Generalized permutative representation of Cuntz algebra. I
—Generalization of cycle type—

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Abstract

We consider a kind of generalization of permutative representation with cycle by Bratteli and Jorgensen. We show their properties, existence, irreducibility and equivalence by using parameter of representation.

1 Introduction

We define a class of representations of Cuntz algebra which is a kind of generalization of permutative representation by [5, 6, 7]. Let $N \geq 2$ and $s_1, \ldots, s_N$ generators of Cuntz algebra $\mathcal{O}_N$. For an element

$$w = w^{(1)} \otimes \cdots \otimes w^{(k)} \in (\mathbf{C}^N)^{\otimes k},$$

$\|w^{(j)}\| = 1$, $j = 1, \ldots, k$, $k \geq 1$, let

$$s(w) \equiv s(w^{(1)}) \cdots s(w^{(k)}), \tag{1.1}$$

$$s(w^{(j)}) \equiv \sum_{i=1}^{N} w_i^{(j)} s_i.$$

We consider a cyclic representation $(\mathcal{H}, \pi)$ of $\mathcal{O}_N$ with the cyclic vector Ω which satisfies an eigen equation:

$$\pi(s(w))\Omega = \Omega. \tag{1.2}$$

Our main results are 1)existence 2)uniqueness 3)equivalence 4)irreducibility about this kind of representations. The remarkable points are followings:

(i) This class is completely reducible, and the uniqueness of irreducible decomposition about this class holds. The uniqueness of irreducible decomposition is very rare in the theory of operator algebra and it has been already stated in [6, 7] for the case of ordinary permutative representation.

(ii) This representation is derived from *the second class gauge transformation of representation of Cuntz algebra.* Correct explanation about this statement is shown in [10]. In subsection 3.2, we show such method by constructing generalized permutative representation from ordinary permutative representation. In this point of view, it is easy to understand actions of several group on the set of representations of $\mathcal{O}_N$.

(iii) This class is properly larger than former class by [5, 6, 7] with *"cycle"*. For example, the following example of representation of $\mathcal{O}_2$ is included in *neither* the class of ordinary permutative representation *nor* that which is rotated by $U(2)$-action on $\mathcal{O}_2$:

$$\frac{1}{\sqrt{2}}\pi\left(s_1(s_1+s_2)\right)\Omega = \Omega \tag{1.3}$$

where $w \equiv \varepsilon_1 \otimes \frac{1}{\sqrt{2}}(\varepsilon_1+\varepsilon_2) \in (\mathbf{C}^2)^{\otimes 2}$ in the equation (1.2), $\varepsilon_1, \varepsilon_2$ are the canonical basis of $\mathbf{C}^2$. The cyclic representation with the cyclic vector Ω which satisfies equation (1.3) is unique up to unitary equivalence and irreducible. This result is shown in subsection 3.3 and 6.2.

This paper is the first of our series of articles. In the succeeding papers [8, 9, 10], we treat 1) periodic case and its irreducible decomposition, (the notion of "periodicity" is explained in the next section), 2) the class of generalization of the case of *"chain"* in [5, 6, 7], 3) the second class gauge transformation of representation of Cuntz algebra.

2 Preparation

In this section, we prepare several notions and lemmata in order to consider generalized permutative representation of Cuntz algebra. We consider a semigroup which consists of all monomials of tensor algebra over a finite dimensional Hilbert space. Our strategy is a characterization of a class of representations with parameter by property of elements in the parameter space.

Let $\mathbf{Z}_k$ be the cyclic group of order k, $k \geq 1$. Assume that $\mathbf{Z}_k$ acts on a set $\{1,\ldots,k\}$ of numbers and $\sigma : \{1,\ldots,k\} \to \{1,\ldots,k\}$ is the generator of $\mathbf{Z}_k$ which is defined by

$$\sigma(1) = 2, \ldots, \sigma(k-1) = k, \quad \sigma(k) = 1. \tag{2.1}$$

We call σ *the shift*.

Let V be a Hilbert space over $\mathbf{C}$ and $V^{\otimes k}$ k-times tensor space of V for $k \geq 1$. For $p \in \mathbf{Z}_k$, define an operator

$$\hat{p} : V^{\otimes k} \to V^{\otimes k}; \quad \hat{p}\left(v^{(1)} \otimes \cdots \otimes v^{(k)}\right) \equiv v^{(p(1))} \otimes \cdots \otimes v^{(p(k))}. \tag{2.2}$$

Then $\hat{\cdot}$ is a unitary action of cyclic group $\mathbf{Z}_k$ on $V^{\otimes k}$.

Fix $N \geq 2$. Let

$$S(\mathbf{C}^N) \equiv \{z \in \mathbf{C}^N : \|z\| = 1\}$$

be the unit complex sphere. Denote

$$TS(\mathbf{C}^N) \equiv \coprod_{k \geq 1} S(\mathbf{C}^N)^{\otimes k},$$

$$S(\mathbf{C}^N)^{\otimes k} \equiv \left\{ z^{(1)} \otimes \cdots \otimes z^{(k)} \in (\mathbf{C}^N)^{\otimes k} : \begin{array}{l} z^{(j)} \in S(\mathbf{C}^N), \\ j = 1, \ldots, k \end{array} \right\}.$$

When $w \in S(\mathbf{C})^{\otimes k}$, we call k *the length of* w. Remark that the description of $w \in TS(\mathbf{C}^N)$ by tensor product is not unique. For example $w = (cw^{(1)}) \otimes w^{(2)} = w^{(1)} \otimes (cw^{(2)})$.

$TS(\mathbf{C}^N)$ is a semigroup by the following operation:

$$TS(\mathbf{C}^N) \times TS(\mathbf{C}^N) \ni (x, y) \longmapsto x \otimes y \in TS(\mathbf{C}^N).$$

The action of $\mathbf{Z}_k$ on $(\mathbf{C}^N)^{\otimes k}$ in (2.2) induces an action of $\mathbf{Z}_k$ on $S(\mathbf{C}^N)^{\otimes k} \subset (\mathbf{C}^N)^{\otimes k}$ naturally. We denote id the unit of $\mathbf{Z}_k$.

Definition 2.1 (i) *$w \in S(\mathbf{C}^N)^{\otimes k}$ is periodic if there is $p \in \mathbf{Z}_k \setminus \{id\}$ such that $\hat{p}(w) = w$.*

(ii) *$w \in S(\mathbf{C}^N)^{\otimes k}$ is non periodic if w is not periodic.*

(iii) *For $w, w' \in S(\mathbf{C}^N)^{\otimes k}$, $w \sim w'$ if there is $p \in \mathbf{Z}_k$ such that $\hat{p}(w) = w'$. We call $\sim$ the cyclic equivalence by $\mathbf{Z}_k$.*

(iv) *For $w, w' \in TS(\mathbf{C}^N)$, $w \sim w'$ if the lengths of w and w' coincide and $w \sim w'$.*

Specially, if $k = 1$, then any element in $S(\mathbf{C}^N)$ is non periodic. w in (1.3) is non periodic. For example, a set

$$S_P(\mathbf{C}^2)^{\otimes 2} = \{v \otimes v \in S(\mathbf{C}^2)^{\otimes 2} : v \in S(\mathbf{C}^2)\}$$

is the set of all periodic elements in $S(\mathbf{C}^2)^{\otimes 2}$.

Note that there is an action of $U(1) \equiv \{c \in \mathbf{C} : |c| = 1\}$ on $S(\mathbf{C}^N)^{\otimes k}$ by scalar multiple:

$$S(\mathbf{C}^N)^{\otimes k} \ni w \longmapsto cw \in S(\mathbf{C}^N)^{\otimes k} \quad (c \in U(1)).$$

Lemma 2.2 *If $w \in S(\mathbf{C}^N)^{\otimes k}$ is periodic, then cw is periodic for each $c \in \mathbf{C}$, $|c| = 1$.*

Proof. Assume that $w = z^{\otimes l}$, $l \geq 2$. Let $\xi \equiv c^{1/l}$. Then $cw = (\xi z)^{\otimes l}$. Hence cw is periodic. ∎

Note that $S(\mathbf{C}^N)^{\otimes k}$ has a map $< \cdot | \cdot >$: $S(\mathbf{C}^N)^{\otimes k} \times S(\mathbf{C}^N)^{\otimes k} \to \mathbf{C}$ which is the restriction of the inner product of $(\mathbf{C}^N)^{\otimes k}$. Furthermore we use the notion of orthogonality for $S(\mathbf{C}^N)^{\otimes k}$ with respect to $< \cdot | \cdot >$.

Lemma 2.3 *For $w, w' \in S(\mathbf{C}^N)^{\otimes k}$, the followings are equivalent:*

(i) *There is $c \in \mathbf{C}$ such that $w' = cw$.*

(ii) $| < w|w' > | = 1$.

(iii) *w and w' are linearly dependent in $(\mathbf{C}^N)^{\otimes k}$.*

By this lemma, we can use a notion of linearly dependence for $TS(\mathbf{C}^N)$.

Lemma 2.4 *Let $w, w' \in S(\mathbf{C}^N)^{\otimes k}$. Then the following equivalence holds:*

$$< w|w' >= 1 \quad \Leftrightarrow \quad w = w'.$$

Proof. By Lemma 2.3, $< w|w' >= 1 \Leftrightarrow$ there is $c \in \mathbf{C}$ such that $w = cw'$ and $c = 1 \Leftrightarrow w = w'$. ∎

Proposition 2.5 (i) *If w is non periodic, then*

$$| < w|\hat{p}(w) > | < 1 \quad (p \in \mathbf{Z}_k \setminus \{id\}).$$

(ii) *If $w \in S(\mathbf{C}^N)^{\otimes k}$ and $v \in S(\mathbf{C}^N)^{\otimes l}$ are non periodic and $l \neq k$, then*

$$| < w^{\otimes l}|v^{\otimes k} > | < 1.$$

(iii) *If $w, v \in S(\mathbf{C}^N)^{\otimes k}$ satisfy $| < w|v > | < 1$, then*

$$| < w^{\otimes l}|v^{\otimes l} > | < 1 \quad (l \geq 1).$$

The proof of Proposition 2.5 is shown in Appendix A.

Note: For the aim of our theory, we consider the quotient space $S(\mathbf{C}^N)^{\otimes k}/\sim$ as the set of invariants of representations of $\mathcal{O}_N$ in subsection 6.3. An element of $S(\mathbf{C}^N)^{\otimes k}/\sim$ is regarded as a set of elements in $S(\mathbf{C}^N)$ which has a cyclic order. In our theory, $TS(\mathbf{C}^N)$ has two roles. The first is a parameter space of a class of representations of Cuntz algebra which is defined in section 3. The second is that some subset of $TS(\mathbf{C}^N)$ is an index set of some complete orthonormal basis of representation of Cuntz algebra which is treated in section 4. This accidental coincidence is interesting. Although we do not know that reason. On the other hand, the theory in [5], the corresponded object with $TS(\mathbf{C}^N)$ is

$$\{\varepsilon_J \in TS(\mathbf{C}^N) : J \in \{1,\ldots,N\}^k, k \geq 1\}.$$

where $\{\varepsilon_i\}_{i=1}^N$ is the canonical basis of $\mathbf{C}^N$ and $\varepsilon_J \equiv \varepsilon_{i_1} \otimes \cdots \otimes \varepsilon_{i_k}$ when $J = (i_1,\ldots,i_k)$. This correspondence is explained in subsection 3.3.

3 GP representation with cycle

In this paper, a word "representation" always means a unital *-representation.

3.1 Definition of generalized permutative representation with cycle

Let $N \geq 2$ and $\mathcal{O}_N$ the Cuntz algebra with generators $s_1,\ldots,s_N$ which satisfy the following relations

$$s_i^* s_j = \delta_{ij} I, \qquad \sum_{i=1}^N s_i s_i^* = I. \tag{3.1}$$

Recall an equation (1.1) for $w = w^{(1)} \otimes \cdots \otimes w^{(k)} \in S(\mathbf{C}^N)^{\otimes k}$. We summarize the simple formulae about $s(w)$ here.

$$s(w)^* = s(w^{(k)})^* \cdots s(w^{(1)})^* \tag{3.2}$$

If $\{\varepsilon_i\}_{i=1}^N$ is the canonical orthonormal basis of $\mathbf{C}^N$, then

$$s(\varepsilon_J) = s_{i_1} \cdots s_{i_k} \tag{3.3}$$

when $\varepsilon_J \equiv \varepsilon_{i_1} \otimes \cdots \otimes \varepsilon_{i_k}$ and $J = (i_1, \ldots, i_k) \in \{1, \ldots, N\}^k$, $k \geq 1$. We often write s_J as $s(\varepsilon_J)$. Then

$$s_J^* = s(\varepsilon_J)^* = s_{i_k}^* \cdots s_{i_1}^*.$$

Specially, $s_i = s(\varepsilon_i)$, $i = 1, \ldots, N$. If $w, w^{'} \in S(\mathbf{C}^N)^{\otimes k}$, then

$$s(w)^* s(w^{'}) = < w|w^{'} > I. \tag{3.4}$$

In general,

$$s(w)s(v) = s(w \otimes v) \tag{3.5}$$

for $w, v \in TS(\mathbf{C}^N)$. Let $\mathrm{Iso}(\mathcal{O}_N) \equiv \{x \in \mathcal{O}_N : x^* x = I\}$ be the semigroup of all isometries in $\mathcal{O}_N$.

Lemma 3.1 *A map $s : TS(\mathbf{C}^N) \to \mathrm{Iso}(\mathcal{O}_N)$ is an injective semigroup homomorphism.*

Proof. Since (3.5), s is a homomorphism of semigroup. Let $w, w^{'} \in TS(\mathbf{C}^N)$. Assume that $s(w) = s(w^{'})$ and $w \in S(\mathbf{C}^N)^{\otimes k}$, $w^{'} \in S(\mathbf{C}^N)^{\otimes l}$. We can assume that $k \geq l$ without loss of generality. Then

$$I = < w|w > I = s(w)^* s(w) = s(w)^* s(w^{'}). \tag{3.6}$$

If $k > l$, then r.h.s. of (3.6) equals to $s(w^{''})^*$ where $w^{''} \in S(\mathbf{C}^N)^{\otimes (k-l)}$ is a suitable element. This equality never holds. Hence $k = l$. Then r.h.s. of (3.6) equals to $< w|w^{'} > I$. Hence $< w|w^{'} >= 1$. By Lemma 2.4, $w = w^{'}$. Therefore s is injective. ∎

In this way, we have a family of isometries in $\mathcal{O}_N$ which are parameterized by $TS(\mathbf{C}^N)$. By this parameterization, we define a representation of $\mathcal{O}_N$ by $w \in TS(\mathbf{C}^N)$ as follows.

Definition 3.2 *$(\mathcal{H}, \pi, \Omega)$ is the GP(= generalized permutative) representation of $\mathcal{O}_N$ with cycle by $w \in S(\mathbf{C}^N)^{\otimes k}$ if $(\mathcal{H}, \pi)$ is a cyclic representation of $\mathcal{O}_N$ with the cyclic unit vector Ω which satisfies the following equation:*

$$\pi(s(w))\Omega = \Omega. \tag{3.7}$$

We denote $GP(w) \equiv (\mathcal{H}, \pi, \Omega)$. The equation (3.7), $\pi(s(w))$, and vector Ω are called GP equation, GP operator and GP vector, respectively. k is called the length of cycle of $(\mathcal{H}, \pi, \Omega)$.

The assumption of $\|\Omega\| = 1$ is used in section 4.

Definition 3.3 (i) *A representation* $(\mathcal{H},\pi)$ *of* $\mathcal{O}_N$ *is GP(= generalized permutative) with cycle if there are* $w \in TS(\mathbf{C}^N)$ *and a (cyclic)vector* $\Omega \in \mathcal{H}$ *such that* $(\mathcal{H},\pi,\Omega) = GP(w)$*, that is, they satisfy the condition (3.7).*

(ii) *For* $w, w^{'} \in TS(\mathbf{C}^N)$, $GP(w) \sim GP(w^{'})$ *if when* $GP(w) = (\mathcal{H},\pi,\Omega)$ *and* $GP(w^{'}) = (\mathcal{H}^{'},\pi^{'},\Omega^{'})$*, then* $(\mathcal{H},\pi)$ *and* $(\mathcal{H}^{'},\pi^{'})$ *are unitarily equivalent.*

(iii) *For a representation* $(\mathcal{H},\pi)$ *of* $\mathcal{O}_N$ *and* $w \in TS(\mathbf{C}^N)$, $(\mathcal{H},\pi) \succeq GP(w)$ *if there is* $\Omega \in \mathcal{H}$ *such that* $\pi(s(w))$ *and* Ω *satisfy (3.7).*

Note that there is no assumption of cyclicity for Ω in Definition 3.3 (iii).

We identify $\pi(s_i)$ and s_i from here when there is no confusion. By using this convention, we often use $s(w)\Omega = \Omega$ instead of the equation (3.7). The notion of "cycle" is taken from [5].

A naive meaning of cycle is the following relation between vectors and operators: for $w = w^{(1)} \otimes \cdots \otimes w^{(k)} \in S(\mathbf{C}^N)^{\otimes k}$,

$$\begin{aligned}
\Omega &\overset{s(w^{(k)})}{\longmapsto} s(w^{(k)})\Omega \\
&\overset{s(w^{(k-1)})}{\longmapsto} s(w^{(k-1)})s(w^{(k)})\Omega \overset{s(w^{(k-2)})}{\longmapsto} \cdots \overset{s(w^{(2)})}{\longmapsto} \left(s(w^{(2)})\cdots s(w^{(k)})\right)\Omega \\
&\overset{s(w^{(1)})}{\longmapsto} \left(s(w^{(1)})\cdots s(w^{(k)})\right)\Omega \\
&= s\left(w^{(1)} \otimes \cdots \otimes w^{(k)}\right)\Omega \qquad (\text{ by } (3.5)) \\
&= s(w)\Omega \\
&= \Omega \qquad (\text{ by } (3.7)).
\end{aligned}$$

In this way, a couple of families which consist same number of operators and vectors is a "cycle".

Remark that a representation π of $\mathcal{O}_N$ on a Hilbert space $\mathcal{H}$ is one-to-one corresponded to a family of operators $\{t_1,\ldots,t_N\}$ on $\mathcal{H}$ which satisfies the relations (3.1) by the relation

$$t_i = \pi(s_i) \quad (i = 1,\ldots,N). \tag{3.8}$$

Therefore we often identify a representation π of $\mathcal{O}_N$ and a family $\{t_1,\ldots,t_N\}$ of operators in this paper. For example, we often use the symbol for the GP representation $(\mathcal{H},\{t_1,\ldots,t_N\},\Omega)$ instead of $(\mathcal{H},\pi,\Omega)$ in the sense of (3.8).

Note: In [6, 7], they treat the free semigroup and its algebra in order to consider representations of Cuntz algebra. On the other hand, $TS(\mathbf{C}^N)$ itself is not a free semigroup because the phase factor of tensor decomposition of $w \in S(\mathbf{C}^N)^{\otimes k}$ brings a freedom of description of w. A subsemigroup $\left\{\varepsilon_I : I \in \{1,\ldots,N\}^k,\ k \geq 1\right\}$ of $TS(\mathbf{C}^N)$ is the free semigroup.

3.2 Existence of GP representation

Fix $N \geq 2$. We show the existence of $GP(w)$ by any $w \in TS(\mathbf{C}^N)$. The proof is given by constructing a concrete representation of $\mathcal{O}_N$ on $l_2(\mathbf{N})$.

Proposition 3.4 *For each $w \in TS(\mathbf{C}^N)$, there exists the GP representation of $\mathcal{O}_N$ by w.*

Proof. Fix $w \in S(\mathbf{C}^N)^{\otimes k}$. We construct the GP representation by w. Assume that $w = w^{(1)} \otimes \cdots \otimes w^{(k)}$, $w^{(j)} \in S(\mathbf{C}^N)$, $j = 1,\ldots,k$. Let $f = \{f_i\}_{i=1}^N$ be a branching function system ([5]) on $\mathbf{N}$ which is defined by

$$f_i : \mathbf{N} \to \mathbf{N} \quad (i = 1,\ldots,N),$$

$$f_1(n) = \begin{cases} \sigma^{-1}(n) & (1 \leq n \leq k), \\ \\ N(n-1)+1 & (n \geq k+1), \end{cases}$$

$$f_i(n) = \begin{cases} (N-1)(n-1)+i-1+k & (1 \leq n \leq k), \\ \\ N(n-1)+i & (n \geq k+1) \end{cases}$$

where $2 \leq i \leq N$ and $\sigma \in \mathbf{Z}_k$ is a shift in (2.1). This function system is represented as follows:

n	$f_1(n)$	$f_2(n)$	$\cdots$	$f_N(n)$
1	k	$k+1$	$\cdots$	$k+N-1$
2	1	$k+N$	$\cdots$	$k+2N-2$
$\vdots$	$\vdots$	$\vdots$	$\vdots\vdots\vdots$	$\vdots$
$k-1$	$k-2$	$N(k-2)+3$	$\cdots$	$N(k-1)+1$
k	$k-1$	$N(k-1)+2$	$\cdots$	Nk
$k+1$	$Nk+1$	$Nk+2$	$\cdots$	$N(k+1)$
$\vdots$	$\vdots$	$\vdots$	$\vdots\vdots\vdots$	$\vdots$

Note that the value of f_1 is quite different in other f_i, $i = 2,\ldots,N$ on $1 \leq n \leq k$. We can check easily the following properties:

$$f_i \text{ is injective}, \quad f_i(\mathbf{N}) \cap f_j(\mathbf{N}) = \emptyset \quad (i \neq j), \quad \coprod_{i=1}^{N} f_i(\mathbf{N}) = \mathbf{N}. \tag{3.9}$$

By the column of $f_1(n)$ in the above tabular,

$$f_1^k(1) = 1 \tag{3.10}$$

where $f_1^k \equiv \underbrace{f_1 \circ \cdots \circ f_1}_{k}$. The permutative representation $(l_2(\mathbf{N}), \pi)$ of $\mathcal{O}_N$ by f is defined by

$$\pi(s_i)e_n = e_{f_i(n)} \quad (i = 1, \ldots, N,\, n \in \mathbf{N}).$$

Note that $(l_2(\mathbf{N}), \pi)$ is not irreducible when $k \geq 2$ ([5]). $(l_2(\mathbf{N}), \pi)$ satisfies

$$\pi(s_1)e_n = e_{\sigma^{-1}(n)} \quad (1 \leq n \leq k).$$

By the equation (3.10),

$$\pi(s_1)^k e_1 = e_1.$$

Denote $t_i \equiv \pi(s_i)$.

Choose a family $\{g(n)\}_{n=1}^k \subset U(N)$ of unitary matrices which satisfy

$$g_{j1}(n) = w_j^{(\sigma^{-1}(n))} \quad (j = 1, \ldots, N,\, n = 1, \ldots, k)$$

where $w_j^{(n)}$ is the j-th component of vector $w^{(n)} \in S(\mathbf{C}^N)$, $j = 1, \ldots, N$. Rewrite $\{s_i\}_{i=1}^N$ a family of operators on $l_2(\mathbf{N})$ which is defined by

$$s_i e_n \equiv \begin{cases} \displaystyle\sum_{j=1}^N g_{ji}^*(n) t_j e_n & (1 \leq n \leq k), \\ t_i e_n & (n \geq k+1) \end{cases}$$

for $i = 1, \ldots, N$. Then $\{s_i\}_{i=1}^N$ satisfies the relation (3.1). Hence $(l_2(\mathbf{N}), \{s_i\}_{i=1}^N)$ is a new representation of $\mathcal{O}_N$. From this, we have

$$t_i e_n \equiv \begin{cases} \displaystyle\sum_{j=1}^N g_{ji}(n) s_j e_n & (1 \leq n \leq k), \\ s_i e_n & (n \geq k+1) \end{cases}$$

for $i = 1, \ldots, N$. Since $t_1 e_n = e_{\sigma^{-1}(n)}$, $1 \le n \le k$,

$$\begin{aligned} e_{\sigma^{-1}(n)} &= t_1 e_n \\ &= \sum_{j=1}^{N} g_{j1}(n) s_j e_n \\ &= \sum_{j=1}^{N} w_j^{(\sigma^{-1}(n))} s_j e_n \\ &= s\left(w^{(\sigma^{-1}(n))}\right) e_n. \end{aligned}$$

Hence $s(w^{(n)}) e_{\sigma(n)} = e_n$ for $1 \le n \le k$. From this,

$$\begin{aligned} s(w)e_1 &= s(w^{(1)}) \cdots s(w^{(k)}) e_{\sigma(k)} \\ &= s(w^{(1)}) \cdots s(w^{(k-1)}) e_k \\ &= s(w^{(1)}) \cdots s(w^{(k-1)}) e_{\sigma(k-1)} \\ &= \cdots \\ &= s(w^{(1)}) e_2 \\ &= s(w^{(1)}) e_{\sigma(1)} \\ &= e_1. \end{aligned}$$

Therefore $s(w)e_1 = e_1$. Hence a representation $(l_2(\mathbf{N}), \{s_1, \ldots, s_N\})$ satisfies the equation (3.7) with respect to w for $\Omega = e_1$. We finish to construct the GP representation $(l_2(\mathbf{N}), \{s_1, \ldots, s_N\}, e_1)$ of $\mathcal{O}_N$ by w. ∎

Note: The proof of existence of GP representation is the method of the second class gauge transformation of representations of Cuntz algebra. The relation between (3.7) and the second class gauge transformation of representation is explained in the next paper [10].

3.3 Relation with permutative representation

We show the relation between GP representation and ordinary permutative representation by [5].

Let $\{\varepsilon_i\}_{i=1}^{N}$ be the canonical orthonormal basis of $\mathbf{C}^N$. If $w = \varepsilon_I \equiv \varepsilon_{i_1} \otimes \cdots \otimes \varepsilon_{i_k} \in S(\mathbf{C}^N)^{\otimes k}$ for $I = (i_1, \ldots, i_k)$, then the equation (3.7) becomes

$$\pi(s_I)\Omega = \Omega.$$

where $s_I \equiv s_{i_1}\cdots s_{i_k}$. On the other hand, the permutative representation $(l_2(\mathbf{N}),\pi_f)$ with cycle by [5] is given by a branching function system $f = \{f_i\}_{i=1}^N$, that is, f is a family which satisfies (3.9). Furthermore the condition of cycle is corresponded to the relation for an element $n_0 \in \mathbf{N}$

$$f_I(n_0) = n_0$$

where $f_I = f_{i_1}\circ\cdots\circ f_{i_k}$ when $I = (i_1,\ldots,i_k)$. Let $\{e_n\}_{n\in\mathbf{N}}$ be the canonical basis of $l_2(\mathbf{N})$ and $\Omega \equiv e_{n_0} \in l_2(\mathbf{N})$. By definition of the permutative representation

$$\begin{aligned}\pi_f(s_I)\Omega = &\ \pi_f(s_{i_1})\cdots\pi_f(s_{i_k})e_{n_0}\\ = &\ e_{f_I(n_0)}\\ = &\ e_{n_0}\\ = &\ \Omega.\end{aligned}$$

Hence $(l_2(\mathbf{N}),\pi_f,\Omega)$ is the GP representation of $\mathcal{O}_N$ by $w = \varepsilon_I$. Consequently, any ordinary permutative representation with cycle is included in the class of GP representation with cycle.

We show the case of chain [5, 6, 7] and decomposition of them in the succeeding paper [8, 9]. The structure of basis and action of generator of $\mathcal{O}_N$ on them are discussed in subsection 4.4.

4 Structure and canonical basis of GP representation

We construct the basis of the representation space of GP representation by the canonical way here. In the original definition of permutative representation [5], it is defined by using a branching function system and the action of $\mathcal{O}_N$ on a complete orthonormal basis(=CONB) of a Hilbert space. In this sense, it is assumed that the existence of such CONB to define a permutative representation. On the other hand, our definition of generalized permutative representation is not assumed the existence of such suitable CONB at the statement of definition. It is shown that such CONB is automatically derived from the equation (3.7). Such CONB is divided into two kinds, "cycle" and "tree". This is an analogy that a graph which consists of vertices = CONB, and edges = operators, looks like trees on roots which are cyclicly connected each other. The meaning of this analogy is cleared in the following subsections.

4.1 Construction of cycle basis

Let $w \in S(\mathbf{C}^N)^{\otimes k}$. Fix a tensor decomposition of w:

$$w = w^{(1)} \otimes \cdots \otimes w^{(k)} \tag{4.1}$$

for $w^{(j)} \in S(\mathbf{C}^N)$, $j = 1, \ldots, k$. Let

$$w_j \equiv \hat{\sigma}^{j-1}(w) \quad (j = 1, \ldots, k).$$

For example,

$$w_1 = w, \quad w_2 = w^{(2)} \otimes \cdots \otimes w^{(k)} \otimes w^{(1)}.$$

Let $GP(w) = (\mathcal{H}, \{s_1, \ldots, s_N\}, \Omega)$ be the GP representation of $\mathcal{O}_N$ by w. By definition, $s(w)\Omega = \Omega$. Let

$$e_j \equiv s(w^{(j)}) \cdots s(w^{(k)})\Omega \quad (j = 1, \ldots, k). \tag{4.2}$$

Since $s(w^{(j)})$ is an isometry for each $j = 1, \ldots, k$ and $\|\Omega\| = 1$, $\|e_j\| = 1$ for each $j = 1, \ldots, k$. Note that there is a freedom of the choice of phase factor of tensor decomposition (4.1). Hence (4.2) depends on the choice of phase of tensor factor $w^{(i)}$, $i = 1, \ldots, k$. We check this freedom at several stages in our paper.

Lemma 4.1 (i) *$s(w^{(j-1)})e_j = e_{j-1}$ for $j = 2, \ldots, k$ and $s(w^{(k)})e_1 = e_k$.*

(ii) *$s(w_j)e_j = e_j$ for $j = 1, \ldots, k$.*

(iii) *$s(w^{(j)})^*\Omega = < w^{(j)}|w^{(1)} > e_2$.*

(iv) *If $v \in S(\mathbf{C}^N)^{\otimes a}$, $0 < a < k$, then there is $c \in \mathbf{C}$ such that*

$$s(v)^*\Omega = c \cdot e_{a+1}.$$

(v) *If $v \in S(\mathbf{C}^N)^{\otimes (lk+a)}$, $l \geq 1$, $0 \leq a < k$, then there is $c \in \mathbf{C}$ such that*

$$s(v)^*\Omega = c \cdot e_{a+1}.$$

Proof.
(i)

$$\begin{aligned} s(w^{(j-1)})e_j &= s(w^{(j-1)})\left(s(w^{(j)}) \cdots s(w^{(k)})\Omega\right) \\ &= s(w^{(j-1)})s(w^{(j)}) \cdots s(w^{(k)})\Omega \\ &= e_{j-1}, \end{aligned}$$

$$\begin{aligned} s(w^{(k)})e_1 &= s(w^{(k)})\Omega \\ &= e_k. \end{aligned}$$

(ii)

$$\begin{aligned} s(w_j)e_j = & \left(s(w^{(j)})\cdots s(w^{(k)})s(w^{(1)})\cdots s(w^{(j-1)})\right)\left(s(w^{(j)})\cdots s(w^{(k)})\Omega\right) \\ = & s(w^{(j)})\cdots s(w^{(k)})\left(s(w^{(1)})\cdots s(w^{(k)})\right)\Omega \\ = & s(w^{(j)})\cdots s(w^{(k)})\Omega \\ = & e_j. \end{aligned}$$

(iii)

$$\begin{aligned} s(w^{(j)})^*\Omega = & s(w^{(j)})^*s(w)\Omega \\ = & < w^{(j)}|w^{(1)} > s(w^{(2)}\otimes\cdots\otimes w^{(k)})\Omega \\ = & < w^{(j)}|w^{(1)} > e_2. \end{aligned}$$

(iv)

$$\begin{aligned} s(v)^*\Omega = & s(v^{(1)}\otimes\cdots\otimes v^{(a)})^*\Omega \\ = & s(v^{(a)})^*\cdots s(v^{(1)})^*\Omega \\ = & s(v^{(a)})^*\cdots s(v^{(2)})^*e_2 \quad (\text{ by (iii) }) \\ = & \cdots \\ = & s(v^{(a)})^*e_a \\ = & e_{a+1} \end{aligned}$$

for each $a = 1, \ldots, k-1$.
(v) If $a = 0$, then $s(v)^*\Omega =< v|w^{\otimes l} > \Omega = ce_1$ where $c \equiv< v|w^{\otimes l} >$. Assume that $1 \le a \le k-1$. Decompose $v = v_1 \otimes v_2$ such that $v_1 \in S(\mathbf{C}^N)^{\otimes lm}$ and $v_2 \in S(\mathbf{C}^N)^{\otimes a}$.

$$\begin{aligned} s(v)^*\Omega = & < v_1|w^{\otimes l} > s(v_2)^*\Omega \\ = & < w^{(j)}|w^{(1)} > c^{'}e_{a+1} \quad (\text{ by (iv) }) \\ = & ce_{a+1} \end{aligned}$$

where $c \equiv< w^{(j)}|w^{(1)} > c^{'}$. ∎

Note $e_1 = \left(s(w^{(1)})\cdots s(w^{(k)})\right)\Omega = s(w)\Omega = \Omega$.

Corollary 4.2 (i) $s(w^{(j)})e_{\sigma(j)} = e_j$ *for* $j = 1, \ldots, k$.

(ii) *If* $(\mathcal{H}, \{s_1, \ldots, s_N\}, \Omega)$ *is the GP representation of* $\mathcal{O}_N$ *by* $w \in S(\mathbf{C}^N)^{\otimes k}$, *then for each* $p \in \mathbf{Z}_k$, *there is a cyclic vector* $\Omega^{'} \in \mathcal{H}$ *such that* $s(\hat{p}(w))\Omega^{'} = \Omega^{'}$.

Proof. (i) This follows from Lemma 4.1 (i). (ii) When $j = 1$, it is trivial. For $p \in \mathbf{Z}_k \setminus \{id\}$, let $j \equiv p(1) \in \{2, \ldots, k\}$. By definition of e_j in Lemma 4.1 (ii),

$$s(w^{(1)})\cdots s(w^{(j-1)})e_j = \Omega.$$

If $\Omega^{'} \equiv e_j$, then $\Omega \in \pi(\mathcal{O}_N)\Omega^{'}$ and

$$s(\hat{p}(w))\Omega^{'} = s(w_j)e_j = e_j = \Omega^{'}.$$

Hence $\Omega^{'}$ is cyclic, too. Hence $(\mathcal{H}, \{s_1, \ldots, s_N\}, \Omega^{'})$ satisfies the condition of the GP representation of $\mathcal{O}_N$ by $\hat{p}(w)$. ∎

Proposition 4.3 *(Cyclic symmetry of GP representation) If $(\mathcal{H}, \pi, \Omega)$ is the GP representation of $\mathcal{O}_N$ by $w \in S(\mathbf{C}^N)^{\otimes k}$, then for each $p \in \mathbf{Z}_k$, there is $\Omega^{'} \in \mathcal{H}$ such that $(\mathcal{H}, \pi, \Omega^{'})$ is the GP representation of $\mathcal{O}_N$ by $\hat{p}(w)$, too.*

Proof. Assume that $(\mathcal{H}, \pi, \Omega)$ is the GP representation of $\mathcal{O}_N$ by $w \in S(\mathbf{C}^N)^{\otimes k}$. Fix $p \in \mathbf{Z}_k$. By Lemma 4.2 (ii), there is a cyclic vector $\Omega^{'} \in \mathcal{H}$ such that $(\mathcal{H}, \pi, \Omega^{'})$ satisfies the condition (3.7) with respect to $\hat{p}(w)$, too. Hence $(\mathcal{H}, \pi, \Omega^{'})$ is the GP representation of $\mathcal{O}_N$ by $\hat{p}(w) \in S(\mathbf{C}^N)^{\otimes k}$, too. ∎

Recall Definition 3.3.

Corollary 4.4 *Let $w \in S(\mathbf{C}^N)^{\otimes k}$. If a representation $(\mathcal{H}, \pi)$ of $\mathcal{O}_N$ is $GP(w)$, then $(\mathcal{H}, \pi)$ is $GP(\hat{p}(w))$ for each $p \in \mathbf{Z}_k$, too.*

The equivalence of two GP representations is discussed in subsection 5.3.

So far, we do not assume the non periodicity of w. From now, we treat only *non periodic* case. We treat about the periodic case in the succeeding our paper.

Lemma 4.5 *If $w \in S(\mathbf{C}^N)^{\otimes k}$ is non periodic, then*

$$< e_j | e_{j^{'}} >= \delta_{jj^{'}} \quad (j, j^{'} = 1, \ldots, k).$$

Proof.

$$\begin{aligned} < e_j | e_{j^{'}} >= & < s(w_j)e_j | s(w_{j^{'}})e_{j^{'}} > \\ = & < e_j | s(w_j)^* s(w_{j^{'}})e_{j^{'}} > \\ = & < w_j | w_{j^{'}} >< e_j | e_{j^{'}} > \quad (\text{ by } (3.4)). \end{aligned}$$

Hence

$$< e_j | e_{j^{'}} >=< w_j | w_{j^{'}} >< e_j | e_{j^{'}} > .$$

If $< e_j | e_{j^{'}} > \neq 0$, then $< w_j | w_{j^{'}} >= 1$. From this, if $| < w_j | w_{j^{'}} > | < 1$, then $< e_j | e_{j^{'}} >= 0$. On the other hand, if $j \neq j^{'}$, then $| < w_j | w_{j^{'}} > | < 1$ by Proposition 2.5 (i). Therefore $< e_j | e_{j^{'}} >= 0$ when $j \neq j^{'}$. ∎

Definition 4.6 *For a non periodic element* $w \in S(\mathbf{C}^N)^{\otimes k}$ *and its tensor decomposition* $\{w^{(j)}\}_{j=1}^k \subset S(\mathbf{C}^N)$, $\{e_j\}_{j=1}^k$ *is called the cycle basis of* $GP(w)$ *with respect to* $\{w^{(j)}\}_{j=1}^k$.

By definition of cycle basis, if $\{w^{(j)}\}_{j=1}^k$ and $\{v^{(j)}\}_{j=1}^k$ are two tensor decompositions of w, then associated cycle basis of them are equal up to phase factor. In this sense, the cycle basis of $GP(w)$ is canonically defined from w with phase freedom.

Note: The orthogonality of cycle basis is automatically induced from the equation (3.7) and the relations (3.1). This shows the importance of condition (3.7) for representation of $\mathcal{O}_N$.

Lemma 4.7 *Let* $w \in S(\mathbf{C}^N)^{\otimes k}$ *and* $(\mathcal{H}, \{s_1, \ldots, s_N\}, \Omega)$ *the GP representation of* $\mathcal{O}_N$ *by* w. *Fix* $\{w^{(i)}\}_{i=1}^k$ *the tensor decomposition of* w *Assume that* $\{e_i\}_{i=1}^k$ *the cycle basis of* w *with respect to* $\{w^{(i)}\}_{i=1}^k$.

If $w' \in S(\mathbf{C}^N)^{\otimes(kn+a)}$, $n \geq 0$ *and* $0 \leq a < k$, *then*

$$s(w')^*\Omega = < w'|\phi_{n,a} > e_{a+1}$$

where $\phi_{n,a} \in S(\mathbf{C}^N)^{\otimes(kn+a)}$ *which is defined by*

$$\phi_{n,a} \equiv \begin{cases} w^{\otimes n} & (a = 0) \\ \\ w^{\otimes n} \otimes w^{(1)} \otimes \cdots \otimes w^{(a)} & (0 < a < k). \end{cases}$$

Proof. By (3.7), $(s(w))^n\Omega = (s(w))^{n-1}\Omega = \cdots = \Omega$. If $0 < a < k$, then

$$\begin{aligned} s(\phi_{n,a})e_{a+1} = & \; s(w^{\otimes n} \otimes w^{(1)} \otimes \cdots \otimes w^{(a)})s(w^{(a+1)}) \cdots s(w^{(k)})\Omega \\ = & \; s(w^{\otimes n})s(w)\Omega \\ = & \; (s(w))^n\Omega \qquad (\text{ by } (3.5)\ ) \\ = & \; \Omega. \end{aligned}$$

The case $a = 0$ follows from the last equality in the above. Hence

$$s(\phi_{n,a})e_{a+1} = \Omega. \tag{4.3}$$

By (4.3) and (3.4),

$$\begin{aligned} s(w')^*\Omega = & \; s(w')^* \left(s(\phi_{n,a})e_{a+1}\right) \\ = & \; \left(s(w')^* s(\phi_{n,a})\right) e_{a+1} \\ = & \; < w'|\phi_{n,a} > e_{a+1}. \end{aligned}$$

$\blacksquare$

Note that the right hand side in the equation of Lemma 4.7 is independent in the choice of tensor decomposition of w.

Let

$$V_w \equiv \text{Lin} < \{e_j : j = 1, \ldots, k\} > .$$

Then V_w is a subspace of $\mathcal{H}$ and its definition is independent in the difference of phase factor of cycle basis of $GP(w)$.

Lemma 4.8 *For each $I \in \bigcup_{k\geq 1}\{1,\ldots,N\}^k$, $s_I^* V_w \subset V_w$.*

Proof. Note

$$s_I^* e_j = \quad s(\varepsilon_I)^* \left(s(w^{(j)}) \cdots s(w^{(k)})\Omega \right).$$

If $l \equiv |I| < k - j + 1$, then

$$\begin{aligned} s_I^* e_j = & \;< \varepsilon_I | w^{(j)} \otimes \cdots \otimes w^{(j+l-1)} > s(w^{(j+l)}) \cdots s(w^{(k)})\Omega \\ = & \;< \varepsilon_I | w^{(j)} \otimes \cdots \otimes w^{(j+l-1)} > e_{j+l} \\ \in & \;V_w. \end{aligned}$$

If $|I| = k - j + 1$, then

$$s_I^* e_j =< \varepsilon_I | w^{(j)} \otimes \cdots \otimes w^{(k)} > \Omega \in V_w.$$

If $l \equiv |I| > k - j + 1$, let $I_1 = (i_1, \ldots, i_k)$ and $I_2 = (i_{k+1}, \ldots, i_l)$. Then

$$\begin{aligned} s_I^* e_j = & \;s(\varepsilon_{I_1} \otimes \varepsilon_{I_2})^* s(w^{(j)} \otimes \cdots \otimes w^{(k)})\Omega. \\ = & \;s(\varepsilon_{I_2})^* s(\varepsilon_{I_1})^* s(w^{(j)} \otimes \cdots \otimes w^{(k)})\Omega. \\ = & \;< \varepsilon_{I_1} | w^{(j)} \otimes \cdots \otimes w^{(k)} > s(\varepsilon_{I_2})^*\Omega. \end{aligned}$$

By Lemma 4.7, $s(\varepsilon_{I_2})^*\Omega \in V_w$. Hence

$$s_I^* e_j =< \varepsilon_{I_1} | w^{(j)} \otimes \cdots \otimes w^{(k)} > s(\varepsilon_{I_2})^*\Omega \in V_w.$$

$\blacksquare$

Corollary 4.9

$$\mathcal{O}_N V_w = \overline{\text{Lin} < \{s_I\Omega,\, \Omega : I \in \{1,\ldots,N\}^k,\, k \geq 1\} >} \tag{4.4}$$

where the overline means the clousure in $\mathcal{H}$.

Proof. Denote $\mathcal{K}$ the set in the right hand side of (4.4). By definition (4.2) of the cycle basis e_j, $e_j \in \mathcal{K}$. Hence $V_w \subset \mathcal{K}$. By Lemma 4.8,

$$s_I s_J^* V_w \in s_I V_w \subset s_I \mathcal{K} \subset \mathcal{K}$$

for each I, J. Since Lin $< \{s_I s_J^* : |I|+|J| \geq 1\} >$ is a dense *- subalgebra of $\mathcal{O}_N$, $\mathcal{O}_N V_w \subset \mathcal{K}$. On the other hand, by the definition of GP representation $(\mathcal{H}, \{s_1, \ldots, s_N\}, \Omega)$, $\mathcal{H}$ is cyclic with respect to Ω. Hence

$$\mathcal{O}_N V_w \subset \mathcal{K} \subset \mathcal{H} = \mathcal{O}_N \Omega.$$

Hence $\mathcal{O}_N V_w = \mathcal{K}$. ∎

Corollary 4.10 $\mathcal{H} = \overline{\mathrm{Lin} < \{s_I\Omega,\, \Omega : I \in \{1, \ldots, N\}^k,\, k \geq 1\} >}$.

¿From this, we can consider the GP representation space as the right hand side in the statement in Corollary 4.10. The characteristic property of generalized permutative representation with cycle is the existence of a finite dimensional subspace V_w. In the case of "chain" in [5], there is *no* such V_w which satisfies the property in Lemma 4.8. In the analogy of tree and root, then V_w is associated with root.

4.2 Property of cycle basis

Assume that $w \in S(\mathbf{C}^N)^{\otimes k}$ is non periodic, $\{w^{(j)}\}_{j=1}^k$ is a tensor decomposition of w and $\{e_j\}_{j=1}^k$ is the cycle basis of $GP(w)$ with respect to $\{w^{(j)}\}_{j=1}^k$.

For $j \in \{1, \ldots, k\}$, let

$$N_j(w) \equiv \{z \in S(\mathbf{C}^N) :< z|w^{(j)} >= 0\}. \tag{4.5}$$

Lemma 4.11 *Let $j, j^{'} = 1, \ldots, k$.*

(i) *If $j \neq j^{'}$, then*

$$< s(z)e_j|e_{j'} >= 0$$

for each $z \in N_{\sigma^{-1}(j)}(w)$.

(ii) *If $j \neq j^{'}$, then*

$$< s(v)s(z)e_j|e_{j'} >= 0$$

for each $v \in TS(\mathbf{C}^N)$ and $z \in N_{\sigma^{-1}(j)}(w)$.

(iii) *If* $j \neq j^{'}$*, then*

$$< s(v)s(z)e_j | s(z^{'})e_{j'} >= 0$$

for each $v \in TS(\mathbf{C}^N)$*,* $z \in N_{\sigma^{-1}(j)}(w)$ *and* $z^{'} \in N_{\sigma^{-1}(j')}(w)$*.*

(iv) *If* $j \neq j^{'}$*, then*

$$< s(v)s(z)e_j | s(v^{'})s(z^{'})e_{j'} >= 0$$

for each $v, v^{'} \in TS(\mathbf{C}^N)$*,* $z \in N_{\sigma^{-1}(j)}(w)$ *and* $z^{'} \in N_{\sigma^{-1}(j')}(w)$*.*

Proof.
(i) By Corollary 4.2 (i),

$$\begin{aligned} < s(z)e_j | e_{j'} >= & < s(z)e_j | s(w^{(j^{'})})e_{\sigma(j')} > \\ = & < z | w^{(j^{'})} >< e_j | e_{\sigma(j')} > \quad (\text{ by } (3.4)) \\ = & < z | w^{(j^{'})} > \delta_{j,\sigma(j')}. \end{aligned}$$

If $j = \sigma(j^{'})$, then $< z | w^{(j^{'})} >=< z | w^{(\sigma^{-1}(j))} >= 0$ by choice of z. Hence $< s(z)e_j | e_{j'} >= 0$.
(ii) Assume that $v \in S(\mathbf{C}^N)^{\otimes(kn+a)}$, $n \geq 0$ and $0 \leq a < k$. Let

$$y \equiv w^{\otimes n} \otimes w^{(j^{'})} \otimes w^{(\sigma(j^{'}))} \otimes \cdots \otimes w^{(\sigma^{a-1}(j^{'}))}.$$

Note "the length of v" = "the length of y". Then $s(y)e_{\sigma^a(j')} = e_{j'}$. Hence

$$\begin{aligned} < s(v)s(z)e_j | e_{j'} >= & < s(v)s(z)e_j | s(y)e_{\sigma^a(j')} > \\ = & < v | y >< s(z)e_j | e_{\sigma^a(j')} > \end{aligned}$$

By (i), if $j \neq \sigma^a(j^{'})$, then $< s(z)e_j | e_{\sigma^a(j')} >= 0$. Hence $< s(v)s(z)e_j | e_{j'} >= 0$.

Assume that $j = \sigma^a(j^{'})$. Then

$$\begin{aligned} < s(v)s(z)e_j | e_{j'} >= & < v | y >< s(z)e_j | e_j > \\ = & < v | y >< s(z)e_j | s(w^j)e_{\sigma(j)} > \\ = & < v | y >< z | w^{(j)} >< e_j | e_{\sigma(j)} > \\ = & < v | y >< z | w^{(j)} > \delta_{j,\sigma(j)} \\ = & 0. \end{aligned}$$

by Corollary 4.2 (i). Hence $< s(v)s(z)e_j | e_{j'} >= 0$.

(iii) Assume that $v \in S(\mathbf{C}^N)^{\otimes l}$, $l \geq 1$. If $l = 1$, then

$$\begin{aligned} < s(v)s(z)e_j | s(z^{'})e_{j'} >= & < v|z^{'} >< s(z)e_j|e_{j'} > \\ = & \ 0 \end{aligned}$$

by (i).

Assume $l \geq 2$ and choose v_1, v_2 such that $v = v_1 \otimes v_2$ and $v_1 \in S(\mathbf{C}^N)$. Then

$$\begin{aligned} < s(v)s(z)e_j | s(z^{'})e_{j'} >= & < v_1|z^{'} >< s(v_2)s(z)e_j|e_{j'} > \\ = & \ 0 \end{aligned}$$

by (ii).

(iv) Assume that $v \in S(\mathbf{C}^N)^{\otimes l}, v^{'} \in S(\mathbf{C}^N)^{\otimes l^{'}}$, $l, l^{'} \geq 1$. We can assume that $l \geq l^{'}$ without loss of generality.

If $l - l^{'} = 0$, then

$$\begin{aligned} < s(v)s(z)e_j | s(v^{'})s(z^{'})e_{j'} >= & < v \otimes z|v^{'} \otimes z^{'} >< e_j|e_{j'} > \\ = & \ 0. \end{aligned}$$

Assume $l - l^{'} \geq 1$ and choose v_1, v_2 such that $v = v_1 \otimes v_2$ and $v_1 \in S(\mathbf{C}^N)^{\otimes l^{'}}$. Then

$$\begin{aligned} < s(v)s(z)e_j | s(v^{'})s(z^{'})e_{j'} >= & < v_1|v^{'} >< s(v_2)s(z)e_j|s(z^{'})e_{j'} > \\ = & \ 0 \end{aligned}$$

by (iii). ∎

Assume that $w \in S(\mathbf{C}^N)^{\otimes k}$ is non periodic. For $GP(w) = (\mathcal{H}, \{s_1, \ldots, s_N\}, \Omega)$, define a family of subspaces of $\mathcal{H}$ by

$$\mathcal{T}_j(w) \equiv \overline{\text{Lin} < \{s(v)s(z)e_j, s(z)e_j, e_j : v \in TS(\mathbf{C}^N),\ z \in N_{\sigma^{-1}(j)}(w)\} >}.$$

for $j = 1, \ldots, k$.

Theorem 4.12 *If $\mathcal{H}$ is the GP representation of $\mathcal{O}_N$ by non periodic $w \in S(\mathbf{C}^N)^{\otimes k}$, then the following decomposition holds:*

$$\mathcal{H} = \bigoplus_{j=1}^{k} \mathcal{T}_j(w).$$

Proof. By Lemma 4.11, $\{\mathcal{T}_j(w)\}_{j=1}^k$ are mutually orthogonal. Hence

$$\bigoplus_{j=1}^k \mathcal{T}_j(w) \subset \mathcal{H}.$$

On the other hand, by Corollary 4.9 and 4.10,

$$\begin{aligned}\mathcal{H} = &\ \mathcal{O}_N V_w \\ = &\ \bigoplus_{j=1}^k \overline{\mathrm{Lin} < \{s_I e_j,\, e_j : I \in \{1,\ldots,N\}^k,\, k \geq 1\} >}. \\ \subset &\ \bigoplus_{j=1}^k \mathcal{T}_j(w).\end{aligned}$$

Hence $\mathcal{H} = \bigoplus_{j=1}^k \mathcal{T}_j(w)$. ∎

Note that a decomposition in Theorem 4.12 is independent in the choice of tensor decomposition in (4.1).

4.3 Tree subspace of GP representation

Assume that $w \in S(\mathbf{C}^N)^{\otimes k}$ is non periodic and we use symbols $\mathcal{T}_j(w)$, $j = 1, \ldots, k$, $N_j(w)$ in subsection 4.2. We consider $\mathcal{T}_j(w)$, $j = 1, \ldots, k$.

Lemma 4.13 *Fix $j \in \{1, \ldots, k\}$.*

(i) $< s(z)e_j|e_j >= 0$ *for* $z \in N_{\sigma^{-1}(j)}(w)$.

(ii) $< s(v)s(z)e_j|e_j >= 0$ *for* $z \in N_{\sigma^{-1}(j)}(w)$ *and* $v \in TS(\mathbf{C}^N)$.

(iii) $< s(v)s(z)e_j|s(z^{'})e_j >= 0$ *for* $z, z^{'} \in N_{\sigma^{-1}(j)}(w)$ *and* $v \in TS(\mathbf{C}^N)$.

(iv) $< s(v)s(z)e_j|s(v^{'})s(z^{'})e_j >= 0$ *for* $z, z^{'} \in N_{\sigma^{-1}(j)}(w)$ *when* $v \in TS(\mathbf{C}^N)$ *and* $v^{'} \in TS(\mathbf{C}^N)$ *are different in length.*

Proof.
(i) By Corollary 4.2 (i),

$$\begin{aligned}< s(z)e_j|e_j >= &\ < s(z)e_j|s(w^{(j)})e_{\sigma(j)} > \\ = &\ < z|w^{(j)} >< e_j|e_{\sigma(j)} > \\ = &\ 0.\end{aligned}$$

(ii) Assume that $v \in S(\mathbf{C}^N)^{\otimes l}$.
If $l = 1$, then

$$\begin{aligned} < s(v)s(z)e_j|e_j >= & < v|w^{(j)} >< s(z)e_j|e_{\sigma(j)} > \\ = & \ 0 \end{aligned}$$

by Corollary 4.2 (i) and Lemma 4.11 (i). If $l \geq 2$, then choose v_1, v_2 such that $v = v_1 \otimes v_2$ and $v_1 \in S(\mathbf{C}^N)$. Then

$$\begin{aligned} < s(v)s(z)e_j|e_j >= & < v_1|w^{(j)} >< s(v_2)s(z)e_j|e_{\sigma(j)} > \\ = & \ 0 \end{aligned}$$

by Corollary 4.2 (i) and Lemma 4.11 (ii).
(iii) Assume that $v \in S(\mathbf{C}^N)^{\otimes l}$. If $l = 1$, then

$$\begin{aligned} < s(v)s(z)e_j|s(z^{'})e_j >= & < v|z^{'} >< s(z)e_j|e_j > \\ = & \ 0 \end{aligned}$$

by (i). If $l \geq 2$, then choose v_1, v_2 such that $v = v_1 \otimes v_2$ and $v_1 \in S(\mathbf{C}^N)$. Then

$$\begin{aligned} < s(v)s(z)e_j|s(z^{'})e_j >= & < v_1|z^{'} >< s(v_2)s(z)e_j|e_j > \\ = & \ 0 \end{aligned}$$

by (ii).
(iv) Assume that $v \in S(\mathbf{C}^N)^{\otimes l}$ and $v^{'} \in S(\mathbf{C}^N)^{\otimes l^{'}}$. We can assume that $l > l^{'}$ without loss of generality. If $l - l^{'} = 1$, then

$$\begin{aligned} < s(v)s(z)e_j|s(v^{'})s(z^{'})e_j >= & < v|v^{'} \otimes z^{'} >< s(z)e_j|e_j > \\ = & \ 0 \end{aligned}$$

by (i). If $l - l^{'} \geq 2$, then choose v_1, v_2 such that $v = v_1 \otimes v_2$ and $v_1 \in S(\mathbf{C}^N)^{\otimes l^{'}}$. Then

$$\begin{aligned} < s(v)s(z)e_j|s(v^{'})s(z^{'})e_j >= & < v_1|v^{'} >< s(v_2)s(z)e_j|s(z^{'})e_j > \\ = & \ 0 \end{aligned}$$

by (iii). ∎

Theorem 4.14 *For each $j = 1, \ldots, k$, we have the following decomposition:*

$$\mathcal{T}_j(w) = \bigoplus_{l \geq 0} \mathcal{F}_j^{(l)}(w)$$

where

$$\begin{aligned}
\mathcal{F}_j^{(0)}(w) \equiv & \ \mathbf{C}e_j, \\
\mathcal{F}_j^{(1)}(w) \equiv & \ \mathrm{Lin} < \{s(z)e_j : z \in N_{\sigma^{-1}(j)}(w)\} >, \\
\mathcal{F}_j^{(l)}(w) \equiv & \ \mathrm{Lin} < \{s(v)s(z)e_j : z \in N_{\sigma^{-1}(j)}(w),\, v \in S(\mathbf{C}^N)^{\otimes(l-1)}\} >
\end{aligned}$$

for $l \geq 2$ and the infinite direct sum means the clousure of algebraic direct sum in $\mathcal{H}$.

Proof. By Lemma 4.13, $\left\{\mathcal{F}_j^{(l)}(w) : l \geq 0\right\}$ are mutually orthogonal. Hence the direct sum decomposition makes sense in $\mathcal{T}_j(w)$. Furthermore $\mathcal{T}_j(w)$ consists of $\left\{\mathcal{F}_j^{(l)}(w) : l \geq 0\right\}$ by definition of $\mathcal{T}_j(w)$. ∎

Note

$$\begin{aligned}
s(z)\mathcal{F}_j^{(0)}(w) \ &\subset \ \mathcal{F}_j^{(1)}(w) \qquad (\, z \in N_{\sigma^{-1}(j)}(w)\,), \\
s_i\mathcal{F}_j^{(l)}(w) \ &\subset \ \mathcal{F}_j^{(l+1)}(w) \qquad (i = 1, \ldots, N,\, l \geq 1)
\end{aligned}$$

for each $j = 1, \ldots, k$.

Theorem 4.15 *Let $GP(w) = (\mathcal{H}, \{s_1, \ldots, s_N\}, \Omega)$ for non periodic $w \in S(\mathbf{C}^N)^{\otimes k}$. Then the following decomposition holds:*

$$\mathcal{H} = \bigoplus_{j=1}^{k} \bigoplus_{l \geq 0} \mathcal{F}_j^{(l)}(w),$$

$$\mathcal{F}_j^{(l+1)}(w) = \bigoplus_{m=1}^{N} s_m \mathcal{F}_j^{(l)}(w) \cong \mathbf{C}^N \otimes \mathcal{F}_j^{(l)}(w) \qquad (l \geq 1),$$

$$\mathcal{F}_j^{(1)}(w) \cong N_{\sigma^{-1}(j)}(w) \otimes \mathcal{F}_j^{(0)}(w) \cong \mathbf{C}^{N-1} \otimes \mathcal{F}_j^{(0)}(w)$$

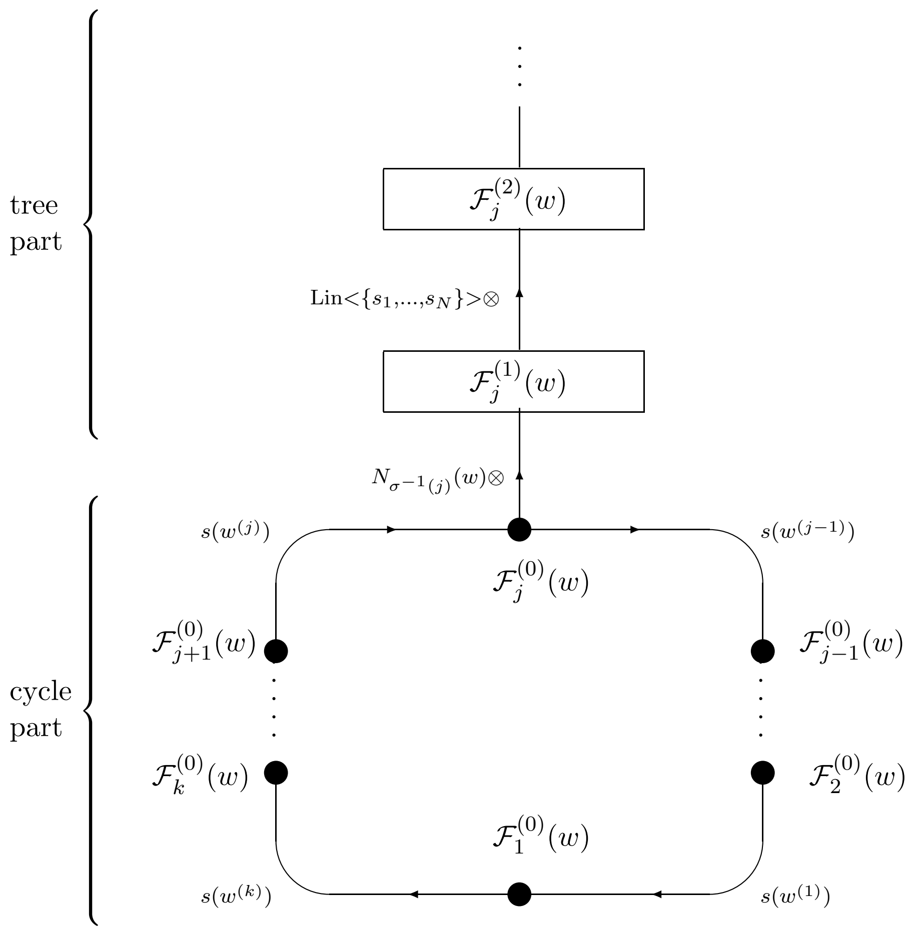
for $j = 1, \ldots, k$. Furthermore

$$s(w^{(j)})\mathcal{F}_{\sigma(j)}^{(0)}(w) = \mathcal{F}_j^{(0)}(w) \quad (j = 1, \ldots, k).$$

Proof. By combining Theorem 4.12 and 4.14, we have the first formula. The second follows from definition of $\mathcal{F}_j^{(l)}(w)$ and orthogonality of s_i and s_j when $i \neq j$. The third follows from definition of $\mathcal{F}_j^{(1)}(w)$. The last formula follows by Corollary 4.2 (i). ∎

We use this decomposition in subsection 4.4.

The following illustration is the decomposition in Theorem 4.15:



Note: By definition of $\mathcal{F}_j^{(l)}(w)$, the decomposition in Theorem 4.15 is independent in the choice of tensor decomposition of w. It is remarkable that only one equation (3.7) induces a direct sum decomposition of the representation space and the meaning of decomposition is clear as the statement in Theorem 4.15.

4.4 Construction of tree basis

The aim of this subsection is to construct a complete orthonormal basis of the GP representation by non periodic $w \in TS(\mathbf{C}^N)$ according to the direct sum decomposition in Theorem 4.15. Our strategy is to construct an orthonormal basis of $\mathcal{F}_j^{(l)}(w)$ for each $j = 1, \ldots, k$, $l \geq 0$. By definition of $\mathcal{F}_j^{(l)}(w)$, it seems that the structure of $\mathcal{H}$ is similar to the full Fock space over $\mathbf{C}^N$. The precise answer of this analogy is obtained by showing the form of basis of $\mathcal{H}$ from here.

Assume that $w \in S(\mathbf{C}^N)^{\otimes k}$ is non periodic, $GP(w) = (\mathcal{H}, \{s_1, \ldots, s_N\}, \Omega)$ and $\{e_j\}_{j=1}^k$ is the cycle basis of $GP(w)$ with respect to a tensor decomposition $\{w^{(j)}\}_{j=1}^k$ in Definition 4.6.

Fix $j \in \{1, \ldots, k\}$. For a component $w^{(j)}$, choose an orthogonal family $\{w^{(j)}[l] : l = 1, \ldots, N\}$ in $S(\mathbf{C}^N)$ such that $w^{(j)}[1] = w^{(j)}$. By definition, $\{w^{(j)}[l] : l = 2, \ldots, N\} \subset N_j(w)$ in (4.5).

Define a subset $\Lambda(w)$ of $TS(\mathbf{C}^N)$ by

$$\Lambda(w) \equiv \coprod_{j=1}^{k} \coprod_{m \geq 0} \Lambda_j^{(m)}(w)$$

where

$$\begin{aligned}
\Lambda_j^{(0)}(w) \equiv & \left\{ w^{(j)} \otimes \cdots \otimes w^{(k)} \right\}, \\
\Lambda_1^{(1)}(w) \equiv & \left\{ w^{(k)}[l] : \ l = 2, \ldots, N \ \right\}, \\
\Lambda_j^{(1)}(w) \equiv & \left\{ w^{(j-1)}[l] \otimes w^{(j)} \otimes \cdots \otimes w^{(k)} : \ l = 2, \ldots, N \ \right\} \quad (j = 2, \ldots, k), \\
\Lambda_j^{(m)}(w) \equiv & \left\{ \varepsilon_I \otimes x : x \in \Lambda_j^{(1)}(w),\, I \in \{1, \ldots, N\}^{m-1} \right\}
\end{aligned}$$

for $m \geq 2$ where $\{\varepsilon_i : i = 1, \ldots, N\}$ is the canonical basis of $\mathbf{C}^N$ and $\varepsilon_I \equiv \varepsilon_{i_1} \otimes \cdots \otimes \varepsilon_{i_m}$ when $I = (i_1, \ldots, i_m) \in \{1, \ldots, N\}^m$. Specially, $\Lambda_1^{(0)} = \{w\}$. The cardinality of these sets are followings

$$\begin{aligned}
\#\Lambda_j^{(0)}(w) = & \ 1, \\
\#\Lambda_j^{(1)}(w) = & \ N - 1, \\
\#\Lambda_j^{(m)}(w) = & \ (N-1)N^{m-1}
\end{aligned}$$

for $m \geq 2$ and $j = 1, \ldots, k$.

Define a family $\{e_x \in \mathcal{H} : x \in \Lambda(w)\}$ of unit vectors in $\mathcal{H}$ by

$$e_x \equiv s(x)\Omega \quad (x \in \Lambda(w)).$$

We distinguish $\{e_x : x \in \Lambda(w)\}$ and the cycle basis in (4.2) by the kind of suffix.

Proposition 4.16 *For non periodic $w \in TS(\mathbf{C}^N)$, $\{e_x \in \mathcal{H} : x \in \Lambda(w)\}$ is a complete orthonormal basis of the GP representation of $\mathcal{O}_N$ by w.*

Proof. Note

$$\{\, e_x : x \in \Lambda(w) \,\} = \coprod_{j=1}^{k} \coprod_{m \geq 0} \left\{ e_x : x \in \Lambda_j^{(m)}(w) \right\}. \tag{4.6}$$

By definition (4.2) of cycle basis, $\{e_x : x \in \Lambda_j^{(0)}(w)\} = \{e_j\} \subset \mathcal{F}_j^{(0)}(w)$. Since $\{w^{(j)}[l] : l = 2, \ldots, k\} \subset N_j(w)$ by (4.5),

$$\left\{ e_x : x \in \Lambda_j^{(1)}(w) \right\} = \left\{ s(w^{(\sigma^{-1}(j))}[l]) e_j : l = 2, \ldots, N \right\} \subset \mathcal{F}_j^{(1)}(w). \tag{4.7}$$

Furthermore

$$\begin{aligned} \left\{ e_x : x \in \Lambda_j^{(m)}(w) \right\} = & \left\{ s_I s(w^{(\sigma^{-1}(j))}[l]) e_j : \begin{array}{l} l = 2, \ldots, N, \\ I \in \{1, \ldots, N\}^{m-1} \end{array} \right\} \\ \subset & \ \mathcal{F}_j^{(m)}(w) \end{aligned}$$

for each $m \geq 2$ and $j = 1, \ldots, k$.

By Theorem 4.15, a decomposition (4.6) is a decomposition of orthogonal families of vectors in $\mathcal{H}$. It is sufficient to show that orthogonality of vectors in each family $\{e_x : x \in \Lambda_j^{(m)}(w)\}$.

<u>When $m = 0$:</u> $\{e_x : x \in \Lambda_j^{(0)}(w)\}$ is a one-point set. Hence there is nothing to show about this case.

<u>When $m = 1$:</u> For $l, l' = 2, \ldots, N$,

$$< s(w^{(j)}[l])\Omega \,|\, s(w^{(j)}[l'])\Omega > = < w^{(j)}[l] \,|\, w^{(j)}[l'] > = \delta_{l,l'}.$$

By (4.7) $\{e_x : x \in \Lambda_j^{(1)}(w)\}$ is an orthogonal family.

<u>When $m \geq 2$:</u> For $I, I' \in \{1, \ldots, N\}^{m-1}$ and $l, l' = 2, \ldots, N$,

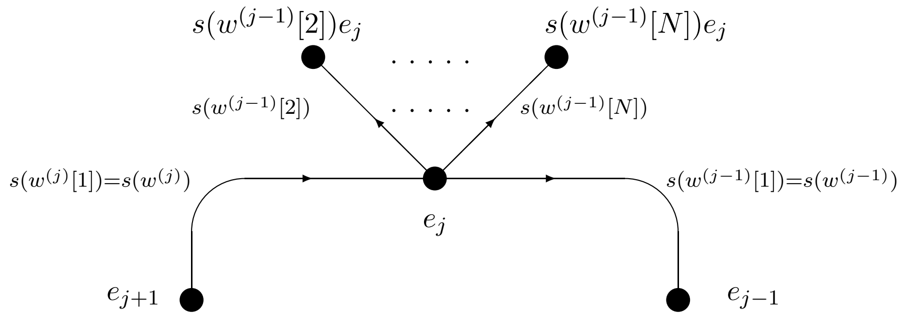
$$< s_I s(w^{(j)}[l])\Omega \,|\, s_{I'} s(w^{(j)}[l'])\Omega > = \delta_{I,I'} \delta_{l,l'}.$$

Hence $\{e_x : x \in \Lambda_j^{(m)}(w)\}$ is an orthogonal family, too. By comparing the number of elements in $\Lambda_j^{(m)}(w)$ and the dimension of $\mathcal{F}_j^{(m)}(w)$, $\{e_x : x \in \Lambda_j^{(m)}(w)\}$ is a basis of $\mathcal{F}_j^{(m)}(w)$. We have

$$\mathrm{Lin} < \{e_x : x \in \Lambda_j^{(m)}(w)\} >= \mathcal{F}_j^{(m)}(w). \tag{4.8}$$

By Theorem 4.15, an orthogonal family $\{e_x : x \in \Lambda(w)\}$ is complete. ∎

We illustrate this basis by the following figure:



Recall Corollary 4.2 (i). In this figure, a vertex and an edge mean a vector and an operator on the representation space, respectively.

We check the action of $\mathcal{O}_N$ on this basis. If $m \geq 1$, then

$$s_i e_x = s_i s(x)\Omega = s(\varepsilon_i \otimes x)\Omega = e_{\varepsilon_i \otimes x} \tag{4.9}$$

for $i = 1, \ldots, N$, $x \in \Lambda_j^{(m)}(w)$. Hence s_i moves tree basis to tree basis except cycle. This action is similar to ordinary permutive representation ([5]). The case $m = 0$ is complicated rather than that of $m \neq 0$. Define a family $\{g(n)\}_{n=1}^k$ of unitaries in $U(N)$ by

$$g_{ij}(n) \equiv w_i^{(\sigma^{-1}(n))}[j] \quad (i, j = 1, \ldots, N,\ n = 1, \ldots, k).$$

Then

$$g(n) = \begin{pmatrix} w_1^{(\sigma^{-1}(n))}[1] & \cdots & w_1^{(\sigma^{-1}(n))}[N] \\ w_2^{(\sigma^{-1}(n))}[1] & \cdots & w_2^{(\sigma^{-1}(n))}[N] \\ \vdots & \vdots\vdots\vdots & \vdots \\ w_N^{(\sigma^{-1}(n))}[1] & \cdots & w_N^{(\sigma^{-1}(n))}[N] \end{pmatrix}$$

for $n=1,\ldots,k$. By choice of $\{\, w^{(\sigma^{-1}(n))}[l]\,\}_{l=1}^{N}$, $g(n)$ is a unitary matrix. By this,

$$s(w^{(n)}[i])=\sum_{j=1}^{N} w_j^{(n)}[i]s_j=\sum_{j=1}^{N} g_{ji}(\sigma(n))s_j=\alpha_{g(\sigma(n))}(s_i)$$

for $i,j=1,\ldots,N$, $n=1,\ldots,k$ where α is the natural $U(N)$ action on $\mathcal{O}_N$. Hence

$$s_i=\alpha_{g(\sigma(n))^*}\left(s(w^{(n)}[i])\right).$$

By using this equation, compute action of s_i:

$$\begin{aligned}
s_i e_{x_{\sigma(m)}} &= s_i e_{w^{(m)}\otimes\cdots\otimes w^{(k)}}\\
&= s_i s(w^{(m)}\otimes\cdots\otimes w^{(k)})\Omega\\
&= s_i e_m\\
&= \alpha_{g(m)^*}\left(s(w^{(\sigma^{-1}(m))}[i])\right)e_m\\
&= \sum_{j=1}^{N}(g(m)^*)_{ji}s(w^{(\sigma^{-1}(m))}[j])e_m\\
&= \overline{g(m)}_{i1}s(w^{(\sigma^{-1}(m))}[1])e_m+\sum_{j=2}^{N}\overline{g(m)}_{ij}s(w^{(\sigma^{-1}(m))}[j])e_m\\
&= \overline{g(m)}_{i1}e_{\sigma^{-1}(m)}\\
&\quad +\sum_{j=2}^{N}\overline{g(m)}_{ij}s(w^{(\sigma^{-1}(m))}[j]\otimes w^{(m)}\otimes\cdots\otimes w^{(k)})\Omega\\
&= \overline{w}_i^{(\sigma^{-1}(m))}e_{x_m}+\sum_{j=2}^{N}\overline{w}_i^{(\sigma^{-1}(m))}[j]s(y_{j,m})\Omega\\
&= \overline{w}_i^{(\sigma^{-1}(m))}e_{x_m}+\sum_{j=2}^{N}\overline{w}_i^{(\sigma^{-1}(m))}[j]e_{y_{j,m}}
\end{aligned}$$

where

$$x_1\equiv w^{(k)},\quad x_m\equiv w^{(\sigma^{-1}(m))}\otimes\cdots\otimes w^{(k)}, \tag{4.10}$$

$$y_{j,1} \equiv w^{(k)}[j], \quad y_{j,m} \equiv w^{(\sigma^{-1}(m))}[j] \otimes w^{(m)} \otimes \cdots \otimes w^{(k)}$$

for $j = 2, \ldots, N$, $m = 2, \ldots, k$. Note

$$x_m \in \Lambda^{(0)}_{\sigma^{-1}(m)}(w), \quad y_{j,m} \in \Lambda^{(1)}_{\sigma^{-1}(m)}(w)$$

for $m = 1, \ldots, k$ and $j = 1, \ldots, N$.

Lemma 4.17 *Under the assumption in Proposition 4.16 and symbols (4.10), the following equation holds:*

$$s_i e_{x_{\sigma(m)}} = \overline{w}_i^{(\sigma^{-1}(m))} e_{x_m} + \sum_{j=2}^{N} \overline{w}_i^{(\sigma^{-1}(m))}[j] e_{y_{j,m}}$$

for $m = 1, \ldots, k$ and $i = 1, \ldots, N$.

Corollary 4.18 *(Ordinary cycle basis notation) Under the assumption in Proposition 4.16, the following equation holds:*

$$s_i e_m = \overline{w}_i^{(\sigma^{-1}(m))} e_{\sigma^{-1}(m)} + \sum_{j=2}^{N} \overline{w}_i^{(\sigma^{-1}(m))}[j] s(w^{(\sigma^{-1}(m))}[j]) e_m$$

for $m = 1, \ldots, k$ and $j = 1, \ldots, N$.

By Lemma 4.17, the action of generators of $\mathcal{O}_N$ on the cycle basis is clarified. For s_i action, the first term in the right hand side is a cycle basis, again. On the other hand, other term is in $\mathcal{F}_i^{(1)}(w)$ and this is "outside" cycle. By checking matrix element of $g(m)$, it is known that $\left(\bar{w}_i^{(\sigma^{-1}(m))}[l]\right)_{l=1}^{N} \in \mathbf{C}^N$ is a unit vector. Hence it seems that an operator s_i is arisen from a branching function system([5]) with weight $\left(\bar{w}_i^{(\sigma^{-1}(m))}[l]\right)_{l=1}^{N}$. In this point of view, GP representation is regarded as a permutative representation by "a *quantum* branching function system".

Note: The definition of the basis in Proposition 4.16 depends on the choice of orthonormal families $\left\{\{w^{(n)}[l] : l = 1, \ldots, N\} : n = 1, \ldots, k\right\}$. Although, the choice of these families is independent in GP representation by w. In the same way, the formula in Lemma 4.17 is determined by only the choice of w and orthonormal families. Conversely, if we define a family $\{s_1, \ldots, s_N\}$ of operators on a Hilbert space $\mathcal{H}$ by Lemma 4.17 and equations (4.9), then we have a representation of $\mathcal{O}_N$. This style of definition of representation is a generalization of permutative representation ([5]).

5 Uniqueness, irreducibility and equivalence

5.1 Uniqueness of GP representation

Lemma 5.1 *Let $(\mathcal{H}, \{s_1, \ldots, s_N\}, \Omega)$ be the GP representation of $\mathcal{O}_N$ by non periodic $w \in S(\mathbf{C}^N)^{\otimes k}$ and $\{e_x : x \in \Lambda(w)\}$ the canonical basis in Proposition 4.16. For $x \in \Lambda(w) \cap S(\mathbf{C}^N)^{\otimes a}$, there are $m \in \mathbf{N}$ and $c \in \mathbf{C}$ such that*

$$(s(w)^*)^{m+M} e_x = \begin{cases} c \cdot e_1 & (a \equiv 0 \ mod\ k), \\ & \\ c \cdot (< w|w_j >)^M e_j & \left(\begin{array}{l} a \not\equiv 0 \ mod\ k \\ j \equiv k - a + 1 \ mod\ k \end{array} \right) \end{cases}$$

for each $M \geq 1$

Proof.
If $x \in \Lambda(w) \cap S(\mathbf{C}^N)^{\otimes a}$, $0 < a < k$, then let $m \equiv 1$ and $j \equiv k - a + 1$.

$$\begin{aligned} (s(w)^*)^m e_x = & \quad s(w)^* s(x)\Omega \\ = & \quad < w^{(1)} \otimes \cdots \otimes w^{(a)}|x > s(w^{(a+1)} \otimes \cdots \otimes w^{(k)})^*\Omega \\ = & \quad c' < w^{(1)} \otimes \cdots \otimes w^{(a)}|x > e_{k-a+1} \\ = & \quad c \cdot e_j \end{aligned}$$

by Lemma 4.1 (iv). Hence

$$(s(w)^*)^{m+M} e_x = c \cdot (s(w)^*)^M e_j = c\cdot < w|w_j >^M e_j$$

for each $M \geq 1$.
If $x \in \Lambda(w) \cap S(\mathbf{C}^N)^{\otimes lk}$, $l \geq 1$, then let $m \equiv l$

$$\begin{aligned} (s(w)^*)^m e_x = & \quad s(w^{\otimes l})^*(s(x)\Omega) \\ = & \quad < w^{\otimes l}|x > \Omega \\ = & \quad ce_1 \end{aligned} \tag{5.1}$$

where $c \equiv < w^{\otimes l}|x >$.
If $x \in \Lambda(w) \cap S(\mathbf{C}^N)^{\otimes (lk+a)}$, $0 < a < k$ and $l \geq 1$, then let $m \equiv l + 1$ and $j \equiv k - a + 1$. Then

$$\begin{aligned} (s(w)^*)^m e_x = & \quad s(w^{\otimes l} \otimes w)^*(s(x)\Omega) \\ = & \quad < \phi_{l,a}|x > s(w^{(a+1)} \otimes \cdots \otimes w^{(k)})^*\Omega \\ = & \quad < \phi_{l,a}|x > c' e_{k-a+1} \quad \text{(by Lemma 4.1 (iv))} \\ = & \quad ce_j \end{aligned}$$

where $\phi_{l,a}$ is in Lemma 4.7 and $c \equiv < \phi_{l,a}|x > c'$. Hence

$$(s(w)^*)^{m+M} e_x = c \cdot (s(w)^*)^M e_j = c \cdot < w|w_j >^M e_j$$

for each $M \geq 1$. ∎

Lemma 5.2 *Let $(\mathcal{H}, \{s_1, \ldots, s_N\}, \Omega)$ be the GP representation of $\mathcal{O}_N$ by non periodic $w \in S(\mathbf{C}^N)^{\otimes k}$. If $v \in \mathcal{H}$ satisfies $< v|\Omega >= 0$, then*

$$\lim_{m\to\infty} (s(w)^*)^m v = 0.$$

Proof. Any cycle and tree basis are written as

$$e_z = s(z)\Omega$$

for $z \in \Lambda(w)$ by Proposition 4.16. It is sufficient to consider the case $v = s(z)\Omega$. By Lemma 5.1, if $z \in S(\mathbf{C}^N)^{\otimes(lk+a)}$, $0 < a < k$, then

$$\begin{aligned}
\lim_{m\to\infty} (s(w)^*)^m v = & \lim_{M\to\infty} (s(w)^*)^{m+M} s(z)\Omega \\
= & \lim_{M\to\infty} c < w|w_{k-a+1} >^M e_{k-a+1} \\
= & 0.
\end{aligned}$$

because if $k-a+1 \neq 1$, then $| < w|w_{k-a+1} > | < 1$. Therefore it is sufficient to consider the case $z \in S(\mathbf{C}^N)^{\otimes lk}$, $l \geq 1$.

$$\begin{aligned}
(s(w)^*)^l e_z = & s(w^{\otimes l})^*(s(z)\Omega) \\
= & < w^{\otimes l}|z > \Omega \\
= & \begin{cases} < w|z > \Omega & (l = 1), \\ < w^{\otimes(l-1)}|z_1 >< w|z_2 > \Omega & (l \geq 2) \end{cases}
\end{aligned} \tag{5.2}$$

where $z = z_1 \otimes z_2$, $z_1 \in S(\mathbf{C}^N)^{\otimes(l-1)}$ and $z_2 \in S(\mathbf{C}^N)$ when $l \geq 2$. If $l = 1$, then

$$\begin{aligned}
z \in \ & S(\mathbf{C}^N)^{\otimes k} \cap \Lambda(w) \setminus \{w\} \\
= & \left\{ \begin{array}{c} w^{(1)}[n] \otimes w^{(2)} \otimes \cdots \otimes w^{(k)}, \\ \varepsilon_i \otimes w^{(2)}[n] \otimes w^{(3)} \otimes \cdots \otimes w^{(k)}, \\ \vdots \\ \varepsilon_{(i_1,\ldots,i_{k-2})} \otimes w^{(k-1)}[n] \otimes w^{(k)}, \\ \varepsilon_{(i_1,\ldots,i_{k-1})} \otimes w^{(k)}[n] \end{array} : \begin{array}{l} n = 2, \ldots, N, \\ i_j = 1, \ldots, N \\ j = 1, \ldots, k-1 \end{array} \right\}.
\end{aligned}$$

where we remove $w \in S(\mathbf{C}^N)^{\otimes k} \cap \Lambda(w)$ under assumption of this Lemma. By 5.2, $(s(w)^*)^l e_z = 0$ because $< w|z >= 0$ by choice of $w^{(i)}[n]$, $n = 2, \ldots, N$. If $l \geq 2$, z_2 is in $S(\mathbf{C}^N)^{\otimes k} \cap \Lambda(w) \setminus \{w\}$ by definition of $\Lambda(w)$. Therefore $(s(w)^*)^l e_z = 0$, too. Hence

$$\begin{aligned}\lim_{m\to\infty}(s(w)^*)^m v = & \lim_{M\to\infty}(s(w)^*)^M(s(w)^*)^l e_z \\ = & \ 0.\end{aligned}$$

∎

Corollary 5.3 *(Uniqueness of GP vector) Assume that $(\mathcal{H}, \pi)$ is a representation of $\mathcal{O}_N$. If $\Omega, \Omega^{'} \in \mathcal{H}$ are cyclic vectors by $\pi(\mathcal{O}_N)$ and satisfy the condition (3.7) with respect to common non periodic $w \in S(\mathbf{C}^N)^{\otimes k}$, then there is $c \in \mathbf{C}$ such that $\Omega = c\Omega^{'}$.*

Proof. We identify $\pi(s_i)$ and s_i here. We can denote

$$\Omega^{'} = c\Omega + y, \quad < \Omega|y >= 0, \quad c \in \mathbf{C}.$$

By assumption, $s(w)\Omega^{'} = \Omega^{'}$. Hence $s(w)^*\Omega^{'} = \Omega^{'}$ and

$$(s(w)^*)^n\Omega^{'} = \Omega^{'} \quad (n \geq 1).$$

By Lemma 5.2,

$$\begin{aligned}\Omega^{'} = & \lim_{n\to\infty}(s(w)^*)^n\Omega^{'} \\ = & \lim_{n\to\infty}c(s(w)^*)^n\Omega + \lim_{n\to\infty}(s(w)^*)^n y \\ = & \ c\Omega + 0.\end{aligned}$$

Hence $\Omega^{'} = c\Omega$. ∎

Recall the equivalence of GP representations in Definition 3.3 (ii).

Proposition 5.4 *(Uniqueness of GP representation) If $w \in TS(\mathbf{C}^N)$ is non periodic, then any two GP representations of $\mathcal{O}_N$ by w are equivalent each other.*

Proof. Assume that both $(\mathcal{H},\pi,\Omega)$ and $(\mathcal{H}^{'},\pi^{'},\Omega^{'})$ are GP representations by w. Fix orthonormal families $\{w^{(j)}[l] : l = 1,\ldots,N\}$, $j = 1,\ldots,k$, which is taken in Proposition 4.16 with respect to w. By Proposition 4.16, there are complete orthonormal bases (=CONB) $\{\pi(s(x))\Omega : x \in \Lambda(w)\}$, $\{\pi^{'}(s(x))\Omega^{'} : x \in \Lambda(w)\}$ of $\mathcal{H}$ and $\mathcal{H}^{'}$, respectively. Define a unitary

$$U : \mathcal{H} \to \mathcal{H}^{'}; \quad U\pi(s(x))\Omega \equiv \pi^{'}(s(x))\Omega^{'} \quad (x \in \Lambda(w)).$$

Then U is well defined because U maps CONB in $\mathcal{H}$ to that in $\mathcal{H}^{'}$. Note the set $\Lambda(w)$ of indexes is independent in the choice of representation.

If $x \in \Lambda(w) \setminus \coprod_{j=1}^{k} \Lambda_j^{(0)}(w)$, then

$$\begin{aligned} U\pi(s_i)U^*\pi^{'}(s(x))\Omega^{'} &= U\pi(s_i)\pi(s(x))\Omega \\ &= U\pi(s_i s(x))\Omega \\ &= U\pi(s(\varepsilon_i)s(x))\Omega \\ &= U\pi(s(\varepsilon_i \otimes x))\Omega \\ &= \pi^{'}(s(\varepsilon_i \otimes x))\Omega^{'} \\ &= \pi^{'}(s_i)\pi^{'}(s(x))\Omega^{'}. \end{aligned}$$

where we use (4.9). Hence $U\pi(s_i)U^* = \pi^{'}(s_i)$, $i = 1,\ldots,N$ on $\left(\oplus_{j=1}^{k}\mathcal{F}_j^{(0)}(w)\right)^{\perp}$.

If $x \in \Lambda_j^{(0)}(w)$, $j = 1,\ldots,k$, then

$$\begin{aligned} U\pi(s_i)U^*\pi^{'}(s(x))\Omega^{'} &= U\pi(s(\varepsilon_i \otimes x))\Omega \\ &= U\left(\sum_m a_m \pi(s(y_m))\Omega\right) \\ &= \sum_m a_m U\pi(s(y_m))\Omega \\ &= \sum_m a_m \pi^{'}(s(y_m))\Omega^{'} \\ &= \pi^{'}(s(\varepsilon_i \otimes x))\Omega^{'} \\ &= \pi^{'}(s_i)\pi^{'}(s(x))\Omega^{'} \end{aligned}$$

where $a_m \in \mathbf{C}$ and $y_m \in \Lambda(w)$ are determined by

$$\pi(s_i)\pi(s(x))\Omega = \sum_m a_m \pi(s(y_m))\Omega$$

in Lemma 4.17.

Hence $U\pi(s_i)U^* = \pi'(s_i)$ for each $i = 1, \ldots, N$. Therefore π and π' are equivalent. ∎

5.2 Irreducibility

Proposition 5.5 *If $w \in S(\mathbf{C}^N)^{\otimes k}$ is non periodic, then the GP representation of $\mathcal{O}_N$ by w is irreducible.*

Proof. Let $(\mathcal{H}, \{s_1, \ldots, s_N\}, \Omega)$ be the GP representation by w. Let $v \in \mathcal{H}$, $v \neq 0$. It is sufficient to show that the cyclic vector Ω is obtained from v by action of $\mathcal{O}_N$.

Because $\mathcal{H}$ is cyclic, there is $x \in \mathcal{O}_N$ such that $< x\Omega|v > \neq 0$. Therefore we can assume that $< \Omega|v >= 1$ by replacing v and x^*v and normalizing it. Denote

$$v = \Omega + y, \quad < \Omega|y >= 0.$$

By Lemma 5.2,

$$\begin{aligned}\lim_{n\to\infty}(s(w)^*)^n v = & \lim_{n\to\infty}(s(w)^*)^n\Omega + \lim_{n\to\infty}(s(w)^*)^n y \\ = & \ \Omega.\end{aligned}$$

Hence

$$\Omega \in \overline{\mathcal{O}_N v}.$$

Therefore $(\mathcal{H}, \{s_1, \ldots, s_N\})$ is irreducible. ∎

In [5], the non periodicity is necessary and sufficient condition of irreducibility of permutative representation. Although, in Definition 3.2, there is an irreducible case for periodic case, too. This difference occurs because of that of definition of permutative representation and GP representation. Under some additional condition, such necessary and sufficient condition holds. We explain the periodic case in the succeeding our paper [8].

5.3 Equivalence of GP representation

For two representations $(\mathcal{H}_1, \pi_1)$ and $(\mathcal{H}_2, \pi_2)$ of $\mathcal{O}_N$, $(\mathcal{H}_1, \pi_1) \sim (\mathcal{H}_2, \pi_2)$ means that $(\mathcal{H}_1, \pi_1)$ and $(\mathcal{H}_2, \pi_2)$ are unitarily equivalent.

Lemma 5.6 *Assume that $(\mathcal{H}, \pi)$ and $(\mathcal{H}', \pi')$ are representations of $\mathcal{O}_N$ and there are $x \in \mathcal{O}_N$ and $\Omega' \in \mathcal{H}'$ such that $\pi'(x)\Omega' = \Omega'$. If $(\mathcal{H}, \pi) \sim (\mathcal{H}', \pi')$, then $\pi(x)$ has eigen value 1.*

Proof. Denote $t_i \equiv \pi(s_i)$ and $t_i^{'} \equiv \pi^{'}(s_i)$ for $i = 1, \ldots, N$. If they are equivalent, then there is a unitary $U : \mathcal{H} \to \mathcal{H}^{'}$ such that

$$Ut_iU^* = t_i^{'} \quad (i = 1, \ldots, N).$$

If $x \in \mathcal{O}_N$ and $\Omega^{'} \in \mathcal{H}^{'}$ satisfy $\pi^{'}(x)\Omega^{'} = \Omega^{'}$, then a vector $\Omega \equiv U^*\Omega^{'} \in \mathcal{H}$ satisfies

$$\begin{aligned}\pi(x)\Omega = & \ U^*\pi^{'}(x)UU^*\Omega^{'} \\ = & \ U^*\pi^{'}(x)\Omega^{'} \\ = & \ U^*\Omega^{'} \\ = & \ \Omega.\end{aligned}$$

Hence $\pi(x)$ has an eigen vector Ω with eigen value 1. ∎

Corollary 5.7 *Let $(\mathcal{H}, \pi)$ and $(\mathcal{H}^{'}, \pi^{'})$ be representations of $\mathcal{O}_N$ and $x \in \mathcal{O}_N$. Assume that $\pi(x)$ has an eigen vector on $\mathcal{H}$. If there is no eigen vector of $\pi^{'}(x)$ on $\mathcal{H}^{'}$, then $(\mathcal{H}, \pi) \not\sim (\mathcal{H}^{'}, \pi^{'})$.*

Recall the notation $\sim$ in $TS(\mathbf{C}^N)$ and $GP(w)$ for $w \in TS(\mathbf{C}^N)$ in Definition 2.1 and 3.2.

Lemma 5.8 *Let $w, v \in TS(\mathbf{C}^N)$ be non periodic. If $w \sim v$, then $GP(w) \sim GP(v)$.*

Proof. Let $(\mathcal{H}, \pi, \Omega)$ and $(\mathcal{H}^{'}, \pi^{'}, \Omega^{'})$ be $GP(w)$ and $GP(v)$, respectively. Assume that $w \in S(\mathbf{C}^N)^{\otimes k}$. If $w \sim v$ then, there is $p \in \mathbf{Z}_k$ such that $v = \hat{p}(w)$. By Proposition 4.3, there is $\Omega^{''} \in \mathcal{H}$ such that $(\mathcal{H}, \pi, \Omega^{''})$ is $GP(\hat{p}(w)) = GP(v)$. Hence both $(\mathcal{H}, \pi, \Omega^{''})$ and $(\mathcal{H}^{'}, \pi^{'}, \Omega^{'})$ are $GP(v)$. By Proposition 5.4, $(\mathcal{H}, \pi)$ and $(\mathcal{H}^{'}, \pi^{'})$ are equivalent. Hence $GP(w) \sim GP(v)$. ∎

Lemma 5.9 *Assume that $v, w \in TS(\mathbf{C}^N)$ are non periodic and $v \not\sim w$. Let $(\mathcal{H}, \{s_1, \ldots, s_N\})$ be a representation of $\mathcal{O}_N$. If $\Omega, \Omega^{'} \in \mathcal{H}$ satisfy $s(w)\Omega = \Omega$ and $s(v)\Omega^{'} = \Omega^{'}$, then $< \Omega|\Omega^{'} >= 0$.*

Proof. Assume that $w \in S(\mathbf{C}^N)^{\otimes k}$ and $v \in S(\mathbf{C}^N)^{\otimes l}$. If $k \neq l$, then

$$< \Omega|\Omega^{'} >=< s(w)^l\Omega|s(v)^k\Omega^{'} >=< w^{\otimes l}|v^{\otimes k} >< \Omega|\Omega^{'} > .$$

By Proposition 2.5 (iii), $| < w^{\otimes l}|v^{\otimes k} > | < 1$. Hence $< \Omega|\Omega^{'} >= 0$.

Assume that $k = l$. If $| < w|v > | < 1$, $< \Omega|\Omega^{'} >= 0$ by the same argument in the above.

If $| < w|v > | = 1$, then we can denote $v = cw$ for $c \in \mathbf{C}$, $c \neq 1$. Then

$$s(v)\Omega = cs(w)\Omega = c\Omega.$$

Hence Ω and $\Omega^{'}$ are eigen vectors of $s(v)$ which have different eigen values each other. Hence $< \Omega|\Omega^{'} >= 0$. ∎

Lemma 5.10 *Assume that $v, w \in TS(\mathbf{C}^N)$ are non periodic. If $v \not\sim w$, then $GP(w) \not\sim GP(v)$.*

Proof. Assume that $w \not\sim v$. By assumption and Proposition 5.5, both $GP(w)$ and $GP(v)$ are irreducible. We can assume that $GP(w)$ and $GP(v)$ are represented on the common separable infinite dimensional Hilbert space $\mathcal{H}$ without loss of generality because $\mathcal{O}_N$ is a separable infinite dimensional C*-algebra. Hence we can denote

$$GP(w) = (\mathcal{H}, \{s_1, \ldots, s_N\}, \Omega), \quad GP(v) = (\mathcal{H}, \{t_1, \ldots, t_N\}, \Omega^{'}). \tag{5.3}$$

We assume that $GP(w) \sim GP(v)$ and imply contradiction. If $GP(w) \sim GP(v)$, then we can assume that $t_i = s_i$ for each $i = 1, \ldots, N$ in (5.3) by Proposition 5.4.

Therefore we can denote

$$GP(w) = (\mathcal{H}, \{s_1, \ldots, s_N\}, \Omega), \quad GP(v) = (\mathcal{H}, \{s_1, \ldots, s_N\}, \Omega^{'}).$$

Note

$$s(w)\Omega = \Omega, \quad s(v)\Omega^{'} = \Omega^{'} \tag{5.4}$$

by Definition 3.2. By the canonical basis of $GP(w)$, we can denote

$$\Omega^{'} = \sum_{i=1}^{k} a_i e_i + y$$

where $e_1 = \Omega$ and $y \in \mathcal{H}$ such that $< e_i|y >= 0$ for each $i = 1, \ldots, k$, $a_j \in \mathbf{C}$. Since $w \not\sim v$, $w_j = \hat{\sigma}^{j-1}(w) \not\sim v$ for $j = 1, \ldots, k$. Note e_j is GP vector for $s(w_j)$. By Lemma 5.9, $< \Omega^{'}|e_j >= 0$. Hence $\Omega^{'} = y$. Therefore $\Omega \in \left(\oplus_{j=1}^{k}\mathcal{F}_j^{(0)}(w)\right)^{\perp}$. By the canonical basis, there is the smallest $m \geq 1$ such that $u \in \mathcal{F}_j^{(m)}(w)$ and $< u|\Omega^{'} >\neq 0$. Then

$$\Omega^{'} \in \bigoplus_{m^{'} \geq m} \bigoplus_{j=1}^{k} \mathcal{F}_j^{(m^{'})}(w).$$

On the other hand

$$s(v)\Omega' = \sum_{i=1}^{N} v_i s_i \Omega' \in \bigoplus_{m' \geq m+1} \bigoplus_{j=1}^{k} \mathcal{F}_j^{(m')}(w)$$

by (4.9). Hence $s(v)\Omega' \neq \Omega'$. This is contradiction. Hence $GP(w) \not\sim GP(v)$. ∎

By combining Lemma 5.8 and 5.10, we have the following statement.

Proposition 5.11 *(Equivalence of GP representation with cycle) Let $w, v \in TS(\mathbf{C}^N)$ be non periodic. There is the following equivalence:*

$$GP(w) \sim GP(v) \quad \Leftrightarrow \quad w \sim v.$$

6 Application

6.1 GP state

In usual theory of operator algebra, the notion of state is often treated rather than representation of algebra. We show the relation between GP representation and state of Cuntz algebra.

Proposition 6.1 *(Representation and state) Let $w \in S(\mathbf{C}^N)^{\otimes k}$ be non periodic.*

The GP representation of $\mathcal{O}_N$ by w is equivalent to the GNS representation by a state ρ of $\mathcal{O}_N$ which satisfies the following equation:

$$\rho(s_I s_J^*) = \begin{cases} \overline{w(I)} \cdot w(J) & (|I| - |J| = 0 \ mod\ k), \\ \\ 0 & (otherwise) \end{cases} \tag{6.1}$$

for each $I, J \in \bigcup_{m \geq 0} \{1, \ldots, N\}^m$ where

$$w(I) \equiv \prod_{j=1}^{m} w_{i_j}^{(\sigma^{j-1}(1))}$$

for $I = (i_1, \ldots, i_m) \in \{1, \ldots, N\}^m$, $m \geq 1$, σ is the shift in $\mathbf{Z}_k$ under the following convention:

$$s_I s_J^* = \begin{cases} s_J^* & (I = \emptyset), \\ \\ s_I & (J = \emptyset), \end{cases}$$

$$w_I = \quad 1 \quad (I = \emptyset).$$

Proof. Let $(\mathcal{H},\pi,\Omega)$ be the GP representation by $w\in S(\mathbf{C}^N)^{\otimes k}$. By Proposition 5.5, $(\mathcal{H},\pi,\Omega)$ is irreducible. Hence any vector state of $\pi(\mathcal{O}_N)$ on $\mathcal{H}$ is pure. Therefore the GNS representation of any pure state associated with vector state on $\mathcal{H}$ is irreducible and unitarily equivalent to $(\mathcal{H},\pi,\Omega)$. A state ρ on $\mathcal{O}_N$ is uniquely determined by the condition (6.1) because it is densely defined on $\mathcal{O}_N$.

Put

$$\rho\equiv <\Omega|\pi(\cdot)\Omega> .$$

It is sufficient to show that ρ satisfies the condition (6.1).

Assume that $|I|=kn+a$ and $|J|=kn^{'}+a^{'}$, $n,n^{'}\geq 0$ and $0\leq a,a^{'}<k$. By Lemma 4.7,

$$\begin{aligned}\rho(s_Is_J^*) =& <\Omega|\pi(s_Is_J^*)\Omega>\\ =& <\pi(s_I^*)\Omega|\pi(s_J^*)\Omega>\\ =& \Big\langle <\varepsilon_I|\phi_{n,a}>e_{a+1}\Big|<\varepsilon_J|\phi_{n^{'},a^{'}}>e_{a^{'}+1}\Big\rangle\\ =& \overline{<\varepsilon_I|\phi_{n,a}>}<\varepsilon_J|\phi_{n^{'},a^{'}}><e_{a+1}|e_{a^{'}+1}>\end{aligned}$$

where we use symbol $\phi_{n,a}$ in Lemma 4.7. When $a\neq 0$,

$$\begin{aligned}<\varepsilon_I|\phi_{n,a}>=& \Big(<\varepsilon_{i_1}|w^{(1)}><\varepsilon_{i_2}|w^{(2)}>\cdots<\varepsilon_{i_k}|w^{(k)}>\Big)\\ &\times\Big(<\varepsilon_{i_{k+1}}|w^{(1)}><\varepsilon_{i_{k+2}}|w^{(2)}>\cdots<\varepsilon_{i_{2k}}|w^{(k)}>\Big)\\ &\times\cdots\\ &\times\Big(<\varepsilon_{i_{(n-1)k+1}}|w^{(1)}><\varepsilon_{i_{(n-1)k+2}}|w^{(2)}>\cdots<\varepsilon_{i_{nk}}|w^{(k)}>\Big)\\ &\times<\varepsilon_{i_{nk+1}}|w^{(1)}><\varepsilon_{i_{nk+2}}|w^{(2)}>\cdots<\varepsilon_{i_{nk+a}}|w^{(a)}>\\ =& w^{(1)}_{i_1}\cdots w^{(k)}_{i_k}\times w^{(1)}_{i_{k+1}}\cdots w^{(k)}_{i_{2k}}\times\cdots\times w^{(1)}_{i_{(n-1)k+1}}\cdots w^{(k)}_{i_{nk}}\\ &\times w^{(1)}_{i_{nk+1}}\cdots w^{(a)}_{i_{nk+a}}\\ =& w(I).\end{aligned}$$

When $a=0$,

$$<\varepsilon_I|\phi_{n,0}>=<\varepsilon_I|w^{\otimes kn}>=<\varepsilon_{i_1,\ldots,i_k}|w>\cdots<\varepsilon_{i_{(n-1)k+1},\ldots,i_{nk}}|w>=w(I).$$

Hence $<\varepsilon_I|\phi_{n,a}>= w(I)$ for each $n \geq 0$ and $0 \leq a < k$. Therefore

$$\begin{aligned}
\rho(s_I s_J^*) = & \ \overline{<\varepsilon_I|\phi_{n,a}>} <\varepsilon_J|\phi_{n',a'}><e_{a+1}|e_{a'+1}> \\
= & \ \overline{w(I)} \cdot w(J) \cdot \delta_{a,a'} \\
= & \begin{cases} \overline{w(I)} \cdot w(J) & (a-a'=0), \\ 0 & (\text{otherwise}), \end{cases} \\
= & \begin{cases} \overline{w(I)} \cdot w(J) & (|I|-|J|'=0 \bmod k), \\ 0 & (\text{otherwise}). \end{cases}
\end{aligned}$$

∎

We call *the GP state of* $\mathcal{O}_N$ *by* w a state which is defined by (6.1).

Corollary 6.2 *Let* $N \geq 2$ *and* $w \in S(\mathbf{C}^N)^{\otimes k}$. *Assume that* ρ_w *is a state of* $\mathcal{O}_N$ *which satisfies the condition (6.1).*

(i) *If* w *is non periodic, then* ρ_w *is pure.*

(ii) *Assume that* w, w' *are non periodic. Then the GNS reprsentations associated with* ρ_w *and* $\rho_{w'}$ *are equivalent if and only if* $w \sim w'$.

(iii) *If* $k=1$, *then* ρ_w *is always pure.*

(iv) *If* $k=1$, *then for any two* w, w', *associated GNS representations by* ρ_w *and* $\rho_{w'}$ *are inequivalent when* $w \neq w'$.

In this way, we obtain many concrete pure states of $\mathcal{O}_N$ from non periodic $w \in TS(\mathbf{C}^N)$.

6.2 Example

Example 6.3 (i) Recall an example which is defined by an equation (1.3) in section 1. By Proposition 5.5, the GP representation in (1.3) is irreducible because $w \in S(\mathbf{C}^2)^{\otimes 2}$ in (1.3) is non periodic. Since any permutative representation of $\mathcal{O}_2$ with cycle is given by the case $w = \varepsilon_I$, $I \in \{1,2\}^k$, (1.3) is not equivalent to any permutative representation with cycle by Proposition 5.11. Furthermore, if α_g is a natural automorphism of $\mathcal{O}_2$ associated with $g = (g_{ij}) \in U(2)$, then the permutative representation $GP(\varepsilon_I)$ associated with ε_I is changed to $GP(v)$ by α_g as following $v \in S(\mathbf{C}^N)^{\otimes k}$:

$$v = v^{(1)} \otimes \cdots \otimes v^{(k)},$$

$$v^{(j)} = g^*_{1i_j}\varepsilon_1 + g^*_{2i_j}\varepsilon_2 \quad (j = 1, \ldots, k)$$

when $I = (i_1, \ldots, i_k)$. Since (1.3) has the length 2, it is sufficient to consider the case

$$v = v^{(1)} \otimes v^{(2)},$$

$$v^{(1)} = g^*_{1i_1}\varepsilon_1 + g^*_{2i_1}\varepsilon_2, \quad v^{(2)} = g^*_{1i_2}\varepsilon_1 + g^*_{2i_2}\varepsilon_2$$

for $I = (i_1, i_2)$. If w in (1.3) and v are equivalent, then $g^*_{21} = 0$ or $g^*_{22} = 0$. Then

$$g^* = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & c_1 \\ c_2 & 0 \end{pmatrix}.$$

Hence v is one of the followings:

$$a\varepsilon_1 \otimes \varepsilon_1, \quad a\varepsilon_1 \otimes \varepsilon_2, \quad a\varepsilon_2 \otimes \varepsilon_1, \quad a\varepsilon_2 \otimes \varepsilon_2$$

where $a \in U(1)$. Hence v is not equivalent to w. Therefore, $GP(w)$ is not equivalent to any permutative representation with cycle which is rotated $U(2)$-action by Proposition 5.11.

(ii) Because any $w \in S(\mathbf{C}^N)$ is non periodic, a cyclic representation of $\mathcal{O}_N$ with the cyclic vector Ω which satisfies $s(w)\Omega = \Omega$ is irreducible by Proposition 5.5. Because any two different elements in $S(\mathbf{C}^N)$ are not equivalent, GP representations associated with them are not equivalent each other by Proposition 5.11.

(iii) For $k \geq 1$, a cyclic representation of $\mathcal{O}_N$ with the cyclic vector Ω which satisfies

$$(s_1 + s_2)(s_1 + \xi s_2)(s_1 + \xi^2 s_2) \cdots (s_1 + \xi^{k-1} s_2)\Omega = 2^{k/2}\Omega$$

is irreducible where $\xi \equiv e^{2\pi\sqrt{-1}/k}$.

6.3 Spectrum of $\mathcal{O}_N$

We summarize our result by the word "spectrum" of $\mathcal{O}_N$. Let Spec$\mathcal{O}_N$ be the set of all unitary equivalence classes of irreducible representations of $\mathcal{O}_N$, that is

$$\mathrm{Spec}\mathcal{O}_N \equiv \mathrm{IrrRep}\mathcal{O}_N/\sim .$$

On the other hand, denote

$$TS_P(\mathbf{C}^N) \equiv \{w \in TS(\mathbf{C}^N) : w \text{ is periodic }\}.$$

Then

$$TS_P(\mathbf{C}^N) = \left\{v^{\otimes k} \in TS(\mathbf{C}^N) : v \in TS(\mathbf{C}^N),\, k \geq 2\right\}.$$

For example, $\varepsilon_1 \otimes \varepsilon_1$, $\varepsilon_1 \otimes \varepsilon_1 \otimes \varepsilon_1$, $\varepsilon_1 \otimes \varepsilon_2 \otimes \varepsilon_1 \otimes \varepsilon_2$, $\varepsilon_1 \otimes \varepsilon_1 \otimes \varepsilon_2 \otimes \varepsilon_1 \otimes \varepsilon_1 \otimes \varepsilon_2$ are in $TS_P(\mathbf{C}^N)$. If $TS_{NP}(\mathbf{C}^N)$ is the set of all non periodic elements in $TS(\mathbf{C}^N)$, then

$$TS_{NP}(\mathbf{C}^N) = TS(\mathbf{C}^N) \setminus TS_P(\mathbf{C}^N)$$

by definition of non periodicity. Recall the equivalence relation $\sim$ on $TS(\mathbf{C}^N)$ in Definition 2.1 (iv).

Theorem 6.4 *There is an injective map*

$$\widetilde{GP} : TS_{NP}(\mathbf{C}^N)/\sim \quad \hookrightarrow \quad \mathrm{Spec}\mathcal{O}_N.$$

Proof. For $w \in TS_{NP}(\mathbf{C}^N)$, $GP(w)$ is irreducible by Proposition 5.5. Hence we have a map

$$GP : TS_{NP}(\mathbf{C}^N) \to \mathrm{IrrRep}\mathcal{O}_N$$

where we do not distinguish the cyclic vector in $GP(w)$. From this map, we have the natural map

$$\widehat{GP} : TS_{NP}(\mathbf{C}^N) \quad \to \quad \mathrm{IrrRep}\mathcal{O}_N/\sim \quad = \quad \mathrm{Spec}\mathcal{O}_N.$$

By Proposition 5.11, this map is well defined on the quotient space $TS_{NP}(\mathbf{C}^N)/\sim$. In this way, we have an injective map

$$\widetilde{GP} : TS_{NP}(\mathbf{C}^N)/\sim \quad \hookrightarrow \quad \mathrm{Spec}\mathcal{O}_N.$$

∎

Here we try to explain a part of $TS_{NP}(\mathbf{C})$ by using geometric realization. Because any element in $S(\mathbf{C}^N)$ is non periodic and any two different elements in $S(\mathbf{C}^N)$ are inequivalent, we can identify $S(\mathbf{C}^N)$ and $S_{NP}(\mathbf{C}^N)/\sim \ \equiv \ \left(S(\mathbf{C}^N) \cap TS_{NP}(\mathbf{C}^N)\right)/\sim$. Hence $\widetilde{GP}([w])$ and $GP(w)$ can be identified for each $w \in S(\mathbf{C}^N)$. Therefore $S(\mathbf{C}^N)$ can be regarded as a (complex)sphere which consists of spectrums of $\mathcal{O}_N$. In other word, $S(\mathbf{C}^N)$ is embedded into Spec$\mathcal{O}_N$.

Although, this can be obtained from ordinary permutative representations ([5]) by rotation of $U(N)$. Furthermore by $U(N)$ action on Spec$\mathcal{O}_N$, $S(\mathbf{C}^N)$ is an orbit of spectrums. $\{\varepsilon_1\} \times \left(S(\mathbf{C}^N) \setminus \{\varepsilon_1\}\right)$ is regarded as a subset of Spec$\mathcal{O}_N$ in the similar reason.

This study is shown in succeeding our paper([10]).

Note: In this paper, we don't treat the case "chain". Hence there are many elements in the spectrum of $\mathcal{O}_N$ except $TS_{NP}(\mathbf{C}^N)/\sim$. Our ultimate aim is to describe any element in Spec$\mathcal{O}_N$ by this way.

6.4 Other topics

There are several applications of permutative representation in quantum field theory [1, 2, 3, 4]. By restricting permutative representation of $\mathcal{O}_2$ on $CAR \equiv \mathcal{O}_2^{U(1)}$, we have many formulae of representation of CAR and its irreducible decomposition formulae.

Furthermore we have a class of endomorphisms of Cuntz algebra by combinatrix method. In order to analyze them, the permutative representation and its theory are useful. We treat this work in the succeeding our papar.

Appendix

A Lemmata

Lemma A.1 *If $w = w^{(1)} \otimes \cdots \otimes w^{(k)}, v = v^{(1)} \otimes \cdots \otimes v^{(k)} \in S(\mathbf{C}^N)^{\otimes k}$ are linearly dependent, then $w^{(i)}$ and $v^{(i)}$ are linearly dependent for each $i = 1, \ldots, k$, too.*

Proof. If v and w are linearly dependent, then there is $c \in \mathbf{C}$, $|c| = 1$, such that $v = cw$ because $\|v\| = 1 = \|w\|$. On the other hand,

$$\begin{aligned} 1 = & \; | < w|v > | \\ = & \; | < w^{(1)} \otimes \cdots \otimes w^{(k)} | v^{(1)} \otimes \cdots \otimes v^{(k)} > | \\ = & \; \left| \prod_{i=1}^{k} < w^{(i)} | v^{(i)} > \right| \\ = & \; \prod_{i=1}^{k} \left| < w^{(i)} | v^{(i)} > \right| . \end{aligned}$$

Because

$$| < w^{(i)} | v^{(i)} > | \leq \|w^{(i)}\| \, \|v^{(i)}\| = 1,$$

$| < w^{(i)} | v^{(i)} > | = 1$. Hence $w^{(i)}$ and $v^{(i)}$ are linearly dependent for each $i = 1, \ldots, k$. ∎

Lemma A.2 *If $w \in S(\mathbf{C}^N)^{\otimes k}$ is non periodic, then $\hat{p}(w)$ and w are linearly independent for each $p \in \mathbf{Z}_k \setminus \{id\}$.*

Proof. Assume that $\hat{p}(w)$ and w are linearly dependent. By Lemma, A.1, $w^{(i)}$ and $w^{(p(i))}$ are linearly dependent for $i = 1, \ldots, k$. Let $c_i \in \mathbf{C}$ by

$$w^{(p(i))} = c_i w^{(i)} \quad (i = 1, \ldots, k).$$

If p is a generator of $\mathbf{Z}_k$, then there are $\{d_i\}_{i=1}^{k}$ such that

$$w^{(i)} = d_i w^{(1)}.$$

Hence

$$w = (d_1 \cdot d_k)(w^{(1)})^{\otimes k}.$$

Therefore w is periodic. This is contradiction.

Assume that there is $0 < m < k$ such that $p^m = id$. Let $M \equiv k/m$. Then

$$w = (c^{'})^M \left(w^{(1)} \otimes \cdots \otimes w^{(m)} \right)^{\otimes M}$$

where $c^{'} \equiv c_1 \cdots c_m$. From this, w is periodic. This is contradiction. ∎

Lemma A.3 *If $w \in S(\mathbf{C}^N)^{\otimes k}$ and $v \in S(\mathbf{C}^N)^{\otimes l}$ are non periodic and $l \neq k$, then $w^{\otimes l}$ and $v^{\otimes k}$ are linearly independent in $(\mathbf{C}^N)^{\otimes lk}$.*

Proof. We can assume that $l > k$ without loss of generality. Then we can denote $l = nk + a$, $0 \leq a < k$, $n \geq 1$. Denote $w = w^{(1)} \otimes \cdots \otimes w^{(k)}$, $v = v^{(1)} \otimes \cdots \otimes v^{(l)}$. If they are linearly dependent, then there is $c \in \mathbf{C}$ such that

$$w^{\otimes l} = cv^{\otimes k}.$$

From this, there are $\{c_m\} \subset \mathbf{C}$ such that

$$w^{(\sigma^m(1))} = c_{m+1} v^{(\eta^m(1))} \quad (0 \leq m \leq kl - 1)$$

where σ and η are shifts of $\mathbf{Z}_k$ and $\mathbf{Z}_l$, respectively. Note

$$v^{(j)} = \bar{c}_j w^{(\sigma^{j-1}(1))} \quad (j = 1, \ldots, l). \tag{A.1}$$

For $0 \leq n^{'} \leq n - 1$,

$$\begin{aligned} v^{(n^{'}k+i)} &= \bar{c}_{n^{'}k+i} w^{(\sigma^{n^{'}k+i-1}(1))} \\ &= \bar{c}_{n^{'}k+i} w^{(\sigma^{i-1}(1))} \quad (\text{ by } \sigma^k = id) \\ &= c_i \bar{c}_{n^{'}k+i} v^{(i)} \qquad (\text{ by (A.1)}). \end{aligned}$$

Therefore for $m = 1, \ldots, kl$, $i = 1, \ldots, k$, there is $c^{'} \in \mathbf{C}$ such that

$$v^{(m)} = c^{'} v^{(i)} \qquad m = i \text{ mod k}.$$

Hence v is periodic. This is contradiction. Therefore v and w are linearly independent. ∎

Lemma A.4 *If $v, w \in S(\mathbf{C}^N)^{\otimes k}$ are linearly independent, then $v^{\otimes l}$ and $w^{\otimes l}$ are linearly independent, too for each $l \geq 2$.*

Proof. If they are linearly dependent, then there is $c \in \mathbf{C}$ such that

$$w^{\otimes l} = cv^{\otimes l}.$$

$$c =< v^{\otimes l} | w^{\otimes l} >= (< v | w >)^l.$$

Hence

$$| < v | w > | = 1.$$

Therefore v and w are linearly dependent. This is contradiction. ∎

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