pick up and fix any rational point  $\varphi = (1, \varphi(A, B)) \in \underline{M}_1(\mathbf{Q}) \subseteq \underline{M}(\mathbf{Q})$ . By restricting the map  $S_{\varphi}$  (5.1.1) to unipotent parts, we get the isomorphism of unipotent pro-algebraic groups  $S_{\varphi} : \underline{GT}_1 \xrightarrow{\sim} \underline{GRT}_1$ . Denote by  $S_{\varphi}^{\sharp} : \mathcal{O}(\underline{GRT}_1) \rightarrow \mathcal{O}(\underline{GT}_1)$  the isomorphism of Hopf algebra (over  $\mathbf{Q}$ ) induced by  $S_{\varphi}$ . The following two diagrams are commutative for each  $g \in \underline{GT}(\mathbf{Q})$ . (for  $\tau$ , see *Notation 1.3.1*).

$$\begin{array}{ccccc} \underline{GT} & \longrightarrow & \mathbb{G}_m & \xleftarrow{\frac{1}{\varpi}} & \mathbb{G}_m & & \mathcal{O}(\underline{GT}_1) & \xrightarrow{\tau(g)=(\iota_g^{-1})^{\sharp}} & \mathcal{O}(\underline{GT}_1) \\ S_{\varphi} \downarrow \simeq & & \frac{1}{\varpi} \downarrow & & id \downarrow & & (S_{\varphi}^{\sharp})^{-1} \downarrow \simeq & & (S_{\varphi}^{\sharp})^{-1} \downarrow \simeq \\ \underline{GRT} & \longrightarrow & \mathbb{G}_m & \xleftarrow{\varpi} & \mathbb{G}_m & & \mathcal{O}(\underline{GRT}_1) & \xrightarrow{\tau(S_{\varphi}(g))=(\iota_{S_{\varphi}(g)}^{-1})^{\sharp}} & \mathcal{O}(\underline{GRT}_1) \end{array}$$

Namely we get a morphism of weighted modules ([HM]§3.3.)

$$(\mathcal{O}(\underline{GT}_1), \mathbb{G}_m, \frac{1}{\varpi}, \underline{GT}) \longrightarrow (\mathcal{O}(\underline{GRT}_1), \mathbb{G}_m, \varpi, \underline{GRT}) \quad ,$$

which means that  $(S_{\varphi}^{\sharp})^{-1} : \mathcal{O}(\underline{GT}_1) \rightarrow \mathcal{O}(\underline{GRT}_1)$  is an isomorphism of modules preserving weight filtrations of  $\mathcal{O}(\underline{GT}_1)$  and  $\mathcal{O}(\underline{GRT}_1)$ . Since  $S_{\varphi}^{\sharp} : \mathcal{O}(\underline{GRT}_1) \rightarrow \mathcal{O}(\underline{GT}_1)$  is a natural isomorphism of Hopf algebras,  $(S_{\varphi}^{\sharp})^{-1}$  is an isomorphism of filtered Hopf algebras. By taking the graded quotient of  $(S_{\varphi}^{\sharp})^{-1}$  by these weight filtrations, we get an isomorphism

$$Gr_{\bullet}^W (S_{\varphi}^{\sharp})^{-1} : Gr_{\bullet}^W \mathcal{O}(\underline{GT}_1) \longrightarrow Gr_{\bullet}^W \mathcal{O}(\underline{GRT}_1)$$

of graded Hopf algebras.

**Lemma 6.3.3.** *The above isomorphism  $Gr_{\bullet}^W (S_{\varphi}^{\sharp})^{-1}$  is canonical in the sense that it does not depend on the choice of rational points  $\varphi \in \underline{M}_1(\mathbf{Q})$ .*

**Proof .** Take another rational point  $\varphi' \in \underline{M}_1(\mathbf{Q})$ . By *Proposition 2.3.4*, there should exist a unique element  $a \in \underline{GRT}_1(\mathbf{Q})$  such that  $\varphi' = a \circ \varphi$ . It follows that  $S_{\varphi'} = \tau_a \circ S_{\varphi}$ , which implies that

$$(S_{\varphi'}^{\sharp})^{-1} = (\tau_{a^{-1}})^{\sharp} \circ (S_{\varphi}^{\sharp})^{-1} : \mathcal{O}(\underline{GT}_1) \xrightarrow{(S_{\varphi}^{\sharp})^{-1}} \mathcal{O}(\underline{GRT}_1) \xrightarrow{(\tau_{a^{-1}})^{\sharp}} \mathcal{O}(\underline{GRT}_1) .$$

Since the  $\underline{GRT}_1$ -action on  $Gr_{\bullet}^W \mathcal{O}(\underline{GRT}_1)$  by  $\tau$  is trivial by *Proposition 1.2.3.(b)*,

$$Gr_{\bullet}^W (S_{\varphi'}^{\sharp})^{-1} = Gr_{\bullet}^W (S_{\varphi}^{\sharp})^{-1} : Gr_{\bullet}^W \mathcal{O}(\underline{GT}_1) \xrightarrow{\sim} Gr_{\bullet}^W \mathcal{O}(\underline{GRT}_1). \quad .$$

Thus we see that this isomorphism  $Gr_{\cdot}^W(S_{\varphi}^{\sharp})^{-1}$  does not depend on the choice of rational points of  $\underline{M}_1(\mathbf{Q})$ .  $\square$

**The rest of the proof of theorem 6.3.2.** : By composing the above isomorphism  $Gr_{\cdot}^W(S_{\varphi}^{-1})^{\sharp}$  (*Lemma 6.3.3*) with the natural isomorphism  $\tilde{s}_0 : Gr_{\cdot}^W \mathcal{O}(\underline{GRT}_1) \rightarrow \mathcal{O}(\underline{GRT}_1)$  (2.1.2), we finally obtain a canonical isomorphism  $q^{\sharp}$  of graded Hopf algebras over  $\mathbf{Q}$

$$q^{\sharp} := \tilde{s}_0 \circ Gr_{\cdot}^W(S_{\varphi}^{-1})^{\sharp} : Gr_{\cdot}^W \mathcal{O}(\underline{GT}_1) \xrightarrow{\sim} \mathcal{O}(\underline{GRT}_1). \quad .$$

$\square$

Put  $\underline{GrGT}_1 := Spec Gr_{\cdot}^W \mathcal{O}(\underline{GT}_1)$ . Then (6.3.1) induces the following isomorphism

$$(6.3.2) \quad q : \underline{GRT}_1 \xrightarrow{\sim} \underline{GrGT}_1$$

of pro-algebraic groups over  $\mathbf{Q}$ .

**Remark 6.3.4.** We can relate the above canonical isomorphism  $q : \underline{GRT}_1 \xrightarrow{\sim} \underline{GrGT}_1$  (6.3.2) with the isomorphism  $p : \underline{GT}_1 \times_{\mathbf{Q}} \mathbf{C} \rightarrow \underline{GRT}_1 \times_{\mathbf{Q}} \mathbf{C}$  (*Proposition 5.1.2*) of pro-algebraic groups, as follows. Recall that the map  $p$  is induced from  $\varphi_{KZ}(A, B) \in \underline{M}_1(\mathbf{C})$ , i.e.  $p = S_{\varphi_{KZ}}$ . As we have seen in the proof of *Theorem 6.3.2*,  $Gr_{\cdot}^W S_{\varphi}^{\sharp}$  does not depend on the choice of points  $\varphi$  on  $\underline{M}_1$ . Therefore  $\tilde{s}_0 \circ Gr_{\cdot}^W p^{\sharp} = q^{\sharp} \otimes_{\mathbf{Q}} id_{\mathbf{C}}$ .

**6.4. Three embeddings on Galois Side.** We describe here the prescription how to associate embeddings of pro-algebraic groups with  $T_l$ ,  $\Phi_{\mathbf{Q}}^{(l)}$  and  $\Phi_{\mathbf{Q}_l}^{(l)}$  (§4.2).

1. Suppose that  $\mathcal{O}(\underline{Gal}_{\mathbf{Q}}^{(l)})$  (resp.  $\mathcal{O}(\underline{Gal}_{\mathbf{Q}_l}^{(l)})$ ) is the regular function ring of the unipotent pro-algebraic group  $\underline{Gal}_{\mathbf{Q}}^{(l)}$  (cf. *Definition 4.2.2*) over  $\mathbf{Q}$  (resp.  $\underline{Gal}_{\mathbf{Q}_l}^{(l)}$  (cf. *Definition 4.2.3*) over  $\mathbf{Q}_l$ ), which is a Hopf algebra over  $\mathbf{Q}$  (resp. over  $\mathbf{Q}_l$ ). With the embeddings  $T_l$ ,  $\Phi_{\mathbf{Q}}^{(l)}$  and  $\Phi_{\mathbf{Q}_l}^{(l)}$  (cf. (4.2.1)~(4.2.3)) of unipotent pro-algebraic groups, we obtain the following surjective homomorphisms of Hopf algebras.

$$(6.4.1) \quad T_l^{\sharp} : \mathcal{O}(\underline{Gal}_{\mathbf{Q}}^{(l)}) \otimes_{\mathbf{Q}} \mathbf{Q}_l \twoheadrightarrow \mathcal{O}(\underline{Gal}_{\mathbf{Q}_l}^{(l)})$$

$$(6.4.2) \quad \Phi_{\mathbf{Q}}^{(l)\sharp} : \mathcal{O}(\underline{GT}_1) \twoheadrightarrow \mathcal{O}(\underline{Gal}_{\mathbf{Q}}^{(l)})$$

$$(6.4.3) \quad \Phi_{\mathbf{Q}_l}^{(l)\sharp} : \mathcal{O}(\underline{GT}_1) \otimes_{\mathbf{Q}} \mathbf{Q}_l \twoheadrightarrow \mathcal{O}(\underline{Gal}_{\mathbf{Q}_l}^{(l)})$$

Notice that  $\Phi_{\mathbf{Q}_l}^{(l)\sharp} = T_l^{\sharp} \circ (\Phi_{\mathbf{Q}}^{(l)\sharp} \otimes_{\mathbf{Q}} id_{\mathbf{Q}_l})$ .

2. By the surjection  $\Phi_{\mathbf{Q}}^{(l)\sharp}$  (resp.  $\Phi_{\mathbf{Q}_l}^{(l)\sharp}$ ), the Hopf algebra  $\mathcal{O}(\underline{Gal}_{\mathbf{Q}}^{(l)})$  over  $\mathbf{Q}$  (resp.  $\mathcal{O}(\underline{Gal}_{\mathbf{Q}_l}^{(l)})$  over  $\mathbf{Q}_l$ ) is equipped with a structure of filtered Hopf algebra over  $\mathbf{Q}$  (resp. over  $\mathbf{Q}_l$ ) by the filtration  $\left\{W_n \mathcal{O}(\underline{Gal}_{\mathbf{Q}}^{(l)})\right\}_{n \in \mathbf{Z}}$  (resp.  $\left\{W_n \mathcal{O}(\underline{Gal}_{\mathbf{Q}_l}^{(l)})\right\}_{n \in \mathbf{Z}}$ ) induced from that of  $\mathcal{O}(\underline{GT}_1)$  (§2.2.2). Let  $Gr^{\cdot W} \mathcal{O}(\underline{Gal}_{\mathbf{Q}}^{(l)})$ ,  $Gr^{\cdot W} \mathcal{O}(\underline{Gal}_{\mathbf{Q}_l}^{(l)})$  and  $Gr^{\cdot W} \mathcal{O}(\underline{GT}_1)$  be the associated graded algebras of  $\mathcal{O}(\underline{Gal}_{\mathbf{Q}}^{(l)})$ ,  $\mathcal{O}(\underline{Gal}_{\mathbf{Q}_l}^{(l)})$  and  $\mathcal{O}(\underline{GT}_1)$  by these filtrations respectively. Note that they are equipped with structures of graded Hopf algebra. Thus from (6.4.1)~(6.4.3), we obtain surjective homomorphisms of graded Hopf algebras as follows:

$$(6.4.4) \quad Gr^{\cdot W} T_l^{\sharp} : Gr^{\cdot W} \mathcal{O}(\underline{Gal}_{\mathbf{Q}}^{(l)}) \otimes_{\mathbf{Q}} \mathbf{Q}_l \twoheadrightarrow Gr^{\cdot W} \mathcal{O}(\underline{Gal}_{\mathbf{Q}_l}^{(l)})$$

$$(6.4.5) \quad Gr^{\cdot W} \Phi_{\mathbf{Q}}^{(l)\sharp} : Gr^{\cdot W} \mathcal{O}(\underline{GT}_1) \twoheadrightarrow Gr^{\cdot W} \mathcal{O}(\underline{Gal}_{\mathbf{Q}}^{(l)})$$

$$(6.4.6) \quad Gr^{\cdot W} \Phi_{\mathbf{Q}_l}^{(l)\sharp} : Gr^{\cdot W} \mathcal{O}(\underline{GT}_1) \otimes_{\mathbf{Q}} \mathbf{Q}_l \twoheadrightarrow Gr^{\cdot W} \mathcal{O}(\underline{Gal}_{\mathbf{Q}_l}^{(l)})$$

Notice that  $Gr^{\cdot W} \Phi_{\mathbf{Q}_l}^{(l)\sharp} = Gr^{\cdot W} T_l^{\sharp} \circ (Gr^{\cdot W} \Phi_{\mathbf{Q}}^{(l)\sharp} \otimes id_{\mathbf{Q}_l})$ .

3. Suppose that  $\underline{GrGal}_{\mathbf{Q}}^{(l)}$  (resp.  $\underline{GrGal}_{\mathbf{Q}_l}^{(l)}$ ) is the pro-algebraic group over  $\mathbf{Q}$  (resp. over  $\mathbf{Q}_l$ ) whose regular function ring is isomorphic to the graded Hopf algebra  $Gr^{\cdot W} \mathcal{O}(\underline{Gal}_{\mathbf{Q}}^{(l)})$  over  $\mathbf{Q}$  (resp.  $Gr^{\cdot W} \mathcal{O}(\underline{Gal}_{\mathbf{Q}_l}^{(l)})$  over  $\mathbf{Q}_l$ ). Recall that  $\underline{GrGT}_1$  is isomorphic to  $\underline{GRT}_1$  by  $q$  (6.3.2). Therefore, from (6.4.4)~(6.4.6), we get the following embeddings of unipotent pro-algebraic groups:

$$(6.4.7) \quad GrT_l : \underline{GrGal}_{\mathbf{Q}_l}^{(l)} \hookrightarrow \underline{GrGal}_{\mathbf{Q}}^{(l)} \times_{\mathbf{Q}} \mathbf{Q}_l$$

$$(6.4.8) \quad Gr\Phi_{\mathbf{Q}}^{(l)} : \underline{GrGal}_{\mathbf{Q}}^{(l)} \hookrightarrow \underline{GRT}_1$$

$$(6.4.9) \quad Gr\Phi_{\mathbf{Q}_l}^{(l)} : \underline{GrGal}_{\mathbf{Q}_l}^{(l)} \hookrightarrow \underline{GRT}_1 \times_{\mathbf{Q}} \mathbf{Q}_l$$

Notice that  $Gr\Phi_{\mathbf{Q}_l}^{(l)} = (Gr\Phi_{\mathbf{Q}}^{(l)} \times id_{\mathbf{Q}_l}) \circ GrT_l$ .

**Remark 6.4.1.** Note that *Conjecture B* (§4.2) implies that  $Gr\Phi_{\mathbf{Q}}^{(l)}$  is an isomorphism and *Conjecture C* (§4.2) implies that  $GrT_l$  is an isomorphism.

**6.5. Comparison between Galois Side and Hodge Side.** By combining *Theorem* 6.3.2 with the embeddings  $Gr\Phi_{\mathbf{Q}}^{(l)}$  (6.4.8) and  $\Phi_{DR}$  (3.2.3), we get Figure 2 in §0. Note that we can compare two objects,  $\underline{GrGal}_{\mathbf{Q}}^{(l)}$  in Galois Side and  $Spec Z/(\pi^2)$  in Hodge Side, in the same box  $\underline{GRT}_1$ . In contrast, we could not make a direct comparison

between two objects in Galois Side and in Hodge Side inside a common box (over  $\mathbf{Q}$ ) in §5.3.

Validities of *Conjecture A* (§3.4), *Conjecture B* (§4.2) and *Conjecture C* (§4.2) would imply the following relationship between Galois Side and Hodge Side.

“ In Galois Side,  $\underline{GRT}_1$  should be a common structure of  $\underline{GrGal}_{\mathbf{Q}_l}^{(l)}$  for all prime  $l$ , i.e.  $\underline{GrGal}_{\mathbf{Q}_l}^{(l)} \cong \underline{GRT}_1 \times_{\mathbf{Q}} \mathbf{Q}_l$ . On the other hand, in Hodge Side,  $\text{Spec } Z./(\pi^2)$  should become a pro-algebraic group  $\underline{GRT}_1$ . Therefore  $\text{Spec } Z./(\pi^2)$  is the common  $\mathbf{Q}$ -structure of  $\underline{GrGal}_{\mathbf{Q}_l}^{(l)}$ , i.e.  $\underline{GrGal}_{\mathbf{Q}_l}^{(l)} \cong \text{Spec } Z./(\pi^2) \times_{\mathbf{Q}} \mathbf{Q}_l$  for all prime  $l$ . ”

Roughly speaking, MZV might be a regular function of the graded Galois image.

**Remark 6.5.1.** The two surjections  $\Psi_{DR} : \mathfrak{D}^* \twoheadrightarrow NZ$ , (for  $NZ$ , see *Proposition 3.4.2*) and  $\Psi_l^* : \mathfrak{D}^* \otimes_{\mathbf{Q}} \mathbf{Q}_l \twoheadrightarrow \mathfrak{g}^{(l)*}$  of graded vector spaces are concerned in [F]§5 (see the picture below).

$$\begin{array}{ccc}
 \boxed{\text{Galois Side}} & & \boxed{\text{Hodge Side}} \\
 & \mathfrak{g}^{(l)*} \xleftarrow{\Psi_l^*} \mathfrak{D}^* \otimes_{\mathbf{Q}} \mathbf{Q}_l & \\
 & \mathfrak{D}^* \xrightarrow{\Psi_{DR}} NZ &
 \end{array}$$

We can deduce these surjections from  $\Phi_{DR}$  (3.2.3) and  $\Phi_{\mathbf{Q}_l}^{(l)}$  (4.2.3) in the following way:

Take the dual of the differential (i.e. the cotangent space) of the embedding  $\Phi_{DR} : \text{Spec } Z./(\pi^2) \hookrightarrow \underline{GRT}_1$  at the unit element  $e$  of  $\underline{GRT}_1$ . Then by (3.4.1) we get a surjection of graded vector spaces  $(d\Phi_{DR})_e^\sharp : \mathfrak{D}^* \twoheadrightarrow NZ$ , which is equal to the above map  $\Psi_{DR}$ . On the other hand, take the dual of the differential of the embedding  $\text{Gr}\Phi_{\mathbf{Q}_l}^{(l)} : \underline{GrGal}_{\mathbf{Q}_l}^{(l)} \hookrightarrow \underline{GRT}_1 \times_{\mathbf{Q}} \mathbf{Q}_l$  (6.4.9) at the unit element  $e$  of  $\underline{GRT}_1$ . Then by (4.3.4) we get a surjection of graded vector spaces  $(d\text{Gr}\Phi_{\mathbf{Q}_l}^{(l)})_e^\sharp : \mathfrak{D}^* \otimes_{\mathbf{Q}} \mathbf{Q}_l \twoheadrightarrow \mathfrak{g}^{(l)*}$ , which is equal to the above map  $\Psi_l^*$ .

**6.6. Chase.** Here we present a few relationships between Galois Side and Hodge Side deduced from *Figure 2*. §6.6.2 and §6.6.3 are just special cases of §6.6.1.

6.6.1. *Non-abelian case.*

**$\underline{GRT}_1$ :** Recall that  $\underline{GRT}_1(\mathbf{Q})$  is defined as a subset of the non-commutative formal power series ring  $\mathbf{Q}\langle\langle A, B \rangle\rangle$  (Definition 2.1.4). Thus we obtain an embedding  $g_1 : \underline{GRT}_1(\mathbf{Q}) \hookrightarrow \mathbf{Q}\langle\langle A, B \rangle\rangle$ . This embedding determines an invertible element  $\Phi_{GRT} = 1 + \sum_{W:\text{words}} d_W W \in \left(\mathcal{O}(\underline{GRT}_1), \langle\langle A, B \rangle\rangle\right)^\times$  of the  $\mathcal{O}(\underline{GRT}_1)$ -coefficient non-commutative formal power series ring with two variables.

**$\underline{GT}_1$ :** Recall that  $\underline{GT}_1(\mathbf{Q})$  is defined as a subset of the pro-algebraic group  $\underline{F}_2(\mathbf{Q})$  (Definition 2.2.3). By sending free generators  $X$  and  $Y$  into  $e^A$  and  $e^B$  respectively, we get an embedding  $g_2 : \underline{F}_2(\mathbf{Q}) \hookrightarrow \mathbf{Q}\langle\langle A, B \rangle\rangle$ .

**Galois Side:** With the composition of embeddings  $g_2 \circ \Phi_{\mathbf{Q}}^{(l)} : \underline{Gal}_{\mathbf{Q}}^{(l)}(\mathbf{Q}) \hookrightarrow \mathbf{Q}\langle\langle A, B \rangle\rangle$ , we associate an invertible element  $\Phi_{Ih}^{(l)} = 1 + \sum_{W:\text{words}} \kappa_W^{(l)} W \in \left(\mathcal{O}(\underline{Gal}_{\mathbf{Q}}^{(l)})\langle\langle A, B \rangle\rangle\right)^\times$ . We call it as the *l-adic Ihara associator*<sup>6</sup>. Note that for each word  $W$  with  $wt W = n$ ,  $\kappa_W^{(l)}$  lies in  $W_n \mathcal{O}(\underline{Gal}_{\mathbf{Q}}^{(l)})$ . Denote by  $Gr \kappa_W^{(l)}$  its quotient class in  $W_n \mathcal{O}(\underline{Gal}_{\mathbf{Q}}^{(l)}) / W_{n-1} \mathcal{O}(\underline{Gal}_{\mathbf{Q}}^{(l)})$ . Put  $Gr \Phi_{Ih}^{(l)} := 1 + \sum_{W:\text{words}} Gr \kappa_W^{(l)} W \in \left(Gr^W \mathcal{O}(\underline{Gal}_{\mathbf{Q}}^{(l)})\langle\langle A, B \rangle\rangle\right)^\times$ , which is also an invertible element associated with the composition  $g_1 \circ Gr \Phi_l$ .

**Hodge Side:** With the composition of embeddings  $g_1 \circ \Phi_{DR}$ , we obtain an invertible element of  $\left(Z/(\pi^2)\right)\langle\langle A, B \rangle\rangle$ . From the definition of  $\Phi_{DR}$  (see Theorem 3.2.5), It is equal to  $\Phi_{KZ} \bmod \pi^2$ , which stands for the image of  $\Phi_{KZ}(A, B) = \varphi_{KZ}(2\pi i A, 2\pi i B) \in Z.\langle\langle A, B \rangle\rangle$  (cf. Definition 2.3.8 and (3.2.4)) by the natural projection  $Z.\langle\langle A, B \rangle\rangle \rightarrow \left(Z/(\pi^2)\right)\langle\langle A, B \rangle\rangle$ .

**Proposition 6.6.1.** *Let  $Gr \Phi_{\mathbf{Q}\langle\langle A, B \rangle\rangle}^{(l)\sharp} : \mathcal{O}(\underline{GRT}_1).\langle\langle A, B \rangle\rangle \twoheadrightarrow Gr^W \mathcal{O}(\underline{Gal}_{\mathbf{Q}}^{(l)})\langle\langle A, B \rangle\rangle$  be the surjection induced from  $Gr^W \Phi_{\mathbf{Q}}^{(l)\sharp}$  (6.4.5) and  $\Phi_{DR\langle\langle A, B \rangle\rangle}^\sharp : \mathcal{O}(\underline{GRT}_1).\langle\langle A, B \rangle\rangle \twoheadrightarrow \left(Z/(\pi^2)\right)\langle\langle A, B \rangle\rangle$  be the surjection induced from  $\Phi_{DR}^\sharp$  (3.2.2). Then*

$$Gr \Phi_{\mathbf{Q}\langle\langle A, B \rangle\rangle}^{(l)\sharp}(\Phi_{GRT}) = Gr \Phi_{Ih}^{(l)} \quad , \quad \Phi_{DR\langle\langle A, B \rangle\rangle}^\sharp(\Phi_{GRT}) = \Phi_{KZ} \bmod \pi^2.$$

**Proof .** This follows from definitions of  $Gr \Phi_{Ih}^{(l)}$  and  $\Phi_{KZ} \bmod \pi^2$ .  $\square$

Thus we get Figure 3(§0).

<sup>6</sup>This should be an appropriate name since essentially the same series was studied in his successive papers [AI88], [AI90] and [Ih86a]~[Ih00].

**Remark 6.6.2.** Recall that *Conjecture B* and *Conjecture A* claim that both  $Gr\Phi_{\mathbf{Q}\langle\langle A, B \rangle\rangle}^{(l)\sharp}$  and  $\Phi_{DR\langle\langle A, B \rangle\rangle}^\sharp$  are isomorphisms. Therefore

“ If *Conjecture A* and *Conjecture B* hold, then  $Gr\Phi_{Ih}^{(l)}$  must be identified with  $\Phi_{KZ}$  mod  $\pi^2$  via  $(\Phi_{DR\langle\langle A, B \rangle\rangle}^\sharp)^{-1} \circ (Gr\Phi_{\mathbf{Q}\langle\langle A, B \rangle\rangle}^{(l)\sharp})^{-1}$  for all prime  $l$ . ”

6.6.2. *Meta-abelian case.*

**Definition 6.6.3.** Suppose that  $R$  is a commutative ring. Let  $\psi$  be any invertible element of the non-commutative formal power series ring  $R\langle\langle A, B \rangle\rangle$  with two variables. Decompose  $\psi^{-1}$  uniquely  $\psi^{-1} = C_\psi + X_\psi A + Y_\psi B$  as in the way of [Ih99b]. Here  $C_\psi \in R$  and  $X_\psi, Y_\psi \in R\langle\langle A, B \rangle\rangle$ . The **meta-abelian quotient** of  $\psi$  is the image of  $C_\psi + X_\psi A$  by the natural surjection  $R\langle\langle A, B \rangle\rangle \rightarrow R[[A, B]]$  into the *commutative* formal power series ring.

**$\underline{GRT}_1$ :** We denote by  $B_{GRT} \in \mathcal{O}(\underline{GRT}_1)[[A, B]]$  the meta-abelian quotient of  $\Phi_{GRT} \in \left(\mathcal{O}(\underline{GRT}_1), \langle\langle A, B \rangle\rangle\right)^\times$ .

**Galois Side:** The meta-abelian quotient of the  $l$ -adic Ihara associator  $\Phi_{Ih}^{(l)} \in \left(\mathcal{O}(\underline{Gal}_{\mathbf{Q}}^{(l)}), \langle\langle A, B \rangle\rangle\right)^\times$  is equal to the  **$l$ -adic universal power series for Jacobi sums**  $B^{(l)}(A, B) \in \mathcal{O}(\underline{Gal}_{\mathbf{Q}}^{(l)})[[A, B]]$  (see [Ih90]§6.4.)<sup>7</sup>, whose presentation is calculated in [A89], [C] and [IKY] as follows:

$$B^{(l)}(A, B) = \exp \left[ \sum_{m \geq 3: \text{odd}} \frac{\kappa_m^{(l)*}}{m!} \{(A+B)^m - A^m - B^m\} \right] \in \mathcal{O}(\underline{Gal}_{\mathbf{Q}}^{(l)})[[A, B]].$$

Here  $\kappa_m^{(l)*} \in \mathcal{O}(\underline{Gal}_{\mathbf{Q}}^{(l)})$  is the  **$l$ -adic  $m$ -th Soulé character** (see [Sou] and also [F]§5.) and we note that  $\kappa_m^{(l)*}$  belongs to  $W_m \mathcal{O}(\underline{Gal}_{\mathbf{Q}}^{(l)})$ .

We denote  $Gr\kappa_m^{(l)*}$  by its quotient class in  $W_m \mathcal{O}(\underline{Gal}_{\mathbf{Q}}^{(l)}) / W_{m-1} \mathcal{O}(\underline{Gal}_{\mathbf{Q}}^{(l)})$  and put

$$Gr_{\cdot}^W B^{(l)}(A, B) := \exp \left[ \sum_{m \geq 3: \text{odd}} \frac{Gr\kappa_m^{(l)*}}{m!} \{(A+B)^m - A^m - B^m\} \right] \in Gr_{\cdot}^W \mathcal{O}(\underline{Gal}_{\mathbf{Q}}^{(l)})[[A, B]].$$

In fact,  $Gr_{\cdot}^W B^{(l)}$  is the meta-abelian quotient of  $Gr\Phi_{Ih}^{(l)} \in \left(Gr_{\cdot}^W \mathcal{O}(\underline{Gal}_{\mathbf{Q}}^{(l)}), \langle\langle A, B \rangle\rangle\right)^\times$  (§6.6.1).

<sup>7</sup>According to the notation in [Ih90],  $B^{(l)}(A, B) = \psi^{\text{ab}}(e^A - 1, e^B - 1)$ .

**Hodge Side:** Recall that the *classical gamma function*  $\Gamma(z)$  has a presentation  $\Gamma(1-z) = \exp \left\{ \gamma z + \sum_{n=2}^{\infty} \frac{\zeta(n)}{n} z^n \right\}$  ( $\gamma$ : Euler constant), from which we can deduce the following formula of the **classical beta function**  $B(x, y)$  :

$$(1-x-y) \cdot B(1-x, 1-y) = \exp \left[ \sum_{m=2}^{\infty} \frac{\zeta(m)}{m} \{x^m + y^m - (x+y)^m\} \right].$$

By expanding this into  $\mathbf{C}$ -coefficient commutative formal power series, we see that  $(1-A-B) \cdot B(1-A, 1-B)$  naturally determines the element of  $Z, [[A, B]]$ . We denote by  $(1-A-B) \cdot B(1-A, 1-B) \bmod \pi^2$  its image by the natural projection  $Z, [[A, B]] \rightarrow (Z/(\pi^2))[[A, B]]$ , which is in fact equal to the meta-abelian quotient of  $\Phi_{KZ} \bmod \pi^2 \in \left( (Z/(\pi^2)) \langle \langle A, B \rangle \rangle \right)^\times$  ([Dr]).

Therefore by taking the meta-abelian quotient part of *Proposition 6.6.1*, we get

**Corollary 6.6.4.** *Let  $Gr\Phi_{\mathbf{Q}[[A, B]]}^{(l)\sharp} : \mathcal{O}(\underline{GRT}_1), [[A, B]] \rightarrow Gr^W \mathcal{O}(\underline{Gal}_{\mathbf{Q}}^{(l)}) [[A, B]]$  be the surjection induced from  $Gr^W \Phi_{\mathbf{Q}}^{(l)\sharp}$  (6.4.5) and  $\Phi_{DR[[A, B]]}^{\sharp} : \mathcal{O}(\underline{GRT}_1), [[A, B]] \rightarrow (Z/(\pi^2))[[A, B]]$  be the surjection induced from  $\Phi_{DR}^{\sharp}$  (3.2.2). Then*

$$\begin{aligned} Gr\Phi_{\mathbf{Q}[[A, B]]}^{(l)\sharp}(B_{GRT}) &= Gr^W B^{(l)}(A, B) \quad , \\ \Phi_{DR[[A, B]]}^{\sharp}(B_{GRT}) &= (1-A-B) \cdot B(1-A, 1-B) \quad \bmod \pi^2. \end{aligned}$$

| Galois Side   | Hodge Side  |
|---|---|
| $Gr^W \mathcal{O}(\underline{Gal}_{\mathbf{Q}}^{(l)}), [[A, B]] \xleftarrow{Gr\Phi_{\mathbf{Q}[[A, B]]}^{(l)\sharp}} \mathcal{O}(\underline{GRT}_1), [[A, B]] \xrightarrow{\Phi_{DR[[A, B]]}^{\sharp}} (Z/(\pi^2))[[A, B]]$ | $Gr^W B^{(l)}(A, B) \longleftarrow B_{GRT}(A, B) \longmapsto (1-A-B) \cdot B(1-A, 1-B) \bmod \pi^2$ |
| <u><math>l</math>-adic universal power series<br/>for Jacobi sums</u>   | <u>classical beta function</u>  |

**Remark 6.6.5.** (1) We recall that  $\prod_{l:\text{prime}} B^{(l)}(A, B)$  was named the adelic beta function as an analogue of the classical beta function in [A87], and has been widely studied in [A87], [A89], [C], [Ich], [Ih86a]~[Ih90] and [IKY] (for a pro-finite group analogue, see also [Ih99b] and [Ih00]).

- (2) We recall that *Conjecture B* and *Conjecture A* claim that both  $Gr\Phi_{\mathbf{Q}[[A,B]]}^{(l)\sharp}$  and  $\Phi_{DR[[A,B]]}^{\sharp}$  are isomorphisms. Therefore  
 “If *Conjecture A* and *Conjecture B* hold, then  $Gr^W B^{(l)}$  must be identified with  $(1 - A - B) \cdot B(1 - A, 1 - B) \pmod{\pi^2}$  via  $(\Phi_{DR[[A,B]]}^{\sharp}) \circ (Gr\Phi_{\mathbf{Q}[[A,B]]}^{(l)\sharp})^{-1}$  for all prime  $l$ .”

We hope that this formulation is the desired direct correspondence.

6.6.3. *Special case.* Especially let  $d_m := d_{A^{m-1}B} \in \mathcal{O}(\underline{GRT}_1)$ , be the coefficient of  $W = A^{m-1}B$  of  $\Phi_{GRT} \in \mathcal{O}(\underline{GRT}_1, \langle\langle A, B \rangle\rangle)$  (§6.6.1) for  $m = 3, 5, 7, \dots$ . Then by restricting the correspondence in *Proposition* 6.6.1 to each coefficient of  $A^{m-1}B$ , we get

**Corollary 6.6.6.** For  $m = 3, 5, 7, \dots$ ,

$$Gr^W \Phi_{\mathbf{Q}}^{(l)\sharp}(d_m) = \frac{1}{(m-1)!} Gr \kappa_m^{(l)*} \quad \text{and} \quad \Phi_{DR}^{\sharp}(d_m) = -\zeta(m) \pmod{\pi^2} .$$

|   |   |
|---|---|
| Galois Side   | Hodge Side  |
| $Gr^W \mathcal{O}(\underline{Gal}_{\mathbf{Q}}^{(l)})$<br>$\xleftarrow{Gr^W \Phi_{\mathbf{Q}}^{(l)\sharp}} \mathcal{O}(\underline{GRT}_1)$<br>$\xrightarrow{\Phi_{DR}^{\sharp}} Z./(\pi^2)$ | $\frac{1}{(m-1)!} Gr \kappa_m^{(l)*} \longleftarrow d_m \longrightarrow -\zeta(m) \pmod{\pi^2}$ |
| <u><math>l</math>-adic Soulé character</u>  | <u>Riemann zeta value</u>   |

Recall that *Conjecture B* and *Conjecture A* claim that both  $Gr^W \Phi_{\mathbf{Q}}^{(l)\sharp}$  and  $\Phi_{DR}^{\sharp}$  are isomorphisms. Therefore

“If *Conjecture A* and *Conjecture B* hold, then Riemann zeta values are identified with Soulé characters via  $(Gr^W \Phi_{\mathbf{Q}}^{(l)\sharp}) \circ (\Phi_{DR}^{\sharp})^{-1}$  .”

**Remark 6.6.7.** We note that each coefficient of  $\Phi_{KZ}$  is calculated in [LM] and [F] and remark that each coefficient of  $\Phi_{Ih}^{(l)}$  can be also calculated in principle by Anderson-Ihara theory ([AI90]), from which we can get analogous correspondence to *Corollary* 6.6.6 associated with any given MZV’s. Therefore the above correspondence is merely one of special corollaries of *Proposition* 6.6.1.

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