Vanishing of Stokes curves

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1 Introduction

No algorithm of describing the complete Stokes geometry is known for higher order ordinary differential equations with a large parameter. However, by virtue of the exact steepest descent method proposed in [AKT3] it now becomes possible to determine whether a Stokes phenomenon for Borel resummed WKB solutions actually occurs at a given point or not with the aid of a computer. In this report, using the exact steepest descent method mainly, we discuss an interesting phenomenon that a Stokes curve emanating from an ordinary turning point may vanish (i.e., a Stokes phenomenon for Borel resummed WKB solutions no longer occurs on it) after crossing other Stokes curves.

In the case of second order equations such a phenomenon of vanishing of a Stokes curve never happens (cf. [V]). In contrast with the second order case, since Stokes

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curves with different types often intersect, it happens to higher order equations that a Stokes curve may vanish or the structure of Stokes phenomena for WKB solutions occurring on it may change after it crosses other Stokes curves. For example, as is explained in [AKT1, Section 2] (cf. [AKT4, Remark 2.1] also), at a crossing point of three Stokes curves of type $(j < k)$, $(k < l)$ and $(j < l)$ (see, e.g., [AKT1] for the terminology frequently used in the exact WKB analysis), the structure of Stokes phenomena for WKB solutions occurring on the Stokes curve of type $(j < l)$ ("non-adjacent Stokes curve") changes in general. In particular, when turning points of an equation or an operator P in question are all simple, that is, when the Borel transform P_B of P is with simple characteristics, (note that an operator with simple discriminant in the sense of [AKT1, Definition 1.1] satisfies this condition,) the non-adjacent Stokes curve is expected to vanish after passing through a crossing point of three Stokes curves unless any degeneracy happens to P. (We will discuss a kind of non-apparent degeneracy in Section 3.) In our opinion the validity of this expectation is closely related to the existence of an algorithm of describing the complete Stokes geometry for higher order equations. The purpose of this report is to examine if a phenomenon of vanishing of a Stokes curve actually occurs to a Stokes curve emanating from an ordinary turning point (we sometimes call such a Stokes curve an "ordinary Stokes curve" to distinguish it from a new Stokes curve) by studying several concrete examples mainly with the aid of a computer.

The report is organized as follows: In Section 2 we discuss an example whose turning points are all simple. For this example it can be confirmed that vanishing of an ordinary Stokes curve really occurs. Next we investigate in Section 3 the case where all turning points are simple but there is a kind of degeneracy that an ordinary turning point and a virtual turning point (which is new terminology for a new turning point of [AKT1]) are merged. In this case vanishing of an ordinary Stokes curve does not occur; instead the structure of Stokes phenomena for WKB solutions on a non-adjacent Stokes curve (more precisely, the value of the Stokes coefficient describing Stokes phenomena) changes at a crossing point of Stokes curves. Section 4 is devoted to the study of an equation with double turning points, which appears in connection with the problem of non-adiabatic transition probabilities in quantum mechanics. In this case also we can conclude that vanishing of an ordinary Stokes curve does not occur in general. In Section 5 we give a summary and conclusions.

Finally, we would like to express our sincere gratitude to Professor T. Kawai for many valuable discussions with him.

2 An equation with simple turning points

In this section we study the following equation with a large parameter $\eta > 0$:

$$P\psi = \left(\frac{d^3}{dx^3} + 6(1+x)\eta^2 \frac{d}{dx} + (2-4ix)\eta^3 \right) \psi = 0. \tag{1}$$

This is an equation obtained by putting $\lambda = i\eta$ in [AKT1, Example 2.5]. The configuration and the type of Stokes curves of (1) is shown in Figure 1, where a wiggly line designates a cut which we have placed to define a characteristic root of (1) as single-valued analytic function.

Figure 1: Stokes curves of (1).

There are three turning points a_0, a_1 and a_2 all of which are simple turning points. As is clear from Figure 1, a Stokes curve emanating from a_0 intersects another Stokes curve emanating from a_1 at an ordered crossing point B. Hence we have to add a new Stokes curve passing through B, which is also included in Figure 1. (In Figure 1 and subsequent figures describing Stokes curves as well, a virtual turning point is designated by a small dot like an ordinary turning point and a broken line indicates that no Stokes phenomenon for WKB solutions occurs on the portion.) Note that the new Stokes curve thus added intersects with two ordinary Stokes curves again at A and a Stokes curve emanating from a_0 is non-adjacent there. Thus we can expect that the Stokes curve emanating from a_0 may vanish after passing through the crossing point A. In what follows we will check this expectation by using two different methods.

2.1 Verification by the steepest descent method

As (1) is a Laplace type equation and the employment of the Laplace transformation $\psi = \int \exp(\eta x \xi)\hat{\psi} d\xi$ provides us with an integral representation of solutions of the

Figure 2: Magnification of Figure 1 near the crossing point A.

form

$$\psi(x) = \int \exp(\eta f(x,\xi)) \frac{1}{6\xi - 4i} d\xi \tag{2}$$

where

$$f(x,\xi) = x\xi + \frac{1}{18}\xi^3 + \frac{i}{18}\xi^2 + \frac{25}{27}\xi + \left(\frac{1}{3} + \frac{50}{81}i\right) \log\left(\xi - \frac{2}{3}i\right), \tag{3}$$

the ordinary steepest descent method is applicable to (1) (cf. [U1], [U2], [T1]). We first use the steepest descent method to examine if the Stokes curve emanating from a_0 vanishes after passing through A.

Figure 3 illustrates the configuration of steepest descent paths of $\mathrm{Re} f(x,\xi)$ passing through saddle points of $f(x,\xi)$ near the Stokes curve in question below the crossing point A, more precisely at $x = x_0$ and $x = x_1$ whose location is indicated in Figure 2. (In Figure 3 and subsequent Figures 4 and 5 a small dot designates a saddle point and a larger dot designates a singular point of the integrand of (2).) Figure 3 clearly shows that the configuration changes when one crosses the Stokes curve below A. This change of the configuration implies that a Stokes phenomenon for Borel resummed WKB solutions really occurs on the Stokes curve (cf. [T1, Proposition 3]).

Figure 3: Steepest descent paths at $x = x_0$ (a) and $x = x_1$ (b).

On the other hand, above the crossing point A the configuration becomes a different one as is illustrated in Figure 4 and its magnification Figure 5 near the unique singular point in Figure 4. We can read from Figs. 4 and 5 the fact that a steepest descent path passing through the lowest saddle point flows into a singular point in both Figs. 4(a) and 4(b) (or 5(a) and 5(b)) and no change of the configuration occurs when one crosses the Stokes curve above A. Hence we can conclude that the Stokes curve in question vanishes after passing through the crossing point A.

Figure 4: Steepest descent paths at $x = x_2$ (a) and $x = x_3$ (b).

Figure 5: Magnification of Figure 4 near the singular point.

2.2 Verification by using the connection formula

Next we try to determine the connection formula on the Stokes curve in question by employing the reasoning used in [BNR] to confirm that it vanishes after passing through A.

Let ψ_j $(j = 0, 1)$ be WKB solutions of (1) with the good normalization in the sense that they satisfy the Airy type connection formula near the simple turning point a_1 (cf. [AKT1, Theorem 1.8]). We first consider the situation near the ordered crossing point B (Figure 6). It follows from the above normalization that ψ_j $(j = 0, 1, 2)$ should satisfy the following connection formula when crossing a Stokes curve $\gamma_{01}^{\pm}$ in the direction indicated by an arrow in Figure 6:

$$\psi_0 \longmapsto \psi_0 + i\psi_1, \quad \psi_1 \longmapsto \psi_1, \quad \psi_2 \longmapsto \psi_2. \tag{4}$$

Similarly, since $\gamma_{12}^{\pm}$ is a Stokes curve emanating from the simple turning point a_0, we can choose an appropriate normalization of ψ_2 so that ψ_j $(j = 0, 1, 2)$ should satisfy

$$\psi_0 \longmapsto \psi_0, \quad \psi_1 \longmapsto \psi_1 + i\psi_2, \quad \psi_2 \longmapsto \psi_2. \tag{5}$$

Figure 6: Schematic illustration of Stokes curves near B.

across $\gamma_{12}^{\pm}$. On the other hand, the explicit form of the connection formula on γ_{02}^{+} is different from that on γ_{02}^{-} as $\gamma_{02}^{\pm}$ is non-adjacent at B. On the side γ_{02}^{-} the connection formula is trivial (i.e., no Stokes phenomenon occurs) since there exists a virtual turning point on this side (cf. [AKT1, p. 77], [V, p. 244]), while the connection formula on γ_{02}^{+} should be of the form

$$\psi_0 \longmapsto \psi_0 + c\psi_2, \quad \psi_1 \longmapsto \psi_1, \quad \psi_2 \longmapsto \psi_2 \tag{6}$$

with some constant c. The constant c in (6) can be determined explicitly by the reasoning used in [BNR] in the following manner: We consider the analytic continuation of ψ_0 from Region I to Region II. If we continue it via the right side ("+" side) of B, we find by (4), (5) and (6) that ψ_0 should become $\psi_0 + i\psi_1 + (c-1)\psi_2$, while ψ_0 continued via the left side ("$-$" side) of B should be $\psi_0 + i\psi_1$ in Region II. Since these two resulting expressions must coincide (as the crossing point B is a regular point of (1)), we obtain $c = 1$.

Having this result in mind, we next consider the situation near the crossing point A (Figure 7). It follows from our normalization of ψ_j ($j = 0, 1$) that we have the following connection formula on $\tilde{\gamma}_{01}^{\pm}$:

$$\psi_0 \longmapsto \psi_0, \quad \psi_1 \longmapsto \psi_1 + i\psi_0, \quad \psi_2 \longmapsto \psi_2. \tag{7}$$

Furthermore, as $\tilde{\gamma}_{02}^{-}$ in Figure 7 is the same Stokes curve with γ_{02}^{+} in Figure 6 and (6) holds with $c = 1$ there, we find that the connection formula

$$\psi_0 \longmapsto \psi_0 + \psi_2, \quad \psi_1 \longmapsto \psi_1, \quad \psi_2 \longmapsto \psi_2 \tag{8}$$

holds on $\tilde{\gamma}_{02}^{\pm}$. Similarly, the same formula with (5) holds on the side $\tilde{\gamma}_{12}^{-}$ of the non-adjacent Stokes curve $\tilde{\gamma}_{12}^{\pm}$. Then the connection formula

$$\psi_0 \longmapsto \psi_0, \quad \psi_1 \longmapsto \psi_1 + \tilde{c}\psi_2, \quad \psi_2 \longmapsto \psi_2. \tag{9}$$

Figure 7: Schematic illustration of Stokes curves near A.

on the opposite side $\tilde{\gamma}_{12}^{+}$ can be determined by the same reasoning as above. As a matter of fact, we can deduce $\tilde{c} = 0$ from the coincidence of the two possible analytic continuations of ψ_1 from Region $\tilde{\text{I}}$ to Region $\tilde{\text{II}}$. We have thus verified that the Stokes curve in question, i.e., $\tilde{\gamma}_{12}^{+}$ really vanishes.

3 A degenerate equation with simple turning points

In a similar manner to the preceding section we study the following concrete equation whose turning points are all simple in this section:

$$P\psi = \left(\frac{d^3}{dx^3} + x^2\eta^2\frac{d}{dx} + (2-i)\eta^3\right)\psi = 0. \tag{10}$$

Figure 8 shows the configuration and the type of Stokes curves of (10).

Equation (10) has two crossing points, which are denoted by A and B in Figure 8, of three ordinary Stokes curves. In what follows we take one of the crossing points (say, A) and examine whether vanishing of the non-adjacent Stokes curve (i.e., a Stokes curve emanating from a_0) occurs there or not.

Figure 8: Stokes curves of (10).

Figure 9: Magnification of Figure 8 near the crossing point A.

3.1 Verification by the exact steepest descent method

First we apply the exact steepest descent method to (10). That is, consider the Laplace transform of (10)

$$\hat{P}\hat{\psi} = \eta \left(\xi \frac{d^2}{d\xi^2} + 2\frac{d}{d\xi} + (\xi^3 + 2 - i)\eta^2 \right) \hat{\psi} = 0, \tag{11}$$

take a WKB solution $\hat{\psi}_{\pm} = \exp \int^{\xi} (-\eta x_{\pm}(\xi) + \cdots) \, d\xi$ of it, where $x_{\pm}(\xi)$ is a root of the equation $\xi^3 + x^2\xi + (2-i) = 0$ (i.e., the characteristic polynomial of the original equation (10)) with respect to x, and investigate the exact steepest descent paths of its inverse Laplace transform

$$\int \exp(\eta x \xi)\hat{\psi}_{\pm} d\xi = \int \exp\left\{\eta\left(x\xi - \int^{\xi} x_{\pm}(\xi)d\xi\right) + \cdots\right\} d\xi. \tag{12}$$

Figures 10 and 11 illustrate the configuration of exact steepest descent paths of (12) (and of Stokes curves of (11) which are necessary to draw exact steepest descent paths) at $x = x_0, \ldots, x_3$ whose location is indicated in Figure 9. (In Figures 10, 11 and subsequent figures describing exact steepest descent paths a solid line and a small dot respectively designate a steepest descent path and a saddle point, while a broken line and a small circle respectively designate a Stokes curve and a turning point for $\hat{P}\hat{\psi} = 0$.)

Figure 10: Exact steepest descent paths at $x = x_0$ (a) and $x = x_1$ (b).

Figure 11: Exact steepest descent paths at $x = x_2$ (a) and $x = x_3$ (b).

In Figure 10 (i.e., across the Stokes curve in question above the crossing point A) we can observe a change of the configuration caused by the degeneracy that a steepest descent path σ, which is obtained by repeated bifurcation from an ordinary steepest descent path passing through a saddle point ξ_1, hits another saddle point ξ_0. On the other hand, in Figure 11 (i.e., across the Stokes curve below A) another change of the configuration can be observed besides the above change; another bifurcated steepest descent path $\tilde{\sigma}$ hits ξ_0 simultaneously. This implies that the structure of Stokes phenomena for Borel resummed WKB solutions on the non-adjacent Stokes curve in question changes at the crossing point A, but vanishing of a Stokes curve does not occur there, although all turning points of (10) are simple.

3.2 Discussion by adding a small perturbation

To clarify the reason why vanishing of a Stokes curve does not occur for (10) at A, we add the following small perturbation to (10):

$$(13) \qquad P_\epsilon \psi = \left(\frac{d^3}{dx^3} + (x^2 + i\epsilon)\eta^2 \frac{d}{dx} + (2 - i)\eta^3 \right) \psi = 0 \quad (\epsilon > 0 : \text{small}).$$

Then the configuration of Stokes curves is slightly perturbed as follows:

Figure 12: Stokes curves of (13).

In Figure 12 the value of ϵ is taken to be 0.1 and some relevant new Stokes curves are also included to resolve ordered crossing points. As is visualized in Figure 12 and subsequent Figure 13, this perturbation splits the Stokes curve for $\epsilon = 0$ in question emanating from a_0 into two Stokes curves when $\epsilon > 0$; one is an ordinary Stokes curve emanating from $\tilde{a}_0$ and the other is a new Stokes curve emanating from a virtual turning point $\tilde{b}$. (Both turning points $\tilde{a}_0$ and $\tilde{b}$ approach a_0 as $\epsilon \to 0$.)

Consequently when $\epsilon > 0$ three crossing points A_j ($j = 0, 1, 2$) appear near the crossing point A for $\epsilon = 0$. Among them only A_2 is an ordered crossing point of (13). We now examine whether a Stokes phenomenon occurs or not near these crossing points by using the exact steepest descent method.

Figure 13: Magnification of Figure 12 near $\{A_j\}_{j=0,1,2}$.

Figure 14: Exact steepest descent paths at $x = x_0$ (a), $x = x_1$ (b) and $x = x_2$ (c).

Figure 15: Exact steepest descent paths at $x = x_3$ (a), $x = x_4$ (b) and $x = x_5$ (c).

Let us take six points $x_0, \ldots, x_5$ near the crossing points (cf. Figure 13) and draw the configuration of exact steepest descent paths at these points x_j ($j = 0, \ldots, 5$); the resulting figures are Figures 14 and 15. There is no topological difference between the configuration of Fig. 14(a) and that of Fig. 14(b), while between Fig. 14(b) and Fig. 14(c) we can observe the same change of the configuration with that in Figure 10. The same change can be found between Fig. 15(b) and Fig. 15(c) also, and between Fig. 15(a) and Fig. 15(b) there occurs another change of the configuration which is the same with one of the changes observed in Figure 11.

Comparison of this observation with that in the preceding subsection leads to the following conclusion: The Stokes curve for $\epsilon = 0$ in question has been split into two Stokes curves when $\epsilon > 0$. Between them an ordinary Stokes curve emanating from $\tilde{a}_0$ passes through no ordered crossing point and the structure of the Stokes phenomenon for Borel resummed WKB solutions on it is unchanged near the crossing points in question. On the other hand, a new Stokes curve emanating from $\tilde{b}$ passes through an ordered crossing point A_2 and there vanishing of a Stokes curve really occurs. The phenomenon observed at the crossing point A for $\epsilon = 0$ can be considered as confluence of these two phenomena. That is, the reason why vanishing of a Stokes curve does not occur for $\epsilon = 0$ is that (10) has a degeneracy that a turning point and a virtual turning point are merged.

4 An equation with double turning points

To see what occurs when we drop the assumption that all turning points are simple we discuss the following 3×3 system in this section:

$$(14) \qquad i\frac{d}{dx}\psi = \eta \left[\begin{pmatrix} \rho_0(x) & 0 & 0 \\ 0 & \rho_1(x) & 0 \\ 0 & 0 & \rho_2(x) \end{pmatrix} + \eta^{-1/2} \begin{pmatrix} 0 & \alpha_{01} & \alpha_{02} \\ \overline{\alpha_{01}} & 0 & \alpha_{12} \\ \overline{\alpha_{02}} & \overline{\alpha_{12}} & 0 \end{pmatrix} \right] \psi,$$

where

$$(15) \qquad \rho_0(x) \equiv \rho_0 = 1 + i, \quad \rho_1(x) = \left(1 - \frac{i}{2}\right) x, \quad \rho_2(x) = \left(2 + \frac{2}{3}i\right) x.$$

System (14) has its origin in a non-adiabatic level crossing problem in quantum mechanics (in that context each $\rho_j(x)$ is assumed to be real) and is equivalent to a third-order single equation of the form

$$(16) \quad P\psi = \left(\frac{d^3}{dx^3} + i\left(\rho_0 + \rho_1 + \rho_2 + \cdots\right)\eta \frac{d^2}{dx^2} \right. \\ \left. - \left(\rho_0\rho_1 + \rho_1\rho_2 + \rho_2\rho_0 + \cdots\right)\eta^2 \frac{d}{dx} - i\left(\rho_0\rho_1\rho_2 + \cdots\right)\eta^3 \right) \psi = 0$$

(where $\cdots$ denotes lower order terms with respect to η). Note that the turning points of an equation of this type are all double. The configuration of ordinary Stokes curves and relevant new Stokes curves of (14) (or equivalently of (16)) is described in Figure 16.

Figure 16: Stokes curves of (14) (or equivalently (16)).

At a crossing point A in Figure 16 an ordinary Stokes curve emanating from a turning point a_{02} is non-adjacent. In what follows we examine whether vanishing of a Stokes curve occurs at A with this ordinary Stokes curve.

4.1 Verification by the exact steepest descent method

We first apply the exact steepest descent method. In this case, since $\rho_0(x) \equiv \rho_0$ is a constant, there exists a characteristic root being independent of x. Hence we have to take into account exact steepest descent paths emanating from a singular point corresponding to a constant characteristic root together with those emanating from saddle points (cf. [KT, Section 4]). Figures 18 and 19 describe the configuration of all such exact steepest descent paths at $x = x_0$, $x = x_1$ (i.e., below A) and at $x = x_2$, $x = x_3$ (i.e., above A) respectively, where x_j $(j = 0, \ldots, 3)$ is a point marked in Figure 17.

A change of the configuration can be observed in Figure 18; a steepest descent path σ_+ emanating from ξ_0, a singular point corresponding to a constant characteristic root $-i\rho_0$, hits a saddle point ξ_2. The same change occurs also in Figure 19, but this time it is "two-ply", that is, not only σ_+ but also another steepest descent path, which a steepest descent path σ_- emanating from ξ_0 bifurcates at its crossing point with a Stokes curve passing through ξ_0, hits ξ_2 simultaneously. (The crossing point of σ_- and the Stokes curve through ξ_0 is passed also by σ_+, and hence the bifurcated steepest descent path overlies σ_+. See [KT, Remark 3.1].)

Figure 17: Magnification of Figure 16 near the crossing point A.

Figure 18: Exact steepest descent paths at $x = x_0$ (a) and $x = x_1$ (b).

Figure 19: Exact steepest descent paths at $x = x_2$ (a) and $x = x_3$ (b).

From this observation we may conclude that the structure of Stokes phenomena for WKB solutions on the non-adjacent ordinary Stokes curve in question changes at A, but vanishing of a Stokes curve does not occur there.

4.2 Discussion by adding a small perturbation

We next check the conclusion obtained in the preceding subsection by adding a small perturbation to (14) or (16) and reducing the problem to that for an equation without any multiple turning point.

As a perturbed equation of (16) we consider the following here:

$$P_\epsilon \psi = \left(P + \epsilon(i-1)\eta^3\right)\psi = 0. \tag{17}$$

The configuration of ordinary Stokes curves and relevant new Stokes curves of (17) for $\epsilon = 0.008$ is given in Figure 20.

Figure 20: Stokes curves of (17).

Figure 21: Magnification of Figure 20 near $\{A_j\}_{j=0,1}$.

The Stokes curve in question for $\epsilon = 0$ is split into two curves when $\epsilon > 0$; one is an ordinary Stokes curve emanating from an ordinary turning point $\tilde{a}_{02}$ and the other is a new Stokes curve emanating from a virtual turning point $\tilde{b}$. The crossing point A accordingly splits into two crossing points A_0 and A_1 (cf. Figure 21). Taking six points $x_0, \ldots, x_5$ near these crossing points (cf. Figure 21) and drawing the configuration of exact steepest descent paths at these points x_j $(j = 0, \ldots, 5)$ with the aid of a computer, we obtain Figures 22 and 23.

Figure 22: Exact steepest descent paths at $x = x_0$ (a), $x = x_1$ (b) and $x = x_2$ (c).

Figure 23: Exact steepest descent paths at $x = x_3$ (a), $x = x_4$ (b) and $x = x_5$ (c).

In Figure 22, a change occurs with a steepest descent path emanating from a saddle point $\tilde{\xi}_0$ between Fig. 22(a) and 22(b), but there is no change of the configuration between Fig. 22(b) and 22(c). (See Fig. 24 below for the configuration of exact steepest descent paths near $\tilde{\xi}_0$; the configuration is not changed near $\tilde{\xi}_0$ between Fig. 22 and Fig. 23.) On the other hand, three figures in Figure 23 are mutually different; a steepest descent path σ_0 emanating from $\tilde{\xi}_0$ and another steepest descent path σ_{bb}, which is obtained from σ_b by bifurcation (cf. Figure 24; σ_b itself is a bifurcated steepest descent path obtained from σ_0), simultaneously hit a saddle point $\tilde{\xi}_2$ between Figs. 23(a) and 23(b), and a steepest descent path which a steepest descent path $\tilde{\sigma}_0$ emanating from $\tilde{\xi}_0$ bifurcates hits $\tilde{\xi}_2$ between Figs. 23(b) and 23(c). We should here notice that the change between Figs. 23(a) and 23(b) is an "apparent" one due to the following delicate cancellation: Locally near the saddle point $\tilde{\xi}_0$ we consider an inverse Laplace transform $\int \exp(\eta x \xi)\hat{\psi}_+ d\xi$ of an appropriate

Figure 24: Magnification of Figures 23 (a) and (b).

WKB solution $\hat{\psi}_+$. However, when one crosses a Stokes curve $\hat{\gamma}_0$ in Figure 24, the Borel resummed WKB solution $\hat{\psi}_+$ becomes $\hat{\psi}_+ + c\hat{\psi}_-$ with some constant c ("Stokes phenomenon"). Similarly, $\hat{\psi}_-$ becomes $\hat{\psi}_- + \tilde{c}\hat{\psi}_+$ with another constant $\tilde{c}$ when one crosses a Stokes curve $\hat{\gamma}_1$. Taking account of these connection formulas and of the configuration of steepest descent paths and Stokes curves, we find that globally we should consider

$$\int_{\sigma_0} \exp(\eta x\xi)\hat{\psi}_+ d\xi + c\int_{\sigma_b} \exp(\eta x\xi)\hat{\psi}_- d\xi + c\tilde{c}\int_{\sigma_{bb}} \exp(\eta x\xi)\hat{\psi}_+ d\xi. \tag{18}$$

This is the core idea of the exact steepest descent method proposed in [AKT3]. Since both Stokes curves $\hat{\gamma}_0$ and $\hat{\gamma}_1$ emanate from the same simple turning point in this case, we can normalize WKB solutions $\hat{\psi}_\pm$ so that $c = \tilde{c} = i$ is satisfied. Hence the integral along the steepest descent path σ_{bb} in (18) is cancelled. (Note that σ_{bb} overlies σ_0 thanks to the same reasoning as in the preceding subsection.) We have thus verified that no Stokes phenomenon occurs between Figs. 23(a) and 23(b).

As a conclusion we can claim that when $\epsilon > 0$ vanishing of a Stokes curve occurs for both of the two Stokes curves that the Stokes curve in question for $\epsilon = 0$ is split into. Furthermore the Stokes phenomena on the Stokes curve for $\epsilon = 0$ can be considered as confluence of the Stokes phenomena on these two Stokes curves for $\epsilon > 0$. The reason why the structure of Stokes phenomena for WKB solutions changes but does not vanish at the crossing point A for $\epsilon = 0$ is that it has these two different origins for $\epsilon > 0$.

4.3 Verification by using the connection formula

Finally we further confirm the conclusion obtained in Section 4.1 by computing the connection formulas on the Stokes curve in question explicitly.

We employ the same argument as in Section 2.2. Let us first consider the situation near the crossing point B (Figure 25).

Figure 25: Schematic illustration of Stokes curves near B.

If we assume that ψ_j $(j = 0, 1, 2)$ should satisfy the connection formula

$$\psi_0 \longmapsto \psi_0, \quad \psi_1 \longmapsto \psi_1 + c_{01}\psi_0, \quad \psi_2 \longmapsto \psi_2 \tag{19}$$

with some constant c_{01} across $\gamma_{01}^{\pm}$ and further assume that they satisfy

$$\psi_0 \longmapsto \psi_0 + c_{02}\psi_2, \quad \psi_1 \longmapsto \psi_1, \quad \psi_2 \longmapsto \psi_2 \tag{20}$$

with another constant c_{02} across $\gamma_{02}^{\pm}$, the same reasoning as in Section 2.2 verifies that they should satisfy

$$\psi_0 \longmapsto \psi_0, \quad \psi_1 \longmapsto \psi_1 + c_{01}c_{02}\psi_2, \quad \psi_2 \longmapsto \psi_2 \tag{21}$$

across γ_{12}^{+}. (Note that the connection formula is trivial on the side γ_{12}^{-} since there exists a virtual turning point on this side.) Then, again by the same reasoning near the crossing point A, we find that ψ_j $(j = 0, 1, 2)$ should satisfy the connection formula

$$\psi_0 \longmapsto \psi_0 + c\psi_2, \quad \psi_1 \longmapsto \psi_1, \quad \psi_2 \longmapsto \psi_2 \tag{22}$$

across $\tilde{\gamma}_{02}^{+}$ with

$$c = c_{02} + c_{01}\tilde{c}_{01}c_{02}, \tag{23}$$

Figure 26: Schematic illustration of Stokes curves near A.

assuming that they should satisfy

$$\psi_0 \longmapsto \psi_0 + \tilde{c}_{01}\psi_1, \quad \psi_1 \longmapsto \psi_1, \quad \psi_2 \longmapsto \psi_2 \tag{24}$$

with a constant $\tilde{c}_{01}$ across $\tilde{\gamma}_{01}^{\pm}$ (cf. Figure 26). Now, since a_{01} is a double turning point and we have a local reduction of (14) to a Weber type equation near a double turning point ([AKT4, Appendix]), the connection formula near a_{01} should be described by that of the Weber equation. In particular, we can choose appropriate normalization of ψ_j so that c_{01} and $\tilde{c}_{01}$ are respectively given by

$$c_{01} = \frac{i\sqrt{2\pi}}{\Gamma(-\kappa+1/2)}e^{i\pi\kappa} + \cdots, \quad \tilde{c}_{01} = \frac{i\sqrt{2\pi}}{\Gamma(\kappa+1/2)} + \cdots \tag{25}$$

(cf. [T2, Section 3]). Here κ is of the form $\kappa = a|\alpha_{01}|^2 + b$ with some constants $a \neq 0$ and b, and $\cdots$ denotes lower order terms with respect to η. The other constant c_{02} has a similar expression also. Hence we obtain

$$\begin{aligned} c &= c_{02}(1 + c_{01}\tilde{c}_{01}) \\ &= c_{02}\left(1 - \frac{2\pi}{\Gamma(-\kappa+1/2)\Gamma(\kappa+1/2)}e^{i\pi\kappa} + \cdots\right) \\ &= -c_{02}(e^{2i\pi\kappa} + \cdots), \end{aligned} \tag{26}$$

which is equal neither to c_{02} nor to 0 for generic values of α_{01} and α_{02}. We have thus verified that the value of the Stokes coefficient describing Stokes phenomena on the non-adjacent ordinary Stokes curve in question changes at A, but vanishing of a Stokes curve does not occur there in general.

5 Conclusions

As we have seen in Section 2, in the case of higher order ordinary differential equations an ordinary Stokes curve may vanish after crossing other Stokes curves. Such a phenomenon of vanishing of a Stokes curve is expected to occur quite generically for equations whose turning points are all simple. In contrast with the case of equations with simple turning points, as the examples in Sections 3 and 4 show, the Stokes coefficient describing Stokes phenomena for WKB solutions only changes its value (not necessarily vanishes) at a crossing point of Stokes curves when a double turning point exists or when there is a degeneracy that an ordinary turning point and a virtual turning point are merged: The confluence of turning points causes vanishing of a Stokes curve not to be observed in these cases.

Both in the example of Section 2 and in the perturbed equations discussed in Sections 3.2 and 4.2 vanishing of a Stokes curve certainly occurs. As these examples are suggesting, we can expect that a non-adjacent Stokes curve should vanish after passing through a crossing point of three Stokes curves for an equation with simple turning points if any two (ordinary or virtual) turning points are not merged or connected by a Stokes curve. In the exact WKB theory for higher order equations it is one of the important problems to prove this expectation rigorously by analyzing the Riemann sheet structure of Borel transformed WKB solutions.

References

[AKT1] T. Aoki, T. Kawai and Y. Takei: New turning points in the exact WKB analysis for higher order ordinary differential equations. Analyse algébrique des perturbations singulières, I; Méthodes résurgentes, Hermann, 1994, pp. 69–84.

[AKT2] ______: On the exact WKB analysis for the third order ordinary differential equations with a large parameter. Asian J. Math., **2**(1998), 625–640.

[AKT3] ______: On the exact steepest descent method — a new method for the description of Stokes curves. J. Math. Phys., **42**(2001), 3691–3713.

[AKT4] ______: Exact WKB analysis of non-adiabatic transition probabilities for three levels. RIMS preprint (No. 1331), to appear in J. Phys. A.

[BNR] H. L. Berk, W. M. Nevins and K. V. Roberts: New Stokes' line in WKB theory. J. Math. Phys., **23**(1982), 988–1002.

[KT] T. Koike and Y. Takei: The effect of new Stokes curves in the exact steepest descent method. In this volume.

[T1] Y. Takei: Integral representation for ordinary differential equations of Laplace type and exact WKB analysis. RIMS Kôkyûroku, No. 1168, 2000, pp. 80–92.

[T2] ______: An explicit description of the connection formula for the first Painlevé equation. Toward the Exact WKB Analysis of Differential Equations, Linear or Non-Linear, Kyoto Univ. Press, 2000, pp. 271–296.

[U1] K. Uchiyama: On examples of Voros analysis of complex WKB theory. Analyse algébrique des perturbations singulières, I, Méthodes résurgentes, Hermann, 1994, pp. 104–109.

[U2] ______: Graphical illustration of Stokes phenomenon of integrals with saddles. Toward the Exact WKB Analysis of Differential Equations, Linear or Non-Linear, Kyoto Univ. Press, 2000, pp. 87–96.

[V] A. Voros: The return of the quartic oscillator — The complex WKB method. Ann. Inst. Henri Poincaré, **39**(1983), 211–338.