

A Unified Scheme for Generalized Sectors based on Selection Criteria.

II. Spontaneously broken symmetries and some basic notions in quantum measurements

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Abstract

Continuing the analysis in a unified scheme for treating generalized superselection sectors based upon the notion of selection criteria for states of relevance in quantum physics, we extend the Doplicher-Roberts superselection theory for recovering the field algebra and the gauge group (of the first kind) from the data of group invariant observables to the situations with spontaneous symmetry breakdown: in use of the machinery proposed, the basic structural features of the theory with spontaneously broken symmetry, are clarified in a clear-cut way, such as the degenerate vacua parametrized by the variable belonging to the relevant homogeneous space, the Goldstone modes and condensates.

1 Introduction

Continuing the analysis started in the previous paper (called I) [1], we apply here our unified method of treating generalized superselection sectors based upon the notion of selection criteria to the situation with spontaneous symmetry breakdown (SSB, for short). While the essence of the *superselection theory* of Doplicher-Haag-Roberts (DHR) [2] and of Doplicher-Roberts (DR) [3, 4] in algebraic quantum field theory (QFT) [5] was already explained briefly in I in a reformulated form convenient for the present context, it may still be meaningful to mention some of more general aspects of it for the sake of explaining the reason why we regard our analysis of SSB as important.

The DHR-DR *superselection theory* gives a general scheme for understanding the relations between a symmetry and its observable consequences in relativistic QFT. It tells us that, *if* the internal symmetry of the theory under consideration is, i) described by a gauge group G of the *1st kind*

(i.e., *global* gauge symmetry), and is, ii) *unbroken*, the basic structure of the standard QFT can be recovered totally from the data encoded in *observables* \mathfrak{A} which are defined as G -invariant combinations of field operators (i.e., $\mathfrak{A} = \mathfrak{F}^G$: fixed-point subalgebra of the field algebra \mathfrak{F} under G) and constitute a *net* $\mathcal{O} \longrightarrow \mathfrak{A}(\mathcal{O})$ of *local subalgebras of observables* satisfying the *local commutativity* (i.e., Einstein causality). This implies, in particular, that the Bose/Fermi statistics of the basic fields is automatically derived without necessity of introducing from the outset *unobservable* field operators such as *fermionic fields* subject to local *anticommutativity violating Einstein causality*, which shows that they are simple mathematical devices for bookkeeping of half-integer spin states. While all the non-trivial space-time behaviours are described here by the observable net $\mathcal{O} \longrightarrow \mathfrak{A}(\mathcal{O})$, the internal symmetry aspects are encoded in the *superselection structure*, which also originates from the observable net.

The symmetry arising from this beautiful theory is, however, found always to be *unbroken*, excluding the situation of spontaneous symmetry breakdown (SSB), which poses a question about the “**stability**” of this method, as remarked by the late Moshé Flato [6]. Indeed we know that many (actually, almost all) of the “sacred symmetries” in nature can be broken (explicitly or spontaneously) in various situations: e.g.,

- SSB’s of chiral symmetry in the electro-weak theory based upon $SU(2) \times U(1)$, electromagnetic $U(1)$ in the superconductivity, and the rotation symmetry $SO(3)$ in ferromagnetism, etc.,
- Lorentz invariance is broken spontaneously in thermal equilibria with $T \neq 0^\circ K$ [7],
- supersymmetry is shown to be unbroken only in the vacuum states [8].

So, the question as to whether or not this theory can incorporate systematically the cases of SSB is a real challenge to it, deserving serious examination, and if the answer is yes, what kind of superselection structure is realized in that case is a non-trivial interesting question. This sort of investigation is expected also to give us some important clues for getting rid of another restriction of *global* gauge symmetries so as to incorporate local gauge symmetries.

In the following, we give an affirmative answer to the above question, revealing very interesting sector structures emerging from SSB: when we start to extend this formalism to the situations with *spontaneous symmetry breakdown*, we encounter the presence of *continuous sectors* (or, “degenerate vacua” in the physicist’s terminology) parametrized by continuous *macroscopic order parameters*, as is seen in Sec.2.3. This requires us to extend the traditional notion of sectors identified as the discrete family of irreducible (or more generally, factorial) representations of the observable algebra \mathfrak{A}

to incorporate the continuous ones. What is more interesting physically and mathematically is the dual or reciprocal relation found between the above degenerate vacua with classical parameters and the quantum Goldstone modes found inside the sectors on a fixed pure vacuum picked up from the degenerate vacua, because it leads us to the point very close to such a physical expression that “the Goldstone modes search the degenerate vacua in a virtual way”. At the same time, this is also related with the mathematical notion of *duality* for homogeneous spaces and their representations, as a natural extension of Tannaka-Krein duality of compact groups. In this way, the basic structural features of the theory with spontaneously broken symmetry, are clarified in a clear-cut way, establishing mutual relationship among degenerate vacua, order parameters, Goldstone modes and condensates responsible for SSB.

As other examples of applications of the method, we examine in Sec.3 also some basic notions supporting the physical and operational meanings of the mathematical framework of quantum theory; the problem of physical implementability of the probabilistic interpretations formulated and examined in a form of the realizability and also that of state preparation as reachability problem, in the general context of control theory, and so on.

2 SSB-vacua as continuous sectors with order parameter whose quantum precursor is Goldstone mode

2.1 Dual net \mathfrak{A}^d and unbroken symmetry H

To treat physically more interesting cases of **spontaneous symmetry breakdown (SSB)**, we need to extend the original DR superselection theory where the internal symmetry is unbroken with unitary implementers as long as the Haag duality $\mathfrak{A}^d(\mathcal{O}) := \pi_0(\mathfrak{A}(\mathcal{O}'))' = \pi_0(\mathfrak{A}(\mathcal{O}))$ (for $\mathcal{O} \in \mathcal{K}$) holds to play the crucial roles. It can be shown that this property is also a necessary condition for the field system with normal statistics and with unbroken symmetry (see, [2, 4]). As pointed out by Roberts [9], SSB does not take place without the breakdown of the Haag duality.

In the previous case with unbroken symmetry, the superselection sectors are parametrized by the *discrete* variables belonging to the dual \hat{G} of a compact group G . In the situation with SSB, one anticipates physically the appearance of *continuous* macroscopic *order parameters*, as typically exemplified by the continuous directions of magnetization in the ferromagnetism, which strongly suggests the appearance of *continuous superselection sectors*, parametrized by macroscopic order parameters. This will be shown actually to be the case in the following.

For the sake of convenience, we change the notation adopted in I; Sec.4

in the unbroken symmetry case, so that the observable algebra \mathfrak{A} and the symmetry group G in I;Sec.4 are replaced, respectively, by the dual net \mathfrak{A}^d (of the genuine observable algebra \mathfrak{A}) and the group H of *unbroken remaining symmetry* in the present context. To begin with, the correspondence between physically relevant states ω around the vacuum ω_0 and such an endomorphism ρ as $\omega = \omega_0 \circ \rho$ can be maintained when all the ingredients here are understood in relation to the dual net \mathfrak{A}^d under the natural assumption of *essential duality*

$$\mathfrak{A}^{dd} = \mathfrak{A}^d \tag{1}$$

which is equivalent to the local commutativity of the dual net and is valid whenever some Wightman fields are underlying the theory [10]. First, in view of the relation $\mathfrak{A}^d(\mathcal{O}')'' = \mathfrak{A}(\mathcal{O}')''$ [9], the starting vacuum state and representation, ω_0 and (π_0, \mathfrak{H}_0) , can safely be extended from \mathfrak{A} to \mathfrak{A}^d (meaning both the local net and the global algebra). Then the DHR selection criterion is understood for the states ω of \mathfrak{A}^d , as $\omega \upharpoonright_{\mathfrak{A}^d(\mathcal{O}')} = \omega_0 \upharpoonright_{\mathfrak{A}^d(\mathcal{O}')}$, and is equivalent to the existence of $\rho \in \mathcal{T} \subset \text{End}(\mathfrak{A}^d)$ such that $\omega = \omega_0 \circ \rho$. On the basis of these items, we can repeat the same procedure of constructing the field algebra \mathfrak{F} and the group H of unbroken symmetry according to the general method [3, 4]:

$$\mathfrak{F} = \mathfrak{A}^d \otimes_{\mathcal{O}_{d_0}^H} \mathcal{O}_{d_0}, \quad H = \text{Gal}(\mathfrak{F}/\mathfrak{A}^d). \tag{2}$$

2.2 Spontaneously broken symmetry

What we are going to show is the following superselection structure of the theory with spontaneously broken symmetry described by the Galois group $G := \text{Gal}(\mathfrak{F}/\mathfrak{A}) \supset H$. First we consider the irreducible H -covariant vacuum representation (π, U, \mathfrak{H}) of the system $\mathfrak{F} \curvearrowright_{\tau} H$, $\pi(\tau_h(F)) = U(h)\pi(F)U(h)^*$ for $\forall F \in \mathfrak{F}$, $\forall h \in H$, containing the original representation (π_0, \mathfrak{H}_0) of \mathfrak{A} and \mathfrak{A}^d as the cyclic fixed-point subspace under $U(H)$: $\mathfrak{H}_0 = \{\xi \in \mathfrak{H}; U(h)\xi = \xi \text{ for } \forall h \in H\}$, $\overline{\pi(\mathfrak{F})\mathfrak{H}_0} = \mathfrak{H}$. Then, according to the DHR sector structure in the unbroken case [2], we have

$$\mathfrak{Z}(\pi(\mathfrak{A}^d)'') = \bigoplus_{\eta \in \hat{H}} \mathbb{C}(\mathbf{1}_{\mathfrak{H}_\eta} \otimes \mathbf{1}_{W_\eta}) = C(\hat{H}) = \mathfrak{Z}(U(H)''). \tag{3}$$

Since this group H is the maximal group of unbroken symmetry in the irreducible vacuum situation, the group G bigger than H cannot be unitarily implemented in the above representation (π, \mathfrak{H}) of \mathfrak{F} , which is just the precise meaning of the SSB of G in the present situation. In more general situations the precise definition can be given by:

Definition 1 A symmetry described by a (strongly continuous) automorphic action τ of G on the field algebra \mathfrak{F} is said to be **unbroken** in a given representation (π, \mathfrak{H}) of \mathfrak{F} if each **factor** subrepresentation $(\sigma, \mathfrak{H}_\sigma)$, $\sigma(\mathfrak{F})' \cap \sigma(\mathfrak{F})'' = \mathbb{C}1_{\mathfrak{H}_\sigma}$, appearing in the central decomposition of (π, \mathfrak{H}) admits a **covariant representation** of the system $G \curvearrowright \mathfrak{F}$ in the sense that there exists a (strongly continuous) unitary representation $(U_\sigma, \mathfrak{H}_\sigma)$ of G verifying the relation $\sigma(\tau_g(F)) = U_\sigma(g)\sigma(F)U_\sigma(g)^*$ for $\forall g \in G, \forall F \in \mathfrak{F}$. If the symmetry is not unbroken, it is said to be **broken spontaneously**.

Note that the above characterization of unbroken symmetry can be reformulated equivalently as the pointwise invariance of the spectrum of centre of $\pi(\mathfrak{F})''$ under G . Therefore, SSB means in short the *conflict between the unitary implementability and the factoriality (=triviality of centres)* [11]. The situation with SSB is seen to exhibit the features of the so-called “infrared instability” under the action of G , because G does not stabilize the spectrum of centre which can be viewed physically as *macroscopic order parameters* emerging in the infrared (=low energy) regions. Since the above definition of SSB still allows the mixture of unbroken and broken *subrepresentations* of a given π , we need to decompose the spectrum of the centre of $\pi(\mathfrak{F})''$ into domains each of which is *ergodic* under G (central ergodicity). Then, π is decomposed into the direct sum (or, direct integral) of unbroken factor representations and broken non-factor representations, each component of which is stable under G . Thus we obtain a *phase diagram* on the spectrum of the centre.

As indicated above, the intuitive physical picture of *order parameters arising from the SSB* from G down to H is realized in connection with the sector structure of the whole theory involving the presence of *continuous sectors* parametrized by $\dot{j} \in H \setminus G$. Here we need to combine the above two formulations of discrete sectors of unbroken internal symmetry (I;Sec.4) and of continuous sectors (I;Sec.2) in the following way. One important point to be mentioned here is that our motivation for treating here the *centres* at various levels of representations is always coming from the natural and inevitable occurrence of *disjoint* representations which leads to the appearance of *macroscopic order parameters to classify different modes* of macroscopic manifestations of microscopic systems; this should be properly contrasted to a mathematical pursuit of generalizing the pre-existing machinery involving factor algebras to non-factorial ones.

According to this formulation, we should find such a covariant representation of the system $(\mathfrak{F} \curvearrowright_\tau G)$ as implementing *minimally* the broken G in the sense of *central ergodicity* under G , which is implied by the factoriality of representation of the crossed product algebra $\mathfrak{F} \rtimes_\tau G$ of \mathfrak{F} with G . Since the subgroup H is unbroken in the irreducible covariant representation (π, U, \mathfrak{H}) of $\mathfrak{F} \curvearrowright_\tau H$, what we seek for can actually be provided

by the representation $(\hat{\pi}, \hat{\mathfrak{H}})$, induced from (π, U, \mathfrak{H}) , of the crossed product $\hat{\mathfrak{F}} := \mathfrak{F} \rtimes_{\hat{\tau}} (H \backslash G) = \Gamma(G \times_H \mathfrak{F})$ of \mathfrak{F} with the homogeneous space $H \backslash G$ (having the right G -action being transitive, and hence, trivially G -ergodic), which can be identified with the algebra of H -equivariant norm-continuous functions $\hat{F} : G \rightarrow \mathfrak{F}$,

$$\hat{F}(hg) = \tau_h(\hat{F}(g)). \quad (4)$$

Denoting $d\xi$ the left-invariant Haar measure on G/H (equipped with the left G -action), we define a Hilbert space $\hat{\mathfrak{H}}$ of L^2 -sections of $\Gamma(G \times_H \mathfrak{H})$ by

$$\hat{\mathfrak{H}} = \int_{\xi \in G/H}^{\oplus} (d\xi)^{1/2} \mathfrak{H} = L^2(\Gamma(G \times_H \mathfrak{H}), d\xi), \quad (5)$$

which can also be identified with the L^2 -space of \mathfrak{H} -valued H -equivariant functions ψ on G ,

$$\psi(gh) = U(h^{-1})\psi(g). \quad (6)$$

On this $\hat{\mathfrak{H}}$, a representation of $\hat{\mathfrak{F}}$ is defined by

$$(\hat{\pi}(\hat{F})\psi)(g) = \pi(\hat{F}(g^{-1}))(\psi(g)) \quad \text{for } \hat{F} \in \hat{\mathfrak{F}}, \psi \in \hat{\mathfrak{H}}, g \in G, \quad (7)$$

which is compatible with the above equivariance condition, (6):

$$\begin{aligned} (\hat{\pi}(\hat{F})\psi)(gh) &= \pi(\hat{F}(h^{-1}g^{-1}))(\psi(gh)) \\ &= U(h^{-1})\pi(\hat{F}(g^{-1}))U(h)U(h^{-1})(\psi(g)) = U(h^{-1})(\hat{\pi}(\hat{F})\psi)(gh). \end{aligned} \quad (8)$$

As is well-known, this representation $(\hat{\pi}, \hat{\mathfrak{H}})$ is equivalent to the covariant representation $(\bar{\pi}, \hat{U}, \hat{\mathfrak{H}})$ of the dynamical system $\mathfrak{F} \curvearrowright_{\tau} G$ defined on $\hat{\mathfrak{H}}$ by $(\hat{U}(g)\psi)(g_1) := \psi(g^{-1}g_1)$, $(\bar{\pi}(F)\psi)(g) := \pi(\tau_{g^{-1}}(F))\psi(g)$ ($\psi \in \hat{\mathfrak{H}}$) and satisfying $\bar{\pi}(\tau_g(F)) = \hat{U}(g)\bar{\pi}(F)\hat{U}(g)^{-1}$, through the relations $\hat{\pi} := \bar{\pi} \rtimes \hat{U}$ (meaning $\hat{\pi}(\hat{F}) = \int dg \bar{\pi}(\hat{F}(g))\hat{U}(g)$) and $\hat{U}(f) := \hat{\pi}(\hat{f})$ with $\hat{f} : G \ni g \mapsto f(g)\mathbf{1} \in \mathfrak{F}$ for $f \in C(G)$ and $\bar{\pi}(F) := s - \lim_{\alpha \rightarrow \infty} \hat{\pi}(F_\alpha)$, $F_\alpha(g) = Ff_\alpha(g)$, $f_\alpha \rightarrow \delta_e$, $f_\alpha \in C(H \backslash G)$. Here also, all the operations are compatible with the constraints of H -equivariance.

2.3 Sector structures and $c \rightarrow q$ channel

What is important for us here is to clarify the *centre* of \mathfrak{A} in the representation $(\hat{\pi} |_{\mathfrak{A}}, \hat{\mathfrak{H}})$ obtained by the restriction of $(\hat{\pi}, \hat{\mathfrak{H}})$ from $\hat{\mathfrak{F}}$ to \mathfrak{A} , which gives the superselection sectors as seen below. The mutual relations among the relevant C^* -algebras can be summarized in the following commuting

diagram:

$$\begin{array}{ccccc}
& & \hat{\mathfrak{F}} = \Gamma(G \times_H \mathfrak{F}) & & \\
& \hat{m}_G \swarrow \nearrow i_G & \circlearrowleft & \hat{m}_{H \setminus G} \swarrow \nearrow i_{H \setminus G} & \\
\hat{\mathfrak{F}}^G = \mathfrak{A}^d = \mathfrak{F}^H & \xleftrightarrow{\quad} & \begin{array}{c} m_H \\ \xleftrightarrow{\quad} \\ i_H \end{array} & \xleftrightarrow{\quad} & \mathfrak{F} = \mathfrak{A}^d \otimes_{\mathcal{O}_{d_0}^H} \mathcal{O}_{d_0} , \\
& m_{G/H} \searrow \swarrow i_{G/H} & \circlearrowright & i_G \nearrow \swarrow m_G & \\
& & \mathfrak{A} = \mathfrak{F}^G & &
\end{array}$$

where the maps i_G and m_G , etc. are, respectively, the embedding maps (of a C^* -algebra into another) and the conditional expectations, such as

$$m_G : \mathfrak{F} \ni F \longmapsto m_G(F) := \int_G dg \tau_g(F) \in \mathfrak{A}. \quad (9)$$

It is interesting to note that the relation between $\hat{\mathfrak{F}}$ and \mathfrak{A}^d is just parallel to that between \mathfrak{F} and \mathfrak{A} , which becomes relevant later to our discussion of the sector structures. Using the relations, $\hat{\pi} \upharpoonright_{\mathfrak{A}} (\mathfrak{A}) = \int_{\dot{g} \in H \setminus G}^{\oplus} d\dot{g} \pi \upharpoonright_{\mathfrak{A}} (\mathfrak{A}) = C(H \setminus G) \otimes \pi \upharpoonright_{\mathfrak{A}} (\mathfrak{A})$ and $\pi \upharpoonright_{\mathfrak{A}^d} (\mathfrak{A}^d) = \oplus_{\eta \in \hat{H}} (\pi_{\eta}(\mathfrak{A}^d) \otimes \mathbf{1}_{W_{\eta}})$, we can show

$$\begin{aligned}
\mathfrak{Z}(\hat{\pi} \upharpoonright_{\mathfrak{A}} (\mathfrak{A})'') &= \left\{ \begin{array}{l} \int_{\dot{g} \in H \setminus G}^{\oplus} d\dot{g} [w - \lim_{\alpha \rightarrow \infty} \pi(A_{\alpha})]; \text{ net } \{A_{\alpha}\} \subset \mathfrak{A}, \\ \text{s.t. } [w - \lim_{\alpha \rightarrow \infty} \pi \upharpoonright_{\mathfrak{A}} (A_{\alpha})] \in \mathfrak{Z}(\pi(\mathfrak{A})'') \end{array} \right\} \\
&= \left\{ \begin{array}{l} \int_{\dot{g} \in H \setminus G}^{\oplus} d\dot{g} [\oplus_{\eta \in \hat{H}} [w - \lim_{\alpha \rightarrow \infty} (\pi_{\eta} \upharpoonright_{\mathfrak{A}} (A_{\alpha}) \otimes \mathbf{1}_{W_{\eta}})]; \\ \text{net } \{A_{\alpha}\} \subset \mathfrak{A} \text{ s.t. } [w - \lim_{\alpha \rightarrow \infty} \pi_{\eta} \upharpoonright_{\mathfrak{A}} (A_{\alpha})] \\ \in \mathfrak{Z}(\pi_{\eta} \upharpoonright_{\mathfrak{A}} (\mathfrak{A})'') \end{array} \right\} \\
&= L^{\infty}(H \setminus G; d\dot{g}) \otimes [\oplus_{\eta \in \hat{H}} \mathfrak{Z}(\pi_{\eta} \upharpoonright_{\mathfrak{A}} (\mathfrak{A})'')]. \quad (10)
\end{aligned}$$

On the basis of these structures of relevant centres of representations, we define a $c \rightarrow q$ channel Ψ as follows:

$$\begin{aligned}
\Psi : \mathfrak{A}^d \ni B &\longmapsto \Psi(B) \in C(H \setminus G) \otimes [\oplus_{\chi \in \hat{H}} \mathfrak{Z}(\pi_{\chi}(\mathfrak{A})'')], \\
\Psi(B) : (H \setminus G) \times \text{Spec}(\oplus_{\chi \in \hat{H}} \mathfrak{Z}(\pi_{\chi}(\mathfrak{A})'')) \ni (\dot{g}, (\eta, \gamma)) &\longmapsto \\
&\rightarrow (\omega_0 \circ \sigma_{\gamma} \circ m_{G/H} \circ \rho_{\eta} \circ m_H)(\tau_{\dot{g}}(B)). \quad (11)
\end{aligned}$$

Here, $\rho_{\eta} \in \mathcal{T}$ is a local endomorphism of \mathfrak{A}^d belonging to the DR-category $\mathcal{T}_{\mathfrak{A}^d}$ on \mathfrak{A}^d and $\gamma \in \text{Spec}(\mathfrak{Z}(\pi_{\eta} \upharpoonright_{\mathfrak{A}} (\mathfrak{A})''))$ is an irreducible representation $\gamma \in \hat{G}$ of G whose restriction to H contains $\eta \in \hat{H}$: i.e., $\exists \eta_1 \in \text{Rep} H$ s.t. $\gamma \upharpoonright_H = \eta \oplus \eta_1$. (Although not fully expressed in either of the above formulae, nor becomes relevant to the context of discussing $c \rightarrow q$ channel Ψ , we need to be careful about such constraints imposed on the components of the *central elements* that all the components c_{γ} 's belonging to the same $\gamma \in \hat{G}$ should be identical even if they appear as subrepresentations $\pi_{\gamma}(\mathfrak{A})$ of $\pi_{\eta} \upharpoonright_{\mathfrak{A}}$'s which are *mutually disjoint as representations of \mathfrak{A}^d* : i.e., $c \in \oplus_{\chi \in \hat{H}} \mathfrak{Z}(\pi_{\chi}(\mathfrak{A})'')$)

can be decomposed as $c = (c_\eta)_{\eta \in \hat{H}}$, $c_\eta = (c_\gamma^{(\eta)})_{\gamma \in \text{Spec}(\mathfrak{Z}(\pi_\eta \upharpoonright_{\mathfrak{A}}(\mathfrak{A}''))}$, where c_{η_1} and c_{η_2} for $\eta_1, \eta_2 \in \hat{H}$, $\eta_1 \neq \eta_2$, are independent up to the constraint that $c_\gamma^{(\eta_1)} = c_\gamma^{(\eta_2)}$ for any $\gamma \in \hat{G}$.

Before a field algebra \mathfrak{F} is constructed, the Doplicher-Roberts method based on the local endomorphisms and the related Cuntz algebras [12] seems to be the only possible path from \mathfrak{A} to the pair of \mathfrak{F} and G *without knowing* either of them, which has necessarily led us to an *unbroken* and *compact* symmetry group. However, in the situation of SSB with massless spectrum, there is *no* reason *nor* guarantee for the broken group $G = \text{Gal}(\mathfrak{F}/\mathfrak{A})$ to be compact, as shown in [10] through the counter-examples. Fortunately, once \mathfrak{F} is so constructed from the dual net \mathfrak{A}^d and the DR category $\mathcal{T}_{\mathfrak{A}^d}$ as to show certain kinds of stability properties (as will be discussed later), we need not any more stick to the original line of thought inherent to the Doplicher-Roberts theory: having at hand the information on the group $G = \text{Gal}(\mathfrak{F}/\mathfrak{A})$, we can control the mutual relations among \mathfrak{F} , G , \mathfrak{A}^d and \mathfrak{A} by means of various versions of crossed products applicable to G , *irrespectively* of whether it is compact or not [13]. However, when $G = \text{Gal}(\mathfrak{F}/\mathfrak{A})$ is compact, as is common in the physical examples of SSB (such as the case of chiral $SU(2) \times SU(2)$ down to the vectorial $SU(2)$), we can get more detailed information as follows along the line suggested in [4], in full use of the corresponding category $\mathcal{T}_{\mathfrak{A}}$ on \mathfrak{A} . First, we have the relation

$$\begin{aligned} & \text{Spec}(\mathfrak{Z}(\pi_\eta \upharpoonright_{\mathfrak{A}}(\mathfrak{A}'')) \\ & = \{\gamma \in \hat{G}; \pi_\gamma = \pi_0 \circ \sigma_\gamma: \text{subrepresentation of } \pi \circ [\rho_\eta] \upharpoonright_{\mathfrak{A}}\}, \end{aligned} \quad (12)$$

where $[\rho_\eta]$ is a minimal $\rho \in \mathcal{T}_{\mathfrak{A}^d}$ s.t. $\rho(\mathfrak{A}) \subset \mathfrak{A}$ and $\rho \succ \rho_\eta$ w.r.t. the ordering defined by

$$\begin{aligned} \rho \succ \rho_1 & \stackrel{\text{def.}}{\iff} \exists \rho_2 \in \mathcal{T}_{\mathfrak{A}^d} \exists w_1, w_2 : \text{isometries} \in \mathfrak{A}^d \\ & \text{s.t. } \rho(B) = w_1 \rho_1(B) w_1^* + w_2 \rho_2(B) w_2^* \text{ for } B \in \mathfrak{A}^d, \end{aligned} \quad (13)$$

which is equivalent to $\pi_\rho = \pi_{\rho_1} \oplus \pi_{\rho_2}$ as representations of \mathfrak{A}^d and to $\eta_\rho = \eta_{\rho_1} \oplus \eta_{\rho_2}$ as representations of H . σ_γ is a local endomorphism of \mathfrak{A} corresponding to $\gamma \in \hat{G}$, which means that $\text{Spec}(\mathfrak{Z}(\pi_\eta \upharpoonright_{\mathfrak{A}}(\mathfrak{A}''))$ consists of the G -charges $\gamma \in \hat{G}$ *contained in an H -sector* having $\eta \in \hat{H}$ among its components. In contrast to the analysis [9, 14] in the opposite direction from the data of a given family of subspaces in the representations of G to determine the corresponding subgroup H in G , here we need to examine the sector structures from the smaller group H to a bigger one G . To be more precise, we can prove and utilize the following results under the assumption of the compactness of $G = \text{Gal}(\mathfrak{F}/\mathfrak{A})$:

1. *Finite-dimensional induction* for a compact pair $H \hookrightarrow G$:

Any finite-dimensional unitary representation (η, W) of H can be extended to a representation (γ, V) of G by taking a direct sum $\gamma|_H \cong$

$\eta \oplus \eta'$ with a suitable representation (η', W') of H (for proof, see [15]). At the level of a field algebra, this kind of induction is sufficient, in contrast to the situations of states for which the genuine Mackey induction is indispensable.

2. Stability and consistency of field algebra construction in SSB:

In use of the above result, one can verify the *stability* of the crossed product construction of the field algebra under the change of Cuntz algebras as the isomorphism between \mathfrak{F} due to the original DR construction from the dual net \mathfrak{A}^d and the crossed product of \mathfrak{A}^d with a Cuntz algebra \mathcal{O}_d for any $d > d_0$:

$$\mathfrak{F} := \mathfrak{A}^d \otimes_{\mathcal{O}_{d_0}^H} \mathcal{O}_{d_0} \cong \mathfrak{A}^d \otimes_{\mathcal{O}_d^H} \mathcal{O}_d, \quad (14)$$

where the isomorphism \cong is due to a joint work [16] with T. Nozawa. While the relation $g(\mathfrak{A}^d) = \mathfrak{A}^d = \mathfrak{F}^H$ for $g \in G$ requires $g \in N_H$, the normalizer of unbroken H in $G = Gal(\mathfrak{F}/\mathfrak{A})$, the equality

$$g(\mathfrak{A}^d) \otimes_{\mathcal{O}_d^{gHg^{-1}}} \mathcal{O}_d = g(\mathfrak{A}^d \otimes_{\mathcal{O}_d^H} \mathcal{O}_d) = \mathfrak{F} \quad (15)$$

can be verified even for such $g \in G$ that $g \notin N_H$, which shows the *consistency* of the construction method with the action of G bigger than H .

While the relation $Gal(\mathfrak{A}^d/\mathfrak{A}) = N_H/H$ was verified in [17], their analysis of degenerate vacua was restricted only to N_H in order to avoid $g(\mathfrak{A}^d) \neq \mathfrak{A}^d$. In the physically interesting situations involving Lie groups, however, the reductivity of a compact Lie group $H \hookrightarrow G$ implies that N_H/H is abelian and/or discrete with a vanishing Lie brackets, which does not seem to be relevant to the physically meaningful contexts.

3. *Duality for homogeneous spaces* and its endomorphism version: For a compact group pair $H \hookrightarrow G$, the definition of $Rep_{G/H}$ and the mutual relations among Rep_G , Rep_H and $Rep_{G/H}$ can be described in terms of a *homotopy-fibre category* Rep_G over Rep_H with $Rep_{G/H}$ as homotopy fibre (S. Maumary [18]): Over $\eta \in Rep_H$ a homotopy fibre (h-fibre for short) is given by a category η/Rep_G (which is called a comma category under η [19] whose objects are pairs (γ, T) of $\gamma \in Rep_G$ and $T \in Rep_H(\eta, \gamma|_H)$ and whose morphisms $\phi : (\gamma, T) \rightarrow (\gamma', T')$ are given

by $\phi \in \text{Rep}_G(\gamma, \gamma')$ s.t. $T' = \phi \circ T$:

$$\begin{array}{ccccc}
& & \eta & & \\
& & \swarrow T & & T' \searrow \\
\gamma|_H & \rightarrow & \xrightarrow{\phi|_H} & \rightarrow & \gamma'|_H \\
i_H \uparrow & & & & \uparrow i_H \\
\gamma & \rightarrow & \xrightarrow{\phi} & \rightarrow & \gamma'
\end{array} \tag{16}$$

(To be more precise, the comma category η/Rep_G is to be understood as η/i_H where the functor $i_H : \text{Rep}_G \rightarrow \text{Rep}_H$ is the restriction of G -representations to the subgroup H of G .)

The h-fibre over the trivial representation $\eta = \iota \in \text{Rep}_H$ of H is nothing but the category of *linear representations of G/H* due to Iwahori-Sugiura [20], to which any other h-fibres can be shown to be homotopically equivalent [18].

The version in terms of endomorphisms dual to the above h-fibre category is given as follows [21]:

$$\begin{aligned}
\text{End}(\mathfrak{A}^d) &\supset \mathcal{T}_{\mathfrak{A}^d} && \longleftrightarrow \text{Rep}_H \\
\text{End}(\mathfrak{A}^d, \mathfrak{A}) &\supset \mathcal{T}_{\mathfrak{A}} = \{\rho \in \mathcal{T}_{\mathfrak{A}^d}; \rho(\mathfrak{A}) \subset \mathfrak{A}\} && \longleftrightarrow \text{Rep}_G \\
\mathcal{T}_{\mathfrak{A}^d}/\mathcal{T}_{\mathfrak{A}} &&& \longleftrightarrow \text{Rep}_{G/H}
\end{aligned}$$

The h-fibre category over $\rho \in \mathcal{T}_{\mathfrak{A}^d}$ is given by $\rho/\mathcal{T}_{\mathfrak{A}}$ [or, more precisely, ρ/\mathcal{D} with \mathcal{D} being the functor $\mathcal{T}_{\mathfrak{A}} \ni \sigma \mapsto \tilde{\sigma} \in \mathcal{T}_{\mathfrak{A}^d}$ extending endomorphisms from \mathfrak{A} to \mathfrak{A}^d] with the object set

$$\{(\sigma, T); \sigma \in \mathcal{T}_{\mathfrak{A}}, T \in (\rho, \tilde{\sigma}) \subset \mathfrak{A}^d\} \tag{17}$$

and with the set of morphisms

$$\{\phi : (\sigma, T) \rightarrow (\sigma', T'); \phi \in \mathcal{T}_{\mathfrak{A}}(\sigma, \sigma') \subset \mathfrak{A}, T' = \phi \circ T \in (\rho, \tilde{\sigma}') \subset \mathfrak{A}^d\} \tag{18}$$

(: semidirect product of $\mathcal{T}_{\mathfrak{A}}$ and \mathfrak{A}^d), where

$$\tilde{\sigma} = \sigma \circ j'_{\mathfrak{A}(\mathcal{O})} \circ j_{\mathfrak{A}^d(\mathcal{O})} \tag{19}$$

gives the extension of endomorphism $\sigma \in \mathcal{T}_{\mathfrak{A}}$ of $\mathfrak{A}(\mathcal{O})$ to $\mathfrak{A}^d(\mathcal{O})$ for $\mathcal{O} = \text{supp}\sigma$ and $j'_{\mathfrak{A}(\mathcal{O})}, j_{\mathfrak{A}^d(\mathcal{O})}$ are the modular conjugations of von Neumann algebras $\mathfrak{A}(\mathcal{O})'$ and $\mathfrak{A}^d(\mathcal{O})$, respectively:

$$\mathfrak{A}^d(\mathcal{O}) \xrightarrow{j_{\mathfrak{A}^d(\mathcal{O})}} \mathfrak{A}^d(\mathcal{O})' \subset \mathfrak{A}(\mathcal{O})' \xrightarrow{j'_{\mathfrak{A}(\mathcal{O})}} \mathfrak{A}(\mathcal{O})'' = \mathfrak{A}(\mathcal{O}). \tag{20}$$

Corresponding to $\mathcal{T}_{\mathfrak{A}} \xrightarrow{i} \mathcal{T}_{\mathfrak{A}^d} \xrightarrow{V} \text{Hilb}$, we have an embedding map

$H = \text{End}_{\otimes}(V) \xrightarrow{j} \text{End}_{\otimes}(V \circ i) \equiv G$, as a result of which the bigger group G suffering from SSB is determined.

\therefore) For any $u \in H = \text{End}_{\otimes}(V)$, $\forall \rho \in \mathcal{T}_{\mathfrak{A}^d}$, $\exists u_{\rho} : V_{\rho} \rightarrow V_{\rho}$ s.t. for $\forall T \in (\rho_1, \rho_2)$ $V_T \circ u_{\rho_1} = u_{\rho_2} \circ V_T$. Then, for any $\sigma \in \mathcal{T}_{\mathfrak{A}}$, $i(\sigma) = \tilde{\sigma} \in \mathcal{T}_{\mathfrak{A}^d}$ and $\forall S \in (\sigma_1, \sigma_2)$ $i(S) \in (i(\sigma_1), i(\sigma_2)) \subset \mathfrak{A} \subset \mathfrak{A}^d$, $V_{i(S)} \circ u_{i(\sigma_1)} = u_{i(\sigma_2)} \circ V_{i(S)}$, which means $j(u) = u_{i(\cdot)} : \mathcal{T}_{\mathfrak{A}} \rightarrow \mathcal{U}(V_{i(\cdot)})$ is a natural unitary transformation from the functor $V \circ i = i^*(V)$ to itself, belonging to $\text{End}_{\otimes}(V \circ i) = G$.

Then, for each $\sigma \in \mathcal{T}_{\mathfrak{A}}$, we obtain $\gamma_{\sigma}|_H = \gamma_{\sigma} \circ j = \eta_{i(\sigma)}$, which states that for each H -representation of the form $\eta_{i(\sigma)}$ ($\sigma \in \mathcal{T}_{\mathfrak{A}}$), there is a G -representation γ_{σ} whose restriction to H is $\eta_{i(\sigma)} = \gamma_{\sigma}|_H$. This is just the categorical dual formulation of the finite-dimensional induction in 1.

4. Generalizing a theorem in [3], we obtain [21]

Proposition 2 *If $\mathfrak{Z}(\mathfrak{A}^d) = \mathbb{C}1$ (as a C^* -algebra), we have the following relations*

$$\mathfrak{A}^d \otimes_{\mathcal{O}_d^G} \mathcal{O}_d = \Gamma(G \times_H (\mathfrak{A}^d \otimes_{\mathcal{O}_d^H} \mathcal{O}_d)) =: \hat{\mathfrak{F}}, \quad (21)$$

$$\hat{\mathfrak{F}}^G = \mathfrak{A}^d = (\mathfrak{A}^d \otimes_{\mathcal{O}_d^H} \mathcal{O}_d)^H, \quad (22)$$

$$\text{Spec}(\mathfrak{Z}(\mathfrak{A}^d \otimes_{\mathcal{O}_d^G} \mathcal{O}_d)) = G/H. \quad (23)$$

On the basis of the above machinery, we can derive the following isomorphisms and the reciprocity relations of Frobenius type:

$$\begin{array}{ccc} \mathcal{T}_{\mathfrak{A}}([\rho_{\eta}], \sigma_{\gamma}) & \simeq & \mathcal{T}_{\mathfrak{A}^d}(\rho_{\eta}, \tilde{\sigma}_{\gamma}) \\ \downarrow & & \downarrow \\ \text{Rep}_G([\eta], \gamma) & \simeq & \text{Rep}_H(\eta, \gamma \upharpoonright_H) \end{array} . \quad (24)$$

In use of this, we see that the condition for $\gamma \in \hat{G}$ to belong to $\text{Spec}(\mathfrak{Z}(\pi_{\eta} \upharpoonright_{\mathfrak{A}} (\mathfrak{A})''))$ is equivalent to the condition that $\pi_{\gamma} = \pi_0 \circ \sigma_{\gamma}$ is a subrepresentation of $\pi \circ [\rho_{\eta}] \upharpoonright_{\mathfrak{A}}$, and hence, that the restriction $\gamma \upharpoonright_H$ of γ to H contains $\eta \in \hat{H}$:

$$\gamma \in \text{Spec}(\mathfrak{Z}(\pi_{\eta} \upharpoonright_{\mathfrak{A}} (\mathfrak{A})'')) \iff \gamma \in \hat{G} \text{ and } \exists \eta_1 \in \text{Rep}H \text{ s.t. } \gamma \upharpoonright_H = \eta \oplus \eta_1, \quad (25)$$

which is also equivalent to the condition that $\exists v_1, v_2$: isometries $\in \mathfrak{A}^d$ s.t. $\tilde{\sigma}_{\gamma}(B) = v_1 \rho_{\eta}(B) v_1^* + v_2 \rho_{\eta_1}(B) v_2^*$ for $B \in \mathfrak{A}^d$ and $\pi_0 \circ \sigma_{\gamma}(\mathfrak{A}) \subset \pi_0 \circ [\rho_{\eta}](\mathfrak{A})$.

2.4 Interpretation of sector structure: degenerate vacua with order parameters, Goldstone modes and condensates

Now the physical meaning of our map, $\Psi : \mathfrak{A}^d \ni B \mapsto \Psi(B)$, $[\Psi(B)](\dot{g}, (\eta, \gamma)) = (\omega_0 \circ \sigma_\gamma \circ m_{G/H} \circ \rho_\eta \circ m_H)(\tau_{\dot{g}}(B))$, is clear, and its dual,

$$\begin{aligned} \Psi^* : M_1(H \setminus G) \otimes M_1\left(\coprod_{\chi \in \hat{H}} \left\{ \gamma \in \hat{G}; \exists \eta_1 \in \text{Rep} H \text{ s.t. } \gamma \upharpoonright_H = \chi \oplus \eta_1 \right\}\right) \\ \rightarrow E_{\mathfrak{A}^d}, \end{aligned}$$

gives a $c \rightarrow q$ channel, whose inverse $(\Psi^*)^{-1}$ exists on the states of \mathfrak{A}^d selected by the DHR criterion, as a $q \rightarrow c$ channel to provide the physical interpretations of such states in terms of the order parameters in $\dot{g} \in H \setminus G$, H -charge $\eta \in \hat{H}$ and G -charge $\gamma \in \hat{G}$ constrained to η by the relation $\gamma \upharpoonright_H \succ \eta$. In view of our starting premise of the observable algebra \mathfrak{A} , however, it looks more natural to take $A \in \mathfrak{A}$ as the argument of Ψ , instead of $B \in \mathfrak{A}^d$. Then, because of G -invariance of $A \in \mathfrak{A} = \mathfrak{F}^G$, $\Psi \upharpoonright_{\mathfrak{A}}$ becomes independent of $\dot{g} \in H \setminus G$, $[\Psi(A)](\dot{g}, (\eta, \gamma)) = (\omega_0 \circ \sigma_\gamma \circ m_{G/H} \circ \rho_\eta \circ m_H)(\tau_{\dot{g}}(A)) = (\omega_0 \circ \sigma_\gamma \circ m_{G/H} \circ \rho_\eta)(A)$, failing to pick up the information on $H \setminus G$.

To settle this matter, we first consider the physical meaning of the obtained order parameters:

- i) $H \setminus G$ as *order parameters* to parametrize the degenerate vacua: in the decomposition of the representation space $\hat{\mathfrak{H}}$ of $\hat{\mathfrak{F}}$ to pure vacuum representations of $\hat{\mathfrak{F}}$ in \mathfrak{H} , we get the centre $L^\infty(H \setminus G, d\dot{g})$ with the spectrum $H \setminus G$ which parametrizes the **degenerate vacua** with minimum energy 0 generated by the SSB of G up to the unbroken remaining H . The physical meaning of this quantity shows up in such forms as the direction of magnetizations in the Heisenberg ferromagnets, or, as the *Josephson effect* where the difference of the phases of Cooper pair condensates between adjacent vacua across a junction exhibits such eminent physical effects as the *Josephson current* (see, e.g., [22]). Mathematically, the Mackey induction from H to G is relevant here. (While the disjointness $\omega \circ \tau_g$ between any pure vacuum ω of $\hat{\mathfrak{F}}$ and $\omega \circ \tau_g$ for any $g \in G$ s.t. $g \notin H$ might puzzle one about possible discontinuous behaviours of the order parameter $\dot{g} \in H \setminus G$ under G , this is unnecessary worry because of the presence of centre $C(H \setminus G)$ in $\hat{\mathfrak{F}}$ in the C^* -version on which G acts continuously.)
- ii) Internal spectrum \hat{H} of *excited states* on a chosen vacuum parametrized by a fixed $\dot{g} = Hg \in H \setminus G$: in the representation space \mathfrak{H} of $\hat{\mathfrak{F}}$, we see the standard picture of sectors $(\pi_\eta, \mathfrak{H}_\eta)$ with respect to \mathfrak{A}^d parametrized by $\eta \in \hat{H}$, which describes the internal symmetry aspects of excited states in terms of the unbroken H .

iii) Description of SSB dual to i) in terms of *Goldstone sectors* within \mathfrak{A}^d : the sectors w.r.t. \mathfrak{A}^d need still be decomposed into the disjoint representations (= \mathfrak{A} -sectors) of our genuine observables \mathfrak{A} , corresponding to the “diagonalizations” of $\mathfrak{Z}(\pi_\eta \upharpoonright_{\mathfrak{A}} (\mathfrak{A})'')$ in each \mathfrak{A}^d -sector with $\eta \in \hat{H}$. If the group G of spontaneously broken symmetry is compact, this process is controlled by the **finite-dimensional induction/reduction** between H and G on the algebra \mathfrak{F} and by its **dual version with respect to local endomorphisms** of \mathfrak{A} and of \mathfrak{A}^d . Since the gap between \mathfrak{A}^d and \mathfrak{A} is essentially due to the Goldstone modes, this may be interpreted as an abstract version of the Goldstone and/or low-energy theorems in the sense that they give a *dual* description of the degenerate vacua in i), in a virtual way *within* a fixed pure vacuum representation.

The remarkable parallel and/or reciprocal relations described above and depicted below, between $[\hat{\mathfrak{F}} \text{ vs. } \mathfrak{F}]$ and $[\mathfrak{A} \text{ vs. } \mathfrak{A}^d]$, or, between $[\mathfrak{F} \text{ vs. } \mathfrak{A}]$ and $[\hat{\mathfrak{F}} \text{ vs. } \mathfrak{A}^d]$,

$$\begin{array}{ccc}
\hat{\mathfrak{F}} & \begin{array}{c} H \setminus G \\ \text{---} \text{---} \text{---} \end{array} & \mathfrak{F} \\
| \begin{array}{c} G \\ \text{---} \end{array} & \begin{array}{c} / \\ H \end{array} & | \begin{array}{c} G \\ \text{---} \end{array} \\
\mathfrak{A}^d & \begin{array}{c} \text{---} \text{---} \text{---} \\ G/H \end{array} & \mathfrak{A} \\
| \begin{array}{c} W_{\eta\rho} \simeq H_\rho \\ \text{---} \end{array} & & | \begin{array}{c} V_{\gamma\sigma} \simeq H_\sigma \\ \text{---} \end{array} \\
\rho_\eta(\mathfrak{A}^d) & \begin{array}{c} \eta \prec \gamma \upharpoonright_H \\ \text{---} \text{---} \text{---} \\ [\pi_\eta](\mathfrak{A}) \supset \pi_\gamma(\mathfrak{A}) \end{array} & \sigma_\gamma(\mathfrak{A}),
\end{array}$$

can naturally be understood from the viewpoint of implicit relevance of the Jones tower structures [23, 24].

On the basis of the above duality between i) and iii), it will be possible to recover the information on the $H \setminus G$ dependence of states, within a pure vacuum representation without going over to the other such, through the closer analysis of the difference term $\eta_1 \in \text{Rep}H$ between $\gamma \upharpoonright_H = \eta \oplus \eta_1$ and $\eta \in \hat{H}$, which can be justified by the structure of G/H involved in the proof [15] of finite-dimensional inductions. The verification of this expectation will fully justify such an heuristic and physical expression that “the Goldstone modes search the degenerate vacua in a virtual way”.

While it would be highly interesting to verify it rigorously along the above iii), we here simply take \mathfrak{A}^d as our *extended observables*. This standpoint is also fully justified by the natural physical meaning of \mathfrak{A}^d as the *maximal local net generated by the original net* \mathfrak{A} , which is just the version, adapted to the observable net, of the notion of the Borchers classes [25] consisting of all the *relatively local* fields to absorb the *arbitrariness* in

the “interpolating fields” [26]: in spite of their G -non-invariance property, the *Goldstone modes* related to the homogeneous space $H\backslash G$ are allowed to appear here with the qualification of such extended observables belonging to \mathfrak{A}^d and they detect the information concerning the position of a pure vacuum $\dot{g} \in H\backslash G$ among the degenerate vacua, as is exhibited through the \dot{g} -dependence of $\Psi(B)$ for $B \in \mathfrak{A}^d$. To be more precise, one needs to be careful about the distinctions and mutual relations among the following four levels involving Goldstone modes and order parameters:

- i) degenerate vacua as continuous sectors parametrized by the *order parameters* $\dot{g} \in H\backslash G$ which is a global notion,
- ii) *Goldstone modes* and *Goldstone sectors* belonging, respectively, to \mathfrak{A}^d and its representation space \mathfrak{H}_γ , whose massless spectrum is responsible for the validity of Goldstone theorem,
- iii) *Goldstone multiplet* belonging to \mathfrak{F} and consisting of Goldstone modes and of condensates responsible for the above i); this field multiplet transforms under G according to a *linear* representation, which is nothing but a “*linear representation of a homogeneous space*” according to the definition of [20]. What is most confusing is the mutual relation between the Goldstone modes and the condensates; in the simplest example of SSB (e.g., Heisenberg ferromagnet) from $G = SO(3)$ to $H = SO(2)$ with $H\backslash G = S^2$, a pure vacuum among degenerate vacua is parametrized and geometrically depicted by a point $p \in S^2$, a condensate by a radius from the centre of the unit ball to p , and the Goldstone modes geometrically expressed by *tangent vectors* at p *tangential to S^2* and *orthogonal to the condensate*. The Goldstone multiplet is an entity in \mathfrak{F} which is behaving as a three-dimensional covariant vector under $SO(3)$.
- iv) There is a useful physical notion called “nonlinear realization” of Goldstone bosons [27], expressing the above situation in a nice geometric way and serving as very effective tools in the derivation of the so-called low energy theorems, such as the soft-pion theorem, to describe the low energy scattering processes involving Goldstone bosons associated with SSB. While its functional role is very akin to our Goldstone modes in ii), it may not be so straightforward to accommodate it literally into the present context, because of the nonlinear transformation law exhibited in its transformation property under G .

As already remarked, we have recourse in the above considerations to the compactness assumption of the spontaneously broken G , which does not hold on the general ground, as shown in [10]. While we have benefited

from this assumption in drawing a detailed and concrete picture of the fiber-structure involved in $Spec(\oplus_{\eta \in \hat{H}} \mathfrak{Z}(\pi_\eta \upharpoonright_{\mathfrak{A}} (\mathfrak{A}'')))$, the essential feature can be expected to survive without this assumption. For instance, the relations

$$\mathfrak{F} = \mathfrak{A} \rtimes_\delta G, \quad \mathfrak{A}^d = \mathfrak{F}^H = \mathfrak{A} \rtimes_\delta (H \backslash G), \quad (26)$$

with a co-action δ of G on $\mathfrak{A} = \mathfrak{F}^G$ are known to hold in their W^* -versions for any locally compact group G [13]. So the verification of their C^* -versions will support the validity of the essential points of our picture described above that the Goldstone degrees of freedom related to $H \backslash G$ are contained in \mathfrak{A}^d to describe the SSB-sector structure within the pure vacuum representations, according to the above iii), etc.

3 Operational meanings of selection criteria in quantum measurements

3.1 Spectral decomposition and probabilistic interpretation

In view of the importance of the interpretations above, we pick up some relevant points here from the quantum measurement processes, in regard to the following basic points:

i) The operator-theoretical notion of spectral decomposition of a self-adjoint observable A to be measured is equivalent to the algebraic homomorphism (so-called the map of “functional calculus”):

$$\begin{aligned} \hat{A} : L^\infty(Spec(A)) \ni f &\mapsto \hat{A}(f) = f(A) \\ &:= \int_{a \in Spec(A)} f(a) E_A(da) \in \mathfrak{A}'' \subset B(\mathfrak{H}), \end{aligned} \quad (27)$$

where \mathfrak{H} is the Hilbert space of the defining representation of the observable algebra \mathfrak{A} to which our observable A belongs. Here we omit the symbol for discriminating the original C^* -algebra \mathfrak{A} and its representation in \mathfrak{H} , and hence, we will freely move between C^* - and W^* -versions without explicit mention. This fits quite well to the common situations of discussing measurements owing to the absence of disjoint representations in the purely quantum side \mathfrak{A} with *finite* degrees of freedom (due to Stone-von Neumann theorem). In such cases, the non-trivial existence of a centre comes only from the classical system coupled to quantum one (, the former of which need to be derived from the quantum system with infinite degrees of freedom at the “ultimate” levels, though).

ii) To give this homomorphism \hat{A} is (almost) equivalent to giving a spectral measure E_A by

$$E_A : \mathcal{B}(Spec(A)) \ni \Delta \mapsto E_A(\Delta) := \hat{A}(\chi_\Delta) = \chi_\Delta(A) \in Proj(\mathfrak{H}), \quad (28)$$

on the σ -algebra $\mathcal{B}(Spec(A))$ on $Spec(A)$ of Borel sets Δ , identified with the indicator function χ_Δ , taking values in the set $\text{Proj}(\mathfrak{H})$ of orthogonal projections in \mathfrak{H} . Then the dual map \hat{A}^* defines a mapping from a quantum state ω to a *probability distribution*, $p^A(\cdot | \omega) : \mathcal{B}(Spec(A)) \ni \Delta \mapsto p^A(\Delta | \omega) = \text{Prob}(A \in \Delta | \omega) := \omega(E_A(\Delta))$, of measured values in the measurements of A performed in the state ω . The above reservation “(almost) equivalent” is due to the fact that the reverse direction from a probability distribution to a spectral decomposition admits a slightly more general notion, positive-operator valued measure (POM), which corresponds to a unital completely positive map instead of a homomorphism and which becomes relevant for treating the set of mutually non-commutative observables. In any case, the operational meaning of the mathematical notion of spectral decomposition is exhibited by this \hat{A}^* (or, the dual of POM) as a simplest sort of $q \rightarrow c$ *channel* providing the familiar probabilistic interpretation.

iii) To implement physically the spectral decomposition, however, we need some *physical interaction processes* between the system and the apparatus through the coupling term of the observable $A \in \mathfrak{A}$ to be measured and an external field J belonging to the apparatus. While one of the most polemic issues in the measurement theory is as to how this “contraction of wave packets” is realized consistently with the “standard” formulaion of quantum theory, we here avoid this issue, simply taking such a “phenomenological” standpoint that our purpose will be attained if the composite system consisting of the object system and the classical system involving J is effectively (Fourier- or Legendre-) transformed through this coupled dynamical process into $\mathfrak{A} \otimes C^*\{A\} =: \mathfrak{A}_A = C(Spec(A), \mathfrak{A})$, the centre of which is just the commutative C^* -algebra $C^*\{A\} \simeq C(Spec(A))$ generated by a self-adjoint operator $A: C^*\{A\} \xrightarrow{\iota} \mathfrak{Z}(\mathfrak{A}_A) \hookrightarrow \mathfrak{A}_A$. So the *superselection structure* comes in here with sectors parametrized by the spectrum of the observable A to be measured. (It was the important contribution of Machida and Namiki [28] that shed a new light on the notion of continuous superselection rules, where the focus was, unfortunately, upon sectors related to *irrelevant unobservable* variables, in sharp contrast to those discussed here.)

3.2 Measurement scheme and its realizability

Then the basic measurement scheme [29] reduces to the requirement that all the information on the probability distribution in ii) should be recorded in and can be read out from this classical part $\{A\}'' = L^\infty(Spec(A))$ as a mathematical representative of the measuring apparatus:

$$\omega(E_A(\Delta)) = p^A(\Delta | \omega) = (\omega \otimes \mu_0)[\hat{\tau}(\mathbf{1} \otimes \chi_\Delta)], \quad (29)$$

where μ_0 is some initial state of $\{A\}''$ and $\hat{\tau} \in \text{Aut}(\mathfrak{A}_A)$ describes the effects of dynamics of the composite system of \mathfrak{A} and $C^*\{A\}$ (or, more generally, a dissipative dynamics of a completely positive map also to be allowed).

We are interested here in examining how the problem of a selection criterion according to our general formulation becomes relevant to the present context. Applying to any state $\hat{\omega} \in E_{\mathfrak{A}_A}$ the uniquely determined *central decomposition*, we have

$$\hat{\omega} = \int_{\text{Spec}(A)} d\mu(a)(\omega_a \otimes \delta_a), \quad (30)$$

with some family of states $\{\omega_a\} \subset E_{\mathfrak{A}}$ (which are universally chosen by $\omega_a(B) := \langle \psi_a | B\psi_a \rangle$ with $A\psi_a = a\psi_a$ if A has only *discrete spectrum*). What plays important roles here is the instrument $\mathcal{I}_{A,\hat{\tau}}$ [30] depending on $A \in \mathfrak{A}$ and on the composite-system dynamics $\hat{\tau}$ defined by

$$\begin{aligned} \mathcal{I}_{A,\hat{\tau}} : \mathfrak{A}_A \ni \hat{B} &\longmapsto \mathcal{I}_{A,\hat{\tau}}(\hat{B}) := \int d\mu_0(a)(\hat{\tau}(\hat{B}))(a) \\ &= \int d\mu_0(a)\delta_a(\hat{\tau}(\hat{B})) \in \mathfrak{A}, \\ \mathcal{J}_{A,\hat{\tau}}(\Delta|\omega)(B) &:= [\mathcal{I}_{A,\hat{\tau}}^*(\omega)](B \otimes \chi_\Delta) = \omega(\mathcal{I}_{A,\hat{\tau}}(B \otimes \chi_\Delta)) \\ &= (\omega \otimes \mu)[\hat{\tau}(B \otimes \chi_\Delta)]. \end{aligned} \quad (31)$$

In terms of these notions, Eq. (29) can be rewritten as

$$\begin{aligned} \hat{A}^*(\omega) &= (\mathcal{I}_{A,\hat{\tau}} \circ \iota')^*(\omega) \\ \implies \hat{A}^* &= \iota'^* \circ \mathcal{I}_{A,\hat{\tau}}^*, \end{aligned} \quad (32)$$

where $\iota'^* : E_{\mathfrak{A}''_A} \rightarrow M_1(\text{Spec}(A))$ defined by the dual of

$$\iota' : \{A\}'' \ni f \longmapsto \mathbf{1} \otimes f \in \mathfrak{A}''_A \quad (33)$$

is the standard (tautological) $q \rightarrow c$ channel to allow the data read-out from the system-apparatus composite system. Eq.(33) selects out an observable A (or its corresponding $q \rightarrow c$ channel \hat{A}^* describing the probabilistic interpretation of A) according to a criterion as to whether it can be factorized into the standard tautological $q \rightarrow c$ channel ι'^* and some instrument $\mathcal{I}_{A,\hat{\tau}}$. In view of the formal similarity between the DHR criterion, $\omega = \omega_0 \circ \rho$, and Eq.(33), it is interesting to note that what are examined here is $q \rightarrow c$ channels, \hat{A}^* and ι'^* , the latter of which is a fixed standard one. This criterion is just for examining whether the measurement of A can actually be materialized by means of the coupling $\hat{\tau}$ between the system containing A and some measuring apparatus constituting the composite system $\mathfrak{A}_A = \mathfrak{A} \otimes \{A\}''$. In this sense, the criterion examines the *realization problem* in the context of control theory [31], asking whether a suitable choice of an apparatus and a choice of dynamical coupling can correctly describe the input-output behaviour of the system. Once this criterion is valid, its experimental observation is most conveniently described by the instrument $\mathcal{J}_{A,\hat{\tau}}(\Delta|\omega)(B)$ whose interpretation is given [29] by

1) the probability distribution of the measured value of A in a state ω is given by $\mathcal{J}_{A,\hat{\tau}}(\Delta|\omega)(\mathbf{1}) = p_A(\Delta|\omega)$,

2) the final state realized (in the repeatable measurement) after the readout $a \in \Delta$ is given by $\frac{\mathcal{J}_{A,\hat{\tau}}(da|\omega)}{p_A(da|\omega)}$,

3) in combination of 1) and 2), the quantity $\mathcal{J}_{A,\hat{\tau}}(\Delta|\omega)(B)$ itself can be regarded as the expectation value of another observable $B \in \mathfrak{A}$ when the initial state ω goes into some final state whose A -values belong to $\Delta(\subset \text{Spec}(A))$.

3.3 Problem of state preparation as reachability problem

In the related context, we need to examine the problem of *reachability* to ask whether there is a controlled way to drive the (composite) system to any desired state starting from some initial state; this is nothing but the problem of **state preparation**, which has not been seriously discussed, in spite of its vital importance in the physical interpretation of quantum theory.

For this purpose, we need to define the $c \rightarrow q$ channel relevant to it. Fixing a family $(\omega_a)_{a \in \text{Spec}(A)} =: \phi$ of states on \mathfrak{A} appearing in the central decomposition (30), we can define a $c \rightarrow q$ channel by

$$C_{A,\phi} : \mathfrak{A}_A \ni \hat{B} \longmapsto (\text{Spec}(A) \ni a \longmapsto \omega_a(\hat{B}(a))) \in C(\text{Spec}(A)), \quad (35)$$

and hence, $C_{A,\phi}^* : M_1(\text{Spec}(A)) \ni \rho \longmapsto C_{A,\phi}^*(\rho) \in E_{\mathfrak{A}_A}$, where

$$\begin{aligned} C_{A,\phi}^*(\rho)(\hat{B}) &= \rho(C_{A,\phi}(\hat{B})) = \int d\rho(a) \omega_a(\hat{B}(a)) = \int d\rho(a) (\omega_a \otimes \delta_a)(\hat{B}), \\ \text{or, } C_{A,\phi}^*(\rho) &= \int d\rho(a) (\omega_a \otimes \delta_a). \end{aligned} \quad (36)$$

In terms of these, the reachability (or, preparability) criterion can be formulated as to examine the validity of

$$\omega = \lim_{t \rightarrow \infty} (\iota^* \circ C_{A,\phi}^*)(\mu_{\hat{\tau}_t}), \quad (37)$$

where $\iota^* : E_{\mathfrak{A}_A} \rightarrow E_{\mathfrak{A}}$ is the dual of $\iota : \mathfrak{A} \ni B \longmapsto B \otimes \mathbf{1} \in \mathfrak{A}_A$, and the measure $\mu_{\hat{\tau}_t}^\omega \in M_1(\text{Spec}(A))$ is defined through the central decomposition of $(\omega \otimes \mu_0) \circ \hat{\tau}_t = \int d\mu_{\hat{\tau}_t}^\omega(a) \omega_a \otimes \delta_a$ valid for such an observable A as with discrete spectrum. If we can find such a suitable coupled dynamics $\hat{\tau}_t$ and an initial and final probability measures $\mu_0, \mu_1 \in M_1(\text{Spec}(A))$ that $\lim_{t \rightarrow \infty} (\omega \otimes \mu_0) \circ \hat{\tau}_t(B \otimes \mathbf{1}) = (\omega \otimes \mu_1)(B \otimes \mathbf{1})$ for each $B \in \mathfrak{A}$, then a state ω can actually be prepared:

$$\begin{aligned} (\iota^* \circ C_{A,\phi}^*)(\mu_{\hat{\tau}})(B) &= \mu_{\hat{\tau}}(C_{A,\phi}(B \otimes \mathbf{1})) = \int d\mu_{\hat{\tau}}(a) \omega_a(B \otimes \mathbf{1}) \\ &= (\omega \otimes \mu_0) \circ \hat{\tau}_t(B \otimes \mathbf{1}) \xrightarrow{t \rightarrow \infty} (\omega \otimes \mu_1)(B \otimes \mathbf{1}) = \omega(B), \end{aligned} \quad (38)$$

in the sense that there is some operational means specified in terms of $A \in \mathfrak{A}$, a coupled dynamics $\hat{\tau}_t$ and an initial and final probability measures $\mu_0, \mu_1 \in M_1(\text{Spec}(A))$.

Here, the assumption of discreteness of the spectrum of A is no problem, since A plays here only a subsidiary role. However, this problem becomes crucial when we start to examine the *repeatability* of the measurement of the observable A itself. We compare the above $q \rightarrow c$ channel $(C_{A,\phi}^*)^{-1}$ with another natural $q \rightarrow c$ channel $(\iota \circ \hat{A})^*$, which can be defined on all the states $\in E_{\mathfrak{A}_A}$, independently of a specific choice of a family $\phi = (\omega_a)_{a \in \text{Spec}(A)}$ of states on \mathfrak{A} , simply as the dual of the composed embedding maps, $C(\text{Spec}(A)) \xrightarrow{\hat{A}} \mathfrak{A} \xrightarrow{\iota} \mathfrak{A}_A$. As is seen from the relation,

$$\begin{aligned} & (\iota \circ \hat{A})^* \left(\int d\mu(a) (\omega_a \otimes \delta_a) \right) (f) \\ &= \int d\mu(a) (\omega_a \otimes \delta_a) \left((\iota \circ \hat{A})(f) \right) = \int d\mu(a) (\omega_a \otimes \delta_a) (f(A) \otimes \mathbf{1}) \\ &= \int d\mu(a) \omega_a(f(A)) = \int d\mu(a) \int \omega_a(dE_A(b)) f(b), \end{aligned} \quad (39)$$

$(\iota \circ \hat{A})^*$ is, in general, not equal to $(C_{A,\phi}^*)^{-1}$, nor has a simple interpretation. If we can choose such a family $(\omega_a)_{a \in \text{Spec}(A)}$ that $\int f(b) \omega_a(dE_A(b)) = f(a)$ for $\forall f \in C(\text{Spec}(A))$, or equivalently, $\omega_a(E_A(\Delta)) = \chi_\Delta(a)$ for $\forall \Delta$: measurable subset of $\text{Spec}(A)$, we can attain the equality between $(C_{A,\phi}^*)^{-1}$ and $(\iota \circ \hat{A})^*$ on the image of $C_{A,\phi}^*$ in $E_{\mathfrak{A}_A}$, which can be extended to the whole $E_{\mathfrak{A}_A}$ by the use of the Hahn-Banach extension. As a result, we can attain universally the state preparations and physical interpretations (in relation to A), independently of a specific choice of the above family $(\omega_a)_{a \in \text{Spec}(A)}$. While such a choice is always possible for observables A with discrete spectrum, its impossibility for those A with *continuous spectra* forces us to consider the *approximate measurement scheme* (see [29]), which involves the essential dependence on the choice of the family $(\omega_a)_{a \in \text{Spec}(A)}$ and the selection of and restriction to preparable and interpretable states.

In this way, we have seen that this approach provides a simple unified scheme based upon instruments and channels for discussing various aspects in the measurement processes without being trapped in the depth of philosophical issues. So, it will be worthwhile to attempt the possible extension of the measurement scheme to more general situations involving QFT. It will be also interesting to examine the problems of state correlations in entanglements, of state estimation, and so on, in use of the notions of mutual entropy, channel capacities [32], Cramér-Rao bounds, etc.

Through the above relation with the spectral decomposition of an observable A and the superselection sectors parametrized by $a \in \text{Spec}(A)$, we

can reconfirm the naturality of our extending the meaning of sectors from their traditional version of discrete one, to the present version including both: in SSB, order parameters of continuous family of disjoint states (of \mathfrak{A}) parametrized by $H \setminus G$ and in thermal situations, (inverse) temperatures $\beta [= (\beta^\mu)]$ discriminating pure thermodynamic phases corresponding also to disjoint KMS states (of \mathfrak{A}), and variety of non-equilibrium local states ([34]). Our way of unifying these various cases is seen to be quite similar to the unified treatment of discrete and continuous spectra of self-adjoint operators in the general theory of spectral decompositions.

4 Outlook

We conclude this paper by mentioning some problems under investigation, which will be reported somewhere.

1. Treatment of a non-compact group of broken internal symmetry as remarked in Sec.2.3 and 2.4.
2. Reformulation of characterization of KMS states: in I;Sec.2 and I;Sec.3, we have just relied on the known simplicial structure of the set of all KMS states. To be consistent with the spirit of the present scheme, we need also to find a version of selection criterion to characterize these KMS states, whose essence should be found in the **zeroth law of thermodynamics** from which the familiar parameter of temperatures arises (in combination with the first and second laws in such a form as the passivity [33]). In any case, such a physically interesting problem as drawing a phase diagram just belongs to the analysis of selection structure in the present context.
3. To substantiate the above consideration, it is necessary to develop a systematic way of treating a chemical potential as one of the order parameters to be added to temperature. This requires the local and systematic treatment of conserved currents such as $T_{\mu\nu}$ and j_μ , extended to thermal situations just in a parallel way to the local thermal observables in [34].
4. It would be worthwhile to examine whether the notion of a field algebra \mathfrak{F} is a simple mathematical device, convenient for making the interpretation easier from the viewpoint laid out by Klein's Erlangen programme and no more than that.

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