Conjugacy Relationship between M-convex and L-convex Functions in Continuous Variables[1]

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September 30, 2002

Abstract

By extracting combinatorial structures in well-solved nonlinear combinatorial optimization problems, Murota (1996,1998) introduced the concepts of M-convexity and L-convexity to functions defined over the integer lattice. Recently, Murota–Shioura (2000, 2001) extended these concepts to polyhedral convex functions and quadratic functions in continuous variables. In this paper, we consider a further extension to more general convex functions defined over the real space, and provide a proof for the conjugacy relationship between general M-convex and L-convex functions.

Keywords: combinatorial optimization, matroid, base polyhedron, convex function, convex analysis.

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[1]This work is supported by Grant-in-Aid of the Ministry of Education, Culture, Sports, Science and Technology of Japan.

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1 Introduction

Combinatorial optimization problems with nonlinear objective functions have been dealt with more often than before due to theoretical interest and needs of practical applications. Extensive studies have been done for revealing the essence of the well-solvability in nonlinear combinatorial optimization problems [1, 3, 5, 6, 7, 11, 12, 20]. By extracting combinatorial structures in well-solved nonlinear combinatorial optimization problems, Murota [8, 9] introduced the concepts of M-convexity and L-convexity for functions defined over the integer lattice; subsequently, their variants called M$^\natural$-convexity and L$^\natural$-convexity were introduced by Murota–Shioura [13] and by Fujishige–Murota [4], respectively. Applications of M-/L-convexity can be found in mathematical economics with indivisible commodities [2, 17, 18], system analysis by mixed polynomial matrices [10], etc. Recently, Murota–Shioura [14, 15] extended these concepts to polyhedral convex functions and quadratic functions defined over the real space. In this paper, we consider a further extension to more general convex functions defined over the real space.

The concepts of M-convexity and L-convexity are defined for polyhedral convex functions and quadratic functions as follows. Let n be a positive integer, and put $N = \{1, 2, \ldots, n\}$. A polyhedral convex function (or quadratic function) $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ is said to be M-convex if $\mathrm{dom}\, f$ is nonempty and f satisfies (M-EXC):

(M-EXC) $\forall x, y \in \mathrm{dom}\, f$, $\forall i \in \mathrm{supp}^+(x-y)$, $\exists j \in \mathrm{supp}^-(x-y)$, $\exists \alpha_0 > 0$:

$$f(x) + f(y) \geq f(x - \alpha(\chi_i - \chi_j)) + f(y + \alpha(\chi_i - \chi_j)) \qquad (\forall \alpha \in [0, \alpha_0]), \tag{1.1}$$

where

$$\mathrm{dom}\, f = \{x \in \mathbf{R}^n \mid f(x) < +\infty\},$$
$$\mathrm{supp}^+(x-y) = \{i \in N \mid x(i) > y(i)\}, \quad \mathrm{supp}^-(x-y) = \{i \in N \mid x(i) < y(i)\},$$

$x(i)$ is the i-th component of a vector $x \in \mathbf{R}^n$ for $i \in N$, and $\chi_i \in \{0,1\}^n$ is the i-th unit vector for $i \in N$. On the other hand, a polyhedral convex function (or quadratic function) $g : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ is said to be L-convex if $\mathrm{dom}\, g \neq \emptyset$ and g satisfies (LF1) and (LF2):

(LF1) $g(p) + g(q) \geq g(p \wedge q) + g(p \vee q) \qquad (\forall p, q \in \mathrm{dom}\, g)$,
(LF2) $\exists r \in \mathbf{R}$ such that $g(p + \lambda \mathbf{1}) = g(p) + \lambda r \quad (\forall p \in \mathrm{dom}\, g,\ \forall \lambda \in \mathbf{R})$,

where $p \wedge q, p \vee q \in \mathbf{R}^n$ are defined by

$$(p \wedge q)(i) = \min\{p(i), q(i)\}, \quad (p \vee q)(i) = \max\{p(i), q(i)\} \quad (i \in N),$$

and $\mathbf{1} \in \mathbf{R}^n$ is the vector with all components equal to one.

To fully cover the well-solved nonlinear combinatorial optimization problems, it is desirable to further extend these concepts to more general convex functions defined over the real space on the basis of (M-EXC), and (LF1) and (LF2), respectively. It can be easily imagined that the previous results of M-/L-convexity for polyhedral convex functions and quadratic functions naturally extend to more general M-/L-convex functions. In particular, it is natural to imagine that the conjugacy relationship holds for general M-convex and L-convex functions over the real space, as in the cases of functions over the integer lattice [9, Th. 4.24], polyhedral convex functions [14, Th. 5.1], and quadratic functions [15, Th. 3.1]. However, the proof cannot be extended so directly to general M-/L-convex functions, but some technical difficulties such as topological issues arise. By taking such technical difficulties into consideration, we define M-convex and L-convex functions over the real space as closed proper convex functions satisfying (M-EXC), and (LF1) and (LF2), respectively. The primary contribution of this paper is to provide a rigorous proof of the following conjugacy relationship between general M-convex and L-convex functions over the real space.

Theorem 1.1. *For* $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ *with* $\mathrm{dom}\, f \neq \emptyset$*, define its conjugate function* $f^\bullet : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ *by*

$$f^\bullet(p) = \sup\{\langle p, x\rangle - f(x) \mid x \in \mathbf{R}^n\} \qquad (p \in \mathbf{R}^n), \tag{1.2}$$

where $\langle p, x\rangle = \sum_{i=1}^{n} p(i)x(i)$*.*
(i) *If* f *is M-convex, then* $f^\bullet$ *is L-convex and* $(f^\bullet)^\bullet = f$*.*
(ii) *If* g *is L-convex, then* $g^\bullet$ *is M-convex and* $(g^\bullet)^\bullet = g$*.*
(iii) *The mappings* $f \mapsto f^\bullet$ $(f$ *: M-convex) and* $g \mapsto g^\bullet$ $(g$ *: L-convex) provide a one-to-one correspondence between the classes of M-convex and L-convex functions, and are the inverse of each other.*

We also show that a conjugate pair of M-convex and L-convex functions arise from the minimum cost flow/tension problems.

The organization of this paper is as follows. Section 2 provides the precise definitions of M-/M$^\natural$-convex and L-/L$^\natural$-convex functions, and shows various examples of these functions. The conjugacy relationship between M-/L-convexity is proven in Section 3.

In this paper, we focus on the conjugacy relationship between M-convex and L-convex functions. See [16] for other properties of M-convex and L-convex functions in continuous variables.

2 M-convex and L-convex Functions over the Real Space

2.1 Definitions of M-convex and L-convex Functions

Let $f : \mathbf{R}^n \to \mathbf{R} \cup \{\pm\infty\}$ be a function. A function f is said to be *convex* if its epigraph $\{(x, \alpha) \in \mathbf{R}^n \times \mathbf{R} \mid \alpha \geq f(x)\}$ is a convex set. A convex function f with $f > -\infty$ is said to be *proper* if $\mathrm{dom}\, f \neq \emptyset$, and *closed* if its epigraph is a closed set. We denote by $\arg\min f$ the set of minimizers of f, i.e., $\arg\min f = \{x \in \mathbf{R}^n \mid f(x) \leq f(y)\ (\forall y \in \mathbf{R}^n)\}$, which can be an empty set. Note that for a closed proper convex function f, any level set $\{x \in \mathbf{R}^n \mid f(x) \leq \eta\}$ $(\eta \in \mathbf{R})$ is a closed set, and $\arg\min f \neq \emptyset$ if $\mathrm{dom}\, f$ is bounded.

We call a function $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ *M-convex* if it is closed proper convex and satisfies (M-EXC). The effective domain $\mathrm{dom}\, f$ of an M-convex function f is contained in a hyperplane $\{x \in \mathbf{R}^n \mid x(N) = r\}$ for some $r \in \mathbf{R}$, where $x(N) = \sum_{i=1}^{n} x(i)$.

Proposition 2.1. *If f is M-convex, then $x(N) = y(N)$ for all $x, y \in \mathrm{dom}\, f$.*

Proof. To the contrary assume $x(N) > y(N)$ for some $x, y \in \mathrm{dom}\, f$. Put

$$S = \{z \in \mathbf{R}^n \mid x \wedge y \le z \le x \vee y,\ f(z) \le \max\{f(x), f(y)\}\},$$

which is a bounded closed set. Let $x_*, y_* \in S$ minimize the value $||x_* - y_*||_1$ among all pairs of vectors in S with $x_*(N) = x(N)$ and $y_*(N) = y(N)$. The property (M-EXC) for x_* and y_* implies

$$f(x_*) + f(y_*) \ge f(x_* - \alpha(\chi_i - \chi_j)) + f(y_* + \alpha(\chi_i - \chi_j))$$

for some $i \in \mathrm{supp}^+(x_* - y_*)$, $j \in \mathrm{supp}^-(x_* - y_*)$, and a sufficiently small $\alpha > 0$. Putting $\widehat{x} = x_* - \alpha(\chi_i - \chi_j)$ and $\widehat{y} = y_* + \alpha(\chi_i - \chi_j)$, we have $\widehat{x} \in S$ or $\widehat{y} \in S$, a contradiction to the choice of x_* and y_* since $||\widehat{x} - y_*||_1 < ||x_* - y_*||_1$ and $||x_* - \widehat{y}||_1 < ||x_* - y_*||_1$. □

Hence, an M-convex function loses no information other than r when projected onto an $(n-1)$-dimensional space. We call a function $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ *M$^\natural$-convex* if the function $\widehat{f} : \mathbf{R}^{n+1} \to \mathbf{R} \cup \{+\infty\}$ defined by

$$\widehat{f}(x, x_{n+1}) = \begin{cases} f(x) & (x \in \mathbf{R}^n,\ x_{n+1} \in \mathbf{R},\ x_{n+1} = -x(N)), \\ +\infty & (\text{otherwise}) \end{cases}$$

is M-convex.

On the other hand, we call a function $g : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ *L-convex* if g is a closed proper convex function satisfying (LF1) and (LF2). Due to the property (LF2), an L-convex function loses no information other than r when restricted to a hyperplane $\{p \in \mathbf{R}^n \mid p(i) = 0\}$ for any $i \in N$. We call a function $g : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ *L$^\natural$-convex* if the function $\widehat{g} : \mathbf{R}^{n+1} \to \mathbf{R} \cup \{+\infty\}$ defined by

$$\widehat{g}(p, p_{n+1}) = g(p - p_{n+1}\mathbf{1}) \quad (p \in \mathbf{R}^n,\ p_{n+1} \in \mathbf{R})$$

is L-convex.

We denote by $\mathcal{M}_n$ (resp. $\mathcal{L}_n$) the class of M-convex (resp. L-convex) functions in n variables:

$$\begin{aligned} \mathcal{M}_n &= \{f \mid f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\},\ \text{M-convex}\}, \\ \mathcal{L}_n &= \{g \mid g : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\},\ \text{L-convex}\}. \end{aligned}$$

We define $\mathcal{M}_n^\natural$ and $\mathcal{L}_n^\natural$ to be the classes of M$^\natural$-convex and L$^\natural$-convex functions, respectively. As is obvious from the definitions, M$^\natural$-convex (resp. L$^\natural$-convex) function is essentially equivalent to M-convex (resp. L-convex) function, whereas the class of M$^\natural$-convex (resp. L$^\natural$-convex) functions contains that of M-convex (resp. L-convex) functions as a proper subclass. These relationships can be summarized as

$$\mathcal{M}_n \subset \mathcal{M}_n^\natural \simeq \mathcal{M}_{n+1}, \qquad \mathcal{L}_n \subset \mathcal{L}_n^\natural \simeq \mathcal{L}_{n+1}.$$

2.2 Examples

M-/M$^\natural$-convex and L-/L$^\natural$-convex functions have rich examples [11, 12, 14, 15].

Example 2.2 (affine functions). For $p_0 \in \mathbf{R}^n$ and $\beta \in \mathbf{R}$, the function $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ given by $f(x) = \langle p_0, x\rangle + \beta$ $(x \in \mathrm{dom}\, f)$ is M-convex or M$^\natural$-convex according as $\mathrm{dom}\, f = \{x \in \mathbf{R}^n \mid x(N) = 0\}$ or $\mathrm{dom}\, f = \mathbf{R}^n$. For $x_0 \in \mathbf{R}^n$ and $\nu \in \mathbf{R}$, the function $g : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ given by $g(p) = \langle p, x_0\rangle + \nu$ $(p \in \mathbf{R}^n)$ is L-convex as well as L$^\natural$-convex. □

We denote by $\mathcal{C}^1$ the class of univariate closed proper convex functions, i.e.,

$$\mathcal{C}^1 = \{\varphi : \mathbf{R} \to \mathbf{R} \cup \{+\infty\} \mid \varphi : \text{ closed proper convex}\}.$$

Recall that the conjugate function $f^\bullet$ of a function f is defined by (1.2).

Example 2.3. For $\varphi, \psi \in \mathcal{C}^1$, the functions $f, g : \mathbf{R}^2 \to \mathbf{R} \cup \{+\infty\}$ given by

$$\begin{aligned} f(x(1), x(2)) &= \begin{cases} \varphi(x(1)) & ((x(1), x(2)) \in \mathbf{R}^n,\ x(1) + x(2) = 0), \\ +\infty & (\text{otherwise}), \end{cases} \\ g(p(1), p(2)) &= \psi(p(1) - p(2)) \quad ((p(1), p(2)) \in \mathbf{R}^2) \end{aligned}$$

are M-convex and L-convex, respectively. Moreover, if φ and ψ are conjugate to each other, then f and g are conjugate to each other. □

Example 2.4 (separable-convex functions). Let $f_i \in \mathcal{C}^1$ $(i \in N)$ be a family of univariate convex functions. The function $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ defined by

$$f(x) = \sum_{i=1}^{n} f_i(x(i)) \qquad (x \in \mathbf{R}^n)$$

is M$^\natural$-convex as well as L$^\natural$-convex. The restriction of f to the hyperplane $\{x \in \mathbf{R}^n \mid x(N) = 0\}$ is M-convex if its effective domain is nonempty.

For functions $g_{ij} \in \mathcal{C}^1$ indexed by $i, j \in N$, the function $g : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ defined by

$$g(p) = \sum_{i=1}^{n} \sum_{j=1}^{n} g_{ij}(p(j) - p(i)) \qquad (p \in \mathbf{R}^n)$$

is L-convex with $r = 0$ in (LF2) if $\mathrm{dom}\, g \neq \emptyset$. □

Example 2.5 (quadratic functions). Let $A = (a(i,j))_{i,j=1}^{n} \in \mathbf{R}^{n \times n}$ be a symmetric matrix. Define a quadratic function $f : \mathbf{R}^n \to \mathbf{R}$ by $f(x) = (1/2)x^{\mathrm{T}}Ax$ $(x \in \mathbf{R}^n)$. Then, f is M$^\natural$-convex if and only if

$$x^{\mathrm{T}} a_i \geq \min\{x^{\mathrm{T}} a_j \mid j \in \mathrm{supp}^-(x)\} \qquad (\forall x \in \mathbf{R}^n,\ \forall i \in \mathrm{supp}^+(x)),$$

where a_i denotes the i-th column of A for $i \in N$. The function f is L$^\natural$-convex if and only if

$$a(i,j) \leq 0 \quad (\forall i, j \in N,\ i \neq j), \qquad \sum_{i=1}^{n} a(i,j) \geq 0 \quad (\forall j \in N).$$

□

Example 2.6 (minimum cost flow/tension problems). M-/L-convex functions arise from the minimum cost flow/tension problems with nonlinear cost functions.

Let $G = (V, A)$ be a directed graph with a specified vertex subset $T \subseteq V$. Suppose that we are given a family of convex functions $f_a \in \mathcal{C}^1$ $(a \in A)$, each of which represents the cost of flow on arc a. A vector $\xi \in \mathbf{R}^A$ is called a flow, and the boundary $\partial\xi \in \mathbf{R}^V$ of a flow ξ is given by

$$\partial\xi(v) = \sum\{\xi(a) \mid \text{ arc } a \text{ leaves } v\} - \sum\{\xi(a) \mid \text{ arc } a \text{ enters } v\} \qquad (v \in V).$$

Then, the minimum cost of a flow that realizes a supply/demand vector $x \in \mathbf{R}^T$ is represented by a function $f : \mathbf{R}^T \to \mathbf{R} \cup \{\pm\infty\}$ defined as

$$f(x) = \inf_{\xi}\{\sum_{a \in A} f_a(\xi(a)) \mid (\partial\xi)(v) = -x(v) \ (v \in T), \ (\partial\xi)(v) = 0 \ (v \in V \setminus T)\}.$$

On the other hand, suppose that we are given another family of convex functions $g_a \in \mathcal{C}^1$ $(a \in A)$, each of which represents the cost of tension on arc a. Any vector $\widetilde{p} \in \mathbf{R}^V$ is called a potential, and the coboundary $\delta\widetilde{p} \in \mathbf{R}^A$ of a potential $\widetilde{p}$ is defined by $\delta\widetilde{p}(a) = \widetilde{p}(u) - \widetilde{p}(v)$ for $a = (u, v) \in A$. Then, the minimum cost of a tension that realizes a potential vector $p \in \mathbf{R}^T$ is represented by a function $g : \mathbf{R}^T \to \mathbf{R} \cup \{\pm\infty\}$ defined as

$$g(p) = \inf_{\eta, \tilde{p}}\{\sum_{a \in A} g_a(\eta(a)) \mid \eta(a) = -\delta\widetilde{p}(a) \ (a \in A), \ \widetilde{p}(v) = p(v) \ (v \in T)\}.$$

It can be shown that both f and g are closed proper convex if $f(x_0)$ and $g(p_0)$ are finite for some $x_0 \in \mathbf{R}^T$ and $p_0 \in \mathbf{R}^T$, which is a direct extension of the results in Iri [5] and Rockafellar [20] for the case of $|T| = 2$. These functions, however, are equipped with different combinatorial structures; f is M-convex and g is L-convex, as follows.

Theorem 2.7. *If f_a and g_a are conjugate to each other for all $a \in A$, then f and g are M-convex and L-convex, respectively, and conjugate to each other, where it is assumed that at least one of the following conditions holds:*

(a) $-\infty < f(x_0) < +\infty$ *for some* $x_0 \in \mathbf{R}^T$,
(b) $-\infty < g(p_0) < +\infty$ *for some* $p_0 \in \mathbf{R}^T$,
(c) $f(x_0) < +\infty, g(p_0) < +\infty$ *for some* $x_0 \in \mathbf{R}^T, p_0 \in \mathbf{R}^T$.

We first prove the closedness of f and g and the conjugacy relationship. For this, we use the following duality theorem for the minimum cost flow/tension problems.

Theorem 2.8 (cf. [20, Sec. 8H]). *Let $G = (V, A)$ be a directed graph with a specified vertex subset $T \subseteq V$. Also, let $f_a, g_a \in \mathcal{C}^1$ $(a \in A)$ and $f_v, g_v \in \mathcal{C}^1$ $(v \in T)$ be conjugate pairs of closed convex functions. Then, we have*

$$\begin{aligned} &\inf_{\xi, x}\left\{ \sum_{a \in A} f_a(\xi(a)) + \sum_{v \in T} f_v(x(v)) \;\middle|\; \begin{array}{l} (\partial\xi)(v) = -x(v) \ (v \in T), \\ (\partial\xi)(v) = 0 \ (v \in V \setminus T) \end{array} \right\} \\ = &\sup_{\eta, \tilde{p}}\left\{ -\sum_{a \in A} g_a(\eta(a)) - \sum_{v \in T} g_v(-\widetilde{p}(v)) \;\middle|\; \eta(a) = -\delta\widetilde{p}(a) \ (a \in A) \right\} \end{aligned}$$

unless $\inf = +\infty$ *and* $\sup = -\infty$.

Lemma 2.9. *Let $x \in \mathbf{R}^n$ and $p \in \mathbf{R}^n$.*
(i) $f(x) = g^\bullet(x)$ *if* $f(x) < +\infty$ *or* $g(p_0) < +\infty$ *for some* $p_0 \in \mathbf{R}^T$.
(ii) $g(p) = f^\bullet(p)$ *if* $g(p) < +\infty$ *or* $f(x_0) < +\infty$ *for some* $x_0 \in \mathbf{R}^T$.

Proof. To prove (i), consider functions $f_v, g_v \in \mathcal{C}^1$ $(v \in T)$ given as

$$f_v(\alpha) = \begin{cases} 0 & (\alpha = x(v)), \\ +\infty & (\alpha \neq x(v)), \end{cases} \qquad g_v(\beta) = x(v)\beta \quad (\beta \in \mathbf{R})$$

for the given $x \in \mathbf{R}^T$. The functions f_v and g_v are conjugate to each other for each $v \in T$. If $f(x) < +\infty$ or $g(p_0) < +\infty$ for some $p_0 \in \mathbf{R}^T$, then Theorem 2.8 implies that

$$\begin{aligned} f(x) &= \inf_{\xi, x} \left\{ \sum_{a \in A} f_a(\xi(a)) + \sum_{v \in T} f_v(x(v)) \;\middle|\; \begin{array}{l} (\partial\xi)(v) = -x(v) \ (v \in T), \\ (\partial\xi)(v) = 0 \ (v \in V \setminus T) \end{array} \right\} \\ &= \sup_{\eta, \tilde{p}} \left\{ \sum_{v \in T} \widetilde{p}(v)x(v) - \sum_{a \in A} g_a(\eta(a)) \;\middle|\; \eta(a) = -\delta\widetilde{p}(a) \ (a \in A) \right\} \\ &= \sup\{\langle p, x\rangle - g(p) \mid p \in \mathbf{R}^T\} \;=\; g^\bullet(x). \end{aligned}$$

The proof for (ii) is similar to that for (i) and therefore omitted. □

We see from Lemma 2.9 that three conditions (a), (b), and (c) are equivalent to each other. Hence, Lemma 2.9 implies that if one of these conditions holds, then both f and g are closed proper convex functions with $f = g^\bullet$, $g = f^\bullet$.

We then prove the M-convexity of f and the L-convexity of g.

[(M-EXC) for f] Let $x, y \in \mathrm{dom}\, f$ and $u \in \mathrm{supp}^+(x - y)$. For any $\varepsilon > 0$ and $z \in \{x, y\}$, there exist $\xi_z \in \mathbf{R}^A$ with

$$\sum_{a \in A} f_a(\xi_z(a)) \le f(z) + \varepsilon, \quad (\partial\xi_z)(v) = -z(v) \ (v \in T), \quad (\partial\xi_z)(v) = 0 \ (v \in V \setminus T).$$

By a standard augmenting path argument, there exist $\pi : \{0, \pm 1\}^A$ and $v \in \mathrm{supp}^-(x-y)$ $(\subseteq T)$ such that

$$\mathrm{supp}^+(\pi) \subseteq \mathrm{supp}^+(\xi_y - \xi_x), \ \mathrm{supp}^-(\pi) \subseteq \mathrm{supp}^-(\xi_y - \xi_x), \ \partial\pi = \chi_u - \chi_v,$$

where we can assume the following inequality with $m = |A|$, $n = |V|$:

$$\min\{|\xi_x(a) - \xi_y(a)| \mid a \in A, \ \pi(a) = \pm 1\} \ge \{x(u) - y(u)\}/m^n.$$

Putting $\alpha_0 = \{x(u) - y(u)\}/m^n$, we have

$$f_a(\xi_x(a) + \alpha\pi(a)) + f_a(\xi_y(a) - \alpha\pi(a)) \le f_a(\xi_x(a)) + f_a(\xi_y(a)) \quad (\alpha \in [0, \alpha_0])$$

for all $a \in A$. Hence follows that

$$\begin{aligned} & f(x - \alpha(\chi_u - \chi_v)) + f(y + \alpha(\chi_u - \chi_v)) \\ &\quad\le \sum_{a \in A}\{f_a(\xi_x(a) + \alpha\pi(a)) + f_a(\xi_y(a) - \alpha\pi(a))\} \\ &\quad\le \sum_{a \in A}\{f_a(\xi_x(a)) + f_a(\xi_y(a))\} \;\le\; f(x) + f(y) + 2\varepsilon \qquad (\alpha \in [0, \alpha_0]). \end{aligned}$$

Since $\varepsilon > 0$ can be chosen arbitrarily and T is a finite set, there exists some $v = v_*$ satisfying

$$f(x - \alpha(\chi_u - \chi_v)) + f(y + \alpha(\chi_u - \chi_v)) \leq f(x) + f(y) \qquad (\alpha \in [0, \alpha_0]),$$

implying (M-EXC) for f.
[L-convexity for g] Let $p, q \in \mathrm{dom}\, g$. For any $\varepsilon > 0$ there exist $\widetilde{p}, \widetilde{q} \in \mathbf{R}^V$ with

$$\sum_{a \in A} g_a(-\delta\widetilde{p}(a)) \leq g(p) + \varepsilon, \quad \widetilde{p}(v) = p(v) \ (v \in T),$$
$$\sum_{a \in A} g_a(-\delta\widetilde{q}(a)) \leq g(q) + \varepsilon, \quad \widetilde{q}(v) = q(v) \ (v \in T).$$

It holds that $(\widetilde{p} \wedge \widetilde{q})(v) = (p \wedge q)(v)$, $(\widetilde{p} \vee \widetilde{q})(v) = (p \vee q)(v)$ for all $v \in T$, and

$$g_a(-\delta(\widetilde{p} \wedge \widetilde{q})(a)) + g_a(-\delta(\widetilde{p} \vee \widetilde{q})(a)) \leq g_a(-\delta\widetilde{p}(a)) + g_a(-\delta\widetilde{q}(a)) \qquad (a \in A)$$

by convexity of g_a. Hence follows that

$$\begin{aligned} g(p \wedge q) + g(p \vee q) &\leq \sum_{a \in A} g_a(-\delta(\widetilde{p} \wedge \widetilde{q})(a)) + \sum_{a \in A} g_a(-\delta(\widetilde{p} \vee \widetilde{q})(a)) \\ &\leq \sum_{a \in A} g_a(-\delta\widetilde{p}(a)) + \sum_{a \in A} g_a(-\delta\widetilde{q}(a)) \ \leq \ g(p) + g(q) + 2\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ can be chosen arbitrarily, we have $g(p) + g(q) \geq g(p \wedge q) + g(p \vee q)$. The property (LF2) for g is immediate from the equation $\delta(\widetilde{p} + \lambda\mathbf{1}) = \delta\widetilde{p}$ for $\widetilde{p} \in \mathbf{R}^V$ and $\lambda \in \mathbf{R}$. □

3 Proof of Conjugacy Relationship

In this section, we prove Theorem 1.1, the main result of this paper.

The conjugacy operation $f \mapsto f^\bullet$ given by (1.2) induces a symmetric one-to-one correspondence in the class of all closed proper convex functions on $\mathbf{R}^n$.

Theorem 3.1 ([19, Th. 12.2]). *For a closed proper convex function $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$, its conjugate $f^\bullet : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ is also a closed proper convex function with $f^{\bullet\bullet} = f$.*

By Theorem 3.1, it remains to show that "$f \in \mathcal{M}_n \Longrightarrow f^\bullet \in \mathcal{L}_n$" and "$g \in \mathcal{L}_n \Longrightarrow g^\bullet \in \mathcal{M}_n$."

3.1 Proof of "$f \in \mathcal{M}_n \Longrightarrow f^\bullet \in \mathcal{L}_n$"

We first show some useful properties.

Proposition 3.2. *Let $f : \mathbf{R}^2 \to \mathbf{R} \cup \{+\infty\}$ be a function in two variables with* $\mathrm{dom}\, f \neq \emptyset$. *If f is supermodular, then its conjugate $f^\bullet : \mathbf{R}^2 \to \mathbf{R} \cup \{+\infty\}$ is submodular.*

Proof. It suffices to show

$$f^\bullet(\lambda,\mu)+f^\bullet(\lambda',\mu')\leq f^\bullet(\lambda,\mu')+f^\bullet(\lambda',\mu) \tag{3.1}$$

for $(\lambda,\mu),(\lambda',\mu')\in\mathbf{R}^2$ with $\lambda\geq\lambda'$ and $\mu\geq\mu'$. We claim that

$$[\lambda\alpha+\mu\beta-f(\alpha,\beta)]+[\lambda'\alpha'+\mu'\beta'-f(\alpha',\beta')]\leq f^\bullet(\lambda,\mu')+f^\bullet(\lambda',\mu) \tag{3.2}$$

holds for any $(\alpha,\beta),(\alpha',\beta')\in\mathbf{R}^2$. The inequality (3.1) is immediate from (3.2), since the supremum of the left-hand side of (3.2) over (α,β) and (α',β') coincides with the left-hand side of (3.1).

We now prove (3.2). If $\alpha\geq\alpha'$ and $\beta\geq\beta'$, we have $f(\alpha,\beta)+f(\alpha',\beta')\geq f(\alpha,\beta')+f(\alpha',\beta)$ by the supermodularity of f, and therefore

$$\text{LHS of (3.2)}\leq[\lambda\alpha+\mu'\beta'-f(\alpha,\beta')]+[\lambda'\alpha'+\mu\beta-f(\alpha',\beta)]\leq\text{RHS of (3.2)}.$$

If $\alpha\leq\alpha'$, we have $\lambda\alpha+\lambda'\alpha'\leq\lambda\alpha'+\lambda'\alpha$ and therefore

$$\text{LHS of (3.2)}\leq[\lambda\alpha'+\mu'\beta'-f(\alpha',\beta')]+[\lambda'\alpha+\mu\beta-f(\alpha,\beta)]\leq\text{RHS of (3.2)}.$$

We can prove (3.2) similarly for the case $\beta\leq\beta'$. □

Proposition 3.3 ([19, Cor. 7.5.1]). *Let $f:\mathbf{R}^n\to\mathbf{R}\cup\{+\infty\}$ be a closed proper convex function. For $x\in\mathbf{R}^n$ and $y\in\mathrm{dom}\,f$, we have*

$$f(x)=\lim_{\lambda\uparrow 1}f(\lambda x+(1-\lambda)y).$$

Proposition 3.4. *Let $f:\mathbf{R}^n\to\mathbf{R}\cup\{+\infty\}$ be a closed proper convex function satisfying the property:*

(P0) $\forall x,y\in\mathrm{dom}\,f$ *with* $x\geq y$, $\forall i\in\mathrm{supp}^+(x-y)$, $\exists\alpha_0>0$*:*

$$f(x)+f(y)\geq f(x-\alpha\chi_i)+f(y+\alpha\chi_i)\qquad(\alpha\in[0,\alpha_0]).$$

Then, f satisfies the supermodular inequality:

$$f(x)+f(y)\leq f(x\wedge y)+f(x\vee y)\qquad(x,y\in\mathbf{R}^n). \tag{3.3}$$

In particular, an M$^\natural$-convex function satisfies the supermodular inequality (3.3).

Proof. Note that an M$^\natural$-convex function satisfies the property (P0). Hence, it suffices to show the former claim only.

We first prove that (P0) implies the following stronger property:

(P1) $\forall x,y\in\mathrm{dom}\,f$ with $x\geq y$, $\forall i\in\mathrm{supp}^+(x-y)$:

$$f(x)+f(y)\geq f(x-\{x(i)-y(i)\}\chi_i)+f(y+\{x(i)-y(i)\}\chi_i). \tag{3.4}$$

Put $\overline{\alpha} = x(i) - y(i)$, and define functions $\varphi_x, \varphi_y : [0, \overline{\alpha}] \to \mathbf{R} \cup \{+\infty\}$ by

$$\varphi_x(\alpha) = f(x - \alpha\chi_i), \quad \varphi_y(\alpha) = f(y + \{\overline{\alpha} - \alpha\}\chi_i) \quad (\alpha \in [0, \overline{\alpha}]).$$

Claim 1. Let $\alpha \in [0, \overline{\alpha}]$.
(i) If $\varphi_x(\alpha) < +\infty$, then $\varphi_x((\alpha + \overline{\alpha})/2) < +\infty$.
(ii) If $\varphi_y(\alpha) < +\infty$, then $\varphi_y(\alpha/2) < +\infty$.

[Proof of Claim 1] We prove (i) only, where we may assume $\alpha < \overline{\alpha}$. Put $\widehat{x} = x - \alpha\chi_i$ and

$$\alpha_* = \sup\{\beta \mid f(\widehat{x} - \beta\chi_i) + f(y + \beta\chi_i) \le f(\widehat{x}) + f(y)\}.$$

Assume, to the contrary, that $\alpha_* < \{\overline{\alpha} - \alpha\}/2$ $(= \{\widehat{x}(i) - y(i)\}/2)$. By Proposition 3.3, we have $f(\widehat{x} - \alpha_*\chi_i) + f(y + \alpha_*\chi_i) \le f(\widehat{x}) + f(y)$. Put $\widetilde{x} = \widehat{x} - \alpha_*\chi_i$, $\widetilde{y} = y + \alpha_*\chi_i$. Then, we have $i \in \mathrm{supp}^+(\widetilde{x} - \widetilde{y})$. Hence, the property (P0) for $\widetilde{x}$ and $\widetilde{y}$ implies that there exists a sufficiently small $\beta > 0$ satisfying

$$f(\widehat{x}) + f(y) \ge f(\widetilde{x}) + f(\widetilde{y}) \ge f(\widetilde{x} - \beta\chi_i) + f(\widetilde{y} + \beta\chi_i),$$

a contradiction to the choice of α_*. Hence, we have $\alpha_* \ge \{\overline{\alpha} - \alpha\}/2$, from which $\varphi_x((\alpha + \overline{\alpha})/2) < +\infty$ follows. [End of Claim 1]

We also define a function $\varphi : [0, \overline{\alpha}] \to \mathbf{R} \cup \{\pm\infty\}$ by $\varphi(\alpha) = \varphi_x(\alpha) - \varphi_y(\alpha)$ $(\alpha \in [0, \overline{\alpha}])$. Since φ_x and φ_y are closed convex functions with $\varphi_x(0) < +\infty$, $\varphi_y(\overline{\alpha}) < +\infty$, we have $\varphi_x(\alpha) < +\infty$ $(0 \le \forall\alpha < \overline{\alpha})$ and $\varphi_y(\alpha) < +\infty$ $(0 < \forall\alpha \le \overline{\alpha})$ by Claim 1. This property, together with Proposition 3.3 for φ_x and φ_y, implies that φ is continuous in the interval $\{\alpha \mid 0 < \alpha < \overline{\alpha}\}$, and $\varphi(0) = \lim_{\alpha\downarrow 0}\varphi(\alpha)$, $\varphi(\overline{\alpha}) = \lim_{\alpha\uparrow\overline{\alpha}}\varphi(\alpha)$. To prove (3.4), it suffices to show that $\varphi(\alpha)$ is nonincreasing on $[0, \overline{\alpha}]$, which follows from Claim 2 below:

Claim 2. $\varphi'(\alpha; 1) \le 0$, $\varphi'(\alpha; -1) \ge 0$ $(0 < \forall\alpha < \overline{\alpha})$.

[Proof of Claim 2] It is noted that the values $\varphi'(\alpha; \pm 1) = \varphi'_x(\alpha; \pm 1) - \varphi'_y(\alpha; \pm 1)$ are well-defined for all α with $0 < \alpha < \overline{\alpha}$. We here prove $\varphi'(\alpha; 1) \le 0$ only since $\varphi'(\alpha; -1) \ge 0$ can be proven similarly. Put $x' = x - \alpha\chi_i$ and $y'_\delta = y + \{\overline{\alpha} - \alpha - \delta\}\chi_i$ for $\delta > 0$. Then, we have $i \in \mathrm{supp}^+(x' - y'_\delta)$. By (P0), there exists some $\beta_0 > 0$ such that $f(x') + f(y'_\delta) \ge f(x' - \beta\chi_i) + f(y'_\delta + \beta\chi_i)$ $(\forall\beta \in [0, \beta_0])$, implying $\varphi'_x(\alpha; 1) \le -\varphi'_y(\alpha + \delta; -1)$. Hence follows $\varphi'_x(\alpha; 1) \le -\lim_{\delta\downarrow 0}\varphi'_y(\alpha + \delta; -1) = \varphi'_y(\alpha; 1)$. [End of Claim 2]

We finally prove the supermodularity of f by using the property (P1). The proof is by induction on the cardinality of the sets $\mathrm{supp}^+(x-y)$ and $\mathrm{supp}^-(x-y)$. If $|\mathrm{supp}^+(x-y)| \le 1$ or $|\mathrm{supp}^-(x-y)| \le 1$, then (3.3) follows immediately from (P1). Hence, we consider the case when $|\mathrm{supp}^+(x-y)| > 1$ and $|\mathrm{supp}^-(x-y)| > 1$. We may assume $x \wedge y, x \vee y \in \mathrm{dom}\, f$, since otherwise (3.3) holds immediately. Let $i \in \mathrm{supp}^+(x-y)$. Then, we have $(x \wedge y) + \{x(i) - y(i)\}\chi_i \in \mathrm{dom}\, f$ by (P1), and the induction assumption implies

$$f(y) - f(x \wedge y) \le f(y + \{x(i) - y(i)\}\chi_i) - f((x \wedge y) + \{x(i) - y(i)\}\chi_i) \le f(x \vee y) - f(x).$$

□

We now assume $f \in \mathcal{M}_n$ and show $f^\bullet \in \mathcal{L}_n$. Put $r = x(N)$ with some $x \in \mathrm{dom}\, f$, which is independent of the choice of x by Proposition 2.1. For $p \in \mathrm{dom}\, f^\bullet$ and $\lambda \in \mathbf{R}$, we have

$$\begin{aligned} f^\bullet(p + \lambda \mathbf{1}) &= \sup\{\langle p + \lambda \mathbf{1}, x\rangle - f(x) \mid x \in \mathrm{dom}\, f\} \\ &= \sup\{\langle p, x\rangle - f(x) \mid x \in \mathrm{dom}\, f\} + \lambda x(N) \;=\; f^\bullet(p) + \lambda r, \end{aligned}$$

implying (LF2) for $f^\bullet$.

To prove the submodularity (LF1) for $f^\bullet$, we first assume that $\mathrm{dom}\, f$ is bounded. Since $\mathrm{dom}\, f^\bullet = \mathbf{R}^n$, the submodularity of $f^\bullet$ is equivalent to the local submodularity (see, e.g., [14, Th. 4.27]):

$$f^\bullet(p + \lambda \chi_i) + f^\bullet(p + \mu \chi_j) \geq f^\bullet(p) + f^\bullet(p + \lambda \chi_i + \mu \chi_j), \tag{3.5}$$

where $p \in \mathbf{R}^n$, $i, j \in N$ are distinct indices, and λ, μ are nonnegative reals. We fix $p \in \mathbf{R}^n$, and define functions $\widetilde{g}, \widetilde{f} : \mathbf{R}^2 \to \mathbf{R} \cup \{+\infty\}$ by

$$\begin{aligned} &\widetilde{g}(\lambda, \mu) = f^\bullet(p + \lambda \chi_i + \mu \chi_j) \qquad (\lambda, \mu \in \mathbf{R}), \\ &\widetilde{f}(\alpha, \beta) = \inf\{f(x) - \langle p, x\rangle \mid x \in \mathrm{dom}\, f,\ x(i) = \alpha,\ x(j) = \beta\} \qquad (\alpha, \beta \in \mathbf{R}). \end{aligned}$$

Claim. $\widetilde{f}$ satisfies the property (P0) in Proposition 3.4.
[Proof of Claim] We may assume $p = \mathbf{0}$ since $f(x) - \langle p, x\rangle$ is also M-convex as a function in x. It suffices to show that for any $(\alpha, \beta), (\alpha', \beta') \in \mathrm{dom}\, \widetilde{f}$ with $\alpha > \alpha'$ and $\beta \geq \beta'$, there exists $\delta_0 > 0$ satisfying

$$\widetilde{f}(\alpha, \beta) + \widetilde{f}(\alpha', \beta') \geq \widetilde{f}(\alpha - \delta, \beta) + \widetilde{f}(\alpha' + \delta, \beta') \qquad (\forall \delta \in [0, \delta_0]).$$

Since f is a closed proper convex function with bounded effective domain, there exist $x, x' \in \mathrm{dom}\, f$ satisfying $x(i) = \alpha$, $x(j) = \beta$, $\widetilde{f}(\alpha, \beta) = f(x)$, and $x'(i) = \alpha'$, $x'(j) = \beta'$, $\widetilde{f}(\alpha', \beta') = f(x')$, respectively. Since $i \in \mathrm{supp}^+(x - x')$, (M-EXC) for f implies that there exist $k \in \mathrm{supp}^-(x - x')$ and $\delta_0 > 0$ satisfying

$$\begin{aligned} \widetilde{f}(\alpha, \beta) + \widetilde{f}(\alpha', \beta') &= f(x) + f(x') \\ &\geq f(x - \delta(\chi_i - \chi_k)) + f(x' + \delta(\chi_i - \chi_k)) \\ &\geq \widetilde{f}(\alpha - \delta, \beta) + \widetilde{f}(\alpha' + \delta, \beta') \qquad (\forall \delta \in [0, \delta_0]), \end{aligned}$$

where it is noted that $k \neq j$ since $j \notin \mathrm{supp}^-(x - x')$. [End of Claim]

Hence, $\widetilde{f}$ is supermodular by Proposition 3.4. Since $\widetilde{g} = (\widetilde{f})^\bullet$, the function $\widetilde{g}$ is submodular by Proposition 3.2, implying the inequality (3.5). This concludes the proof of (LF1) for $f^\bullet$ when $\mathrm{dom}\, f$ is bounded.

Finally, we consider the case when $\mathrm{dom}\, f$ is unbounded. For a fixed vector $x_0 \in \mathrm{dom}\, f$, we define $f_k : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ $(k = 1, 2, \dots)$ by

$$f_k(x) = \begin{cases} f(x) & (x \in \mathbf{R}^n,\ |x(i) - x_0(i)| \leq k \text{ for all } i \in N), \\ +\infty & (\text{otherwise}). \end{cases}$$

Since $f_k \in \mathcal{M}_n$ and $\mathrm{dom}\, f_k$ is bounded, $f_k^\bullet$ fulfills (LF1). Hence, for any $p, q \in \mathrm{dom}\, f^\bullet$ we have

$$\begin{aligned} f^\bullet(p) + f^\bullet(q) &= \lim_{k\to\infty} \{f_k^\bullet(p) + f_k^\bullet(q)\} \\ &\geq \lim_{k\to\infty} \{f_k^\bullet(p \wedge q) + f_k^\bullet(p \vee q)\} = f^\bullet(p \wedge q) + f^\bullet(p \vee q). \end{aligned}$$

3.2 Proof of "$g \in \mathcal{L}_n \Longrightarrow g^\bullet \in \mathcal{M}_n$"

We will use the following characterizations of M-convex functions. Define

$$f'(x; j, i) = \lim_{\alpha \downarrow 0} \{f(x + \alpha(\chi_j - \chi_i)) - f(x)\}/\alpha \quad (x \in \mathrm{dom}\, f,\ i, j \in N).$$

(M-EXC$_\mathbf{s}$) $\forall x, y \in \mathrm{dom}\, f$, $\forall i \in \mathrm{supp}^+(x - y)$, $\exists j \in \mathrm{supp}^-(x - y)$:

$$f(x) + f(y) \geq f(x - \alpha(\chi_i - \chi_j)) + f(y + \alpha(\chi_i - \chi_j)) \quad (\forall \alpha \in [0, \{x(i) - y(i)\}/2t]),$$

where $t = |\mathrm{supp}^-(x - y)|$.
(M-EXC$'$) $\forall x, y \in \mathrm{dom}\, f$, $\forall i \in \mathrm{supp}^+(x - y)$, $\exists j \in \mathrm{supp}^-(x - y)$:

$$f'(x; j, i) < +\infty,\ f'(y; i, j) < +\infty, \text{ and } f'(x; j, i) + f'(y; i, j) \leq 0.$$

Theorem 3.5. *For a closed proper convex function $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$,* (M-EXC) $\Longleftrightarrow$ (M-EXC$_\mathrm{s}$) $\Longleftrightarrow$ (M-EXC$'$).

Since the implications "(M-EXC$_\mathrm{s}$) $\Longrightarrow$ (M-EXC)" and "(M-EXC) $\Longrightarrow$ (M-EXC$'$)" are obvious, we prove below the reverse implications.

Proof. [(M-EXC) $\Longrightarrow$ (M-EXC$_\mathrm{s}$)] Let $x_0, y_0 \in \mathrm{dom}\, f$, and $i \in \mathrm{supp}^+(x_0 - y_0)$. Put $\mathrm{supp}^-(x_0 - y_0) = \{j_1, j_2, \ldots, j_t\}$. For $h = 1, 2, \ldots, t$, we recursively define a function $\varphi_h : \mathbf{R} \to \mathbf{R} \cup \{+\infty\}$, $\alpha_h \in \mathbf{R}$, and $x_h, y_h \in \mathbf{R}^n$ by

$$\begin{aligned} &\varphi_h(\alpha) = f(x_{h-1} - \alpha(\chi_i - \chi_{j_h})) + f(y_{h-1} + \alpha(\chi_i - \chi_{j_h})) \qquad (\alpha \in \mathbf{R}), \\ &\alpha_h \ = \ \sup\{\alpha \mid \varphi_h(\alpha) \leq \varphi_h(0),\ \alpha \leq \min[x_{h-1}(i) - y_{h-1}(i), y_{h-1}(j_h) - x_{h-1}(j_h)]/2\}, \\ &x_h = x_{h-1} - \alpha_h(\chi_i - \chi_{j_h}),\ y_h = y_{h-1} + \alpha_h(\chi_i - \chi_{j_h}). \end{aligned}$$

Since each φ_h is closed proper convex, Proposition 3.3 implies

$$x_h, y_h \in \mathrm{dom}\, f,\ f(x_h) + f(y_h) \leq f(x_{h-1}) + f(y_{h-1}) \quad (h = 1, 2, \ldots, t). \tag{3.6}$$

Assume, to the contrary, that $\sum_{h=1}^{t} \alpha_h < \{x_0(i) - y_0(i)\}/2$. Since $i \in \mathrm{supp}^+(x_t - y_t)$, there exist some $j_h \in \mathrm{supp}^-(x_t - y_t) \subseteq \mathrm{supp}^-(x_0 - y_0)$ and a sufficiently small $\alpha > 0$ such that

$$f(x_t) + f(y_t) \geq f(x_t - \alpha(\chi_i - \chi_{j_h})) + f(y_t + \alpha(\chi_i - \chi_{j_h})). \tag{3.7}$$

Putting $\widetilde{x}_h = x_h - \alpha(\chi_i - \chi_{j_h})$ and $\widetilde{x}_t = x_t - \alpha(\chi_i - \chi_{j_h})$, we have

$$x_h(k) = \min\{\widetilde{x}_h(k), x_t(k)\}, \qquad \widetilde{x}_t(k) = \max\{\widetilde{x}_h(k), x_t(k)\} \qquad (\forall k \in N \setminus \{i\}).$$

Therefore, Proposition 3.4 implies

$$f(x_h - \alpha(\chi_i - \chi_{j_h})) + f(x_t) \leq f(x_h) + f(x_t - \alpha(\chi_i - \chi_{j_h})). \tag{3.8}$$

Similarly, we have

$$f(y_h + \alpha(\chi_i - \chi_{j_h})) + f(y_t) \leq f(y_h) + f(y_t + \alpha(\chi_i - \chi_{j_h})). \tag{3.9}$$

From (3.7), (3.8), and (3.9) follows $f(x_h - \alpha(\chi_i - \chi_{j_h})) + f(y_h + \alpha(\chi_i - \chi_{j_h})) \leq f(x_h) + f(y_h)$, a contradiction to the definition of x_h and y_h. Hence, we have $\sum_{h=1}^{t} \alpha_h = \{x_0(i) - y_0(i)\}/2$. Let s be the index with $\alpha_s = \max\{\alpha_h \mid 1 \leq h \leq t\}$. For $\alpha \in [0, \alpha_s]$, we have

$$\begin{aligned} &\{f(x - \alpha(\chi_i - \chi_{j_s})) - f(x)\} + \{f(y + \alpha(\chi_i - \chi_{j_s})) - f(y)\} \\ &\quad \leq \ \{f(x_{s-1} - \alpha(\chi_i - \chi_{j_s})) - f(x_{s-1})\} + \{f(y_{s-1} + \alpha(\chi_i - \chi_{j_s})) - f(y_{s-1})\} \ \leq \ 0, \end{aligned}$$

where the first inequality is by Proposition 3.4 and the second by (3.6) and convexity of f. This shows (M-EXC$_{\mathrm{s}}$) for f since $\alpha_s \geq \{x(i) - y(i)\}/2t$.

[(M-EXC$'$) $\Longrightarrow$ (M-EXC)] Let $x, y \in \mathrm{dom}\, f$ and $i \in \mathrm{supp}^+(x - y)$. We prove that there exist some $j \in \mathrm{supp}^-(x - y)$ and $\alpha_0 > 0$ satisfying

$$f'(x - \alpha(\chi_i - \chi_j); j, i) + f'(y + \alpha(\chi_i - \chi_j); i, j) \leq 0 \qquad (\forall \alpha \in [0, \alpha_0]), \tag{3.10}$$

which, together with convexity of f, yields the desired inequality (1.1).

Put $x_* = x - |J|\beta\chi_i + \beta\chi_J$ and $y_* = y + |J|\beta\chi_i - \beta\chi_J$ with a sufficiently small $\beta > 0$ and

$$J = \{j \in \mathrm{supp}^-(x - y) \mid f'(x; j, i) < +\infty, \ f'(y; i, j) < +\infty\}.$$

By the convexity of $\mathrm{dom}\, f$, we have $x_*, \ y_* \in \mathrm{dom}\, f$. By (M-EXC$'$) applied to x_*, y_* and $i \in \mathrm{supp}^+(x_* - y_*)$, there exists $j_0 \in \mathrm{supp}^-(x_* - y_*)$ with $f'(x_*; j_0, i) + f'(y_*; i, j_0) \leq 0$. Since $f'(x_*; j_0, i) < +\infty$, we have $x' = x_* + \alpha(\chi_{j_0} - \chi_i) \in \mathrm{dom}\, f$ for some $\alpha > 0$. Since $j_0 \in \mathrm{supp}^+(x' - x)$ and $\mathrm{supp}^-(x' - x) = \{i\}$, the property (M-EXC$'$) for $\mathrm{dom}\, f$ implies $x + \alpha'(\chi_{j_0} - \chi_i) \in \mathrm{dom}\, f$ for a sufficiently small $\alpha' > 0$, from which $f'(x; j_0, i) < +\infty$ follows. Similarly, we have $f'(y; i, j_0) < +\infty$. Hence, $j_0 \in J$.

The inequality $f'(x_*; j_0, i) + f'(y_*; i, j_0) \leq 0$, together with the convexity of f, implies

$$f'(x_*; i, j_0) + f'(y_*; j_0, i) \geq 0. \tag{3.11}$$

For $\alpha \in [0, \beta/2]$, we put $x_\alpha = x - \alpha(\chi_i - \chi_{j_0}) \in \mathrm{dom}\, f$ and $y_\alpha = y + \alpha(\chi_i - \chi_{j_0}) \in \mathrm{dom}\, f$. The property (M-EXC$'$) implies

$$f'(x_*; i, j_0) + f'(x_\alpha; j_0, i) \leq 0 \tag{3.12}$$

since $j_0 \in \mathrm{supp}^+(x_* - x_\alpha)$ and $\mathrm{supp}^-(x_* - x_\alpha) = \{i\}$. Similarly, we have

$$f'(y_\alpha; i, j_0) + f'(y_*; j_0, i) \leq 0. \tag{3.13}$$

From (3.11), (3.12), and (3.13) follows (3.10) with $j = j_0$ and $\alpha_0 = \beta/2$. □

For $x \in \mathbf{R}^n$, we define a function $g[-x] : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ by $g[-x](p) = g(p) - \langle p, x \rangle$ ($p \in \mathbf{R}^n$).

Lemma 3.6. *Let $g \in \mathcal{L}_n$, and $x, y \in \mathbf{R}^n$ be vectors with* $\arg\min g[-x] \neq \emptyset$ *and* $\arg\min g[-y] \neq \emptyset$. *Then, for any $i \in \mathrm{supp}^+(x-y)$, there exists $j \in \mathrm{supp}^-(x-y)$ such that*

$$p(j) - p(i) \leq q(j) - q(i) \quad (\forall p \in \arg\min g[-x], \forall q \in \arg\min g[-y]). \tag{3.14}$$

Proof. First we note that $x(N) = y(N) = r$, where $r \in \mathbf{R}$ is the value in (LF2) for g. It is easy to see that we have

$$p,\ q \in D \Longrightarrow p \wedge q,\ p \vee q \in D, \qquad p \in D,\ \lambda \in \mathbf{R} \Longrightarrow p + \lambda \mathbf{1} \in D$$

for $D = \arg\min g[-x]$ and $D = \arg\min g[-y]$. Therefore, the inequality (3.14) can be rewritten as $p(j) \leq q(j)$ ($\forall p \in D_x,\ \forall q \in D_y$), where

$$\begin{array}{rcl} D_x & = & \{p \in \mathbf{R}^n \mid p \in \arg\min g[-x],\ p(i) = 0\}, \\ D_y & = & \{p \in \mathbf{R}^n \mid p \in \arg\min g[-y],\ p(i) = 0\}. \end{array}$$

Assume, to the contrary, that for any $j \in \mathrm{supp}^-(x-y)$, there exists a pair of vectors $p_j \in D_x$, $q_j \in D_y$ such that $p_j(j) > q_j(j)$. Putting

$$p_x = \bigvee \{p_j \mid j \in \mathrm{supp}^-(x-y)\}, \qquad q_y = \bigwedge \{q_j \mid j \in \mathrm{supp}^-(x-y)\},$$

we have $p_x \in D_x$, $q_y \in D_y$, and $\mathrm{supp}^-(x-y) \subseteq \mathrm{supp}^+(p_x - q_y)$. We also put $S^+ = \mathrm{supp}^+(p_x - q_y)$, $\lambda = \min\{p_x(j) - q_y(j) \mid j \in S^+\}$ (> 0). Then, L-convexity of g implies

$$\begin{array}{rcl} g(p_x) + g(q_y) & = & g(p_x - \lambda \mathbf{1}) + g(q_y) + \lambda r \\ & \geq & g((p_x - \lambda \mathbf{1}) \vee q_y) + g((p_x - \lambda \mathbf{1}) \wedge q_y) + \lambda r \\ & = & g((p_x - \lambda \mathbf{1}) \vee q_y) + g(p_x \wedge (q_y + \lambda \mathbf{1})). \end{array} \tag{3.15}$$

Since

$$\begin{array}{rcl} ((p_x - \lambda \mathbf{1}) \vee q_y)(j) & = & \begin{cases} p_x(j) - \lambda & (j \in S^+), \\ q_y(j) & (j \in N \setminus S^+), \end{cases} \\ (p_x \wedge (q_y + \lambda \mathbf{1}))(j) & = & \begin{cases} q_y(j) + \lambda & (j \in S^+), \\ p_x(j) & (j \in N \setminus S^+), \end{cases} \end{array}$$

we have

$$\begin{array}{l} \langle (p_x - \lambda \mathbf{1}) \vee q_y, x \rangle + \langle p_x \wedge (q_y + \lambda \mathbf{1}), y \rangle - \langle p_x, x \rangle - \langle q_y, y \rangle \\ \quad = \ \lambda \sum_{j \in S^+} \{y(j) - x(j)\} + \sum_{j \in N \setminus S^+} \{q_y(j) - p_x(j)\}\{x(j) - y(j)\} \\ \quad \geq \ \lambda \sum_{j \in S^+} \{y(j) - x(j)\} \\ \quad \geq \ \lambda \sum_{v \in N \setminus \{i\}} \{y(j) - x(j)\} \ = \ \lambda \{x(i) - y(i)\} \ > \ 0, \end{array} \tag{3.16}$$

where the inequalities follow from $\mathrm{supp}^-(x-y) \subseteq S^+$. From (3.15) and (3.16) follows

$$g[-x]((p_x - \lambda \mathbf{1}) \vee q_y) + g[-y](p_x \wedge (q_y + \lambda \mathbf{1})) < g[-x](p_x) + g[-y](q_y),$$

a contradiction to the fact that $p_x \in \arg\min g[-x]$, $q_y \in \arg\min g[-y]$. $\square$

Let $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ be a convex function and $x \in \mathrm{dom}\, f$. The subdifferential of f at x, denoted by $\partial f(x)$, is defined as

$$\partial f(x) = \{p \in \mathbf{R}^n \mid f(y) \geq f(x) + \langle p, y - x \rangle \ (\forall y \in \mathbf{R}^n)\}.$$

For $y \in \mathbf{R}^n$, the directional derivative of f at x w.r.t. y is defined by

$$f'(x; y) = \lim_{\alpha \downarrow 0} \{f(x + \alpha y) - f(x)\}/\alpha.$$

Note that $f'(x; j, i) = f'(x; \chi_j - \chi_i)$ for $i, j \in N$.

Proposition 3.7 (cf. [19, Th. 23.4]). *Let* $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ *be a convex function with* $\mathrm{dom}\, f = \{x \in \mathbf{R}^n \mid x(N) = r\}$ *for some* $r \in \mathbf{R}$*. Then, for any* $x \in \mathrm{dom}\, f$ *we have* $\partial f(x) \neq \emptyset$ *and* $f'(x; y) = \sup\{\langle p, y \rangle \mid p \in \partial f(x)\}$ $(y \in \mathbf{R}^n)$.

We now assume $g \in \mathcal{L}_n$ and show $g^\bullet \in \mathcal{M}_n$. It is easy to see that the conjugate function $g^\bullet$ satisfies $\mathrm{dom}\, g^\bullet \subseteq \{x \in \mathbf{R}^n \mid x(N) = r\}$, where $r \in \mathbf{R}$ is the value in (LF2) for g. We firstly consider the case when $\mathrm{dom}\, g^\bullet = \{x \in \mathbf{R}^n \mid x(N) = r\}$. Let $x, y \in \mathrm{dom}\, g^\bullet$ and $i \in \mathrm{supp}^+(x-y)$. Since $\arg\min g[-x] = \partial g^\bullet(x)$ and $\arg\min g[-y] = \partial g^\bullet(y)$ hold, it follows from Proposition 3.7 that $\arg\min g[-x] \neq \emptyset$ and $\arg\min g[-y] \neq \emptyset$. By Lemma 3.6, there exists $j \in \mathrm{supp}^-(x-y)$ satisfying (3.14), implying

$$\begin{aligned}
&(g^\bullet)'(x; j, i) + (g^\bullet)'(y; i, j) \\
&= \sup\{p(j) - p(i) \mid p \in \arg\min g[-x]\} + \sup\{q(i) - q(j) \mid q \in \arg\min g[-y]\} \leq 0,
\end{aligned}$$

where the equality is by Proposition 3.7. This shows (M-EXC$'$) for $g^\bullet$, which, together with Theorem 3.5, yields M-convexity of $g^\bullet$.

We then consider the general case. For fixed $j_0 \in N$ and $q \in \mathrm{dom}\, g$ with $q(j_0) = 0$, we define $g_k : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ $(k = 1, 2, \dots)$ by

$$g_k(p) = \begin{cases} g(p) & (p \in \mathbf{R}^n,\ |p(i) - p(j_0) - q(i)| \leq k \text{ for all } i \in N), \\ +\infty & (\text{otherwise}). \end{cases}$$

It can be easily shown that each g_k is an L-convex function with $\mathrm{dom}\, g_k^\bullet = \{x \in \mathbf{R}^n \mid x(N) = r\}$. Therefore, the discussion above shows that each $g_k^\bullet$ is M-convex, and therefore satisfies (M-EXC$_\mathrm{s}$) by Theorem 3.5. For $x, y \in \mathrm{dom}\, g^\bullet$ $(\subseteq \mathrm{dom}\, g_k^\bullet)$ and $i \in \mathrm{supp}^+(x-y)$, there exists some $j_k \in \mathrm{supp}^-(x-y)$ such that

$$g_k^\bullet(x) + g_k^\bullet(y) \geq g_k^\bullet(x - \alpha(\chi_i - \chi_{j_k})) + g_k^\bullet(y + \alpha(\chi_i - \chi_{j_k})) \qquad (\forall \alpha \in [0, \{x(i) - y(i)\}/2t])$$

with $t = |\mathrm{supp}^-(x-y)|$. Since $\mathrm{supp}^-(x-y)$ is a finite set, we may assume that $j_k = j_*$ $(k = 1, 2, \dots)$ for some $j_* \in \mathrm{supp}^-(x-y)$. Then, for any $\alpha \in [0, \{x(i) - y(i)\}/2t]$ we have

$$\begin{aligned} g^\bullet(x) + g^\bullet(y) &= \lim_{k\to\infty} \{g_k^\bullet(x) + g_k^\bullet(y)\} \\ &\geq \lim_{k\to\infty} \{g_k^\bullet(x - \alpha(\chi_i - \chi_{j_*})) + g_k^\bullet(y + \alpha(\chi_i - \chi_{j_*}))\} \\ &= g^\bullet(x - \alpha(\chi_i - \chi_{j_*})) + g^\bullet(y + \alpha(\chi_i - \chi_{j_*})). \end{aligned}$$

Thus, (M-EXC$_\mathrm{s}$) holds for $g^\bullet$, which shows M-convexity of $g^\bullet$ by Theorem 3.5.

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