Gaussian and Poisson White Noises with Related Characterization Theorems*

Nobuhiro Asai†, Izumi Kubo‡, and Hui-Hsiung Kuo§

Dedicated to Professor Leonard Gross on the occasion of his 70th birthday.

Abstract

Let μ_G and μ_P be a Gaussian measure and a Poisson measure on E^* , respectively. Let a_t and a_t^* be respectively annihilation and creation operators at a point $t \in \mathbb{R}$. In the theory of quantum white noise, it is known that a_t is a continuous linear operator from $\Gamma_u(E_{\mathbb{C}})$ into itself and a_t^* is a continuous linear operator from $\Gamma_u(E_{\mathbb{C}})^*$ into itself. In paticular, $a_t + a_t^*$ and $a_t + a_t^* + a_t^* a_t + I$ are called the quantum Gaussian white noise and the quantum Poisson white noise, respectively. The main purpose of this work is to realize quantum Gaussian and Poisson white noises in terms of multiple Wiener-Itô integrals, and show that such realizations cannot be achieved by J-transform and its holomorphy, but can be done by S_X-transform depending on the exponential function ϕ_ξ^X, which determines a unitary isomorphism between Boson Fock space and $L^2(E^*, \mu_X)$, $X = G,\ P$. In Appendix A, some connections between [6][7] and [9] will be discussed.

1 Introduction

Let E^* be the dual space of a nuclear space E. The complexification of E is denoted by $E_{\mathbb{C}}$. Let μ_G and μ_P be a Gaussian measure and a Poisson measure on E^* , respectively. It can be shown that the spaces of generalized functions $\Gamma_u(E_{\mathbb{C}})^*$ and test functions $\Gamma_u(E_{\mathbb{C}})$ derived from Boson Fock space are characterized in terms of analyticity and growth conditions on their J-transforms, Theorem 2.2 and 2.3, respectively. It is important to notice that the J-transform can be introduced independently from the structure of measures μ_X on E^*. In addition, we need not refer to the isomorphism

*May 9, 2002.

†Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan.

‡Department of Mathematics, Graduate School of Science, Hiroshima University, Higashi-Hiroshima, 739-8526, Japan.

§Department of Mathematics, Louisiana State University, Baton Rouge, Louisiana 70803, USA.

U_X between the Boson Fock space and $L^2(E^*, \mu_X)$ in order to characterize spaces $\Gamma_u(E_{\mathbb{C}})^*$ and $\Gamma_u(E_{\mathbb{C}})$ in terms of their J-transforms. In fact, the essential tools to prove Theorems 2.2 and 2.3 are the Cauchy integral formula for entire holomorphic functions of several variables, Legendre transform, dual function, Schwartz kernel theorem, and properties of the nuclear space. Hence the measure μ_X "seems" to play no role in the theory of generalized functions on infinite dimensional space. So we have the following two questions:

Q1. What is the role of $L^2(E^*, \mu_X)$?

Q2. How do we realize quantum Gaussian and Poisson white noises in terms of multiple Wiener-Itô integrals ?

Unfortunately, J-transform cannot give us exact answers to **(Q1)(Q2)** due to the lack of the isomorphism U_X.

To answer the natural questions above, it is necessary to construct the Gel'fand triple in terms of multiple Wiener-Itô integrals associated with μ_X on E^*. Of course, the spaces of generalized functions $[E]^*_{u,X}$ and test functions $[E]_{u,X}$ originated from $L^2(E^*, \mu_X)$, $X = G$ or P, can be characterized in terms of analyticity and growth conditions on their S_X-transforms, Theorems 4.1 and 4.3, respectively. As a result, one can see in Theorem 4.5 the correspondences between classical and quantum white noises of Gaussian and Poisson types. That is, a quantum Gaussian white noise is a Fock space realization of a classical Gaussian white noise $\dot{B}(t)$ and a quantum Poisson white noise is of a classical Poisson white noise $\dot{P}(t)$.

2 Gel'fand Triple in Terms of Boson Fock Space

Consider the special Hilbert space $H \equiv L^2(\mathbb{R}, dt)$ with norm $|\cdot|_0$. Let A be an operator in H such that there exists an orthonormal basis $\{e_j\}_{j=1}^{\infty}$ satisfying the conditions:

1. $Ae_j = \lambda_j e_j$,
2. $1 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots$,
3. $\sum_{n=0}^{\infty} \lambda_j^{-2\alpha} < \infty$ for some positive constant α.

For each $p \geq 0$, define the norm $|\xi|_p = |A^p \xi|_0$ and let

$$E_p = \{\xi \in H; |\xi|_p < \infty\}.$$

It can be shown that $E_p \subset E_q$ for any $p \geq q \geq 0$ and the inclusion map $i_{p+\alpha,p} : E_{p+\alpha} \hookrightarrow E_p$ is a Hilbert-Schmidt operator for any $p \geq 0$. Let $E = \operatorname{proj\,lim}_{p\to\infty} E_p$ and E^* be the dual space of E. Then E is a nuclear space and we obtain a base Gel'fand triple $E \subset H \subset E^*$ with the following continuous inclusions:

$$E \subset E_p \subset H \equiv E_0 \subset E_p^* \subset E^*, \qquad p \geq 0,$$

where the norm on E_p^* is given by $|f|_{-p} = |A^{-p} f|_0$.

Let $C_{+,1/2}$ denote the collection of all positive continuous functions u on $[0, \infty)$ satisfying

$$\lim_{r \to \infty} \frac{\log u(r)}{\sqrt{r}} = \infty. \tag{2.1}$$

For $u \in C_{+,1/2}$, the dual function u^* of u is given by

$$u^*(r) = \sup_{s>0} \frac{e^{2\sqrt{rs}}}{u(s)}, \quad r \in [0, \infty). \tag{2.2}$$

For later use, we introduce the following additional conditions on u.

(G1) $\inf_{r \geq 0} u(r) = 1$,

(G2) $\displaystyle\limsup_{r \to \infty} \frac{\log u(r)}{r} < \infty$,

(G3) $\log u(x^2)$ is a convex function for $x \in [0, \infty)$.

We denote the complexification of a real space K by $K_{\mathbb{C}}$. It is well-known that the Boson Fock space over $H_{\mathbb{C}}$, denoted by $\Gamma(H_{\mathbb{C}})$, is a Hilbert space consisting of sequences $(f_n)_{n=0}^{\infty}$, where $f_n \in H_{\mathbb{C}}^{\widehat{\otimes} n}$ and $\sum_{n=0}^{\infty} n! |f_n|_0^2 < \infty$. For $(f_n) \in \Gamma(H_{\mathbb{C}})$, $p \geq 0$, and a given function $u \in C_{+,1/2}$ satisfying the conditions $(G1)(G2)(G3)$, define

$$\|(f_n)\|_{\Gamma_u(E_{p,\mathbb{C}})} := \left(\sum_{n=0}^{\infty} \frac{1}{\ell_u(n)} |f_n|_p^2 \right)^{\frac{1}{2}}$$

where ℓ_u is the Legendre transform of u given by

$$\ell_u(t) = \inf_{r>0} \frac{u(r)}{r^t}, \quad t \in [0, \infty). \tag{2.3}$$

Technical details of Equations (2.2)(2.3) and $(G1)(G2)(G3)$ can be found in [6]. See also Appendix A. Let $\Gamma_u(E_{p,\mathbb{C}}) = \{(f_n)_{n=0}^{\infty} \in \Gamma(H_{\mathbb{C}})\ ;\ \|(f_n)\|_{\Gamma_u(E_{p,\mathbb{C}})} < \infty\}$ for each $p \geq 0$ and $\Gamma_u(E_{\mathbb{C}})$ be the space of test functions, which is the projective limit of the family $\{\Gamma_u(E_{p,\mathbb{C}})\ ;\ p \geq 0\}$. Hence $\Gamma_u(E_{\mathbb{C}}) \subset \Gamma(H_{\mathbb{C}})$ and the condition (b) implies that $\Gamma_u(E_{\mathbb{C}})$ is a nuclear space. The dual space $\Gamma_u(E_{\mathbb{C}})^*$ is called the space of generalized functions. By identifying $\Gamma(H_{\mathbb{C}})$ with its dual we get the following continuous inclusions:

$$\Gamma_u(E_{\mathbb{C}}) \hookrightarrow \Gamma_u(E_{p,\mathbb{C}}) \hookrightarrow \Gamma(H_{\mathbb{C}}) \hookrightarrow \Gamma_u(E_{p,\mathbb{C}})^* \hookrightarrow \Gamma_u(E_{\mathbb{C}})^*$$

and $\Gamma_u(E_{\mathbb{C}}) \subset \Gamma(H_{\mathbb{C}}) \subset \Gamma_u(E_{\mathbb{C}})^*$ is a Gel'fand triple. Note that we have used condition $(G2)$ in order to have the continuous inclusion $\Gamma_u(E_{p,\mathbb{C}}) \hookrightarrow \Gamma(H_{\mathbb{C}})$. The canonical bilinear form on $\Gamma_u(E_{\mathbb{C}})^* \times \Gamma_u(E_{\mathbb{C}})$ is denoted by $\langle\langle \cdot, \cdot \rangle\rangle_\Gamma$. For each $\Phi \in \Gamma_u(E_{\mathbb{C}})^*$, there exists a unique $F_n \in (E_{\mathbb{C}}^{\otimes n})^*_{symm}$ with

$$\|(F_n)\|_{\Gamma_u(E_{p,\mathbb{C}})^*} := \left(\sum_{n=0}^{\infty} (n!)^2 \ell_u(n) |F_n|_{-p}^2 \right)^{\frac{1}{2}} < \infty$$

for some $p \geq 0$ such that

$$\langle\langle (F_n), (f_n) \rangle\rangle_\Gamma = \sum_{n=0}^{\infty} n! \langle F_n, f_n \rangle.$$

Let δ_t be the Dirac delta function at $t \in \mathbb{R}$. In order to deal with annihilation and creation operators in the framework of white noise theory, we assume, from now on, additional hypothesises on the base triple $E \subset H \subset E^*$ as follows:

(H1) each function $\xi \in E$ has a continuous version $\widetilde{\xi}$,

(H2) $\delta_t \in E^*$ for all $t \in \mathbb{R}$ so that $\langle \delta_t, \xi \rangle = \tilde{\xi}(t)$,

(H3) the mapping $t \mapsto \delta_t$ is continuous with the strong topology for E^*,

(H4) E is an algebra.

Thus functions in E will be regarded to be continuous, and $\widetilde{\xi}$ be simply denoted by ξ throghout this paper.

Now we are ready to define an *annihilation operator* a_t by

$$a_t \big(f_n\big)_{n=0}^{\infty} = \big(n \delta_t \otimes_1 f_n\big)_{n=1}^{\infty}, \quad a_t \Omega = 0, \quad f_n \in E_{\mathbb{C}}^{\widehat{\otimes} n}$$

where $\otimes_1$ is the contraction of tensor product and Ω is the Fock vacuum. It is easy to show that a_t is a continuous linear operator from $\Gamma_u(E_{\mathbb{C}})$ into itself. The adjoint operator a_t^* of a_t is called a *creation operator*, given by

$$a_t^* \big(F_n\big)_{n=0}^{\infty} = \big(\delta_t \widehat{\otimes} F_n\big)_{n=0}^{\infty}, \quad F_n \in (E_{\mathbb{C}}^{\otimes n})^*_{symm},$$

and is a continous linear operator from $\Gamma_u(E_{\mathbb{C}})^*$ into itself. These operators satisfy the following canonical commutation relations:

$$[a_s, a_t] = 0, \quad [a_s^*, a_t^*] = 0, \quad [a_s, a_t^*] = \delta_s(t) I.$$

Note that operators a_t and a_t^* are not operator-valued distributions. In quantum stochastic calculus (cf. [11][24]), $a_t + a_t^*$ is called the *quantum Gaussian white noise* and $a_t + a_t^* + a_t^* a_t + I$ is called the *quantum Poisson white noise* acting on the same space $\Gamma_u(E_{\mathbb{C}})^*$ of generalized functions. We remark that we need the condition in $(H4)$ in order to discuss quantum Poisson white noise and exponential vector associated with Poisson measure given by (3.2). In addition, $(H4)$ will be necessary to prove Theorem 4.5 (2).

Since $u \in C_{+,1/2}$ satisfies $(G3)$, $\left(\frac{\xi^{\otimes n}}{n!}\right) \in \Gamma_u(E_{\mathbb{C}})$ for all $\xi \in E_{\mathbb{C}}$. So let us introduce the J-transform for the characterization of $\Gamma_u(E_{\mathbb{C}})$ and $\Gamma_u(E_{\mathbb{C}})^*$.

Definition 2.1 (J-transform). For $(F_n)_{n=0}^{\infty} \in \Gamma_u(E_{\mathbb{C}})^*$, *$J$-transform* is defined to be the function

$$(J(F_n))(\xi) = \left\langle\left\langle (F_n), \left(\frac{\xi^{\otimes n}}{n!}\right) \right\rangle\right\rangle_{\Gamma}, \quad \xi \in E_{\mathbb{C}}. \tag{2.4}$$

The next Theorems 2.2 and 2.3 claim respectively that $\Gamma_u(E_{\mathbb{C}})^*$ and $\Gamma_u(E_{\mathbb{C}})$ are characterized in terms of analyticity and growth order of J-transforms.

Theorem 2.2. *Suppose $u \in C_{+,1/2}$ satisfies conditions $(G1)(G2)(G3)$. Then a $\mathbb{C}$-valued function F on $E_{\mathbb{C}}$ is the J-transform of a generalized function in $\Gamma_u(E_{\mathbb{C}})^*$ if and only if it satisfies the conditions:*

(a) For any $\xi, \eta \in E_{\mathbb{C}}$, the function $F(z\xi + \eta)$ is an entire holomorphic function of $z \in \mathbb{C}$.

(b) There exist constants $K, a, p \geq 0$ such that

$$|F(\xi)| \leq K u^*(a|\xi|_p^2)^{1/2}, \quad \forall \xi \in E_{\mathbb{C}}.$$

Theorem 2.3. *Suppose $u \in C_{+,1/2}$ satisfies conditions $(G1)(G2)(G3)$. Then a $\mathbb{C}$-valued function F on $E_{\mathbb{C}}$ is the J-transform of a test function in $\Gamma_u(E_{\mathbb{C}})$ if and only if it satisfies the conditions:*

(a) For any $\xi, \eta \in E_{\mathbb{C}}$, the function $F(z\xi + \eta)$ is an entire holomorphic function of $z \in \mathbb{C}$.

(b)' For any constants $a, p \geq 0$, there exists a constant $K \geq 0$ such that

$$|F(\xi)| \leq K u(a|\xi|_{-p}^2)^{1/2}, \quad \forall \xi \in E_{\mathbb{C}}.$$

Since Theorems 2.2 and 2.3 can be obtained with the same technique given in [7], we omit their proofs. The reader should notice that $(H1)(H2)(H3)(H4)$ are not used at all to prove them. The well-known examples for u will be mentioned later in Example 3.3, Remarks 4.2 and 4.4.

Questions: Note that up to here we have not yet fixed any isomorphism U_X between $\Gamma(H_{\mathbb{C}})$ and $L^2(E^*, \mu_X)$, which will be given in the next Section 3. Hence, the measure μ_X plays virtually no role in the definition of J-transform. Moreover we do not necessarily quote U_X to characterize spaces

$\Gamma_u(E_{\mathbb{C}})$ and $\Gamma_u(E_{\mathbb{C}})^*$ in terms of their J-transforms. However, it does not mean that the considerations of J-transform and its holomorphy are enough to study our questions **(Q1)(Q2)**. A crucial point is as follows. It is easy to see that the flow Φ_t given by

$$\Phi_t = \begin{cases} (0, 1_{[0,t]}, 0, \cdots) & \text{for each } t \geq 0 \\ (0, -1_{[t,0]}, 0, \cdots) & \text{for each } t < 0, \end{cases}$$

is an element of $\Gamma(H_{\mathbb{C}})$. Then the tangent vector $\dot{\Phi}_t$ is $(0, \delta_t, 0, \cdots)$ and belongs to $\Gamma_u(E_{\mathbb{C}})^*$ with $u(r) = e^r$ and $J\dot{\Phi}_t(\xi) = \xi(t)$. However, Theorems 2.2 and 2.3 cannot distinguish wheather $\dot{\Phi}_t$ is the tangent vector corresponding to Brownian motion or (compensated) Poisson process in both classical and quantum contexts, for example. That is,

$$\dot{\Phi}_t \leftrightarrow \text{Which type of white noises ?}$$

This is not the problem on topology and holomorphy, but is on sample functions and measures on E^*. We will make comments on this point in Sections 3 and 4.

3 Gel'fand Triples in Terms of Multiple Wiener-Itô Integrals Associated with Gaussian and Poisson Measures

To answer our questions **(Q1)(Q2)**, we shall construct the Gel'fand triple in terms of multiple Wiener-Itô integrals associated with Gaussian and Poisson measures on E^*. One can see the correspondence between classical and quantum white noises of Gaussian and Poisson types. In the following, we quickly summarize the essence of Gaussian white noise theory from [19][20][23] and Poisson white noise theory from [13]. From now on, we always suppose $(H1)(H2)(H3)$ to discuss the Gaussian part. On the other hand, we also assume $(H4)$ in addition to $(H1)(H2)(H3)$ when the Poisson part is discussed.

Let μ_G be the standard Gaussian measure on E^* given by

$$\int_{E^*} \exp[i\langle x, \xi\rangle] d\mu_G(x) = \exp\Big[-\frac{1}{2}|\xi|_0^2\Big], \quad \xi \in E$$

and μ_P be the Poisson measure on E^* by

$$\int_{E^*} \exp[i\langle x, \xi\rangle] d\mu_P(x) = \exp\Big[\int_{\mathbb{R}} (e^{i\xi(t)} - 1) dt\Big], \quad \xi \in E.$$

Let us denote the complex Hilbert space $L^2(E^*, \mu_X)$ by $(L^2)_X$, $X = G,\ P$ and the multiple Wiener-Itô integrals with respect to a measure μ_X by

$I_n^X(f_n)$ for $f_n \in H_{\mathbb{C}}^{\widehat{\otimes} n}$. Then each $\varphi \in (L^2)_X$ is uniquely decomposed as

$$\varphi(x) = \sum_{n=0}^{\infty} I_n^X(f_n), \quad f_n \in H_{\mathbb{C}}^{\widehat{\otimes} n}.$$

It is important to notice that there exist unitary isomorphisms U_X between $(L^2)_X$, $X = G, P$, and $\Gamma(H_{\mathbb{C}})$ determined uniquely by the *exponential functions (vectors)*

$$\phi_\xi^G(x) \equiv \exp\Big[\langle x, \xi\rangle - \frac{1}{2}|\xi|_0^2\Big] \longleftrightarrow \left(\frac{\xi^{\otimes n}}{n!}\right)_{n=0}^{\infty} \equiv e(\xi), \quad \xi \in E_{\mathbb{C}} \tag{3.1}$$

when $X = G$ and

$$\phi_\xi^P(x) \equiv \exp\Big[\langle x, \log(1+\xi)\rangle - \int_{\mathbb{R}} \xi(t)dt\Big] \longleftrightarrow e(\xi), \quad \xi \in E_{\mathbb{C}} \tag{3.2}$$

when $X = P$, respectively. We remark that the Poisson case is not addressed in [3][4][5][7][8][9][18][20][23]. It is known that the linear span of the set $\{\phi_\xi^X\ ; \xi \in E_{\mathbb{C}}\}$ is dense in $[E]_{u,X}$ and $\{e(\xi); \xi \in E_{\mathbb{C}}\}$ does the same for $\Gamma_u(E_{\mathbb{C}})$. In those cases, it holds that the $(L^2)_X$-norm of φ is given by

$$\|\varphi\|_0^2 = \int_{E^*} |\varphi(x)|^2 d\mu_X(x) = \sum_{n=0}^{\infty} n!|f_n|_0^2.$$

The S_X-transform of $\varphi \in (L^2)_X$, given by

$$(S_X\varphi)(\xi) := \int_{E^*} \varphi(x)\phi_\xi^X(x)d\mu_X(x), \quad \xi \in E_{\mathbb{C}},$$

is an isomorphism from $(L^2)_X$ onto the Hilbert space $\mathcal{K}$ of holomorphic functions F on $E_{\mathbb{C}}$ with a reproducing kernel $\exp[\langle \xi, \eta\rangle]$, $\xi, \eta \in E$. We remark here that S_G has been recognized as an extension of the *Segal-Bargmann transform* to generalized functions on infinite dimensional space [10].

Let

$$[E_p]_{u,X} = \Big\{\varphi \in (L^2)_X\ ;\ \|\varphi\|_{p,u} := \sum_{n=0}^{\infty} \frac{1}{\ell_u(n)}|f_n|_p^2 < \infty\Big\}$$

and $[E]_{u,X}$ be the space of test functions, which is the projective limit of the family $\{[E_p]_{u,X}\ ;\ p \geq 0\}$. Hence $[E]_{u,X} \subset (L^2)_X$ by condition $(G2)$, and the condition (b) implies that $[E]_{u,X}$ is a nuclear space. The dual space $[E]_{u,X}^*$ is called the space of generalized functions. Then we obtain the following continuous inclusions:

$$[E]_{u,X} \hookrightarrow [E_p]_{u,X} \hookrightarrow (L^2)_X \hookrightarrow [E_p]_u^* \hookrightarrow [E]_{u,X}^*,$$

and $[E]_{u,X} \subset (L^2)_X \subset [E]^*_{u,X}$ is a Gel'fand triple.

Let $\partial_{t,G}$ be the *Gâteaux derivative* in the direction of δ_t, so-called *Hida derivative* and $\partial^*_{t,G}$ be the adjoint operator. In addition, let $\partial_{t,P}$ be the *difference operator*, given by $\partial_{t,P}\varphi(x) = \varphi(x+\delta_t) - \varphi(x), \varphi \in [E]_{u,P}$, and $\partial^*_{t,P}$ be the adjoint operator. Then it can be shown that $\partial_{t,X}$ is a continuous linear operator from $[E]_{u,X}$ into itself and $\partial^*_{t,X}$ is a continuous linear operator from $[E]^*_{u,X}$ into itself.

It is known that the Brownian motion $B(t)$ is represented by

$$B(t) = \begin{cases} I_1^G(1_{[0,t]}) & \text{if } t \geq 0 \\ -I_1^G(1_{[t,0]}) & \text{if } t < 0. \end{cases} \tag{3.3}$$

Similarly, the compensated Poisson process is given by

$$P(t) - t = \begin{cases} I_1^P(1_{[0,t]}) & \text{if } t \geq 0 \\ -I_1^P(1_{[t,0]}) & \text{if } t < 0. \end{cases} \tag{3.4}$$

Since characteristic functions $1_{[0,t]}$ and $1_{[t,0]}$ are elements of H, $B(t)$ and $P(t)-t$ are in $(L^2)_G$ and $(L^2)_P$, respectively. Hence we obtain $U_G \Phi_t U_G^{-1} = B(t)$ and $U_P \Phi_t U_P^{-1} = P(t) - t$.

On the other hand, the distributional derivative of $B(t)$ with respect to t, so-called *Gaussian white noise* $\dot{B}(t)$, has the form $\dot{B}(t) = I_1^G(\delta_t)$ for each $t \in \mathbb{R}$. Similarly, the *Poisson white noise* $\dot{P}(t)$ has the expression $\dot{P}(t) - 1 = I_1^P(\delta_t)$ for each $t \in \mathbb{R}$. In those cases, since δ_t is in E^*, $\dot{B}(t)$ and $\dot{P}(t) - 1$ belong to $[E]^*_{u,G}$ and $[E]^*_{u,P}$, respectively (A function u will be chosen in the proof of Theorem 4.5). Thus, we get the relationship between the vector $\dot{\Phi}_t$ and classical Gaussian and Poisson white noises as follows. This is the partial answer to the question brought up in the end of Section 2.

Proposition 3.1. *The following equalities hold*
(1) $U_G \dot{\Phi}_t U_G^{-1} = \dot{B}(t)$,
(2) $U_P \dot{\Phi}_t U_P^{-1} = \dot{P}(t) - 1$.

Next, since $u \in C_{+,1/2}$ satisfies $(G3)$, the exponential function $\phi_\xi^X(x) \in [E]_{u,X}$ for any $\xi \in E_{\mathbb{C}}$. Hence the S_X-transform can be extended to a continuous linear functional on $[E]^*_{u,X}$ as follows.

Definition 3.2 (S_X-transform). For $\Phi \in [E]^*_{u,X}$, S_X*-transform* is defined by

$$(S_X\Phi)(\xi) = \langle\langle \Phi, \phi_\xi^X \rangle\rangle_X, \quad \xi \in E_{\mathbb{C}}, \tag{3.5}$$

where $\langle\langle \cdot, \cdot \rangle\rangle$ is the bilinear pairing of $[E]^*_{u,X}$ and $[E]_{u,X}$.

This transform plays essential role to study connections between classical and quantum white noises.

Example 3.3. The Gel'fand triple $[E]_{u,X} \subset (L^2)_X \subset [E]^*_{u,X}$ becomes
(1) the *Hida-Kubo-Takenaka space* [19][20][23] if $X = G$ and $u(r) = e^r$, and the *Ito-Kubo space* [13] if $X = P$ and $u(r) = e^r$,
(2) the *Kondratiev-Streit space* [17] if $X = G$ and $u(r) = \exp[(1+\beta)r^{\frac{1}{1+\beta}}]$ for $0 \le \beta < 1$,
(3) the *Cochran-Kuo-Sengupta (CKS) space* of Bell numbers with degree k if $X = G$ and $u^*(r) = \exp_k(r)/\exp_k(0)$, where $\exp_k(r)$ is the k-th iterated exponential function [8]. Consult papers [3][4][5][6][8] for more general construction of CKS space and [2][18] for more details on Bell numbers.

Remark 3.4. We exclude the case of the Kondratiev-Streit space of $\beta = 1$ [15]. It is because the function $u(r) = \exp[2\sqrt{r}]$ does not satisfy Equation (2.1). Hence the exponential functions (3.1)(3.2) do not make sense for $\xi \in E_{\mathbb{C}}$ with $2|\xi|_p \ge 1$, so that J and S_X-transforms of generalized functions are defined only for $\xi \in E_{\mathbb{C}}$ with $2|\xi|_p < 1$.

4 Characterization Theorems and Quantum White Noises

Now we come to the characterization of $[E]^*_{u,X}$ associated with μ_X, $X = G, P$, in a single statement. The proof is almost the same as that in [7], but it is under $(G1)(G2)^*(G3)$ only with μ_G. The condition $(G2)^*$ is given in Appendix A.

Theorem 4.1. *Let a measure μ_X on E^* be given. Suppose $u \in C_{+,1/2}$ satisfies conditions $(G1)(G2)(G3)$. Then a $\mathbb{C}$-valued function F on $E_{\mathbb{C}}$ is the S_X-transform of a generalized function in $[E]^*_{u,X}$ if and only if it satisfies the conditions:*

(a) For any $\xi, \eta \in E_{\mathbb{C}}$, the function $F(z\xi + \eta)$ is an entire holomorphic function of $z \in \mathbb{C}$.

(b) There exist constants $K, a, p \ge 0$ such that

$$|F(\xi)| \le K u^*(a|\xi|_p^2)^{1/2}, \quad \forall \xi \in E_{\mathbb{C}}.$$

Remark 4.2. Theorem 4.1 was first proved by Potthoff-Streit [25] in case of $X = G$ and $u^*(r) = e^r$. It was extended to the case of $X = G$ and $u^*(r) = \exp[(1-\beta)r^{\frac{1}{1-\beta}}]$ by Kondratiev-Streit [16][17]. Moreover, Cochran et al. [8] proved the case when $X = G$ and the growth condition (b) is determined by the exponential generating function $G_\alpha(r) = \sum \frac{\alpha(n)}{n!} r^n$. Asai et al. [4][6][7] minimized conditions on sequences $\{\alpha(n)\}$ of positive real numbers in such a way that the theorem holds.

Similarly, the characterization of $[E]_{u,X}$ associated with μ_X, $X = G, P$, is stated below in a single statement. The proof is almost the same as that in [7], but it is under $(G1)(G2)^*(G3)$ only with μ_G .

Theorem 4.3. *Let a measure μ_X on E^* be given. Suppose $u \in C_{+,1/2}$ satisfies conditions $(G1)(G2)(G3)$. Then a $\mathbb{C}$-valued function F on $E_{\mathbb{C}}$ is the S_X-transform of a test function in $[E]_{u,X}$ if and only if it satisfies the conditions:*

(a) For any $\xi, \eta \in E_{\mathbb{C}}$, the function $F(z\xi + \eta)$ is an entire holomorphic function of $z \in \mathbb{C}$.

(b)' For any constants $a, p \geq 0$, there exists a constant $K \geq 0$ such that

$$|F(\xi)| \leq K u(a|\xi|_{-p}^2)^{1/2}, \quad \forall \xi \in E_{\mathbb{C}}.$$

Remark 4.4. Theorem 4.3 was proved by Kuo et al. [21] in case of $X = G$ and $u(r) = e^r$. It was extended to the case of $X = G$ and $u(r) = \exp[(1+\beta)r^{\frac{1}{1+\beta}}]$ by Kondratiev-Streit [17]. Moreover, Asai et al. [3] proved the case when $X = G$ and the growth condition (b)' is determined by the exponential generating function $G_{1/\alpha}(r) = \sum \frac{1}{\alpha(n)n!} r^n$. Asai et al. [4][6][7] minimized conditions on sequences $\{\alpha(n)\}$ of positive real numbers in such a way that the theorem holds.

In the next Theorem 4.5, the exponential functions (vectors) given by Equations (3.1) (3.2) will play essential roles to characterize the type of white noise. Before stating and proving this theorem, let us point out that the multiplication operator by $\dot{B}(t)$ has the expression [19],

$$\dot{B}(t)\cdot = \partial_{t,G} + \partial_{t,G}^*, \tag{4.1}$$

and the multiplication operator by $\dot{P}(t)$ has the form [13],

$$\dot{P}(t)\cdot = \partial_{t,P} + \partial_{t,P}^* + \partial_{t,P}^*\partial_{t,P} + I. \tag{4.2}$$

Remember that $\partial_{t,X}$ and $\partial_{t,X}^*$ are the operators in the stage of *Schrödinger representation*. On the other hand, a_t and a_t^* are the operators in the stage of *Fock space representation*.

Theorem 4.5. *(1) The quantum Gaussian white noise $a_t + a_t^*$ can be realized as a classical Gaussian white noise $\dot{B}(t)$ in $(E)_G^*$.*
(2) The quantum Poisson white noise $a_t + a_t^* + a_t^* a_t + I$ can be realized as a classical Poisson white noise $\dot{P}(t)$ in $(E)_P^*$.

Proof. First, we consider the Gaussian case $X = G$. For any $\xi, \eta \in E_{\mathbb{C}}$,

$$\begin{aligned} (J(a_t + a_t^*)e(\xi))(\eta) &= \langle\langle (a_t + a_t^*)e(\xi), e(\eta)\rangle\rangle_\Gamma \\ &= (\xi(t) + \eta(t))e^{\langle \xi, \eta \rangle}. \end{aligned} \tag{4.3}$$

On the other hand, by Equation (4.1), we have

$$\begin{aligned}(S_G\dot{B}(t)\phi_\xi^G)(\eta) &= (S_G[\partial_{t,G}+\partial_{t,G}^*]\phi_\xi^G)(\eta)\\ &= \langle\langle[\partial_{t,G}+\partial_{t,G}^*]\phi_\xi^G,\phi_\eta^G\rangle\rangle_G\\ &= (\xi(t)+\eta(t))e^{\langle\xi,\eta\rangle}. \qquad (4.4)\end{aligned}$$

Due to Equations (4.3)(4.4), we have $U_G(a_t+a_t^*)U_G^{-1}=\dot{B}(t)$, where $\dot{B}(t)$ is considered as a multiplication operator. In fact, $(S_G\dot{B}(t))(\xi)=\langle\delta_t,\xi\rangle=\xi(t)$ satisfies the condition (b) with $u(r)=\exp(r)$ in Theorem 4.1. Hence we get $\dot{B}(t)\in(E)_G^*$. Therefore, we have finished to prove our first assertion.

Next consider the Poisson case $X=P$. For any $\xi,\eta\in E_{\mathbb{C}}$,

$$\begin{aligned}(J(a_t+a_t^*+a_t^*a_t+I)e(\xi))(\eta) &= \langle\langle(a_t+a_t^*+a_t^*a_t+I)e(\xi),e(\eta)\rangle\rangle_\Gamma\\ &= (\xi(t)+\eta(t)+\eta(t)\xi(t)+1)e^{\langle\xi,\eta\rangle}. \qquad (4.5)\end{aligned}$$

Note that the function $\eta\xi$ above makes sense as a member of $E_{\mathbb{C}}$ due to $(H4)$. On the other hand, by Equation (4.2), we have

$$\begin{aligned}(S_P\dot{P}(t)\phi_\xi^P)(\eta) &= (S_P[\partial_{t,P}+\partial_{t,P}^*+\partial_{t,P}^*\partial_{t,P}+I]\phi_\xi^P)(\eta)\\ &= \langle\langle[\partial_{t,P}+\partial_{t,P}^*+\partial_{t,P}^*\partial_{t,P}+I]\phi_\xi^P,\phi_\eta^P\rangle\rangle_P\\ &= (\xi(t)+\eta(t)+\eta(t)\xi(t)+1)e^{\langle\xi,\eta\rangle}. \qquad (4.6)\end{aligned}$$

By Equations (4.5)(4.6), we have $U_P(a_t+a_t^*+a_t^*a_t+I)U_P^{-1}=\dot{P}(t)$, where $\dot{P}(t)$ is considered as a multiplication operator. In fact, $(S_P\dot{P}(t))(\xi)=\langle\delta_t,\xi\rangle+1=\xi(t)+1$ satisfies the condition (b) with $u(r)=\exp(r)$ in Theorem 4.1. Hence we get $\dot{P}(t)\in(E)_P^*$. Thus we have proved the second claim. □

Therefore, the quantum Gaussian white noise $a_t+a_t^*$ and the quantum Poisson white noise $a_t+a_t^*+a_t^*a_t+I$ are realized on the common generalized function space $\Gamma_u(E_{\mathbb{C}})^*$ with $u(r)=e^r$. To get such beautiful realizations of white noises on Fock space, the choice of exponential functions is one of essential points. Consult the explanation in the end of Appendix A to make this point clearer.

A Relationships with the Work by Gannoun et al. [9]

In the rest of this paper, we shall discuss some of similarities and differences between our papers [6][7] and Gannoun et al. [9] (GHOR for simplicity). We refer the readers to consult the papers [4][6][7] for more technical and delicate differences, which will not be mentioned in this paper.

First, the basic equalities are

$$u(r)=e^{2\theta(\sqrt{r})},\quad u^*(r)=e^{2\theta^*(\sqrt{r})}$$

where $\theta^*(s) = \sup_{t>0}\{st - \theta(t)\}$ is adopted in GHOR. In the following table we give the correspondence between our G-conditions and their θ-conditions.

	u	θ
$C_{+,1/2}$	$\lim_{r\to\infty} \frac{\log u(r)}{\sqrt{r}} = \infty$	$\lim_{r\to\infty} \frac{\theta(r)}{r} = \infty$
$(G1)$	$\inf_{r\geq 0} u(r) = 1$	
$(G1)^*$	u is increasing and $u(0) = 1$	θ is increasing and $\theta(0) = 0$
$(G2)$	$\limsup_{r\to\infty} \frac{\log u(r)}{r} < \infty$	
$(G2)^*$	$\lim_{r\to\infty} \frac{\log u(r)}{r} < \infty$	$\lim_{r\to\infty} \frac{\theta(r)}{r^2} < \infty$
$(G3)$	u is (log, x^2)-convex	θ is convex

Assume that $u \in C_{+,1/2}$ satisfies $(G1)(G2)(G3)$ conditions. For $p \geq 0$, let $\mathcal{A}_{p,u}$ consist of all functions φ on $E_{p,\mathbb{C}}$ satisfying the conditions:

(a) φ is an analytic function on $E^*_{p,\mathbb{C}}$.

(b) There exists a constant $C \geq 0$ such that

$$|\varphi(x)| \leq Cu(|x|^2_{-p})^{1/2}, \quad \forall x \in E^*_{p,\mathbb{C}}.$$

For each $\varphi \in \mathcal{A}_{p,u}$, we define

$$\|\varphi\|_{\mathcal{A}_{p,u}} = \sup_{x\in E^*_{p,\mathbb{C}}} |\varphi(x)|u(|x|^2_{-p})^{-1/2}. \tag{A.1}$$

Then $\mathcal{A}_{p,u}$ is a Banach space with norm $\|\cdot\|_{\mathcal{A}_{p,u}}$. Let $\mathcal{A}_u$ be the projective limit of $\{\mathcal{A}_{p,u}; p \geq 0\}$ and $\mathcal{A}^*_u$ be the dual space of $\mathcal{A}_u$. This construction is motivated by the analytic extension of test functions in Gaussian white noise theory done by Lee [22] (See also [14][20]). This direct construction is useful to characterize generalized measures [1][5][20][22], for example.

GHOR defined the same intrinsic topology as (A.1), independently, and proved the topological equivalence between $\mathcal{A}_u$ and $\Gamma_u(E_\mathbb{C})$ for $u \in C_{+,1/2}$ satisfying $(G1)^*(G2)^*(G3)$. On the other hand, Asai et al. examined the equivalence between $\mathcal{A}_u$ and $[E]_{u,G}$ for $u \in C_{+,1/2}$ satisfying $(G1)(G2)^*(G3)$. Actually, $(G2)^*$ is slightly stronger than $(G2)$. However, $(G2)$ is strong enough to guarantee that the nuclear spaces $\Gamma_u(E_\mathbb{C})$ and $[E]_{u,X}$ are the subspaces of $\Gamma(H_\mathbb{C})$ and $(L^2)_X$, respectively. Moreover, although $(G1)$ is weaker than $(G1)^*$, by Lemma 3.1 in [7] we can construct an equivalent function satisfying $(G1)^*$ even if we begin with $(G1)$. Thus we have the following Theorem A.1 under slightly weaker assumptions on u.

Theorem A.1. *Suppose $u \in C_{+,1/2}$ satisfies conditions (G1)(G2)(G3). Then the families of norms $\{\|\cdot\|_{p,u}; p \geq 0\}$ and $\{\|\cdot\|_{\mathcal{A}_{p,u}}; p \geq 0\}$ are equivalent.*

In [4][7], not only the general construction of spaces, but also the minimal conditions on u are examined to carry out white noise operator theory. This consideration is quite important to discuss the continuity of various operators, Wick products and so on. See also [2][18][23]. This matter is not addressed in the paper by GHOR. We emphasize again that GHOR concerned topological equivalences among various spaces of test and generalized functions in terms of different representation spaces. Therefore, one cannot find the answers to our questions **(Q1)(Q2)** in GHOR because these matters are not consequences of topological aspects.

The following diagram shows relations with several triples and horizontal arrows indicate continous inclusions.

$$\begin{array}{ccccc}
\Gamma_u(E_{\mathbb{C}}) = F_\theta(E_{\mathbb{C}}) & \longrightarrow & \Gamma(H_{\mathbb{C}}) & \longrightarrow & \Gamma_u(E_{\mathbb{C}})^* = F_\theta(E_{\mathbb{C}})^* \cong G_\theta(E_{\mathbb{C}}^*) \\
{\scriptstyle U_X}\Big\downarrow & & {\scriptstyle U_X}\Big\downarrow & & {\scriptstyle U_X}\Big\downarrow \\
[E]_{u,X} & \longrightarrow & (L^2)_X & \longrightarrow & [E]^*_{u,X} \\
{\scriptstyle S_X}\Big\downarrow & & {\scriptstyle S_X}\Big\downarrow & & {\scriptstyle S_X}\Big\downarrow \\
\mathcal{A}_u = \mathcal{F}_\theta(E_{\mathbb{C}}^*) & \longrightarrow & \mathcal{K} = \mathcal{F}ock(H_{\mathbb{C}}) & \longrightarrow & \mathcal{A}_u^* = \mathcal{F}_\theta(E_{\mathbb{C}}^*)^* \cong \mathcal{G}_{\theta^*}(E_{\mathbb{C}})
\end{array}$$

where $\mathcal{F}ock(H_{\mathbb{C}})$ denotes the holomorphic function's realization of $\Gamma(H_{\mathbb{C}})$ and notations containing θ were used by GHOR. Moreover, $\cong$ means a topological equivalence. The mappings U_X and S_X are unitary. The first row is the Gel'fand triple in terms of the *Fock space representation*, the second row is of the *Schrödinger representation* and the third row is of the *Segal-Bargmann representation.*

The Laplace transform $\mathcal{L}$ is used in GHOR. We know the following relationships between our S_G, S_P-transforms and $\mathcal{L}$:

$$S_G\Phi(\xi) = \exp\Big(-\frac{1}{2}|\xi|_0^2\Big)\mathcal{L}\Phi(\xi), \quad S_P\Phi(e^\xi - 1) = \exp\Big(-\int_{\mathbb{R}}(e^{\xi(t)} - 1)dt\Big)\mathcal{L}\Phi(\xi)$$

$\xi \in E_{\mathbb{C}}$ for $\Phi \in [E]_u^*((F_n) \in \Gamma_u(E_{\mathbb{C}})^*)$. In this situation [12] S_P is an isomorphism from $(L^2)_P$ onto a reproducing kernel Hilbert space with kernel

$$\exp\langle e^\xi - 1, e^\eta - 1\rangle, \quad \xi, \eta \in E.$$

This reproducing kernel Hilbert space is different from $\mathcal{K}$. So if the set of exponential functions

$$\left\{\exp\left[\langle x, \xi\rangle - \int_{\mathbb{R}}(e^{\xi(t)} - 1)\, dt\right] \Big| \, \xi \in E\right\}$$

is taken to determine the isomorphism between $\Gamma(H_{\mathbb{C}})$ and $(L^2)_P$, it can be proved that the quantum Poisson white noise $a_t + a_t^* + a_t^* a_t + I$ is realized on $\Gamma_u(E_{\mathbb{C}})^*$ with $u(r) = \exp[e^r - 1]$. Hence the Laplace transform $\mathcal{L}$ is not appropriate to represent quantum Gaussian and Poisson white noises on the common Fock space $\Gamma_u(E_{\mathbb{C}})^*$ with $u(r) = e^r$.

Acknowledgments

Parts of the results in this paper were based upon Asai's talk in the AMS special session "Analysis on Infinite Dimensional Spaces (in honor of L. Gross)" during the AMS-MAA Joint Mathematics Meetings in New Orleans, USA, January 10–13, 2001. He wants to give his deepest appreciation to Professors H.-H. Kuo and A. Sengupta for the warm hospitality during his visit to Louisiana State University (January 8–16, 2001).

References

[1] N. Asai, *A note on general setting of white noise analysis and positive generalized functions*, RIMS Kokyuroku (Kyoto) **1139** (2000), 19–29.

[2] N. Asai, I. Kubo, and H.-H. Kuo, *Bell numbers, log-concavity, and log-convexity*, Acta. Appl. Math., **63** (2000), 79–87.

[3] N. Asai, I. Kubo, and H.-H. Kuo, *Characterization of test functions in CKS-space*, in: "Mathematical Physics and Stochastic Analysis: in honor of L. Streit", S. Albeverio et al. (eds.) World Scientific, (2000), pp. 68–78.

[4] N. Asai, I. Kubo, and H.-H. Kuo, *CKS-space in terms of growth functions*, in: "Quantum Information II", T. Hida and K. Saitô. (eds.) World Scientific, (2000), pp. 17–27.

[5] N. Asai, I. Kubo, and H.-H. Kuo, *Characterization of Hida measures in white noise analysis*, in: "Infinite Dimensional Harmonic Analysis", H. Heyer et al. (eds.) D.+M. Gräbner, (2000), pp. 70–83.

[6] N. Asai, I. Kubo, and H.-H. Kuo, *Roles of log-concavity, log-convexity, and growth order in white noise analysis*, Infinite Dimensional Analysis, Quantum Probability and Related Topics, **4** (2001), 59–84.

[7] N. Asai, I. Kubo, and H.-H. Kuo, *General characterization theorems and intrinsic topologies in white noise analysis*, Hiroshima Math. J., **31** (2001), 299–330.

[8] W. G. Cochran, H.-H. Kuo, and A. Sengupta, *A new class of white noise generalized functions*, Infinite Dimensional Analysis, Quantum Probability and Related Topics, **1** (1998), 43–67.

[9] R. Gannoun, R. Hachaichi, H. Ouerdiane, and A. Rezgui, *Un théorème de dualite entré espaces de fonctions holomorphes à croissance exponentiele*, J. Funct. Anal., **171** (2000), 1–14.

[10] L. Gross and P. Malliavin, *Hall's transform and the Segal-Bargmann map.* in: Itô's Stochastic Calculus and Probability Theory, M. Fukushima et al. (eds.) Springer-Verlag, (1996), pp. 73–116.

[11] R. L. Hudson and K. R. Parthasarathy, *Quantum Ito's formula and stochastic evolutions*, Comm. Math. Phys., **93** (1984), 301–323.

[12] Y. Ito, *Generalized Poisson functionals*, Prob. Th. Rel. Fields, **77** (1988), 1–28.

[13] Y. Ito and I. Kubo, *Calculus on Gaussian and Poisson white noises*, Nagoya Math. J., **111** (1988), 41–84.

[14] Yu. G. Kontratiev, *Nuclear spaces of entire functions in problems of infinite dimensional analysis*, Soviet Math. Dokl., **22** (1980), 588–592.

[15] Yu. G. Kondratiev, P. Leukert, and L. Streit, *Wick calculus in white noise analysis*, Acta Appl. Math., **44** (1996), 269–294.

[16] Yu. G. Kondratiev and L. Streit, *A remark about a norm estimate for white noise distributions*, Ukrainian Math. J., **44** (1992), 832–835.

[17] Yu. G. Kondratiev and L. Streit, *Spaces of white noise distributions: Constructions, Descriptions, Applications. I*, Reports on Math. Phys., **33** (1993), 341–366.

[18] I. Kubo, H.-H. Kuo, and A. Sengupta, *White noise analysis on a new space of Hida distributions*, Infinite Dimensional Analysis, Quantum Probability and Related Topics, **2** (1999), 315–335.

[19] I. Kubo and S. Takenaka, *Calculus on Gaussian white noise I, II, III, IV*, Proc. Japan Acad., **56A** (1980), 376–380, **56A** (1980), 411–416, **57A** (1981), 433–437, **58A** (1982), 186–189.

[20] H.-H. Kuo, *"White Noise Distribution Theory"*, CRC Press, 1996.

[21] H.-H. Kuo, J. Potthoff, and L. Streit, *A characterization of white noise test functionals*, Nagoya Math. J., **121** (1991), 185–194.

[22] Y.-J. Lee, *Analytic version of test functionals, Fourier transform, and a characterization of measures in white noise calculus*, J. Funct. Anal., **100** (1991), 359–380.

[23] N. Obata, *"White Noise Calculus and Fock Space"*, Lecture Notes in Math. **1577**, Springer-Verlag, 1994.

[24] N. Obata, *Generalized quantum stochastic processes on Fock space*, Publ. of RIMS Kyoto Univ., **31** (1995), 667–702.

[25] J. Potthoff and L. Streit, *A characterization of Hida distributions*, J. Funct. Anal., **101** (1991), 212–229.