Categorical Representation of Locally Noetherian Log Schemes

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Contents:

Introduction

Let $X^{\log}$ be a *fine* (cf. [Kato1], §2.3) *saturated* (cf. (the evident étale generalization of) [Kato2], §1.5) *log scheme* (cf. [Kato1], §1.2) whose underlying scheme X is *locally noetherian*. Let us denote by

$$\mathrm{Sch}(X)$$

the category whose *objects* are *morphisms of finite type* $Y \to X$, where Y is a *noetherian scheme*, and whose *morphisms* (from an object $Y_1 \to X$ to an object $Y_2 \to X$) are *morphisms of finite type* $Y_1 \to Y_2$ lying over X, and by

$$\mathrm{Sch}^{\log}(X^{\log})$$

the category whose *objects* are *morphisms of fine saturated log schemes* $Y^{\log} \to X^{\log}$, where Y is a *noetherian scheme*, and the underlying morphism of schemes $Y \to X$ is *of finite type*, and whose *morphisms* (from an object $Y_1^{\log} \to X^{\log}$ to an object $Y_2^{\log} \to X^{\log}$) are *morphisms of finite type* $Y_1^{\log} \to Y_2^{\log}$ (i.e., morphisms for which the underlying morphism of schemes $Y_1 \to Y_2$ is of finite type) lying over $X^{\log}$.

Our *main results* (which correspond to Theorems 1.7, 2.19, in the text) are the following:

Theorem A. (Categorical Reconstructibility of Locally Noetherian Schemes) *The locally noetherian scheme X may be reconstructed category-theoretically from $\mathrm{Sch}(X)$, in a fashion that is* **functorial** *with respect to X — cf. Theorem 1.7 for more details.*

Theorem B. (Categorical Reconstructibility of Locally Noetherian Log Schemes) *The locally noetherian fine, saturated log scheme $X^{\log}$ may be reconstructed category-theoretically from $\mathrm{Sch}^{\log}(X^{\log})$, in a fashion that is* **functorial** *with respect to $X^{\log}$ — cf. Theorem 2.19 for more details.*

These results are partially motivated by the *anabelian philosophy of Grothendieck* — cf., e.g., [Mzk], [NTM], for more details. In essence, the *difference* is that in the *anabelian case*, instead of considering the category $\mathrm{Sch}(X)$ of (roughly speaking) *all* schemes of finite type over X, one considers the category $\acute{\mathrm{E}}\mathrm{t}(X)$ of *finite étale coverings* of X.

Another important motivating circle of ideas for the author was the work of [Bell1–4], [Lwv1–2]. The main idea here is (roughly speaking) that instead of working with *set-theoretic objects* — such as *schemes* or *log schemes* — one should regard *categories* as the *"fundamental, primitive objects" of mathematics discourse*. From this point of view, it is thus of interest to know — cf., e.g., [John], Theorem 7.24, for the case of *sober topological spaces* — whether or not schemes/log schemes may be *"represented"* by categories (such as $\mathrm{Sch}(X)$, $\mathrm{Sch}^{\log}(X^{\log})$). Theorems A and B provide one *natural* (though most probably *non-unique*!) *affirmative answer* to this question.

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Notations and Conventions:

We will denote by $\mathbb{N}$ the set of *natural numbers*, by which we mean the set of integers $n \geq 0$, and by $\mathbb{Z}$ the *ring of rational integers*.

Section 1: Locally Noetherian Schemes

Let X be a *locally noetherian scheme*. Let us denote by

$$\mathrm{Sch}(X)$$

the category whose *objects* are *morphisms of finite type* $Y \to X$, where Y is a *noetherian scheme*, and whose *morphisms* (from an object $Y_1 \to X$ to an object $Y_2 \to X$) are *morphisms of finite type* $Y_1 \to Y_2$ lying over X. To simplify terminology, we shall often refer to the *domain* Y of an arrow $Y \to X$ which is an object of $\mathrm{Sch}(X)$ as an "object of $\mathrm{Sch}(X)$." The purpose of the following discussion (cf. Theorem 1.7 below) is to show that the scheme X may be *reconstructed purely category-theoretically* from the category $\mathrm{Sch}(X)$.

In the following discussion, we shall often speak of various properties of objects and morphisms of $\mathrm{Sch}(X)$ as being *"category-theoretic."* By this, we mean that they are *preserved by arbitrary equivalences of categories*

$$\mathrm{Sch}(X) \xrightarrow{\sim} \mathrm{Sch}(X')$$

(where X' is another locally noetherian scheme). To simplify notation, however, we omit explicit mention of this equivalence, of X', and of the various "primed" objects and morphisms corresponding to the original objects and morphisms in $\mathrm{Sch}(X)$.

Proposition 1.1. **(Characterization of One-Pointed Schemes)** *Let us refer to schemes whose underlying topological spaces consist of precisely one element as* **one-pointed***.*

(i) If X is nonempty, then there exists an **immersion** $Y \hookrightarrow X$ *from a reduced one-pointed Y into X.*

(ii) The reduced one-pointed objects Y of $\mathrm{Sch}(X)$ *(i.e., objects which are spectra of fields) may be category-theoretically characterized as the* **"minimal objects,"** *i.e., the nonempty objects for which any monomorphism $Z \rightarrowtail Y$ in* $\mathrm{Sch}(X)$ *(where Z is nonempty) is necessarily an isomorphism.*

(iii) The one-pointed objects Y of $\mathrm{Sch}(X)$ *may be category-theoretically characterized as the objects for which there exists (up to isomorphism) precisely one monomorphism $Z \rightarrowtail Y$ in* $\mathrm{Sch}(X)$ *(namely, $Y_{\mathrm{red}} \hookrightarrow Y$) from a reduced one-pointed object Z to Y.*

Proof. We begin with *assertion* (i). By replacing X by an open affine in X, we may assume that X is the spectrum of a noetherian ring A. Then assertion (i) follows from the fact that such a ring possesses at least one maximal ideal.

Next, we turn to *assertion* (ii). Since an immersion is a monomorphism, the sufficiency portion of assertion (ii) follows formally from assertion (i). As for necessity, let $Z \rightarrowtail Y$ be a monomorphism to a reduced one-pointed Y. Thus, the diagonal $Z \to Z \times_Y Z$ is an isomorphism (since $Z \to Y$ is a monomorphism), so $Z \to Y$ is *unramified*. Since it is also clearly *flat* (since Y is the spectrum of a field), we thus conclude that $Z \to Y$ is an *étale* monomorphism. Since Z is nonempty, we thus conclude that $Z \to Y$ is an isomorphism, as desired.

Finally, we verify *assertion* (iii). Necessity follows from the existence of the monomorphism $Y_{\mathrm{red}} \rightarrowtail Y$; the fact that every monomorphism $Z \rightarrowtail Y$ (where Z is a reduced one-pointed object) necessarily factors through Y_{red}; and the necessity portion of assertion (ii). To prove sufficiency, observe that the condition stated in assertion (iii) implies that every nonempty open subscheme of Y is equal to Y (cf. assertion (i), applied to the open subscheme and its complement). In particular, the underlying topological space of Y is *connected* and of *dimension zero*. Since Y is a noetherian scheme, this implies that Y is one-pointed, as desired. ○

Corollary 1.2. **(Characterization of Smooth Morphisms)** *The smooth morphisms $Y_1 \to Y_2$ of* $\mathrm{Sch}(X)$ *may be characterized category-theoretically as those morphisms which satisfy the following property: Let $Z_0 \rightarrowtail Z$ be a monomorphism of one-pointed schemes. Then any commutative diagram*

$$\begin{array}{ccc} Z_0 & \longrightarrow & Y_1 \\ \downarrow & & \downarrow \\ Z & \longrightarrow & Y_2 \end{array}$$

admits a morphism $Z \to Y_1$ such that both of the resulting triangular diagrams commute.

Proof. One verifies immediately that a monomorphism of one-pointed schemes is necessarily a *closed immersion*. Thus, Corollary 1.2 is a formal consequence of Proposition 1.1, (i), (iii), and a well-known characterization of *smoothness* (cf., e.g., [EGA IV], Corollary 12.1.7, Proposition 17.14.2). ○

Corollary 1.3. **(Characterization of Open Immersions and Coverings)** *The* **open immersions** *$Y_1 \to Y_2$ of* $\mathrm{Sch}(X)$ *may be characterized category-theoretically as the smooth monomorphisms. A collection $Z_\alpha \to Y$ (for α ranging over the elements of some index set A) of open immersions of* $\mathrm{Sch}(X)$ *is a* **covering** *if and only if every monomorphism $P \rightarrowtail Y$, where P is a reduced, one-pointed scheme, admits a factorization through some Z_α.*

Proof. This is a formal consequence of Corollary 1.2; Proposition 1.1, (i) (applied to the complement of the union of the images of the Z_α), (ii); and [EGA IV], Theorem 17.9.1. ○

Next, let us recall that a *sober* topological space is one for which every irreducible closed subset has exactly one generic point (cf. [John], Definition 7.21, (ii)). If T is a *topological space*, then we shall denote the *category of sheaves on* T by:

$$\mathrm{Shv}(T)$$

Here, we implicitly wish to think of $\mathrm{Shv}(T)$ as a *"topos over some fixed category of sets* $\mathfrak{Ens}$*"* (cf. [John], p. 113). In fact, since the natural geometric morphism of topoi $\mathrm{Shv}(T) \to \mathfrak{Ens}$ is *unique up to canonical isomorphism* (cf. [John], Proposition 4.41), we shall take the liberty of omitting explicit mention of the "structure morphisms to $\mathfrak{Ens}$" in the following discussion.

If $\mathcal{T}_1$, $\mathcal{T}_2$ are *topoi*, then let us denote by

$$\mathfrak{Mor}(\mathcal{T}_1, \mathcal{T}_2)$$

the *category of geometric morphisms of topoi* $\mathcal{T}_1 \to \mathcal{T}_2$ (cf. [John], Definition 1.16), and by

$$\mathrm{Mor}(\mathcal{T}_1, \mathcal{T}_2)$$

the *set of isomorphism classes of geometric morphisms of topoi* $\mathcal{T}_1 \to \mathcal{T}_2$ (i.e., the set of isomorphism classes of objects of the category $\mathfrak{Mor}(\mathcal{T}_1, \mathcal{T}_2)$). Then *sober topological spaces* admit the following interesting property (cf. [John], Theorem 7.24):

Theorem 1.4. (Categorical Reconstructibility of Sober Topological Spaces) *Let* T_1*,* T_2 *be* **sober** *topological spaces. Denote by* $\mathrm{Mor}(T_1, T_2)$ *the set of continuous maps* $T_1 \to T_2$*. Then the natural morphism*

$$\mathrm{Mor}(T_1, T_2) \to \mathrm{Mor}(\mathrm{Shv}(T_1), \mathrm{Shv}(T_2))$$

is a bijection.

Corollary 1.5. (Categorical Reconstructibility of Underlying Topological Spaces) *If* Y *is an object of* $\mathrm{Sch}(X)$*, then denote the underlying topological space of* Y *by* $|Y|$*. Then* $|Y|$ *may be categorically reconstructed (up to canonical isomorphism) from the data* $(\mathrm{Sch}(X), Y)$ *(i.e., of a category and an object in this category).*

Proof. Indeed, (since $|Y|$ is *sober* — cf. [John], p. 230) this is a formal consequence of Corollary 1.3, Theorem 1.4, since the category of sheaves $\mathrm{Shv}(|Y|)$ may be reconstructed from $\mathrm{Sch}(X)$ as soon as one knows the subcategory of $\mathrm{Sch}(X)$ consisting of open immersions into Y, together with the information of which collections of open immersions are *coverings*. ○

Ultimately, we would like to reconstruct not just the topological space $|Y|$ but the *scheme structure of* Y category-theoretically from the data $(\mathrm{Sch}(X), Y)$. Thus, to do this, it remains to reconstruct the *structure sheaf* $\mathcal{O}_Y$ of Y. Since this structure sheaf is represented by the *ring scheme* $\mathbb{A}^1_Y$ (i.e., the affine line, equipped with its usual ring scheme structure) over Y, it thus suffices to show that we can reconstruct this ring scheme category-theoretically.

Proposition 1.6. (Canonical Open Subschemes of the Projective Line) *Suppose (for simplicity) that* Y *is* **connected**.

(i) The projective line $\mathbb{P}^1_Y$ *over* Y*, together with its* **labelled** *sections* 0_Y, 1_Y, ∞_Y *over* Y*, may be characterized category-theoretically (up to* **unique** *isomorphism) from the data* $(\mathrm{Sch}(X), Y)$.

(ii) The scheme $(\mathbb{G}_m)_Y$ *over* Y*, together with its group scheme structure and section* 1_Y *over* Y*, may be characterized category-theoretically (up to an isomorphism, which is* **unique** *— up to the inversion morphism on this group scheme) from the data* $(\mathrm{Sch}(X), Y)$.

(iii) The scheme $\mathbb{A}^1_Y$ *over* Y*, together with its ring scheme structure and sections* 0_Y, 1_Y *over* Y*, may be characterized category-theoretically (up to canonical isomorphism) from the data* $(\mathrm{Sch}(X), Y)$.

Proof. Note that, in light of Corollary 1.5, the *proper* morphisms of $\mathrm{Sch}(X)$ may be characterized category-theoretically as those which are *universally closed* and give rise to *closed diagonal morphisms.* Then $\mathbb{P}^1_Y$ may be characterized as the unique (up to possibly noncanonical isomorphism) *smooth* (cf. Corollary 1.2), *proper* $Z \to Y$ whose *fibers* over reduced one-pointed objects of $\mathrm{Sch}(X)$ have underlying topological spaces (cf. Corollary 1.5) which are *connected* and *one-dimensional*, and which, moreover, admit a *section* $\sigma : Y \to Z$ with the property that the *cardinality* of the Y'-linear automorphisms of the data $(Z \to Y, \sigma)$ *after base-change* to some $Y' \to Y$ *cannot be bounded* (by a finite cardinal) independently of Y'. Thus, *assertion (i)* follows from the fact that automorphisms of $\mathbb{P}^1_Y$ that fix three non-intersection sections are necessarily equal to the identity.

Assertion (ii) follows formally from assertion (i) by thinking of $(\mathbb{G}_m)_Y$ as *representing the functor* that assigns to $Y' \to Y$ the set of *automorphisms* of $\mathbb{P}^1_{Y'}$ which fix 0_Y, ∞_Y.

Finally, *assertion (iii)* follows formally from assertions (i), (ii) by observing that the *addition operation* on $\mathbb{A}^1_Y$ may be characterized as the unique morphism

$$\mathbb{A}^1_Y \times_Y \mathbb{A}^1_Y \to \mathbb{A}^1_Y$$

which has the expected restrictions to 0_Y, 1_Y and is compatible with the action of $(\mathbb{G}_m)_Y$ on all three copies of $\mathbb{A}^1_Y$. (Note that this compatibility is simply the *"distribuitivity"* property of the addition and multiplication operations of the ring structure.) ○

If $\mathcal{C}_1$, $\mathcal{C}_2$ are *categories*, then let us denote by

$$\mathfrak{Isom}(\mathcal{C}_1, \mathcal{C}_2)$$

the *category of equivalences* $\mathcal{C}_1 \xrightarrow{\sim} \mathcal{C}_2$, and by

$$\mathrm{Isom}(\mathcal{C}_1, \mathcal{C}_2)$$

the *set of isomorphism classes of equivalences* $\mathcal{C}_1 \xrightarrow{\sim} \mathcal{C}_2$ (i.e., the set of isomorphism classes of objects of the category $\mathrm{Isom}(\mathcal{C}_1, \mathcal{C}_2)$).

We are now ready to state the *main result* of the present §:

Theorem 1.7. (Categorical Reconstructibility of Locally Noetherian Schemes) *Let X, X' be* **locally noetherian schemes***.*

(i) Let $f : X \to X'$ be a **quasi-compact** *morphism of schemes. Then the* **functor**

$$\mathrm{Sch}(f) : \mathrm{Sch}(X') \to \mathrm{Sch}(X)$$

induced by base-change by f has no nontrivial automorphisms.

(ii) Denote the set of isomorphisms of schemes $X \xrightarrow{\sim} X'$ by $\mathrm{Isom}(X, X')$*. Then the natural map*

$$\mathrm{Isom}(X, X') \to \mathrm{Isom}(\mathrm{Sch}(X'), \mathrm{Sch}(X))$$

given by $f \mapsto \mathrm{Sch}(f)$ is **bijective***.*

Proof. Observe that, in *assertion (i)*, it is necessary to assume that f be *quasi-compact* in order to ensure that base-change by f preserves the property of being *noetherian*. To complete the proof of *assertion (i)*, it suffices to show that there do not exist any nontrivial collections of automorphisms $\alpha_Y : Y \xrightarrow{\sim} Y$ which are *functorial* in Y, as Y varies among the objects in the essential image $\mathrm{Im}(\mathrm{Sch}(f))$ of $\mathrm{Sch}(f)$. (Here, the functoriality is also with respect to morphisms in the image of $\mathrm{Sch}(f)$.) Let $Y \hookrightarrow X$ be an *open immersion* in this essential image $\mathrm{Im}(\mathrm{Sch}(f))$, arising, say, from an open immersion $Y' \hookrightarrow X'$ of $\mathrm{Sch}(X')$. Since $Y \hookrightarrow X$ is a *monomorphism*, it follows that α_Y is the identity. Write $P_Y \stackrel{\mathrm{def}}{=} \mathbb{P}^1_Y$. Let C_Y be the *stable curve* (cf., e.g., [DM]) over Y obtained by gluing together two copies of P_Y along the copies of 0_Y, 1_Y, ∞_Y. Then since any automorphism of C_Y necessarily *preserves the (scheme-theoretic) nodes* — thought of, for instance, as the support locus of the coherent sheaf $\mathcal{E}xt^1(\Omega_{C_Y/Y}, \mathcal{O}_{C_Y})$ (which is functorial in automorphisms of C_Y) — we conclude that α_{P_Y} fixes (scheme-theoretically) the sections 0_Y, 1_Y, ∞_Y, up to a *permutation* $\in \mathfrak{S}_3$ (the symmetric group on three letters). But by the *functoriality* of $Z \mapsto \alpha_Z$, we thus conclude that this permutation lies in the *center* of $\mathfrak{S}_3$ (which is trivial), hence that α_{P_Y} is the *identity*. From this, we conclude (by considering the evident open immersion) that $\alpha_{\mathbb{A}^1_Y}$ is the identity, hence (by considering fibered products over Y) that $\alpha_{\mathbb{A}^n_Y}$ is the identity, for all $n \geq 1$. But this implies that α_Z is the identity for any $Z = \phi(Z')$, where $Z' \to Y'$ is *affine*. Thus, by the functoriality of $Z \mapsto \alpha_Z$, we conclude that α_Z is the identity, for all objects Z of $\mathrm{Im}(\mathrm{Sch}(f))$, as desired.

Next, we turn to *assertion (ii)*. Suppose that we are given an *equivalence*:

$$\phi : \mathrm{Sch}(X') \xrightarrow{\sim} \mathrm{Sch}(X)$$

Let Y' be an object of $\mathrm{Sch}(X')$, write $Y \stackrel{\mathrm{def}}{=} \phi(Y')$. Then by Corollary 1.5 above, we obtain a natural *homeomorphism*

$$|Y| \xrightarrow{\sim} |Y'|$$

induced by ϕ, together with, by Proposition 1.6, (iii), a compatible isomorphism $\mathcal{O}_{Y'} \xrightarrow{\sim} \mathcal{O}_Y$ of structure sheaves. That is to say, we obtain an *isomorphism of schemes*

$$\phi_Y : Y \xrightarrow{\sim} Y'$$

which is *functorial* in Y'.

Now, let us observe that the objects of $\mathrm{Sch}(X)$ which are *open immersions* $U \hookrightarrow X$ *into* X may be characterized category-theoretically as follows: First of all, the objects of $\mathrm{Sch}(X)$ given by monomorphisms $Y \rightarrowtail X$ may be characterized by the property that any arrow $Z \to Y$ in $\mathrm{Sch}(X)$ is the unique arrow from Z to Y. Among such objects $Y \to X$ of $\mathrm{Sch}(X)$, the open immersions are those for which, for every $Z_0 \hookrightarrow Z$ as in Corollary 1.2, any Z_0-point of Y lifts to a (unique) Z-point of Y. Thus, by taking Y' equal to *various open subschemes* $U' \subseteq X'$, we obtain (by gluing) an isomorphism

$$\phi_X : X \xrightarrow{\sim} X'$$

which satisfies (by the functoriality of ϕ_Y): $\mathrm{Sch}(\phi_X) = \phi$ (where the "$=$" makes sense, in light of assertion (i)). Finally, when $\phi = \mathrm{Sch}(f)$ for some $f : X \xrightarrow{\sim} X'$, it is clear that $\phi_X = f$. This completes the proof. $\bigcirc$

Finally, before proceeding, we note the following *partial strengthening* of Theorem 1.7, (i):

Theorem 1.8. **(Further Rigidity Property)** *Let X be a* **locally noetherian scheme**. *Suppose that for every object $Y \to X$ of $\mathrm{Sch}(X)$, one is given an automorphism $\alpha_Y : Y \xrightarrow{\sim} Y$ —* **not necessarily over** *X! — with the property that for every morphism $Y_1 \to Y_2$ of $\mathrm{Sch}(X)$, one has a commutative diagram:*

$$\begin{array}{ccc} Y_1 & \xrightarrow{\alpha_{Y_1}} & Y_1 \\ \downarrow & & \downarrow \\ Y_2 & \xrightarrow{\alpha_{Y_2}} & Y_2 \end{array}$$

Then all of the α_Y are equal to the **identity**.

Proof. By considering morphisms as in Proposition 1.1, (i), one sees that every α_Y induces the *identity* on the *underlying topological space* $|Y|$ of Y. Next, observe that the *stable curve* C_Y of the proof of Theorem 1.7, (i), is, in fact, *defined over* $\mathbb{Z}$. Thus, there is a *natural isomorphism* $\beta_{C_Y} : C_Y \xrightarrow{\sim} C_Y$ lying over α_Y (i.e., the product over $\mathbb{Z}$ of α_Y with the identity on the evident natural model for C_Y over $\mathbb{Z}$). As in the proof of Theorem 1.7, (i), α_{C_Y} induces the automorphism α_{P_Y} of P_Y, so α_{P_Y} preserves the sections 0_Y, 1_Y, ∞_Y (up to a permutation). Moreover, just as in the proof of Theorem 1.7, (i), this permutation is *necessarily the identity.* Thus, we conclude that α_{P_Y} is equal to the isomorphism β_{P_Y} induced on P_Y by β_{C_Y}, hence that $\alpha_{\mathbb{A}^1_Y}$ is equal to the isomorphism $\beta_{\mathbb{A}^1_Y}$ induced on $\mathbb{A}^1_Y$ by β_{P_Y}. But, by considering sections of $\mathbb{A}^1_Y \to Y$ (i.e., morphisms $Y \to \mathbb{A}^1_{\mathbb{Z}}$), this implies that α_Y induces the identity not only on $|Y|$, but also on *sections of the structure sheaf* $\mathcal{O}_Y$, i.e., that α_Y is the identity, as desired. ○

Section 2: Log Structures

In this §, we discuss the *logarithmic analogue* of the theory of §1. Let $X^{\log}$ be a *fine* (cf. [Kato1], §2.3) *saturated* (cf. (the evident étale generalization of) [Kato2], §1.5) *log scheme* (cf. [Kato1], §1.2) whose underlying scheme X is *locally noetherian*. Let us denote by

$$\mathrm{Sch}^{\log}(X^{\log})$$

the category whose *objects* are *morphisms of fine saturated log schemes* $Y^{\log} \to X^{\log}$, where Y is a *noetherian scheme*, and the underlying morphism of schemes $Y \to X$ is *of finite type*, and whose *morphisms* (from an object $Y_1^{\log} \to X^{\log}$ to an object $Y_2^{\log} \to X^{\log}$) are *morphisms of finite type* $Y_1^{\log} \to Y_2^{\log}$ lying over $X^{\log}$. To simplify terminology, we shall often refer to the *domain* $Y^{\log}$ of an arrow $Y^{\log} \to X^{\log}$ which is an object of $\mathrm{Sch}^{\log}(X^{\log})$ as an "object of $\mathrm{Sch}^{\log}(X^{\log})$." Note that by associating to an object $Y \to X$ of $\mathrm{Sch}(X)$ the object $Y^{\log} \to X^{\log}$ of $\mathrm{Sch}^{\log}(X^{\log})$ obtained by equipping Y with the log structure obtained by pulling back the log structure on $X^{\log}$ via $Y \to X$, we obtain an *embedding*

$$\mathrm{Sch}(X) \hookrightarrow \mathrm{Sch}^{\log}(X^{\log})$$

— which thus allows us to regard $\mathrm{Sch}(X)$ as a *subcategory* of $\mathrm{Sch}^{\log}(X^{\log})$.

Let $Y^{\log}$ be a *log scheme*. Then we shall denote its *underlying scheme* (respectively, the *morphism of monoids defining its log structure*) by Y (respectively, $\exp_Y : M_Y \to \mathcal{O}_Y$). Thus, we have an *exact sequence of étale monoids on* Y

$$0 \to \mathcal{O}_Y^{\times} \to M_Y \to P_Y \to 0$$

— where the *"characteristic"* P_Y is defined so as to make the sequence exact. If $Y^{\log}$ is *fine* (hence *integral*), then we have *injections*

$$P_Y \hookrightarrow P_Y^{\mathrm{gp}}; \quad M_Y \hookrightarrow M_Y^{\mathrm{gp}}$$

(where the superscript "gp" denotes the group associated to the monoid in question).

If Y is *reduced* (respectively, *one-pointed* — cf. Proposition 1.1), then we shall say that $Y^{\log}$ is *reduced* (respectively, *one-pointed*). Suppose that $Y^{\log}$ is *reduced and one-pointed*, i.e., Y is equal to the spectrum of a field k. Then one may think of P_Y as the data of a (discrete) monoid equipped with a continuous action of the *absolute Galois group* G_k of k. When this action is *trivial*, we shall say that the log structure on $Y^{\log}$ is *split*. In this case, we shall denote (by abuse of notation) $\Gamma(Y, P_Y)$ by P_Y.

Proposition 2.1. **(Local Structure of Monoids)** *Let $Y^{\log}$ be a reduced, one-pointed fine saturated log scheme with split log structure. Then P_Y is a finitely generated, torsion-free, integral saturated monoid, with no nonzero invertible elements. In particular, P_Y^{gp} is a finitely generated torsion-free abelian group.*

Proof. Indeed, since *torsion elements* of P_Y are necessarily *invertible*, and the properties *"finitely generated," "integral,"* and *"saturated"* follow from the definitions, it suffices to verify that P_Y has no nonzero invertible elements. Suppose that $f_P \in P_Y$ is invertible. Then since $\mathcal{O}_Y^{\times}$ is a group, any lifting (which exists étale locally on Y) f_M of f_P to M_Y is invertible. On the other hand, by the definition of a log structure (cf. [Kato1], §1.2), the invertibility of $\exp_Y(f_M)$ implies that $f_M \in \mathcal{O}_Y^{\times}$, so $f_P = 0$, as desired. ○

Lemma 2.2. **(Pointwise Nature of Log Structures)** *Let $Z^{\log} \to Y^{\log}$ be a morphism of fine log schemes. Let $\Sigma \subseteq |Z|$ be a subset of the underlying topological space $|Z|$ of Z such that every open subset of $|Z|$ containing Σ is equal to $|Z|$ itself. Suppose further that $Z \xrightarrow{\sim} Y$, and that for every geometric point ζ of Z whose image in Z lies in Σ, we have $P_{Y,\zeta} \xrightarrow{\sim} P_{Z,\zeta}$. Then $Z^{\log} \to Y^{\log}$ is an isomorphism.*

Proof. Indeed, since we have $P_{Y,\zeta} = M_{Y,\zeta}/\mathcal{O}_{Y,\zeta}^{\times}$, $P_{Z,\zeta} = M_{Z,\zeta}/\mathcal{O}_{Z,\zeta}^{\times}$, it follows that $P_{Y,\zeta} \xrightarrow{\sim} P_{Z,\zeta}$ implies that $M_{Y,\zeta} \xrightarrow{\sim} M_{Z,\zeta}$. Thus, our hypothesis on Σ, together with the *coherence* of the log structures involved implies that $M_Y \xrightarrow{\sim} M_Z$, as desired. ○

Proposition 2.3. **(Monomorphisms of Fine Saturated Log Schemes)** *A morphism $Z^{\log} \to Y^{\log}$ in $\mathrm{Sch}^{\log}(X^{\log})$ is a* **monomorphism** *(in $\mathrm{Sch}^{\log}(X^{\log})$) if and only if $Z \to Y$ is a monomorphism in the category of schemes, and, moreover, for every geometric point ζ of Z, the induced morphism $P_{Y,\zeta}^{\mathrm{gp}} \to P_{Z,\zeta}^{\mathrm{gp}}$ is surjective.*

Proof. *Sufficiency* is a formal consequence of the definitions (and the fact that for any fine log scheme $S^{\log}$, $M_S \to M_S^{\mathrm{gp}}$ is *injective*). Moreover, the *necessity* of the condition that $Z \to Y$ be a monomorphism (in $\mathrm{Sch}(X)$, which is easily verified to be the same as a monomorphism in the category of all schemes) is a formal consequence of the definitions. Thus, to complete the proof of necessity, it suffices — by applying an appropriate base-change — to consider the case where $Z = Y = \mathrm{Spec}(k)$ (where k is a field) and the log structures on $Z^{\log}$, $Y^{\log}$ are *split*. If $P_Y^{\mathrm{gp}} \to P_Z^{\mathrm{gp}}$ *fails to be surjective*, then there exists an *artinian k-algebra A* (of finite type), together with a *nontrivial character* $\chi : P_Z^{\mathrm{gp}}/P_Y^{\mathrm{gp}} \to \boldsymbol{\mu}_N(A) \stackrel{\mathrm{def}}{=} \{a \in A^{\times} \mid a^N = 1\}$ (for some integer $N \geq 1$). If we equip $W \stackrel{\mathrm{def}}{=} \mathrm{Spec}(A)$ with the log structure pulled back from $Z^{\log}$, then we obtain a morphism $W^{\log} \to Z^{\log}$ in $\mathrm{Sch}^{\log}(X^{\log})$. In particular, since $P_Z \xrightarrow{\sim} P_W$, we may think of χ as a character on M_W^{gp} which vanishes on M_Y^{gp}. Thus, the automorphism $\alpha : W^{\log} \to W^{\log}$ which is the identity on W and which maps a section $f \in M_W$ to $f \cdot \chi(f)$ has a *nontrivial composite* with $W^{\log} \to Z^{\log}$, but a *trivial composite* with $W^{\log} \to Z^{\log} \to Y^{\log}$. This shows that $Z^{\log} \to Y^{\log}$ is not a monomorphism, thus completing the proof of necessity. ○

Proposition 2.4. **(Minimal Objects)** *An object* $Y^{\log}$ *of* $\mathrm{Sch}^{\log}(X^{\log})$ *will be called* **minimal** *if it is nonempty and satisfies the property that any monomorphism* $Z^{\log} \rightarrowtail Y^{\log}$ *(where* $Z^{\log}$ *is nonempty) in* $\mathrm{Sch}^{\log}(X^{\log})$ *is necessarily an isomorphism.*

(i) A minimal object $Y^{\log}$ *is necessarily reduced and one-pointed.*

(ii) If $Y^{\log}$ *is reduced and one-pointed, and its log structure is* **trivial**, *then* $Y^{\log}$ *is minimal.*

(iii) If $Y^{\log}$ *is reduced and one-pointed, and its log structure is given by the chart* $\mathbb{N} \ni 1 \mapsto 0$, *then* $Y^{\log}$ *is minimal.*

Proof. Assertions (i) and (ii) are a formal consequence of the definitions; Proposition 1.1, (ii); and Proposition 2.3. Assertion (iii) is a formal consequence of the definitions; Proposition 1.1, (ii); Proposition 2.3; and the following elementary observation: Any quotient of integral monoids $\mathbb{N} \twoheadrightarrow Q$ for which Q has no nonzero invertible elements (cf. Proposition 2.1), and $\mathbb{N} \ni 1$ does not map to $0 \in Q$, is an *isomorphism.* ○

Before proceeding, we review the following well-known

Lemma 2.5. **(Monoids and Cones)** *Let* $V_{\mathbb{Z}}$ *be a finitely generated free* $\mathbb{Z}$*-module. Write* $V_{\mathbb{R}} \stackrel{\mathrm{def}}{=} V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}$*;* $V_{\mathbb{Z}}^{\vee} \stackrel{\mathrm{def}}{=} \mathrm{Hom}_{\mathbb{Z}}(V_{\mathbb{Z}}, \mathbb{Z})$*;* $V_{\mathbb{R}}^{\vee} \stackrel{\mathrm{def}}{=} V_{\mathbb{Z}}^{\vee} \otimes_{\mathbb{Z}} \mathbb{R}$*. If* $\sigma \subseteq V_{\mathbb{R}}^{\vee}$ *is a* **strongly convex rational polyhedral cone** *(cf., e.g., [Oda], p. 5), then let us write* $\sigma^{\vee} \stackrel{\mathrm{def}}{=} \{v \in V_{\mathbb{R}} \mid \langle v, w\rangle \geq 0, \ \forall\ w \in \sigma\}$*. Then:*

(i) The correspondence $\sigma \mapsto P \stackrel{\mathrm{def}}{=} \sigma^{\vee} \bigcap V_{\mathbb{Z}}$ *defines a bijection between the set of strongly convex rational polyhedral cones* σ *in* $V_{\mathbb{R}}^{\vee}$ *and the set of finitely generated saturated monoids* $P \subseteq V_{\mathbb{Z}}$ *which generate* $V_{\mathbb{Z}}$ *as a group.*

(ii) Let $P \subseteq V_{\mathbb{Z}}$ *be a finitely generated monoid which generates* $V_{\mathbb{Z}}$ *as a group. Then its saturation* $P^{\mathrm{sat}} \stackrel{\mathrm{def}}{=} \{v \in V_{\mathbb{Z}} = P^{\mathrm{gp}} \mid \exists\ n \in \mathbb{Z}_{>0} \text{ such that } n \cdot v \in P\}$ *is also finitely generated.*

(iii) Let $P \subseteq V_{\mathbb{Z}}$ *be a finitely generated saturated monoid which generates* $V_{\mathbb{Z}}$ *as a group and satisfies* $P \bigcap (-P) = 0$*; let* $a, b \in P$ *be* **distinct** *elements. Then there exists a morphism of monoids*

$$\phi : P \to \mathbb{N}$$

such that $\phi(a) \neq \phi(b)$*, and* $\phi(c) \neq 0$*, for all nonzero* $c \in P$*.*

(iv) Let $P_1 \subseteq P_2$ $(\subseteq V_{\mathbb{Z}})$ *be finitely generated saturated monoids which generate* $V_{\mathbb{Z}}$ *as a group and satisfy* $P_1 \neq P_2$, $P_1 \bigcap (-P_1) = 0$*. Then there exists a morphism of monoids*

$$\phi : P_1 \to \mathbb{N}$$

such that $\phi(c) \neq 0$, *for all nonzero* $c \in P_1$, *and the induced map* $\phi^{\mathrm{gp}} : P_1^{\mathrm{gp}} = V_{\mathbb{Z}} \to \mathbb{Z}$ *satisfies* $\phi^{\mathrm{gp}}(P_2) \not\subseteq \mathbb{N}$.

Proof. Indeed, *assertion (i)* is stated in [Oda], p. 9. *Assertion (ii)* follows, for instance, from the fact that the *normalization* of $\mathbb{C}[P]$ is *finite* (by elementary commutative algebra) over $\mathbb{C}[P]$ and also clearly of the form $\mathbb{C}[Q]$ for some monoid $Q \in V_{\mathbb{Z}}$. Since by a well-known argument (cf. [Oda], p. 14), we have $Q = P^{\mathrm{sat}}$, this finiteness implies the finite generation of P^{sat}.

Next, we turn to *assertion (iii)*. The condition $P \bigcap (-P) = 0$ implies that the cone σ corresponding to P (cf. (i)) *generates* $V_{\mathbb{R}}^{\vee}$. In particular, there exists an element $\lambda \in \sigma$ such that $\lambda(b - a) \neq 0$. Moreover, λ can be chosen to be *lie in the interior of* σ — which means that σ contains an open ball in $V_{\mathbb{R}}^{\vee}$ containing λ — and (since $V_{\mathbb{Q}}^{\vee}$ is dense in $V_{\mathbb{R}}^{\vee}$) to be *integral* (i.e., $\lambda \in V_{\mathbb{Z}}^{\vee}$). These conditions imply that the map $\lambda : V_{\mathbb{Z}} \to \mathbb{Z}$ determined by λ maps nonzero elements of P to nonzero elements of $\mathbb{N}$ (and a, b to distinct elements), hence defines a map $\phi : P \to \mathbb{N}$ with the desired properties.

Finally, we condition *assertion (iv)*. Let $\sigma_2 \subseteq \sigma_1$ be the cones corresponding to P_1, P_2, respectively. Since $\sigma_1 \not\subseteq \sigma_2$, and both σ_1, σ_2 are *closed* subsets of $V_{\mathbb{R}}^{\vee}$, we conclude that the interior of σ_1 is not contained in σ_2, hence that there exists an element λ in the *interior* of σ_1 such that $\lambda \notin \sigma_2$. Moreover, (since $V_{\mathbb{Q}}^{\vee}$ is dense in $V_{\mathbb{R}}^{\vee}$) λ may be chosen to belong to $V_{\mathbb{Z}}^{\vee}$. Since $P_1 \bigcap (-P_1) = 0$, it follows that the interior of σ_1 is *open* in $V_{\mathbb{R}}^{\vee}$, so the map $\lambda : V_{\mathbb{Z}} \to \mathbb{Z}$ maps nonzero elements of P_1 to nonzero elements of $\mathbb{N}$. Since $\lambda \notin \sigma_2$, it follows that there exist elements of P_2 on which λ takes *negative values*. Thus, λ defines a map $\phi : P_1 \to \mathbb{N}$ with the desired properties. ○

Lemma 2.6. **(Fibered Products of Fine Saturated Log Schemes)** *Let* $Y_1^{\log} \to Z^{\log}$, $Y_2^{\log} \to Z^{\log}$ *be morphisms in* $\mathrm{Sch}^{\log}(X^{\log})$. *Write*

$$W_{\mathrm{Sch}^{\log}}^{\log} \stackrel{\mathrm{def}}{=} Y_1^{\log} \times_{Z^{\log}}^{\mathrm{Sch}^{\log}} Y_2^{\log}; \quad W_{\mathrm{fs}}^{\log} \stackrel{\mathrm{def}}{=} Y_1^{\log} \times_{Z^{\log}}^{\mathrm{fs}} Y_2^{\log}$$
$$W_{\mathrm{fine}}^{\log} \stackrel{\mathrm{def}}{=} Y_1^{\log} \times_{Z^{\log}}^{\mathrm{fine}} Y_2^{\log}; \quad W_{\log}^{\log} \stackrel{\mathrm{def}}{=} Y_1^{\log} \times_{Z^{\log}}^{\log} Y_2^{\log}$$

for the fibered products in the category $\mathrm{Sch}^{\log}(X^{\log})$*; the category of all fine, saturated log schemes; the category of all fine log schemes; the category of all log schemes, respectively. Then:*

(i) We have natural morphisms

$$W_{\mathrm{Sch}^{\log}}^{\log} \to W_{\mathrm{fs}}^{\log} \to W_{\mathrm{fine}}^{\log} \to W_{\log}^{\log}$$

of which the first is an **isomorphism**. *Moreover,* $W_{\log}$ *may be identified with the fibered product* $Y_1 \times_Z Y_2$ *in the category of schemes.*

(ii) The underlying morphism of schemes $W_{\mathrm{fine}} \to W_{\log}$ *of the third morphism of (i) is a* **closed immersion** *which induces an* **isomorphism** $(W_{\mathrm{fine}})_{\mathrm{red}} \to (W_{\log})_{\mathrm{red}}$. *The underlying morphism of monoids* $P_{W_{\log}} \to P_{W_{\mathrm{fine}}}$ *is surjective.*

(iii) The underlying morphism of schemes $W_{\mathrm{fine}} \to W_{\mathrm{fs}}$ of the second morphism of (i) is **finite** *and* **surjective**. *The underlying morphism of monoids $P_{W_{\mathrm{fine}}} \to P_{W_{\mathrm{fs}}}$ is injective and induces an isomorphism $P^{\mathrm{sat}}_{W_{\mathrm{fine}}} \xrightarrow{\sim} P_{W_{\mathrm{fs}}}$.*

Proof. *Assertion (i)* (except for the isomorphism assertion) is a formal consequence of the definitions and [Kato1], §1.6. *Assertion (ii)* is a formal consequence of the definitions and [Kato1], §2.6, 2.8 (cf. also the proof of [Kato1], Proposition 2.7). *Assertion (iii)* (together with the isomorphism assertion of (i)) follows by applying the same argument as that of the proof of [Kato1], Proposition 2.7, to the *"saturation"* of a chart, i.e., if $P \to M_{W_{\mathrm{fine}}}$ is a local chart for $W^{\log}_{\mathrm{fine}}$, then $W^{\log}_{\mathrm{fs}}$ is obtained (étale locally) from $W^{\log}_{\mathrm{fine}}$ by base-changing by $\mathbb{Z}[P] \to \mathbb{Z}[P^{\mathrm{sat}}]$, where $\mathrm{Spec}(\mathbb{Z}[P])$ (respectively, $\mathrm{Spec}(\mathbb{Z}[P^{\mathrm{sat}}])$) is regarded as being equipped with the log structure associated to the evident pre-log structure $P \to \mathbb{Z}[P]$ (respectively, $P^{\mathrm{sat}} \to \mathbb{Z}[P^{\mathrm{sat}}]$). ○

Proposition 2.7. **(Minimal Hulls)** *Let $Y^{\log}$ be a* **one-pointed object** *of the category $\mathrm{Sch}^{\log}(X^{\log})$. Then a monomorphism $H^{\log} \rightarrowtail Y^{\log}$ will be called a* **hull** *for $Y^{\log}$ if every morphism $S^{\log} \to Y^{\log}$ from a* **minimal** *object $S^{\log}$ to $Y^{\log}$ factors (necessarily uniquely!) though $H^{\log}$. A hull $H^{\log} \rightarrowtail Y^{\log}$ will be called a* **minimal hull** *if every monomorphism $H_1^{\log} \rightarrowtail H^{\log}$ for which the composite $H_1^{\log} \rightarrowtail Y^{\log}$ is a hull is necessarily an isomorphism. A one-pointed object $H^{\log}$ will be called a* **minimal hull** *if the identity morphism $H^{\log} \to H^{\log}$ is a minimal hull for $H^{\log}$.*

(i) There exists a morphism $S^{\log} \to Y^{\log}$, for some minimal object $S^{\log}$.

(ii) If $H_1^{\log} \rightarrowtail Y^{\log}$, $H_2^{\log} \rightarrowtail Y^{\log}$ are hulls, then the product $H_1^{\log} \times_{Y^{\log}} H_2^{\log}$ (in $\mathrm{Sch}^{\log}(X^{\log})$) is also a hull. In particular, any two minimal hulls are isomorphic (via a unique isomorphism over $Y^{\log}$).

(iii) $Y^{\log}_{\mathrm{red}} \rightarrowtail Y^{\log}$ is a minimal hull.

Proof. First, observe that *assertion (ii)* is a formal consequence of the definitions.

Next, recall that by Proposition 2.4, (i), every minimal object is reduced and one-pointed. Thus, every morphism $S^{\log} \to Y^{\log}$ (where $S^{\log}$ is minimal) factors (uniquely) though $Y^{\log}_{\mathrm{red}}$. In particular, $Y^{\log}_{\mathrm{red}} \rightarrowtail Y^{\log}$ is a *hull* for $Y^{\log}$. Thus, for the remainder of the proof, we may assume (without loss of generality) that $Y^{\log}$ is *reduced*. By Proposition 2.4, (ii), we may also assume (without loss of generality) that the log structure on $Y^{\log}$ is *nontrivial*.

Next, let us observe that if $Y' \to Y$ is *finite étale*, then the result $H^{\log} \times_Y Y' \rightarrowtail Y^{\log} \times_Y Y'$ of base-changing a *hull* $H^{\log} \rightarrowtail Y^{\log}$ is again a hull. Indeed, this follows immediately from the definitions. In particular, to show that any (reduced) $Y^{\log}$ is a minimal hull, we may assume (without loss of generality) that the log structure on $Y^{\log}$ is *split*.

Now let $H^{\log} \rightarrowtail Y^{\log}$ be a *hull*. Then I *claim* that the morphism $P^{\mathrm{gp}}_Y \to P^{\mathrm{gp}}_H$ is *bijective*. Indeed, *surjectivity* follows from Proposition 2.3, while *injectivity* follows from the following observation: By Lemma 2.5, (iii) (cf. also Proposition 2.1), for any two distinct elements $a, b \in P_Y$, there exists a *morphism of monoids*

$$P_Y \to \mathbb{N}$$

that maps a, b to *distinct elements* of $\mathbb{N}$ and nonzero elements of P_Y to nonzero elements of $\mathbb{N}$. In particular, this morphism of monoids determines a morphism from a *minimal object* of the type described in Proposition 2.4, (iii), to $Y^{\log}$. Thus, the desired injectivity follows from the existence of such a morphism (and the definition of a hull). Note that the existence of such a morphism also completes the proof of *assertion (i)*.

Thus, to complete the proof of *assertion (iii)*, it suffices to show that the injection $P_Y \hookrightarrow P_H$ is *surjective*. But this follows by constructing a suitable morphism from a *minimal object* of the type described in Proposition 2.4, (iii), to $Y^{\log}$ (cf. the preceding paragraph), by applying Lemma 2.5, (iv). ○

Corollary 2.8. **(Classification of Split Minimal Objects)** *Every minimal object with split log structure is one of the two types described in Proposition 2.4, (ii), (iii).*

Proof. Indeed, this follows by constructing a suitable morphism from a *minimal object* of the type described in Proposition 2.4, (iii), to $Y^{\log}$ (cf. the proof of Proposition 2.7), by applying Lemma 2.5, (iii). ○

Corollary 2.9. **(Characterization of One-Pointed Objects)** *The one-pointed objects $Y^{\log}$ of $\mathrm{Sch}^{\log}(X^{\log})$ may be characterized category-theoretically as the nonempty objects which satisfy the following property: For any two morphisms $S_i^{\log} \to Y^{\log}$ (for $i = 1, 2$), where $S_i^{\log}$ is a* **minimal** *object, the product $S_1^{\log} \times_{Y^{\log}} S_2^{\log}$ (in $\mathrm{Sch}^{\log}(X^{\log})$) is* **nonempty**.

Proof. If an object $Y^{\log}$ of $\mathrm{Sch}^{\log}(X^{\log})$ is *not* one-pointed, then by Proposition 1.1, (iii), there exist *non-isomorphic monomorphisms* $T_1, T_2 \rightarrowtail Y$ (in $\mathrm{Sch}(X)$), for some reduced one-pointed T_1, T_2. Thus, T_1, T_2 necessarily have *non-intersecting images* in Y. If we equip T_1, T_2 with the log structures pulled back from Y, then we obtain morphisms $T_1^{\log}, T_2^{\log} \to Y^{\log}$ in $\mathrm{Sch}^{\log}(X^{\log})$ such that $T_1^{\log} \times_{Y^{\log}} T_2^{\log}$ is *empty* (cf. Proposition 2.6, (i), (ii), (iii)). Thus, if we choose (for $i = 1, 2$) a morphism $S_i^{\log} \to T_i^{\log}$, where $S_i^{\log}$ is *minimal* — cf. Proposition 2.7, (i) — we obtain a contradiction to the condition of Corollary 2.9.

Conversely, if an object $Y^{\log}$ of $\mathrm{Sch}^{\log}(X^{\log})$ *is* one-pointed, and $S_i^{\log} \to Y^{\log}$ (for $i = 1, 2$) are as in the statement of Corollary 2.9, then by Proposition 2.6, (i), (ii), (iii), $S_1^{\log} \times_{Y^{\log}} S_2^{\log}$ is *nonempty*, as desired. ○

Corollary 2.10. **(Characterization of Reduced One-Pointed Objects)** *The reduced one-pointed objects* $Y^{\log}$ *of* $\mathrm{Sch}^{\log}(X^{\log})$ *may be characterized category-theoretically as the one-pointed objects which are* **minimal hulls** *(cf. Proposition 2.7).*

Proof. Indeed, this is a formal consequence of Proposition 2.7, (iii), and Corollary 2.9. ○

Definition 2.11.

(i) A morphism $Y_1^{\log} \to Y_2^{\log}$ of objects of $\mathrm{Sch}^{\log}(X^{\log})$ will be called *log-like* if the underlying morphism of schemes $Y_1 \to Y_2$ is an isomorphism.

(ii) A morphism $Y_1^{\log} \to Y_2^{\log}$ of objects of $\mathrm{Sch}^{\log}(X^{\log})$ will be called *scheme-like* if the log structure on $Y_1^{\log}$ is the pull-back of the log structure on $Y_2^{\log}$ via the underlying morphism of schemes $Y_1 \to Y_2$.

Corollary 2.12. **(Characterization of Log-like and Scheme-like Morphisms of Reduced One-Pointed Objects)** *Let* $Y_1^{\log} \to Y_2^{\log}$ *be a morphism of reduced one-pointed objects of* $\mathrm{Sch}^{\log}(X^{\log})$*. Then we have the following* **category-theoretic criteria** *for this morphism to be log-like/scheme-like:*

(i) $Y_1^{\log} \to Y_2^{\log}$ *is* **log-like** *if and only if it factors as the composite of a monomorphism* $Y_1^{\log} \rightarrowtail Y_3^{\log}$*, where* $Y_3^{\log}$ *is a reduced one-pointed object of* $\mathrm{Sch}^{\log}(X^{\log})$*, with a morphism* $Y_3^{\log} \to Y_2^{\log}$ *which admits a section* $Y_2^{\log} \to Y_3^{\log}$ *(i.e., such that* $Y_2^{\log} \to Y_3^{\log} \to Y_2^{\log}$ *is the identity).*

(ii) $Y_1^{\log} \to Y_2^{\log}$ *is* **scheme-like** *if and only if the* **category of factorizations** $Y_1^{\log} \to Y_3^{\log} \to Y_2^{\log}$*, where* $Y_3^{\log}$ *is a reduced one-pointed object of* $\mathrm{Sch}^{\log}(X^{\log})$*, and* $Y_1^{\log} \to Y_3^{\log}$ *is* **log-like** *— i.e., whose objects are such factorizations and whose morphisms are commutative diagrams*

$$\begin{array}{ccccc} Y_1^{\log} & \longrightarrow & Y_3^{\log} & \longrightarrow & Y_2^{\log} \\ \downarrow & & \downarrow & & \downarrow \\ Y_1^{\log} & \longrightarrow & Y_4^{\log} & \longrightarrow & Y_2^{\log} \end{array}$$

of morphisms in $\mathrm{Sch}(X^{\log})$*, of which* $Y_1^{\log} \to Y_1^{\log}$ *and* $Y_2^{\log} \to Y_2^{\log}$ *are the identity morphisms — admits the factorization* $Y_1^{\log} = Y_1^{\log} \to Y_2^{\log}$ *as a* **terminal object***.*

Proof. First, we consider *assertion (i)*. The *sufficiency* of the given condition follows from Proposition 2.3 (and the fact that the section $Y_2^{\log} \to Y_3^{\log}$ is necessarily a monomorphism). To prove *necessity*, we construct $Y_3^{\log}$ as follows. First, let $Y' \to Y \stackrel{\mathrm{def}}{=} Y_1 \cong Y_2$ be a finite étale Galois covering (with Y' connected) so that the pull-backs of the log structures on $Y_1^{\log}$, $Y_2^{\log}$ to Y' are *split*. Write $G \stackrel{\text{def}}{=} \mathrm{Gal}(Y'/Y)$. Suppose that $P_{Y_1 \times_Y Y'}$ is generated by r elements. Write

$$P_G$$

for the monoid given by taking the direct product of $[Y' : Y]$ copies of $\mathbb{N}$, one indexed by each element of G. Thus, G acts naturally on P_G. If we equip the scheme Y' with the log structure defined by the pre-log structure which sends nonzero elements of P_G^r (the direct product of r copies of P_G) to $0 \in \mathcal{O}_{Y'}$ and then *descend* (by using the G-action on P_G^r) to Y, we obtain a *log scheme* $Y_4^{\log}$ such that $P_{Y_4 \times_Y Y'} = P_G^r$.

Next, observe that (by the definition of r) there exists a *G-equivariant surjection of monoids*:

$$P_G^r \twoheadrightarrow P_{Y_1 \times_Y Y'}$$

Thus, by descent, this surjection determines a *monomorphism* $Y_1^{\log} \rightarrowtail Y_4^{\log}$.

Now set $Y_3^{\log} \stackrel{\text{def}}{=} Y_4^{\log} \times_Y Y_2^{\log}$. Note that this product yields the *same log scheme*, whether taken in $\mathrm{Sch}^{\log}(X^{\log})$ or in the category of all log schemes. Moreover, the morphisms $Y_1^{\log} \to Y_2^{\log}$, $Y_1^{\log} \rightarrowtail Y_4^{\log}$ determine a *monomorphism*

$$Y_1^{\log} \rightarrowtail Y_3^{\log}$$

whose composite with the projection $Y_3^{\log} \to Y_2^{\log}$ is the original morphism $Y_1^{\log} \to Y_2^{\log}$. Thus, to complete the proof of assertion (i), it suffices to prove the existence of a section of the projection $Y_3^{\log} \to Y_2^{\log}$. But the existence of such a section follows from the (readily verified) existence of a *G-equivariant morphism of monoids*:

$$P_G^r \to P_{Y_2 \times_Y Y'}$$

(which is not necessarily surjective).

Finally, we verify *assertion (ii)*. Consider the factorization $Y_1^{\log} \to Y_4^{\log} \to Y_2^{\log}$, where $Y_4 \stackrel{\text{def}}{=} Y_1$, and its log structure is the log structure pulled back from $Y_2^{\log}$. One checks easily that *this* factorization is a terminal object in the category of factorizations and that $Y_4^{\log} \to Y_2^{\log}$ is (by construction) *scheme-like*. If $Y_1^{\log} \to Y_2^{\log}$ is *scheme-like*, then it follows from the definitions that the factorization $Y_1^{\log} \to Y_4^{\log} \to Y_2^{\log}$ is isomorphic to the factorization $Y_1^{\log} = Y_1^{\log} \to Y_2^{\log}$. This proves *necessity*. On the other hand, if the factorization $Y_1^{\log} = Y_1^{\log} \to Y_2^{\log}$ is a *terminal object*, then since a terminal object is *unique* (up to isomorphism), we thus conclude that $Y_1^{\log} = Y_1^{\log} \to Y_2^{\log}$ is isomorphic to $Y_1^{\log} \to Y_4^{\log} \to Y_2^{\log}$, so $Y_1^{\log} \to Y_2^{\log}$ is *scheme-like*. This proves *sufficiency*. ○

Corollary 2.13. (Characterization of Restriction of the Log Structure to a Point) *Let $S^{\log} \rightarrowtail Y^{\log}$ be a monomorphism of objects of* $\mathrm{Sch}^{\log}(X^{\log})$*. Suppose that $S^{\log}$ is reduced and one-pointed. Then $S^{\log} \to Y^{\log}$ is* **scheme-like** *if and only if it is a* **terminal object** *among the arrows $T^{\log} \to Y^{\log}$ over $Y^{\log}$ for which $T^{\log}$ is reduced and one-pointed, and, moreover, $T^{\log} \times_{Y^{\log}} S^{\log}$ is nonempty.*

Proof. Indeed, this is a formal consequence of the observation that the condition that $T^{\log} \times_{Y^{\log}} S^{\log}$ be *nonempty* is equivalent to the condition that T and S have the *same image* in Y (cf. Corollary 2.9). ○

Corollary 2.14. (Characterization of Arbitrary Scheme-like Morphisms) *Let $Y_1^{\log} \to Y_2^{\log}$ be a morphism of objects of* $\mathrm{Sch}^{\log}(X^{\log})$*. Then we have the following* **category-theoretic criterion** *for this morphism to be scheme-like: $Y_1^{\log} \to Y_2^{\log}$ is* **scheme-like** *if and only if for every commutative diagram*

$$\begin{array}{ccc} S_1^{\log} & \rightarrowtail & Y_1^{\log} \\ \downarrow & & \downarrow \\ S_2^{\log} & \rightarrowtail & Y_2^{\log} \end{array}$$

of morphisms in $\mathrm{Sch}(X^{\log})$*, where (for $i = 1, 2$) $S_i^{\log}$ is reduced and one-pointed, and the horizontal morphisms are* **scheme-like** *monomorphisms, it holds that the morphism $S_1^{\log} \to S_2^{\log}$ is also* **scheme-like***.*

Proof. Indeed, this is a formal consequence of Corollaries 2.12, (ii); 2.13; Lemma 2.2; and Proposition 1.1, (i) (which implies that any open subset of Y_1 that contains the images of all scheme-like monomorphisms $S_1^{\log} \to Y_1^{\log}$ is equal to Y_1). ○

Corollary 2.15. (Reconstruction of the Underlying Scheme) *Let $Y^{\log}$ be an object of* $\mathrm{Sch}^{\log}(X^{\log})$*. Then the functor*

$$\mathrm{Sch}(Y) \to \mathrm{Sch}^{\log}(Y^{\log}) \qquad (\subseteq \mathrm{Sch}^{\log}(X^{\log}))$$
$$\{Z \to Y\} \to \{Z^{\log} \to Y^{\log}\}$$

— defined by equipping Z with the log structure pulled back from Y — determines an **equivalence of categories** *between* $\mathrm{Sch}(Y)$ *and the subcategory of* $\mathrm{Sch}^{\log}(Y^{\log})$ *of objects $Z^{\log} \to Y^{\log}$ for which the morphism $Z^{\log} \to Y^{\log}$ is* **scheme-like** *and morphisms $Z_1^{\log} \to Z_2^{\log}$ (over $Y^{\log}$) which are scheme-like. In particular, (cf. Theorem 1.7) one can reconstruct the scheme Y category-theoretically from the data* $(\mathrm{Sch}^{\log}(X^{\log}), Y^{\log})$ *(i.e., of a category and an object in the category) in a fashion which is* **functorial** *in $Y^{\log}$.*

Proof. This is a formal consequence of Corollary 2.14 and Theorem 1.7. ○

Remark 2.15.1. Note that the above proof of Corollary 2.15 furnishes an *interesting application* of Theorem 1.7, i.e., an interesting instance of a *natural situation* in which the category "$\mathrm{Sch}(Y)$" may appear *"in disguise"* (i.e., as a certain subcategory of $\mathrm{Sch}^{\log}(Y^{\log})$). Another (similar) example of the category $\mathrm{Sch}(Y)$ appearing in disguise is the (classical) *theory of* (say, faithfully flat) *descent*: Indeed, suppose that Y is an S-scheme of finite type (where S is noetherian), and that $T \to S$ is *faithfully flat*. Then $\mathrm{Sch}(Y)$ "appears in disguise" as the *category of objects of* $\mathrm{Sch}(Y \times_S T)$ *equipped with descent data for* $T \to S$.

Thus, in order to prove the *logarithmic analogue of Theorem 1.7*, it remains only to *reconstruct* (in a category-theoretic fashion) the *log structure* on an object $Y^{\log}$ of $\mathrm{Sch}^{\log}(X^{\log})$. To do this, we use the object $\mathbb{A}^1_Y$ (as in §1), which we equip with *two distinct log structures*, as follows: Write

$$\mathbb{A}_{Y^{\log}} \stackrel{\mathrm{def}}{=} \mathbb{A}^1_{\mathbb{Z}} \times_{\mathbb{Z}} Y^{\log}; \quad \mathbb{A}^{\log}_{Y^{\log}} \stackrel{\mathrm{def}}{=} (\mathbb{A}^1_{\mathbb{Z}})^{\log} \times_{\mathbb{Z}} Y^{\log}$$

where $(\mathbb{A}^1_{\mathbb{Z}})^{\log}$ is defined to be the affine line $\mathbb{A}^1_{\mathbb{Z}} = \mathrm{Spec}(\mathbb{Z}[T])$ over $\mathbb{Z}$ equipped with the log structure determined by the *divisor* $V(T)$ (i.e., "the origin"). Thus, (one verifies easily that) we have a natural morphism

$$\exp_{\mathbb{A}} : \mathbb{A}^{\log}_{\mathbb{Z}} \to \mathbb{A}_{\mathbb{Z}}$$

whose induced map on $Y^{\log}$-valued points may be *naturally identified* with:

$$\exp_Y : M_Y \to \mathcal{O}_Y$$

Moreover, (one verifies easily that) the morphism $\mathbb{A}_Y \times_Y \mathbb{A}_Y \to \mathbb{A}_Y$ that defines the *multiplication operation* on the ring scheme $\mathbb{A}_Y \to Y$ admits a *unique extension* to a *morphism of log schemes over* $Y^{\log}$:

$$\mathbb{A}^{\log}_{Y^{\log}} \times_{Y^{\log}} \mathbb{A}^{\log}_{Y^{\log}} \to \mathbb{A}^{\log}_{Y^{\log}}$$

This morphism induces (on $Y^{\log}$-valued points) the *monoid operation* on M_Y.

Lemma 2.16. (Characterization of the Log Structure on the Affine Line) *Let $Y^{\log}$ be an object of $\mathrm{Sch}^{\log}(X^{\log})$. Then the arrow $\mathbb{A}^{\log}_{Y^{\log}} \to Y^{\log}$ of $\mathrm{Sch}^{\log}(X^{\log})$ may be category-theoretically characterized as the* **unique** *(up to canonical isomorphism) arrow $Z^{\log} \to Y^{\log}$ equipped with an identification of the underlying morphism of schemes (cf. Corollary 2.15) $Z \to Y$ with $\mathbb{A}_Y \to Y$ satisfying the following properties:*

(i) Away from the zero section of $Z \to Y$, the morphism $Z^{\log} \to Y^{\log}$ is **scheme-like**.

(ii) Let $Z_1^{\log} \rightarrowtail Z^{\log}$, $Z_2^{\log} \rightarrowtail Y^{\log}$ *be scheme-like monomorphisms, where* $Z_1^{\log}$, $Z_2^{\log}$ *are reduced, one-pointed. Suppose that the composite* $Z_1^{\log} \to Z^{\log} \to Y^{\log}$ *factors through (necessarily uniquely!)* $Z_2^{\log}$, *and that the image of* Z_1 *lies in the zero section of* $Z \to Y$. *Then if the log structure on* $Z_2^{\log}$ *is* **trivial**, *we assume that* $Z_1^{\log} \to Z_2^{\log}$ *is* **not** *an isomorphism. On the other hand, if the log structure on* $Z_2^{\log}$ *is* **nontrivial**, *we assume that* $Z_1^{\log} \to Z_2^{\log}$ *satisfies the (category-theoretic) condition of Lemma 2.17, (ii), below.*

(iii) Let $T_0^{\log} \rightarrowtail T^{\log}$ *be a monomorphism of one-pointed objects of* $\mathrm{Sch}^{\log}(X^{\log})$. *Then any commutative diagram*

$$\begin{array}{ccc} T_0^{\log} & \longrightarrow & Z^{\log} \\ \downarrow & & \downarrow \\ T^{\log} & \longrightarrow & Y^{\log} \end{array}$$

admits a morphism $T^{\log} \to Z^{\log}$ *such that both of the resulting triangular diagrams commute.*

(iv) There exists a $Y^{\log}$*-morphism* $Z^{\log} \times_{Y^{\log}} Z^{\log} \to Z^{\log}$ *in* $\mathrm{Sch}^{\log}(X^{\log})$ *whose induced morphism on underlying schemes is equal to the morphism* $\mathbb{A}_Y \times_Y \mathbb{A}_Y \to \mathbb{A}_Y$ *defining the multiplication operation on* $\mathbb{A}_Y$.

Finally, assuming that all of these conditions (i) — (iv) are **satisfied**, *the morphism* $Z^{\log} \times_{Y^{\log}} Z^{\log} \to Z^{\log}$ *of (iv) is the unique* $Y^{\log}$*-morphism in* $\mathrm{Sch}^{\log}(X^{\log})$ *whose induced morphism on underlying schemes is equal to the morphism* $\mathbb{A}_Y \times_Y \mathbb{A}_Y \to \mathbb{A}_Y$ *defining the multiplication operation on* $\mathbb{A}_Y$.

Proof. First, we observe that condition (ii) of Lemma 2.16 is *category-theoretic* — cf. Lemma 2.17, (i), (ii), below. Next, we observe that it suffices to determine the *log structure in a formal neighborhood of the zero section* of $Z \to Y$. Thus, it suffices to *replace* Z, Y by *étale localizations* of Z, Y such that the zero section $Y \to Z$ is *compatible* with these étale localizations. To keep the notation *simple*, we shall denote (for the remainder of this proof) these étale localizations (by abuse of notation) by Z, Y. Thus, we have a morphism of log schemes $Z^{\log} \to Y^{\log}$, together with a *"zero section"* $Y^{\log} \to Z^{\log}$. Also, in the following discussion, we *fix a point* $z \in Z$ lying in this zero section which is the image of a morphism of finite type from a reduced, one-pointed scheme to Z. Write $y \in Y$ for the image of z in Y.

Since we have allowed ourself to pass to étale localizations, we may assume that the morphism $Z^{\log} \to Y^{\log}$ admits a *chart* $Q \to P$ — where we may assume that Q (respectively, P) maps *bijectively* onto $P_{Y,y}$ (respectively, $P_{Z,z}$) (cf. [Kato2], Lemma 1.6, (2)) — i.e., that we have a *commutative diagram*

$$\begin{array}{ccccc} Z^{\log} & \longrightarrow & W^{\log} & \longrightarrow & \mathrm{Spec}(\mathbb{Z}[P])^{\log} \\ & \searrow & \downarrow & & \downarrow \\ & & Y^{\log} & \longrightarrow & \mathrm{Spec}(\mathbb{Z}[Q])^{\log} \end{array}$$

where $W^{\log}$ is defined as the fibered product that makes the square *cartesian*; *all* of the *horizontal* morphisms are *scheme-like*; and $\mathrm{Spec}(\mathbb{Z}[P])^{\log}$, $\mathrm{Spec}(\mathbb{Z}[Q])^{\log}$ are equipped with the log structures associated to the evident pre-log structures $P \to \mathrm{Spec}(\mathbb{Z}[P])$, $Q \to \mathrm{Spec}(\mathbb{Z}[Q])$. Note that $z \mapsto w \in W$ such that the *residue fields* $k(z)$, $k(w)$, $k(y)$ all *coincide*. Let us denote this field by k.

Now let us write $\widehat{W}^{\log}$ for the *formal completion* of $W^{\log}$ at w. Write $W_0^{\log} \subseteq \widehat{W}^{\log}$ for $\mathrm{Spec}(k(w))$, equipped with the log structure pulled back from $W^{\log}$. Then it follows formally from the above discussion that the monomorphism $W_0^{\log} \rightarrowtail W^{\log}$ *factors* (uniquely) through $Z^{\log} \to W^{\log}$, and that the composite of $W_0^{\log} \to Z^{\log}$ with $Z^{\log} \to Y^{\log}$ *coincides* with the composite of $W_0^{\log} \rightarrowtail W^{\log}$ with $W^{\log} \to Y^{\log}$. It thus follows from *condition (iii)* of Lemma 2.16 that we have a morphism

$$\widehat{W}^{\log} \to Z^{\log}$$

whose *composite* $\kappa : \widehat{W}^{\log} \to W^{\log}$ with $Z^{\log} \to W^{\log}$ *restricts to the natural monomorphism* $W_0^{\log} \rightarrowtail W^{\log}$ on $W_0^{\log}$, hence induces a morphism $\widehat{\kappa} : \widehat{W}^{\log} \to \widehat{W}^{\log}$ (which is the identity on $W_0^{\log}$). Moreover, (by *condition (iii)* of Lemma 2.16) we may choose $\widehat{W}^{\log} \to Z^{\log}$ so that the composite of κ with $W^{\log} \to Y^{\log}$ is the natural morphism $\widehat{W}^{\log} \to Y^{\log}$. It thus follows formally that the *scheme-like* endomorphism

$$\widehat{\kappa} : \widehat{W}^{\log} \to \widehat{W}^{\log}$$

is a *closed immersion*, hence (by the elementary commutative algebra fact that surjective endomorphisms of noetherian rings are necessarily bijective) an *isomorphism*. In particular, we conclude that if we denote by $\widehat{Z}^{\log}$ the *formal completion* of $Z^{\log}$ at z, then the morphism $\widehat{W}^{\log} \to Z^{\log}$ gives rise to a *closed immersion*

$$\widehat{W}^{\log} \hookrightarrow \widehat{Z}^{\log}$$

whose composite with the natural morphism $\widehat{Z}^{\log} \to \widehat{W}^{\log}$ is $\widehat{\kappa}$.

Next, let us denote by $\widehat{W}_y^{\log}$, $\widehat{Z}_y^{\log}$ the *fibers* of $\widehat{W}^{\log}$, $\widehat{Z}^{\log}$ over y. Then by *condition (ii)* of Lemma 2.16, we have the following *inequalities*:

$$\begin{aligned} 1 = \dim(\widehat{Z}_y) \geq \dim(\widehat{W}_y) &\geq \dim(\mathrm{Spec}(\mathbb{Z}[P])) - \dim(\mathrm{Spec}(\mathbb{Z}[Q])) \\ &= \mathrm{rk}(P^{\mathrm{gp}}) - \mathrm{rk}(Q^{\mathrm{gp}}) \geq 1 \end{aligned}$$

Thus, $1 = \dim(\widehat{Z}_y) = \dim(\widehat{W}_y)$. Moreover, since $\widehat{Z}_y$ is the formal spectrum of a power series ring in one variable over the field k (a ring which has no nontrivial dimension one quotients), we thus conclude that the closed immersion $\widehat{W}_y \hookrightarrow \widehat{Z}_y$ is, in fact, an *isomorphism*. In particular, if we interpret this fact (cf. the definition of $W^{\log}$) in terms of the *"power series rings"* $k[[Q]]$, $k[[P]]$ (where we note that it makes sense to consider such power series since Q, P have *no nonzero invertible elements* — cf. Proposition 2.1), we conclude that the *fiber* over the closed point of the range of the morphism

$$\mathrm{Spf}(k[[P]]) \to \mathrm{Spf}(k[[Q]])$$

is *isomorphic to* $\mathrm{Spf}(k[[T]])$ (where T is an indeterminate). That is to say, there exists a *surjection* (of complete noetherian rings)

$$k[[Q]][[T]] \twoheadrightarrow k[[P]]$$

(where T is an indeterminate). Since $k[[Q]][[T]]$ is a *domain* — indeed, $k[Q]$ is an *excellent normal domain*, so its completion $k[[Q]]$, being *local* and *normal*, is necessarily a *domain* — and $\dim(k[[Q]][[T]]) = \dim(k[[P]])$, we thus conclude that this surjection is, in fact, an *isomorphism*.

In particular, it follows that the morphism

$$k[[Q]] \to k[[P]]$$

is *flat*. This implies — cf. [Kato1], the proof of the implication (iii) $\Longrightarrow$ (v) of Proposition 4.1, in which it is clear that "$k[[P]]$," "$k[[Q]]$" may be substituted for "$k[P]$," "$k[Q]$" — that the morphism of monoids $Q \to P$ is *integral*, i.e., satisfies the conditions of [Kato1], Proposition 4.1. Moreover, by *condition (ii)* of Lemma 2.16; Lemma 2.17, (ii), below, the morphism $Q^{\mathrm{gp}} \to P^{\mathrm{gp}}$ is *injective*, with *nonzero, torsion-free cokernel*. Put another way, we have shown that the morphism

$$\mathrm{Spec}(k[P])^{\log} \to \mathrm{Spec}(k[Q])^{\log}$$

is *integral* (in the sense of [Kato1], Definition 4.3), *log smooth*, and of *relative dimension* 1. Moreover, the scheme-theoretic fiber of this morphism over the k-point of the range defined by sending all the elements of Q to 0 is *smooth* over k. Thus, by the theory of [KatoF] (cf. especially, [KatoF], Theorem 1.1, (2)), we conclude that the morphism of monoids $Q \to P$ may be *identified with the natural inclusion*:

$$Q \hookrightarrow Q \times \mathbb{N}$$

Finally, by Lemma 2.18 below (and *condition (iv)* of Lemma 2.16), we conclude that the *divisor* in Z determined by considering the image under $M_Z \to \mathcal{O}_Z$ of the inverse image in M_Z of the element $1 \in \mathbb{N} \subseteq Q \times \mathbb{N} = P = P_Z$ is equal (scheme-theoretically) to the *zero section*. This completes the proof of Lemma 2.16 (including the assertion at the end of the statement of Lemma 2.16). $\bigcirc$

Lemma 2.17. (Characterization of Trivial and Relatively Nontrivial Log Structures)

(i) Suppose that $Z^{\log}$ *is a reduced, one-pointed object of* $\mathrm{Sch}^{\log}(X^{\log})$. *Then the condition that the log structure on* $Z^{\log}$ *be* **trivial** *may be categeory-theoretically characterized by the condition that a morphism* $W^{\log} \to Z^{\log}$ *in* $\mathrm{Sch}^{\log}(X^{\log})$ *is completely determined by the underlying morphism of schemes (cf. Corollary 2.15).*

(ii) Let $Z_1^{\log} \to Z_2^{\log}$ *be a* **log-like** *morphism between reduced, one-pointed objects of* $\mathrm{Sch}^{\log}(X^{\log})$ *with nontrivial log structures. Suppose that the morphism* $W_1^{\log} \to W_2^{\log}$ *obtained by base-changing this morphism by some* **scheme-like** *morphism* $W_2^{\log} \to Z_2^{\log}$ *between reduced, one-pointed objects of* $\mathrm{Sch}^{\log}(X^{\log})$ *admits* **at least two sections**. *Then the morphism* $P_{Z_2}^{\mathrm{gp}} \to P_{Z_1}^{\mathrm{gp}}$ *is injective and has a nonzero, torsion-free cokernel.*

Proof. *Assertion (i)* is a formal consequence of the definitions and the observation that if the log structure on $Z^{\log}$ is *split* and *nontrivial*, then $Z^{\log}$ has *lots of endomorphisms* which induce the identity on the underlying scheme Z — given by *multiplication by a positive integer* on P_Z. As for *assertion (ii)*, the *injectivity* of $P_{Z_2}^{\mathrm{gp}} \to P_{Z_1}^{\mathrm{gp}}$, as well as the fact that its cokernel is *torsion-free*, is a formal consequence of the existence of sections of $W_1^{\log} \to W_2^{\log}$ (and Proposition 2.1). Finally, if this cokernel were *zero*, then $W_1^{\log} \to W_2^{\log}$ would be a *monomorphism* (by Proposition 2.3), hence could not admit more than one section. This completes the proof of assertion (ii). ○

Lemma 2.18. (Characterization of the Origin) *Let* A *be an* **artinian local ring**, *with maximal ideal* $\mathfrak{m}_A$. *Then any* $\delta \in \mathfrak{m}_A$ *such that one has*

$$T_1 \cdot T_2 - \delta \in (T_1 - \delta)(T_2 - \delta) \cdot A[[T_1, T_2]]^{\times}$$

(where T_1, T_2 *are indeterminates) is equal to* 0.

Proof. Indeed, by induction on the *length* of A, we may assume (without loss of generality) that $\delta \in I$, for some *ideal* $I \subseteq A$ such that $I^2 = 0$. Then for some *unit* $u \in A[[T_1, T_2]]^{\times}$, we have:

$$\begin{aligned}(T_1 \cdot T_2 - \delta) &= (T_1 - \delta)(T_2 - \delta) \cdot u \\ &= (T_1 \cdot T_2 - \delta \cdot T_1 - \delta \cdot T_2) \cdot u \in A[[T_1, T_2]]\end{aligned}$$

Thus, projecting by $A[[T_1, T_2]] \twoheadrightarrow A$ (where $T_1, T_2 \mapsto 0$) yields $\delta = 0$, as desired. ○

We are now ready to state the *main result* of the present §:

Theorem 2.19. (Categorical Reconstructibility of Locally Noetherian Log Schemes) *Let* $X^{\log}$, $(X')^{\log}$ *be* **fine saturated log schemes**, *whose underlying schemes are* **locally noetherian**.

(i) Let $f^{\log} : X^{\log} \to (X')^{\log}$ *be a morphism of log schemes, whose underlying morphism of schemes is* **quasi-compact**. *Then the* **functor**

$$\mathrm{Sch}^{\log}(f^{\log}) : \mathrm{Sch}^{\log}((X')^{\log}) \to \mathrm{Sch}^{\log}(X^{\log})$$

induced by base-change by $f^{\log}$ *has no nontrivial automorphisms.*

(ii) Denote the set of isomorphisms of log schemes $X^{\log} \xrightarrow{\sim} (X')^{\log}$ *by:*

$$\mathrm{Isom}(X^{\log}, (X')^{\log})$$

Then the natural map

$$\mathrm{Isom}(X^{\log}, (X')^{\log}) \to \mathrm{Isom}(\mathrm{Sch}^{\log}((X')^{\log}), \mathrm{Sch}(X^{\log}))$$

given by $f^{\log} \mapsto \mathrm{Sch}^{\log}(f^{\log})$ *is* **bijective**.

Proof. First, we verify *assertion (i)*. It is a formal consequence of Theorem 1.7, (i), that any functorial automorphism of the objects of the essential image $\mathrm{Im}(\mathrm{Sch}^{\log}(f^{\log}))$ of $\mathrm{Sch}^{\log}((X')^{\log})$ is the identity on the *underlying schemes*. Moreover, since every automorphism of a log scheme of the form given in Proposition 2.4, (iii), which induces the *identity* on the underlying scheme necessarily induces the identity on the *characteristic* of the log scheme, and the morphisms from such log schemes (i.e., of the form given in Proposition 2.4, (iii)) to an arbitrary object $(Y')^{\log}$ of $\mathrm{Sch}^{\log}((X')^{\log})$ are *sufficiently abundant* to *"separate points"* (cf. Lemma 2.5, (iii)) of the geometric fibers of the characteristic $P_{Y'}$ of $(Y')^{\log}$, we conclude that the induced automorphism on the *characteristic* $P_{Y'}$ is also the *identity*. Thus, by *functoriality* (and the discussion preceding Lemma 2.16), it follows that it suffices to prove that any $Y^{\log}$-linear automorphism $\beta_{Z^{\log}}$ of an arrow $Z^{\log} \to Y^{\log}$ satisfying the conditions of Lemma 2.16 such that $\beta_{Z^{\log}}$ induces the identity on the *underlying schemes and characteristics* is necessarily the identity on $Z^{\log}$. But since the *subobject* $\mathcal{L}^{\times}$ of M_Z given by the *sheaf of ideals* of the zero section — where we note that this subobject $\mathcal{L}^{\times}$ is stabilized by $\beta_{Z^{\log}}$ since $\beta_{Z^{\log}}$ induces the identity on the characteristic P_Z — clearly maps *injectively* via $\exp_Z$ into $\mathcal{O}_Z$ (and $\beta_{Z^{\log}}$ induces the identity on $\mathcal{O}_Z$), we thus conclude that such an $\beta_{Z^{\log}}$ is the *identity on* $Z^{\log}$, as desired.

Finally, *assertion (ii)* follows formally — cf. the proof of Theorem 1.7, (ii) — from assertion (i); Theorem 1.7, (ii); and Lemma 2.16 (cf. also the discussion preceding Lemma 2.16). ○

Finally, we consider the *logarithmic analogue* of Theorem 1.8:

Theorem 2.20. **(Further Rigidity Property)** *Let* $X^{\log}$ *be a* **fine saturated log scheme***, whose underlying scheme is* **locally noetherian***. Suppose that for every object* $Y^{\log} \to X^{\log}$ *of* $\mathrm{Sch}^{\log}(X^{\log})$*, one is given an automorphism* $\alpha_{Y^{\log}} : Y^{\log} \xrightarrow{\sim} Y^{\log}$ *—* **not necessarily over** $X^{\log}$*! — with the property that for every morphism* $Y_1^{\log} \to Y_2^{\log}$ *of* $\mathrm{Sch}^{\log}(X^{\log})$*, one has a commutative diagram:*

$$\begin{array}{ccc} Y_1^{\log} & \xrightarrow{\alpha_{Y_1^{\log}}} & Y_1^{\log} \\ \downarrow & & \downarrow \\ Y_2^{\log} & \xrightarrow{\alpha_{Y_2^{\log}}} & Y_2^{\log} \end{array}$$

Then all of the $\alpha_{Y^{\log}}$ *are equal to the* **identity***.*

Proof. By Theorem 1.8, every $\alpha_{Y^{\log}}$ induces the identity on the *underlying scheme* Y. In particular, $\alpha_{\mathbb{A}^{\log}_{Y^{\log}}}$ induces the identity on the underlying scheme $\mathbb{A}_Y$. Moreover, just as in the proof of Theorem 2.19, (i), since the *characteristic* of a log scheme of the form given in Proposition 2.4, (iii), has *no nontrivial automorphisms*, one concludes — by applying Lemma 2.5, (iii), to *"separate points"* — that $\alpha_{Y^{\log}}$ induces the *identity on the characteristic* P_Y. Thus, we conclude (cf. the proof of Theorem 2.19, (i)) that $\alpha_{Y^{\log}}$ is the identity, as desired. ○

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